NOVIKOV HOMOLOGY, TWISTED ALEXANDER POLYNOMIALS AND THURSTON CONES

A.V.PAJITNOV

Abstract. Let M be a connected CW complex, let G denote the fundamental group of M. Let π be an epimorphism of G onto a free finitely generated abelian group H, let $\xi : H \to \mathbf{R}$ be a homomorphism and ρ be an anti-homomorphism of G to the group $GL(V)$ of automorphisms of a free finitely generated R-module V (where R is a commutative factorial ring).

We associate to these data the *twisted Novikov homology* of M which is a module over the Novikov completion of the ring $\Lambda = R[H]$. The twisted Novikov homology provides the lower bounds for the number of zeros of any Morse form which cohomology class equals $\xi \circ \pi$. This construction generalizes the work by H.Goda and the author [11].

In the case when M is a compact connected 3-manifold M with zero Euler characteristic we obtain a criterion of vanishing of the twisted Novikov homology of M in terms of the corresponding twisted Alexander polynomial of the group G.

We discuss the relations of the twisted Novikov homology with the Thurston norm on the 1-cohomology of M.

1. Introduction

Let M be a closed manifold, $f : M \to S^1$ be a circle-valued Morse function on M. Let $m_k(f)$ denote the number of critical points of f of index k. The Morse-Novikov theory provides lower bounds for the numbers $m_k(f)$ which are computable in terms of the homotopy type of M and the homotopy class of f. The general schema of obtaining such bounds is as follows (see [19], [6], [21], [23]). Consider a regular covering $\mathcal{P} : \bar{M} \to M$ with structure group G, such that the function $f \circ \mathcal{P} : \bar{M} \to S^1$ is homotopic to zero (or equivalently, f lifts to a Morse function $\bar{M} \to \mathbf{R}$). The induced homomorphism $f_* : \pi_1(M) \to \mathbf{Z}$ can be factored through a homomorphism $\xi = \xi(f) : G \to \mathbf{Z}$.

Applying the standard method of counting gradient flow lines (see [39]) one obtains a chain complex $\mathcal{N}_*$ (*the Novikov complex*) over a certain completion $\widehat{\mathcal{L}}_\xi$ of the group ring $\mathcal{L} = \mathbf{Z}[G]$. [1]) The Novikov

[1]See the definition of $\widehat{\mathcal{L}}_\xi$ in Subsection 2.1.

complex is freely generated over $\widehat{\mathcal{L}}_\xi$ by the critical points of ω, and its homology is isomorphic to the homology of the tensor product

$$\widehat{\mathcal{C}}_*(\bar{M},\xi) = \widehat{\mathcal{L}}_\xi \underset{\mathcal{L}}{\otimes} \mathcal{C}_*(\bar{M}), \tag{1}$$

where $\mathcal{C}_*(\bar{M})$ is the cellular chain complex of the covering $\bar{M}$. In particular, if the chain complex (1) is not contractible, the function f must have at least one critical point. Developing further this observation, one can obtain lower bounds for the numbers $m_k(f)$ in terms of the numerical invariants of the homology of the chain complex (1).

Let us discuss different possible choices of the covering $\mathcal{P}$. The universal covering $\widetilde{M} \to M$ contains certainly the maximum of information. The disadvantage is that the corresponding ring $\widehat{\mathcal{L}}_\xi$, being a completion of the group ring of the fundamental group may be very complicated, and it can be difficult to extract the necessary numerical data.

Another obvious possibility is the infinite cyclic covering

$$\mathcal{P}_f : \bar{M}_f \to M$$

induced by f from the universal covering $\mathbf{R} \to S^1$. Here the ring $\widehat{\mathcal{L}}_\xi$ is a principal ideal domain, and the explicit lower bounds for the numbers of the critical points are easy to deduce (see Subsection 2.1 for more details).

An intermediate choice is the maximal free abelian covering

$$\mathcal{P}_{ab} : \overline{M}_{ab} \to M.$$

The structure group of this covering is equal to $H_1(M)/Tors$. The Novikov ring in this case is a completion of the Laurent polynomial ring in several variables, and its homology properties are in general rather complicated. But this choice has an advantage that for any function $f : M \to S^1$ the function $f \circ \mathcal{P}_{ab}$ is homotopic to zero. This allows to study the dependance of the Novikov homology on the class

$$\xi = \xi(f) \in \mathrm{Hom}(H_1(M)/Tors, \mathbf{Z}) \approx H^1(M,\mathbf{Z}).$$

In particular one can get some information about the set of all ξ such that the Novikov homology

$$\widehat{H}^{ab}_*(M,\xi) = H_*\Big(\widehat{\mathcal{C}}_*(\overline{M}_{ab},\xi)\Big)$$

vanishes and, therefore, obtain an information about the set of classes in $H^1(M,\mathbf{Z}) = [M,S^1]$ representable by fibrations.

Definition 1.1. Let H be a finitely generated free abelian group. Put

$$H_{\mathbf{R}} = H \otimes \mathbf{R}, \quad H'_{\mathbf{R}} = \mathrm{Hom}(H,\mathbf{R}) = \mathrm{Hom}(H,\mathbf{Z}) \otimes \mathbf{R}.$$

A *closed cone* in $H'_{\mathbf{R}}$ is a closed subset C such that $v \in C \Rightarrow \lambda \cdot v \in C$ for every $\lambda \geqslant 0$.

An *open cone* in $H'_{\mathbf{R}}$ is an open subset D such that $v \in C \Rightarrow \lambda \cdot v \in C$ for every $\lambda > 0$.

An *integral hyperplane* of the vector space $H'_{\mathbf{R}} = \mathrm{Hom}(H, \mathbf{R})$ is a vector subspace of codimension 1 in $H'_{\mathbf{R}}$ having a basis formed by elements of $\mathrm{Hom}(H, \mathbf{Z})$.

A connected component of the complement to a given hyperplane Γ will be called *open half-space* corresponding to Γ. The closure of an open half-space will be called *closed half-space* corresponding to Γ.

A closed cone which is the intersection of a finite family of closed half-spaces corresponding to integral hyperplanes is called *closed polyhedral cone.*

An open cone which is the intersection of a finite family of open half-spaces corresponding to integral hyperplanes is called *open polyhedral cone.*

A subset $C \subset H'$ is called *open polyhedral conical subset,* if it is empty or equals $H' \setminus \{0\}$ or is a finite disjoint union of open polyhedral cones.

A subset $A \subset H'$ is called *quasi-polyhedral conical subset* if there is an open polyhedral conical subset C and a finite union D of integral hyperplanes, such that $D \cup C = D \cup A$.

The next theorem follows from the main theorem of my paper [22], it is based on earlier results of J.-Cl. Sikorav (see [21]).

Theorem 1.2. *The set of all classes $\xi \in H^1(M, \mathbf{Z})$ such that the Novikov homology $\widehat{H}^{ab}_*(M, \xi)$ vanishes is the intersection with $H^1(M, \mathbf{Z})$ of a quasi-polyhedral conical subset of $H^1(M, \mathbf{R})$.* [2]

Now let us proceed to non-abelian coverings. In the joint work with H.Goda [11] we introduced a new version of the Novikov homology. We call it *twisted Novikov homology.* The input data for the construction is: a connected CW complex M, a homomorphism $\xi : \pi_1(M) \to \mathbf{Z}$, and an anti-homomorphism $\rho : \pi_1(M) \to GL(n, \mathbf{Z})$. The resultant twisted Novikov homology groups are modules over the principal ideal domain $\mathbf{Z}((t))$, so the numerical invariants are easily extracted from the homological data. On the other hand the non-abelian homological algebra of the universal covering of M is encoded in this homology via the representation ρ. The construction of the twisted Novikov homology is

[2] The results of [21] and [22] pertain actually to a more general case of arbitrary homomorphisms $\pi_1(M) \to \mathbf{R}$, and not only homomorphisms $\pi_1(M) \to \mathbf{Z}$, see the discussion below.

motivated by the notion of *twisted Alexander polynomial* for knots and links. (See the papers [15] of X.S.Lin and [37] of M.Wada for the definition and properties of the twisted Alexander polynomials, and the paper [9] of H.Goda, T.Kitano, T.Morifuji for applications of twisted Alexander polynomials to fibering obstructions for knots and links).

The definition of the twisted Novikov homology generalizes immediately to the case of arbitrary cohomology classes $\xi \in H^1(M, \mathbf{R})$. The input data for this construction is as follows. Let R be a commutative ring. Let V be a finitely generated free left R-module. Denote by $GL_R(V)$ the group of all automorphisms of V over R. Let $\rho : G \to GL_R(V)$ be an anti-homomorphism (that is, $\rho(ab) = \rho(b)\rho(a)$ for all $a, b \in G$; ρ will also be called *right representation*). Let $\pi : G \to H$ be an epimorphism of G onto a free finitely generated abelian group H. Let $\xi : H \to \mathbf{R}$ be a group homomorphism.

To this data we associate the twisted Novikov homology as follows. Let $\Lambda = R[H]$; put $V^H = \Lambda \underset{R}{\otimes} V$; define a right representation $\rho_\pi : G \to GL_\Lambda(V^H)$ as follows:

$$\rho_\pi(g)\Big(\sum_i \lambda_i \otimes v_i\Big) = \sum_i \big(\pi(g)\lambda_i\big) \otimes \rho(g)v_i. \tag{2}$$

Form the tensor product of the cellular chain complex $\mathcal{C}_*(\widetilde{M})$ of the universal covering with the right $\mathbf{Z}G$-module V^H:

$$\widetilde{\mathcal{C}}_*(M, \rho_\pi) = V^H \underset{\mathbf{Z}G}{\otimes} \mathcal{C}_*(\widetilde{M}).$$

This is a chain complex of left free Λ-modules (observe that $\mathrm{rk}_{\Lambda}(\widetilde{\mathcal{C}}_k(M, \rho_\pi)) = n \cdot \mathrm{rk}\, \mathcal{C}_k(\widetilde{M})$, where n is the rank of V over R). Apply the tensor product with $\widehat{\Lambda}_\xi$ to obtain the chain complex

$$\widehat{\mathcal{C}}_*(M, \rho_\pi, \xi) = \widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} \widetilde{\mathcal{C}}_*(M, \rho_\pi). \tag{3}$$

Its homology

$$\widehat{H}_*(M, \rho_\pi, \xi) = H_*(\widehat{\mathcal{C}}_*(M, \rho_\pi, \xi)) \tag{4}$$

is called *twisted Novikov homology.* The twisted Novikov homology of our paper [11] corresponds to the particular case when $H = \mathbf{Z}$, and the homomorphism $\xi : H \to \mathbf{R}$ above is the inclusion $\mathbf{Z} \hookrightarrow \mathbf{R}$.

The present generalization makes simpler the statements of several theorems below. It has also a geometrical background; it corresponds to Morse forms while the framework of our paper [11] was related to circle-valued Morse functions. (Recall that a closed 1-form on a manifold M

is called *Morse form*, if locally it is a differential of a Morse function.) Namely we have the following theorem:

Theorem 1.3. *Let ω be a Morse form on a closed connected manifold M. Assume that the cohomology class $[\omega] \in H^1(M, \mathbf{R}) = Hom(\pi_1(M), \mathbf{R})$ can be factored as $[\omega] = \xi \circ \pi$, where $\pi : \pi_1(M) \to H$ is an epimorphism onto a free abelian group, and $\xi : H \to \mathbf{R}$ is a homomorphism. Then there is a chain complex $\mathcal{N}_*^\rho$ such that*

1) *$\mathcal{N}_k^\rho$ is a free $\widehat{\Lambda}_\xi$-module with $n \cdot m_k(\omega)$ free generators in degree k (where $m_k(\omega)$ stands for the number of zeros of the form ω of index k).*
2) *$\mathcal{N}_*^\rho$ is chain homotopy equivalent to $\widehat{\mathcal{C}}_*(\widetilde{M}, \rho_\pi, \xi)$.*

In particular, if the cohomology class $[\omega]$ contains a nowhere vanishing 1-form, then the twisted Novikov homology $\widehat{H}_*(M, \rho_\pi, \xi)$ equals to zero.

The most natural choice of the epimorphism $\pi : \pi_1(M) \to H$ is the projection $\pi_1(M) \to H_1(M)/tors$ onto the integral homology group of M modulo its torsion subgroup. The corresponding twisted Novikov homology will be denoted by $\widehat{H}_*(M, \rho, \xi)$.

Definition 1.4. A non-zero homomorphism

$$\xi \in Hom(H_1(M), \mathbf{R}) = \mathrm{Hom}(H_1(M)/Tors, \mathbf{R}) = H^1(M, \mathbf{R})$$

is called *ρ-acyclic*, if the ρ-twisted Novikov homology $\widehat{H}_*(M, \rho, \xi)$ vanishes. The set of all ρ-acyclic classes will be denoted $\mathcal{V}_{alg}(M, \rho)$.

The reason for studying the ρ-acyclic classes is that any class ξ containing a nowhere vanishing closed 1-form is ρ-acyclic for any representation ρ. An immediate generalization of Theorem 1.2 leads to the following result about the algebraic structure of the set of ρ-acyclic classes.

Theorem 1.5. *For a given right representation ρ of $\pi_1(M)$ the set $\mathcal{V}_{alg}(M, \rho)$ of all ρ-acyclic classes ξ is a quasi-polyhedral conical subset.*

In general we do not know whether the set of all ρ-acyclic classes is an open polyhedral conical subset. In other words, we can describe the structure of the set of all ρ-acyclic classes only up to some finite union of hyperplanes in $H^1(M, \mathbf{R})$. However in the case when M is a 3-manifold, we have a much stronger assertion, which is the main result of the present paper (see Section 5):

Theorem 1.6. *Let M be a connected compact three-dimensional manifold (maybe with a non-empty boundary), such that $\chi(M) = 0$. Let ρ be a right representation of $\pi_1(M)$. Then:*

1. *The set $\mathcal{V}_{alg}(M,\rho)$ of all ρ-acyclic classes ξ is an open polyhedral conical subset.*

2. *This subset is entirely determined by the twisted Alexander polynomial associated to the group $\pi_1(M)$ and the representation ρ.*

The open polyhedral cones forming the subset $\mathcal{V}_{alg}(M,\rho)$ will be called *ρ-acyclicity cones.* The theorem above implies that the set of all ρ-acyclic classes depends only on the group $\pi_1(M)$ and the representation ρ.

Along with the twisted Alexander polynomials the proof of the above theorem uses another polynomial invariants of the chain complexes, which we introduce in Section 3 and call *the Fitting invariants.* These invariants are defined as the GCDs of the minors of the second boundary operator of the chain complex (3). We show that they are directly related to the Novikov homology, and in many cases the Novikov homology in degree one can be computed from the sequence of the Fitting invariants. The twisted Alexander polynomial is a much more sophisticated invariant, but it turns out that the image of the twisted Alexander polynomial in the Novikov completion $\widehat{\Lambda}_\xi$ is essentially the same as the image of the corresponding Fitting invariant.

Theorem 1.6 is related to the famous Thurston's theorem [34], which implies that the set $\mathcal{V}(M)$ of all classes $\xi \in H^1(M,\mathbf{R})$ representable by closed nowhere vanishing 1-forms is a finite union of open polyhedral cones, namely the cones on certain faces of the unit ball of the Thurston norm on $H^1(M,\mathbf{R})$. We shall call these cones *Thurston cones.* For every right representation ρ of $\pi_1(M)$ in $GL(\mathbf{Z}^n)$ the set $\mathcal{V}(M)$ is contained in the set $\mathcal{V}_{alg}(M,\rho)$ of all ρ-acyclic classes:

$$\mathcal{V}(M) \subset \mathcal{V}_{alg}(M,\rho)$$

so each of the Thurston cones is contained in one of the ρ-acyclicity cones. The ρ-acyclicity cones are computable from the twisted Alexander polynomial, which is in its turn computable from the Alexander matrix associated to any finite presentation of the group $\pi_1(M)$. Thus the set $\mathcal{V}_{alg}(M,\rho)$ is in a sense a computable upper bound for the set $\mathcal{V}(M)$. Let

$$\mathcal{V}_{alg}(M) = \bigcap_{\rho} \mathcal{V}_{alg}(M,\rho)$$

where ρ ranges over the set of all right representations ρ of $\pi_1(M)$ in $GL(\mathbf{Z}^n)$. We have then:

$$\mathcal{V}(M) \subset \mathcal{V}_{alg}(M). \tag{5}$$

It is an interesting problem to investigate the relationship between the two sets, and in particular answer the next question:

For which manifolds the equality $\mathcal{V}(M) = \mathcal{V}_{alg}(M)$ holds?

We do not know examples of manifolds for which $\mathcal{V}(M) \neq \mathcal{V}_{alg}(M)$. On the other hand, it is easy to construct manifolds for which

$$\mathcal{V}(M) \neq \bigcap_{\rho \in \mathcal{R}_f} \mathcal{V}_{alg}(M, \rho),$$

where $\mathcal{R}_f$ is the set of all right representations of G over finite fields (see Section 5.3). For such manifolds the right representations over finite fields are not sufficient to detect all the cohomology classes representable by non-singular 1-forms

A certain amount of computations will be necessary to clarify the relation between $\mathcal{V}(M)$ and $\mathcal{V}_{alg}(M)$ and answer the question above. The recent progress in the software related to the computations of invariants of knots and links, especially the Kodama's KNOT program allows us to hope that such computations can be carried out.

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Notes on the terminology.

Ring means an associative ring with a unit. For a ring R we denote by $R^\bullet$ the multiplicative group of all invertible elements of R. *Module* means always a left module, if the contrary is not stated explicitly. The homology of a space X with integral coefficients will be denoted by $H_*(X)$. For a left module V over a commutative ring R we denote by $GL_R(V)$ the group of all R-automorphisms of V. When the ring R is clear from the context, we use the abbreviated notation $GL(V)$. For a ring R the symbol $GL(n, R)$ denotes the multiplicative group of all invertible $n \times n$-matrices with coefficients in R.

2. Twisted Novikov homology

2.1. **Novikov homology.**

Definition 2.1. Let G be a group, and R be a commutative ring with a unit. Put $\mathcal{L} = R[G]$. Let $\xi : G \to \mathbf{R}$ be a group homomorphism. Let $\widehat{\widehat{\mathcal{L}}}$ be the set of all formal linear combinations (infinite in general) of the form $\lambda = \sum_{g\in G} n_g g$ where $n_g \in R$. For $\lambda \in \widehat{\widehat{\mathcal{L}}}$ and $C \in \mathbf{R}$ put

$$\text{supp } (\lambda, C) = \{g \in G \mid n_g \neq 0 \quad \& \quad \xi(g) > C\}.$$

The Novikov ring is defined as follows:

$$\widehat{\mathcal{L}}_\xi = \{\lambda \in \widehat{\widehat{\mathcal{L}}} \mid \text{supp } (\lambda, C) \quad \text{is finite for every} \quad C \in \mathbf{R}\}.$$

(It is easy to show that the subset $\widehat{\mathcal{L}}_\xi \subset \widehat{\widehat{\mathcal{L}}}$ has indeed a natural structure of a ring, containing $\mathcal{L}$ as a subring.)

Example 2.2. Let G be an abelian finitely generated group. A homomorphism $\xi : G \to \mathbf{R}$ is called *rational* if $\xi = \lambda \cdot \xi_0$, where $\xi_0 : G \to \mathbf{Z}$ is a homomorphism, and $\lambda \in \mathbf{R}$. Equivalently, ξ is rational, if $\xi = 0$ or $\xi(G) \approx \mathbf{Z}$. When ξ is non-zero and rational, we have an isomorphism

$$\widehat{\mathcal{L}}_\xi \approx K[[t]][t^{-1}] \quad \text{where} \quad K = R[\text{Ker } \xi]$$

so that the ring $\widehat{\mathcal{L}}_\xi$ is a localization of the ring of the power series $K[[t]]$. In particular $\widehat{\mathcal{L}}_\xi$ is Noetherian.

Example 2.3. Let G be a free abelian finitely generated group. In this case $\mathcal{L} = R[G]$ is isomorphic to the ring of Laurent polynomials in k variables (where $k = \text{rk } G$) with coefficients in R.

When $\xi : G \to \mathbf{R}$ is injective (such homomorphisms are also called *totally irrational*), the algebraic properties of the ring $\widehat{\mathcal{L}}_\xi$ are suprisingly simple:

Theorem 2.4. *If ξ is totally irrational, and R is a principal ideal domain, then the ring $\widehat{\mathcal{L}}_\xi$ is also a principal ideal domain.*

For the case $R = \mathbf{Z}$ this theorem is due to J.-Cl. Sikorav (his proof was published in [21]). The proof in the general case is similar. Observe that if R is a field, then $\widehat{\mathcal{L}}_\xi$ is also a field.

The main topological applications of these constructions are in the theory of Morse forms. Let M be a closed connected C^∞ manifold, let ω be a Morse form on M. Let $[\omega] \in H^1(M, \mathbf{R})$ denote the de Rham cohomology class of ω; then $[\omega]$ can also be identified with a homomorphism $G = \pi_1(M) \to \mathbf{R}$. Let $\widehat{\mathfrak{L}}_{[\omega]}$ denote the corresponding Novikov completion of the ring $\mathfrak{L} = \mathbf{Z}G$. The next theorem relates the homotopy type of the completed chain complex

$$\widehat{\mathcal{C}}_*(\widetilde{M}, [\omega]) = \widehat{\mathfrak{L}}_{[\omega]} \underset{\mathfrak{L}}{\otimes} \mathcal{C}_*(\widetilde{M}) \tag{6}$$

(where $\mathcal{C}_*(\widetilde{M})$ denote the cellular chain complex of the universal covering of M) and the geometrical properties of the form ω. Denote by $S_k(\omega)$ the set of zeros of ω of index k, and by $S(\omega)$ the set of all zeros.

Theorem 2.5. *There is a chain complex $\mathcal{N}_*$ over the ring $\widehat{\mathfrak{L}}_{[\omega]}$ such that:*

1. *$\mathcal{N}_k$ is freely generated over $\widehat{\mathfrak{L}}_{[\omega]}$ by $S_k(\omega)$.*
2. *The chain complexes $\mathcal{N}_*$ and $\widehat{\mathcal{C}}_*(M, [\omega])$ are chain homotopy equivalent.*

For the case of integral classes $[\omega] \in H^1(M, \mathbf{Z})$ the theorem follows from the existence of the Novikov complex $\mathcal{N}_*(f, v)$, associated to a circle-valued Morse map f such that $df = \omega$ and any transverse f-gradient v (see [23]). The case of rational classes $[\omega] \in H^1(M, \mathbf{Q})$ follows immediately. For the case of 1-forms belonging to arbitrary cohomology classes see the later paper by F.Latour [14], and also the works of D.Schütz [30], [31].

2.2. **Twisted Novikov homology.** In this subsection we begin our study of the twisted Novikov homology. Let M be a connected CW complex. Let $\rho : G \to GL_R(V)$ be any right representation, $\pi : G \to H$ an epimorphism, $\xi : H \to \mathbf{R}$ a homomorphism. In the introduction we have associated to these data a chain complex $\widehat{\mathcal{C}}_*(M, \rho_\pi, \xi)$ over $\widehat{\Lambda}_\xi$ (where $\Lambda = R[H]$). When $\xi : H \to \mathbf{R}$ is a monomorphism and R is a principal ideal domain, the ring $\widehat{\Lambda}_\xi$ is also a principal ideal domain. In this case put

$$\widehat{b}_i(M, \rho_\pi, \xi) = \operatorname{rk}_{\widehat{\Lambda}_\xi} H_i(\widehat{\mathcal{C}}_*(M; \rho_\pi, \xi)),$$

$$\widehat{q}_i(M, \rho_\pi, \xi) = \text{t.n.}_{\widehat{\Lambda}_\xi} H_i(\widehat{\mathcal{C}}_*(M; \rho_\pi, \xi)).$$

(where t.n. stands for the torsion number of the module, that is, the minimal possible number of generators of its torsion submodule). Observe that if R is a field, then all the numbers $\widehat{q}_i(M, \rho_\pi, \xi)$ vanish.

Theorem 2.6. *For a given right representation ρ and a given homomorphism $\pi : G \to H$ the numbers $b_k(M, \rho_\pi, \xi)$ do not depend on the monomorphism ξ. There is a set $\Gamma \subset H'_{\mathbf{R}} = Hom(H, \mathbf{R})$ which is a finite union of integral hyperplanes such that in every connected component of the complement $H'_{\mathbf{R}} \setminus \Gamma$ the numbers $\widehat{q}_k(M, \rho_\pi, \xi)$ do not depend on ξ.*

In the particular case of the trivial representation this theorem is due to J.-Cl.Sikorav (see [21]); the proof in the general case is similar.

Corollary 2.7. *There is an open polyhedral conical subset $S \subset H'_{\mathbf{R}} = \mathrm{Hom}(H, \mathbf{R})$ such that a monomorphism $\xi : H \to \mathbf{R}$ is in S if and only if the Novikov homology $\widehat{H}_*(M; \rho_\pi, \xi)$ vanishes.*

It is natural to ask, whether we can drop the condition of injectivity of ξ in the Corollary above and still keep the conclusion of the Corollary. I do not know if the answer is positive in general, but it is the case when M is a compact connected 3-manifold with $\chi(M) = 0$ and $\pi : G \to H$ is the projection onto $H = H_1(M, \mathbf{Z})/Tors$ (see Section 5).

Let us proceed to the applications of the twisted Novikov homology in the theory of Morse forms. Now we shall assume that M is a closed connected C^∞ manifold. Let ω be a Morse form on M. The de Rham cohomology class $[\omega] \in H^1(M, \mathbf{R})$ of ω can be identified with a homomorphism $G = \pi_1(M) \to \mathbf{R}$. Let $\pi : G \to H$ be an epimorphism, such that $[\omega]$ factors through π, so that $[\omega] = \xi \circ \pi$, where $\xi : H \to \mathbf{R}$ is a homomorphism.

Theorem 2.8. *There is a chain complex $\mathcal{N}^\rho_*$ over the ring $\widehat{\Lambda}_\xi$ such that:*

1. *$\mathcal{N}^\rho_k$ is a free $\widehat{\Lambda}_\xi$-module with $n \cdot m_k(\omega)$ free generators.*
2. *The $\widehat{\Lambda}_\xi$-modules $H_*(\mathcal{N}^\rho_*)$ and $\widehat{H}_*(M; \rho_\pi, \xi)$ are isomorphic.*

Proof. Let $\widehat{V}^H = \widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} V^H$; the composition of the right representation ρ_π with the natural inclusion $GL_\Lambda(V^H) \hookrightarrow GL_{\widehat{\Lambda}_\xi}(\widehat{V}^H)$ determines a right representation $\widehat{\rho}_\pi$ of G in $GL_{\widehat{\Lambda}_\xi}(\widehat{V}^H)$. Using the factorization $[\omega] = \xi \circ \pi$ it is not difficult to check that the representation $\widehat{\rho}_\pi$ extends to a structure of a right $\widehat{\mathfrak{L}}_{[\omega]}$-module on $\widehat{V}^H$, and we have:

$$\widehat{\mathcal{C}}_*(M; \rho_\pi, \xi) = \widehat{V}^H \underset{\widehat{\mathfrak{L}}_{[\omega]}}{\otimes} \widehat{\mathcal{C}}_*(\widetilde{M}, [\omega]). \tag{7}$$

Now our theorem follows immediately from Theorem 2.5. □

Corollary 2.9. *Let ω be a closed 1-form without zeros. Assume that*

$$[\omega] = \xi \circ \pi \in \mathrm{Hom}(G, \mathbf{R}) = H^1(M, \mathbf{R})$$

where $\xi : H \to \mathbf{R}$ is a homomorphism, and $\pi : G \to H$ is an epimorphism. Then $\widehat{H}_*(M; \rho_\pi, \xi) = 0$. □

Corollary 2.10. *Let ω be a Morse form. Assume that*

$$[\omega] = \xi \circ \pi \in \mathrm{Hom}(G, \mathbf{R}) = H^1(M, \mathbf{R})$$

where $\xi : H \to \mathbf{R}$ is a monomorphism, and $\pi : G \to H$ is an epimorphism. Assume that R is a principal ideal domain. Then

$$m_i(\omega) \geqslant \widehat{b}_i(M, \rho_\pi, \xi) + \widehat{q}_i(M, \rho_\pi, \xi) + \widehat{q}_{i-1}(M, \rho_\pi, \xi). \quad (8)$$ □

3. Fitting invariants of chain complexes

Let A be a finitely generated module over a commutative ring Q. Let

$$0 \longleftarrow A \longleftarrow C_0 \xleftarrow{\lambda} C_1$$

be a presentation for A, with C_0, C_1 free finitely generated modules. By definition the k-th Fitting ideal of A is the ideal generated by all $(n-k) \times (n-k)$-minors of the matrix of the homomorphism λ, where $n = \mathrm{rk}\ C_0$ (see [5], §20.2).

In this section we give a generalization of this construction and for every chain complex C_* of free finitely generated Q-modules we define a family of ideals of Q (*the Fitting ideals of* C_*) which are chain homotopy invariants of C_*. If Q is a factorial ring, each ideal has its greatest common divisor, thus we derive from the family of the Fitting ideals a family of elements of Q, which are called *the Fitting invariants of* C_*. When Q is a principal ideal domain, these invariants determine the homology of C_* (see Subsection 3.3).

If C_* is a chain complex over a *non-commutative* ring $\mathfrak{L}$, and V is a left Q-module, which is also a right $\mathfrak{L}$-module, we can form the tensor product $V \underset{\mathfrak{L}}{\otimes} C_*$ and consider the Fitting invariants of the resultant complex. A particular case of this construction leads to the well known *knot polynomials* $\Delta_k(t)$ of [4], Ch. 8. We discuss this and similar constructions in Subsections 3.4 and 3.6.

While the definition of the Fitting ideals for chain complexes is apparently new, many similar constructions exist already in the literature. Let us mention for example the invariants of knots, deduced from the representation spaces of the fundamental group of the knot (see the paper [33] of Le Ty Quok Thang), and the twisted Alexander-Fox polynomials of V.Turaev (see [36]).

3.1. **Matrices of homomorphisms: terminology.** This subsection is purely terminological: we describe the conventions with which we shall be working. Let R be a ring (non-commutative in general). Let A, B be free finitely generated left modules over R. Choose a finite basis $\{e_i\}_{1\leqslant i\leqslant k}$ in A, and a finite basis $\{f_j\}_{1\leqslant j\leqslant m}$ in B. Let $\phi : A \to B$ be a module homomorphism. Write

$$\phi(e_i) = \sum_j M_{ij} f_j. \quad \text{with} \quad M_{ij} \in R. \tag{9}$$

The matrix (M_{ij}) will be denoted $M(\phi)$ and called *the matrix of the homomorphism* ϕ with respect to the chosen bases. Thus the coordinates of the images $\phi(e_i)$ of the basis elements of A in B are *the rows* of the matrix $M(\phi)$ (which has k rows and m columns). Here is the composition formula:

$$M(\phi \circ \psi) = M(\psi) \cdot M(\phi),$$

(where $\psi : C \to A,\ \phi : A \to B$ are homomorphisms of left modules, and $\cdot$ stands for the usual matrix product). This way of associating a matrix to a module homomorphism will be called *row-wise*. For a free module A with k free generators the map

$$\phi \mapsto M(\phi) : \mathrm{Hom}(A, A) \to Mat(k \times k, R)$$

is therefore an anti-homomorphism.

In many recent textbooks on linear algebra one finds another convention:

$$\phi(e_i) = \sum_j \widetilde{M}_{ji} f_i, \tag{10}$$

so that $\widetilde{M}(\phi) = M(\phi)^T$, and the coordinates of the images of the basis elements of A in B form the columns of the matrix $\widetilde{M}$. This way of associating the matrix to a module homomorphism will be called *column-wise*.

When the ring R is commutative the convention (10) leads to the following composition formula:

$$\widetilde{M}(\phi \circ \psi) = \widetilde{M}(\phi) \cdot \widetilde{M}(\psi),$$

therefore the map

$$\phi \mapsto \widetilde{M}(\phi), \quad Hom(A, A) \to Mat(k \times k, R)$$

is a ring homomorphism.

3.2. **The Fitting invariants of chain complexes over commutative rings.**

Definition 3.1. A chain complex

$$C_* = \{\ldots \longleftarrow C_k \xleftarrow{\partial_{k+1}} C_{k+1} \ldots\}$$

of left modules over some ring Q is called *regular* if every C_i is a finitely generated free Q-module, and $C_i = 0$ for $i < 0$.

In this subsection Q is a commutative factorial ring.

Definition 3.2. Let C_* be a regular chain complex of Q-modules, and $k \in \mathbf{N}$. Choose any finite bases in the modules C_k, C_{k+1} and let $M(\partial_{k+1})$ be the matrix of ∂_{k+1} with respect to these bases.

Let $I_s(\partial_{k+1})$ denote the ideal in Q generated by all $s \times s$-minors of $M(\partial_{k+1})$. (Here we assume that s is an integer with $0 < s \leqslant \min(\text{rk } C_k, \text{rk } C_{k+1})$. If $s > \min(\text{rk } C_k, \text{rk } C_{k+1})$, then put by definition $I_s(\partial_{k+1}) = 0$, and for $s \leqslant 0$ put $I_s(\partial_{k+1}) = Q$.)

The next lemma is a well-known consequence of the Binet-Cauchy formula (see for example [17], p. 25).

Lemma 3.3. *The ideal $I_s(\partial_{k+1})$ does not depend on the particular choice of bases in C_k and C_{k+1}.* □

It is clear that the ideal $I_s(\partial_{k+1})$ is not in general an invariant of the homotopy type of the chain complex. However we can re-index the sequence $I_s(\partial_{k+1})$ and obtain homotopy invariants.

Definition 3.4. Put

$$J_m^{(k)}(C_*) = I_{\text{rk } C_k - \text{rk } C_{k-1} - m+1}(\partial_{k+1}). \quad ^{3})$$

Proposition 3.5. *For every m, k the ideal $J_m^{(k)}(C_*)$ is a homotopy invariant of the regular chain complex C_*.*

Proof. Let F be a free finitely generated Q-module. Let $T_*(i, F)$ denote the chain complex

$$T_*(i,F) = \{0 \longleftarrow \cdots 0 \longleftarrow F \xleftarrow{\text{id}} F \longleftarrow 0 \longleftarrow \cdots\}$$

concentrated in degrees $i, i+1$. A chain complex isomorphic to a direct sum of complexes $T_*(i, F)$ for some $i \geqslant 0$ and some F will be called *trivial.* The next lemma is one of the versions of the Cockroft-Swan theorem [2], the proof is similar to the the proof of Lemma 1.8 in [24].

[3] This re-indexing may seem arbitrary, but we shall see that it fits with the usual notation for the knot polynomials.

Lemma 3.6. *Let C_*, D_* be chain homotopy equivalent complexes. Then there are trivial chain complexes T_*, T'_* such that $C_* \oplus T_* \approx D_* \oplus T'_*$.*

Proof. Let

$$K_* = \bigoplus_{i \in \mathbf{Z}} T(i, C_i).$$

This is a trivial chain complex. Let $\phi : C_* \to D_*$ be a chain homotopy equivalence. Define a chain map $\psi : C_* \to D_* \oplus K_*$ as follows:

$$\psi(c) = (\phi(c), c, \partial c) \in D_i \oplus C_i \oplus C_{i-1} \quad \text{for} \quad c \in C_i. \tag{11}$$

It is clear that ψ is a chain homotopy equivalence which is a split monomorphism. The quotient chain complex $(D_* \oplus K_*)/\text{Im}\ \psi$ is a contractible chain complex of free finitely generated modules. Thus we obtain an exact sequence

$$0 \longrightarrow C_* \xrightarrow{\psi} D_* \oplus K_* \longrightarrow S_* \longrightarrow 0, \tag{12}$$

where S_* is a regular acyclic chain complex. Such a sequence splits (see for example [3], 13.2), and we obtain an isomorphism

$$C_* \oplus S_* \approx D_* \oplus K_*.$$

It is easy to prove that there is a free trivial chain complex R_*, such that $S_* \oplus R_*$ is trivial, and this completes the proof of the lemma with $T_* = S_* \oplus R_*,\ T'_* = K_* \oplus R_*$. □

Now let us return to the proof of our proposition. In view of the preceding lemma it suffices to check the following easily proved assertion: the ideal $J_m^{(k)}(C_*)$ does not change if we add to the C_* the chain complex $T_*(i, F)$ where F is a finitely generated free Q-module, and i equals one of the numbers $k-1, k$ or $k+1$. □

Definition 3.7. The ideal $J_m^{(k)}(C_*)$ is called *the m-th Fitting ideal of* C_*. The GCD (= *the greatest common divisor*) of all the non-zero elements in the ideal $J_m^{(k)}(C_*)$ will be denoted by $F_m^{(k)}(C_*)$ and called *Fitting invariant of* C_*. This element is well-defined up to multiplication by invertible elements of Q. The sequence

$$\dots J_m^{(k)}(C_*) \subset J_{m+1}^{(k)}(C_*) \dots \tag{13}$$

of the Fitting ideals will be called *the Fitting sequence of C_* in degree k*. The subsequence of (13) formed by all non-trivial ideals, is called *reduced Fitting sequence of C_* in degree k.*

Recall that an ideal $I \subset Q$ is called *non-trivial* if $I \neq 0,\ I \neq Q$.

Observe that the length of the reduced Fitting sequence in degree k is not more than rk C_k.

Lemma 3.8. *Let $S \subset Q^\bullet$ be a multiplicative subset of Q. Then up to invertible elements of $S^{-1}Q$ we have:*

$$F_m^{(k)}(C_*) = F_m^{(k)}(S^{-1}C_*).$$

Proof. It suffices to recall that the GCD of elements of a factorial ring does not change when the ring is localized. □

Remark 3.9. The Fitting invariants $F_m^{(k)}$ for $k \leqslant 2$ can be defined in a slightly more general framework:

Definition 3.10. A chain complex C_* of left Q-modules will be called *2-regular* if $C_i = 0$ for $i > 0$ and C_0, C_1, C_2 are finitely generated free Q-modules.

Using the same procedure as above we can define the Fitting invariants $F_m^{(i)}$ with $i \leqslant 2$ for any 2-regular chain complex over a commutative ring Q.

3.3. **The case when Q is principal.** Assume that Q is a principal ideal domain. We shall show that in this case the sequence of Fitting invariants determines the homology of the chain complex. We shall give only the statements of the theorems; the proofs are obtained applying the standard results about the structure of the modules over principal ideal domains.

Let us begin with a theorem which shows how to compute the Betti numbers

$$b_k(C_*) = \text{rk }_Q H_k(C_*)$$

and the torsion numbers

$$q_k(C_*) = \text{t.n.}_Q H_k(C_*)$$

from the Fitting invariants. (Recall that the *torsion number of a module X* is the minimal possible number of generators of the torsion submodule of X.) Let C_* be a regular chain complex of Q-modules, let $\gamma_k = \text{rk } C_k$. Consider the subsequence of the Fitting sequence starting with $J^{(k)}_{-\gamma_{k-1}+1} = I_{\gamma_k}(\partial_{k+1})$:

$$J_{-\gamma_{k-1}+1},\ J_{-\gamma_{k-1}+2}, \ldots$$

Let A_k be the number of zero ideals in this sequence. Let $\varkappa_k$ be the cardinality of the reduced Fitting sequence of C_* in degree k.

Theorem 3.11. *We have*

$$b_k(C_*) = A_k + A_{k-1} - \gamma_{k-1}; \tag{14}$$

$$q_k(C_*) = \varkappa_k. \tag{15}$$

□

The torsion submodule of the homology is also determined by the Fitting invariants. Write the reduced Fitting sequence in degree k as follows:

$$I_1, \dots, I_{\varkappa_k}. \tag{16}$$

Let θ_s be the GCD of all the elements of I_s then we have: $\theta_{s+1} \mid \theta_s$ for every s. Put $\lambda_s = \theta_s/\theta_{s+1}$.

Theorem 3.12. *The elements $\lambda_s \in Q$ are non-invertible, for every s we have: $\lambda_{s+1} \mid \lambda_s$ and*

$$Tors\ H_k(C_*) \approx \bigoplus_{i=1}^{\varkappa_k} Q/\lambda_i Q. \qquad \square$$

3.4. **The twisted Fitting invariants of $\mathbf{Z}G$-complexes.** In this subsection we apply the Fitting invariants of the preceding subsection to construct invariants of chain complexes over non-commutative rings. The most important for us are modules over group rings, and we limit ourselves to this case, although there are obvious generalizations.

Let G be a group and let C_* be a regular chain complex over $\mathfrak{L} = \mathbf{Z}G$. Let $\theta : G \to GL_Q(W)$ be a right representation, where W is a finitely generated free Q-module over some commutative factorial ring Q. Form the tensor product

$$C_*(\theta) = W \underset{Q}{\otimes} C_*;$$

the Fitting invariants $F_m^{(k)}(W \underset{Q}{\otimes} C_*)$ of this complex will be denoted by $\delta_m^{(k)}(C_*, \theta)$ and called *twisted Fitting invariants of C_**.

Remark 3.13. Similarly to Remark 3.9 we obtain the Fitting invariants $\delta_m^{(k)}(C_*, \theta) \in Q$ where $k \leqslant 2$ and C_* is a 2-regular chain complex.

In this paper we shall be interested mainly in the case when the ring Q is the group ring of some free abelian group, or a Novikov completion of such group ring, or a localization of such group ring.

Let H be a free abelian finitely generated group, and R a commutative factorial ring. The group ring $R[H]$ will be denoted by Λ. Let V be a finitely generated free left R-module. Let $\rho : G \to GL_R(V)$ be a right representation of G and $\pi : G \to H$ is an epimorphism. Recall that in Introduction we have associated to this data a right representation $\rho_\pi : G \to GL(V^H)$ where $V^H = \Lambda \underset{R}{\otimes} V$.

Definition 3.14. Let C_* be a regular chain complex over $\mathbf{Z}G$. The Fitting invariant $\delta_m^{(k)}(C_*, \rho_\pi) \in \Lambda$ will be called *the twisted Fitting invariant of C_* with respect to (ρ, π)*.

In the rest of this subsection we investigate the behaviour of the Fitting invariants of C_* with respect to certain completions and localizations of the representation ρ_π.

Definition 3.15. Let $\xi : H \to \mathbf{R}$ be a non-zero homomorphism. An element $x \in \Lambda = R[H]$ is called *ξ-monic* if

$$x = x_0 h_0 + \sum_{i=1}^{s} x_i h_i \quad \text{with} \quad x_i \in R, h_i \in H,$$
$$\text{where} \quad x_0 \in R^\bullet \quad \text{and} \quad \xi(h_i) < \xi(h_0) \quad \text{for every} \quad i \neq 0.$$

The multiplicative subset of all ξ-monic elements will be denoted S_ξ. The ring $S_\xi^{-1}\Lambda$ will be also denoted by $\Lambda_{(\xi)}$.

The next proposition is immediate.

Proposition 3.16. *An element $x \in \Lambda$ is ξ-monic if and only if it is invertible in $\widehat{\Lambda}_\xi$.* □

Therefore the ring $\Lambda_{(\xi)}$ can be considered as a subring of $\widehat{\Lambda}_\xi$. Let $\widetilde{\rho}_{\pi,\xi}$ denote the composition

$$G \xrightarrow{\rho_\pi} GL_\Lambda(V^H) \hookrightarrow GL_{\Lambda_{(\xi)}}(\Lambda_{(\xi)} \underset{\Lambda}{\otimes} V^H).$$

This is a right representation of G, and we obtain the corresponding twisted Fitting invariants $\delta_m^{(k)}(C_*, \widetilde{\rho}_{\pi,\xi})$. It is clear that

$$\delta_m^{(k)}(C_*, \rho_{\pi,\xi}) = \delta_m^{(k)}(C_*, \widetilde{\rho}_{\pi,\xi}). \tag{17}$$

Similarly, let $\widehat{\rho}_{\pi,\xi}$ denote the composition

$$G \xrightarrow{\rho_\pi} GL_\Lambda(V^H) \hookrightarrow GL_{\widehat{\Lambda}_\xi}(\widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} V^H).$$

This is a right representation of G, and we obtain the corresponding twisted Fitting invariants $\delta_m^{(k)}(C_*, \widehat{\rho}_{\pi,\xi})$.

In the rest of this subsection we restrict ourselves to the particular case $R = \mathbf{Z}$, although some of the results can be proved in a more general setting.

Proposition 3.17. *Let $\xi : H \to \mathbf{R}$ be a monomorphism. Let $R = \mathbf{Z}$. Then*

$$\delta_m^{(k)}(C_*, \rho_\pi) = \delta_m^{(k)}(C_*, \widehat{\rho}_{\pi,\xi}).$$

Proof. We shall reduce the proof to the equality (17).

Proposition 3.18 ([21])**.** *If $\xi : H \to \mathbf{R}$ is monomorphic, then $\Lambda_{(\xi)}$ is a principal ideal domain.*

Corollary 3.19. *For every two elements $a, b \in \Lambda$ we have*

$$GCD_\Lambda(a,b) = GCD_{\Lambda_{(\xi)}}(a,b) = GCD_{\widehat{\Lambda}_\xi}(a,b). \qquad \square$$

The proposition follows immediately. $\square$

Now we can explain how to compute the twisted Novikov homology of a chain complex in terms of its twisted Fitting invariants. Let C_* be a regular chain complex over $\mathfrak{L} = \mathbf{Z}G$. Let $\pi : G \to H$ an epimorphism onto a free finitely generated abelian group, and $\rho : G \to GL(V)$ be a right representation, where $V \approx \mathbf{Z}^n$ is a free finitely generated module over $\mathbf{Z}$. Let $\xi : H \to \mathbf{R}$ be a monomorphism, so that $\widehat{\Lambda}_\xi$ is a principal ideal domain. Applying the results of Subsection 3.3 we obtain the following description of the twisted Novikov homology.

Let $\gamma_k = n \cdot \text{rk } C_k$, where $n = \text{rk }_{\mathbf{Z}} V$, and rk C_k is the number of free generators of C_k. Consider the segment of the Fitting sequence of $C_*(\rho_\pi) = V^H \underset{\Lambda}{\otimes} C_*$ in degree k starting with $J^{(k)}_{-\gamma_{k-1}+1} = I_{\gamma_k}(\partial_{k+1})$:

$$J_{-\gamma_{k-1}+1},\ J_{-\gamma_{k-1}+2},\ \ldots$$

Let A_k be the number of zero ideals in this segment. Let $I_1 \subset \ldots \subset I_{B_k}$ be the reduced Fitting sequence of $C_*(\rho_\pi)$ in degree k, let $\lambda_s \in \Lambda$ be the GCD of the ideal I_s. Then $\lambda_i \mid \lambda_j$ for $i \geqslant j$. Let $\varkappa_k(\xi)$ be the number of *non-ξ-monic* elements λ_j.

Theorem 3.20. *We have*

$$\widehat{b}_k(X, \rho_\pi, \xi) = A_k + A_{k-1} - \gamma_{k-1}; \tag{18}$$

$$\widehat{q}_k(X, \rho_\pi, \xi) = \varkappa_k(\xi). \qquad \square \tag{19}$$

As for the torsion submodule in degree k, let $\theta_s = \lambda_s / \lambda_{s+1} \in \Lambda$.

Theorem 3.21. *The elements $\theta_s \in \Lambda_{(\xi)}$ are non-invertible. For every s*

$$\theta_{s+1} \quad \textit{divides} \quad \theta_s \quad \textit{in} \quad \Lambda_{(\xi)}$$

and

$$Tors\ \widehat{H}_k(X, \rho_\pi, \xi) \approx \bigoplus_{i=1}^{\varkappa_k(\xi)} \widehat{\Lambda}_\xi / \theta_i \widehat{\Lambda}_\xi. \qquad \square$$

3.5. **The twisted Fitting invariants of $\mathbf{Z}G$-modules.** Let G be a group, and N be a left $\mathbf{Z}G$-module, admitting a free finitely generated resolution

$$0 \longleftarrow R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \ldots$$

over $\mathbf{Z}G$ (so that the homology of R_* vanishes in all dimensions except zero, and $H_0(R_*) \approx N$.) Let Q be a commutative factorial ring, and V

a finitely generated free left Q-module. Let $\theta : G \to GL(V)$ be a right representation of G. Consider the chain complex

$$V \underset{\mathbf{Z}G}{\otimes} R_* \tag{20}$$

of free left Q-modules (the module V is endowed with the structure of a right $\mathbf{Z}G$-module via the representation θ). Observe that the twisted Fitting invariants

$$F_m^{(k)}(V \underset{\mathbf{Z}G}{\otimes} R_*) \tag{21}$$

depend only on N and θ, but not on the particular choice of the resolution R_* (indeed, any two resolutions are chain homotopy equivalent). The most important for us are the Fitting invariants corresponding to the second boundary operator.

Definition 3.22. The element $F_m^{(2)}(V \underset{\mathbf{Z}G}{\otimes} R_*) \in Q$ will be called *m-th Fitting invariant* (or *det-invariant*) of the pair (G, θ) and denoted by

$$\delta_m(G, N, \theta) = F_m^{(2)}(V \underset{\mathbf{Z}G}{\otimes} R_*) \in Q.$$

This element is well-defined up to multiplication by an invertible element of the ring Q.

Similarly, we obtain the Fitting invariants $\delta_m(G, N, \theta)$ for the case when N is a free left $\mathbf{Z}G$-module which admits a 2-regular resolution. (see Remarks 3.9, 3.13).

Example 3.23. Let K be an oriented knot; put $G = \pi_1(S^3 \setminus K)$. Let N be an open tubular neighbourhood of K. Choose any finite CW decomposition of $S^3 \setminus N$. The cellular chain complex $\mathcal{C}_*(\widetilde{S^3 \setminus N})$ is a free $\mathbf{Z}G$-resolution of the module $H_0(\widetilde{S^3 \setminus N}) \approx \mathbf{Z}$. The canonical epimorphism $\varepsilon : G \to \mathbf{Z}$, sending each positively oriented meridian to 1 extends to a ring homomorphism $\theta : \mathbf{Z}G \to (\mathbf{Z}[\mathbf{Z}])^\bullet = GL(1, \mathbf{Z}[\mathbf{Z}])$. The Fitting invariant $\delta_1(G, \mathbf{Z}, \theta) \in \mathbf{Z}[\mathbf{Z}]$ equals clearly the Alexander polynomial of the knot. More generally, $\delta_i(G, \mathbf{Z}, \theta)$ is the knot polynomial $\Delta_i(t)$ (in the terminology of [4], Ch. 8).

The previous example has a natural generalisation. Let G be a finitely presented group. The abelian group $\mathbf{Z}$ endowed with the trivial action of G admits a 2-regular free resolution. Let V be a free finitely generated left R-module, where R is a commutative factorial ring, and $\rho : G \to GL_R(V)$ be a right representation. Let $\pi : G \to H$ be a homomorphism of G to a free abelian finitely generated group. We have then the twisted Fitting invariants corresponding to the right representation $\rho_\pi : G \to GL_\Lambda(V^H)$, where $\Lambda = R[H]$.

Definition 3.24. The twisted Fitting invariant $\delta_m(G, \mathbf{Z}, \rho_\pi) \in \Lambda$ will be denoted also by $\delta_m(G, \rho_\pi)$. The first Fitting invariant $\delta_1(G, \mathbf{Z}, \rho_\pi)$ will be denoted also by $A(G, \rho_\pi)$.

It turns out that the Fitting invariants with non-positive indices vanish:

Lemma 3.25.

$$\delta_m(G, \rho_\pi) = 0 \quad \textit{for} \quad m \leqslant 0.$$

Proof. Pick a presentation of a group G, let $g_1, ..., g_s$ be the generators, and $r_1, ..., r_l$ be the relations. Write a 2-regular free resolution for $\mathbf{Z}$ over $\mathfrak{L} = \mathbf{Z}G$ as follows:

$$F_* = \{0 \longleftarrow \mathfrak{L} \xleftarrow{\partial_1} \mathfrak{L}^s \xleftarrow{\partial_2} \mathfrak{L}^l \longleftarrow ...\};$$

here the free generators $e_1, ..., e_s$ of the module $F_1 = \mathfrak{L}^s$ correspond to the generators $g_1, ..., g_s$ of G and the homomorphism ∂_1 is given by $\partial_1(e_i) = 1 - g_i$. We can assume that $l \geqslant s$, and that $t = \pi(g_1)$ is one of the free generators of the abelian group H. Let us consider the image of our Fitting invariant in the field of fractions $\mathcal{R}$ of $R[H]$. This element coincides with the Fitting invariant of the chain complex

$$\mathcal{F}_* = \mathcal{R}^n \underset{\mathbf{Z}G}{\otimes} F_* = \{0 \longleftarrow \mathcal{R}^n \xleftarrow{\widetilde{\partial}_1} \mathcal{R}^{ns} \xleftarrow{\widetilde{\partial}_2} \mathcal{R}^{nl} \longleftarrow ...\}.$$

Observe that $\widetilde{\partial}_1$ is an epimorphism. Indeed, the restriction of $\widetilde{\partial}_1$ to the first direct summand $\mathcal{R}^n$ of $\mathcal{R}^{ns}$ equals $1 - t\rho(g_1) : \mathcal{R}^n \to \mathcal{R}^n$, and the determinant of this map is non-zero, therefore invertible in $\mathcal{R}$. Thus the rank of the matrix $\widetilde{\partial}_2$ is not more than $n(s-1)$, and the first Fitting invariant which can be non-zero is

$$J_1^{(2)}(\mathcal{F}_*) = I_{\mathrm{rk}\,\mathcal{F}_1 - \mathrm{rk}\,\mathcal{F}_0}(\partial_2).$$

□

The Fitting invariants of modules are useful for computations of the homology with twisted coefficients. Let X be a connected finite CW complex, put $G = \pi_1(X)$. Let $\pi : G \to H$ an epimorphism onto a free finitely generated abelian group, $\rho : G \to GL(V)$ be a right representation, where $V \approx \mathbf{Z}^n$ is a free finitely generated module over $\mathbf{Z}$. Let $\mathcal{C}_*(\widetilde{X})$ denote the cellular chain complex of the universal covering of X.

Proposition 3.26. $\delta_i(\mathcal{C}_*(\widetilde{X}), \rho_\pi) = \delta_i(G, \rho_\pi)$.

Proof. We have:

$$H_0(\widetilde{X}) \approx \mathbf{Z}, \quad H_1(\widetilde{X}) = 0.$$

The module $H_2(\widetilde{X})$ can be non-zero, and therefore the chain complex $\mathcal{C}_*(\widetilde{X})$ fails in general to be a resolution of the module $\mathbf{Z}$. Choose any subset $S \subset \mathcal{C}_2(\widetilde{X})$ generating the $\mathbf{Z}G$-submodule $Z_2(\widetilde{X})$ of 2-cycles in the complex $\mathcal{C}_*(\widetilde{X})$. Put $\mathfrak{L} = \mathbf{Z}G$ and let $\mathfrak{L}^S$ be the free $\mathfrak{L}$-module generated by the set S. Extend the identity map $S \xrightarrow{\text{id}} S$ to a $\mathfrak{L}$-module map $\phi : \mathfrak{L}^S \to \mathcal{C}_2(\widetilde{X})$. Put

$$\mathcal{C}'_3 = \mathcal{C}_3(\widetilde{X}) \oplus \mathfrak{L}^S, \quad \partial'_3 = (\partial_3, \phi) : \mathcal{C}'_3 \to \mathcal{C}_2(\widetilde{X}).$$

The chain complex $\mathcal{C}'_*$ is 2-regular and its second homology vanishes. Applying the same procedure to the third, fourth, etc. homology modules, we obtain an acyclic 2-regular resolution D_* of the module $\mathbf{Z}$ such that

$$\mathcal{C}_*(\widetilde{X}) \subset D_* \quad \text{and} \quad D_i/\mathcal{C}_i(\widetilde{X}) = 0 \quad \text{for} \quad i \leqslant 2.$$

Thus

$$\delta_i(\mathcal{C}_*(\widetilde{X}), \rho_\pi) = \delta_i(D_*, \rho_\pi); \quad \delta_i(G, \rho_\pi) = \delta_i(D_*, \rho_\pi). \qquad \square$$

3.6. **Fitting invariants of knots and links.** Let L be an oriented link. Put $G = \pi_1(S^3 \setminus L)$. The group G is finitely presented, therefore the module $\mathbf{Z}$ has a 2-regular free resolution over $\mathbf{Z}G$. Thus for any epimorphism $\pi : G \to H$ and any right representation $\rho : G \to GL_R(V)$ we obtain a sequence of elements

$$\delta_1(G, \rho_\pi),\ \delta_2(G, \rho_\pi), ... \in \Lambda = R[H],$$

defined up to multiplication by an invertible element of R.

Definition 3.27. The elements

$$\delta_i(G, \rho_\pi) = \delta_i(\pi_1(S^3 \setminus L), \rho_\pi)$$

are called *the Fitting invariants of the link L.*

We shall discuss these invariants and their relations with Novikov homology in more details in Section 5.

4. Twisted Alexander polynomials

We begin with a recollection of M.Wada's definition of the twisted Alexander polynomial (see [37] and also [12]); this occupies the first two subsections. In the rest of the section we discuss the relations between the Fitting invariants and twisted Alexander polynomials. The one-variable case and the multi-variable case are slightly different from each other and are considered separately.

4.1. **W-invariant of a matrix.** Let $\mathfrak{L}$ be a ring with a unit (non-commutative in general); let $\lambda : \mathfrak{L} \to Mat(n \times n, \Lambda)$ be a ring homomorphism, where Λ is a commutative factorial ring.

For any matrix $\mathcal{A}$ over $\mathfrak{L}$ we can substitute instead of each of its matrix entries $\mathcal{A}_{ij}$ its image with respect to λ. The result of this operation will be denoted by $\psi(\mathcal{A})$; the size of $\psi(\mathcal{A})$ is n times the size of $\mathcal{A}$. We have:

$$\psi(\mathcal{A}_1\mathcal{A}_2) = \psi(\mathcal{A}_1)\psi(\mathcal{A}_2),$$

if the number of rows of $\mathcal{A}_2$ equals the number of columns of $\mathcal{A}_1$.

Now let $\mathcal{B}$ be an $l \times s$ - matrix with coefficients in $\mathfrak{L}$. Assume that $l \geqslant s - 1$. Let

$$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix}$$

be a column of elements of $\mathfrak{L}$, such that

$$\mathcal{B} \cdot \alpha = 0. \tag{22}$$

We are going to associate to these data an element

$$W = W(\mathcal{B}, \alpha, \lambda) \in \Lambda.$$

For an integer j with $1 \leqslant j \leqslant s$ denote by $\mathcal{B}_j$ the $l \times (s-1)$-matrix obtained from $\mathcal{B}$ by suppressing the j-th column. We have the $nl \times n(s-1)$-matrix $\psi(\mathcal{B}_j)$. Let $S = i_1 < i_2 < ... < i_{n(s-1)}$ be a sequence of integers in $[1, nl]$. Let $\psi(\mathcal{B}_j)^S$ denote the square matrix formed by all the matrix entries of $\psi(\mathcal{B}_j)$ contained in the rows with indices in S. It is easy to deduce from the condition (22) that for every i, j with $1 \leqslant i, j \leqslant s$ we have

$$\det\Big(\psi(\mathcal{B}_j)^S\Big) \cdot \det\big(\ \psi(a_i)\ \big) = \det\Big(\psi(\mathcal{B}_i)^S\Big) \cdot \det\big(\ \psi(a_j)\ \big). \tag{23}$$

Let $Q_j(\mathcal{B})$ denote the GCD of the elements $\det\Big(\psi(\mathcal{B}_j)^S\Big)$ over all S. This element is defined up to multiplication by an invertible element of Λ.

Definition 4.1. Assume that there exists j with $\det\big(\ \psi(a_j)\ \big) \neq 0$. The element

$$W(\mathcal{B}, \alpha, \lambda) = \frac{Q_j(\mathcal{B})}{\det\big(\ \psi(a_j)\ \big)}$$

of the fraction field of Λ will be called W-invariant of the matrix $\mathcal{B}$.

If $l \leqslant s - 1$ we set by definition $W(\mathcal{B}, \alpha, \lambda) = 0$.

4.2. **Twisted Alexander polynomials: definition.** Now we apply the construction of the previous subsection to define the twisted Alexander polynomial. Let G be a finitely presented group. Let $\pi : G \to H$ be an epimorphism of G onto a finitely generated free abelian group H. Let R be a commutative factorial ring, and let $\lambda : G \to GL(n, R)$ be a group homomorphism. The tensor product of this homomorphism with $\pi : G \to R[H]^\bullet$ gives rise a ring homomorphism $\lambda_\pi : \mathfrak{L} = \mathbf{Z}G \to Mat(n \times n, \Lambda)$, where $\Lambda = R[H]$.

Pick a finite presentation $p = (g_1, ..., g_s \mid h_1, ...h_l)$ of the group G, with generators g_j and relators h_i. Let $\frac{\partial h_i}{\partial g_j} \in \mathbf{Z}G$ denote the corresponding Fox derivative.

Definition 4.2. The matrix

$$\mathcal{A} = \mathcal{A}(G, p) = \Big(\frac{\partial h_i}{\partial g_j}\Big) \in Mat(l \times s, \mathfrak{L})$$

will be called *Alexander matrix* of the presentation p.

Let

$$\alpha = \begin{pmatrix} 1 - g_1 \\ \vdots \\ 1 - g_s \end{pmatrix} \in \mathfrak{L}^s.$$

We have:

$$\mathcal{A} \cdot \alpha = 0,$$

therefore the constructions of the previous subsection apply and we obtain the W-invariant $W(\mathcal{A}, \lambda_\pi, \alpha)$ in the fraction field of $R[H]$. The ring $R[H]$ being isomorphic to the ring of Laurent polynomials in k variables with coefficients in R (where $k = \text{rk } H$), the W-invariant can be considered as a rational function of k variables with coefficients in the fraction field of $R[H]$. M.Wada proves in [37] that this element does not depend on the particular choice of the presentation p (up to multiplication by an invertible element of Λ) and is therefore determined by λ and π.

Definition 4.3. The element $W(\mathcal{A}, \lambda_\pi, \alpha)$ of the fraction field of $R[H]$ is denoted by $\Delta_{G,\lambda}$ and called *twisted Alexander polynomial of G associated to λ and π.*

4.3. **Relation with the Fitting invariants: the one-variable case.** In the case when $H \approx \mathbf{Z}$ the ring $\Lambda = R[H]$ is isomorphic to $R[t, t^{-1}]$. By definition the twisted Alexander polynomial is an element of the field of fractions of the ring $R[t]$. Consider a multiplicative subset $\Sigma \subset \Lambda$ consisting of all Laurent polynomials of the form $a_i t^i + ... + a_j t^j$, where $i, j \in \mathbf{Z},\ i \leqslant j$ and a_i, a_j are invertible elements of R.

Proposition 4.4. $\Delta_{G,\lambda}(t) \in \Sigma^{-1}\Lambda$.

Proof. We can assume that the first generator g_1 satisfies $\pi(g_1) = t \in R[t, t^{-1}]$. The first coefficient of the polynomial $\det(1 - t\lambda(g_1))$ equals 1 and the last coefficient equals $\pm\det(\rho(g_1))$, which is an invertible element of R. $\square$

Now we can proceed to the comparison of Fitting invariants and twisted Alexander polynomials. Let $\rho : G \to GL_R(V)$ be a right representation of G, where V is a finitely generated free left R-module. Choose a basis in V, then we obtain a homomorphism

$$\overline{\rho} : G \to GL(n, R); \quad \overline{\rho}(g) = M(\rho(g)).$$

Proposition 4.5. *The Fitting invariant $A(G, \rho_\pi)$ divides the twisted Alexander polynomial $\Delta_{G,\overline{\rho}}$ in the ring $\Sigma^{-1}\Lambda$.*

Proof. Let $g_1, ..., g_s$ be generators of G and $h_1, ..., h_l$ be relators. The free resolution of $\mathbf{Z}$ over $\mathfrak{L} = \mathbf{Z}G$ can be constructed as follows:

$$F_* = \{0 \longleftarrow \mathfrak{L} \xleftarrow{\partial_1} \mathfrak{L}^s \xleftarrow{\partial_2} \mathfrak{L}^l \longleftarrow ... \tag{24}$$

where

$$M(\partial_1) = \begin{pmatrix} 1 - g_1 \\ \vdots \\ 1 - g_s \end{pmatrix}$$

and $M(\partial_2)$ equals the Alexander matrix $\mathcal{A} = \mathcal{A}(G, p)$ corresponding to the presentation $p = (g_1, \dots, g_s \mid h_1, \dots, h_l)$. (see [1], Ch. 9). The polynomial $A(G, \rho_\pi)$ is computed from the chain complex $\mathcal{F}_* = \Lambda^n \underset{\mathbf{Z}G}{\otimes} F_*$. The matrix $M(\widetilde{\partial}_2)$ of the second boundary operator in the chain complex $\mathcal{F}_*$ is equal to the matrix $\psi(\mathcal{A})$ of Subsection 4.1, where $\psi = \overline{\rho_\pi}$. Now the proposition follows easily, since for every j the element $Q_j(\mathcal{A})$ from Definition 4.1 is the GCD of a certain family of $(s-1)n \times (s-1)n$-minors of $M(\widetilde{\partial}_2)$, and $A(G, \rho_\pi)$ is the GCD of *all* $(s-1)n \times (s-1)n$-minors. $\square$

Proposition 4.6. *If the group G has a presentation with s generators and $s - 1$ relations (for example, G is the fundamental group of the complement to a link in S^3), then the elements*

$$A(G, \rho_\pi), \quad \Delta_{G,\overline{\rho}} \in \Sigma^{-1}\Lambda$$

are equal up to multiplication by an invertible element of $\Sigma^{-1}\Lambda$.

Proof. As before we can assume that $\pi(g_1) = t \in R[t, t^{-1}]$. Suppressing the first n columns of the matrix $\psi(\mathcal{A})$ we obtain an $n(s-1) \times n(s-1)$-matrix $\mathcal{M}'$. Up to invertible elements of $\Sigma^{-1}\Lambda$ we have:

$$\det \mathcal{M}' = \Delta_{G,\overline{\rho}}. \tag{25}$$

The boundary operator ∂_1' in the localized complex

$$\mathcal{F}_*' = \Sigma^{-1}\mathcal{F}_* = \Sigma^{-1}\Lambda \underset{\mathbf{Z}G}{\otimes} F_*$$

is an epimorphism (see (24) for the definition of F_*), and it is easy to deduce that the second boundary operator ∂_2' in this complex is isomorphic to a homomorphism $(0,\phi) : \Lambda^{n(s-1)} \to \Lambda^n \oplus \Lambda^{n(s-1)}$ where the matrix of ϕ equals $\mathcal{M}'$. Thus up to invertible elements of $\Sigma^{-1}\Lambda$ we have:

$$\det \mathcal{M}' = I_{n(s-1)}(\partial_2) = J_1^{(2)}(\mathcal{F}_*') = J_1^{(2)}(\mathcal{F}_*) = A(G, \rho_\pi), \tag{26}$$

and the proof is completed. □

4.4. **Relation with Fitting invariants: the multi-variable case.** In the case when $H \approx \mathbf{Z}^k$ with $k \geqslant 2$ the ring $\Lambda = R[H]$ is isomorphic to $R[t_1, t_1^{-1}, ..., t_k, t_k^{-1}]$. The twisted Alexander polynomial in this case is an element of the ring Λ itself (cf. [37], p.253). Indeed, choose a presentation $p = (g_1, \dots, g_s \mid h_1, \dots, h_l)$ of G in such a way, that $\pi(g_1) = t_1, \pi(g_2) = t_2$. Let $\rho : G \to GL_R(V)$ be a right representation of the group G where V is a free finitely generated R-module. Put

$$P_1 = \det(1 - t_1\rho(g_1)), \quad P_2 = \det(1 - t_2\rho(g_2)).$$

Then

$$P_1 = 1 + a_1t_1 + ... + a_nt_1^n, \quad \text{where} \quad a_n = \pm \det \rho(g_1); \tag{27}$$

$$P_2 = 1 + b_1t_2 + ... + b_nt_2^n, \quad \text{where} \quad b_n = \pm \det \rho(g_2). \tag{28}$$

Let $\mathcal{A} = \mathcal{A}(G, p)$ denote the Alexander matrix for the presentation p. For any sequence $S = i_1 < i_2 < ... < i_{n(s-1)}$ of integers put

$$\alpha_1^S = \det \psi(\mathcal{A}_1)^S, \quad \alpha_2^S = \det \psi(\mathcal{A}_2)^S.$$

The property (23) implies

$$P_2\alpha_1^S = P_1\alpha_2^S \quad \text{for every} \quad S. \tag{29}$$

The polynomials P_1, P_2 are relatively prime in Λ, therefore $P_1 \mid \alpha_1^S$ in Λ, and

$$\Delta_{G,\bar{\rho}} = \frac{Q_1}{P_1} \in \Lambda, \quad \text{where} \quad Q_1 = \mathrm{GCD}_S\ \alpha_1^S,$$

as we have claimed above. Moreover, it is easy to deduce from (29) the following :

$$\mathrm{GCD}_S\big(\alpha_1^S, \alpha_2^S\big) \;\Big|\; \frac{Q_1}{P_1},$$

and we obtain

Proposition 4.7. *The Fitting invariant* $A(G, \rho_\pi)$ *divides the twisted Alexander polynomial* $\Delta_{G,\overline{\rho}}$ *in the ring* $\Lambda = R[H]$. □

Now we proceed to an analog of Proposition 4.6 for the multi-variable case. Let us introduce the corresponding localization.

Definition 4.8. Let $\mu : H \to \mathbf{R}$ be any non-trivial group homomorphism.

1) We say that an element $x \in \Lambda$ has *μ-monic ends* if

$$x = x_0 h_0 + \sum_{i=1}^{a-1} x_i h_i + x_a h_a, \qquad \text{with} \qquad x_i \in R, h_i \in H,$$

where $x_0, x_a \in R^\bullet$ and $\mu(h_0) < \mu(h_i) < \mu(h_a) \quad \forall i : 0 < i < a.$

2) The multiplicative subset of all the elements of Λ with μ-monic ends will be denoted $\Sigma_\mu \subset \Lambda$.

Proposition 4.9. *Assume that the group G has a presentation with s generators and $s-1$ relators. Then the images of the elements* $A(G, \rho_\pi)$ *and* $\Delta_{G,\overline{\rho}}$ *in the ring* $\Sigma_\mu^{-1}\Lambda$ *are equal up to invertible elements of this ring.*

Proof. Similar to the proof of Proposition 4.6. □

It is natural to ask whether the Fitting invariant and twisted Alexander polynomial are equal up to invertible elements of Λ, at least in the case of knot groups. I do not know if it is true.

5. Three-dimensional manifolds

In this section we study the particular case case of C^∞ compact three-dimensional manifolds M of zero Euler characteristic. We prove Theorem 5.5 which gives a criterion for vanishing of the twisted Novikov homology of M in terms of the twisted Alexander polynomial of $\pi_1(M)$ or, equivalently, in terms of the first Fitting invariant of $\pi_1(M)$. The second main result of this section is Theorem 5.7. This theorem describes for a given right representation ρ of $\pi_1(M)$ and for a given epimorphism $\pi_1(M) \to H$ the structure of all classes $\xi \in H^1(M, \mathbf{R})$ such that the ρ-twisted Novikov homology $\widehat{H}_*(M, \rho_\pi, \xi)$ vanishes. It turns out that this set is an open conical polyhedral subset of $H^1(M, \mathbf{R})$. In Subsection 5.3 we discuss the relation of the twisted Novikov homology and the Thurston norm; we suggest a natural question about this relation.

5.1. **The twisted Novikov homology and the twisted Alexander polynomial for 3-manifolds.** Let M be a compact C^∞ three-dimensional manifold with $\chi(M) = 0$. Write $G = \pi_1(M)$, $\mathfrak{L} = \mathbf{Z}G$. We begin with two lemmas which describe the homotopy type of the chain complex of the universal covering of M.

Lemma 5.1. *Let M be a closed connected 3-manifold. Then the cellular chain complex of its universal covering is chain homotopy equivalent to the following one:*

$$C_* = \{0 \longleftarrow \mathfrak{L} \xleftarrow{\partial_1} \mathfrak{L}^l \xleftarrow{\partial_2} \mathfrak{L}^l \xleftarrow{\partial_3} \mathfrak{L} \longleftarrow 0\} \tag{30}$$

where:

1) *The matrix of ∂_1 equals*

$$\begin{pmatrix} 1 - g_1 \\ 1 - g_2 \\ \vdots \\ 1 - g_l \end{pmatrix}$$

and the elements $g_1, \ldots, g_l \in G$ generate the group G.

2) *The matrix of ∂_3 equals*

$$\begin{pmatrix} 1 - \varepsilon_1 h_1, & 1 - \varepsilon_2 h_2, & \ldots & , 1 - \varepsilon_l h_l \end{pmatrix}$$

where $h_i \in G$ and $\varepsilon_i = 1$ if the loop h_i conserves the orientation, $\varepsilon_i = -1$ if h_i reverses the orientation. The elements $h_1, \ldots, h_l \in G$ generate the group G.

3) *The matrix of ∂_2 is the Alexander matrix associated to some presentation of the group G.*

Proof. The lemma follows immediately from the existence of the Hegaard decomposition for closed three-dimensional manifolds. □

Lemma 5.2. *Let M be a connected compact 3-manifold with non-empty boundary and $\chi(M) = 0$. Then the cellular chain complex of the universal covering $\widetilde{M}$ is chain homotopy equivalent to the following one:*

$$C_* = \{0 \longleftarrow \mathfrak{L} \xleftarrow{\partial_1} \mathfrak{L}^l \xleftarrow{\partial_2} \mathfrak{L}^{l-1} \longleftarrow 0\} \tag{31}$$

where the matrix of ∂_1 equals

$$\begin{pmatrix} 1 - g_1 \\ 1 - g_2 \\ \vdots \\ 1 - g_l \end{pmatrix}$$

and the elements $g_1, \dots, g_l \in G$ generate the group G. The matrix of ∂_2 is the Alexander matrix associated to some presentation of the group G.

Proof. The Morse theory guarantees the existence of a Morse function $f : M \to \mathbf{R}$ such that

$$f|\partial M = const = \max_{x \in M} f(x)$$

and the number $m_i(f)$ of critical points of index i satisfies the following:

$$m_0(f) = 1,\ m_1(f) = m_2(f),\ m_3(f) = 0.$$

The cellular decomposition corresponding to f, satisfies the requirements of the lemma. □

We shall use these lemmas to study the twisted Alexander polynomial and the Fitting invariants of M. Let V be a free finitely generated R-module (where R is a commutative factorial ring) and $\rho : G \to GL_R(V)$ be a right representation. Let $\pi : G \to H$ be an epimorphism of G onto a free abelian finitely generated group H. Let $\xi : H \to \mathbf{R}$ be any non-trivial homomorphism. Denote $R[H]$ by Λ, and let Σ_ξ be the multiplicative subset of Laurent polynomials with ξ-monic ends (see Definition 4.8). Let

$$\widetilde{C}_* = \Sigma_\xi^{-1}\Big(V^H \underset{\mathfrak{L}}{\otimes} C_*\Big),$$

where V^H is endowed with the structure of a right $\mathfrak{L}$-module determined by the representation $\rho_\pi : G \to GL(V^H)$. Put $\Lambda_{[\xi]} = \Sigma_\xi^{-1}\Lambda$, and denote rk V by n. We have a natural isomorphism

$$\widetilde{C}_1 \approx \Lambda_{[\xi]}^{n(l-1)} \oplus \Lambda_{[\xi]}^n$$

(where the first summand of the direct sum correspond to the elements $e_1, ..., e_{l-1}$ of the $\mathfrak{L}$-basis of C_1, and the second corresponds to e_l). The projection of $\widetilde{C}_1$ onto the direct summand $\Lambda_{[\xi]}^{n(l-1)}$ will be denoted by p_1. Similarly, the module $\widetilde{C}_2$ is naturally isomorphic to $\Lambda_{[\xi]}^{n(l-1)}$ in the case $\partial M \neq \varnothing$ and to the direct sum $\Lambda_{[\xi]}^{n(l-1)} \oplus \Lambda_{[\xi]}^n$ in the case of closed manifolds. Let

(32) $$\mathcal{D} = p_1 \circ \Big(\partial_2 | \Lambda_{[\xi]}^{n(l-1)}\Big).$$

Proposition 5.3. *Assume that*

(33) $$\xi(g_l) < 0,$$

and in the case $\partial_1(M) = \varnothing$ assume moreover that

(34) $$\xi(h_l) < 0.$$

Then the chain complex $\widetilde{C}_*$ *is chain homotopy equivalent to a free* $\Lambda_{[\xi]}$*-chain complex*

$$0 \longleftarrow \Lambda_{[\xi]}^{n(l-1)} \overset{\mathcal{D}}{\longleftarrow} \Lambda_{[\xi]}^{n(l-1)} \longleftarrow 0, \tag{35}$$

concentrated in dimensions 1 and 2.

Proof. Let us do the case of closed manifolds, the case $\partial M \neq 0$ is similar. Observe that the homomorphisms

$$1 - \rho_\pi(g_l),\ 1 - \rho_\pi(h_l) : \Lambda_{[\xi]}^n \to \Lambda_{[\xi]}^n$$

are invertible. Therefore, using a standard basis change in the chain complex $\widetilde{C}_*$ we can split off from $\widetilde{C}_*$ two trivial chain complexes

$$A_* \approx T_*(1, \Lambda_{[\xi]}^n), \quad B_* \approx T_*(2, \Lambda_{[\xi]}^n),$$

in such a way, that the resulting chain complex is isomorphic to (35). □

Now we can establish the relation between the twisted Alexander polynomial and the Fitting invariant of a three-manifold.

Proposition 5.4. *Let G be the fundamental group of a compact three-dimensional C^∞ manifold M with $\chi(M) = 0$. Let $\rho : G \to GL_R(V)$ be a right representation, where V is a free finitely generated R-module over a factorial commutative ring R. Let $\pi : G \to H$ be an epimorphism, where H is a free abelian finitely generated group. Let $\xi : H \to \mathbf{R}$ be any non-trivial homomorphism.*

Then the Fitting invariant $A(G, \rho_\pi)$ and the twisted Alexander polynomial $\Delta_{G,\overline{\rho}}$ are equal up to multiplication by an invertible element of $\Lambda_{[\xi]}$.

Proof. We shall do the case of closed manifolds, and the case $\partial M \neq \varnothing$ is even simpler, and will be omitted.

The Fitting invariant of a chain complex depends only on its homotopy type and does not change when we localize the base ring. Therefore we can use the chain complex (35) for computation of the image of $A(G, \rho_\pi)$ in the ring $\Lambda_{[\xi]}$ (we assume that the conditions (33), (34) hold, which is easy to arrange by a permutation of the elements of the basis). Thus the image of $A(G, \rho_\pi)$ in the ring $\Lambda_{[\xi]}$ equals $\det \mathcal{D}$ (up to invertible elements of this ring).

To compute the twisted Alexander polynomial we shall use the matrix of the boundary operator ∂_2 from (30). The $nl \times nl$-matrix $\mathcal{D}' = \psi(\partial_2)$ satisfies $\mathcal{H} \circ \mathcal{D}' = 0$, where $\mathcal{H} = \psi\big((1 - h_1, ...1 - h_l)\big)$.

Let $\mathcal{D}''$ denote the $nl \times n(l-1)$-matrix obtained by suppressing the last n columns of the matrix of ∂_2. Using the invertibility over $\Lambda_{[\xi]}$ of the matrix $1-\rho_\pi(h_l)$ and the condition $\mathcal{D}''\circ\mathcal{H}=0$, we deduce that the last n rows of the matrix $\mathcal{D}''$ are linear combinations of the first $n(l-1)$ rows. This implies that the GCD of the $n(l-1)\times n(l-1)$-minors of the matrix $\mathcal{D}''$ equals to the determinant of $\mathcal{D}$. We obtain therefore the following equality:

$$\Delta_{G,\bar\rho} = \frac{\det \mathcal{D}}{\det(1-\rho_\pi(g_l))} \ ,$$

and the proof is over. □

Theorem 5.5. *Let M be a connected compact C^∞ manifold of dimension 3 with $\chi(M)=0$. Let $G=\pi_1(M)$, and let $\rho : G\to GL_R(V)$ be a right representation of the group G, where V is a finitely generated free module over a commutative factorial ring R. Let $\pi : G\to H$ be an epimorphism onto a free abelian finitely generated group, and $\xi : H\to \mathbf{R}$ be a non-zero homomorphism. Then the three following conditions are equivalent:*

1) *The twisted Novikov homology $\widehat{H}_i(M,\rho_\pi,\xi)$ vanishes for all i.*
2) *The first twisted Novikov homology module $\widehat{H}_1(M,\rho_\pi,\xi)$ vanishes.*
3) *The Fitting invariant $A(G,\rho_\pi)$ is ξ-monic.*
4) *The twisted Alexander polynomial $\Delta_{G,\bar\rho}$ is ξ-monic.*

Proof. Observe that the two last conditions are equivalent in view of Proposition 5.4. Therefore it suffices to prove that the first three conditions are equivalent. Proposition 5.3 implies that the twisted Novikov homology $\widehat{H}_*(M,\rho_\pi,\xi)$ is isomorphic to the homology of the chain complex

$$0 \longleftarrow \widehat{\Lambda}_\xi^{n(l-1)} \xleftarrow{\mathrm{id}\otimes\mathcal{D}} \widehat{\Lambda}_\xi^{n(l-1)} \longleftarrow 0, \tag{36}$$

concentrated in dimensions 1 and 2, and $\det\mathcal{D}$ equals to the Fitting invariant of M with respect to ρ,ξ. The homology of the chain complex (36) vanishes for every i if and only if it vanishes for $i=1$, and both these conditions are equivalent to the invertibility of $\det\mathcal{D}$ in $\widehat{\Lambda}_\xi$. This last condition holds if and only if this determinant is is invertible in $\Lambda_{(\xi)}$. We have seen during the proof of Proposition 5.4 that the elements $\det\mathcal{D}$ and $A(G,\rho_\pi)$ are equal up to multiplication by invertible elements of $\Lambda_{(\xi)}$. Therefore the twisted Novikov homology vanishes if and only if the element $A(G,\rho_\pi)$ is ξ-monic. □

The previous theorem allows to describe the structure of the set of all ξ such that the Novikov homology $\widehat{H}_*(M,\rho_\pi,\xi)$ vanishes.

Definition 5.6. Let $G=\pi_1(M)$, and let $\rho : G\to GL_R(V)$ be a right representation of the group G. Let $\pi : G\to H$ be an epimorphism onto a free abelian finitely generated group. A non-zero cohomology class $\xi \in H^1(M, \mathbf{R})$ will be called (ρ, π)-*acyclic*, if the twisted Novikov homology $\widehat{H}_1(M, \rho_\pi, \xi)$ vanishes. When the homomorphism π is clear from the context, we shall say that ξ is ρ-acyclic.

Theorem 5.7. *For a given right representation $\rho : G \to GL_R(V)$ and a given epimorphism $\pi : G \to H$ the set of all (ρ, π)-acyclic classes $\xi \in H^1(M, \mathbf{R})$ is an open polyhedral conical subset of $H^1(M, \mathbf{R})$. If R is a field, then the set of all (ρ, π)-acyclic classes is either empty, or equals $H^1(M, \mathbf{R}) \setminus \{0\}$ or is the complement in $H^1(M, \mathbf{R})$ to a finite union of integral hyperplanes.*

Proof. The Fitting invariant $A(G, \rho_\pi)$ is an element of $R[H]$. The group H is isomorphic to the integral lattice $\mathbf{Z}^k \subset \mathbf{R}^k \approx H \otimes \mathbf{R}$, and $A(G, \rho_\pi)$ is then identified with a Laurent polynomial $\mathcal{A}$ in variables $t_1, ..., t_k$ with coefficients in R. If $\mathcal{A} = 0$, then no class ξ is ρ-acyclic. If $\mathcal{A}$ is a monomial, $\mathcal{A} = \alpha \cdot h$, where $\alpha \in R$, and $h \in H$, then either

(1) α is invertible, and in this case all classes ξ are ρ-acyclic, or
(2) α is non-invertible, and in this case there are no ρ-acyclic classes.

Now let us consider the non-degenerate case, when the Newton polytope $\mathcal{P}$ of the polynomial $\mathcal{A}$ contains more than one point. For a homomorphism $\xi : H \to \mathbf{R}$ the polynomial $\mathcal{A}$ is ξ-monic if and only if the polytope $\mathcal{P}$ has a vertex v such that:

1) the coefficient a_v of $\mathcal{A}$ corresponding to this vertex is an invertible element of R,
2) for every other vertex v' of $\mathcal{P}$ we have $\xi(v) > \xi(v')$.

For a given vertex v the set Γ_v of all ξ satisfying the conditions 1) and 2) above is an open polyhedral cone. Indeed, for a pair of vertices v, v' of $\mathcal{P}$ put

$$\Gamma_{v,v'} = \{\xi \in H^1(M, \mathbf{R}) \mid \xi(v) = \xi(v')\},$$
$$\Gamma^+_{v,v'} = \{\xi \in H^1(M, \mathbf{R}) \mid \xi(v) > \xi(v')\},$$

so that $\Gamma^+_{v,v'}$ is one of the two open half-spaces corresponding to $\Gamma_{v,v'}$. Then

$$\Gamma_v = \bigcap_{v'} \Gamma^+_{v,v'}.$$

The sets Γ_v are open polyhedral cones which are disjoint for different v.

Now let R be a field. The cases when $\mathcal{A} = 0$ or $\mathcal{A}$ is invertible are done as above. When the Newton polytope $\mathcal{P}$ of $\mathcal{A}$ contains more than one vertex, and the set of all (ρ, π)-acyclic classes is the complement in $H^1(M, \mathbf{R})$ to the union of all the hyperplanes $\Gamma_{v,v'}$. $\square$

5.2. **Detecting fibred links.** In this section we give a necessary condition for a link in S^3 to be fibred. Let us first recall the definition of a fibred link and related notions.

Definition 5.8. **1.** Let V be a compact topological $n-1$-manifold with $\partial V \neq \varnothing$ and let $h : V \to V$ be a homeomorphism which restricts to the identity on ∂V. Forming a mapping torus V_h, and identifying $(x,t) \sim (x,t')$ for each $x \in \partial V, t, t' \in S^1$ we obtain a closed topological manifold. This manifold is denoted $B(V,h)$.

A closed manifold M is called *an open book decomposition* if it is homeomorphic to $B(V,h)$ for some V and h. The images in M of fibers $V \times \{t\}$ $(t \in S^1)$ are called *pages* of the open book, and the image of $\partial V \times \{t\}$ in M is called *binding* .

2. A C^∞-embedding of the disjoint union of several copies of an oriented circle into S^3 is called *oriented link.*

3. An oriented link L is called *fibred* if there is an orientation preserving homeomorphism

$$\phi : S^3 \to B(F^2, h)$$

where F^2 is a compact oriented surface and the restriction of ϕ to L is a preserving orientation homeomorphism onto the binding ∂F of the open book.

For an oriented link L let $G = \pi_1(S^3 \setminus L)$. Let $\eta : G \to \mathbf{Z}$ be a homomorphism, which sends every positive meridian of L to $1 \in \mathbf{Z}$. Form the corresponding completion $\widehat{\mathfrak{L}}_\eta$ of the group ring $\mathfrak{L} = \mathbf{Z}G$, and let $\mathcal{C}_*(\widetilde{S^3 \setminus L})$ be the cellular chain complex of the universal covering of the link complement.

If the link L is fibred then the complement $S^3 \setminus L$ admits a fibration over a circle. Although the manifold $S^3 \setminus L$ is not compact it turns out that the corresponding analog of Theorem 2.5 holds; this is the subject of the next proposition.

Proposition 5.9. *If the link L is fibred, then*

$$H_*\Big(\widehat{\mathfrak{L}}_\eta \underset{\mathfrak{L}}{\otimes} \mathcal{C}_*(\widetilde{S^3 \setminus L})\Big) = 0.$$

Proof. It is clear that $S^3 \setminus L$ is homotopy equivalent to the mapping torus F^2_h where h is a homeomorphism. The space F^2_h is homotopy equivalent to the mapping torus F^2_g where $g : F^2 \to F^2$ is a cellular map (see [28], Proposition 6.1). Thus our proposition follows from the next theorem.

Theorem 5.10. *Let X be a finite connected CW-complex. Let $g : X \to X$ be a cellular map, and let X_g be the mapping torus. Let $G = \pi_1(X_g)$ and let $\eta : G \to \mathbf{Z}$ be the homomorphism induced by the projection $f : X_g \to S^1$. Put $\mathfrak{L} = \mathbf{Z}G$ and let $\widehat{\mathfrak{L}}_\eta$ be the corresponding Novikov completion. Then*

$$H_*\Big(\widehat{\mathfrak{L}}_\eta \underset{\mathfrak{L}}{\otimes} \mathcal{C}_*(\widetilde{X_g})\Big) = 0.$$

Proof. Let

$$H = \ker \eta, \;\; R = \mathbf{Z}H$$

$$P = \{\lambda \in \mathfrak{L} \mid \text{supp } \lambda \in \eta^{-1}(]-\infty, 0])$$

$$\widehat{P} = \{\lambda \in \widehat{\mathfrak{L}}_\eta \mid \text{supp } \lambda \in \eta^{-1}(]-\infty, 0])\}.$$

Pick any $t \in G$ such that $\eta(t) = -1$. Then the ring P is isomorphic to the twisted polynomial ring $R_\theta[t]$, and the ring $\widehat{P}$ is isomorphic to the twisted power series ring $R_\theta[[t]]$ (where θ is the isomorphism of the ring R defined by $x \mapsto txt^{-1}$.) Denote by $Y \to X_g$ the infinite cyclic covering induced by the natural projection $f : X_g \to S^1$ from the universal covering $\mathbf{R} \to S^1$. The function f lifts to a continuous function $F : Y \to \mathbf{R}$; for each $k \in \mathbf{Z}$ the space $F^{-1}([k, k+1])$ is homeomorphic to the mapping cylinder Z_g of the map $g : X \to X$. Let $Y_k = F^{-1}(]-\infty, k])$. For every $m < k$ the inclusion $Y_m \subset Y_k$ is a homotopy equivalence. Let $\widetilde{Y}$ be the universal covering for Y, which is also a universal covering for X_g. Let $\widetilde{Y}_k$ be the inverse image of Y_k in $\widetilde{Y}$.

The cellular chain complex $\mathcal{C}_*(\widetilde{Y}_k)$ is a free finitely generated P-module. We have for every k

$$\widehat{\mathfrak{L}}_\eta \underset{\mathfrak{L}}{\otimes} \mathcal{C}_*(\widetilde{Y}) = \widehat{\mathfrak{L}}_\eta \underset{\widehat{P}}{\otimes} \Big(\widehat{P} \underset{P}{\otimes} \mathcal{C}_*(\widetilde{Y}_k)\Big)$$

thus it remains to show that the chain complex $\widehat{P} \underset{P}{\otimes} \mathcal{C}_*(\widetilde{Y}_0)$ is acyclic. Observe that since the modules $\mathcal{C}_*(\widetilde{Y}_k)$ are free finitely generated P-modules, we have

$$\widehat{P} \underset{P}{\otimes} \mathcal{C}_*(\widetilde{X}_0) = \varprojlim \mathcal{C}_*(\widetilde{X}_0, \widetilde{X}_{-k})$$

(where $k \in \mathbf{N}$). Since the inclusion $X_{-k} \hookrightarrow X_0$ is a homotopy equivalence, all chain complexes $\mathcal{C}_*(\widetilde{X}_0, \widetilde{X}_{-k})$ are acyclic.

Applying Theorem A.19 of [16], Appendix we obtain:

$$H_*\big(\varprojlim \mathcal{C}_*(\widetilde{Y}_0, \widetilde{Y}_{-k})\big) \approx \varprojlim H_*(\mathcal{C}_*(\widetilde{Y}_0, \widetilde{Y}_{-k})) = 0,$$

and this completes the proof. $\square$

Now let us apply this result to the twisted Novikov homology of the complement to a fibered link. Let R be a commutative ring. Denote by ξ the inclusion $\mathbf{Z} \hookrightarrow \mathbf{R}$, and put $\Lambda = R[\mathbf{Z}]$. The ring $\widehat{\Lambda}_\xi$ is then identified with the ring $R((t))$, and the ring $\Lambda_{(\xi)}$ is identified with the ring

$$R\langle t\rangle = \left\{ \frac{P(t)}{t^n \cdot (1+tQ(t))} \mid n \in \mathbf{Z} \quad \text{and} \quad P, Q \in R[t] \right\} \subset R((t)).$$

The ξ-monic elements of Λ are identified with polynomials of the form $\mu(1+tQ(t))$ where $Q(t) \in R[t]$, and $\mu \in R$ is invertible. Invertible elements of $\Lambda_{(\xi)} = R\langle t\rangle$ are also called *monic.* Let $\rho : G \to GL_R(V)$ be any right representation.

Proposition 5.11. *If L is fibred, then the twisted Novikov homology $\widehat{H}_*(S^3 \setminus L, \rho_\pi, \xi)$ equals to zero.*

Proof. The proposition is deduced from Theorem 5.10 in the same way as Theorem 2.8 is deduced from Theorem 2.5. □

Observe that the space $S^3 \setminus L$ is homotopy equivalent to a compact 3-manifold with boundary. Indeed, let N be an open tubular neighbourhood of L in S^3. Then $S^3 \setminus L$ has the same homotopy type as $S^3 \setminus N$. Theorem 5.5 implies then the following result.

Corollary 5.12. *Let G denote the group $\pi_1(S^3 \setminus L)$. If the link L is fibred, then:*

1. *The Fitting invariant $A(G, \rho_\pi) \in R[t, t^{-1}]$ is monic.*
2. *The twisted Alexander polynomial $\Delta_{G,\overline{\rho}} \in R\langle t\rangle$ is monic.*

This result is related to the theorem due to H.Goda, T.Kitano, and T.Morifuji [9]. Their theorem says that if a knot K is fibred, and $\rho : \pi_1(S^3 \setminus K) \to SL(n, F)$ is a representation (where F is a field), then the leading coefficient of the twisted Alexander polynomial associated to ρ, equals to 1.

Our theorem is valid in more general setting: it allows representations in $GL(n, R)$, where R is a factorial ring, and not only in $SL(n, F)$. On the other hand, for the representations in $SL(n, F)$ the theorem of [9] gives much more information since it guarantees that the leading coefficient of the twisted Alexander polynomial equals 1, and Corollary 5.12 asserts only that this coefficient is non-zero, that is, the twisted Alexander polynomial does not vanish.

5.3. **Thurston cones and ρ-acyclicity cones.** Let M be a closed three-dimensional C^∞ manifold. Put $G = \pi_1(M)$, $\mathfrak{L} = \mathbf{Z}G$. Let

$\mathcal{V}(M) \subset H^1(M, \mathbf{R})$ be the subset of all the cohomology classes, representable by closed 1-forms without zeros. Let

$$\mathcal{V}_h(M) = \{\xi \in H^1(M, \mathbf{R}) \mid H_*(\widehat{\mathfrak{L}}_\xi \underset{\mathfrak{L}}{\otimes} \mathcal{C}_*(\widetilde{M})) = 0\}.$$

For a given right representation $\rho : \pi_1(M) \to GL(\mathbf{Z}^n)$ let $\mathcal{V}_{alg}(M, \rho)$ be the subset of all (ρ, π)-acyclic cohomology classes ξ, where π is the projection $\pi_1(M) \to H_1(M)/Tors$. It follows from the results of the previous section, that

$$\mathcal{V}(M) \subset \mathcal{V}_h(M) \subset \mathcal{V}_{alg}(M) = \bigcap_{\rho \in \mathcal{R}} \mathcal{V}_{alg}(M, \rho), \tag{37}$$

where $\mathcal{R}$ is the set of all right representations $\pi_1(M) \to GL(\mathbf{Z}^n)$. The Thurston theorem [34] implies that the set $\mathcal{V}(M)$ is an open conical polyhedral subset of $H^1(M, \mathbf{R})$. The set $\mathcal{V}_{alg}(M, \rho)$ is also an open conical polyhedral subset of $H^1(M, \mathbf{R})$, as it follows from the results of the previous section.

Question. For which manifolds M the inclusions (37) are equalities?

For every ρ the set $\mathcal{V}_{alg}(M, \rho)$ is effectively computable from the twisted Alexander polynomial or from the first Fitting invariant. Thus, investigating the inclusions (37) will give a considerable amount of information on the structure of the set $\mathcal{V}(M)$. We think that computer experiments can help here, and can clarify the problem (see the paper [11] for an example of application of the Kodama's program KNOTS to a similar question). In the rest of this section we shall show that the properties of the inclusion (37) are quite sensitive to the class of the representations which we consider. Let $\mathcal{R}_{\mathcal{F}}$ be the set of all representations $G \to GL(\mathbf{F}^n)$ where $\mathbf{F}$ is a finite field. We are going to show that there are closed manifolds M with

$$\mathcal{V}(M) \neq \bigcap_{\rho \in \mathcal{R}_{\mathcal{F}}} \mathcal{V}_{alg}(M, \rho).$$

Let M be any 3-manifold with a following property (L):

L1) There is an open subset $U \subset H^1(M, \mathbf{R})$ such that every element of U does not contain a nowhere vanishing 1-form.

L2) There is a non-zero class $\xi_0 \in H^1(M, \mathbf{R})$ which contains a nowhere vanishing 1-form.

Existence of such manifolds is indicated in [34], p. 125-127. If M has property (L), then for every ρ the set $\mathcal{V}(M, \rho)$ of all ρ-acyclic classes is non-empty, and by Theorem 5.7 this set equals $H^1(M, \mathbf{R}) \setminus \{0\}$ or is the complement in $H^1(M, \mathbf{R})$ to a finite union of integral hyperplanes (the case $\mathcal{V}(M, \rho) = \varnothing$ is excluded by the property L2)). Therefore

$\mathcal{V}(M,\rho)$ is open and dense in $H^1(M,\mathbf{R})$ and the intersection I of all the sets $\mathcal{V}_{alg}(M,\rho)$ over all right representations in $GL(\mathbf{F}^n)$ (where $\mathbf{F}$ is a finite field), is a residual subset. Thus I intersects every open subet of $H^1(M,\mathbf{R})$, in particular $U\cap I \neq \emptyset$, and therefore there exist cohomology classes $\xi \in H^1(M,\mathbf{R})$ which are ρ-acyclic for every representation ρ in $GL(\mathbf{F}^n)$, but not representable by a nowhere vanishing closed 1-form.

References

[1] G.Burde, H.Zieschang, *Knots*, de Gruyter, 1985.
[2] W.H.Cockroft, R.G.Swan, *On the homotopy type of 2-dimensional complexes* J.London Math. Soc, **11**, 1 (1961) 306-311.
[3] M.M.Cohen, *A course in Simple-Homotopy theory*, Springer, 1972.
[4] M.H.Crowell, R.H.Fox, *Introduction to knot theory*, Ginn and Company, 1963. Springer, 1977.
[5] D.Eisenbud *Commutative Algebrs with a View Toward Algebraic Geometry* , Springer, 1994.
[6] M.Farber *Exactness of Novikov inequalities* , Functional Analysis and Applications **19**, 1985.
[7] H.Goda, *Heegaard splitting for sutured manifolds and Murasugi sum* , Osaka J.Math. **29**, (1992) 21 – 40.
[8] H.Goda, *On handle number of Seifert surfaces in S^3,* Osaka J.Math. **30**, (1993) 63 – 80
[9] H.Goda, T.Kitano, T.Morifuji, *Reidemeister torsion, twisted Alexander polynomial and fibered knots,* e-print: math.GT/0311155.
[10] H.Goda, T.Morifuji, *Twisted Alexander polynomial for $SL(2,\mathbf{C})$-representations and fibered knots,* C. R. Math. Acad. Sci. Soc. R. Can. 25 (2003) 97-101.
[11] H.Goda, A.Pajitnov, *Twisted Novikov homology and circle-valued Morse theory for knots and links,* e-print: arXiv.math.GT/0312374, Journal publication: Osaka Journal of Mathematics, v.**42** No. 3, 2005.
[12] B.Jiang, S.Wang *Twisted topological invariants associated with representations*, in: Topics in Knot Theory, Kluwer, 1993.
[13] T.Kitano, *Twisted Alexander polynomial and Reidemeister torsion,* Pacific J.Mat, **174,** 1996, p. 431 – 442.
[14] F.Latour, *Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham*, Publ. IHES **80** (1995),
[15] X.S.Lin, *Representations of knot groups and twisted Alexander polynomials* , preprint 1990, publication: Acta Math. Sin. **17** (2001), pp. 361–380.
[16] W.Massey, *Homology and cohomology theory* , Marcel Dekker, 1978.
[17] B.R.McDonald, *Linear Algebra Over Commutative Rings,* Marcel Dekker, 1984, 544 pages.
[18] C.T.McMullen, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology* , Ann. Sci. Ec. Norm. Super, **35** No. 2, 153-171 (2002), http://abel.math.harvard.edu/ ctm/papers.
[19] S.P.Novikov, *Many-valued functions and functionals. An analogue of Morse theory* , Doklady AN SSSR, **260** (1981), 31-35 (in Russian), English translation: Sov.Math.Dokl. **24** (1981), 222-226.

[20] S.P.Novikov, *The hamiltonian formalism and a multivalued analogue of Morse theory,* Uspekhi Mat. Nauk, **37** (1982), 3-49(in Russian), English translation: Russ. Math. Surveys, **37** (1982), 1–56.

[21] A.Pajitnov, *O tochnosti neravenstv tipa Novikov dlya mnogoobrazii so svobodnnoi abelevoi fundamental'noi gruppoi*, Mat. Sbornik, (1989) no. 11.
English translation:
A.V.Pazhitnov, *On the sharpness of Novikov-type inequalities for manifolds with free abelian fundamental group.*, Math. USSR Sbornik, **68** (1991), 351 - 389.

[22] A.V.Pazhitnov, *Modules over some localizations of the ring of Laurent polynomials*, Mathematical Notes, **46**, 1989, no. 5.

[23] A.V.Pajitnov, *On the Novikov complex for rational Morse forms*, preprint: Institut for Matematik og datalogi, Odense Universitet Preprints 1991, No 12, Oct. 1991; journal article: Annales de la Faculté de Sciences de Toulouse **4** (1995), 297–338. The pdf-files available at http://www.math.sciences.univ-nantes.fr/ pajitnov/

[24] A.V.Pajitnov, *Surgery on the Novikov Complex*, Preprint: Rapport de Recherche CNRS URA 758, Nantes, 1993; the pdf-file available at: http://www.math.sciences.univ-nantes.fr/ pajitnov/
Journal article: K-theory **10** (1996), 323-412.

[25] A.V.Pajitnov, *On the asymptotics of Morse numbers of finite covers of manifold* , E-print: math.DG/9810136, 22 Oct 1998, journal article: Topology, **38**, No. 3, pp. 529 – 541 (1999).

[26] A.V.Pajitnov, A.Ranicki, *The Whitehead group of the Novikov ring*, E-print: math.AT/0012031, 5 dec 2000, journal article: K-theory, Vol. 21 No. 4, 2000.

[27] A.V.Pajitnov, C.Weber, L.Rudolph, *Morse-Novikov number for knots and links*, Algebra i Analiz, **13**, no.3 (2001), (in Russian), English translation: Sankt-Petersbourg Mathematical Journal. **13**, no.3 (2002), p. 417 – 426.

[28] A.Ranicki, *The algebraic theory of torsion. I*, Proc. 1983 Rutgers Topology Conference, Springer Lecture Notes, No. 1126, 199-237

[29] D.Rolfsen, *Knots and Links*, Publish or Perish (1976, 1990).

[30] D.Schütz, *One-parameter fixed point theory and gradient flows of closed 1-forms*, e-print: math.DG/0104245, journal article: K-theory, 25(2002), 59-97.

[31] D.Schütz, *Gradient flows of closed 1-forms and their closed orbits*, e-print: math.DG/0009055, journal article: Forum Math. 14(2002) 509–537.

[32] J.-Cl. Sikorav, *Points fixes de difféomorphismes symplectiques, intersections de sous-variétés lagrangiennes, et singularités de un-formes fermées* Thése de Doctorat d'Etat Es Sciences Mathématiques, Université Paris-Sud, Centre d'Orsay, 1987

[33] Thang T.Q. Le, *Varieties of representations and their subvarieties of cohomology jumps for knot groups*, (Russian) Mat. Sb. **184** (1993), p. 57-82. English translation: Russian Acad. Sci. Sb. Math. **78** (1994), p. 187-209.

[34] W.Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. **59**, (1986), pp. 99 – 130.

[35] V.Turaev, *The Alexander polynomial of a three-dimensional manifold*, Math. USSR Sbornik, v. 26 (1975), No. 3, p. 313-329.

[36] V.Turaev, *A norm for the cohomology of 2-complexes*, Algebraic and Geometric topology, Vol. 2(2002), 137-155.

[37] M.Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology, **33** (1994), 241 – 256.
[38] F.Waldhausen, *Algebraic K-theory of generalized free products, I, II*, Ann. of Math., **108** (1978) p. 135-204
[39] E.Witten, *Supersymmetry and Morse theory*, Journal of Diff. Geom., **17** (1985) no. 2.

Laboratoire Mathématiques Jean Leray UMR 6629, Université de Nantes, Faculté des Sciences, 2, rue de la Houssinière, 44072, Nantes, Cedex

E-mail address: pajitnov@math.univ-nantes.fr