THE ALGEBRAIC DETERMINATION OF THE TOPOLOGICAL TYPE OF THE COMPLEMENT OF A KNOT

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The group system of a tame knot [1] consists of the fundamental group of the complement of the knot, along with the conjugacy class of subgroups determined by the fundamental group of the boundary of a small tube around the knot. If the knot is nontrivial, these subgroups are free abelian of rank 2, as was correctly pointed out, but incorrectly proved by M. Dehn in 1910 [2]. In fact this was the purpose of Dehn's Lemma, which was not properly proven until 1957, when Papakyriakopoulos gave a complete proof [3].

It has long been believed that the group system of a knot in S^3 determines the topological type of the complement. Here it will be shown that *if the commutator subgroup of a knot group is finitely generated, then the group system determines the topological type of the complement.*

Let k denote a tame knot in S^3, and let $M(k)$ denote the closure of the complement of a regular neighborhood of k. Then it is well known that the topological type of $M(k)$ depends only on the type of k. (See for example [4].) *Now suppose k_1, k_2, two tame knots in S^3 are given, and there exists an isomorphism p, mapping $\pi_1(M(k_1))$ onto $\pi_1(M(k_2))$ and mapping the conjugacy class of $\pi_1(\dot{M}(k_1))$ onto that of $\pi_1(\dot{M}(k_2))$.* We may assume without loss of generality that the base points, x_1, x_2, for $\pi_1(M(k_1))$, $\pi_1(M(k_2))$ are located on the boundary in each case; also by a suitable automorphism of $\pi_1(M(k_2))$, p may be assumed to map $\pi_1(\dot{M}(k_1)) \subset \pi_1(M(k_1))$ onto $\pi_1(\dot{M}(k_2)) \subset \pi_1(M(k_2))$.

Let m_1, l_1, denote a meridian and longitude of $\dot{M}(k_1)$, (by a meridian is meant a simple representative generating $H_1(M(k_1))$, and homotopic to 0 in $S^3 - \text{int}(M(k_1))$, and by a longitude is meant a simple curve which bounds in $M(k_1)$, and with m_1 generates $\pi_1(\dot{M}(k_1))$, let m_2, l_2 denote a meridian and longitude of $M(k_2)$. Then

$$p[m_1] = \pm [m_2] + r[l_2],$$

and

$$p[l_1] = \pm [l_2].$$

It is easy to see that there exists a homeomorphism f from $\dot{M}(k_1)$ to $\dot{M}(k_2)$ such that $f^\# = p$.

Received by the editors November 18, 1960.

[1] The author held an N.S.F. Postdoctoral fellowship (49164).

Suppose that $[\pi_1, \pi_1]$ *is finitely generated.*

Let S_1', S_2' denote surfaces of minimal genus [5] spanning k_1, k_2 respectively and chosen so that $\dot{M}(k_i)\cap S_i' = l_i$. According to [6] these surfaces are of the same genus. $M(k_i)\cap S_i'$ will be denoted S_i.

We now wish to extend f to a homeomorphism f_3 from $M(k_1)$ to $M(k_2)$: we first extend f to a homeomorphism f_1 from $\dot{M}(k_1)\cup S_1$ to $\dot{M}(k_2)\cup S_2$, such that $(f_1|\, S_1)^\# = p|\, [\pi_1, \pi_1]$, where this last equation has meaning by virtue of [6], where it is proven that $\pi_1(S_1)$ generates $[\pi_1, \pi_1]$ if $[\pi_1, \pi_1]$ is finitely generated. More formally, f_1 will be constructed so that the following diagram is commutative:

$$\begin{array}{ccc} [\pi_1, \pi_1] & \xrightarrow{p|\,[\pi_1, \pi_1]} & p[\pi_1, \pi_1] \\ i\uparrow & & \uparrow j \\ \pi_1(S_1) & \xrightarrow{(f_1|\,S_1)^\#} & \pi_1(S_2). \end{array}$$

Here i and j are isomorphisms by [6].

Let ϕ denote any homeomorphism of S_1 to S_2 which agrees with f on $S_1\cap \dot{M}(k_1) = \dot{S}_1$. (Such exists since S_1 and S_2 are of the same genus.) Now $\phi^\# \circ i^{-1} \circ p^{-1} \circ j$ defines an automorphism of $\pi_1(S_2)$ which leaves $\pi_1(S_2\cap \dot{M}(k_2)) \subset \pi_1(S_2)$ fixed, so by attaching a disc D to S_2 along $\dot{S}_2$ and applying Nielson's theorem [7] to the induced automorphism on $S_2\cup D$ it is easily seen that there exists a homeomorphism $h: S_2\cup D \to S_2\cup D$ leaving D fixed such that $(h|\, S_2)^\# = \phi^\# \circ i^{-1} \circ p^{-1} \circ j$. Now define $f_1|\, S_1 = h^{-1} \circ \phi$. Thus $(f_1|\, S_1)^\# = j^{-1} \circ p \circ i$. Since $h^{-1} \circ \phi = f$ on $S_1\cap \dot{M}(k_1)$, $h^{-1} \circ \phi$ may be extended to a homeomorphism f_1 on $\dot{M}(k_1)\cup S_1$, by defining f_1 on $\dot{M}(k_1)$ to be equal to f, and f_1 on S_1 to equal $h^{-1} \circ \phi$. It is easy to see that f_1 may be extended to a homeomorphism f_2 from a nice small neighborhood N of $\dot{M}(k_1)\cup S_1$ to a nice small neighborhood of $\dot{M}(k_2)\cup S_2$. Now $\dot{N} - \dot{M}(k_1)$ is a 2-manifold, and clearly the component of $S^3 - (\dot{N} - \dot{M}(k_1))$ which does not contain k_1 is homeomorphic to $S^3 - S_1$, but by [6] $\pi_1(S^3 - S_1)$ is free, and an easy application of the Loop theorem [8] and the Dehn Lemma [3] shows that the component of $S^3 - (\dot{N} - \dot{M}(k_1))$ not containing k_1 is a solid torus.

The problem has now been reduced to extending f_2, a homeomorphism defined on the boundary of a solid torus to a homeomorphism of the solid torus, more explicitly, f_2 maps N into a homeomorphic neighborhood of $\dot{M}(k_2)\cup S_2$, and the complements of N, and its heomeomorph are solid tori, whose boundaries T_1, T_2 are homeomorphic to the map f_2.

An observation of Smale which may be applied here is the following:

A homeomorphism H from the boundary of one solid torus R_1 to another, R_2, may be extended to the interior if and only if $H^{\#}$ maps the kernel of the map $\iota^{\#}: \pi_1(\dot{R}_1)\rightarrow\pi_1(R_1)$ onto the kernel of the map $\eta^{\#}: \pi_1(\dot{R}_2)\rightarrow\pi_1(R_2)$.[2]

First we show that f_2 maps loops (on T_1) homotopic to 0 in S^3-N into loops (on T_2) homotopic to 0 in $S^3-f_2(N)$.

Suppose α is a loop on T_1 not homotopic to 0 on T_1, but homotopic to 0 in S^3-N. If $f_2(\alpha)$ is not homotopic to 0 in $S^3-f_2(N)$, $f_2(\alpha)$ determines a nontrivial conjugacy class of $p([\pi_1, \pi_1])$ by [6]. But by assumption α is homotopic to 0 in S^3-N, hence it determines the trivial conjugacy class in $[\pi, \pi]$ and so the trivial conjugacy class in $p([\pi, \pi])$. Arguing similarly with f_2^{-1}, allows the above observation to be applied, so that f_2 *may be extended to* f_3, *which maps* $M(k_1)$ *homeomorphically onto* $M(k_2)$.

REMARK. If the appropriate statements are made concerning the orientation of $\dot{M}(k_1)$ and $\dot{M}(k_2)$, and the isomorphism p, then $M(k_1)$ may be mapped homeomorphically onto $M(k_2)$ by a map preserving the orientation of the boundaries.

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[2] This follows easily from Dehn's Lemma.