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ABSTRACT

The work of Sullivan and Wall on surgery theory, as extended to the topological case by Kirby and Siebenmann, showed the existence of an exact sequence of pointed sets for the surgery theory of a compact oriented manifold with boundary $(M,\partial M)$ of dimension $m \geq 6$:

(*) $$s_{TOP}(M,\partial M) \to n_{TOP}(M,\partial M) \overset{\theta}{\to} L_m(\pi_1(M))$$

We address the problem of group structure in the surgery exact sequence (*) and its extension to the left. When G/TOP is given the infinite loop space structure arising from Quinn's theory of surgery spaces, it is shown:

<u>Theorem</u>: There is a long exact sequence of abelian groups and homomorphisms:

$$\to L_{m+1}(\pi_1(M)) \to s_{TOP}(M,\partial M) \to H^0(M,\partial M;G/TOP) \overset{\theta}{\to} L_m(\pi_1(M))$$

where θ is the surgery obstruction map.

Moreover, it is shown that this sequence is natural with respect to induction and restriction for a covering projection of finite index, or more generally, for oriented bundles with fiber a closed oriented manifold. A corrected version of Siebenmann's periodicity theory for $s_{TOP}(M,\partial M)$ is also obtained.

Finally, Dress induction and localization are applied to the surgery exact sequence to prove:

<u>Theorem</u>: Suppose G is a finite group of homeomorphisms acting freely on pair $(M,\partial M)$. Let $s(H) = s_{TOP}(M/H,\partial M/H)$ for a subgroup H of G. Also let C be a class of subgroups of G and A a subring of the rational

numbers. Then the sum of the induction maps $\oplus_{H\in C}\, s(H) \otimes A \to s(G) \otimes A$ is surjective and product of the restriction maps $S(G) \otimes A \to \pi_{H\in C} s(H) \otimes A$ is injective in the cases:

1. C is the class of cyclic subgroups of G, A = Q.
2. C is the class of 2-hyperelementary subgroups of A, A = Z[1/3, 1/5,...].
3. $\pi_1(M)$ is finite, C is the class of p-elementary subgroups of G, p odd, and A = Z[1/2].
4. $\pi_1(M)$ is finite, C is the union of the classes in 2 and 3, and A = Z.

This memoir is a slightly revised version of the author's doctoral dissertation (Princeton, 1979).

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INTRODUCTION

Let CAT be one of the categories DIFF, PL, TOP of smooth, piecewise linear, and topological manifolds respectively. The work of Sullivan and Wall on surgery theory as extended to the topological case by Kirby and Siebenmann (see [KS], [Su 1], [Wa 1]) showed the existence of an exact sequence of pointed sets for a compact oriented CAT manifold with boundary $(M,\partial M)$, $\dim M = m \geq 6$:

$$(*) \qquad s_{CAT}(M,\partial M) \to n_{CAT}(M,\partial M) \to L_m(\pi_1(M))$$

where $s_{CAT}(M,\partial M)$ is the set of s-cobordism classes of CAT simple homotopy equivalences $N \to M$ relative to ∂M, $n_{CAT}(M,\partial M)$ is the set of normal cobordism classes of CAT normal maps over M relative to ∂M, and θ is the surgery obstruction map. Moreover, $L_{m+1}(\pi_1(M))$ acts on the set $s_{CAT}(M,\partial M)$ and there is a long exact sequence of groups:

$$\to s_{CAT}((M,\partial M)\times(\Delta^1,\partial\Delta^1)) \to n_{CAT}((M,\partial M)\times(\Delta^1,\partial\Delta^1)) \to L_{m+1}(\pi_1(M))$$

The set $n_{CAT}(M,\partial M)$ can be identified with set $[M,\partial M;G/CAT,*]$ of homotopy classes of maps into G/CAT. The space G/CAT is the homotopy fiber of the natural map BCAT $\to$ BG where BCAT is the classifying spaces for stable CAT bundles and BG is the classifying space for stable spherical fibrations.

We address the problem of group structure in the surgery exact sequence (*) and its extension to the left in the case CAT = TOP. We also comment on the case CAT = PL. When G/TOP is given the infinite loop space structure arising from Quinn's theory of surgery spaces, we show:

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Theorem: There is a long exact sequence of abelian groups and homomorphisms:

$$\to L_{m+1}(\pi_1(M)) \to s_{TOP}(M,\partial M) \to H^0(M,\partial M;G/TOP) \overset{\theta}{\to} L_m(\pi_1(M))$$

where θ is the surgery obstruction.

Moreover, we show that this sequence is natural with respect to induction and restriction for a covering projection of finite index, or more generally, for oriented bundles with fiber a closed oriented manifold. A corrected version of Siebenmann's periodicity theorem for $s_{TOP}(M,\partial M)$ ([KS, p. 283]) is also obtained.

Finally, we apply Dress induction and localization to the surgery exact sequence to prove:

Theorem: Suppose G is a finite group of homeomorphisms acting freely on the pair $(M,\partial M)$. Let $s(H) = s_{TOP}(M/H,\partial M/H)$ for a subgroup H of G. Also let C be a class of subgroups of G and A a subring of the rational numbers. Then:

I. The sum of the induction maps

$$\oplus_{H\in C}\, s(H) \otimes A \to s(G) \otimes A \quad \text{is surjective}$$

II. The product of the restriction maps

$$s(G) \otimes A \to \pi_{H\in C}\, s(H) \otimes A \quad \text{is surjective}$$

in the cases:

1. C is the class of cyclic subgroups of G, $A = Q$.
2. C is the class of 2-hyperelementary subgroups of G, $A = Z[1/3, 1/5,\ldots]$.
3. $\pi_1(M)$ is finite, C is the class of p-elementary subgroups of G, p odd, and $A = Z[1/2]$.
4. $\pi_1(M)$ is finite, C is the union of classes in 2 and 3, and $A = Z$.

An easy corollary of the theorem is:

Corollary: A simple homotopy equivalence $f\colon N \to M$ relative to ∂M, $\pi_1(M)$ finite, is homotopic to a homeomorphism relative to ∂M if and only if the lifting $\tilde{f}\colon \tilde{N} \to \tilde{M}$ of f to each covering $\tilde{M}$ of M with $\pi_1(\tilde{M})$ 2-hyper-elementary or p-elementary, p odd, is homotopic to a homeomorphisms relative to ∂M.

Our discussion is organized as follows:

The theory of n-ads is briefly reviewed in Section 1 of Chapter 1. In Section 2 Δ-objects in a category are defined and ordered simplicial complexes and their geometric product are also discussed. Sections 3 and 4 contain an exposition of the homotopy theory of Δ-sets which will be needed in the sequel.

In Chapter 2 we begin be defining certain types of surgery problems in Section 1. These will be used as the basic building blocks for the geometric constructions of that chapter and of Chapter 3. Section 2 contains the definition and a discussion of the surgery space $\mathbb{L}_q(B)$ and of the restricted surgery space $\mathbb{L}_q'(B)$ where B is a CW r-ad. The approach to surgery spaces which we present is a modification of that due to Quinn in his thesis [Q]. Let CAT denote one of the categories TOP or PL of topological and piecewise linear manifolds respectively. In Section 3 we define the Δ-sets $S_{CAT}(X,\partial_0 X)$ and $N_{CAT}(X,\partial_0 X)$ for a CAT manifold r-ad X. $S_{CAT}(X,\partial_0 X)$ is the Δ-set of homotopy CAT structures on X relative to $\partial_0 X$, and $N_{CAT}(X,\partial_0 X)$ is the Δ-set of CAT normal maps over X relative to $\partial_0 X$. Finally we show in Theorem 2.3.4 that $S_{CAT}(X,\partial_0 X)$ can be identified with the homotopy fiber over the 0-component of the geometrically defined surgery obstruction map $F\colon N_{CAT}(X,\partial_0 X) \to \mathbb{L}_m(\delta_0 X)$ (see definition 2.3.3) where $m = \dim X \geq r + r$ and $\delta_0 X$ is the (r-1)-ad obtained from X by deleting the zero-th face, $\partial_0 X$.

In Section 1 of Chapter 3 we show that the surgery spaces $\mathbf{L}_q(B)$ for a CW r-ad B determine a periodic Ω-spectrum $\mathbf{L}(B)$ with zero-th space $\mathbb{L}_8(B)$. The corresponding 0-connected spectrum with zero-th space $\mathbf{L}_8(B)_0$, $\mathbb{L}(B)\ (1,\ldots,\infty)$ in the notation of Adams [A], will also be of interest since when B is a point $\mathbb{L}_8(B)_0$ has the homotopy type of the classifying space G/TOP. The notion of a surgery mock bundle is introduced in Section 2 and the properties of these objects are discussed. This notion will be useful because the cohomology theory with coefficients in the spectrum $\mathbb{L}(B)$ coincides with the cobordism theory of surgery mock bundles and the group $H^0(X;\mathbb{L}(B)\ (1,\ldots,\infty))$ has a similar description. In addition, various geometric maps such as the assembly map, which is the subject of Section 3, and the induction and restriction maps of Chapter 4 are easily defined in terms of surgery mock bundles.

Chapter 4 contains the construction of induction and restriction maps at the Δ-set level for the various terms of the surgery exact sequence. In Section 1 a transfer map $tr(p)\colon \underline{M}^q_{\underline{B}}(E,F) \to M^{q+w}_B(K,L)$ is defined for an oriented simplicial w-mock bundle $p\colon E \to K$, $F = p^{-1}(L)$ (see definition 4.1.1). $\underline{M}^q_{\underline{B}}(E,F)$ and $M^q_B(K,L)$ are Δ-sets isomorphic to the function spaces $\Delta(E,F;\mathbb{L}_q(B)_0,\phi)$ and $\Delta(K,L:\mathbf{L}_q(B),\phi)$ respectively and are described in terms of surgery mock bundles (see definition 3.2.4). In the case p is a triangulation of a covering projection of finite index, the transfer $tr(p)$ gives rise to a transfer map for the cohomology theory $H^*(\ ;\mathbb{L}(B))$ and also for $H^0(\ ;\mathbb{L}(B)(1,\ldots,\infty))$ which coincides with the cohomology transfer of [A] or [Rsh]. If E^e and K^k are oriented simplicial manifold s-ads and $p\colon E \to K$ is an oriented 0-mock bundles triangulating a covering projection of finite index and $p^{-1}(\partial_i K) = \partial_i E$ $i = 0,\ldots,s-2$, then there is a commutative square:

$$\begin{array}{ccc} \underline{M}^q(E,\partial_0 E) & \longrightarrow & \mathbb{L}_{q+e}(\delta_0|E|) \\ \big\downarrow \mathrm{tr}\,(P) & & \big\downarrow \mathbb{L}_{q+e}(|P|) \\ \underline{M}^q(K,\partial_0 K) & \longrightarrow & \mathbb{L}_{q+e}(\delta_0|K|) \end{array}$$

where the horizontal maps are the assembly maps. In Chapter 5 it is shown that the homotopy fiber of the assembly map is homotopy equivalent to the Δ-set of homotopy TOP structures under the appropriate conditions. Hence the induced map of homotopy fibers in the diagram above yields an induction map $I_*(p)\colon S_{TOP}(|E|,\partial_0|E|) \to S_{TOP}(|K|,\partial_0|K|)$. More generally, if p is an oriented simplicial w-mock bundle there is an induction map:

$$I_*(p)\colon S_{TOP}(|E|,\partial_0|E|) \to S_{TOP}((|K|,\partial_0|K| \times (\Delta^w,\partial\Delta^w)) .$$

In Section 2 we show how to obtain restriction maps for surgery spaces and for $S_{TOP}(\ )$. Suppose $p\colon E \to B$ is a map of CW r-ads which is an oriented topological bundle with fiber a compact oriented topological manifold of dimension w such that $p^{-1}(\partial_i B) = \partial_i E$ $i = 0,\dots,r-2$. The pullback with respect to p defines a transfer map for surgery spaces $\mathrm{tr}(p)$: $\mathbb{L}_q(B) \to \mathbb{L}_{q+w}(E)$. This construction is due to Quinn [Q], and is based on the geometrically defined transfer of [Wa 1]. If B and E are also oriented manifolds then the pullback defines a restriction map: $I^*(p)\colon S_{TOP}(B,\partial_0 B) \to S_{TOP}(E,\partial_0 E)$. This map can also be obtained as an induced map of homotopy fibers of the assembly map.

Chapter 5 begins with an exposition of the homotopy equivalence $G/TOP \simeq \mathbb{L}_8(pt)_0$ of Quinn and Siebenmann. This equivalence endows G/TOP with an infinite loop space structure since $\mathbb{L}_8(pt)_0$ is the zero-th space of the spectrum $\mathbb{L}(pt)(1,\dots,\infty)$. Let K be a triangulation of a compact oriented PL-manifold s-ad with $m \geq s+4$. Assume for every subset c of $\{1,\dots,s-2\}\partial_c K = \cap\{\partial_j K|\ j \in c\}$ is connected and non-empty. $\partial_0 K$ may be empty of disconnected. We then establish a homotopy commutative square:

$$\begin{array}{ccc} \underline{M}^{4r}(K,\partial_0 K) & \xrightarrow{\underline{A}} & \mathbb{L}_{m+4r}(\delta_0|K|) \\ \uparrow & & \uparrow \\ N_{TOP}(|K|,\partial_0|K|) & \xrightarrow{F} & \mathbb{L}_m(\delta_0|K|) \end{array}$$

where $\underline{A}$ is the assembly map, F is the surgery obstruction map, and the vertical maps are homotopy equivalences. The homotopy fiber of $\underline{A}$, $E(\underline{A})$, which is a H-group (i.e., a homotopy associative H-space with a homotopy inverse), is then homotopy equivalent to the homotopy fiber of F which is in turn homotopy equivalent to $S_{TOP}(|K|,\partial_0|K|)$. The homotopy sequence of the homotopy fibration $E(\underline{A}) \to \underline{M}^{4r}(K,\partial_0 K) \to \mathbb{L}_{4r+m}(\partial_0|K|)$, which is a long exact sequence of groups, maps isomorphically to the homotopy sequence of the homotopy fibration:

$$S_{TOP}(|K|,\partial_0|K|) \to N_{TOP}(|K|,\partial_0|K|) \to \mathbb{L}_m(\delta_0|K|)$$

yielding a long exact sequence of groups:

$$\to L_{m+1}(\pi_1(\delta_0 K)) \to s_{TOP}(|K|,\partial_0|K|) \to H^0(|K|,\partial_0|K|;G/TOP) \to L_m(\pi_1(\delta_0 K)) \;.$$

This sequence is independent of the triangulation of $M = |K|$. Our analysis also enables us to prove a corrected version of the periodicity theory of Siebenmann [KS, p. 283]:

<u>Theorem</u>: Let $M = |K|$ be as above. Then:

1. $s_{TOP}(M,\partial_0 M) \cong s_{TOP}((M,\partial_0 M) \times (\Delta^4,\partial\Delta^4))$ if $\partial_0 M$ is non-empty.
2. if $\partial_0 M$ is empty then there is an exact sequence of groups

$$0 \to s_{TOP}(M) \to s_{TOP}(M \times \Delta^4, M \times \partial\Delta^4) \to L_0(1) \;.$$

Since $L_0(1) = Z$, $s_{TOP}(M \times \Delta^4, M \times \partial\Delta^4)$ is isomorphic to $s_{TOP}(M)$ or to $s_{TOP}(M) \oplus Z$; moreover, both cases can occur.

Using the results of Chapter 4, the surgery exact sequence is shown to be natural with respect to induction and restriction for a covering projection of finite index and, more generally, for oriented bundles with fiber a closed oriented PL manifold. Finally, we comment on how to extend the above results to non-triangulable topological manifolds and how to make the PL surgery exact sequence:

$$\to s_{PL}(M,\partial_0 M) \to [M,\partial_0 M;G/PL,*] \to L_m(\pi_1(\partial_0 M))$$

an exact sequence of groups and homomorphisms, M a PL manifold s-ad, $\dim M \geq s+4$.

In Section 1 of Chapter 6 the apparatus of induction theory, i.e., Green functors and their modules, is described. The properties of the trivial Green functor and its modules are also discussed. In Section 2 Dress induction and localization are applied to the surgery exact sequence to obtain our induction theorem for $s_{TOP}(\ )$.

1. PRELIMINARIES

1.1 N-ads

Notation: For a finite indexing set A let $\|A\|$ denote the cardinality of A.

Definition (see [Wa 1, Ch. 0]): Let 2^{n-1} be the category whose objects are subsets of $\{0,1,\ldots,n-2\}$ and whose morphisms are the inclusion maps. Let $\underline{C}$ be a subcategory of the category of sets and maps. A n-ad in $\underline{C}$ is an intersection preserving functor $2^{n-1} \to \underline{C}$.

If X is a n-ad, denote $|X| = X(\{0,\ldots,n-2\})$ and for a subset c of $\{0,\ldots,n-2\}$ $\partial_c X = X(\{0,\ldots,n-2\} - c)$. Observe that $\partial_c X = \cap \{\partial_j X \mid j \in c\}$ and hence $|X|$ and $\partial_j X$ determine the n-ad X. The notation $(X;\partial_0 X,\ldots,\partial_{n-2} X)$ will be used for the n-ad X. $\partial_c X$ is naturally a $(n - \|c\|)$-ad, also denoted by $\partial_c X$:

Let $\{0,\ldots,n-2\} - c = \{i(0),\ldots,i(k)\}$ $i(0) < i(1) < \ldots < i(k)$

Then $|\partial_c X| = \partial_c X$, $\partial_j(\partial_c X) = \partial_{c \cup \{i(j)\}} X = \partial_c X \cap \partial_{i(j)} X$.

A map of n-ads in $\underline{C}$, $f\colon X \to Y$, is a morphism $f\colon |X| \to |Y|$ which for every subset c of $\{0,\ldots,n-2\}$ restricts to a morphism $\partial_c X \to \partial_c Y$. If X is a n-ad then the (n-1)-ad $\delta_j X$ is obtained from X by omitting the j-th face, $\partial_j X$. Let M be a m-ad in $\underline{C}$ and N a n-ad in $\underline{C}$. Suppose that $\underline{C}$ is equipped with a suitable product, for example, the Cartesian product when $\underline{C}$ is the category of sets and maps or of spaces and continuous maps. $M \times N$ is the $(m+n-1)$-ad:

$$(M \times N; M \times \partial_0 N,\ldots,M \times \partial_{n-2} N, \partial_0 M \times N,\ldots,\partial_{m-2} M \times N) .$$

Let CAT denote one of the categories:

1. TOP of topological manifolds and continuous maps.
2. PL of piecewise linear manifolds and piecewise linear maps.

Definition: A CAT manifold n-ad is a topological n-ad M such that for every subset c of $\{0,\ldots,n-2\}$ M(c) is a CAT manifold with bounary given by $\partial M(c) = \cup M(b) | b \subset c, b \neq c\}$. $\partial_c M$ is allowed to be empty, the empty set being viewed as a CAT manifold of any dimension. Note that $\dim |\partial_c M| = \dim |M| - \|c\|$.

1.2 Δ-objects

Δ-objects will be used extensively in the geometric constructions of Chapters 2 and 3. The basic references on the subject are [RS2] and [RS3].

Definition: Let Δ be the category with objects Δ^n, the standard n-simplex, $n = 0,1,\ldots$ and whose morphisms are injective order preserving simplicial maps $\Delta^m \to \Delta^n$. A Δ-object in a category $\underline{C}$ is a contravariant functor $\Delta \to \underline{C}$. A morphism of Δ-objects is a natural transformation of functors.

Equivalently a Δ-object X in $\underline{C}$ can be defined as a sequence of objects $X(k)$ $k = 0,1,\ldots$ in $\underline{C}$ together with morphisms $\partial_i\colon X(k) \to X(k-1)$ $0 \leq i \leq k$ satisfying $\partial_i\partial_j = \partial_{j-1}\partial_i$ for $i < j$. The ∂_i's are called face maps and the elements of X(k), in the case $\underline{C}$ is a subcategory of the category of sets and maps, are called k-simplices. A morphism of Δ-objects $X \to Y$ is a sequence of morphisms $f_k\colon X(k) \to Y(k)$ such that $f_k\partial_j = \partial_j f_{k+1}$. Δ-objects in the category of sets and maps are called Δ-sets.

Simplicial complexes are closely related to Δ-sets. Let K be a simplicial complex viewed as a collection of closed linear simplices lying in R^{∞}. K is said to be ordered if the vertices of K are given a partial order which induces a total order on the vertices of each simplex of K. If K is ordered then it can be viewed as a Δ-set, also denoted by K, in a natural manner: the k-simplices of the Δ-set K are the k-simplices of the simplicial complex K and the face maps are determined by the ordering of K. An order preserving simplicial map $f: K \to L$ between ordered simplicial complexes which is injective on each simplex of K is equivalent to a Δ-map $K \to L$, also denoted by f.

The geometric product of two ordered simplicial complexes H and K, denoted $H \otimes K$, is the ordered simplicial complex defined as follows: Let $H^0 = \{v(i) | i \in I\}$ and $K^0 = \{w(j) | j \in J\}$ be the vertex sets of H and K respectively. The vertex set of $H \otimes K$ is $H^0 \times K^0$ ordered lexicographically. A typical r-simplex, σ^r, or $H \otimes K$ has the form $\sigma^r = ((v(i_0),w(j_0)),\dots,(v(i_r),w(j_r)))$ where:

1. $\{v(i_0),\dots,v(i_r)\}$ and $\{w(j_0),\dots,w(j_r)\}$ span a simplex of H and K respectively and v(i_0) <=, ..., <= v(i_r) and w(j_0) <=, ..., <= w(j_r).
2. for every s, $0 \leq s < r$, $(v(i_s),w(j_s))$ is strictly less than $(v(i_{s+1}),w(j_{s+1}))$ in the lexicographic ordering of $H^0 \times K^0$.

The space underlying $H \otimes K$ is $|H| \times |K|$. More generally the geometric product can be defined for arbitrary Δ-sets but this will not be needed. Let H, K, A, B be ordered simplicial complexes and $f: H \to B$ order preserving simplicial maps which are injective on each simplex, i.e., f and g are Δ-maps. The order preserving simplicial map $f \otimes g: H \otimes K \to A \otimes B$ defined on vertices by $f \otimes g((v,w)) = ((f(v),g(w)))$ is clearly a Δ-map.

1.3 The Homotopy Theory of Δ-Sets

We now discuss some of the homotopy theory of Δ-sets which will be applied in subsequent chapters. Let $\Lambda^{n,k} = \Delta^n - (\text{int } \Delta^n \cup \text{int } \partial_k \Delta^n)$ viewed as a simplicial complex with ordering induced from Δ^n.

Definition: A Δ-set X is said to be Kan if every Δ-map $f: \Lambda^{n,k} \to X$ $n,k \geq 0$ extends to a Δ-map $F: \Delta^n \to X$.

Equivalently, X is Kan if for every collection of $n-1$ n-simplices of X x_j $j \in \{0,\ldots,n\}$ $j \neq k$, such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$ $i,j \neq k$, there exists a n-simplex x with $\partial_j x = x_j$ $j \neq k$. A sub Δ-set of X with exactly one n-simplex, v_n, for each $n \geq 0$ will be called a point subcomplex of X. When X is Kan the homotopy groups of the pair (X,v) can be defined in purely combinatorial manner:

$$\pi_m(X,v) = \{x \in X(m) \mid \partial_i x = v_{m-1} \quad i = 0,\ldots,m\}/\text{relation} \quad m \geq 0 .$$

The relation is defined as follows: x is equivalent to y if there exists $z \in X(m+1)$ with $\partial_m z = x$, $\partial_{m+1} z = y$, $\partial_i z = v_i$ for $i = 0,\ldots,m-1$. This is an equivalence relation and there is a natural isomorphism $\pi_m(X,v) \cong \pi_m(|X|,|v|)$ $m \geq 0$ where $|X|$ is the geometric realization of X (see [RS2, Ch. 6]).

Remarks:

1. Relative homotopy groups can also be defined (see [Ma, p. 7] and [RS2, Ch. 6]).
2. The set $\pi_0(X,v)$ when viewed as an unbased set does not depend on v and one writes $\pi_0(X)$. $\pi_0(X)$ is called the set of path components of X.

3. Let (X,v) be a pointed Δ-set. The 0-component of X, denoted X_0, is the pointed Δ-set defined by:

$$X_0(0) = \{x \in X(0) \mid \text{there exists } y \in X(1) \text{ with } \partial_0 y = x,\ \partial_1 y = v_0\}$$
$$X_0(j) = \{x \in X(j) \mid \partial_k x \in X_0(j-1)\ \ k = 0,\dots,j\}\ \ j \geq 1 .$$

Face maps are obtained by restricting the face maps of X. Clearly X_0 is Kan if X is Kan.

4. The Cartesian product of two Δ-sets X and Y is the Δ-set defined by $(X \times Y)(j) = X(j) \times Y(j)$ and with face maps $\partial_k(x,y) = (\partial_k x, \partial_k y)$. When X and Y are Kan $|X \times Y|$ and $|X| \times |Y|$ are homotopy equivalent (see [RS2]).

The following version of J. H. C. Whitehead's theorem is valid for Δ-sets (see [RS2, Ch. 6]):

<u>Theorem 1.3.1</u>: Let X,Y be Kan Δ-sets and $f\colon X \to Y$ a Δ-map. Suppose for every point subcomplex v of X $\pi_j(f)\colon \pi_j(X,v) \to \pi_j(Y,f(v))$ is an isomorphism for every $j \geq 0$. Then f is a homotopy equivalence.

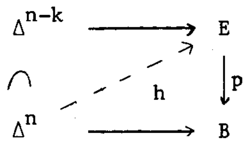
By Remark 6.7 of [RS2] the previous theorem yields a criterion for an inclusion of Kan Δ-sets to be a deformation retraction:

<u>Proposition 1.3.2</u>: Let $X \subset Y$ be an inclusion of Kan Δ-sets and suppose for each n,k every Δ-map $f\colon \Lambda^{n,k} \to Y$ with $f(\partial\Lambda^{n,k}) \subset X$ extends to a Δ-map $F\colon \Delta^n \to Y$ such that $F(\partial_k \Delta^n) \subset X$. Then $X \subset Y$ is a deformation retraction.

There is a notion of fibration for Δ-maps:

<u>Definition 1.3.3</u>: A Δ-map $p\colon E \to B$ is called a Kan fibration if given the

following commutative square of Δ-maps:



A Δ-map $h: \Delta^n \to E$ can be found making the resulting diagram commute. Equivalently p is a Kan fibration if for every collection of n $(n-1)$-simplices x_j $j \in \{0,\ldots,n\}$ $j \neq k$ of E with $\partial_i x_j = \partial_{j-1} x_i$ for $i \leq j$ $i,j \neq k$ and for every n-simplex y of B with $\partial_i y = p(x_i)$ $i \neq k$, there exists a n-simplex x of E such that $\partial_i x = x_i$ for $i \neq k$ and $p(x) = y$.

Let X be a Kan pointed Δ-set. Denote the base point of $X(j)$ by v_j. Define pointed Δ-sets Λx and ΩX by:

$$(\Lambda X)(n) = \{\sigma \in X(n+1) \mid (\partial_0)^{n+1}\sigma = v_0\}$$
$$(\Omega X)(n) = \{\sigma \in X(n+1) \mid (\partial_0)^{n+1} = v_0,\ \partial_{n+1}\sigma = v_n\}\ .$$

The face maps of ΛX and ΩX come from the face maps of X and the base-point of $(\Lambda X)(n)$ and $(\Omega X)(n)$ is v_{n+1}. ΛX is called the Δ-set of paths on X originating at v and ΩX is called the Δ-set of loops on X based at v. Clearly ΛX and ΩX are Kan. Define $p_n: (\Lambda X)(n) \to X(n)$ by $p_n(x) = \partial_{n+1} x$. Then $p_n(\partial_j x) = \partial_{n+1}\partial_j x = \partial_j \partial_{n+2} x = \partial_j p_{n+1}(x)$ $j = 0,\ldots,n+1$. Hence $p: \Lambda X \to X$ is a Δ-map.

<u>Proposition 1.3.4</u>: $p: \Lambda X \to X$ is a Kan fibration.

<u>Proof</u>: Let $x_j \in (\Lambda X)(n-1)$ $j \in \{0,\ldots,n\}$ $j \neq k$ with $\partial_i x_j = \partial_{j-1} x_i$ for $i \leq j$ $i,j \neq k$ and let $y \in X(n)$ $\partial_i y = p(x_i)$ $i \neq k$. View the x_i's as elements of $X(n)$ and let $x_{n+1} = y$. Then $\partial_i x_{n+1} = p(x_i) = \partial_n x_i$ $i \neq k$. Since X is Kan there exists $x \in X(n+1)$ with $\partial_i x = x_i$ $i \neq k$. Clearly $(\partial_0)^{n+1} x = x_0$ and thus x is an n-simplex of ΛX. $p(x) = \partial_{n+1} x = x_{n+1} = y$.

If Z is a space, the singular complex of Z, denoted SZ, is the Δ-set whose j-simplices are continuous maps from Δ^j into Z and whose face maps are given by restriction to $\partial_k\Delta^j$ $k = 0,\dots,j$. The singular complex defines a functor from spaces to Δ-sets; furthermore, if Z is a CW complex and X is a Kan Δ-set then there are natural homotopy equivalences $\phi(Z)\colon SZ \to Z$ and $\psi(X)\colon X \to S|X|$ (see [RS2]). The path space on a pointed space (Z,z), denoted PZ, is the space of continuous maps from the unit interval to Z $h\colon I \to Z$ such that $h(0) = z$. The path fibration $e\colon PZ \to Z$ is defined by $e(h) = h(1)$.

Proposition 1.3.5: Let X be a Kan pointed Δ-set. There is a commutative diagram:

$$\begin{array}{ccccc} \Omega|X| & \longrightarrow & P|X| & \longrightarrow & |X| \\ \uparrow{\scriptstyle L(X)} & & \uparrow{\scriptstyle M(X)} & & \uparrow{\scriptstyle N(X)} \\ |\Omega X| & \longrightarrow & |\Lambda X| & \xrightarrow{|p|} & |X| \end{array}$$

where the vertical maps are natural based homotopy equivalences

Proof: We mimic the proof of Lemma 5.1, p. 36 of [BRS]. Suppose Z is a pointed space. The homeomorphisms $(\Delta^n \times I)/\Delta^n \times 0 \to \Delta^{n+1}$, $(t_0,\dots,t_n;s) \mapsto (st_0,\dots,st_n,1-s)$ $n \geq 0$ induce a natural isomorphism of Δ-sets $G(Z)\colon \Lambda SZ \to SPZ$; furthermore, the diagram:

$$\begin{array}{ccc} \Lambda SZ & \xrightarrow{G(Z)} & SPZ \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Se} \\ SZ & \xrightarrow{p} & SZ \end{array}$$

is commutative and $G(Z)$ restricts to an isomorphism $G'(Z)\colon \Omega SZ \to S\Omega Z$. L, M, N are defined as the composites:

$$L(X) = \phi(\Omega|X|) \cdot |G'(S|X|)| \cdot |\Omega\psi(X)|$$

$$M(X) = \phi(P|X|) \cdot |G(S|X|)| \cdot |\Lambda\psi(X)|$$

$$N(X) = \phi(|X|) \cdot |\psi(X)| .$$

Since each of the maps on the right are natural based homotopy equivalences, so are L, M, and N.

Let (X,x) and (Y,y) be Kan pointed Δ-sets and $f: X \to Y$ a base point preserving Δ-map.

<u>Definition 1.3.6</u>: The homotopy fiber of f over the basepoint y is the pointed Δ-set, $E(f)$, given by:

$$E(f)(n) = \{(x,y) \in X(n) \times Y(n+1) |\ (\partial_0)^{n+1} y = y_0,\ f_n(x) = \partial_{n+1} y\}$$
$$\partial_j(x,y) = (\partial_j x, \partial_j y) \qquad (x,y) \in E(f)(n) \qquad j = 0,\dots,n .$$

The basepoint of $E(f)(n)$ is (x_n, y_{n+1}). There is a pullback square of basepoint preserving Δ-maps:

1.3.7
$$\begin{array}{ccc} E(f) & \xrightarrow{U} & \Lambda Y \\ \downarrow V & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are the natural projections. Using the fact that X is Kan and that p is a Kan fibration it is easy to verify that E(f) is Kan. There is also the following fiber mapping sequence:

1.3.8
$$\dots \longrightarrow \Omega E(f) \longrightarrow \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{R} E(f) \xrightarrow{V} X \xrightarrow{f} Y$$

where
$$R_n(y) = (x_{n+1}, y) \qquad y \in (\Omega Y)(n)$$
$$(\Omega f)_n(x) = f_{n+1}(x) \qquad x \in (\Omega X)(n) .$$

Recall that in the category of pointed spaces and basepoint preserving continuous maps the homotopy fiber, $F(g)$, of $g: (A,a) \to (B,b)$ is defined by the pullback diagram:

$$\begin{array}{ccc} F(g) & \longrightarrow & PB \\ \downarrow & & \downarrow e \\ A & \xrightarrow{g} & B \end{array}$$

Applying the geometric realization functor to diagram 1.3.7 and using Proposition 1.3.5, one obtains a natural based homotopy equivalence $E(f) \simeq F(|f|)$.

<u>Proposition 1.3.9</u>: There is a based homotopy equivalence $\Omega E(f) \simeq E(\Omega f)$.

<u>Proof</u>: Observe that $|\Omega E(f)| \simeq \Omega|E(f)| \simeq \Omega F(|f|)$ and that $|E(\Omega f)| \simeq F(|\Omega f|) \simeq F(\Omega|f|)$ and, furthermore, $\Omega F(|f|)$ and $F(\Omega|f|)$ are homeomorphic.

Given a commutative square of Δ-maps of pointed Δ-sets:

1.3.10
$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow a & & \downarrow b \\ X' & \xrightarrow{g} & Y' \end{array}$$

there is an induced Δ-map of homotopy fibers $h\colon E(f) \to E(g)$ defined by: $h_n(x,y) = (a_n(x), b_{n+1}(y))$ $(x,y) \in E(f)(n)$.

We now discuss the analog of H-spaces in the Δ-set category. Let (Y,e) be a Kan pointed Δ-set. A H-structure on (Y,e) is a Δ-map $g\colon (Y \times Y, (e,e)) \to (Y,e)$ such that the maps $g(\ ,e)$, $g(e,\ )\colon (Y,e) \to (Y,e)$ defined by $g(\ ,e)(x) = g(x,e)$ and $g(e,\ )(x) = g(e,x)$ are homotopic to the identity. Note that $|Y|$ is an H-space with homotopy unit $|e|$ and multiplication given by the composite $|Y| \times |Y| \simeq |Y \times Y| \xrightarrow{|g|} |Y|$. The triple (Y,e,g) is said to be homotopy associative if the H-space $|Y|$ is homotopy associative and a H-group if $|Y|$ is a H-group (i.e., a homotopy associative H-space with a homotopy inverse). A Δ-map $f\colon (Y,e) \to (Y',e')$ between two Kan Δ-sets with H-structures (Y,e,g) and (Y',e',g') is said to be a H-map if $|f|$ is a H-map. f will be called a H-homomorphism if

$f(g(x,y)) = g'(f(x),f(y))$ for each n-simplex (x,y) of $Y \times Y$.

In the case f is a H-homomorphism, a H-structure $m: E(f) \times E(f) \to E(f)$ is defined by $m((x,y),(u,v)) = (g_j(x,u), g'_{j+1}(y,v))$ where (x,y) and (u,v) are j-simplices of $E(f)$. Note that the maps $R: \Omega Y' \to E(f)$, $V: E(f) \to Y$ of 1.3.8 are H-homomorphisms where $\Omega Y'$ has the H-structure induced by the H-structure of Y'. If diagram 1.3.10 is a commutative square of H-homomorphisms, then the induced map $E(f) \to E(g)$, is clearly a H-homomorphism.

More generally, we have the following theorem concerning the H-space properties of the homotopy fiber:

<u>Theorem 1.3.11</u>: Let (Y,e) and (Y',e') be Kan pointed Δ-sets with given H-structures and $f: (Y,e) \to (Y',e')$ a H-map. Then if $E(f)$ is the homotopy fiber of f over e':

1. $E(f)$ is a H-space with the natural basepoint as homotopy unit.
2. The namps $R: Y' \to E(f)$, $V: E(f) \to Y$ of 1.3.8 are H-maps.
3. If Y and Y' are homotopy associative and f is homotopy associative with respect to given associating homotopies for Y and Y' then $E(f)$ is homotopy associative.
4. If in addition to the conditions of 3 above, $\pi_0(Y,e)$ and $\pi_0(Y',e')$ are groups, then $\pi_0(E(f),*)$ is a group.

<u>Proof</u>: Recall that $|E(f)|$ is naturally homotopy equivalent to $F(|f|)$, the homotopy fiber of $|f|$ over the basepoint. 1 and 3 are then a direct consequence of Theorem 1 and Theorem 15 respectively of [St]. 2 follows from the definition of the H-space multiplication in $F(|f|)$ in the proof of Theorem 1 of [St]. Assume the hypotheses of 4. The fiber mapping sequence for f yields an exact sequence of pointed sets:

$$\longrightarrow \pi_1(Y',e') \xrightarrow{\pi_0(R)} \pi_0(E(f),*) \xrightarrow{\pi_0(V)} \pi_0(Y,e) \xrightarrow{\pi_0(f)} \pi_0(Y',e') \ .$$

By 3 $\pi_0(E(f),*)$ is a monoid and by 2 the maps $\pi_0(R)$ and $\pi_0(V)$ are monoid homomorphisms. 4 is a consequence of the following lemma with $B = \text{Im}\ \pi_0(V) = \ker \pi_0(f)$:

<u>Lemma</u>: Let $A \xrightarrow{a} M \xrightarrow{b} B \to 0$ be an exact sequence of monoids and suppose A and B are groups. Then M is a group.

<u>Proof</u>: It is sufficient to show that each $x \in M$ has a left and right inverse. Let $x \in M$. Since b is surjective there exists $y \in M$ such that $b(y) = b(x)^{-1}$. Observe $b(xy) = 1$ and thus $a(z) = xy$ for some $z \in A$. Then $ya(z^{-1})$ is a right inverse for x. Similarly x has a left inverse.

<u>Remark</u>: The conditions of Theorem 1.3.11 (4) are satisfied when Y and Y' are loop spaces and f is a loop map.

We conclude this section with a theorem which will be useful in the sequel:

<u>Theorem 1.3.12</u>: Let the following square be a homotopy commutative diagram of Kan pointed Δ-sets:

$$\begin{array}{ccc} (Y,e) & \xrightarrow{f} & (Y',e') \\ \downarrow U & & \downarrow V \\ (A,a) & \xrightarrow{g} & (B,b) \end{array}$$

Suppose (Y,e), (Y',e'), and f satisfy the hypotheses of Theorem 1.3.11 (4) and that U and V are homotopy equivalences. Then there is a homotopy equivalence of homotopy fibers $E(f) \to E(g)$.

<u>Proof</u>: Since homotopy equivalent maps have homotopy equivalent homotopy fibers it can be assumed, without loss of generality, that the diagram above is strictly commutative. Then by 1.3.10 there is an induced map of homotopy

fibers $h: E(f) \to E(g)$. The homotopy equivalences U and V impose H-structures on A and B respectively so that (A,a), (B,b), and g satisfy the hypotheses of Theorem 1.3.11 (4). The homotopy sequence of the homotopy fibration $E(f) \to Y \overset{f}{\to} Y'$ maps into that of $E(g) \to A \overset{g}{\to} B$:

$$\begin{array}{ccccccccccc} \to & \pi_1(Y,e) & \to & \pi_1(Y',e') & \to & \pi_0(E(f),*) & \to & \pi_0(Y,e) & \to & \pi_0(y',e') \\ & \downarrow \pi_1(U) & & \downarrow \pi_1(V) & & \downarrow \pi_0(h) & & \downarrow \pi_0(U) & & \downarrow \pi_0(V) \\ \to & \pi_1(A,a) & \to & \pi_1(B,b) & \to & \pi_0(E(g),*) & \to & \pi_0(A,a) & \to & \pi_0(B,b) \end{array}$$

By Theorem 1.3.11 (4) the two horizontal sequences are exact sequences of groups and homomorphisms; furthermore, $\pi_0(h)$ is a homomorphisms since h is a H-map. Hence by the five lemma $\pi_0(h)$ is an isomorphism. Similarly $\pi_j(h)\colon \pi_j(E(f),v) \to \pi_j(E(g),h(v))$ is an isomorphism for any point complex v and $j \geq 1$. The theorem follows from Whitehead's theorem (Theorem 1.3.1).

1.4 Function Spaces

Let (K,L) be an ordered simplicial pair and (X,Y) a Δ-set pair. The function space $\Delta(K,L;X,Y)$ is the Δ-set whose j-simplices are Δ-maps $g\colon K \otimes \Delta^j \to X$ with $g(L \otimes \Delta^j)$ contained in Y and whose face maps are given by restriction to $K \otimes \partial_k\Delta^j$ for $k = 0,\dots,j$. A Δ-map $f\colon (K,L) \to (H,J)$ between ordered simplicial pairs induces a Δ-map:

$$f^{\#}\colon \Delta(H,J;X,Y) \to \Delta(K,L;X,Y) \quad \text{given by} \quad f^{\#}(g\colon H \otimes \Delta^j \to X) = g(f \otimes \mathrm{id}).$$

A Δ-map $h\colon (X,Y) \to (W,Z)$ induces a Δ-map:

$$h_{\#}\colon \Delta(K,L;X,Y) \to \Delta(K,L;W,Z) \quad \text{given by} \quad h_{\#}(g\colon K \otimes \Delta^j \to X) = hg\ .$$

Remarks:

1. If (X,Y) is a Kan pair then $\Delta(K,L;X,Y)$ is Kan.
2. If the pointed Δ-set (Y,e) has a H-structure $m\colon Y \times Y \to Y$ with

homotopy unit e then $\Delta(K,L;Y,e)$ inherits a H-structure from Y by means of the induced map

$$\Delta(K,L;Y,e) \times \Delta(K,L;Y,e) = \Delta(K,L;Y \times Y,(e,e)) \xrightarrow{m_{\#}} \Delta(K,L;Y,e) \ .$$

Given pointed spaces A and X and subspaces B of A and Y of X containing the base points, let $\mathrm{Map}(A,B;X,Y)$ be the space of continuous basepoint preserving maps $(A,B) \to (X,Y)$ with the compact-open topology. Suppose (K,L) is an ordered simplicial pair with a basepoint $k \in L$. Assume that K is locally finite and that k is maximal in the ordering of the vertices of K. The Δ-set $S\mathrm{Map}(|K|,|L|;X,Y)$, where S denotes the singular complex, will be naturally identified with the Δ-set whose j-simplices are continuous maps $f\colon (|K \otimes \Delta^j|;|L \otimes \Delta^j|,|k \otimes \Delta^j|) \to (X;Y,*)$ and whose face maps are given by restriction to $|K \otimes \partial_k \Delta^j|$ $k = 0,\dots,j$. In the case (K,L) is an unbased locally finite ordered simplicial pair we replace (K,L) by $(K \cup +, L \cup +)$ where $+$ is a disjoint vertex ordered so that every vertex of K precedes it.

Lemma 1.4.1: Let (K,L) be as above and let (Z,z) be a pointed space. Then there is a natural homotopy equivalence:

$$A\colon \Omega\, S\mathrm{Map}\,(|K|,|L|;Z,z) \to S\mathrm{Map}\,(|K|,|L|;\Omega Z,*) \ .$$

Proof: The Δ-map A is defined as follows: Let $g\colon |K \otimes \Delta^{j+1}| \to Z$ be a j-simplex of $\Omega\, S\mathrm{Map}\,(|K|,|L|;Z,z)$. Then $A_j(g)\colon |K \otimes \Delta^j| \to \Omega Z$ is the map given by $A_j(g)(x)(t) = g(tx + (1-t)v)$ where $x \in |\sigma^r|$, $\sigma^r \in K \otimes \Delta^j$ (identified with $K \otimes \partial_{j+1}\Delta^{j+1}$) and v is the vertex (k,v_{j+1}) of $K \otimes \Delta^{j+1}$, and $0 \le t \le 1$.

A homotopy inverse B for A is defined as follows: Let $f\colon |K \otimes \Delta^j| \to \Omega Z$ be a j-simplex of $S\mathrm{Map}\,(|K|,|L|;Z,*)$. Every $x \in |\sigma|$, where σ is an s-simplex of $K \otimes \Delta^{j+1}$ not in $K \otimes v_{j+1}$, can be uniquely expressed in the form $x = tw + (1-t)v'$ $t \neq 0$, $w \in |\sigma| \cap |K \otimes \partial_{j+1}\Delta^{j+1}|$, and $v' \in |\sigma| \cap |K \otimes v_{j+1}|$. $B_j(f)\colon |K \otimes \Delta^{j+1}| \to Z$ is the map given by $B_j(f)(x) = f(w)(t)$ for x as above and $B_j(f)(x) = *$ for $x \in K \otimes v_{j+1}$. Then $B_j(f)$ is continuous and B is a Δ-map; furthermore it is readily verified that AB and BA are homotopic to the respective identities.

Let (K,L) be as in Lemma 1.4.1 and let $(A,*)$ be a Kan pointed Δ-set. A natural Δ-map $\mu\colon \Omega\Delta(K,L;A,*) \to \Delta(K,L;\Omega A,*)$ is defined as follows: let $f\colon K \otimes \Delta^{j+1} \to A$ be a j-simplex of $\Omega\Delta(K,L;A,*)$ and let v denote the vertex (k,v_{j+1}) of $K \otimes \Delta^{j+1}$. If σ is a r-simplex of $K \otimes \Delta^j$ let $\sigma \cdot v$ be the $(r+1)$-simplex of $K \otimes \Delta^{j+1}$ spanned by σ and v where $K \otimes \Delta^j$ is identified with $K \otimes \partial_{j+1}\Delta^{j+1}$. Define $\mu_j(f)\colon K \otimes \Delta^j \to \Omega A$ to be the Δ-map by $\mu_j(f)(\sigma) = f(\sigma \cdot v)$.

<u>Proposition 1.4.2</u>: μ is a homotopy equivalence.

<u>Proof</u>. Given a Kan Δ-set (B,b) there is a natural homotopy equivalence $n\colon \Delta(K,L;B,b) \to \mathrm{SMap}(|K|,|L|;|B|,|b|)$ given by $n_j(g\colon K \otimes \Delta^j \to B) = |g|$. There is a commutative diagram:

$$\begin{array}{ccccc}
\Omega\Delta(K,L;A,*) & \xrightarrow{\Omega n} & \Omega\,\mathrm{SMap}(|K|,|L|;|A|,*) & \xrightarrow{A} & \mathrm{SMap}(|K|,|L|;|\Omega A|,*) \\
\downarrow{\scriptstyle\mu} & & & & \downarrow{\scriptstyle id} \\
\Delta(K,L;\Omega A,*) & \xrightarrow{n} & \mathrm{SMap}(|K|,|L|;|\Omega A|,*) & \xrightarrow{L_\#} & \mathrm{SMap}(|K|,|L|;|\Omega A|,*)
\end{array}$$

$L_\#$ is the homotopy equivalence induced by the homotopy equivalence $L(A)\colon |\Omega A| \to \Omega|A|$ of Proposition 1.3.5. The Δ-map A is the homotopy equivalence of Lemma 1.4.1. Then μ is a homotopy equivalence since it is a composite of homotopy equivalences.

2. SURGERY SPACES

2.1 Definitions

R^∞ will denote the set of sequences in the real numbers R $(x_0, x_1, \ldots)$ such that $x_i = 0$ for all but finitely many i. R^{n+1} is included in R^∞ by $(x_0, \ldots, x_n) \to (x_0, \ldots, x_n, 0, \ldots)$.

Let B be a topological r-ad and q, j non-negative integers. The objects of the next definition will be used extensively in the sequel:

Definition 2.1.1: A surgery problem of type (q,j) over B consists of the following data:

1. Compact oriented topological manifold $(j+r+2)$-ads M and X of dimension $q+j$ embedded in $\Delta^j \times R^s \subset \Delta^j \times R^\infty$ for some s so that $M \cap \partial_k \Delta^j \times R^s = \partial_k M$ $k = 0, \ldots, j$ and $M - (\cup_{k=0}^{j} \partial_k M) \subset \operatorname{int} \Delta^j \times R^s$ and M has a normal microbundle in $\Delta^j \times R^s$. Similarly for X.
2. A degree 1 map $f\colon M \to X$ of $(j+r+2)$-ads (see [Wal, Ch. 2]) such that $\partial_{j+1} f\colon \partial_{j+1} M \to \partial_{j+1} X$ is a simple homotopy equivalence of $(j+r+1)$-ads.
3. A TOP microbundle, η, over X and a map $b\colon \nu_M \to \eta$ of microbundles covering f where ν_M is a normal microbundle of M in $\Delta^j \times R^s$.
4. A continuous map of $(j+r+2)$-ads, called the reference map $h\colon X \to (B \times \Delta^j; B \times \partial_0 \Delta^j, \ldots, B \times \partial_j \Delta^j, B \times \Delta^j, \partial_0 B \times \Delta^j, \ldots, \partial_{r-2} B \times \Delta^j)$ such that the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \Delta^j \times R^\infty \\ \big\downarrow h & & \big\downarrow \\ B \times \Delta^j & \longrightarrow & \Delta^j \end{array}$$

commutes where i is the inclusion and the unlabelled arrows are the projections.

The notation $M \xrightarrow{f} X \xrightarrow{h} B \times \Delta^j$ b,n will be used for a surgery problem of type (q,j) over B as in the previous definition.

Let $x = M \longrightarrow X \longrightarrow B \times \Delta^j$ b,n be as above. $\partial_k x$ for $k = 0,\ldots,j$ will denote the surgery problem of type $(q,j-1)$ over B given by restriction over the k-th face:

$$\partial_k M \xrightarrow{\partial_k f} \partial_k X \xrightarrow{\partial_k h} B \times \partial_k \Delta^j \quad b|\partial_k M, n|\partial_k X \ .$$

Similarly, $\partial^k x$ is the surgery problem of type $(q-1,j)$ over the $(r-1)$-ad $\partial_k B$ given by restriction over the $(k+j+2)$-th face for $k = 0,\ldots,r-2$. $-x$ is the surgery problem of type (q,j) over B obtained by reversing the orientations of M and X.

Let $N^p \subset R^s$, s large, be a closed oriented topological manifold and suppose N has a normal microbundle v_N. If x is as above then $x \times N$ is defined to be the surgery problem of type $(q+p,j)$ over B given by:

$$M \times N \xrightarrow{f \times id} X \times N \xrightarrow{h\pi} B \times \Delta^j \quad b',n'$$

where $b' = b \times id$ and n' is the microbundle $n \times v_N$.

2.2 Surgery Spaces

Surgery spaces in the non-simple connected case were first constructed by Quinn (see [Q]). In this section we will only consider the untwisted case.

Let B be a CW r-ad.

Definition 2.2.0: The surgery space $\mathbb{L}_q(B)$ is the pointed Δ-set defined as follows:

> $\mathbb{L}_q(B)(j)$ is the set of all surgery problems of type (q,j) over B. The base point, denoted ϕ_j, is the empty surgery problem over B. The face maps are the maps ∂_k $k = 0,\ldots,j$ defined in the previous section.

Let $f\colon B_1 \to B_2$ be a map of CW r-ads. Composing f with the reference map of each simplex of $\mathbb{L}_q(B_1)$ yields a map of pointed Δ-sets $\mathbb{L}_q(f)\colon \mathbb{L}_q(B_1) \to \mathbb{L}_q(B_2)$. $\mathbb{L}_q$ is a covariant functor from the category of CW r-ads to the category of pointed Δ-sets. Orientation reversal and the maps ∂^k $k = 0,\ldots,r-2$ (see Section 2.1) define natural basepoint preserving Δ-maps $-\colon \mathbb{L}_q(B) \to \mathbb{L}_q(B)$ and $\partial_k\colon \mathbb{L}_q(B) \to \mathbb{L}_{q-1}(\partial_k B)$.

Disjoint union provides a natural H-structure for $\mathbb{L}_q(B)$. To be precise, let $t \in R$ and consider the map $d(t)$ defined by $d(t)(x_0,x_1,\ldots) = (t,x_0,x_1,\ldots)$. If $x = M \to X \to B \times \Delta^j$ b,n is a surgery problem of type (q,j) over B then by applying $\mathrm{id} \times d(t)$, where $\mathrm{id}\colon \Delta^j \to \Delta^j$ is the identity, to the ambient space in which M and X are embedded, one obtains in the obvious manner another surgery problem of type (q,j) over B which will be denoted by $D(t)(x)$. Define a basepoint preserving Δ-map $\mu\colon \mathbb{L}_q(B) \times \mathbb{L}_q(B) \to \mathbb{L}_q(B)$ by $\mu_j(x,y) = D(0)(x) \cup D(1)(y)$ where x and y are j-simplices of $\mathbb{L}_q(B)$. The symbol "$\cup$" denotes the union of the underlying sets and maps. The maps $D(t)$ $t = 0,1$ ensures that this is a disjoint union. The basepoint ϕ, generated by the empty surgery problems over B, serves as a homotopy unit.

Orientation reversal, the maps $\partial^k\colon \mathbb{L}_q(B) \to \mathbb{L}_{q-1}(\partial_k B)$, and $\mathbb{L}_q(f)$ where $f\colon B_1 \to B_2$ is a map of CW r-ads, are easily seen to be H-homomorphisms with respect to the H-structure defined by μ.

The properties of $\mathbb{L}_q(B)$ will now be investigated.

Proposition 2.2.1: $\mathbb{L}_q(B)$ is Kan.

Proof: Let $x_j = M_j \to X_j \to B \times \Delta^{n-1}$ b_j, n_j $j \in \{0,\dots,n\}$ $j \neq k$ be a collection of n $(n-1)$-simplies of $\mathbb{L}_q(B)$ with $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$ $i,j \neq k$. Make the canonical identification of Δ^{n-1} with $\partial_j \Delta^n$ to obtain $u =$ $M \xrightarrow{f} X \xrightarrow{h} B \times \Lambda^{n,k}$ b,n where $M,X \subset \Lambda^{n,k} \times R^\infty$ and so that u restricted to $\partial_j \Lambda^{n,k} \times R^\infty$ is x_j for $j \neq k$. By Theorem 3.3.2 u can be viewed as surgery problem of type $(q+n-1,0)$ over the $(r+1)$-ad $B \times (\Lambda^{n,k};\partial\Lambda^{n,k})$ with $\partial_0 M$ and $\partial_0 X$ empty. The above "assembly" process is explained in more detail in Section 3.3. Let I be the unit interval and let the map $g\colon \Lambda^{n,k} \times I \to \Delta^n$ be a homeomorphism with $g \mid \Lambda^{n,k} \times 0$ the natural inclusion $\Lambda^{n,k} \to \Delta^n$ and $g(\partial\Lambda^{n,k} \times I \cup \Lambda^{n,k} \times 1) \subset \partial_k \Delta^n$. Let $X = M \times I \xrightarrow{f \times id} X \times I \xrightarrow{h'} B \times \Delta^n$ b',n' where $h' = (id \times g)(h \times id)$, $b' = b \times id$ $n' = n \times v_I$ where v_I is a normal microbundle to $I \subset R^1$. View $M \times I$ as being embedded in $\Delta^n \times R^\infty$ via g with the manifold $(n+r+2)$-ad structure

$$\partial_j(M \times I) = M_j \times 0 \qquad j \in \{0,\dots,n\} \quad j \neq k$$
$$\partial_k(M \times I) = \partial_1 M \times I \cup (\cup\{M_j \times 1 \mid \ j \in \{0,\dots,n\} \quad j \neq k\})$$
$$\partial_{n+j}(M \times I) = \partial_{j+1} M \times I \qquad j \geq 1$$

and similarly for $X \times I$. Then x is seen to be a surgery problem of type (q,n) over B with $\partial_j x = x_j$ $j \neq k$. Hence the Δ-set $\mathbb{L}_q(B)$ is Kan.

Let $s\colon \Delta^j \to \Delta^{j+1}$ be defined by $s(v) = (1/2)v + (1/2)v_{j+1}$ where v_{j+1} is the $(j+1)$-th vertex of Δ^{j+1}. Define a map $e_q^j\colon \mathbb{L}_q(B)(j) \to (\Omega\mathbb{L}_{q-1}(B))(j)$ as follows: Suppose $x = M \to X \xrightarrow{h} B \times \Delta^j$ b,n is a surgery problem of type (q,j) over B. Embed $M \subset \Delta^j \times R^\infty$ in $\Delta^{j+1} \times R^\infty$ using s and give M the manifold $(j+r+3)$-ad structure obtained from the given manifold $(j+r+2)$-ad structure of M by inserting an empty face in the $(j+1)$-th position, i.e., so that with the new ad structure $\partial_{j+1} M$ is empty. Similarly, do this for X. Then $e_q^j(x)$ is the surgery problem of type $(q-1,j+1)$ over B given by $M \to X \xrightarrow{h'} B \times \Delta^{j+1}$ b,n where $h' = (id \times s)h$. Note that $e_q^j(x) \in (\Omega\mathbb{L}_{q-1}(B))(j)$. The e_q^j's define a Δ-map $e_q\colon \mathbb{L}_q(B) \to \Omega\mathbb{L}_{q-1}(B)$.

<u>Proposition 2.2.2</u>: $e_q\colon \mathbb{L}_q(B) \to \Omega\mathbb{L}_{q-1}(B)$ is a homotopy equivalence, $q \geq 1$.

<u>Proof</u>: View e_q as an inclusion. By Proposition 1.3.2 it is sufficient to show that every Δ-map $f\colon \Lambda^{n,k} \to \Omega\mathbb{L}_{q-1}(B)$ with $f(\partial\Lambda^{n,k}) \subset e_q(\mathbb{L}_q(B))$ extends to a Δ-map $F\colon \Delta^n \to \Omega\mathbb{L}_{q-1}(B)$ with $F(\partial_k\Delta^n) \subset e_q(\mathbb{L}_q(B))$. F is constructed by assembling the image of f as in the proof of Proposition 2.2.1, taking the product with the unit interval, and then giving what results the appropriate $(n+r+2)$-ad structure. An explicit homotopy inverse, d_q: $\Omega\mathbb{L}_{q-1}(B) \to \mathbb{L}_q(B)$, for e_q can be constructed as follows: Each $y \in \Delta^{j+1} - v_{j+1}$ can be uniquely expressed in the form $tv + (1-t)v_{j+1}$ $t \neq 0$, $v \in \partial_{j+1}\Delta^{j+1}$. Define the map $u\colon \Delta^{j+1} - v_{j+1} \to \Delta^j$ by $u(tv + (1-t)v_{j+1}) = v$. Let $x = M \to X \overset{h}{\to} B \times \Delta^{j+1}$ b,n be a j-simplex of $\Omega\mathbb{L}_{q-1}(B)$. $(\partial_0)^{j+1}x = \phi_0$ implies $h(X) \subset B \times (\Delta^{j+1} - v_{j+1})$ and that $M, X \subset (\Delta^{j+1} - v_{j+1}) \times R^\infty$. Then define $(d_q)_j(x)$ to be $M \longrightarrow X \xrightarrow{h'} B \times \Delta^j$ b,n where $h' = (\mathrm{id} \times u)h$, and M and X are given the manifold $(j+r+2)$-ad structure obtained by deleting the empty $(j+1)$-th face and where M and X are viewed as embedded in $\Delta^j \times R^\infty$ by means of the map

$$(\Delta^{j+1} - v_{j+1}) \times R^\infty \to \Delta^{j+1} \times R^\infty (tv + (1-t)v_{j+1}, x) \to (v,t,x).$$

If $f\colon B_1 \to B_2$ is a map of CW r-ads it is easily verified that $e_q\mathbb{L}_q(f) = (\Omega\mathbb{L}_{q-1}(f))e_q$. Also e_q commutes with orientation reversal and the maps ∂^k $k = 0,\ldots,r-2$.

The disjoint union H-structure on $\mathbb{L}_{q-1}(B)$ induces a H-structure on $\Omega\mathbb{L}_{q-1}(B)$; furthermore, it is clear that e_q is a H-homomorphism. In general if K is a H-space then the multiplication on ΩK arising from the loop space structure is homotopic to the multiplication on ΩK induced by the H-space structure of K. So in particular, Proposition 2.2.2 implies that $(\mathbb{L}_q(B)$, disjoint union) is a H-group.

Orientation reversal coincides with the group inverse in homotopy:

Proposition 2.2.3: $\pi_j(-): \pi_j(\mathbb{L}_q(B),\phi) \to \pi_j(\mathbb{L}_q(B),\phi)$ is the group inverse for $j \geq 0$.

Proof: Let $x = M \to X \to B \times \Delta^j$ b,n represent an element of $\pi_j(\mathbb{L}_q(B),\phi)$. Then $\partial_i x$ is the empty surgery problem for $i = 0,\ldots,j$. Take the product of x with the unit interval to obtain a surgery problem y of type $(q,j+1)$ over B where $M \times I$ has the $(n+j+3)$-ad structure:

$$\partial_0(M \times I) = M \cup -M, \qquad \partial_i(M \times I) \text{ is empty for } i = 0,\ldots,j+1,$$
$$\partial_{i+1}(M \times I) = (\partial_i M) \times I \qquad i \geq j+2$$

and similarly for $X \times I$. Thus y is a homotopy $(x \cup -x) \sim \phi_j$.

Now suppose that B is a CW r-ad such that for every subset c of $\{1,\ldots,r-2\}$ $\partial_c B$ is connected and non-empty and $\pi_1(\partial_c|B|)$ is finitely presented.

Proposition 2.2.4: For $j+q-r \geq 5$ there is a functorial isomorphism $\pi_j(\mathbb{L}_q(B),\phi) \cong L_{q+j}(\pi_1(B))$ where $L_{q+1}(\pi_1(B))$ is Wall's algebraic L-group of the group r-lattice $\pi_1(B)$.

Proof: Let $L^1_{q+j}(B)$ be the geometric surgery group defined by Wall in [Wa 1, Ch. 9]. A comparison with Wall's definition (see [Wa 1, pp. 86-87]) shows that a j-simplex x of $\mathbb{L}_q(B)$ such that $\partial_i x \neq \phi_{j-1}$ for $i = 0,\ldots,j$ is the same, after deleting the $j+1$ empty faces, as an object representing an element of $L^1_{q+j}(B)$. Moreover, it is clear that Wall's equivalence relation coincides with the homotopy equivalence relation in the definition of $\pi_j(\mathbb{L}_q(B),\phi)$. Hence $\pi_j(\mathbb{L}_q(B),\phi) = L^1_{q+j}(b)$ and naturality in B is a direct consequence of the definitions. The proposition now follows from Corollary 9.4.1 of [Wa 1].

Proposition 2.2.5 (Periodicity): For $q-r \geq 5$ there is a natural homotopy equivalence $\theta_k\colon \mathbb{L}_q(B) \to \mathbb{L}_{q+4k}(B)$ defined by taking products with the k-fold Cartesian product of 2-dimensional complex projective space.

Proof: Let $P = (CP^2)^k$, the k-fold Cartesian product of 2-dimensional complex projective space. The map $\theta_k = \mathbb{L}_q(B) \to \mathbb{L}_{q+4k}(B)$ is defined on j-simplices by $x \to x \times P$ (see Section 2.1). θ_k is clearly a natural H-homomorphism with respect to the disjoint union H-structure; furthermore, it commutes with the maps e_q, ∂^k $k = 0,\ldots,r-2$, and orientation reversal. The proposition follows from Wall's periodicity theorem (Theorem 9.10 of [Wa 1], Proposition 2.2.4 and Whitehead's theorem (Theorem 1.3.1).

Remarks:

1. The connectivity assumption on B in the two previous propositions could be dropped if we were willing to consider fundamental groupoids.
2. We have only considered the untwisted case in our discussion of surgery spaces. There are difficulties with Wall's construction of geometric surgery groups in the twisted case which have been rectified by Farrell and Hsiang in [FHs].

Let B be a CW r-ad as in Proposition 2.2.4. A surgery problem of type (q,j) over B $M \to x \to B \times \Delta^j$ b,n is said to be restricted if for every subset c of $\{0,\ldots,j\}$:

1. $\partial_c X$ is connected.
2. $\pi_1(\partial_c h)\colon \pi_1(\partial_c X) \to \pi_1(B \times \partial_c \Delta^j)$ is an isomorphism whenever $\partial_c X$ is non-empty.

The next definition is due to Siebenmann in the 1-ad case (see [K]) and is based on [Wa 1, Ch. 9] and [Q];

Definition 2.2.6: The restricted surgery space $\mathbb{L}'_q(B)$ is the Δ-set whose j-simplices are restricted surgery problems of type (q,j) over B and whose face maps are defined as for $\mathbb{L}_q(B)$.

$\mathbb{L}'_q(B)$ is Kan. This is proved by the same argument used in the proof of Proposition 2.2.1. The condition that the reference map of the assembled simplex x of Proposition 2.2.1 induces an isomorphism on fundamental groups is a consequence of Van Kampen's theorem.

There is a natural inclusion $\mathbb{L}'_q(B) \subset \mathbb{L}_q(B)$. The utility of $\mathbb{L}'_q(B)$ is due to the following proposition:

Proposition 2.2.7: For $q-r \geq 4$ the natural inclusion $i: \mathbb{L}'_q(B) \to \mathbb{L}_q(B)$ is a homotopy equivalence.

Proof: Comparison with [Wa 1, Ch. 9] shows that $\pi_0(\mathbb{L}'_q(B)) = L^2_q(B)$ where L^2_q is defined in [Wa 1, p. 88]. Theorem 9.4 of [Wa 1] is then the statement that $\pi_0(i): \pi_0(\mathbb{L}'_q(B)) \to \pi_0(\mathbb{L}_q(B))$ is a bijection. By an easy extension of Wall's argument, one has that $\pi_j(i): \pi_j(\mathbb{L}'_q(B),v) \to \pi_j(\mathbb{L}_q(B),v)$ is a bijection for $j \geq 1$ and any point complex v. The proposition is then a consequence of Whitehead's theorem (Theorem 1.3.1).

Remark: PL surgery problems could have been used in the place of TOP surgery problems in the definitions of the surgery spaces to obtain Δ-sets homotopy equivalent to the ones already defined.

2.3 Spaces of Homotopy Structures and Normal Maps

Throughout this section $X \subset R^\infty$ will be a compact oriented CAT manifold r-ad of dimension n, where CAT = TOP or PL.

Definition 2.3.1: A CAT normal map of type j over X relative to $\partial_0 X$ consists of the following data:

1. A compact oriented CAT manifold $(j+r+1)$-ad of dimension $n+j$ $M \subset \Delta^j \times R$, for some s, so that $M \cap \partial_k \Delta^j \times R^s = \partial_k M$ $k = 0,\ldots,j$ and $M - (\cup\{\partial_k M | \; k = 0,\ldots,j\}) \subset \operatorname{int} \Delta^j \times R^s$, and M has a CAT normal microbundle in $\Delta^j \times R^s$.
2. A degree 1 map $f\colon M \to X \times \Delta^j$ of manifold $(j+r+1)$-ads such that $\partial_{j+1} f$ is a CAT isomorphism of $(j+r)$-ads.
3. A CAT microbundle, n, over $X \times \Delta^j$ and a microbundle map $b\colon v_M \to n$ covering f where v_M is a normal microbundle of M in $\Delta^j \times R^s$.

The notation $M \to X \times \Delta^j$ b,n will be used for the object defined above.

The Δ-set $N_{CAT}(X,\partial_0 X)$, called the Δ-set of CAT normal maps over X relative to $\partial_0 X$, is the ∂-set whose j-simplices are normal maps of type j over X relative to $\partial_0 X$ and whose face maps arise from the $(j+r+1)$-ad structure of each simplex (compare with Definition 2.2.1).

Definition 2.3.2: A CAT simple homotopy equivalence of type j over X relative to $\partial_0 X$ consists of:

1. A CAT manifold $(j+r+1)$-ad M of dimension $n+j$ as in 1 of Definition 2.3.1.
2. A simple homotopy equivalence $f\colon M \to X \times \Delta^j$ of $(j+r+1)$-ads such that $\partial_{j+1} f$ is a CAT isomorphism of $(j+r)$-ads.

The Δ-set $S_{CAT}(X,\partial_0 X)$, called the Δ-set of homotopy CAT structures on X relative to $\partial_0 X$, is the Δ-set whose j-simplices are CAT simple homotopy equivalences of type j over X relative to $\partial_0 X$ and whose face maps are the obvious ones.

The definitions above are a modification of the definitions of Rourke,

[Ro], and Quinn [Q]. Both $N_{CAT}(X,\partial_0 X)$ and $S_{CAT}(X,\partial_0 X)$ are Kan. The proof of this fact is similar to the proof of Proposition 2.2.1. Note that the identity map $X \to X$ determines a basepoint for $S_{CAT}(X,\partial_0 X)$.

The geometric version of the surgery obstruction map can now be defined:

Definition 2.3.3: The surgery obstruction map is the Δ-map $F: N_{CAT}(X,\partial_0 X) \to \mathbb{L}_n(\delta_0 X)$, $n = \dim X$, given by:

$$F_j(M \to X\times\Delta^j\ b,v) = M \to X\times\Delta^j \xrightarrow{id} \delta_0 X\times\delta^j\ b,v\ .$$

A comparison of definitions immediately shows that $\pi_j(F)$: $\pi_j(N_{CAT}(X,\partial_0 X),*) \to \pi_j(\mathbb{L}_n(\delta_0 X),\phi)$ coincides with the geometrically defined surgery obstruction map of Wall (see [Wa 1, p. 107]).

The following theorem, which is stated for the 2-ad case in [KS], will be important in the sequel. This result is due to Quinn (see [Q]) in a weaker form.

Theorem 2.3.4: Suppose that $\dim X \geq r+4$ and for every subset c of $\{1,\dots,r-2\}$ $\partial_c X$ is non-empty and connected. $\partial_0 X$ is allowed to be empty or disconnected. Then there is a homotopy equivalence $S_{CAT}(X,\partial_0 X) \to E(X,\partial_0 X)$ where $E(X,\partial_0 X)$ is the homotopy fiber of the surgery map F of Definition 2.3.3 over the basepoint ϕ.

Proof: Assume X is embedded in R^p where p is large compaed to $n = \dim X$. The connectivity assumption on X implies that $F(N_{CAT}(X,\partial_0 X)) \subset \mathbb{L}'_n(\delta_0 X)$. Let $h: \Delta^j\times I \to \Delta^{j+1}$ be the embedding $(v,t) \to (1-t/2)v + (t/2)v_{j+1}$. Then $h(\partial_k\Delta^j\times I) \subset \partial_k\Delta^{j+1}$ $k = 0,\dots,j$ and $h|\Delta^j\times 0$ is the natural inclusion $\Delta^j \subset \partial_{j+1}\Delta^{j+1}$. Note that the space $\mathbb{L}'_n(\delta_0 X)$ has ϕ, the point complex of empty surgery problems, as a basepoint. Define a Δ-map $U: S_{CAT}(X,\partial_0 X) \to N_{CAT}(X,\partial_0 X)$ by $U_j(x) = M \to X\times\Delta^j\ b(x),n(x)$ where $x = M \xrightarrow{f} X\times\Delta^j$ is a j-simplex of $S_{CAT}(X,\partial_0 X)$ and where the normal data $b(x)$, $n(x)$ is chosen inductively as follows: If $x_0 = M \xrightarrow{f} X$ is a 0-simplex of $S_{CAT}(X,\partial_0 X)$ let v_M be a normal microbundle of M in R^s and choose a homotopy inverse g for f so that $(\partial_1 f)^{-1} = \partial_1 g$. Let $n(x) = g^*(v_M)$ be the induced microbundle and choose a microbundle map $b(x)\colon v \to n$ covering f. If $b(x)$ and $n(x)$ have been defined on j-simplices then for each (j+1)-simplex y use the above procedure to construct normal data $b(y)$ and $n(y)$ while ensuring that $b(\partial_i y) = b(y)|\partial_i M$ and $n(\partial_i y) = n(y)|\ X \times \partial_i \Delta^j$. It is easily verified that U is uniquely defined up to homotopy. Define a Δ-map V: $S_{CAT}(X,\partial_0 X) \to \Lambda \mathbb{L}_n'(\delta_0 X)$ as follows: Let $x = M \xrightarrow{f} X$ be a j-simplex of $S_{CAT}(X,\partial_0 X)$. Then

$$V_j(x) = M \times I \xrightarrow{f \times id} X \times \Delta^j \times I \xrightarrow{id \times h} \delta_0 X \times \Delta^{j+1} \quad b'(x),\ n'(x)$$

where $n'(x) = n(x) \times v_I$, $b'(x) = b(x) \times id$ and v_I is the normal microbundle of $I \subset R^1$. $M \times I$ and $X \times \Delta^j \times I$ are viewed as being embedded in $\Delta^{j+1} \times R^\infty$ via h and have the manifold ((j+1) + (r-1) + 2)-ad structures:

$$\begin{array}{lll}
\partial_k(X \times \Delta^j \times I) = X \times \partial_k \Delta^j \times I & \partial_k(M \times I) = \partial_k M \times I & k = 0,\dots,j \\
= X \times \Delta^j \times 0 & = M \times 0 & k = j+1 \\
= X \times \Delta^j \times 1 \cup \partial_0 X \times \Delta^j \times I & = M \times 1 \cup \partial_{j+1} M \times I & k = j+2 \\
= \partial_{k-j-2} X \times \Delta^j \times I & = \partial_{k-1} M \times I & k \geq j+3
\end{array}$$

Clearly $(\partial_0)^{j+1} V_j(x) = \phi_0$ and $\partial_{j+1} V_j(x) = F_j U_j(x)$. Hence there is a commutative square of Δ-maps:

$$\begin{array}{ccc}
S_{CAT}(X,\partial_0) & \xrightarrow{U} & N_{CAT}(X,\partial_0 X) \\
\downarrow V & & \downarrow F \\
\Lambda \mathbb{L}_n'(\delta_0 X) & \xrightarrow{p} & \mathbb{L}_n'(\delta_0 X)
\end{array}$$

This yields a Δ-map $S = (U,V)\colon S_{CAT}(X,\partial_0 X) \to E'(X,\partial_0 X)$ where $E'(X,\partial_0 X)$ is the homotopy fiber of $F\colon N_{CAT}(X,\partial_0 X) \to \mathbb{L}_n'(\delta_0 X)$ over the basepoint ϕ. It will now be demonstrated that S induces a bijection $\pi_0(S_{CAT}(X,\partial_0 X)) \to \pi_0(E'(X,\partial_0 X))$. To avoid cumbersome notation, the details will be given in the case X is a manifold 2-ad $(X,\partial X)$.

Injectivity of $\pi_0(S)$:

Let $w_i = M \to X$ $i = 0,1$ represent two elements of $\pi_0(S_{CAT}(X,\partial X))$ and suppose that for $i = 0,1$ $S(w_i)$ represent the same element in $\pi_0(E'(X,\partial X))$. Then there exists $y \in E'(X,\partial X)(1)$ such that $\partial_i y = S(w_i)$ $i = 0,1$. Explicitly, let $y = (W \to X \times \Delta^1,\ P \xrightarrow{f} Q \to X \times \Delta^2)$. Although the normal data has been omitted from the notation, it is to be understood. Since $y \in E'(X,\partial X)(1)$ it follows that:

$$\partial_2 P \to \partial_2 Q = W \to X \times \Delta^1$$

$$\partial_2 P \to \partial_3 Q \text{ is a simple homotopy equivalence of 4-ads.}$$

The condition $\partial_i y = S(w_i)$ $i = 0,1$ gives

$$\partial_i P \to \partial_i Q = V_0(w_i) \qquad \partial_i W \to X \times \partial_i \Delta^1 = w_i \qquad i = 0,1$$

Note that P and $\partial_2 P$ are connected and there is a commutative square:

$$\begin{array}{ccc} \pi_1(\partial_2 P) & \xrightarrow{\cong} & \pi_1(X \times \partial_2 \Delta^2) \\ \downarrow & & \downarrow \cong \\ \pi_1(P) & \xrightarrow{\cong} & \pi_1(X \times \Delta^2) \end{array}$$

where $\cong$ indicates isomorphism.

The following version of Wall's $\pi - \pi$ theorem will be used:

Theorem 2.3.5 (see [Wa 1, Ch. 4]): Let $G: N \to Y$ b,n be a CAT normal map of m-ads. Suppose $\dim Y \geq 6$ and $G|\ \partial_i N$ $i = 1,\ldots,m-2$ is a simple homotopy equivalence of (m-1)-ads and the inclusion $\partial_0 Y \subset Y$ induces an isomorphism of fundamental groupoids. Then G is normally bordant relative to $\partial_1 N \cup \ldots \cup \partial_{m-2} N$ to a simple homotopy equivalence of m-ads, the normal bordism having a (m+2)-ad structure.

The theorem above implies that $f: P \to Q$ is normally bordant relevant to

$\partial_0 P \cup \partial_1 P \cup \partial_3 P$ to a simple homotopy equivalence of 5-ads $f'\colon P' \to Q$. $\partial_i P' = \partial_i P$ and $f'|\ \partial_i P' = f|\ \partial_i P$ for $i = 0,1,3$. Note that $z = \partial_2 P' \xrightarrow{\partial_2 f'} \partial_2 Q = X \times \Delta^1$ is a 1-simplex of $S_{CAT}(X,\partial X)$ with $\partial_i x = w_i$ $i = 0,1$. Hence w_0 and w_1 represent the same element in $\pi_0(S_{CAT}(X,\partial X))$.

<u>Surjectivity of</u> $\pi_0(S)$:

Let $y = (M \overset{f}{\to} X,\ P \overset{F}{\to} Q \to X \times \Delta^1)$ represent an element of $\pi_0(E'(X,\partial X))$. Note that:

$$\partial_0 P \text{ and } \partial_0 Q \text{ are empty, } \partial_1 P \to \partial_1 Q = f\colon M \to X,$$
$$\partial_2 P \to \partial_2 Q \text{ is a simple homotopy equivalence, and}$$
$$\pi_1(\partial_1 P) \cong \pi_1(X) \cong \pi_1(X \times \Delta^1) = \pi_1(P).$$

By Theorem 2.3.5 (the $\pi - \pi$ theorem) $F\colon P \to Q$ is normally bordant relative $\partial_2 P$ to a simple homotopy equivalence of 4-ads $F'\colon P' \to Q$ where $\partial_0 P'$ is empty. Hence there is a normal map:

$$G\colon W \to (Q \times [o,1/2];\ Q \times 0,\ Q \times 1/2,\ \partial_1 Q \times [0,1/2],\ \partial_2 Q \times [0,1/2])$$

such that:

$$\partial_0 G = \partial_0 W \to Q \times 0 \qquad = F\colon P \to Q$$

$$\partial_1 G = \partial_1 W \to Q \times 1/2 \qquad = F'\ P' \to Q$$

$$\partial_3 G = \partial_3 W \to \partial_2 Q \times [0,1/2] = \partial_2 P \times [0,1/2] \xrightarrow{F|\partial_2 P \times \mathrm{id}} \partial_2 Q \times [0,1/2]$$

Let $w = \partial_1 P' \to \partial_1 Q$. Note that $\partial_1 Q = X$ and that w is a 0-simplex of $S_{CAT}(X,\partial X)$. Attach at the level-1/2 and a normal bordism to Q which has the form

$$\partial_1 W \times [1/2,1] \to Q \times [1/2,1] \qquad \text{(obtinaed by a homotopy of } F'\text{).}$$

See Diagram 2.3.6. Give the resulting normal map $T \to S$ the following 5-ad structure:

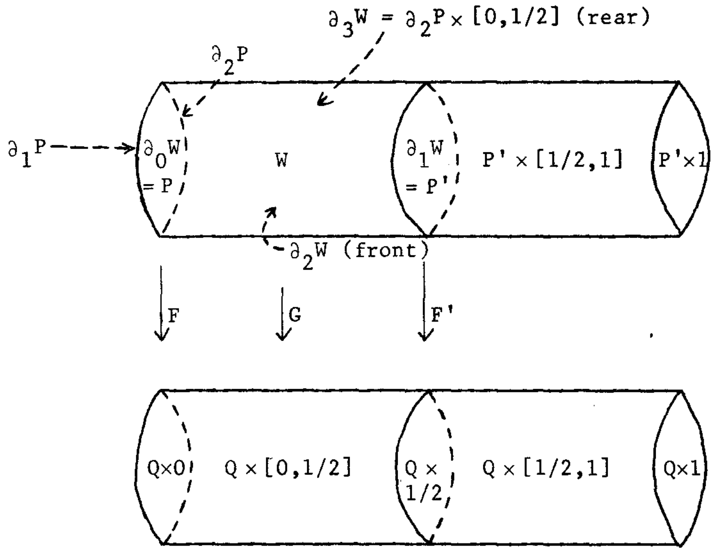


Diagram 2.3.6

$$\partial_0 T \to \partial_0 S = F\colon P \to Q$$

$$\partial_1 T \to \partial_1 S = \partial_1 P' \times [1/2,1] \to \partial_2 Q \times [0,1/2] = V_0(w)$$

$$\partial_2 T \to \partial_2 S = \partial_2 W \to \partial_1 Q \times [0,1/2]$$

$$\partial_3 T \to \partial_3 S = \partial_3 W \cup \partial_2 P \times [1/2,1] \cup \partial_1 W \times 1 \to \partial_2 Q \times [0.1] \cup Q \times 1$$

A reference maps $S \to X \times \Delta^2$ is easily constructed giving $T \to S \to X \times \Delta^2$ the structure of surgery problem of type $(m,2)$ over the 1-ad δX. Let $z = (\partial_2 W \to \partial_1 Q \times [0,1/2],\ T \to S \to X \times \Delta^2)$. Then z is a 1-simplex of $E'(X,\partial X)$ such that $\partial_0 z = y$ and $\partial_1 z = S(w)$. This proves $\pi_0(S)$ is surjective.

To show that $S\colon S_{CAT}(X,\partial X) \to E'(X,\partial X)$ induces a bijection $\pi_j(S_{CAT}(X,\partial X),v) \to \pi_j(E'(X,\partial,X),S(v))$ for all $j \geq 1$ and any point complex v one employs a straightforward extension of the above argument together with the following easy extension of Theorem 2.3.5 (the $\pi - \pi$ theorem):

Theorem 2.3.7: Let $G: N \to Y$ b,n be a CAT normal map of $(m+2)$-ads with $\dim Y \geq s+4$, $s \geq 2$. Suppose for every subset c of $\{0,\ldots,s-2\}$ such that $\partial_c Y$ is non-empty, the inclusion $\partial_c Y \to Y$ induces an isomorphism of fundamental groupoids $\pi_1(\partial_c Y) \to \pi_1(Y)$ and that for each j, $s-1 \leq j \leq m+s-1$, $\partial_j F: \partial_j N \to \partial_j Y$ is a simple homotopy equivalence of $(m+s-1)$-ads. Then G is normally bordant relative to $\partial_{s-1}N \cup \cdots \cup \partial_{m+s-2}N$ to a simple homotopy equivalence of $(m+2)$-ads, the normal bordism having a $(m+s+2)$-ad structure.

By Whitehead's theorem (Theorem 1.3.1) the map $S: S_{CAT}(X,\partial X) \to E'(X,\partial X)$ is a homotopy equivalence. As a consequence of Proposition 2.2.7, the inclusion $E'(X,\partial X) \to E(X,\partial X)$ is a homotopy equivalence. This proves Theorem 2.3.4.

3. SURGERY MOCK BUNDLES AND ASSEMBLY

3.1 Surgery Spectra

Let B be a CW r-ad such that for every subset c of $\{0,\ldots,r-2\}$ $\partial_c B$ is connected and $\pi_1(\partial_c|B|)$ is finitely presented. Define $\mathbb{L}(B) = \{(A_n, a_n\colon A_n \to \Omega A_{n+1}) \mid n \geq 0\}$ as follows:

$$A_{4k+j} = \mathbb{L}_{8-j}(B) \qquad k \geq 0 \qquad j = 0,1,2,3$$

For $j = 0,1,2$ and $k \geq 0$ $a_{4k+j}\colon A_{4k+j} \to \Omega A_{4k+j+1}$ is the map $e_{8-j}\colon \mathbb{L}_{8-j}(B) \to \Omega\mathbb{L}_{7-j}(B)$ of Proposition 2.2.2 and a_{4k+3} is the composite $e_9\theta_1\colon \mathbb{L}_5(B) \to \mathbb{L}_9(B) \to \Omega\mathbb{L}_8(B)$ where θ_1 is the periodicity map of Proposition 2.2.5. Propositions 2.2.2 and 2.2.5 imply that $\mathbb{L}(B)$ is a Ω-spectrum. It is clearly periodic with period 4. The spectrum $\mathbb{L}(B)$ gives rise to a generalized cohomology theory defined by: $H^q(X,Y;\ \mathbb{L}(B)) = [X,Y;|\mathbb{L}_{i-r}(B)|,|\phi|]$ where (X,Y) is a finite CW pair and $q = r \bmod 4$. By the periodicity of $\mathbb{L}(B)$ there is an isomorphism $H^q(X,Y;\mathbb{L}(B)) \cong H^q((X,Y)\times(\Delta^p,\partial\Delta^p);\mathbb{L}(B))$ for $p = 4m \geq 0$.

We will be mainly concerned with the case $B = pt$, where "pt" will be used to denote the space consisting of one point. Note that $\mathbb{L}(pt)$ is not a 0-connected spectrum since $\pi_0(\mathbb{L}_8(pt),\phi) = L_0(1) = Z$, the infinite cyclic group. The associated 0-connected spectrum with zero-th space $\mathbb{L}_8(pt)_0$, $\mathbb{L}(pt)(1,\ldots,\infty)$ in the notation of Adams [A], will also be of interest since it will be shown in Chapter 5 that $\mathbb{L}_8(pt)_0$ has the homotopy type of the space G/TOP.

3.2 Surgery Mock Bundles

The notion of a surgery mock bundle is an adaptation of the mock bundle idea of [BRS].

Definition 3.2.1: Let K be an ordered simplicial complex, B a CW r-ad, and q a non-negative integer. A (q,B) surgery mock bundle over K,ξ, consists of the following data:

1. Closed subspaces M and X of $|K| \times R^p \subset |K| \times R^\infty$ for some p.
2. Microbundles v over M and n over X.
3. A collection, denoted $|\xi|$, of continuous maps

$$M \xrightarrow{f} X \xrightarrow{h} B \times |K|, \qquad b: v \to n$$

where b is a map of microbundles covering f and if σ is a j-simplex of K and $M_\sigma = M \cap \sigma \times R^p$, $X_\sigma = X \cap \sigma \times R^p$, then $|\xi|$ restricts to a surgery problem of type (q,j) over B, denoted by $\xi(\sigma)$:

$$M_\sigma \to X_\sigma \to B \times (\sigma) \; b_\sigma, n_\sigma$$

where $f_\sigma = f|M_\sigma$, $h_\sigma = h|X_\sigma$, $b_\sigma = b|M_\sigma$, $n_\sigma = n|X_\sigma$.

Let L be a subcomplex of K. A (q,B) surgery mock bundle over K relative to L is a (q,B) surgery mock bundle over K,ξ, such that for each simplex σ of L $\xi(\sigma)$ is the empty surgery problem.

Recall that an ordered simplicial complex is said to be oriented if there is a given function u from the simplices of K to $\{-1,1\}$. If τ is a j-simplex of K and $\sigma = \partial_k \tau$ then the incidence number $e(\sigma,\tau)$ is defined by:

$$e(\sigma,\tau) = (-1)^k u(\sigma) u(\tau).$$

Now suppose X^m and Y^{m-1} are connected oriented manifolds with Y included in the boundary of X. The incidence number $e(Y,X)$ is defined by comparing the given orientation of Y with the orientation of Y induced by X:

$$e(Y,X) = 1 \quad \text{if these orientations agree}$$
$$= -1 \quad \text{otherwise}$$

Suppose in Definition 3.2.1 that the complex K is oriented. Then is said to be an oriented (q,B) surgery mock bundle over K if for each pair (σ,τ) where σ is a $(j-1)$-face of the j-simplex τ, $e(\sigma,\tau) = e(X_\sigma,X_\tau) = e(M_\sigma,M_\tau)$. This is to hold for each component if X_σ or M_σ is not connected.

$-\xi$ is defined to be the oriented surgery mock bundle over K given by reversing the orientation of each M_τ and X_τ. If L is a subcomplex of K then the restriction of K to L, denoted $\xi|L$ is defined in the obvious manner.

<u>Definition 3.2.2</u>: An oriented (q,B) surgery mock bundle over K,ξ is said to be special if for every j-simplex σ of K $\xi(\sigma)$ is a j-simplex of $\mathbb{L}_q(B)_0$ where the subscript 0 denotes the 0-component.

There is a notion of induced bundle for surgery mock bundles:

<u>Definition 3.2.3</u>: Let K and H be ordered simplicial complexes, $g\colon H \to K$ a Δ-map, and ξ a (q,B) surgery mock bundle over K (notation as in Definition 3.2.1). The pullback of ξ by g is the (q,B) surgery mock bundle over H, denoted $g^*\xi$:

$$|g^*\xi| = g^*M \to g^*X \to B \times |H| \quad g^*b,\ g^*n$$
$$(g^*\xi)(\sigma) = (g^*M)_\sigma \to (g^*X)_\sigma \to B \times |\sigma| \quad g^*b_\sigma,\ g^*n_\sigma$$

is defined by the following pullback diagram of spaces and continuous maps:

$$\begin{array}{ccc} g^*M & \xrightarrow{V} & M \\ \downarrow & & \downarrow \\ g^*X & \xrightarrow{U} & X \\ \downarrow & & \downarrow \\ B \times |H| & \xrightarrow{id \times |g|} & B \times |K| \end{array}$$

g^*b is the induced microbundle map $V^*(v) \to U^*(n) = g^*n$

which restricts over each simplex σ of H to the pullback diagram:

$$\begin{array}{ccc} (g^*M)_\sigma & \xrightarrow{V_\sigma} & M_{g(\sigma)} \\ \downarrow & & \downarrow \\ (g^*X)_\sigma & \xrightarrow{U_\sigma} & X_{g(\sigma)} \\ \downarrow & & \downarrow \\ B \times |\sigma| & \xrightarrow{id \times |g|} & B \times |g(\sigma)| \end{array}$$

Since g induces a simplicial isomorphism $|\sigma| \to |g(\sigma)|$, V_σ and U_σ are homeomorphisms and hence $(g^*\xi)(\sigma)$ has the structure of a surgery problem of type $(q, \dim \sigma)$ over B.

If H and K are oriented with orientation functions w and w' respectively, define a new orientation function for H by: $p(\sigma) = w(\sigma)w'(g(\sigma))$. Suppose ξ is an oriented surgery mock bundle, as above. Then $g^*\xi$ is an oriented surgery mock bundle where $(g^*M)_\sigma$ is given $p(\sigma)$ times the orientation induced by $M_{g(\sigma)}$ and the homeomorphism V_σ and similarly for $(g^*X)_\sigma$.

The same considerations apply in the relative case and to special surgery mock bundles.

We now investigate the problem of representing the cohomology theory $H^*(\ ;\mathbb{L}(B))$ in terms of surgery mock bundles. The following definitions will be basic to our discussion.

Definition 3.2.4: Let (K,L) be an ordered and oriented simplicial pair. Then $M^q_B(K,L)$, called the Δ-set or oriented (q,B) surgery mock bundles over K relative to L, is the pointed Δ-set defined as follows: $M^q_B(K,L)(j)$ is the set of all oriented (q,B) surgery mock bundles over $K \otimes \Delta^j$ relative to $L \otimes \Delta^j$. The empty surgery mock bundle over $K \otimes \Delta^j$ serves as a base point. Face maps are defined by restriction over $K \otimes \partial_k\Delta^j$ $k = 0,\ldots,j$.

Using special oriented surgery mock bundles (see Definition 3.2.2) one defines the pointed Δ-set $\underline{M}^q_B(K,L)$ as in the previous definition. This Δ-set will be used to obtain a geometric representation for $H^0(\ ;\mathbb{L}(B)(1,\ldots\wp))$.

Let $f\colon (K,L) \to (H,J)$ be a Δ-map of ordered and oriented simplicial pairs. Then f induces Δ-maps:

$$f^{\#}\colon M^q_B(H,J) \to M^q_B(K,L) \quad \text{and} \quad f^{\#}\colon \underline{M}^q_B(H,J) \to \underline{M}^q_B(K,L)$$

given by $(f^{\#})_k(\xi) = (f \otimes id)^*\xi$ where ξ is a k-simplex of $M^q_B(H,J)$ or $\underline{M}^q_B(H,J)$ and $id\colon \Delta^j \to \Delta^j$ is the identity.

Remark: When $B = pt$ the symbol "B" will be dropped from the notation above.

The disjoint union of two oriented (q,B) surgery mock bundles over K relative to L is defined with the aid of the maps $id \times d(t)\colon |K| \times R^\infty \to |K| \times R^\infty$ $t = 0,1$ $d(t)(x_0,x_1,\ldots) = (t,x_0,x_1,\ldots)$ essentially as in Section 2.2. This operation provides a natural H-structure in $M^q_B(K,L)$ and in $\underline{M}^q_B(K,L)$ with the empty surgery mock bundle over K serving as the homotopy unit. Note that if $f\colon (K,L) \to (H,J)$ is a Δ-map then the induced maps $f^{\#}$ are H-homomorphisms. Orientation reversal induces a natural H-homomorphism $M^*_B \to M^*_B$ and $\underline{M}^*_B \to \underline{M}^*_B$.

If $N^n \subset R^p$ is a closed oriented manifold and ξ is an oriented (q,B)

surgery mock bundle over K relative to L then an oriented $(q+n,B)$ surgery mock bundle over K relative to $L, \xi \times N$, is defined by taking Cartesian products with N in the obvious manner (compare with Section 2.1). This yields a natural H-homomorphism $M_B^q(K,L) \to M_B^{q+n}(K,L)$ and similarly a natural H-homomorphism $\underline{M}_B^q(K,L) \to \underline{M}_B^{q+n}(K,L)$. When $N = (CP^2)^k$, the k-fold Cartesian product of 2-dimensional complex projective space, this map will be denoted by Θ_k and will be called the periodicity map.

Let K be an ordered simplicial complex which is oriented with orientation function w. Let L be a subcomplex of K. The glue map $G\colon \Delta(K,L;\mathbb{L}_q(B),\phi) \to M_B^q(K,L)$ is the Δ-isomorphism defined as follows: Let $h\colon K \otimes \Delta^j \to \mathbb{L}_q(B)$ be a j-simplex of $\Delta(K,L;\mathbb{L}_q(B),\phi)$. If σ is a k-simplex of $K \otimes \Delta^j$ let $h(\sigma) = M_\sigma \to X_\sigma \to B \times \Delta^k$ b_σ, n_σ where $M_\sigma, X_\sigma \subset \Delta^k \times R^\infty$. M_σ and X_σ can be viewed as being embedded in $|\sigma| \times R^\infty \subset |K \otimes \Delta^j| \times R^\infty$ for each k-simplex σ of $K \otimes \Delta^j$ if σ is identified with Δ^k using the unique order preserving simplicial isomorphism $\Delta^k \to \sigma$. Since h is a Δ-map the union of $h(\sigma)$'s for $\sigma \in K \otimes \Delta^j$ yields:

$$M \to X \to B \times |K \otimes \Delta^j| \quad b,n \qquad M,X \subset |K \otimes \Delta^j| \times R^\infty$$

which restricts to $h(\sigma)$ over $|\sigma| \times R^\infty$. Reorienting each M_σ and X_σ using w, one obtains an oriented (q,B) surgery mock bundle over $K \otimes \Delta^j$ relative to $L \otimes \Delta^j$ which defines $G_j(h)$. G is clearly a Δ-map.

Similarly there is a glue map:

$$\underline{G}\colon \Delta(K,L;\mathbb{L}_q(B),\phi) \to \underline{M}_B^q(K,L).$$

Both G and $\underline{G}$ have obvious inverses given by "disassembling" a surgery mock bundle into its component surgery problems. Hence G and $\underline{G}$ are isomorphisms.

The Δ-sets $\Delta(K,L;\mathbb{L}_q(B),\phi)$ and $\Delta(K,L;\mathbb{L}_q(B)_0,\phi)$ inherit H-structures from $\mathbb{L}_q(B)$ and $\mathbb{L}_q(B)_0$ respectively. Both are H-groups since $\mathbb{L}_q(B)$ and $\mathbb{L}_q(B)_0$ are H-groups. It is clear that G and $\underline{G}$ are H-homomorphisms and that they are natural with respect to Δ-maps, i.e., if $f: (K,L) \to (H,J)$ is a Δ-map then there is a commutative diagram:

3.2.5
$$\begin{array}{ccc} \Delta(H,J;\mathbb{L}_q(B),\phi) & \xrightarrow{G} & M_B^q(H,J) \\ \downarrow f^{\#} & & \downarrow f^{\#} \\ \Delta(K,L;\mathbb{L}_q(B),\phi) & \xrightarrow{G} & M_B^q(K,L) \end{array}$$

and similarly for $\underline{G}$. There is also a commutative square:

3.2.6
$$\begin{array}{ccc} \Delta(K,L;\mathbb{L}_q(B),\phi) & \xrightarrow{G} & M_B^q(K,L) \\ \downarrow (\theta_k)_{\#} & & \downarrow \Theta_k \\ \Delta(K,L;\mathbb{L}_{q+4k}(B),\phi) & \xrightarrow{G} & M_B^{q+4k}(K,L) \end{array} \qquad q \geq 5$$

where $\theta_k: \mathbb{L}_q(B) \to \mathbb{L}_{q+4k}(B)$ and Θ_k are the periodicity maps defined by taking Cartesian products with $(CP^2)^k$. By Proposition 2.2.5 θ_k is a homotopy equivalence and thus $(\theta_k)_{\#}$ is also a homotopy equivalence. It follows Θ_k is homotopy equivalence. The analogous result holds for $\underline{M}_B^*$.

The natural inclusion $i: \mathbb{L}_q(B)_0 \to \mathbb{L}_q(B)$ induces a H-homormorphism $i: \Delta(K,L;\mathbb{L}_q(B)_0,\phi) \to \Delta(K,L;\mathbb{L}_q(B),\phi)$ and thus also a H-homormorphism $j: \underline{M}_B^q(K,L) \to M_B^q(K,L)$ which can be viewed as a natural inclusion. The map j has the following properties which will be of interest in the sequel:

<u>Proposition 3.2.7</u>: Let (K,L) be an ordered and oriented simplicial pair. Then

1. $\pi_k(j): \pi_k(\underline{M}_B^q(K,L),\phi) \to \pi_k(M_B^q(K,L),\phi)$ is an isomorphism for $k \geq 1$ and an injection for $k = 0$.
2. If in addition K is connected, there is an exact sequence of groups:

$$0 \longrightarrow \pi_0(\underline{M}_B^q(K),\phi) \xrightarrow{\pi_0(j)} \pi_0(M_B^q(K),\phi) \longrightarrow \pi_0(\mathbb{L}_q(B),\phi) .$$

Proof: Let X be the Δ-group with $X(j) = \pi_0(\mathbb{L}_q(B),\phi)$ for all j and with face maps $\partial_s: X(j) \to X(j-1)$ given by the identity map. Note that $\pi_0(X,*) = \pi_0(X,*) = \pi_0(\mathbb{L}_q(B),\phi)$ and that $|X|$ is discrete. Define a Δ-map $p: \mathbb{L}_q(B) \to X$ by sending a j-simplex y of $\mathbb{L}_q(B)$ to the equivalence class of $(\partial_0)^j y$. The homotopy fiber of p over the base point is easily seen to be isomorphic to $\mathbb{L}_q(B)_0$ and so the sequence $\mathbb{L}_q(B)_0 \to \mathbb{L}_q(B) \to X$ is a fibration sequence. Hence the sequence:

$$(*) \qquad \Delta(K,L;\mathbb{L}_q(B)_0,\phi) \to \Delta(K,L;\mathbb{L}_q(B),\phi) \to \Delta(K,L;X,*)$$

is also a fibration sequence (see Theorem 7.8 of [Ma]). Let $Y = \Delta(K,L;X,*)$. Then $|Y|$ is discrete since $|X|$ is discrete. The fiber mapping sequence of 1.3.8 for (*) becomes:

$$(**) \qquad \dots \to \Omega Y \to \Delta(K,L;\mathbb{L}_q(B)_0,\phi) \to \Delta(K,L;\mathbb{L}_q(B),\phi) \to Y .$$

The maps in the above sequence are all H-maps. Since $|Y|$ is discrete $\Omega^k Y$ is a point complex for $k \geq 1$. Hence applying π_0 to (**) yields the first part of the proposition. If K is connected and L is empty note that $Y = \Delta(K;X)$ is isomorphic as a Δ-group to X since $|X|$ is discrete. Applying π_0 to (**) then yields the second part of the proposition.

We now discuss the relationship between surgery mock bundles and the cohomology theory $H^*(\ ;\mathbb{L}(B))$.

Let (K,L) be an ordered and oriented simplicial pair. For $p \geq 0$ define groups $m_B^p(K,L)$ by:

$$3.2.8 \qquad m_B^p(K,L) = \pi_0(M_B^{8-j}(K,L),\phi) \qquad p = j \bmod 4 .$$

Since a 1-simplex of $M_B^q(K,L)$ can be viewed as a cobordism of (q,B) surgery mock bundles over K relative to L, $m_B^p(K,L)$ is the cobordism group of oriented $(8-j,B)$ surgery mock bundles over K relative to L. There is a sequence of isomorphisms:

3.2.9
$$\begin{aligned} m_B^p(K,L) &\cong \pi_0(\Delta(K,L;\mathbb{L}_{8-j}(B),\phi)) && \text{via the glue map } G \\ &\cong [|K|,|L|;|\mathbb{L}_{i-j}(B)|,*] && \text{by [RS2, Ch. 6]} \\ &\cong H^p(|K|,|L|;\mathbb{L}(B)) \ . \end{aligned}$$

In particular, 3.2.9 implies $m_B^*(K,L)$ does not depend on the ordering or orientation of (K,L).

If $f: (K,L) \to (H,J)$ is a Δ-map between ordered and oriented simplicial pairs, the induced homomorphism $f^p: m_B^p(H,J) \to m_B^p(K,L)$ is given by $\pi_0(f^\#)$ where $f^\#: M_B^{8-j}(H,J) \to M_B^{8-j}(K,L)$ has been defined previously and $p = j \bmod 4$. If f is an order preserving simplicial map which is not necessarily injective on each simplex, the induced homomorphism $f^*: m_B^*(H,J) \to m_B^*(K,L)$ is defined using the simplicial mapping cylinder of f (see [Sp, p. 151]): Let (M,N) be the simplicial mapping cylinder of f and $i: (K,L) \to (M,N)$, $j: (H,J) \to (M,N)$ the natural inclusions. Since j is a homotopy equivalence $\pi_0(j^\#)$ is an isomorphism. f^* is given by $f^* = \pi_0(i^\#)\pi_0(j^\#)^{-1}$. It follows from 3.2.9 that $f^* = H^*(|f|)$ where $H^*(|f|)$ is the induced map in the cohomology theory $H^*(\ ;\mathbb{L}(B))$.

Consider the commutative diagram:

3.2.10
$$\begin{array}{ccc} \Delta(K,L;\mathbb{L}_{q+1}(B),\phi) & \xrightarrow{\ G\ } & M_B^{q+1}(K,L) \\ \uparrow{\scriptstyle (d_{q+1})_\#} & & \\ \Delta(K,L;\Omega\mathbb{L}_q(B),\phi) & & \uparrow{\scriptstyle \Sigma_q} \\ \uparrow{\scriptstyle \mu} & & \\ \Omega\Delta(K,L;\mathbb{L}_q(B),\phi) & \xrightarrow{\ \Omega G^{-1}\ } & \Omega M_B^q(K,L) \end{array}$$

where μ is the natural homotopy equivalence of Proposition 1.4.2, $d_{q+1}: \Omega\mathbb{L}_q(B) \to \mathbb{L}_{q+1}(B)$ is the homotopy equivalence defined in the proof of Proposition 2.2., G and G^{-1} are the glue map and its inverse respectively, and finally Σ_q is defined to be the composite. Σ_q will be called the geometric suspension. Σ_q is a homotopy equivalence since it is a composite of homotopy equivalences.

It is readily observed that:

$$\pi_0(\Omega M_B^q(K,L),\phi) \cong \pi_0(M_B^q(K \otimes \Delta^1, K \otimes \partial\Delta^1 \cup L \otimes \Delta^1),\phi) .$$

Define homomorphisms $S_p\colon m^{p+1}(K \otimes \Delta^1, K \otimes \partial\Delta^1 \cup L \otimes \Delta^1) \to m^p(K,L)$ by:

$$\begin{aligned} S_p &= \pi_0(\Sigma_{8-j}) && \text{if } p = j \bmod 4 \quad j = 1,2,3 \\ &= \pi_0(\Theta_1)^{-1}\pi_0(\Sigma_8) && \text{if } p = 0 \bmod 4 \end{aligned}$$

where Θ_1 is the periodicity map of 3.2.6. S_p is an isomorphism since Σ and Θ are homotopy equivalences. By 3.2.9 and the definition of the spectrum $\mathbb{L}(B)$, S_p is seen to coincide with the suspension in the chomology theory $H^*(\ ;\mathbb{L}(B))$. Thus we have shown that m_B^* provides a geometric representation for this theory.

Similarly, $H^0(\ ;\mathbb{L}(B)(1,\ldots,\infty))$ can be geometrically represented as a cobordism group of special oriented surgery mock bundles (see Definition 3.2.2):

$$\underline{m}_B^0(K,L) = \pi_0(\underline{M}_B^8(K,L),\phi)$$

with induced homormophisms defined as for m_B^0. As in 3.2.9 there is a natural isomorphism:

$$\underline{m}_B^0(K,L) = H^0(|K|,|L|;\mathbb{L}(B)(1,\ldots,\infty)) .$$

3.3 Assembly

In this section the notion of assembly is duscussed and the assembly maps are defined. The assembly maps will be important in the sequel since it will be shown in Chapter 5 that under appropriate conditions the surgery obstruction map is given by an assembly map.

Definition 3.3.1: Let K be an ordered and oriented simplicial s-ad. K will be called an oriented simplicial manifold s-ad of dimension m if K is a combinatorial triangulation of a compact oriented PL manifold s-ad M^m with $\partial_j M$ corresponding to the subcomplex $\partial_j K$ of K $j = 0,\ldots,s-2$.

The next theorem, whose proof is postponed to the end of this section, will enable us to define the assembly map:

Theorem 3.3.2 (Assembly): Let B be a CW r-ad and K be as in the above definition. Suppose that ξ is an oriented (q,B) surgery mock bundle over K:

$$|\xi| = M \overset{f}{\to} X \overset{h}{\to} B \times |K| \;\; b,n$$

$$\xi(\sigma) = M_\sigma \to X_\sigma \to B \times |\sigma| \;\; b_\sigma, n_\sigma \quad .$$

Let M have the (s+r)-ad structure:

$$\partial_j M = \cup \{M_\sigma | \; \sigma \in \partial_j K\} \qquad j = 0,\ldots,s-2$$

$$\partial_{s-1+i} M = \cup \{\partial_{m+1+i} M_\sigma | \; \sigma^m \in K\} \qquad i = 0,\ldots,r-1$$

and similarly for X.

Then M and X are compact oriented TOP manifold (s+r)-ads of dimension m+q and $f\colon M \to X$ b,n is a degree 1 normal map of (s-r)-ads such that $\partial_{s-1} f$ is a simple homotopy equivalence of (s+r-1)-ads.

The definition below of the assembly map is an adaptation of Quinn's definition in [Q]:

Definition 3.3.3: Let K be as in Definition 3.3.1 and let B be a CW r-ad. The assembly map $A\colon M_B^q(K,\partial_0 K) \to \mathbb{L}_{q+m}(B \times \delta_0 |K|)$ is the Δ-map defined as follows: Let ξ be a j-simplex of $M_B^q(K,\partial_0 K)$, i.e., an oriented (q,B) surgery mock bundle over $K \otimes \Delta^j$ relative to $\partial_0 K \otimes \Delta^j$. By the previous theorem $|\xi|$ has the structure of a surgery problem of type $(q+m,j)$ over the $(s+r-2)$-ad $B\times\delta_0|K|$. Then A is defined on j-simplices by $A_j(\xi) = |\xi|$. Thus the effect of the assembly map on the surgery mock bundle ξ is essentially to forget its simplicial structure and to view the total space $|\xi|$ of ξ as a surgery problem.

Similarly there is an assembly map:

$$\underline{A}\colon \underline{M}^q_B(K,\partial_0 K) \to \mathbb{L}_{q+m}(B\times\delta_0|K|)\,.$$

Both A and $\underline{A}$ are clearly H-homormorphisms with respect to the disjoint union H-structures. In addition A and $\underline{A}$ commute with the periodicity maps, i.e., there is a commutative diagram:

3.3.4
$$\begin{array}{ccc} M^{q+4p}_B(K,\partial_0 K) & \xrightarrow{A} & \mathbb{L}_{q+m+4p}(B\times\delta_0|K|) \\ \uparrow{\scriptstyle\Theta_p} & & \uparrow{\scriptstyle\theta_p} \\ M^q_B(K,\partial_0 K) & \xrightarrow{A} & \mathbb{L}_{q+m}(B\times\delta_0|K|) \end{array}$$

and similarly for $\underline{A}$. Θ_p and θ_p are the periodicity maps defined by taking Cartesian products with $(CP^2)^p$.

An inspection of the definitions also reveals that there is a commutative diagram:

3.3.5
$$\begin{array}{ccc} M^{q+1}_B(K,\partial_0 K) & \xrightarrow{A} & \mathbb{L}_{q+1+m}(B\times\delta_0|K|) \\ \uparrow{\scriptstyle\Sigma_q} & & \uparrow{\scriptstyle d_q} \\ \Omega M^q_B(K,\partial_0 K) & \xrightarrow{A} & \Omega\mathbb{L}_{q+m}(B\times\delta_0|K|) \end{array}$$

where Σ_q is the geometric suspensis (see 3.2.10) and d_q is defined in the proof of Proposition 2.2.2.

The remainder of this section will be devoted to the proof of Theorem 3.3.2.

Let K be a locally finite ordered simplicial complex, viewed as a collection of closed linear simplices lying in R^{∞}. If σ is a simplex of K, the following notation will be used:

$$B(\sigma,K) = \{\mu \in K \mid \sigma \text{ is a face of } \mu\}$$
$$B(\sigma,K)(j) = \{\mu \in B(\sigma,K) \mid \dim \mu \leq \dim \sigma + j\}$$
$$|B(\sigma,K)|(j) = \cup \{\mu \in B(\sigma,K)(j)\} .$$

Observe that $|B(\sigma,K)|(\infty)$ is just $|\text{star}(\sigma,K)|$, the geometric realization of the star of σ in K.

Lemma 3.3.6. Suppose $q \geq 0$ and K is as above. Let B be a space and $\{B(\tau) \mid \tau \in B(\sigma,K)\}$ a collection of subspaces such that:

1. $B(\tau)$ is a topological ball of dimension $q + \dim \tau$.
2. If μ is a face of τ then $B(\mu) \subset \partial B(\tau)$.
3. $B(\tau) \cap B(\tau') = B(\tau \cap \tau')$ for each $\tau,\tau' \in B(\sigma,K)$.
4. $B = \cup \{B(\tau) \mid \tau \in B(\sigma,K)\}$.

Then B is a homeomorphic to $E^q \times |\text{star}(\sigma,K)|$ where E^q is the standard q-ball.

Proof: Let $B(j) = \cup \{B(\tau) \mid \tau \in B(\sigma,K)(j)\}$ and let $P(j)$ be the proposition:

$P(j)$: There exists a homeomorphism $h_j\colon E^q \times |B(\sigma,K)|(j) \to B(j)$ such that $h_j(E^q \times \tau) = B(\tau)$ for each $\tau \in B(\sigma,K)(j)$.

$P(0)$ is obvious because $B(\sigma,K)(0) = \{\sigma\}$. Assume $P(j)$ and let:

$$\tau \in B(\sigma,K) \ \dim \tau = j + 1 + \dim \sigma$$
$$D(\tau) = \cup \{\mu \in B(\sigma,K) \mid \mu \text{ is a face of } \tau,\ \dim \mu = j + \dim \sigma\}$$
$$D'(\tau) = \cup \{B(\mu) \mid \mu \in D(\tau)\} .$$

$D(\tau)$ is just the union of codimension 1 faces of τ which contain σ and thus $D(\tau)$ is a ball of dimension $j+\dim\sigma$. It follows that $E^q \times D(\tau)$ is a $q+j+\dim\sigma$ ball contained in $\partial(E^q \times \tau)$. By the next lemma (Lemma 3.3.7) $D'(\tau)$ is a $q+j+\dim\sigma$ ball contained in $\partial B(\tau)$. Hence $h: E^q \times E(\tau) \to D'(\tau)$ extends to a homeomorphism $H: E^q \times \tau \to B(\tau)$. Doing this for each $j+1+\dim\sigma$ simplex of $B(\sigma,K)$ yields a homeomorphism h_{j+1} satisfying $P(j+1)$. The lemma follows by induction and from the fact that K is locally finite.

Lemma 3.3.7: Let the notation be as above. Suppose that $\tau \in B(\sigma,K)$ where $\dim\tau = j+1+\dim\sigma$ and $C(\tau)$ is a non-empty subcollection of the codimension one faces of τ which have σ as a face. Let $D = \cup\{B(\mu) \mid \mu \in C(\tau)\}$. Then D is a $j+q+\dim\sigma$ ball with $D \subset \partial B(\tau)$.

Proof: We use induction j.

Assertion (j): The lemma is true for j.

If $j = 0$ then $C(\tau) = \{0\}$ and thus assertion (0) is obvious. Now inductively assume assertion (j). Let $\tau \in B(\sigma,K)$ with $\dim\tau = j+2+\dim\sigma$ and let $C(\tau) = \{\sigma_1,\ldots,\sigma_p\}$ be a non-empty subcollection of the codimension one faces of τ which have σ as a face.

Claim (r): $E(r) = \cup\{B(\sigma_i) \mid 1 \le i \le r\}$ is a $j+1+q+\dim\sigma$ ball. This is obvious when $r = 1$. Inductively assume the claim for r.

$$E(r) \cap B(\sigma_{r+1}) = \cup\{B(\sigma_i \cap \sigma_{r+1}) \mid 1 \le i \le r\} .$$

Since σ_i and σ_{r+1} are codimension one faces of τ each $\sigma_i \cap \sigma_{r+1}$ is a codimension one face of σ_{r+1}. By the induction hypothesis, assertion (j), $E(r) \cap B(\sigma_{r+1})$ is a $q+j+\dim\sigma$ ball and clearly $E(r) \cap B(\sigma_{r+1}) \subset E(r) \cap \partial B(\sigma_{r+1})$. Hence $E(r+1) = E(r) \cup B(\sigma_{r+1})$ is a $q+j+\dim\sigma$ ball since the union of two balls along a common face is a ball. By induction on r claim (p) is true which proves assertion (j+1). The lemma follows by induction on j.

Proposition 3.3.8: Let K^m be a simplicial manifold s-ad and $X \subset |K| \times R^\infty$ a subspace such that for each j-simplex σ of K $X(\sigma) = X \cap (|\sigma| \times R^\infty)$ is a TOP manifold (j+r+2)-ad of dimension j+q with $\partial_i X(\sigma) = X(\partial_i \sigma)$ $i = 0,\dots,j$ and $\partial_i X(\sigma) \subset (\mathrm{int}|\sigma|) \times R^\infty$ for $i > j$. Then X is a TOP manifold (s+r)-ad of dimension m+q. The ad structure of X is given by:

$$\partial_i X = X \cap (|\partial_i K| \times R^\infty) \qquad i = 0,\dots,s-2$$
$$\partial_{s-1+i} X = \cup\{\partial_{m+1+i} X(\sigma) \mid \sigma^m \in K\} \qquad i = 0,\dots,r-1$$

Proof: For simplicity assume $s = r = 1$. The conclusion is then X is a TOP manifold of dimension $m+q$ with boundary X given by $\partial X = \cup\ \{\partial_{m+1} X(\sigma) \mid \sigma^m \in K\}$. The general case of the proposition can be obtained by induction on r and s.

Remark: If τ is a k-simplex of K note that

$$\partial_i(\partial_{k+1} X(\tau)) = \partial X \cap X(\partial_i \tau) \qquad i = 0,\dots,k$$

Suppose $x \in X$ then $x \in X(\sigma)$ with $p = \dim \sigma$ minimal. Since p is minimal either $x \in \mathrm{int}\, X(\sigma)$ or $x \in \mathrm{int}\, \partial_{p+1} X(\sigma)$. The second case occurs if and only if $x \in \partial X$.

Case 1. $x \in \mathrm{int}\, X(\sigma)$. Since $X(\sigma)$ is a p+q dimensional manifold there exists a (p+q)-ball $B(\sigma) \subset \mathrm{int}\, X(\sigma)$ with $x \in \mathrm{int}\, B(\sigma)$. Let $C(0) = \{B(\sigma)\}$. Suppose the collection $C(j) = \{B(\tau) \mid \tau \in B(\sigma,K)(j)\}$ has been constructed so that the following is true:

1. $B(\tau) \subset X(\tau)$ is a ball of dimension $q + \dim \tau$
2. $B(\tau) \cap B(\tau') = B(\tau \cap \tau')$ for $\tau,\tau' \in B(\sigma,K)(j)$
3. If τ' is a face of τ then $B(\tau') \subset \partial B(\tau) \cap \partial B(\tau) \cap \partial X(\tau)$.

Let $\tau \in B(\sigma,K)$ $\dim \tau = j+1+p$ and let $D = \cup\ \{B(\mu) \mid \mu$ is a codimension 1 face of τ containing $\sigma\}$. Note that $D \subset \partial X(\tau)$ and that by Lemma 3.3.7

D is a ball of dimension $p+q+j$. Since $\partial X(\tau)$ is collared in $X(\tau)$ there is a neighborhood $B(\tau)$ of D in $X(\tau)$ such that $B(\tau)$ is homeomorphic to $D \times I$ and $B(\tau) \cap \partial X(\tau) = D$. Find such a $B(\tau)$ for each (j+1+p)-simplex in $B(\sigma,K)$. This yields a collection $C(j+1)$ satisfying conditions 1 - 3 above. By induction we obtain a sequence of such collections $C(j)$ $j = 0,\ldots,m-p$. Lemma 3.3.6 implies that $B = \cup\{B(\tau) | B(\tau) \in C(m-p)\}$ is homeomorphic to $E^q \times |star(\sigma,K)|$ (E^q is the standard q-ball). Since K^m is a conbinatorial triangulation of a PL manifold $|star(\sigma,K)|$ is a m-ball and thus B is a (q+m)-ball. B is clearly a neighborhood of x in A and $x \in int\, B$.

<u>Case 2</u>. $x \in int\, \partial_{p+1}X(\sigma)$. Let $F \subset int\, \partial_{p+1}X(\sigma)$ be a (q+p-1)-ball with $x \in F$. Since $\partial X(\sigma)$ is collared in $\dot{X}(\sigma)$ there is a neighborhood $B(\sigma)$ of F in $X(\sigma)$ such that $B(\sigma)$ is homeomorphic to $F \times I$ and $B(\sigma) \cap X(\sigma) = F$. Proceeding as in Case 1, we obtain a neighborhood B of x in X homeomorphic to the (q+m)-ball $E^q \times |star(\sigma,K)|$ and such that $x \in \partial B$.

X is thus a manifold of dimension $m+q$ with boundary X since each point of X has a neighborhood of the appropriate type.

Let M^m be a compact connected orientable manifold with boundary. An orientation for M is a choice of a generator for the infinite cyclic group $H_m(M,\partial M)$. This homology class will be called the orientation class and will be denoted by $[M]$. If M is not connected then an orientation of M is a choice of orientation for each component. If $M = \cup M(j)$ is a disjoint union of components then $H_m(M,\partial M) = \oplus H_m(M(j),\partial M(j))$ and $[M]$ is defined to be $\oplus [M(j)]$. If M is a manifold s-ad then the orientation class $[M]$ determines orientation classes $[\partial_k M] \in H_{m-1}(\partial_k M, \partial\partial_k M)$ $k = 0,\ldots,s-2$ by means of the map given by the composite:

3.3.9 $$H_m(M,\partial M) \to H_{m-1}(\partial M) \to H_{m-1}(\partial H_{m-1}(\partial M,\Lambda M) \cong \oplus_k H_{m-1}(\partial_k M,\partial(\partial_k M))$$

where $\Lambda M = \cup \{\partial_c M| \ ||c|| = 2\}$.

Proposition 3.3.10: Suppose in addition to the hypotheses of Proposition 3.3.8 that

1. K is an oriented manifold.
2. Each $X(\sigma)$ is oriented.
3. For every pair of simplices (σ,τ) of K with σ a codimension one face of τ the incidence numbers satisfy:

$$E(\sigma,\tau) = e(X(\sigma),X(\tau)) .$$

Then X is an oriented TOP manifold (s+r)-ad.

Proof: By Proposition 3.3.8 X is a TOP manifold (s+r)-ad. Without loss of generality it can be assumed that X is connected. Suppose $s = r = 1$. Then by considering the double of X along ∂X we are reduced to the case $r = 0$, $s = 1$. The case of general s,r can be reduced to the case above by doubling along a face of X and by induction.

Let $F(j) = \cup \{X(\sigma)| \dim \sigma \leq j\}$. Then $F(m) = X$ and the exact sequence of the triple $(F(m),F(m-1),F(m-2))$ yields:

$$0 \to H_{m+q}(F(m),F(m-2)) \to H_{m+q}(F(m),F(m-1)) \xrightarrow{\partial} H_{m+q-1}(F(m-1),F(m-2))$$

$H_{m+q}(F(m),F(m-2)) = X$ by the exact sequence of the pair $(F(m),F(m-2))$ and $H_j(F(i),F(i-1)) = \oplus_{\dim \sigma = i} H_j(X(\sigma),\partial X(\sigma))$ by excision. The exact sequence above becomes:

$$0 \to H_{m+q}(X) \to \oplus_{\dim \tau = m} H_{m+q}(X(\tau),\partial X(\tau)) \xrightarrow{\sum \partial_\tau} \oplus_{\dim \sigma = m-1} H_{m+q-1}(X(\sigma),\partial X(\sigma))$$

where ∂_τ is the map of 3.3.9.

Let $[X(\sigma)] \in H_{m+q}(X(\sigma),\partial X(\sigma))$ be the orientation class, $\dim \sigma = m$.

3.3.11 Define $[X] = \sum_{\dim \sigma = m} [X(\sigma)]$.

3.3.12
$$\partial[X] = \sum_{\dim\sigma=m} \partial_\sigma[X(\sigma)]$$
$$= \sum_{\dim\sigma=m} \sum_{j=0}^{m} e(\partial_j\sigma,\sigma)[X(\sigma_j)] \ .$$

Since K is a closed oriented simplicial manifold, every $(m-1)$-simplex σ of K is the face of exactly two m-simplex τ,τ' of K; furthermore the incidence numbers satisfy $e(\sigma,\tau) = -e(\sigma,\tau')$. Hence 3.3.12 implies that $\partial[X] = 0$. It is easily verified that $[X]$ generates $\ker\partial = H_{m+q}(X)$. Hence X is orientable with orientation class $[X]$.

Proof of Theorem 3.3.2: We have $|\xi| = M \to X \to B \times |K|$ $b\colon v \to n$. By Proposition 3.3.10 M and X are oriented manifold $(s+r)$-ads of dimension $m+q$. By 3.3.11 and the fact that each f_σ has degree 1: $f_*[M] = \sum_{\dim\sigma=m} (f_\sigma)_*[M(\sigma)] = \sum_{\dim\sigma=m}[X(\sigma)] = [X]$. Hence f has degree 1. In order to show that $\partial_{s-1}f$ is a simple homotopy equivalence of $(s+r-1)$-ads one uses induction on the skeleta of K and the sum theorem for simple homotopy equivalences (see [C, p. 76]).

4. INDUCTION AND RESTRICTION

4.1 The Transfer in Surgery Mock Bundle Theory

Let $p: E \to K$ be an order preserving simplicial map between ordered and oriented simplicial complexes and let $w \geq 0$.

Definition 4.1.1: p is an oriented simplicial w-mock bundle if:

1. For every j-simplex σ of K $p^{-1}(\sigma)$ is an oriented simplicial manifold (j+2)-ad of dimension $j+w$ such that as unoriented manifolds $\partial_k p^{-1}(\sigma) = p^{-1}(\partial_k \sigma)$ $k = 0, \ldots, j$.
2. For every j-simplex τ of K if σ is a (j-1)-face of τ then $e(\sigma,\tau) = e(p^{-1}(\sigma), p^{-1}(\tau))$ where the $e(\ ,\ )$'s are the incidence numbers.

Let p be as above, L a subcomplex of K and $F = p^{-1}(L)$. A transfer map $tr(p): M_B^q(E,F) \to M_B^{q+w}(K,L)$ will now be defined: Suppose ξ is a (q,B) surgery mock bundle over E relative to F where B is a CW r-ad

$$|\xi| = M \xrightarrow{f} X \xrightarrow{h} B \times |E| \quad b,n$$

$$\xi(\sigma) = M_\sigma \to X_\sigma \to B \times \sigma \quad b_\sigma, n_\sigma$$

The transfer of ξ with respect to p, denoted $tr(p)(\xi)$, is the $(q+w,B)$ surgery mock bundle over K relative to L given by:

4.1.2 $$|tr(p)(\xi)| = M \xrightarrow{f} X \xrightarrow{h'} B \times |K| \quad b,n \qquad \text{where } h' = (id \times p)h .$$

For a j-simplex σ of K, $tr(p)(\xi)(\sigma)$ is obtained as follows: Let $\xi|p^{-1}(\sigma) = N \to Y \xrightarrow{g} B \times |p^{-1}(\sigma)|$ $b|N$, $n|Y$. Then by Theorem 3.3.2 (Assembly) $x = N \to Y \xrightarrow{g'} B \times \sigma$ $b|N$, $n|Y$, where $g' = (id \times |p|)g$, has the structure of a surgery problem of type $(q+w,j)$ over B. Let $tr(p)(\xi)(\sigma) = x$.

<u>Remark</u>: We view E as being embedded in R^n. Let $c: M \to K$ be the composite $|p|\pi hf$ where π is projection onto $|E|$. Then in 4.1.2 M is viewed as being embedded in $|K| \times R^\infty$ by $M \to |K| \times R^\infty$ $m \to (c(m),m) \in |K| \times |E| \times R^\infty \subset |K| \times R^\infty \subset |K| \times R^\infty$ and similarly for X.

The transfer defines a Δ-map $tr(p): M_B^q(E,F) \to M_B^{q+w}(K,L)$ given on a j-simplex ξ by $tr(p)_j\xi = tr(p \otimes id)(\xi)$ where $id: \Delta^j \to \Delta^j$ is the identity. $tr(p)$ will be called the transfer with respect to p. Composition with the inclusion $\underline{M}_B^q(E,F) \to M_B^q(E,F)$ yeilds a transfer map:

$$\underline{tr}(p): \underline{M}_B^q(E,F) \to M_B^{q+w}(K,L) .$$

Suppose that $w = 0$ and $|p|: |E| \to |K|$ is a covering projection. At least in this case it is possible to define a transfer map $\underline{tr}(p): \underline{M}_B^q(E,F) \to \underline{M}_B^q(K,L)$. Let ξ be a j-simplex of $\underline{M}_B^q(E,F)$. Then for any s-simplex τ of E $\xi(\tau)$ is an s-simplex of $\mathbb{L}_q(B)_0$. Since for each i-simplex σ of K, $p^{-1}(\sigma)$ is a disjoint union of oriented i-simplices of E, we have $tr(p \otimes id)(\xi)(\sigma)$ is an i-simplex of $\mathbb{L}_q(B)_0$. Hence the image of $\underline{tr}(p)$ defined in the previous paragraph lies in $\underline{M}_B^q(K,L)$.

The transfer maps we have defined are clearly H-homomorphisms with respect to the disjoint union H-structures; furthermore, they commute with orientation reversal and the periodicity maps. The transfer is also functorial over pullbacks. This means that if $p': H \to J$ and $p: E \to K$ are oriented simplicial w-mock bundles and $g: E \to H$ and $f: K \to J$ are Δ-maps so that the square:

$$\begin{array}{ccc} |E| & \xrightarrow{|g|} & |H| \\ {\scriptstyle |p|}\downarrow & & \downarrow{\scriptstyle |p'|} \\ |K| & \xrightarrow{|f|} & |J| \end{array}$$

is a pullback square then the following square is commutative:

4.1.3
$$\begin{array}{ccc} M_B^q(H) & \xrightarrow{tr(p')} & M_B^{q+w}(J) \\ \downarrow g^{\#} & & \downarrow f^{\#} \\ M^q(E) & \xrightarrow{tr(p)} & M_B^{q+w}(K) \end{array}$$

This is a direct consequence of the definitions and the fact pullback squares are composable. The corresponding relative version and the version for $\underline{M}_B^*$ are also valid.

It is easily verified that there is a commutative square:

(4.1.4
$$\begin{array}{ccc} \Omega M_B^{q-1}(E,F) & \xrightarrow{\Sigma} & M_B^q(E,F) \\ \downarrow \Omega tr(p) & & \downarrow tr(p) \\ \Omega M_B^{q-1}(K,L) & \xrightarrow{\Sigma} & M_B^q(K,L) \end{array}$$

where $p\colon E \to K$ is an oriented w-mock bundle with $F= p^{-1}(L)$ and Σ is the geometric suspension defined by diagram 3.2.10.

Suppose $p\colon \tilde{X} \to X$ is a covering of finite index where X is a finite polyhedron. Let $t\colon |K| \to X$ be a triangulation of X where K is an ordered and oriented simplicial complex. The triangulation t can be canonically lifted to a triangulation $\tilde{t}\colon |\tilde{K}| \to \tilde{X}$ of $\tilde{X}$ so that there is a simplicial map $q\colon \tilde{K} \to K$ and a commutative square:

4.1.5
$$\begin{array}{ccc} |\tilde{K}| & \xrightarrow{\tilde{t}} & \tilde{X} \\ |q| \downarrow & & \downarrow p \\ |K| & \xrightarrow{t} & X \end{array}$$

(see [Sp, p. 144]). In addition, $\tilde{K}$ can be ordered and oriented so that $q\colon \tilde{K} \to K$ is an oriented simplicial 0-mock bundle. From the transfer map $tr(q)\colon M_B^*(\tilde{K}) \to M_B^*(K)$ one obtains a homomorphism $\pi_0(tr(q))\colon \pi_0(M_B^*(\tilde{K}),\phi) \to \pi_0(M_B^*(K),\phi)$. Then 3.2.9 and 4.1.5 combine to define a transfer homomorphism:

$H^j(\tilde{X};\mathbb{L}(B)) \to H^j(X;\mathbb{L}(B))$ which will temporarily be denoted by TRS(q). This homomorphism does not depend on the choice of triangulation of X. Suppose $t'\colon |L| \to X$ is another PL compatible triangulation of X, i.e., there is a PL homeomorphism $g\colon |K| \to |L|$ such that $t' = tg$. By Lemma 2.5 of [RS1] there is a simplicial complex P containing K and L as subcomplexes and a triangulation: $T\colon (|P|;|K|,|L|) \to (X \times I; X \times 0, X \times 1)$ with $T|\ |K| = t$ and $T|\ |L| = t'$. T lifts to a triangulation $\tilde{T}\colon (|\tilde{P}|;|\tilde{K}|,|\tilde{L}|) \to (X \times I; X \times 0, X \times 1)$ with $\tilde{T}|\ |\tilde{K}| = \tilde{t}$ and $T|\ |L| = \tilde{t}'$. Let $q\colon \tilde{K} \to K$, $q'\colon \tilde{L} \to L$, and $Q\colon \tilde{P} \to P$ be the corresponding simplicial covering projections. Consider the commutative diagram:

$$\begin{array}{ccc}
H^j(\tilde{X};\mathbb{L}(B)) & \xrightarrow{TRS(q)} & H^j(X;\mathbb{L}(B)) \\
\uparrow I_0 & & \uparrow J_0 \\
H^j(\tilde{X}\times I;\mathbb{L}(B)) & \xrightarrow{TRS(Q)} & H^j(X\times I;\mathbb{L}(B)) \\
\downarrow I_1 & & \downarrow J_1 \\
H^j(X;\mathbb{L}(B)) & \xrightarrow{TRS(q')} & H^j(X;\mathbb{L}(B))
\end{array}$$

where the vertical maps are induced by the maps $X \to X \times I$, $\tilde{X} \to \tilde{X} \times I$, $y \to (y,k)$ $k = 0,1$. Note that $I_k = H^j(\tilde{\pi})^{-1}$ and $J_k = H^j(\pi)$ where π and $\tilde{\pi}$ are the projections. It follows that TRS(q) = TRS(q') and hence the homomorphism which we have defined is independent of the triangulation of X. The notation tr(p) will be used for this homomorphism.

Using $\underline{M}_B^*$ in place of M_B^* a transfer homomorphism: $\underline{tr}(p)$: $H^0(X;\mathbb{L}(B)(1,\ldots,\infty)) \to H^0(X;\mathbb{L}(B)(1,\ldots,\infty))$ as above.

More generally, if $p\colon Y \to X$ is a map of polyhedra which can be triangulated as an oriented simplicial w-mock bundle $q\colon E \to K$ then the transfer $tr(q)\colon M_B^*(E) \to M_B^{*+w}(K)$ induces a transfer homomorphism: $tr(p)\colon H^j(Y;\mathbb{L}(B)) \to H^j(X \times \Delta^w, X \times \partial\Delta^w;\mathbb{L}(B))$. To see that this homomorphism does not depend on the given triangulation of p one constructs, given another PL compatible triangulation $q'\colon E' \to K'$ of p, a triangulation of $p \times id\colon Y \times I \to X \times I$

which restricts to the given triangulations of p on $Y \times i \to X \times i$ $i = 0,1$. This can be accomplished by using the mock bundle subdivision theorem of [BRS]. Then arguing as in the discussion which accompanies diagram 4.1.6 shows that tr(p) is uniquely defined.

In any cohomology theory, in particular $H^*(;\mathbb{L}(B))$, there is a transfer defined for finite coverings of finite CW complexes (see [A] or [Rsh]). We now show that this transfer, which we call the cohomology transfer, coincides with our geometrically defined transfer. F. W. Roush has proved the following theorem:

Theorem ([Rsh, p. 5]): In any cohomology theory U^* defined on the category of finite CW pairs, there exists a unique map t, called the transfer, of degree 0 defined on coverings of finite index $p: (X,A) \to (Y,B)$, $t: U^*(X,A) \to U^*(Y,B)$ satisfying the following axioms:

1. t is functorial over pullbacks.
2. t commutes with the coboundary operators for pairs or triples of coverings.
3. t for the identity covering $(X,A) \to (X,A)$ is the identity.
4. For a covering which splits as a topological sum of subspaces each covering the base, transfer for the total covering is the sum of the transfers for the component coverings.

Roush's theorem implies that our geometrically defined transfer for $H^*(;\mathbb{L}(B))$, tr(p), agrees with the cohomology transfer: tr(p) clearly satisfies axioms 3 and 4, axiom 1 follows from 4.1.3 and axiom 2 follows from 4.1.4.

The transfer can also be defined with the aid of the Thom isomorphism. In analogy with Chapter 2 of [BRS], the Thom isomorphism in the cohomology theory $H^*(;\mathbb{L}(B))$ can be easily described in terms of surgery mock bundles: Let $p: E(u) \to K$ be a simplicial triangulation of an oriented r-block bundle u over K, with zero section $s\colon K \to E(u)$. Composition with s defines a Δ-map:

$$T\colon M_B^q(K) \to M_B^{q-r}(E(u),E(\dot{u})) \qquad (q \geq r)$$

Explicitly, let ξ be a zero simplex of $M_B^q(K)$:

$$|\xi| = M \to X \overset{h}{\to} B\times|K|\ b,n \quad \text{and} \quad \xi(\tau) = M_\tau \longrightarrow X_\tau \overset{h_\tau}{\longrightarrow} B\times\tau\ b_\tau, n_\tau\ .$$

Define $T_0(\xi)$ by

$$|T_0(\xi)| = M \to X \overset{h'}{\to} B\times|E(u)|\ b,n \qquad h' = (\mathrm{id}\times s)h$$

$$T_0(\xi)(\sigma) = M_{s(\sigma)} \longrightarrow X_{s(\sigma)} \overset{h'_\sigma}{\longrightarrow} B\times s(\sigma)\ b_{s(\sigma)}, n_{s(\sigma)} \qquad h'_\sigma = (\mathrm{id}\times s)h$$

and similarly for higher simplices of $M_B^q(K)$. Then T induces an isomorphism:

$$T\colon H^*(|K|;\mathbb{L}(B)) \longrightarrow H^{*+r}(|E(u)|,|E(\dot{u})|;\mathbb{L}(B))$$

whose inverse is given by $tr(p)$.

Now suppose $c\colon \tilde{X} \to X$ is a finite covering of finite polyhedra and that X is embedded in $X\times \mathrm{int}(\Delta^s)$ with normal block bundle V^s. The geometrically defined transfer $tr(c)\colon H^*(\tilde{X};\mathbb{L}(B)) \to H^*(X;\mathbb{L}(B))$ is seen to coincide with the composite:

$$(*) \quad H^*(\tilde{X};\mathbb{L}(B)) \overset{T}{\to} H^{*+s}(E(V),E(\dot{V});\mathbb{L}(B)) \overset{i^*}{\to} H^{*+s}(X\times(\Delta^s,\partial\Delta^s);\mathbb{L}(B)) \overset{\Sigma}{\to} H^*(X;\mathbb{L}(B))$$

where Σ is given by s-fold suspension (compare [BRS], p. 25). By the appendix of [BeS] the composite (*) coincides with the cohomology transfer of Roush, thus providing an alternate proof of the agreement of the geometrically defined transfer and the cohomology transfer.

Let $p\colon (X,A) \to (Y,B)$ be a finite covering of finite complexes. Since the inclusion map of zero-th spaces $\mathbb{L}_8(B)_0 \subset \mathbb{L}_8(B)$ extends to a map of spectra $J: \mathbb{L}(B)\langle 1,\ldots,\infty\rangle \to \mathbb{L}(B)$ (see [A, p. 145]) there is a commutative diagram:

$$\begin{array}{ccc} H^0(X,A;\mathbb{L}(B)\langle 1,\ldots,\infty\rangle) & \xrightarrow{J_\#} & H^0(X,A);\mathbb{L}(B)) \\ \downarrow t' & & \downarrow t \\ H^0(Y,B;\mathbb{L}(B)\langle 1,\ldots,\infty\rangle) & \xrightarrow{J_\#} & H^0(Y,B;\mathbb{L}(B)) \end{array}$$

where the vertical maps are the cohomology transfers and the horizontal maps are induced by J.

It is an easy consequence of our definitions that the diagram:

$$\begin{array}{ccc} H^0(X,A;\mathbb{L}(B)\langle 1,\ldots,\infty\rangle) & \xrightarrow{J_\#} & H^0(X,A;\mathbb{L}(B)) \\ \downarrow \underline{tr}(p) & & \downarrow tr(p) \\ H^0(Y,B;\mathbb{L}(B)\langle 1,\ldots,\infty\rangle) & \xrightarrow{J_\#} & H^0(Y,B;\mathbb{L}(B)) \end{array}$$

is commutative where the vertical maps are the geometrically defined transfers. By Proposition 3.2.7 (1) the horizontal maps in the two diagrams above are injective. It has already been shown that $tr(p) = t$. Hence it follows that $\underline{tr}(p) = t'$, i.e., $\underline{tr}(p)$ coincides with the cohomology transfer.

We now investigate the relationship between the assembly maps and the transfer. Let $p: E \to K$ be an oriented simplicial w-mock bundle. Suppose, in addition that E and K are oriented simplicial manifold s-ads with $\partial_j E = p^{-1}(\partial_j K)$ for $j = 0,\ldots,s-2$. Let $\dim E = e$ and $\dim K = k$. Then $e = w + k$ and an inspection of the definitions reveals that there is a commutative square:

4.1.7
$$\begin{array}{ccc} \underline{M}^q_B(E,\partial_0 E) & \xrightarrow{\underline{A}_q(E)} & \mathbb{L}_{q+e}(B\times\delta_0|E|) \\ \downarrow tr(p) & & \downarrow \mathbb{L}_{q+e}(id\times|p|) \\ M^{q+w}_B(K,\partial_0 K) & \xrightarrow{A_{q+w}(K)} & \mathbb{L}_{q+w+k}(B\times\delta_0|K|) \end{array}$$

where $\underline{A}_q(E)$ and $A_{q+w}(K)$ are the assembly maps, id is the identity

$B \to B$, and $\mathbb{L}_{q+e}(\mathrm{id} \times |p|)$ was defined in Section 2.2. When $w = 0$ and $|p|$ is a covering projection there is also a commutative diagram:

4.1.8
$$\begin{array}{ccc} \underline{M}_B^q(E, \partial_0 E) & \xrightarrow{\underline{A}_q(E)} & \mathbb{L}_{q+e}(B \times \delta_0 |E|) \\ \Big\downarrow \mathrm{tr}(p) & & \Big\downarrow \mathbb{L}_{q+e}(\mathrm{id} \times |p|) \\ \underline{M}_B^q(K, \partial_0 K) & \xrightarrow{\underline{A}_q(K)} & \mathbb{L}_{q+e}(B \times \delta_0 |E|) \end{array}$$

The commutative square 4.1.7 induces a H-homomorphism of homotopy fibers of the assembly maps over the basepoint ϕ:

4.1.9
$$E(\underline{A}_q(E)) \to E(A_{q+w}(K)) .$$

Now let $B = pt$, $q = 4n > 5$, $\dim K$ $s+4$ and suppose that for every subset c of $\{1,\ldots,s-2\}$ $\partial_c E$ and $\partial_c K$ are connected and non-empty. $\partial_0 E$ and $\partial_0 K$ may be empty or disconnected. It is proved in Chapter 5 (see 5.1.9 and 5.1.10) that there are homotopy equivalences:

4.1.10
$$E(\underline{A}_{4n}(E)) \simeq S_{TOP}(|E|, \partial_0 |E|) \qquad \text{and}$$
$$E(A_{4n+w}(K)) \simeq S_{TOP}((|K|, \partial_0 |K| \times (\Delta^w, \partial\Delta^w)) \qquad \text{if } w \geq 1$$

Hence if $w \geq 1$, 4.1.9 and 4.1.10 define, up to homotopy, an induction map:

4.1.11
$$I_*(p)\colon S_{TOP}(|E|, \partial_0 |E|) \to S_{TOP}((|K|, \partial_0 |K|) \times (\Delta^w, \partial\Delta^w)) .$$

If $S_{TOP}(\ )$ is given the H-structure imposed by the homotopy equivalences of 4.1.10 then $I_*(p)$ is a H-map.

Similarly, if $w = 0$ and $|p|$ is a covering projection we have, using 4.1.8 in place of 4.1.7, an induction map:

4.1.12
$$I_*(p)\colon S_{TOP}(|E|, \partial_0 |E|) \to S_{TOP}(|K|, \partial_0 |K|) .$$

4.2 The Transfer in the Theory of Surgery Spaces

Let $p: E \to B$ be a map of CW r-ads with $\partial_i E = p^{-1}(\partial_i B)$ for $i = 0,\dots,r-2$. Assume that p is an oriented topological bundle with fiber a closed oriented manifold of dimension w. The construction presented below of the transfer map:

$$tr(p): \mathbb{L}_q(B) \to \mathbb{L}_{q+w}(E) \quad \text{is that of Quinn (see [Q]).}$$

Let $x = M \to X \to B \times \Delta^j$ b,n be a j-simplex of $\mathbb{L}_q(B)$, i.e., x is a surgery problem of type (q,j) over B. Form the pullback diagram:

$$\begin{array}{ccccc} \tilde{M} & \longrightarrow & \tilde{X} & \longrightarrow & E \times \Delta^j \\ \downarrow U & & \downarrow V & & \downarrow p \times id \\ M & \longrightarrow & X & \longrightarrow & B \times \Delta^j \end{array}$$

Then $x' = \tilde{M} \to \tilde{X} \to E \times \Delta^j$ $\tilde{b},\tilde{n}$ is seen to be a j-simplex of $\mathbb{L}_{q+w}(E)$ where $\tilde{b}$ is the induced microbundle map $\tilde{b}: U^*(v) \to V^*(n) = \tilde{n}$. Defining $tr(p)_j x = x'$, we obtain a Δ-map $tr(p): \mathbb{L}_q(N) \to \mathbb{L}_{q+w}(E)$.

The transfer map is clearly a H-homomorphism and commutes with orientation reversal and the periodicity map. Recall that for $j + q \geq 5$ $\pi_j(\mathbb{L}_q(B),\phi) = L_{j+q}(\pi_1(B))$ where $L_{j+q}(\pi_1(B))$ is Wall's algebraic L-group (see Proposition 2.2.3). If p is a covering projection of finite index then the transfer $\pi_j(tr(p))$; $\pi_j(\mathbb{L}_q(B),\phi) \to \pi_j(\mathbb{L}_q(E),\phi)$ coincides with the algebraically defined restriction map: $I^*: L_{j+q}(\pi_1(B)) \to L_{j+q}(\pi_1(E))$ where $j + q \geq 5$ and $I = \pi_1(p)$.

Let $p: H \to K$ be a Δ-map between ordered simplicial complexes and B a CW s-ad. Suppose that H and K are oriented simplicial manifold r-ads of dimension m with $p^{-1}(\partial_i K) = \partial_i H$ $i = 0,\dots,r-2$ and assume that $|p|: |H| \to |K|$ is a covering projection of finite index. It follows from the definitions that there is a commutative square:

4.2.1
$$\begin{array}{ccc} \underline{M}^q_B(K,\partial_0 K) & \xrightarrow{\underline{A}_q(K)} & \mathbb{L}_{q+m}(B \times \delta_0|K|) \\ \downarrow p^{\#} & & \downarrow \mathrm{tr}(\mathrm{id} \times |p|) \\ \underline{M}^q_B(H,\partial_0 H) & \xrightarrow{\underline{A}_q(H)} & \mathbb{L}_{q+m}(B \times \delta_0|H|) \end{array}$$

where the horizontal maps are the assembly maps and $p^{\#}$ is the induced map of Section 3.2. (Recall that $p^{\#}$ is also constructed as a pullback.) There is also a commutative square with M^q_B in place of $\underline{M}^q_B$. The square 4.2.1 induces a H-homomorphism of the homotopy fibers of the assembly maps over the basepoint ϕ:

4.2.2
$$E(\underline{A}_q(K)) \to E(\underline{A}_q(H)) .$$

Now let $B = pt$, $q = 4n \geq 5$, $\dim K \geq r+4$, and suppose that for every subset c of $\{1,\ldots,r-2\}$ $\partial_c H$ and $\partial_c K$ are connected and non-empty. $\partial_0 K$ and $\partial_0 H$ may be empty or disconnected. It is proved in Chapter 5 (see 5.1.9) that there are homotopy equivalences:

4.2.3
$$E(\underline{A}_{4n}(X)) \simeq S_{TOP}(X,\partial_0 X) \qquad X = |K| \text{ or } |H| .$$

Then 4.2.2 and 4.2.3 define, up to homotopy, a restriction map:

4.2.4
$$I^*(p)\colon S_{TOP}(|K|,\partial_0|K|) \to S_{TOP}(|H|,\partial_0|H|) .$$

If $S_{TOP}(\ )$ is given the H-structure imposed by the homotopy equivalences of 4.2.3 then $I^*(p)$ is a H-map.

The restriction map for $S_{TOP}(\ )$ can also be defined directly: Let $x = M \to |K| \times \Delta^j$ be a j-simplex of $S_{TOP}(|K|,\partial_0|K|)$. Form the pullback diagram:

4.2.5
$$\begin{array}{ccc} \tilde{M} & \longrightarrow & |H| \times \Delta^j \\ \downarrow & & \downarrow |p| \times \mathrm{id} \\ M & \longrightarrow & |K| \times \Delta^j \end{array}$$

and define $I^*(p)_j x = \tilde{M} \to |H| \times \Delta^j$ giving a Δ-map

$$I^*(p)\colon S_{TOP}(|K|,\partial_0|K|) \to S_{TOP}(|H|,\partial_0|H|) .$$

Since the vertical maps in 4.2.1 are obtained as pullbacks and the homotopy equivalences of 4.2.4 are natural with respect to pullbacks, it follows that the two definitions of the restriction map agree up to homotopy.

More generally, let $p\colon H \to K$ be an order preserving simplicial map which triangulates an oriented topological bundle with fiber a closed oriented manifold of dimension w. Also suppose that H^{m+v} and K^m are oriented simplicial manifold r-ads with $p^{-1}(\partial_i K) = \partial_i H$ $i = 0,\ldots,r-2$. Let C be the simplicial mapping cylinder of p and $i\colon H \to C$, $j\colon K \to C$ the natural inclusions. Then there is a homotopy commutative diagram:

4.2.6
$$\begin{array}{ccc}
\underline{M}^q(K,\partial_0 K) & \xrightarrow{\underline{A}_q(K)} & \mathbb{L}_{q+m}(\delta_0|K|) \\
\uparrow j^{\#} & & \\
\underline{M}^q(C,\partial_0 C) & & \downarrow \mathrm{tr}(|p|) \\
\downarrow i^{\#} & & \\
\underline{M}^q(H,\partial_0 H) & \xrightarrow{\underline{A}_q(H)} & \mathbb{L}_{q+m+w}(\delta_0|H|)
\end{array}$$

where the horizontal maps are the assembly maps. $j^{\#}$ is a homotopy equivalence since j is a homotopy equivalence. Hence there is an induced map of homotopy fibers: $E(\underline{A}_q(K)) \to E(\underline{A}_q(H))$. When $\dim K \geq r+4$, $q = 4n \geq 5$, and H and K satisfy the 0-connectivity conditions of 4.2.3, this yields a restriction map defined up to homotopy:

4.2.7
$$I^*(p)\colon S_{TOP}(|K|,\partial_0|K|) \to S_{TOP}(|H|,\partial_0|H|) .$$

This restriction map can also be defined by the pullback construction of 4.2.5.

5. GROUP STRUCTURE IN THE SURGERY EXACT SEQUENCE

Notation: As in Chapter 1, SX will denote the singular complex of a space X. Let $P^r = (CP^2)^r$, the r-fold Cartesian product of 2-dimensional complex projective space.

The spaces G/CAT, CAT = TOP or PL, are defined to be the homotopy fiber of the natural map between classifying spaces $BCAT \to BG$ (see [Wa 1], Ch. 10). The following theorem of Quinn and Siebenmann relates G/TOP with the surgery spaces $\mathbb{L}_q(pt)$:

Theorem 5.1.1 ([KS, p. 297]): There are homotopy equivalences:

1. $a(0,r): S(G/TOP) \to \mathbb{L}_{4r}(pt)_0 \qquad r \geq 2$.
2. $a(k,r): S(\Omega^k(G/TOP)) \to \mathbb{L}_{4r+k}(pt) \qquad 4r+k \geq 5,\ k \geq 1$.

Discussion: $a(k,r)$ is defined to be the composite

5.1.2 $$S(\Omega^k(G/TOP)) \xrightarrow{B} SMap(P^r,\Omega^k(G/TOP)) \xrightarrow{C} N_{TOP}(P^r \times \Delta^k, P^r \times \partial\Delta^k)$$

$$\xrightarrow{\mathbb{L}_{4r+k}(\lambda)F} \mathbb{L}_{4r+k}(pt) \qquad k \geq 0$$

$SMap(P^r,\Omega_k(G/TOP))$ is identified with the Δ-set whose j-simplices are continuous maps: $(P^r \times \Delta^k \times \Delta^j, P^r \times \partial\Delta^k \times \Delta^j) \to (G/TOP,*)$ and whose face maps are given by restriction to $P^r \times \Delta^k \times \partial_p\Delta^j$ for $p = 0,\ldots,j$. Similarly, $S(\Omega^k(G/TOP))$ can be viewed as the Δ-set whose j-simplices are continuous maps $(\Delta^k \times \Delta^j, \partial\Delta^k \times \Delta^j)$ $(G/TOP,*)$ and whose face maps are the obvious ones. The Δ-map B takes a j-simplex $u: \Delta^k \times \Delta^j \to G/TOP$ of $S(\Omega^k(G/TOP))$ to u where $\pi: P^r \times \Delta^k \times \Delta^j \to \Delta^k \times \Delta^j$ is the projection. F is the surgery obstruction map of Definition 2.3.3 and for any space X $\lambda: X \to pt$ is the unique map.

The Δ-map C is a special case of the following general construction: Let M be a compact oriented topological manifold s-ad with $m \geq s+4$. We inductively construct a Δ-map $C(M,\partial_0 M)\colon SMap(M,\partial_0 M;G/TOP,*) \to N_{TOP}(M,\partial_0 M)$ simplex by simplex using topological transversality. This map is constructed in the PL case in [Ro], the main difference being that PL transversality is employed in [Ro]. Explicitly: For each 0-simplex $u\colon (M,\partial_0 M) \to (G/TOP,*)$ of $SMap(M,\partial_0 M;G/TOP,*)$ use topological transversality to obtain a normal map of s-ads $x = (N \to M\ b,n)$ relative to $\partial_0 M$ representing u. Recall that this is accomplished by obtaining a fiber homotopy trivialized topological bundle over M,ξ, representing u and then making the collapse map $t\colon S^{m+i} \to T(\xi)$, where $T(\xi)$ is the Thom space of ξ, transverse to $M \subset T(\xi)$ (this requires $m \geq s+4$). Let $C(M,\partial_0 M)_0(u) = x$. Suppose $C(M,\partial_0 M)$ is defined on q-simplices $q \leq j$. For each (j+1)-simplex $u\colon (M \times \Delta^{j+1}, \partial_0 M \times \Delta^{j+1}) \to (G/TOP,*)$ apply (relative) topological transversality to obtain a normal map x of type $j+1$ over M relative to $\partial_0 M$ representing u and such that $\partial_p x = C(M,\partial_0 M)_j(\partial_p u)$ for $p = 0,\dots,j$. Let $C(M,\partial_0 M)_{j+1}(u) = x$. Then by induction we obtain the Δ-map $C(M,\partial_0 M)$. $C(M,\partial_0 M)$ is a homotopy equivalence. This is shown in the PL case in [Ro]; the TOP case follows by substituting TOP transversality for PL transversality used in the argument of [Ro]. Note that the definition of $C(M,\partial_0 M)$ involves an inductive choice for each j-simplex u of $SMap(M,\partial_0 M;G/TOP,*)$ of a normal map representing u. If $C'(M,\partial_0 M)$ is a Δ-map obtained by making different choices of representatives then (relative) transversality can be used to construct a homotopy between $C(M,\partial_0 M)$ and $C'(M,\partial_0 M)$. Hence $C(M,\partial_0 M)$ is uniquely defined up to homotopy.

Returning to 5.1.2, the Δ-map C is given by $C = C(P^r \times \Delta^k, P^r \times \partial\Delta^k)$. Then the map:

$$\pi_j(a(k,r))\colon \pi_j(\Omega^k(G/TOP,*) \to \pi_j(\mathbb{L}_{4r+k}(pt),\phi))$$

is the surgery obstruction map $\pi_{k+j}(G/TOP,*) \to L_{k+j}(1)$ which by the computations of Kirby and Siebenmann (see [KS, p. 267]) is an isomorphism when $k+j \geq 1$ and the zero map when $k+j = 0$. In particular, the image of $a(0,r)$, $r \geq 2$, lies in $\mathbb{L}_{4r}(pt)_0$. Theorem 5.1.1 then follows from Whitehead's theorem.

The homotopy equivalence $S(G/TOP) \simeq \mathbb{L}_8(pt)_0$ of Theorem 5.1.1 (1) gives G/TOP an infinite loop space structure since $\mathbb{L}_8(pt)_0$ is the zero-th space of the spectrum $\mathbb{L}(pt)$ $(1,\ldots,\infty)$ (see Section 3.1).

Let K^m be an oriented simplicial manifold s-ad, $m \geq s+4$. We will now establish homotopy commutative squares:

5.1.3
$$\begin{array}{ccc} \underline{M}^{4r}(K,\partial_0 K) & \xrightarrow{\underline{A}_{4r}} & \mathbb{L}_{4r+m}(\delta_0|K|) \\ \uparrow V & & \uparrow \theta_r \\ N_{TOP}(|K|,\partial_0|K|) & \xrightarrow{F} & \mathbb{L}_m(\delta_0|K|) \end{array} \qquad r \geq 2$$

5.1.4
$$\begin{array}{ccc} M^{4r+k}(K,\partial_0 K) & \xrightarrow{A_{4r+k}} & \mathbb{L}_{4r+k+m}(\partial_0|K|) \\ \uparrow V' & & \uparrow \theta_r \\ N_{TOP}((|K|,\partial_0|K|)\times(\Delta^k,\partial\Delta^k)) & \xrightarrow{F} & \mathbb{L}_{m+k}(\partial_0|K|) \end{array} \qquad 4r+k \geq 5,\ k \geq 1$$

such that the Δ-maps V and V' are homotopy equivalences. In the diagrams above, θ_r is the periodicity map obtained by taking Cartesian products with P^r. $\underline{A}_{4r}$ and A_{4r+k} are the assembly maps, F is the surgery obstruction map (see Definitions 2.3.3) and the maps V and V' are the subject of the discussion below.

Let Q_p be the Kan subcomplex of $\mathbb{L}_p(pt)_0$ whose j-simplices are surgery problems of type (p,j) $x = (M \xrightarrow{f} X \to \Delta^j\ b,n)$ such that f is a homeomorphism or x is the empty surgery problem. It is easily seen that the base point ϕ of $\mathbb{L}_p(pt)$ is a deformation retract of Q_p, in particular Q_p is contractible. Since $\mathbb{L}_{4r}(pt)_0$ is a H-space and Q_{4r} is contractible, Theorem 5.1. (1) gives that $a(0,r)\colon (S(G/TOP),*) \to (\mathbb{L}_{4r}(pt)_0,Q_{4r})$ is a homotopy equivalence of pairs, $r \geq 2$. Hence the induced map

$a(0,r)_{\#}\colon \Delta(K,\partial_0K;S(G/TOP),*) \to \Delta(K,\partial_0K;\mathbb{L}_{4r}(pt)_0,Q_{4r})$ is also a homotopy equivalence.

If X is a pointed CW complex let $n(x)\colon \Delta(K,\partial_0K;SX,*) \to SMap(|K|,\partial_0|K|;X,*)$ be the natural homotopy equivalence given by $n(X)_j(f) = \phi(X)|f|$ where $|f|$ is the geometric realization of f and $\phi(X)\colon |SX| \to X$ is the natural map. Define a Δ-map:

$$W\colon \Delta(K,\partial_0K;S(G/TOP),*) \to N_{TOP}(|K|,\partial_0|K|)$$

by $W = C(|K|,\partial_0|K|)n(G/TOP)$. Then W is a homotopy equivalence since it is a composite of homotopy equivalences.

Consider the following diagram:

5.1.5
$$\begin{array}{ccc}
\Delta(K,\partial_0K;\mathbb{L}_{4r}(pt)_0,\phi) & \xrightarrow{A_1} & \mathbb{L}_{m+4r}(\partial_0|K|) \\
\big\downarrow{\scriptstyle I_{\#}} & & \big\downarrow{\scriptstyle id} \\
\Delta(K,\partial_0K;\mathbb{L}_{4r}(pt)_0,Q_{4r}) & \xrightarrow{A_2} & \mathbb{L}_{m+4r}(\delta_0|K|) \\
\big\uparrow{\scriptstyle a(0,r)_{\#}} & & \big\uparrow \\
\Delta(K,\partial_0K;S(G/TOP),\#) & & \big|{\scriptstyle \theta_r} \\
\big\downarrow{\scriptstyle W} & & \big| \\
N_{TOP}(|K|,\partial_0|K|) & \xrightarrow{F} & \mathbb{L}(\delta_0|K|)
\end{array}$$

In the above diagram A_1 is the composite

$$\Delta(K,\partial_0K;\mathbb{L}_{4r}(pt)_0,\ ) \xrightarrow{\underline{G}} \underline{M}^{4r}(K,\partial_0K) \xrightarrow{\underline{A}} \mathbb{L}_{4r+m}(\delta_0|K|)$$

where $\underline{G}$ is the glue isomorphism and $\underline{A}$ is the assembly map. The Δ-map A_2 is similarly defined. θ_r is the periodicity map and the maps $a(0,r)_{\#}$ and W have been defined previously. $I_{\#}$ is induced by the inclusion $I\colon (\mathbb{L}_{4r}(pt)_0,\phi) \to (\mathbb{L}_{4r}(pt)_0,Q_{4r})$. $I_{\#}$ is a homotopy equivalence since I is a homotopy equivalence of pairs, Q_{4r} being contractible.

The top square in diagram 5.1.5 is strictly commutative. To see that the bottom square in 5.1.5 is homotopy commutative, i.e., that $A_2a(0,r)_{\#}$

and $\theta_r FW$ are homotopic, observe the following:

First, $\mathrm{Map}(|K| \times P^r, \partial_0|K| \times P^r; G/TOP,*) = \mathrm{Map}(|K|, \partial_0|K|; \mathrm{Map}(P;G/TOP),*)$ and there is a homotopy commutative diagram:

5.1.6
$$\begin{array}{ccc}
\Delta(K,\partial_0 K: S(G/TOP),*) & \xrightarrow{B_\#} & \Delta(K,\partial_0 K: SMap(P^r;G/TOP),*) \\
\Big\downarrow{\scriptstyle n(G/TOP)} & & \Big\downarrow{\scriptstyle n(Map(P^r;G/TOP))} \\
SMap(|K|,\partial_0|K|;G/TOP,*) & \xrightarrow{\pi^*} & SMap(|K| \times P^r, \partial_0|K| \times P^r; G/TOP,*) \\
\Big\downarrow{\scriptstyle C(|K|,\partial_0|K|)} & & \Big\downarrow{\scriptstyle C(|K| \times P^r, \partial_0|K| \times P^r)} \\
N_{TOP}(|K|,\partial_0|K|) & \xrightarrow{H=(xP^r)} & N_{TOP}(|K| \times P^r, \partial_0|K| \times P^r)
\end{array}$$

where $B: S(G/TOP) \to SMap(P^r;G/TOP)$ is the map of 5.1.2, π^* is induced by the projection $\pi: |K| \times P^r \to |K|$. Note that the composite of the maps in the first column is W.

Second, let $\underline{Q}$ be the Kan subcomplex of $N_{TOP}(P^r)$ whose j-simplices are normal maps of type j over P^r, $n = (M \xrightarrow{f} P^r \times \Delta^j\ b,n)$ such that f is a homeomorphism. Define the Δ-map $G': \Delta(K,\partial_0 K; N_{TOP}(P^r),\underline{Q}) \to N_{TOP}(|K| \times P^r, \partial_0|K| \times P^r)$ by gluing together the image of a j-simplex $g: K \otimes \Delta^j \to N_{TOP}(P^r)$ of $\Delta(K,\partial_0 K; N_{TOP}(P^r),\underline{Q})$ and then applying Theorem 3.3.2 (Assembly). There is a commutative diagram:

5.1.7
$$\begin{array}{ccc}
\Delta(K,\partial_0 K; N_{TOP}(P^r),\underline{Q}) & \xrightarrow{(\mathbb{L}_{4r}(\lambda)F)_\#} & \Delta(K,\partial_0 K; \mathbb{L}_{4r}(pt), Q_{4r}) \\
\Big\downarrow{\scriptstyle G'} & & \Big\downarrow{\scriptstyle A_2} \\
N_{TOP}(|K| \times P^r, \partial_0|K| \times P^r) & \xrightarrow{\mathbb{L}_{4r+m}(\pi)F} & \mathbb{L}_{4r+m}(\delta_0|K|)
\end{array}$$

where $\mathbb{L}_{4r}(\lambda)F: N_{TOP}(P^r) \to \mathbb{L}_{4r}(pt)$ is the map of 5.1.2 and $\pi: |K| \times P^r \to |K|$ is projection.

Finally, there is a homotopy commutative diagram:

5.1.8

$$\begin{array}{ccc} \Delta(K,\partial_0 K;\mathrm{SMap}(P^r;G/TOP),*) & \xrightarrow{C_\#} & \Delta(K,\partial_0 K;N_{TOP}(P^r),\underline{Q}) \\ & {\scriptstyle W'}\searrow \quad \swarrow{\scriptstyle G'} & \\ & N_{TOP}(|K|\times P^r,\partial_0|K|\times P^r) & \end{array}$$

where W' is the composite of the maps in the second column of diagram 5.1.6 and $C = C(P^r)$: $\mathrm{SMap}(P^r,G/TOP) \to N_{TOP}(P^r)$. Diagram 5.1.8 above is seen to be homotopy commutative as follows: Construct a homotopy inverse $T: N_{TOP}(|K|\times P^r,\partial_0|K|\times P^r) \to \Delta(K,\partial_0 K;N_{TOP}(P^r),\underline{Q})$ for G' using transversality: If $x = (M \overset{f}{\to} |K \otimes \Delta^j| \times P^r\ b,n)$ is a j-simplex of $N_{TOP}(|K|\times P^r,\partial_0|K|\times P^r)$ then the Δ-map $T_j(x)$: $K \otimes \Delta^j \to N_{TOP}(P^r)$ is defined inductively by making f transverse to $|\sigma|\times P^r \subset |K\otimes\Delta^j|\times P^r$ for each simplex σ of $K\otimes\Delta^j$. Then it is easily seen that TW' and C are homotopic.

Now recall that the image of $a(0,r)$ lies in $\mathbb{L}_{4r}(pt)_0$ and observe that $\mathbb{L}_{4r+m}(\pi)FHW = \theta_r FW$. Then combining diagrams 5.1.6, 5.1.7, and 5.1.8 we obtain that the bottom square of diagram 5.1.5 is homotopy commutative.

Let $(I_\#)^{-1}$ and W^{-1} be homotopy inverses for the maps $I_\#$ and W respectively in diagram 5.1.5 and let $\underline{G}$: $\Delta(K,\partial_0 K;\mathbb{L}_{4r}(pt)_0,\phi) \to \underline{M}^{4r}(K,\partial_0 K)$ be the glue isomorphism. Define the Δ-map V by $V = \underline{G}(I_\#)^{-1}a(0,r)_\# W^{-1}$. V is a homotopy equivalence since it is a composite of homotopy equivalences. Diagram 5.1.5 then yields diagram 5.1.3. Similarly, using the maps $a(k,r)$ $k \geq 1$ we obtain diagram 5.1.4.

We now discuss group structure in $\pi_0(S_{TOP}(|K|,\partial_0|K|),*)$ and the surgery exact sequence in the topological case.

Recall from Chapter 3 that the assembly map $\underline{A}_{4r}$: $\underline{M}^{4r}(K,\partial_0 K) \to \mathbb{L}_{4r+m}(\delta_0|K|)$ is a H-homormorphism with respect to the disjoint union H-structures and that $\underline{M}^{4r}(K,\partial_0 K)$ and $\mathbb{L}_{4r+m}(\delta_0|K|)$ are H-groups. Let $E(\underline{A}_{4r})$ be the homotopy fiber of $\underline{A}_{4r}$ over the basepoint ϕ. For the remainder of this chapter it will be assumed that for every subset c of $\{1,\ldots,s-2\}$ $\partial_c K$ is connected and non-empty. $\partial_0 K$ will be permitted to be disconnected or empty unless otherwise stated. By Theorem 2.3.4 there is a homotopy equivalence $S_{TOP}(|K|,\partial_0|K|) \to E(F)$ where $E(F)$ is the homotopy fiber of the surgery obstruction map $F: N_{TOP}(|K|,\partial_0|K|) \to \mathbb{L}_m(\delta_0|K|)$ over the basepoint ϕ. Also recall that diagram 5.1.3 gives a homotopy commutative square:

5.1.3
$$\begin{array}{ccc} \underline{M}^{4r}(K,\partial_0 K) & \longrightarrow & \mathbb{L}_{4r+m}(\delta_0|K|) \\ \uparrow V & & \uparrow \theta_r \\ N_{TOP}(|K|,\partial_0|K|) & \longrightarrow & \mathbb{L}(\delta_0|K|) \end{array} \qquad r \geq 2$$

where V is a homotopy equivalence. By Proposition 2.2.5 the periodicity map θ_r is also a homotopy equivalence. Applying Theorem 1.2.12 to Diagram 5.1.3, we obtain a homotopy equivalence:

5.1.9
$$E(\underline{A}_{4r}) \longrightarrow S_{TOP}(|K|,\partial_0|K|) \ .$$

Theorem 1.3.11 (4) implies that $E(\underline{A}_{4r})$ is a homotopy associative H-space such that $\pi_0(E(\underline{A}_{4r}),*)$ is a group. Then the homotopy equivalence of 5.1.9 imposes a homotopy H-structure on $S_{TOP}(|K|,\partial_0|K|)$ such that $\pi_0(S_{TOP}(|K|,\partial_0|K|), *)$ is a group. It will be shown later that this group structure is independent of the triangulation of $M = |K|$.

<u>Notational Convention</u>: The group $\pi_0(S_{TOP}(|K|,\partial_0|K|),*)$ will be denoted by $s_{TOP}(|K|,\partial_0|K|)$ when it is conentient to do so.

Arguing as above with diagram 5.1.4 in place of diagram 5.1.3 we obtain for $4r+k \geq 5$ and $k \geq 1$ a homotopy equivalence:

5.1.10
$$E(A_{4r+k}) \longrightarrow S_{TOP}((|K|,\partial_0|K|)\times(\Delta^k,\partial\Delta^k)) \ .$$

By diagram 5.1.3 and the proof of Theorem 1.3.12, the homotopy sequence of the homotopy fibration $E(\underline{A}_{4r}) \to \underline{M}^{4r}(K,\partial_0 K) \to \mathbb{L}_{4r+m}(\delta_0|K|)$ yielding a commutative ladder -- diagram 5.1.11:

5.1.11

$$\begin{array}{ccccccc}
\cdots \to & \pi_0(E(\underline{A}_{4r}),*) & \to & \pi_0(\underline{M}^{4r}(K,\partial_0K),\phi) & \to & \pi_0(\mathbb{L}_{4r+m}(\delta_0|K|),\phi) \\
& \downarrow & & \downarrow \pi_0(V^{-1}) & & \downarrow \pi_0(\theta_r^{-1}) \\
\cdots \to & \pi_0(S_{TOP}(|K|,\partial_0|K|),*) & \to & \pi_0(N_{TOP}(|K|,\partial_0|K|),*) & \to & \pi_0(\mathbb{L}_m(\delta_0|K|),\phi)
\end{array}$$

where the top sequence is a long exact sequence of groups and homomorphisms and the vertical maps are isomorphisms. V^{-1} and θ_r^{-1} are homotopy inverse for V and θ_r respectively.

If X is any compact oriented TOP manifold s-ad then it is a direct consequence of the definitions that:

$$\pi_j(S_{TOP}(X,\partial_0X),*) = \pi_0(S_{TOP}((X,\partial_0X)\times(\Delta^j,\partial\Delta^j)),*)$$

and

$$\pi_j(N_{TOP}(X,\partial_0X),*) = \pi_0(S_{TOP}((X,\partial_0X)\times(\Delta^j,\partial\Delta^j)),*).$$

In addition a comparison with the definitions of [Wa 1], Chapter 10 shows that the sequence

$$\pi_0(S_{TOP}(X,\partial_0X),*) \to \pi_0(N_{TOP}(X,\partial_0X),*) \to \pi_0(\mathbb{L}_m(\delta_0X),\phi)$$

is precisely the surgery exact sequence of [Wa 1], p. 197.

By Proposition 2.2.4 $\pi_j(\mathbb{L}_{m+4r}(\delta_0|K|),\phi) = L_{m+j}(\pi_1(\delta_0|K|))$. Let $r = 2$ and recall the following from sections 2 and 3 of Chapter 3:
$\pi_0(\underline{M}^8(K,\partial_0K),\phi) = H^0(|K|,\partial_0|K|;\underline{G/TOP})$ where $\underline{G/TOP}$ is the notation we are using for the spectrum $\mathbb{L}(pt)\ (1,\ldots,\infty)$ with zero-th space $\mathbb{L}_8(pt)_0 \simeq G/TOP$. Also

$$\pi_0(\underline{M}^8(K,\partial_0K),\phi) = H^0((|K|,\partial_0|K|)\times(\Delta^j,\partial\Delta^j);\underline{G/TOP}) .$$

Remark: Proposition 3.2.7 (1) implies that for $j \geq 1$, $\pi_j(\underline{M}^8(K,\partial_0K),\phi) = \pi_j(M^8(K,\partial_0K),\phi) = H^0((|K|,\partial_0|K|)\times(\Delta^j,\partial\Delta^j);\mathbb{L}(pt))$.

The commutative ladder 5.1.11 then yields:

Theorem 5.1.12: There is a long exact sequence of groups and homomorphisms:

$$\cdots \longrightarrow s_{TOP}((|K|,\partial_0|K|)\times(\Delta^1,\partial\Delta^1)) \longrightarrow H^0((|K|,\partial_0|K|)\times(\Delta^1,\partial\Delta^1);\underline{G/TOP})$$

$$\xrightarrow{A_1} L_{m+1}(\pi_1(\delta_0|K|)) \longrightarrow s_{TOP}(|K|,\partial_0|K|) \longrightarrow H^0(|K|,\partial_0|K|;G/TOP)$$

$$\xrightarrow{A_0} L_m(\pi_1(\delta_0|K|))$$

where A_j corresponds to the surgery obstruction map under the isomorphism of 5.1.11.

Remark: In contrast, suppose G/TOP is given the Whitney sum infinite loop space structure. In that case it is known that the surgery obstruction $\theta\colon [|K|,\partial_0|K|;G/TOP] \to L_m(\pi_1(\delta_0|K|))$ is not a homomorphism in general (see [Wa 1, p. 110]).

The periodicity theorem for the group $s_{TOP}(|K|,\partial_0|K|)$ will now be derived. It will also be shown that this group is abelian.

In the special case that $\partial_0 K$ is non-empty, observe that $\Delta(K,\partial_0 K;\mathbb{L}_q(pt)_0,\phi) = \Delta(K,\partial_0 K;\mathbb{L}_q(pt),\phi)$ or equivalently $\underline{M}^q(K,\partial_0 K) = M^q(K,\partial_0 K)$ since if K is connected and $\partial_0 K$ is non-empty then the image of any Δ-map $f\colon (K \otimes \Delta^j,\partial_0 K \otimes \Delta^j) \to (\mathbb{L}_q(pt),\phi)$ must lie in $\mathbb{L}_q(pt)_0$, the component of $\mathbb{L}_q(pt)$ containing the basepoint ϕ. $\underline{M}^q(K,\partial_0 K) = M^q(K,\partial_0 K)$ implies that $E(\underline{A}_q) = E(A_q)$ where $E(\underline{A}_q)$ and $E(A_q)$ are the homotopy fibers of the respective assembly maps over the basepoint ϕ. The periodicity theorem of Siebenmann ([KS, p. 283]) can now be recovered in the case $\partial_0 K$ is non-empty:

Theorem 5.1.13: Let K be as above, $\partial_0 K$ non-empty. Then there is a H-map inducing a homotopy equivalence:

$$S_{TOP}(|K|,\partial_0|K|) \simeq S_{TOP}((|K|)\times(\Delta^4,\partial\Delta^4)) .$$

Proof: For $r \geq 1$ there are H-maps inducing homotopy equivalences:

$$E(\underline{A}_{4(r+1)}) \simeq S_{TOP}(|K|,\partial_0|K|) \qquad \text{by 5.1.9}$$

$$E(A_{4r+4}) \simeq S_{TOP}((|K|,\partial_0|K|) \times (\Delta^4,\partial\Delta^4)) \quad \text{by 5.1.10}$$

Since $E(\underline{A}_{4r+4}) = E(A_{4r+4})$ when K is connected and $\partial_0 K$ is non-empty, the result follows.

Before proceeding to the case $\partial_0 K$ is empty we show that $S_{TOP}(|K|,\partial_0|K|)$ is a loop space when $\partial_0 K$ is non-empty.

Recall from diagram 3.3.5 that there is a homotopy commutative square:

$$\begin{array}{ccc} M^q(K,\partial_0 K) & \xrightarrow{A_q} & \mathbb{L}_{m+q}(\delta_0|K|) \\ \uparrow & & \uparrow \\ \Omega M^{q-1}(K,\partial_0 K) & \xrightarrow{\Omega A_{q-1}} & \Omega\mathbb{L}_{m+q-1}(\delta_0|K|) \end{array}$$

where the vertical arrows are homotopy equivalences. Since $\partial_0 K$ is non-empty and K is connected we can replace M^* by $\underline{M}^*$ and A_* by $\underline{A}_*$ in the diagram above. Applying Theorem 1.3.12 to the resulting diagram yields a H-map inducing a homotopy equivalence $E(\Omega\underline{A}_{q-1}) \simeq E(\underline{A}_q)$. By Proposition 1.3.9 there is a H-map inducing a homotopy equivalence $E(\Omega\underline{A}_{q-1}) \to \Omega E(\underline{A}_{q-1})$. Letting $q = 4r+1$, $r = 2$ and applying 5.1.9 proves:

Proposition 5.1.14: Let K be as above and $\partial_0 K$ non-empty. Then there is a H-map inducing a homotopy equivalence:

$$S_{TOP}((|K|,\partial_0|K|) \times (\Delta^1,\partial\Delta^1)) \simeq \Omega S_{TOP}(|K|,\partial_0|K|) .$$

Remark: Combining 5.1.3 and 5.1.4 we have a H-map inducing a homotopy equivalence:

$$\Omega^4 S_{TOP}(|K|,\partial_0|K|) \simeq S_{TOP}(|K|,\partial_0|K|) \qquad \partial_0 K \text{ non-empty.}$$

Hence $S_{TOP}(|K|,\partial_0|K|)$ is an infinite loop space and in particular $S_{TOP}(|K|,\partial_0|K|)$ is abelian when $\partial_0 K$ is non-empty.

Now suppose that $\partial_0 K$ is empty. K can be viewed as an oriented simplicial manifold (s-1)-ad by deleting the empty zero-th face. The corresponding periodicity result for the group $s_{TOP}(|K|)$ is:

Theorem 5.1.15: There is an exact sequence of groups:

$$0 \to s_{TOP}(|K|) \to s_{TOP}(|K| \times \Delta^4, |K| \times \partial\Delta^4) \to L_0(1).$$

Since $L_0(1) = Z$, the infinite cyclic group and the image of a homomorphism into Z is either zero or infinite cyclic we have the following corollary:

Corollary 5.1.16: The group $s_{TOP}(|K| \times \Delta^4, |K| \times \partial\Delta^4)$ is isomorphic to $s_{TOP}(|K|)$ or to $s_{TOP}(|K|) \oplus Z$.

Remark: Let T^n be the n-torus and S^n the n-sphere. Consider the computations:

1. $s_{TOP}(T^n \times \Delta^k, T^n \times \partial\Delta^k) = 0 \quad n+k \geq 5$ (see [KS, p. 275])
2. $s_{TOP}(S^n) = 0 \quad n \geq 5$ (by the generalized Poincare conjecture) and $s_{TOP}(S^n \times \Delta^4, S^n \times \partial\Delta^4) = Z$ (using the surgery exact sequence). Hence both possibilities in Corollary 5.1.16 can occur. As a consequence Theorem C.5 of [KS, p. 283] is incorrectly stated.

Proof of Theorem 5.1.15: The fiber mapping sequence of $\underline{A}_q: \underline{M}^q(K) \to \mathbb{L}_{m+q}(|K|)$ (see 1.3.8) maps by natural inclusion into the fiber mapping sequence of $A_q: M^q(K) \to \mathbb{L}_{m+q}(|K|)$ yielding a commutative ladder:

$$\begin{array}{ccccccccc}
\longrightarrow \Omega\underline{M}^q(K) & \longrightarrow & \Omega\mathbb{L}_{m+q}(|K|) & \longrightarrow & E(\underline{A}_q) & \longrightarrow & \underline{M}^q(K) & \longrightarrow & \mathbb{L}_{m+q}(|K|) \\
\downarrow \Omega j & & \downarrow id & & \downarrow u & & \downarrow j & & \downarrow id \\
\longrightarrow \Omega M^q(K) & \longrightarrow & \Omega\mathbb{L}_{m+q}(|K|) & \longrightarrow & E(A_q) & \longrightarrow & M^q(K) & \longrightarrow & \mathbb{L}_{m+q}(|K|)
\end{array}$$

where j is the inclusion. Note that $\Omega\Delta(K;\mathbb{L}_q(pt)_0) = \Omega\Delta(K;\mathbb{L}_q(pt))$ or equivalently $\Omega\underline{M}^q(K) = \Omega M^q(K)$ since the image of any Δ-map: $F: (K \otimes \Delta^{j+1}, K \otimes (\partial_0)^{j+1}\Delta^{j+1}) \to (\mathbb{L}_q(pt),\phi)$ must lie in $\mathbb{L}_q(pt)_0$, the component containing the basepoint ϕ, if K is connected. Hence Ωj is the identity. Let $\underline{M}_j = \pi_j(\underline{M}^q(K),\phi)$, $M_j = \pi_j(M^q(K),\phi)$, and $L_j = L_{m+q+j}(\pi_1(|K|))$. Then applying π_0 to the diagram above yields a commutative ladder of exact sequences:

$$\begin{array}{ccccccccc}
\underline{M}_1 & \longrightarrow & L_1 & \longrightarrow & \pi_0(E(\underline{A}_q),*) & \longrightarrow & \underline{M}_0 & \longrightarrow & L_0 \\
\downarrow id & & \downarrow id & & \downarrow u_* & & \downarrow j_* & & \downarrow id \\
M_1 & \longrightarrow & L_1 & \longrightarrow & \pi_0(E(A_q),*) & \longrightarrow & M_0 & \longrightarrow & L_0
\end{array}$$

Let $q = 8$. By Proposition 3.2.7 (2) there is an exact sequence

$$(*) \qquad 0 \longrightarrow \underline{M}_0 \longrightarrow M_0 \longrightarrow L_0(1) \; .$$

Hence j_* is injective. A variant of the 5-lemma shows that u_* must be injective. A further diagram chase reveals that the cokernel of u_* injects into the cokernel of j_*. By (*) the cokernel of j_* injects into $L_0(1)$. This yields an exact sequence:

$$0 \longrightarrow \pi_0(E(\underline{A}_q),\phi) \longrightarrow \pi_0(E(A_q),\phi) \longrightarrow L_0(1) \; .$$

The theorem follows from 5.1.9 and 5.1.10.

<u>Remark</u>: Theorem 5.1.15 also implies that $s_{TOP}(|K|)$ is abelian since $s_{TOP}(|K| \times \Delta^4, |K| \times \partial\Delta^4)$ is abelian.

We now show that the group structure of $s_{TOP}(|K|,\partial_0|K|)$ is independent of the triangulation of the PL manifold s-ad $M = |K|^m$. For clarity this will be proved in the case M is a closed manifold.

Let K' be another oriented simplicial manifold triangulating M. Assume that the triangulations are PL compatible. By Lemma 2.5 of [RS1] there is a triangulation P of $M \times I$ such that P restricted to $M \times 0$ is K and P restricted to $M \times 1$ is K'. Let $i: K \to P$ and $i': K' \to P$ be the inclusions. Consider the following commutative diagram:

5.1.17
$$\begin{array}{ccccc}
E(\underline{A}(K)) & \longrightarrow & \underline{M}^8(K) & \xrightarrow{\underline{A}(K)} & \mathbb{L}_{m+8}(|K|) \\
\uparrow{\scriptstyle U} & & \uparrow{\scriptstyle i^{\#}} & & \uparrow{\scriptstyle \partial^0} \\
E(\underline{A}(P)) & \longrightarrow & \underline{M}^8(P) & \xrightarrow{\underline{A}(P)} & \mathbb{L}_{m+9}((|P|;|K|,|K'|) \\
\downarrow{\scriptstyle W} & & \downarrow{\scriptstyle (i')^{\#}} & & \downarrow{\scriptstyle \partial^1} \\
E(\underline{A}(K')) & \longrightarrow & M^8(K') & \xrightarrow{\underline{A}(K')} & \mathbb{L}_{m+8}(|K'|)
\end{array}$$

where $E(\underline{A}(\ ))$ is the homotopy fiber of the assembly map $\underline{A}(\ )$, $i^{\#}$ and $(i')^{\#}$ are the induced maps, ∂^0 and ∂^1 were defined in Section 2.1, and U and W are the induced maps of homotopy fibers. Since i and i' are homotopy equivalences, $i^{\#}$ and $(i')^{\#}$ are also homotopy equivalences. To see that ∂^k $k = 0,1$ are homotopy equivalences we use the following theorem of Wall:

<u>Theorem 5.1.18</u> ([Wa 1, p. 93]): For any CW n-ad X and $0 \le k \le n-2$ there is a natural exact sequence:

$$\longrightarrow L^1_{p+1}(\partial_k X) \xrightarrow{i_*} L^1_{p+1}(\delta_k X) \longrightarrow L^1_{p+1}(X) \xrightarrow{\partial^k} L^1_p(\partial_k X) \longrightarrow$$

where $L^1_{p+j}(Y) = \pi_j(\mathbb{L}_p(Y),\phi)$ $p+j \ge 5$ and i_* is induced by the inclusion.

Applying the theorem above with $X = (|P|;|K|)$ and observing that i_* is an isomorphism since $i: K \to P$ is a homotopy equivalence gives that $\pi_j(\mathbb{L}_p(|P|;|K|),\phi) = 0$ $p+j \ge 5$. Another application of the theorem with

$X = (|P|,|K|,|K'|)$ shows that $\partial^0: \pi_j(\mathbb{L}_{p+1}(X),\phi) \to \pi_j(\mathbb{L}_p(\partial_k X),\phi)$ is an isomorphism for $j \geq 0$, $p \geq 5$. Thus ∂^0 is a homotopy equivalence by Whitehead's theorem and similarly for ∂^1. By Theorem 1.3.12 U and W are homotopy equivalences. Hence there is a H-map inducing a homotopy equivalence $E(\underline{A}(K)) \to E(\underline{A}(K'))$. Then 5.1.9 gives a H-map inducing a homotopy equivalence $S_{TOP}(|K|) \simeq S_{TOP}(|K'|)$ and in particular a group isomorphism $s_{TOP}(|K|) \cong s_{TOP}(|K'|)$.

Using the results of Chapter 4, it is now easy to show that the surgery exact sequence of 5.1.12 is natural with respect to induction and restriction. We first make the following:

Observation 5.1.19: Given a commutative square of basepoint preserving Δ-maps of Kan pointed Δ-sets:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

the fiber mapping sequence of f (see 1.3.8) maps into the fiber mapping sequence of g yielding a commutative ladder:

$$\begin{array}{ccccccccc} \longrightarrow & \pi_1(B,*) & \longrightarrow & \pi_0(E(f),*) & \longrightarrow & \pi_0(A,*) & \longrightarrow & \pi_0(B,*) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \longrightarrow & \pi_1(D,*) & \longrightarrow & \pi_0(E(g),*) & \longrightarrow & \pi_0(C,*) & \longrightarrow & \pi_0(D,*) \end{array}$$

where $E(f)$ and $E(g)$ are the homotopy fibers of f and g respectively over the basepoints.

Suppose that N^n and M^m are compact oriented PL manifold s-ads such that for every subset c of $\{1,\ldots,s-2\}$ $\partial_c N$ and $\partial_c M$ are connected and non-empty. $\partial_0 N$ and $\partial_0 M$ may be empty or disconnected. Also assume that $m,n \geq s+4$.

Let $p: N \to M$ be a PL map with $p^{-1}(\partial_i M) = \partial_i N$ for $i = 0,\ldots,s-2$ which can be triangulated as an oriented simplicial w-mock bundle $p': E \to K$ (see Definition 4.1.1). Note that $n = m+w$. An example of such a map p is an oriented PL bundle over M with fiber a compact oriented PL manifold of dimension w. When $w = 0$ we will assume that p is a covering projection. Applying Observation 5.1.19 and Theorem 5.1.12 to the commutative square 4.1.7 with $q = 8$ (or to 4.1.8 if $w = 0$) we obtain

Theorem 5.1.20: There is a commutative ladder of exact sequences:

$$\begin{array}{ccccccc}
\longrightarrow & s_{TOP}((N,\partial_0 N)) & \longrightarrow & H^0(N,\partial_0 N;\underline{G/TOP}) & \longrightarrow & L_n(\pi_1(\delta_0 N)) \\
 & \downarrow{\scriptstyle I_*(p)} & & \downarrow{\scriptstyle tr(p)} & & \downarrow{\scriptstyle p_*} \\
\longrightarrow & s_{TOP}((M,\partial_0 M)\times(\Delta^w,\partial\Delta^w)) & \longrightarrow & H^0((M,\partial_0 M)\times(\Delta^w,\partial\Delta^w);\underline{G/TOP}) & \longrightarrow & L_n(\pi_1(\delta_0 M))
\end{array}$$

where $I_*(p)$ is induced by the induction map of 4.1.11 (or of 4.1.12 if $w = 0$) and $p_* = \pi_0(\mathbb{L}_n(p))$.

When $w = 0$ it was shown in Chapter 4 that $tr(p)$ coincides with the cohomology transfer. For all w it was also shown in Chapter 4 that $tr(p)$ is independent of the triangulation of p. The same extendibility argument used there or in 5.1.17 can be used to show that $I_*(p)$ is independent of the given triangulation of p.

Now suppose that $p: N \to M$ is a covering projection. Applying Observation 5.1.19 and Theorem 5.1.12 to the commutative square 4.2.1, we obtain

Theorem 5.1.21: There is a commutative ladder of exact sequences:

$$\begin{array}{ccccc}
\longrightarrow s_{TOP}(M,\partial_0 M) & \longrightarrow & H^0(M,\partial_0 M;\underline{G/TOP}) & \longrightarrow & L_m(\pi_1(\delta_0 M)) \\
\downarrow I^*(p) & & \downarrow H^0(p) & & \downarrow p^* \\
\longrightarrow s_{TOP}(N,\partial_0 M) & \longrightarrow & H^0(N,\partial_0 N;\underline{G/TOP}) & \longrightarrow & L_n(\pi_1(\delta_0 N))
\end{array}$$

where $I^*(p)$ is induced by the restriction map of 4.2.4 or 4.2.5. p^* is given by $\pi_0(tr(p))$ where $tr(p)$ is the transfer in the theory of surgery spaces (see Section 4.2).

More generally, if $p: N \to M$ is an oriented PL bundle with fiber a closed oriented PL manifold of dimension w, then using diagram 4.2.6 in place of 4.2.1 yields a commutative ladder of exact sequences:

$$\begin{array}{ccccc}
\longrightarrow s_{TOP}(M,\partial_0 M) & \longrightarrow & H^0(M,\partial_0 M;\underline{G/TOP}) & \longrightarrow & L_m(\pi_1(\delta_0(M)) \\
\downarrow I^*(p) & & \downarrow H^0(p) & & \downarrow tr(p) \\
\longrightarrow s_{TOP}(N,\partial_0 N) & \longrightarrow & H^0(N,\partial_0 N;\underline{G/TOP}) & \longrightarrow & L_n(\pi_1(\delta_0 N))
\end{array}$$

where $I^*(p)$ is the restriction map of 4.2.7 and $tr(p)$ is induced by the transfer in the theory of surgery spaces (see Section 4.2).

We close this chapter by making some comments on some extensions of the theory hitherto developed.

1. In the topological case we have shown the existence of group structure in the surgery exact sequence for triangulable manifolds. As remarked by Siebenmann ([KS, pp. 280-288]) the theory can be extended to non-triangulable topological manifold s-ads X with $\dim X \geq s+4$ by triangulating the total space D of a stable normal disk bundle of X in Euclidean space and then giving consideration to the corresponding assembly map for D. We will not, however, pursue this in our discussion.

2. The surgery exact sequence can also be made an exact sequence of groups and homomorphisms in the PL case, i.e., if M is a compact oriented PL

manifold s-ad $m \geq s+4$, then there is an exact sequence of groups:

$$(*) \qquad \longrightarrow \pi_0(S_{PL}(M,\partial_0 M),*) \longrightarrow [M,\partial_0 M;G/PL] \longrightarrow L_m(\pi_1(\delta_0 M)) .$$

This is accomplished as follows: The H-space structure on G/TOP arising from the homotopy equivalence $G/TOP \simeq \mathbb{L}_8(pt)_0$ can also be described using the characteristic variety theorem of Sullivan (see [J] for an exposition of the characteristic variety theorem for surgery spaces). The characteristic variety theorem for G/PL (see [Su 2]) provides a natural group structure in [X,G/PL], for X a finite complex, and hence a H-group structure for G/PL. The natural map $i: G/PL \to G/TOP$ is then a H-map when G/PL and G/TOP are given their characteristic variety H-space structures. Let K be an oriented simplicial manifold triangulating M. Then the composite

$$\Delta(K,\partial_0 K;S(G/PL),*) \xrightarrow{i_\#} \Delta(K,\partial_0 K;S(G/TOP),*) \xrightarrow{A_2 a(0,r)} \mathbb{L}_{m+4r}(\delta_0|K|)$$

is a H-map where $A_2 a(0,r)_\#$ is the map of diagram 5.1.5. Call the composite J. In analogy to diagram 5.1.3 there is a homotopy commutative square:

$$\begin{array}{ccc} \Delta(K,\partial_0 K;S(G/PL),*) & \xrightarrow{J} & \mathbb{L}_{m+4r}(\delta_0|K|) \\ \uparrow & & \uparrow \\ N_{PL}(|K|,\partial_0|K|) & \xrightarrow{F} & \mathbb{L}_m(\delta_0|K|) \end{array}$$

where the vertical maps are homotopy equivalences. Let E(J) be the homotopy fiber of J over the basepoint. By Theorem 1.3.11 (4) E(J) is a homotopy associative H-space such that $\pi_0(E(J),*)$ is a group. Theorem 1.3.12 implies that E(J) is homotopy equivalent to the homotopy fiber of the surgery obstruction map $F: N_{PL}(|K|,\partial_0|K|) \to \mathbb{L}_m(\delta_0|K|)$ over the basepoint which by Theorem 2.3.4 is homotopy equivalent to $S_{PL}(|K|,\partial_0|K|)$. The proof of Theorem 1.3.12 also shows that the homotopy sequence of the homotopy fibration $E(J) \to \Delta(K,\partial_0 K;S(G/PL),*) \to \mathbb{L}_{m+4r}(\delta_0 M)$ maps isomorphically into the homotopy sequence of the homotopy fibration $S_{PL}(|K|,\partial_0|K|) \to N_{PL}(|K|,\partial_0|K|) \to \mathbb{L}_m(\delta_0|K|)$. As in the case of Theorem 5.1.12, this yields the exact sequence of groups (*).

The method above fails in the smooth case because of the lack of a characteristic variety theorem for G/O. Thus there remains the unresolved:

Question: Is the surgery exact sequence in the smooth case naturally an exact sequence of groups and homomorphisms?

6. INDUCTION THEOREMS

6.1 Algebraic Preliminaries

Given a finite group G, let $\underline{G}$ be the category whose objects are subgroups of G and whose morphisms $H \to K$ are triples (H,g,K) such that $g \in G$ and gHg^{-1} is a subgroup of K. A morphism $I: H \to K$, $I = (H,g,K)$ can be thought of as the composite of the group homomorphism $H \to gHg^{-1}$ given by conjugation by g followed by the inclusion homomorphism of gHg^{-1} into K. Composition in $\underline{G}$ is defined by $(K,g',L)(H,g,K) = (H,g'g,L)$ and the identity $H \to H$ is (H,e,H) where e is the identity element of G.

In this chapter Ab will denote the category of abelian groups. Suppose $M = (m,m'): \underline{G} \to Ab$ is a pair of functors where m is covariant and m' is contravariant and $m(H) = m'(H)$ for all objects H of $\underline{G}$. If $I: H \to K$ is a morphism of $\underline{G}$, the notation $I_* = m(I)$ and $I^* = m'(I)$ and $M(H) = m(H) = m'(H)$ will be used. I_* is called induction with respect to I and I^* is called restriction with respect to I.

Definition 6.1.1: $M: \underline{G} \to Ab$ as above is a Mackey functor if:

1. For any isomorphism $I: H \to K$, I^*I_* is the identity $M(H) \to M(H)$.
2. For any inner conjugation $I = (H,h,H)$ $h \in H$ I_* and I^* are the identity $M(H) \to M(H)$.
3. The double coset formula holds: Let L and L' be subgroups of the subgroup H of G. Suppose H has a double coset decomposition $H = \cup \{Lg_iL' \mid i = 1,\ldots,n\}$ $g_i \in H$. Then:

$$(L,e,H)^*(L',e,H)_* = \sum_{i=1}^{n} (L \cap (g_iL'g_i^{-1}),e,L)_*(L \cap (g_iL'g_i^{-1}),g_i^{-1},L')^* .$$

Definition 6.1.2: Let $M,N: \underline{G} \to Ab$ be Mackey functors. A morphism of

Mackey functors $f: M \to N$ consists of a collection of homomorphisms $f(H): M(H) \to N(H)$ for each object H of $\underline{G}$ such that for any morphism $I: H \to K$ in $\underline{G}$ there are commutative diagrams:

$$\begin{array}{ccc} M(H) & \xrightarrow{f(H)} & N(H) \\ {\scriptstyle I_*}\downarrow & & \downarrow{\scriptstyle I_*} \\ M(K) & \xrightarrow{f(K)} & N(K) \end{array} \qquad\qquad \begin{array}{ccc} M(H) & \xrightarrow{f(H)} & N(H) \\ {\scriptstyle I^*}\uparrow & & \uparrow{\scriptstyle I^*} \\ M(K) & \xrightarrow{f(K)} & N(K) \end{array}$$

If $f: M \to N$ is a morphism of Mackey functors then $\ker(f)$, $\operatorname{coker}(f)$, $\operatorname{im}(f)$, $\operatorname{coim}(f): \underline{G} \to Ab$ are the Mackey functors defined by:

$$\begin{aligned} (\ker(f))(H) &= \ker(f(H): M(H) \to N(H)) \\ (\operatorname{coker}(f))(H) &= \operatorname{coker}(\quad\text{"}\quad) \\ (\operatorname{im}(f))(H) &= \operatorname{im}(\quad\text{"}\quad) \\ (\operatorname{coim}(f))(H) &= \operatorname{coim}(\quad\text{"}\quad) = M(H)/\ker f(H) \end{aligned}$$

where H is a subgroup of G. Induction and restriction maps are induced by the commutative diagrams of Definition 6.1.2. The conditions of Definition 6.1.1 are trivial to verify. The category of Mackey functors and their morphisms is an abelian category in the obvious manner.

<u>Definition 6.1.3</u>: Let $M: \underline{G} \to Ab$ be a Mackey functor. M is called a Green functor if:

1. For each object H of $\underline{G}$ $M(H)$ is a ring with unit and for each morphism I of $\underline{G}$ I^* is a ring homomorphism.
2. Frobenius reciprocity holds: For each morphism $I: H \to K$ and $y \in M(H)$, $x \in M(K)$ one has:

$$I_*(I^*(x) \cdot y) = x \cdot I_*(y) \quad \text{and} \quad I_*(y \cdot I^*(x)) = I_*(y) \cdot x .$$

A morphism of Green functors $M \to M'$ is a morphism of Mackey functors $f: M \to M'$ such that for each object H of $\underline{G}$ $f(H)$ is a ring homomorphism.

Definition 6.1.4: Let $M: \underline{G} \to Ab$ be a Green functor and $N: \underline{G} \to Ab$ be a Mackey functor. N is said to be a left M-module if:

1. For each object H of $\underline{G}$ $N(H)$ is a left module over the ring $M(H)$.
2. For each morphism $I: H \to K$ in $\underline{G}$ and $v \in M(H)$, $x \in M(K)$, $y \in N(H)$, $z \in N(K)$ one has:

$$I^*(x \cdot z) = I^*(x) \cdot I^*(z) \quad \text{and}$$
$$I_*(I^*(x) \cdot y) = x \cdot I_*(y) \quad \text{and} \quad I_*(v \cdot I^*(z)) = I_*(v) \cdot z \ .$$

Now suppose $M: \underline{G} \to Ab$ is a Green functor and N and P are left M-modules. A morphism of Mackey functors $f: N \to P$ is a M-module morphism if for each object H of $\underline{G}$ $f(H)$ is a $M(H)$-module homomorphism. If $f: N \to P$ is a M-module morphism then $\ker(f)$, $\operatorname{coker}(f)$, $\operatorname{im}(f)$, and $\operatorname{coim}(f)$ are M-modules in the obvious manner.

The tensor product of an abelian group A and a Mackey functor $N: \underline{G} \to Ab$ is the Mackey functor $N \otimes A: \underline{G} \to Ab$ given on objects by $(N \otimes A)(H) = N(H) \otimes A$ and for a morphism I of $\underline{G}$ induction and restriction are given by $I_* \otimes id$ and $I^* \otimes id$ respectively where $id: A \to A$ is the identity. A morphism of Mackey functors $f: N \to P$ and a homomorphism of abelian groups $g: A \to B$ induces a morphism of Mackey functors $f \otimes g: N \otimes A \to P \otimes B$ given by $(f \otimes g)(H) = f(H) \otimes g$.

If A is a ring with unit and $M: \underline{G} \to Ab$ is a Green functor then $M \otimes A$ is also a Green functor where multiplication in the ring $(M \otimes A)(H)$ is given by $(x \otimes a)(y \otimes b) = (xy) \otimes (ab)$ where $x,y \in M(H)$ and $a,b \in A$ and the unit is $1 \otimes 1$. In our application A will be a subring of the rational numbers.

Remark 6.1.5: Suppose M is a Green functor, N is a M-module, A is a ring with unit, and B is an A-module. Then $N \otimes B$ is a $(M \otimes A)$-module where the action of $(M \otimes A)(H)$ on $(N \otimes B)(H)$ is given by:

$$(m \otimes a) \cdot (n \otimes b) = (m \cdot n) \otimes (a \cdot b) \qquad m \in M(H),\ n \in N(H),\ a \in A,\ b \in B$$

Clearly if $g: B \to C$ is an A-module homomorphism then $id \otimes g: N \otimes B \to N \otimes C$ is a $(M \otimes A)$-module morphism.

Given a finite group G, the trivial Green functor, $triv: \underline{G} \to Ab$ is defined as follows: For every object H of $\underline{G}$ $triv(H) = Z$, the ring of integers. If $I = (H,g,K)$ is a morphism of $\underline{G}$, the induction map $I: Z \to Z$ is given by multiplication by $[K: gHg^{-1}]$, the index of gHg^{-1} in K, and the restriction map $I^*: Z \to Z$ is the identity. The double coset formula follows from the fact if L and L' are subgroups of H and $H = \dot{\cup}_i\, Lg_iL'$ is a double coset decomposition then $[H:L] = \sum_i [L: L \cap (g_iL'g_i^{-1})]$. The other defining conditions are also trivially verified.

<u>Proposition 6.1.6</u>: Let $\{H_j |\ j = 1,\ldots,n\}$ be a collection of subgroups of the finite group H and let P be a (possibly empty) subset of Z (the integers) consisting of primes. Suppose for each prime p dividing the order of H and not in P, some H_j contains a Sylow p-group of H. Then the sum of the induction maps: $I = \sum_{j=1}^n (H_j,e,H)_*: \oplus_{j=1}^n triv(H_j) \otimes Z[P] \to triv(H) \otimes Z[P]$ is surjective where $Z[P]$ is the subring of the rationals generated by P.

<u>Proof</u>: Let p be a prime dividing the order of H and not in P. If p divides each $[H: H_j]$ then no H_j can contain a Sylow p-group of H contrary to hypothesis. This implies that the greatest common divisor of the numbers $[H: H_j]$ is equal to a product of primes in P (or equal to 1 if P is empty). Thus there are $m_j \in Z[P]$ such that $\sum_j m_j[H: H_j] = 1$. Hence 1 is the image of I and the result follows.

We now characterize those Mackey functors which are modules over the trivial Green functor:

Proposition 6.1.7: Let $N: \underline{G} \to Ab$ be a Mackey functor. Then N is a module over the trivial Green functor if and only if for each morphism $I = (H,g,K)$ of G the composite $I_*I^*: N(K) \to N(K)$ is multiplication by $[K: gHg^{-1}]$.

Proof: Suppose N is a module over the trivial Green functor. Then for $x \in N(K)$, $1 \in triv(H)$, and $I = (H,g,K)$ we have:

$$I_*I^*(x) = I_*(1 \cdot I^*(x)) = I_*(1) \cdot x = [K: gHg^{-1}]x .$$

Conversely, let the action of $triv(H)$ on $N(H)$ be just the natural Z-module action. The conditions of Definition 6.1.4 follow directly from the fact that I_*I^* is multiplication by the index.

The following remark will be useful in Section 2:

Remark 6.1.8: Let N be a module over the trivial Green functor $triv: \underline{G} \to Ab$ and A a ring with unit. Suppose that $N(H)$ is an A-module for each object H of $\underline{G}$ and for each morphism I of $\underline{G}$, I_* and I^* are A-module homomorphisms. Then N is a $(triv \otimes A)$-module where the action of the ring $(triv \otimes A)(H)$ on $N(H)$ is given by:

$$(x \otimes a) \cdot y = x \cdot (ay) \qquad x \in triv(H),\ a \in A,\ y \in N(H)$$

Note that $x \cdot (ay) = a(x \cdot y)$ since the action of $triv(H)$ is just the natural Z-module action.

The propositions above together with the induction theorems of Dress in [Dr] yield many useful exact sequences for modules over the trivial Green functor. Modules over the trivial Green functor can arise in the following manner:

<u>Proposition 6.1.9</u>: Let $f\colon M \to N$ and $g\colon N \to M$ be morphisms of Mackey functors and suppose N is a module over the trivial Green functor. Then M is a module over the trivial Green functor if one of the following is true:

1. f is a monomorphism.
2. g is an epimorphism.

<u>Proof</u>: Let $I = (H,g,K)$ be a morphism of $\underline{G}$. There are commutative diagrams:

$$\begin{array}{ccccc} 0 & \longrightarrow & M(K) & \xrightarrow{f(K)} & N(K) \\ & & \downarrow{\scriptstyle W} & & \downarrow{\scriptstyle V} \\ 0 & \longrightarrow & M(K) & \xrightarrow{f(K)} & N(K) \end{array} \qquad \begin{array}{ccccc} N(K) & \xrightarrow{g(K)} & M(K) & \longrightarrow & 0 \\ \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle W} & & \\ N(K) & \xrightarrow{g(K)} & M(K) & \longrightarrow & 0 \end{array}$$

where $V = I_*I^*\colon N(K) \to N(K)$ and $W = I_*I^*\colon M(K) \to M(K)$. By Proposition 6.1.7 V is multiplication by $m = [K\colon gHg^{-1}]$.

1. Suppose f is a monomorphism. Then

$$f(K)W(x) = Vf(K)(x) = m \cdot f(K)(x) = f(K)(mx) .$$

Hence $W(x) = mx$ since $f(K)$ is injective. Proposition 6.1.7 implies M is a module over the trivial Green functor.

2. Suppose g is an epimorphism. If $x \in M(K)$ then $g(K)(y) = x$ for some y since $g(y)$ is surjective. Then

$$W(x) = Wg(K)(y) = g(K)V(y) = g(K)(my) = mx .$$

Proposition 6.1.7 implies that M is a module over the trivial Green functor.

<u>Corollary 6.1.10</u>: Let $h\colon N \to P$ be a morphism of Mackey functors where N

is a module over the trivial Green functor. Then ker(h), im(h), and coim(h) are modules over the trivial Green functor.

Proof: The corollary follows from Proposition 6.1.9 and the exact sequences of Mackey functors:

$$0 \to \ker(h) \to N \to \operatorname{coim}(h) \to 0 \quad \text{and} \quad N \to \operatorname{im}(h) \to 0.$$

6.2 Induction Theorems for $s_{TOP}(\ )$

In this section Dress induction theory and localization are applied to the surgery exact sequence in order to study the group $s_{TOP}(X,\partial_0 X) = \pi_0(S_{TOP}(X,\partial_0 X),*)$ of homotopy manifold structures on the manifold s-ad X relative to $\partial_0 X$.

We will need to discuss how a generalized cohomology theory gives rise to Mackey functors:

Let (X,Y) be a finite CW pair and G a finite group acting freely on the pair (X,Y) on the right. Let $g \in G$ and H,K be two subgroups of G such that gHg^{-1} is a subgroup of K. Define the map $p(H,g,K)$: $X/H \to X/K$ to be the composite of the homeomorphism $X/H \to X/gHg^{-1}$, $xH \to (xg^{-1})(gHg^{-1})$ followed by the covering projection $X/gHg^{-1} \to X/K$, $x(gHg^{-1}) \to xK$. Clearly $p(H,g,K)$ is a covering projection of finite index. If $\underline{C}$ is a spectrum and $H^*(\ ;\underline{C})$ is the corresponding cohomology theory, define functors M_j: $\underline{G} \to Ab$ as follows:

$$M_j(K) = H^j(X/K,Y/K;\underline{C}) \quad \text{for an object } K \text{ of } \underline{G}$$

and if $I = (H,g,K)$ is a morphism of $\underline{G}$ define homomorphisms I_*: $M(H) \to M(K)$ and I^*: $M(K) \to M(H)$ by:

6.2.1 $$I_* = H^j(p(H,g,K)) \quad \text{and} \quad I^* = tr^j(p(H,g,K))$$

where tr^j is the cohomology transfer of [Rsh] or [A], defined for the covering $p(H,g,K)$ of finite index.

Proposition 6.2.2: For $j \geq 0$, $M_j: \underline{G} \to Ab$ is a Mackey functor.

Proof: The double coset formula for the transfer was proved by Roush ([Rsh, p. 89]). The other conditions of Definition 6.1.1 are easily verified.

Now suppose X^m is a compact oriented manifold s-ad such that $m \geq s+4$ and for every subset c of $\{1,\ldots,s-2\}$ $\partial_c X$ is connected and non-empty. $\partial_0 X$ is allowed to be empty or disconnected. Let G be a finite group of orientation preserving homeomorphisms of the s-ad X which act freely on X. We now define functors s_j, h_j, $\underline{L}_j: \underline{G} \to Ab$ for $j \geq 0$.

Definition of $s_j: \underline{G} \to Ab$:

For a subgroup H of G define $s_j(H) = \pi_j(S_{TOP}(X/H,\partial_0 X/H),*)$. Let $I = (H,g,K)$ be a morphism of $\underline{G}$. The map $p(H,g,K)$ described previously is a covering of finite index. The induction map of 4.1.12 then defines an induction homomorphism $I_*: s_j(H) \to s_j(K)$ and the restriction map of 4.2.4 defines a restriction homomorphism $I^*: s_j(K) \to s_j(H)$.

Definition of $h_j: \underline{G} \to Ab$:

Let $\underline{G/TOP}$ denote the spectrum $\mathbb{L}(pt)(1,\ldots,\infty)$ with zero-th space $\mathbb{L}_8(pt)_0 \simeq G/TOP$ (see Chapter 5). Then by Proposition 6.2.2 the functors $h: \underline{G} \to Ab$ given by

$$h_j(H) = H^0((X/H,\partial_0 X/H) \times (\Delta^j,\partial\Delta^j);\underline{G/TOP})$$

with induction and restriction defined by 6.2.1 are Mackey functors. Note that if $\partial_0 X$ is non-empty or $j > 0$ then

$$h_j(H) = H^0((X/H,\partial_0 X/H) \times (\Delta^j,\partial\Delta^j);\mathbb{L}(pt)) .$$

Definition of $\underline{L}_j : \underline{G} \to Ab$:

For a subgroup H of G let $\underline{L}_j(H) = \pi_j(\mathbb{L}_m X/H),\phi)$. Then induction and restriction are respectively defined by:

$$I_* = \pi_j(\mathbb{L}_m(p(H,g,K))) \quad \text{and} \quad I^* = \pi_j(tr(p(H,g,K)))$$

where $tr(p(H,g,K))$ is the transfer map defined for $\mathbb{L}$-spaces in Section 4.2. Recall that $\pi_j(\mathbb{L}_m(\delta_0 S/H),\phi)$ coincides with the algebraically defined L-group $L_{m+j}(\pi_1(\delta_0 X/H))$ of the group lattice $\pi_1(\delta_0 X/H)$ and the geometrically defined induction and restriction maps coincide with the algebraically defined induction and restriction maps. By [Dr, p. 302] the functors $\underline{L}_j$ $j \geq 0$ are Mackey functors.

For any subgroup H of G the surgery exact sequence of 5.1.12 becomes:

6.2.3 $$\longrightarrow s_j(H) \longrightarrow h_j(H) \xrightarrow{A_j} \underline{L}_j(H) \longrightarrow \cdots \longrightarrow s_0(H) \longrightarrow h_0(H) \xrightarrow{A_0} \underline{L}_0(H) .$$

Since by 5.1.20 and 5.1.21 the surgery exact sequence is natural with respect to induction and restriction, we obtain for a morphism $I = (H,g,K)$ of $\underline{G}$ two commutative ladders:

6.2.4
$$\begin{array}{ccccccccccccc}
\longrightarrow & s_j(H) & \longrightarrow & h_j(H) & \xrightarrow{A_j} & \underline{L}_j(H) & \longrightarrow \cdots \longrightarrow & s_0(H) & \longrightarrow & h_0(H) & \longrightarrow & \underline{L}_0(H) \\
 & \downarrow{\scriptstyle I_*} & & \downarrow{\scriptstyle I_*} & & \downarrow{\scriptstyle I_*} & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & s_j(K) & \longrightarrow & h_j(K) & \xrightarrow{A_j} & \underline{L}_j(K) & \longrightarrow \cdots \longrightarrow & s_0(K) & \longrightarrow & h_0(K) & \longrightarrow & \underline{L}_0(K)
\end{array}$$

6.2.5
$$\begin{array}{ccccccccccccc}
\longrightarrow & s_j(H) & \longrightarrow & h_j(H) & \xrightarrow{A_j} & \underline{L}_j(H) & \longrightarrow \cdots \longrightarrow & s_0(H) & \longrightarrow & h_0(H) & \longrightarrow & \underline{L}_0(H) \\
 & \uparrow{\scriptstyle I^*} & & \uparrow{\scriptstyle I^*} & & \uparrow{\scriptstyle I^*} & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & s_j(K) & \longrightarrow & h_j(K) & \xrightarrow{A_j} & \underline{L}_j(K) & \longrightarrow \cdots \longrightarrow & s_0(K) & \longrightarrow & h_0(K) & \longrightarrow & \underline{L}_0(K)
\end{array}$$

<u>Remark</u>: Although we have not proved that the functors s_j are Mackey functors, all that will be needed is that they possess induction and restriction homomorphisms compatible with those of h_j and $\underline{L}_j$, as already demonstrated.

Diagrams 6.2.4 and 6.2.5 imply that the surgery obstruction map $A_j\colon h_j \to \underline{L}_j$ $j \geq 0$ is a morphism of Mackey functors. Hence $U_j =$ coker (A_{j+1}) and $V_j = \ker(A_j)$ $j \geq 0$ are also Mackey functors and there are short exact sequences:

6.2.6 $$0 \to U_j(H) \to s_j(H) \to V_j(H) \to 0$$

for each subgroup H of G; moreover, 6.2.4 and 6.2.5 imply that the sequences of 6.2.6 are natural with respect to induction and restriction for morphisms of $\underline{G}$.

Let C be a collection of subgroups of G and $M\colon \underline{G} \to Ab$ a Mackey functor. Suppose for each pair H,H' in C we are given a double coset decomposition $G = \bigcup_i Hg_iH'$. The following notation will be used:

$$M(\cdot) = M(G)$$

$$M(C) = \oplus_{H\in C} M(H)$$

$$M(C\times C) = \oplus_{H,H',g} M(H \cap (gH'g^{-1}))$$ where for fixed (H,H') in $C\times C$, g runs over the double coset representatives of H,H'.

There are homomorphisms:

$$I_* : M(C) \to M(\cdot) \quad \text{and} \quad I^*\colon M(\cdot) \to M(C)$$

$$I_* - J_*\colon M(C\times C) \to M(C) \quad \text{and} \quad I^* - J^*\colon M(C) \to M(C\times C)$$

defined by $I_* = \sum_H (H,e,G)_*$ and $I^* = \pi_H(H,e,G)^*$.

$$I_* - J_* = \sum_{H,H',g} (H \cap (gH'g^{-1}),e,H)_* - (H \cap (gH'g^{-1}),g^{-1},H')_*$$

$$I^* - J^* = \pi_{H,H',g} (H \cap (gH'g^{-1}),e,H)^* - (H \cap (gH'g^{-1}),g^{-1},H')^*$$

The notational convention above can be reformulated more conceptually by defining Mackey functors on the category of finite G-sets as done by Dress in [Dr]. These definitions also make sense for the functors s_j $j \geq 0$.

The following theorem of Dress ([Dr, Proposition 1.2, p. 305]) will be important to our discussion:

Theorem 6.2.7 (the Dress lemma): Let $M: \underline{G} \to Ab$ be a Green functor and $N: \underline{G} \to Ab$ a M-module. Suppose C is a collection of subgroups of G and $I_*: M(C) \to M(\cdot)$ is surjective. Then there are exact sequences:

$$1. \quad 0 \longrightarrow N(\cdot) \xrightarrow{I^*} N(C) \xrightarrow{I^*-J^*} N(C \times C) .$$

$$2. \quad 0 \longleftarrow N(\cdot) \xleftarrow{I_*} N(C) \xleftarrow{I_*-J_*} N(C \times C) .$$

Given a Mackey functor $N: \underline{G} \to Ab$ and a collection C of subgroups of G define $\varprojlim N(C)$ and $\varinjlim N(C)$ by:

$$\varprojlim N(C) = \ker(I^* - J^*) \quad \text{and} \quad \varinjlim N(C) = \operatorname{coker}(I_* - J_*) .$$

The condition that I^* induce an isomorphism $N(G) \to \varprojlim N(C)$ or that I_* induces an isomorphism $\varinjlim N(C) \to N(G)$ is equivalent respectively to the exactness of sequence 1 or of sequence 2 of Theorem 6.2.7.

We now investigate the induction properties of the Mackey functors U_j and V_j $j \geq 0$ of 6.2.6.

The following lemma is implicit in Section 1 of [M1]:

Lemma 6.2.8: Let P be a set of primes and C a collection of subgroups of the finite group G such that for each prime p not in P and dividing the order of G some subgroup of G in C contains a Sylow p-group of G. Let $M_j: \underline{G} \to Ab$ $j \geq 0$ be the Mackey functors associated to any cohomology theory $H^*(\ ;\underline{A})$ for a given free action of G on a finite CW pair (X,Y) (see Proposition 6.2.2). Then there are exact sequences:

$$0 \longrightarrow M_j(\cdot) \otimes Z[P] \xrightarrow{I^*} M_j(C) \otimes Z[P] \xrightarrow{I^*-J^*} M_j(C \times C) \otimes Z[P]$$

$$0 \longleftarrow M_j(\cdot) \otimes Z[P] \xleftarrow{I_*} M_j(C) \otimes Z[P] \xleftarrow{I_*-J_*} M_j(C \times C) \otimes Z[P] .$$

Proof: Let $\omega_j: \underline{G} \to Ab$ be the Mackey functor associated to stable cohomotopy theory. Madsen ([M1, Section 1]) shows that ω_0 is a Green functor, where multiplication is given by the cup product, and the M_j are ω_0-modules. Let H be a subgroup of G, $t = (H,e,G)$, and $n = [G:H]$. By [M1, p. 312], the map $t_* t^*: \omega_0(G) \otimes Z[1/n] \to \omega_0(G) \otimes Z[1/n]$ is an isomorphism. It easily follows from this fact and the hypotheses on C that the sum of the induction maps $\omega_0(C) \otimes Z[P] \to \omega_0(\cdot) \otimes Z[P]$ is surjective. Noting that $M_j \otimes Z[P]$ is a $(\omega_0 \otimes Z[P])$-module, the conclusion of the lemma is then a consequence of the Dress lemma (Theorem 6.2.7).

Proposition 6.2.9: Let A be a subring of the rational numbers and C a collection of subgroups of G. Suppose one of the following is true:

1. C = class of cyclic subgroups of G, $A = Q$ (the rationals).
2. C = class of 2-hyperelementary subgroups of G

$$A = Z_{(2)} = Z[1/3, 1/5, \ldots].$$

3. C = class of p-elementary subgroups of G, p odd, $A = Z[1/2]$.
4. C = union of the classes in 2 and 3 above. $A = Z$.

Then in cases 1-4, there are exact sequences, $j \geq 0$:

$$0 \longrightarrow V_j(\cdot) \otimes A \xrightarrow{I^*} V_j(C) \otimes A \xrightarrow{I^*-J^*} V_j(C \times C) \otimes A$$

$$0 \longleftarrow U_j(\cdot) \otimes A \xleftarrow{I_*} U_j(C) \otimes A \xleftarrow{I_*-J_*} U_j(C \times C) \otimes A$$

Equivalently I^* induces an isomorphism $V_j(G) \to \varprojlim V_j(C)$ and I_* induces an isomorphism $\varinjlim U_j(C) \to U_j(G)$.

Proof: Recall that $V_j = \ker(A_j: h_j \to \underline{L}_j)$ and that $U_j =$ $\operatorname{coker}(A_{j+1}: h_{j+1} \to \underline{L}_{j+1})$. Since a subring of the rationals is flat as a Z-module, $V_j \otimes A = (\ker(A_j)) \otimes A = \ker(A_j \otimes A)$ and $U_j \otimes A = \operatorname{coker}(A_{j+1} \otimes A)$.

Note that for a subgroup H of G there is a short exact sequence, natural in H, $0 \to \pi_1(\delta_0 X) \to \pi_1(\delta_0 X/H) \to H \to 0$ which comes from the homotopy sequence of the fibration $X \to X/H$. The equivariant Witt ring of Dress, [Dr], define a Green functor $W\colon \underline{G} \to Ab$. By Theorem 2.3 of [FHs] the functors $\underline{L}_j$ $j \geq 0$ are W-modules. Theorem 2, p. 295 of [Dr] is precisely the statement that the sum of the induction maps $I_*\colon W(C) \otimes A \to W(\cdot) \otimes A$ is surjective in the cases 1-4. Hence by the Dress lemma (Theorem 6.2.7) there are exact sequences in the cases 1-4:

6.2.10 $$0 \longrightarrow \underline{L}_j(\cdot) \otimes A \xrightarrow{I^*} \underline{L}_j(C) \otimes A \xrightarrow{I^*-J^*} \underline{L}_j(C \times C) \otimes A$$

and

$$0 \longleftarrow \underline{L}_j(\cdot) \otimes A \xleftarrow{I_*} \underline{L}_j(C) \otimes A \xleftarrow{I_*-J_*} \underline{L}_j(C \times C) \otimes A$$

For an early application of induction theory to L-groups localized away from 2 see [Th].

Lemma 6.2.8 yields exact sequences as in 6.2.10 above with h in place of $\underline{L}$ in the cases 1-4.

Consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & V_j(\cdot) \otimes A & \longrightarrow & h_j(\cdot) \otimes A & \xrightarrow{A_j \otimes A} & \underline{L}_j(\cdot) \otimes A \\
 & & \downarrow{\scriptstyle I^*} & & \downarrow & & \downarrow \\
0 & \longrightarrow & V_j(C) \otimes A & \longrightarrow & h_j(C) \otimes A & \longrightarrow & \underline{L}_j(C) \otimes A \\
 & & \downarrow{\scriptstyle I^*-J^*} & & \downarrow & & \\
0 & \longrightarrow & V_j(C \times C) \otimes A & \longrightarrow & H_j(C \times C) \otimes A & &
\end{array}$$

Then the rows and the second and third columns are exact. A straightforward diagram chase shows that the first column must be exact. There is also a commutative diagram:

$$\begin{array}{ccccccc}
 & & \underline{L}_{j+1}(C\times C)\otimes A & \longrightarrow & U_j(C\times C)\otimes A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow I_*-J_* & & \\
h_{j+1}(C)\otimes A & \longrightarrow & \underline{L}_{j+1}(C)\otimes A & \longrightarrow & U_j(C)\otimes A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow I_* & & \\
h_{j+1}(\cdot)\otimes A & \longrightarrow & \underline{L}_{j+1}(\cdot)\otimes A & \longrightarrow & U_j(\cdot)\otimes A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The rows and the first and second columns are exact. Another diagram chase reveals that the third column must be exact. Hence Proposition 6.2.9 has been proved.

The induction properties of the functors s_j $j \geq 0$ will now be examined. Recall that $s_0(H) = s_{TOP}(X/H, \partial_0 X/H)$. We deduce the following theorems:

<u>Theorem 6.2.11</u>: Let C be the class of cyclic subgroups of G. Then for $j \geq 0$:

1. The sum of induction maps $\oplus_{H\in C}\, s_j(H) \otimes Q \to s_j(G) \otimes Q$ is surjective.
2. The product of the restriction maps $s_j(G) \otimes Q \to \pi_{H\in C}\, s_j(H) \otimes Q$ is injective.

<u>Theorem 6.2.12</u>: Let C be the class of 2-hyperelementary subgroups of G. Then for $j \geq 0$:

1. The sum of the induction maps $\oplus_{H\in C}\, s_j(H) \otimes Z_{(2)} \to s_j(G) \otimes Z_{(2)}$ is surjective.
2. The product of the restriction maps $s_j(G) \otimes Z_{(2)} \to \pi_{H\in C}\, s_j(H) \otimes Z_{(2)}$ is injective.

<u>Proof of Theorem 6.2.11</u>: For any spectrum $\underline{B}$, $H^*(\ ;\underline{B}) \otimes Q$ is a direct sum of ordinary cohomology theories. It is a well-known property of ordinary

cohomology theory that if $p: \tilde{X} \to X$ is a covering of finite index of finite connected CW complexes then restriction with respect to p followed by induction with respect to p is multiplication by the index $[\pi_1(X): p_*\pi_1(\tilde{X})]$. Hence by Proposition 6.1.7 $h_j \otimes Q: \underline{G} \to Ab$ is a module over the trivial Green functor, triv: $\underline{G} \to Ab$. Corollary 6.1.10 implies that $V_j \otimes Q = \ker(A_j \otimes Q: h_j \otimes Q \to \underline{L}_j \otimes Q)$ is also a module over the trivial Green functor. By Remark 6.1.8, $V_j \otimes Q$ is a (triv $\otimes$ Q)-module. Proposition 6.1.6 together with the Dress lemma (Theorem 6.2.7) imply that the sum of the induction maps $I_*: V_j(C) \otimes Q \to V_j(\cdot) \otimes Q$ is surjective where C is the class of cyclic subgroups of G.

By Proposition 6.2.9 (1) the sum of the induction maps $I_*: U_j(C) \otimes Q \to U_j(\cdot) \otimes Q$ is surjective. From the exact sequences of 6.2.6 tensored with Q (which remain exact since a subring of Q is flat as a Z-module) and from the fact that these sequences are natural with respect to induction and restriction one obtains a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_j(C) \otimes Q & \longrightarrow & s_j(C) \otimes Q & \longrightarrow & V_j(C) \otimes Q & \longrightarrow & 0 \\ & & \downarrow I_* & & \downarrow I_* & & \downarrow I_* & & \\ 0 & \longrightarrow & U_j(\cdot) \otimes Q & \longrightarrow & s_j(\cdot) \otimes Q & \longrightarrow & V_j(\cdot) \otimes Q & \longrightarrow & 0 \end{array}$$

where the rows are exact and the two outer vertical maps are surjective. A diagram chase shows that $I_*: s_j(C) \otimes Q \to s_j(\cdot) \otimes Q$ must be surjective, i.e., the sum of the induction maps $\oplus_{H \in C}\, s_j(H) \otimes Q \to s_j(G) \otimes Q$ is surjective. This proves the first part of Theorem 6.2.11.

Claim: The product of the restriction maps $I^*: U_j(\cdot) \otimes Q \to U_j(C) \otimes Q$ is injective where C is the class of cyclic subgroups of G.

Let $N_j: \underline{G} \to Ab$ be the Mackey functor $N_j = \mathrm{coim}(A_j: h_j \to \underline{L}_j)$. There are short exact sequences of Mackey functors:

$$0 \longrightarrow N_{j+1} \xrightarrow{\underline{A}_{j+1}} L_{j+1} \longrightarrow U_j \longrightarrow 0$$

where $\underline{A}_{j+1}$ is induced by A_{j+1}.

Since $h_j \otimes Q$ is a module over the trivial Green functor triv: $\underline{G} \to Ab$, $N_j \otimes Q = \mathrm{coim}(A_j \otimes Q\colon h_j \otimes Q \to \underline{L}_j \otimes Q)$ is also a module over the trivial Green functor by Corollary 6.1.10. By Remark 6.1.8, $N_j \otimes Q$ is a (triv $\otimes$ Q)-module. Proposition 6.1.6 and the Dress lemma (Theorem 6.2.7) yield an exact sequence:

$$0 \longrightarrow N_j(\cdot) \otimes Q \xrightarrow{I^*} N_j(C) \otimes Q \longrightarrow N_j(C \times C) \otimes Q .$$

By 6.2.10 there is also an exact sequence:

$$0 \longrightarrow \underline{L}_j(\cdot) \otimes Q \xrightarrow{I^*} \underline{L}_j(C) \otimes Q \longrightarrow \underline{L}_j(C \times C) \otimes Q$$

Consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{j+1}(\cdot) \otimes Q & \longrightarrow & \underline{L}_{j+1}(\cdot) \otimes Q & \longrightarrow & U_j(\cdot) \otimes Q \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle I^*} \\
0 & \longrightarrow & N_{j+1}(C) \otimes Q & \longrightarrow & \underline{L}_{j+1}(C) \otimes Q & \longrightarrow & U_j(C) \otimes Q \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{j+1}(C \times C) \otimes Q & \longrightarrow & \underline{L}_{j+1}(C \times C) \otimes Q & &
\end{array}$$

The rows and the first two columns are exact. A diagram chase (see Lemma 6.2.15) shows the map I^* in the third column must be injective, proving the claim.

By Proposition 6.2.9 (1), $I^*\colon V_j(\cdot) \otimes Q \to V_j(C) \otimes Q$ is injective. The exact sequences of 6.2.6 yield a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_j(\cdot) \otimes Q & \longrightarrow & s_j(\cdot) \otimes Q & \longrightarrow & V_j(\cdot) \otimes Q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I^*} & & \downarrow{\scriptstyle I^*} & & \downarrow{\scriptstyle I^*} & & \\
0 & \longrightarrow & U_j(C) \otimes Q & \longrightarrow & s_j(C) \otimes Q & \longrightarrow & V_j(C) \otimes Q & \longrightarrow & 0
\end{array}$$

where the rows are exact and the two outer vertical maps are injections.

Another diagram chase shows that $I^*: s_j(\cdot) \otimes Q \to s_j(C) \otimes Q$ must be injective, i.e., the product of the restriction maps $s_j(G) \otimes Q \to \pi_{H \in C}\, s_j(H) \otimes Q$ is injective. Hence the second part of Theorem 6.2.11 has been proven.

Proof of Theorem 6.2.12: If we could show that $h_j \otimes Z_{(2)}$ was a module over the trivial Green functor then the proof of Theorem 6.2.11, with all modules tensored with $Z_{(2)}$ in place of Q and Proposition 6.2.9 (2) substituted for Proposition 6.2.9 (1), would provide a proof of Theorem 6.2.12.

Recall that $h_j(H) = H^0((X/H, \partial_0 X/H) \times (\Delta^j, \partial\Delta^j); \underline{G/TOP})$ where $\underline{G/TOP}$ is the spectrum $\mathbb{L}(pt)(1,\ldots,\infty)$ with zero-th space $\mathbb{L}_8(pt)_0 \simeq G/TOP$. The work of Taylor and Williams (see [TW]) shows that the spectrum $\underline{G/TOP}$ localized at the prime 2 is a product of Eilenberg-MacLane spectra. Hence $H^*(\ ;G/TOP) \otimes Z_{(2)}$ is a sum of ordinary cohomology theories. It follows that $h_j \otimes Z_{(2)}$ is a module over the trivial Green functor.

Remark: In contrast, suppose G/TOP is given the infinite loop space structure arising from the Whitney sum, denoted by $(G/TOP)^{\oplus}$. It is known that the spectrum $(G/TOP)^{\oplus}$ localized at 2 is not a product of Eilenberg-MacLane spectra (see [M2]).

In order to obtain an analog of Theorem 6.2.12 for odd primes, our method will require the additional hypothesis that the fundamental group of the s-ad X is finite.

Theorem 6.2.13: Suppose the fundamental group of X is finite and let C be the class of p-elementary subgroups of G, p odd. Then for $j \geq 0$:

1. The sum of the induction maps $\oplus_{H \in C}\, s_j(H) \otimes Z[1/2] \to s_j(G) \otimes Z[1/2]$ is surjective.

2. The product of the restriction maps $s_j(G) \otimes Z[1/2] \to \pi_{H\varepsilon C}\, s_j(H) \otimes Z[1/2]$ is injective.

<u>Proof</u>: It is known that the torsion subgroup of the L-group of a finite group consists only of 2-torsion (see [Wa2]). Hence the map $id \otimes j$: $\underline{L}_j(H) \otimes Z[1/2] \to \underline{L}_j(H) \otimes Q$ is injective for each H where $j: Z[1/2] \to Q$ is the inclusion. Consider the following commutative diagram of morphisms of Mackey functors:

$$\begin{array}{ccc} h_j \otimes Z[1/2] & \xrightarrow{A_j \otimes Z[1/2]} & \underline{L}_j \otimes Z[1/2] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ h_j \otimes Q & \xrightarrow{A_j \otimes Q} & \underline{L}_j \otimes Q \end{array}$$

where $b = id \otimes j$ and $c = id \otimes j$, $id: h_j \to h_j'$, $id: \underline{L}_j \to \underline{L}_j$ are the respective identities. It follows that there is an exact sequence of Mackey functors:

$$0 \to \ker(b) \to \ker((A_j \otimes Q)b) \xrightarrow{b} h_j \otimes Q \ .$$

Let R be the Mackey functor $R = \mathrm{im}(b: \ker((A_j \otimes Q)b) \to h_j \otimes Q)$. Note $V_j \otimes Z[1/2] = \ker(A_j \otimes Z[1/2]) = \ker((A_j \otimes Q)b)$ since c is a monomorphism. The exact sequence above becomes:

6.2.14 $$0 \to \ker(b) \to V_j \otimes Z[1/2] \to R \to 0 \ .$$

Recall that $h_j \otimes Q$ is a module over the trivial Green functor (see the proof of Theorem 6.2.11). By Corollary 6.1.10 R is also a module over the trivial Green functor. Remark 6.1.8 implies that R is a $(\mathrm{triv} \otimes Z[1/2])$-module. Proposition 6.1.6 and the Dress lemma (Theorem 6.2.7) imply that the sum of the induction maps $I_*: R(C) \to R(\cdot)$ is surjective where C is the class of p-elementary subgroups of G, p odd.

Recall from Lemma 6.2.8 and its proof that h_j is a module over a Green functor $\omega: \underline{G} \to Ab$ (arising from stable cohomotopy theory) which has the property that $I_*: \omega(C) \otimes Z[1/2] \to \omega(\cdot) \otimes Z[1/2]$ is surjective. By Remark 6.1.5 $b = id \otimes j: h_j \otimes Z[1/2] \to h_j \otimes Q$ is a $(\omega \otimes Z[1/2])$-module morphism. Hence ker(b) is a $(\omega \otimes Z[1/2])$-module and thus the Dress lemma (Theorem 6.2.7) gives that $I_*: (\ker(b))(C) \to (\ker(b))(\cdot)$ is surjective. The exact sequence 6.2.14 yields a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\ker(b))(C) & \longrightarrow & V_j(C) \otimes Z[1/2] & \longrightarrow & R(C) & \longrightarrow & 0 \\ & & \downarrow I_* & & \downarrow I_* & & \downarrow I_* & & \\ 0 & \longrightarrow & (\ker(b))(\cdot) & \longrightarrow & V_j(\cdot) \otimes Z[1/2] & \longrightarrow & R(\cdot) & \longrightarrow & 0 \end{array}$$

where the rows are exact and the two outer vertical maps have been shown to be surjective. The usual diagram chase shows that $I_*: V_j(C) \otimes Z[1/2] \to V_j(\cdot) \otimes Z[1/2]$ is surjective.

By Proposition 6.2.9 (3) the sum of the induction maps $I_*: U_j(C) \otimes Z[1/2] \to U_j(\cdot) \otimes Z[1/2]$ is surjective. Then the exact sequences of 6.2.6 give a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_j(C) \otimes Z[1/2] & \longrightarrow & s_j(C) \otimes Z[1/2] & \longrightarrow & V_j(C) \otimes Z[1/2] & \longrightarrow & 0 \\ & & \downarrow I_* & & \downarrow I_* & & \downarrow I_* & & \\ 0 & \longrightarrow & U_j(\cdot) \otimes Z[1/2] & \longrightarrow & s_j(\cdot) \otimes Z[1/2] & \longrightarrow & V_j(\cdot) \otimes Z[1/2] & \longrightarrow & 0 \end{array}$$

where the rows are exact and the two outer vertical maps are surjective. Hence $I_*: s_j(C) \otimes Z[1/2] \to s_j(\cdot) \otimes Z[1/2]$ is surjective, proving the first part of Theorem 6.2.13.

In order to prove the second part of Theorem 6.2.13 we will need the following homological lemma:

<u>Lemma 6.2.15</u>: Suppose we are given a commutative diagram of morphisms in a small abelian category:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow q & & \downarrow r & & \\
0 & \longrightarrow & A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & & \\
 & & \downarrow p' & & \downarrow q' & & & & \\
 & & A'' & \xrightarrow{a''} & B'' & & & &
\end{array}
$$

such that the rows and columns are exact. Then there is a monomorphism $\ker(r) \to \ker(a'')$ (and hence a monomorphism $\ker(r) \to A''$).

Proof: By the full embedding theorem for small abelian categories it can be assumed that the diagram above consists of homomorphisms of modules over a ring.

Let $x \in \ker(r)$. Since b is surjective $b(y) = x$ for some $y \in B$. $b'q(y) = rb(y) = 0$. Thus $q(y) = a'(z)$ for some $z \in A'$. $a''p'(z) = q'a'(z) = q'q(y) = 0$. Hence $p'(z) \in \ker(a'')$. Define $R\colon \ker(r) \to \ker(a'')$ by $R(x) = p'(z)$, x and z as above. To see that R is well-defined suppose $b(y') = x$ and $q(y') = a'(z')$. Since $b(y - y') = 0$, $y - y' = a(u)$ for some $u \in A$. Then $a'(z - z') = q(y - y') = qa(u) = a'p(u)$. Hence $z - z' = p(u)$ since a' is injective. $p'p(u) = 0$ implies $p'(z) = p(z')$.

R is clearly a homomorphism. If $R(x) = p'(z) = 0$ then $z = p(s)$ for some $s \in A$. $qa(s) = a'(s) = a'(z) = q(y)$. Thus $a(s) = y$ because q is injective. $x = b(y) = ba(s) = 0$ and hence R is injective.

The lemma is used to prove the following:

Claim: The product of the restriction maps $I^*\colon U_j(\cdot) \otimes Z[1/2] \to U_j(C) \otimes Z[1/2]$ is injective.

Proof of the Claim: Let $N_j\colon \underline{G} \to Ab$ be the Mackey functor

$N_j = \mathrm{coim}(A_j\colon h_j \to \underline{L}_j)$. There is a short exact sequence of Mackey functors: $0 \longrightarrow N_{j+1} \xrightarrow{\underline{A}_{j+1}} \underline{L}_{j+1} \longrightarrow U_j \longrightarrow 0$ where $\underline{A}_{j+1}$ is induced by $A_{j+1}\colon h_{j+1} \to \underline{L}_{j+1}$. Consider the following commutative diagram of morphisms of Mackey functors:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 \longrightarrow & N_{j+1} \otimes Z[1/2] & \longrightarrow & \underline{L}_{j+1} \otimes Z[1/2] & & \longrightarrow & U_j \otimes Z[1/2] & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \downarrow f = \mathrm{id} \otimes j & \\
0 \longrightarrow & N_{j+1} \otimes Q & \longrightarrow & \underline{L}_{j+1} \otimes Q & & \longrightarrow & U_j \otimes Q & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & & \\
 & N_{j+1} \otimes Q/(Z[1/2]) & \longrightarrow & \underline{L}_{j+1} \otimes Q/(Z[1/2]) & & & & \\
 & \downarrow & & \downarrow & & & & \\
 & 0 & & 0 . & & & &
\end{array}$$

where the vertical columns come from tensoring with the exact sequence $0 \to Z[1/2] \xrightarrow{j} Q \to Q/(Z[1/2]) \to 0$. The first and second rows are exact because $Z[1/2]$ and Q are flat Z-modules. The first and second columns are exact because the tensor product is right exact and because $\mathrm{id} \otimes j\colon \underline{L}_{j+1} \otimes Z[1/2] \to \underline{L}_{j+1} \otimes Q$ is a monomorphism (see the proof of the first part of Theorem 6.2.13). Applying Lemma 6.2.15 to the above diagram we obtain a monomorphism: $\ker(f) \to N_{j+1} \otimes Q/(Z[1/2])$ where $f = \mathrm{id} \otimes j$. Recall from the proof of Theorem 6.2.11 that $N_{j+1} \otimes Q$ is a module over the trivial Green functor. Since $N_{j+1} \otimes Q \to N_{j+1} \otimes Q/(Z[1/2])$ is an epimorphism, Proposition 6.1.9 (2) implies that $N_{j+1} \otimes Q/(Z[1/2])$ is also a module over the trivial Green functor. Proposition 6.1.9 (1) then gives that $\ker(f)$ is module over the trivial Green functor, $\mathrm{triv}\colon \underline{G} \to \mathrm{Ab}$. By Remark 6.1.8, $\ker(f)$ is a $(\mathrm{triv} \otimes Z[1/2])$-module. From Proposition 6.1.7 and the Dress lemma (Theorem 6.2.7) we conclude that the product of the restrictions maps $I^*\colon (\ker(f))(\cdot) \to (\ker(f))(C)$ is injective.

From 6.2.10 there is an exact sequence:

$$0 \longrightarrow \underline{L}_j(\cdot) \otimes Z[1/2] \xrightarrow{I^*} \underline{L}_j(C) \otimes Z[1/2] \xrightarrow{I^*-J^*} \underline{L}_j(C \times C) \otimes Z[1/2] \longrightarrow 0$$

Tensoring the above sequence with Q and identifying $Z[1/2] \otimes Q$ with Q via the isomorphism $Z[1/2] \otimes Q \to Q$ $a \otimes b \to ab$, we obtain an exact sequence:

$$0 \longrightarrow \underline{L}_j(\cdot) \otimes Q \xrightarrow{I^*} \underline{L}_j(C) \otimes Q \longrightarrow \underline{L}_j(C \times C) \otimes Q \longrightarrow 0$$

$N_j \otimes Q$, a module over the trivial Green functor, is a $(\text{triv} \otimes Z[1/2])$-module by Remark 6.1.8. Proposition 6.6.6 and the Dress lemma combine to give an exact sequence:

$$0 \longrightarrow N_j(\cdot) \otimes Q \xrightarrow{I^*} N_j(C) \otimes Q \longrightarrow N_j(C \times C) \otimes Q \longrightarrow 0$$

Then arguing as in the proof of the second part of Theorem 6.2.11 we obtain that $I^*: U_j(\cdot) \otimes Q \to U_j(C) \otimes Q$ is injective.

The exact sequence of Mackey functors: $0 \to \ker(f) \to U_j \otimes Z[1/2] \to U_j \otimes Q$ yields a commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & (\ker(f))(\cdot) & \longrightarrow & U_j(\cdot) \otimes Z[1/2] & \longrightarrow & U_j(\cdot) \otimes Q \\ & \downarrow I^* & & \downarrow I^* & & \downarrow I^* \\ 0 \longrightarrow & (\ker(f))(C) & \longrightarrow & U_j(C) \otimes Z[1/2] & \longrightarrow & U_j(C) \otimes Q \end{array}$$

The rows are exact and the two outer vertical maps have been shown to be injective. A diagram chase reveals $I^*: U_j(\cdot) \otimes Z[1/2] \to U_j(C) \otimes Z[1/2]$ is injective, proving the claim.

By Proposition 6.2.9 (3) the product of the restriction maps $I^*: V(\cdot) \otimes Z[1/2] \to V_j(C) \otimes Z[1/2]$ is injective. The exact sequence of 6.2.6 yields a commutative diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & U_j(\cdot) \otimes Z[1/2] & \longrightarrow & s_j(\cdot) \otimes Z[1/2] & \longrightarrow & V_j(\cdot) \otimes Z[1/2] & \longrightarrow 0 \\ & \downarrow I^* & & \downarrow I^* & & \downarrow I^* & \\ 0 \longrightarrow & U_j(C) \otimes Z[1/2] & \longrightarrow & s_j(C) \otimes Z[1/2] & \longrightarrow & V_j(C) \otimes Z[1/2] & \longrightarrow 0 \end{array}$$

where the rows are exact and, as demonstrated, the two outer vertical maps are injective. The usual diagram chase shows that $I^*: s_j(\cdot) \otimes Z[1/2] \to s_j(C) \otimes Z[1/2]$ is injective which is the conclusion of the second part of Theorem 6.2.13.

Combining Theorems 6.2.12 and 6.2.13 we have

Corollary 6.2.16: Suppose the fundamental group of the manifold s-ad X is finite. Let C be the union of the class of p-elementary subgroups of G, p odd, and the 2-hyperelementary subgroups of G. Then for $j \geq 0$:

1. The sum of the induction maps $\oplus_{H\epsilon C}\, s_j(H) \to s_j(G)$ is surjective.
2. The product of the restriction maps $s_j(G) \to \pi_{H\epsilon C}\, s_j(H)$ is injective.

A simple application of the corollary is:

Corollary 6.2.17: Let $f: M \to X$ be a simple homotopy equivalence of compact oriented manifold pairs such that ∂f is a homeomorphism. Suppose X is connected, $m \geq 6$, and the fundamental group of X is finite. Then f is homotopic to a homeomorphism relative to ∂X if and only if for every covering $\tilde{X}$ of X with $\pi_1(\tilde{X})$ p-elementary, p odd, or 2-hyperelementary, the lifting $\tilde{f}: \tilde{M} \to \tilde{X}$ of f is homotopic to a homeomorphism relative $\partial\tilde{X}$.

Proof: The "only if" portion of the corollary is trivial. Let $\bar{X}$ be the universal cover of X and $G = \pi_1(X)$. Then $s_0(G) = s_{TOP}(\bar{X}/G, \bar{X}/G) = s_{TOP}(X, \partial X)$. Noting that $f: M \to X$ is homotopic to a homeomorphism relative to ∂X if and only if f represents the zero element in the group $s_{TOP}(X, \partial X)$, the result follows from Corollary 6.2.16 (2).

The corollary above generalizes to s-ads in the obvious manner.

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