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Classifying pairs of lagrangians in a hermitian vector space

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Abstract

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A new elementary geometric proof, exploiting the positive curvature of complex projective space, of a basic lemma in the theory of lagrangian pairs in hermitian vector space is presented. Applications to the classification of such pairs and to symplectic vector bundles possessing a pair of lagrangian subbundles are given.

Keywords: Lagrangian subspace, Souriau matrix, positive curvature, symplectic vector bundle.

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Introduction

In this paper we study pairs of lagrangian subspaces in a finite dimensional hermitian vector space.

Given a pair (L_1, L_2) of lagrangian subspaces in hermitian vector space $(V, \langle\cdot\,,\cdot\rangle)$ together with orthonormal bases $B_1=\{e_1,\dots,e_n\}$ and $B_2=\{f_1,\dots,f_n\}$ for L_1 and L_2 respectively, a basic invariant of the pair (L_1, L_2) is its *Souriau matrix*, the $n\times n$ complex matrix AA^{t} where $A_{ij}=\langle f_j, e_i\rangle$ and A^{t} is the transpose of A. The conjugacy class of AA^{t} is independent of the choice of orthonormal bases and so we define the *characteristic polynomial* of the pair (L_1, L_2) to be the characteristic polynomial of this matrix.

While the unitary group of V permutes the set of lagrangian subspaces transitively, this action is not doubly transitive.

Theorem 1.9 asserts that pairs of lagrangian subspaces in $(V, \langle\cdot\,,\cdot\rangle)$ are determined up to unitary equivalence by their characteristic polynomials. As a consequence a homogeneous space description of the set of all pairs of lagrangian subspaces with a fixed characteristic polynomial is obtained in Theorem 1.11.

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The starting point for this classification is Theorem 1.4, the "diagonalization lemma", which asserts that for a given pair of lagrangian subspaces (L_1, L_2) there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for L_1 and unit complex numbers α_i, $i = 1, \ldots, n$ such that $\{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is an orthonormal basis for L_2. Note that the Souriau matrix is diagonal with respect to these bases. We offer a new elementary geometric proof of this basic lemma which exploits the positive curvature of complex projective space. Some immediate applications of Theorem 1.4 to symplectic geometry are also given in Section 1 (see Corollaries 1.5, 1.7, 1.8 and Proposition 1.6).

In Section 2 we consider a pair of lagrangian subspaces (L_1, L_2) in a symplectic vector space (E, ω). Theorem 2.5 provides a homogeneous space description of the set $C_P^+(E; L_1, L_2)$ of all positive compatible complex structures J on (E, ω) such that the characteristic polynomial of the pair (L_1, L_2) with respect to the hermitian structure in E given by $g(x, y) = \omega(Jx, y) + \iota\omega(x, y)$ is a fixed complex polynomial P. Applications of these results to symplectic vector bundles are given in Section 3.

The theory expounded here grew from the need to study intersections of real subvarieties in a singular complex projective variety. I am grateful to C. Frohman for useful discussions.

1. The geometry lagrangian pairs

A hermitian vector space is a triple $(V, J, \langle\cdot\,,\cdot\rangle)$ where V is a real vector space, J is a complex structure on V, and $\langle\cdot\,,\cdot\rangle$ is a hermitian form on the complex vector space (V, J). In this section it will be convenient to omit J from the notation since we consider a fixed complex structure on the underlying vector space. A real vector subspace L of V is said to be *totally real* if $\langle u, v\rangle \in \mathbb{R}$ for all $u, v \in L$. L is a *lagrangian subspace* if it is totally real and maximal with respect to this property. In the case V is finite dimensional, which will be assumed henceforth, the dimension of a lagrangian subspace is one-half the real dimension of V.

The real part of $\langle\cdot\,,\cdot\rangle$, denoted $\langle\cdot\,,\cdot\rangle_r$ is a Euclidean inner product on V. Evidently, if $L \subset V$ is a lagrangian subspace, then any real basis for L which is orthonormal with respect to $\langle\cdot\,,\cdot\rangle_r$ is also a complex basis for V which is orthonormal with respect to the hermitian inner product $\langle\cdot\,,\cdot\rangle$.

Let (L_1, L_2) be a pair of lagrangian subspaces of $(V, \langle\cdot\,,\cdot\rangle)$ and suppose $B_1 = \{e_1, \ldots, e_n\}$ and $B_2 = \{f_1, \ldots, f_n\}$ are orthonormal bases for L_1 and L_2 respectively. Let A be the $n \times n$ complex matrix $A_{ij} = \langle f_j, e_i\rangle$. Clearly A is a unitary matrix as it is the matrix with respect to B_1 of the unitary transformation $T: V \to V$ defined by $Te_j = f_j$.

Definition 1.1 (see [2]). The *Souriau matrix* of (L_1, L_2) with respect to (B_1, B_2) is AA^t where A^t is the transpose of A.

Note that AA^t is both unitary and symmetric. Now suppose that $B_1' = \{e_1', \ldots, e_n'\}$ and $B_2' = \{f_1', \ldots, f_n'\}$ are other orthonormal bases for L_1 and L_2 respectively and

$A'(A')^t$ the corresponding Souriau matrix where $A' = \langle f'_j, e'_i \rangle$. Let P, Q be the $n \times n$ matrices $P_{ij} = \langle e_i, e'_j \rangle$, $Q_{ij} = \langle f'_i, f_j \rangle$. Since L_1 and L_2 are lagrangian subspaces P and Q are real orthogonal matrices; furthermore, $A = PA'Q$. Hence

$$AA^t = (PA'Q)(PA'Q)^t = PA'(A')^t P^t.$$

Thus AA^t is conjugate to $A'(A')^t$ via an orthogonal matrix. It follows that the characteristic polynomial of a Souriau matrix for (L_1, L_2) is independent of the choice of orthonormal bases (B_1, B_2). Accordingly, we make the following definition.

Definition 1.2. The *characteristic polynomial* of the pair (L_1, L_2), denoted $\sigma(L_1, L_2)$, is the characteristic polynomial of a Souriau matrix for (L_1, L_2).

In particular $\sigma(L_1, L_2)$ is a monic complex polynomial of degree equal to the complex dimension of V; furthermore, since a Souriau matrix for (L_1, L_2) is unitary its roots must lie on the unit circle in $\mathbb{C}$.

The multiplicative group of nonzero complex numbers acts on the nonzero vectors in V by scalar multiplication. The complex projective space, $\mathrm{CP}(V)$, is the quotient of this action. It is a familiar example of a compact complex manifold of complex dimension $\dim_{\mathbb{C}} V - 1$. Similarly, if W is a real vector space, the real projective space $\mathrm{RP}(W)$ is obtained as the quotient of the action of the multiplicative group of nonzero real numbers on the nonzero vectors of W; it is a compact manifold of dimension $\dim W - 1$. Points of $\mathrm{CP}(V)$ will be written as $[v]$ where $v \in V$ while points of $\mathrm{RP}(W)$ will be written as $[w]_{\mathbb{R}}$ where $w \in W$. If $W \subset V$ is a real vector subspace there is a natural map $\mathrm{RP}(W) \to \mathrm{CP}(V)$ given by $[w]_{\mathbb{R}} \to [w]$. When $W \subset V$ is totally real this map is an embedding: If $[w_1] = [w_2]$ where $w_1, w_2 \in W$, then $\lambda w_1 = w_2$ for some nonzero complex number λ. Since W is totally real $\langle w_2, w_1 \rangle = \lambda \langle w_1, w_1 \rangle \in \mathbb{R}$ and consequently λ is real and so $[w_1]_{\mathbb{R}} = [w_2]_{\mathbb{R}}$. In the case W is totally real we will identify $\mathrm{RP}(W)$ with its image in $\mathrm{CP}(V)$. When $L \subset V$ is a lagrangian subspace $\mathrm{RP}(L) \subset \mathrm{CP}(V)$ will be called a *projective lagrangian*.

Lemma 1.3. *Any two projective lagrangians in* $\mathrm{CP}(V)$ *must have nonempty intersection.*

Proof. Endow $\mathrm{CP}(V)$ with the Fubini-Study metric. For any lagrangian subspace $L \subset V$ the projective lagrangian $\mathrm{RP}(L)$ is a totally geodesic submanifold of $\mathrm{CP}(V)$. Indeed, $\mathrm{RP}(L)$ is the fixed point set of the isometry of $\mathrm{CP}(V)$ which is induced by complex conjugation with respect to L, i.e., the $\mathbb{R}$-linear map $\sigma: V \to V$ uniquely defined by $\sigma(w_1 + Jw_2) = w_1 - Jw_2$ for $w_1, w_2 \in L$.

It is well known that $\mathrm{CP}(V)$ has positive sectional curvature. By a theorem of Frankel [1, Theorem 1, p. 169] any two compact totally geodesic submanifolds of complementary dimension in a connected complete manifold of positive sectional curvature must have nonempty intersection. In particular this applies to any two projective lagrangians in $\mathrm{CP}(V)$. □

We now prove a "diagonalization lemma" for a Souriau matrix of a pair of lagrangian subspaces.

Theorem 1.4 (Diagonalization lemma). *Let (L_1, L_2) be a pair of lagrangian subspaces in a hermitian vector space $(V, \langle\cdot,\cdot\rangle)$ of complex dimension n. There exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for L_1 and unit complex numbers α_i, $i=1,\ldots,n$ such that $\{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is an orthonormal basis for L_2. Furthermore, the numbers $\alpha_1^2, \ldots, \alpha_n^2$ (including multiplicities) are precisely the roots of the characteristic polynomial of the pair (L_1, L_2).*

To justify the name given to this result, note that the Souriau matrix of (L_1, L_2) with respect to the bases provided by the theorem is the matrix $\mathrm{diag}(\alpha_1^2, \ldots, \alpha_n^2)$, the diagonal matrix with diagonal entries $\alpha_1^2, \ldots, \alpha_n^2$.

Proof. By Lemma 1.3 the projective lagrangians $\mathrm{RP}(L_1)$ and $\mathrm{RP}(L_2)$ must have nonempty intersection in $\mathrm{CP}(V)$. Hence there is a nonzero vector $v \in L_1$ and a nonzero complex number λ such that $\lambda v \in L_2$. Let $e_1 = v/\|v\|$ and $\alpha_1 = \lambda/|\lambda|$. Then $e_1 \in L_1$ is of unit length and α_1 is a unit complex number such that $\alpha_1 e_1 \in L_2$. Let V' be the hermitian orthogonal complement of e_1, i.e., V' is the complex hyperplane in V defined by $V' = \{u \in V \mid \langle u, e_1\rangle = 0\}$. Let $L_i' = L_i \cap V'$ for $i = 1, 2$. L_1' is a lagrangian subspace of V' because $\{e_1\}$ can be extended to an orthonormal basis $\{e_1, u_2, \ldots, u_n\}$ for L_1 and L_1' is the real span of $\{u_2, \ldots, u_n\}$. Similarly L_2' is also a lagrangian subspace of V'. Applying mathematical induction, we obtain an orthonormal basis $\{e_1, \ldots, e_n\}$ for L_1 and unit complex numbers α_i, $i=1,\ldots,n$ such that $\{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is an orthonormal basis for L_2. The Souriau matrix of (L_1, L_2) with respect to these bases is the diagonal matrix $\mathrm{diag}(\alpha_1^2, \ldots, \alpha_n^2)$. Its characteristic polynomial is $\sigma(L_1, L_2) = \prod_{i=1}^n (X - \alpha_i^2)$ and thus the numbers $\alpha_1^2, \ldots, \alpha_n^2$ (including multiplicities) are precisely the roots of $\sigma(L_1, L_2)$. □

One immediate consequence of the proof of Theorem 1.4 is:

Corollary 1.5. *Let (L_1, L_2) be a pair of lagrangian subspaces in a hermitian vector space $(V, \langle\cdot,\cdot\rangle)$. There exists a complex hyperplane H such that $(L_1 \cap H, L_2 \cap H)$ is a pair of lagrangian subspaces in H.*

Given a pair of lagrangian subspaces in a hermitian vector space $(V, \langle\cdot,\cdot\rangle)$, we obtain the following *canonical orthogonal decomposition* of L_1. Let $\alpha_{i_1}^2, \ldots, \alpha_{i_m}^2$ be the *distinct* roots of the characteristic polynomial $\sigma(L_1, L_2)$. Define subspaces W_k of L_1 for $k = 1, \ldots, m$ by

$$W_k = \{v \in L_1 \mid \alpha_{i_k} v \in L_2\}.$$

Note that W_k is independent of the choice of square root of $\alpha_{i_k}^2$.

Proposition 1.6 (Canonical orthogonal decomposition). *L_1 decomposes as an orthogonal direct sum $L_1 = W_1 \oplus \cdots \oplus W_m$. The dimension of W_k is the multiplicity of the root $\alpha_{i_k}^2$ of $\sigma(L_1, L_2)$.*

Proof. Let $\alpha=\alpha_{i_s}$, $\beta=\alpha_{i_t}$ and suppose $u\in W_s$, $v\in W_t$. Then αu, $\beta v\in L_2$. Since L_1 and L_2 are lagrangian subspaces the quantities $\langle u, v\rangle$ and $\langle \alpha u, \beta v\rangle$ are real. If $\langle u, v\rangle\neq 0$, then the equality $\langle \alpha u, \beta v\rangle=\alpha\bar{\beta}\langle u, v\rangle$ implies that $\alpha\bar{\beta}\in\mathbb{R}\cap S^1=\{\pm 1\}$. Consequently $\alpha^2=\beta^2$ and so $W_s=W_t$. Thus W_s and W_t are orthogonal if $s\neq t$.

By Theorem 1.4 there is an orthonormal basis $\{e_1,\ldots,e_n\}$ for L_1 such that $\alpha_i e_i\in L_2$. Let $I(j)=\{k\mid \alpha_k^2=\alpha_{i_j}^2\}$. For $v\in L_1$ let

$$w_j=\sum_{k\in I(j)}\langle v, e_k\rangle e_k.$$

Then $w_j\in W_j$ and $v=\sum_{j=1}^m w_j$. Hence $W_1\oplus\cdots\oplus W_m$ spans L_1. The set $\{e_k\mid k\in I(j)\}$ is a basis for W_j and thus dim W_j is equal to the cardinality of $I(j)$ which is also the multiplicity of α_{i_j} as a root of $\sigma(L_1, L_2)$. □

Note that for (L_1, L_2) as above, L_2 also has a natural decomposition as an orthogonal direct sum:

$$L_2=\alpha_{i_1}W_1\oplus\cdots\oplus\alpha_{i_m}W_m.$$

Corollary 1.7. *The intersection of two projective lagrangians* $\mathrm{RP}(L_1)$ *and* $\mathrm{RP}(L_2)$ *in* $\mathrm{CP}(V)$ *is a disjoint union of real projective planes of dimensions* $k_1-1,\ldots,k_m-1$ *where* $k_1,\ldots,k_m$ *are the multiplicities of the roots of the characteristic polynomial* $\sigma(L_1, L_2)$.

Proof. Let $L_1=W_1\oplus\cdots\oplus W_m$ be the canonical orthogonal decomposition given by Proposition 1.6. The W_k are totally real subspaces of V and

$$\mathrm{RP}(L_1)\cap\mathrm{RP}(L_2)=\bigcup_{k=1}^{m}\mathrm{RP}(W_k)$$

is a disjoint union. □

Given a hermitian vector space $(V, \langle\cdot\,,\cdot\rangle)$ and a lagrangian subspace L we define the *unitary group*, $U(V)$, to be the group of all unitary transformations of V and the *orthogonal group* with respect to L, $O_L(V)$, to be the subgroup of $U(V)$ which preserves L. If (L_1, L_2) is a pair of lagrangian subspaces let $L_1=W_1\oplus\cdots\oplus W_m$ be the corresponding canonical orthogonal decomposition. Define $V_k=W_k\oplus JW_k$ for $k=1,\ldots,m$. Then V_k is a complex subspace of V, W_k is a lagrangian subspace of V_k and $V=V_1\oplus\cdots\oplus V_m$ is an orthogonal decomposition of V as a hermitian vector space. An element $\phi\in O_{L_1}(V)\cap O_{L_2}(V)$ must preserve these decompositions of L_1 and V respectively. Conversely any $\phi\in U(V)$ which preserves these decompositions belongs to $O_{L_1}(V)\cap O_{L_2}(V)$. We conclude:

Corollary 1.8. *There is an isomorphism* $O_{L_1}(V)\cap O_{L_2}(V)\cong\prod_{k=1}^m O_{W_k}(V_k)$ *given by* $\phi\mapsto(\phi|_{V_1},\ldots,\phi|_{V_m})$.

We now show that pairs of lagrangian subspaces in a hermitian vector space $(V, \langle\cdot\,,\cdot\rangle)$ are classified up to unitary equivalence by their characteristic polynomial.

Theorem 1.9 (Classification theorem). *Suppose (L_1, L_2) and (L_1', L_2') are pairs of lagrangian subspaces in hermitian vector space $(V, \langle \cdot, \cdot \rangle)$. There exist a unitary transformation $\phi \in U(V)$ such that $\phi(L_i) = L_i'$ for $i = 1, 2$ if and only if $\sigma(L_1, L_2) = \sigma(L_1', L_2')$.*

Proof. Suppose ϕ exists. By Theorem 1.4 there is an orthonormal basis $B_1 = \{e_1, \ldots, e_n\}$ for L_1 and unit complex numbers α_i where $i = 1, \ldots, n$ such that $B_2 = \{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is an orthonormal basis for L_2. Since ϕ is unitary $B_1' = \{\phi(e_1), \ldots, \phi(e_n)\}$ is an orthonormal basis for L_1' and $B_2' = \{\alpha_1 \phi(e_1), \ldots, \alpha_n \phi(e_n)\}$ is an orthonormal basis for L_2'. The Souriau matrix of (L_1, L_2) with respect to (B_1, B_2) is $\operatorname{diag}(\alpha_1^2, \ldots, \alpha_n^2)$ which is also the Souriau matrix of (L_1', L_2') with respect to (B_1', B_2'). Hence $\sigma(L_1, L_2) = \sigma(L_1', L_2')$.

Conversely, suppose $\sigma(L_1, L_2) = \sigma(L_1', L_2')$. Let $\alpha_1, \ldots, \alpha_n$ be the roots of this polynomial (including multiplicities). By Theorem 1.4 there is an orthonormal basis $\{e_1, \ldots, e_n\}$ for L_1 and an orthonormal basis $\{f_1, \ldots, f_n\}$ for L_1' such that $\{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is an orthonormal basis for L_2 and $\{\alpha_1 f_1, \ldots, \alpha_n f_n\}$ is an orthonormal basis for L_2'. Define $\phi : V \to V$ by $\phi e_j = f_j, j = 1, \ldots, n$. Then ϕ is unitary and $\phi(L_i) = L_i'$ for $i = 1, 2$. □

In what follows P will be a monic complex polynomial of degree n whose roots lie on the unit circle in $\mathbb{C}$ and let $(V, \langle \cdot, \cdot \rangle)$ be a hermitian vector space of complex dimension n.

We will use the symbol $\mathscr{L}$ to denote the set of all lagrangian subspaces of V. Define

$$\mathscr{L}_P = \{(L_1, L_2) \in \mathscr{L} \times \mathscr{L} \mid \sigma(L_1, L_2) = P\}.$$

$\mathscr{L}_P$ is the set of all pairs of lagrangian subspaces of V with characteristic polynomial P. $\mathscr{L}_P$ is not empty, indeed:

Lemma 1.10. *Suppose 1 is a root of P of multiplicity m where $m = 0$ means 1 is not a root. Given $L_1 \in \mathscr{L}$ and a subspace $W \subset L_1$ with* dim $W = m$ *there exists an $L_2 \in \mathscr{L}$ such that $L_1 \cap L_2 = W$ and $\sigma(L_1, L_2) = P$.*

Proof. Factor P as $P(X) = (X-1)^m \prod_{j=m+1}^{n} (X - \alpha_j^2)$ where $\alpha_j^2 \neq 1$ for $j = m+1, \ldots, n$. Let $B = \{e_1, \ldots, e_m\}$ be an orthonormal basis for W (B is empty if $m = 0$). Extend B to an orthonormal basis $\{e_1, \ldots, e_n\}$ for L_1. Define L_2 to be the real span of $\{e_1, \ldots, e_m, \alpha_{m+1} e_{m+1}, \ldots \alpha_n e_n\}$. Then L_2 is a lagrangian subspace of V and $\sigma(L_1, L_2) = P$. Clearly $W \subset L_1 \cap L_2$. Suppose $v \in L_1 \cap L_2$. Then $v = w + \sum_{j=m+1}^{n} t_j \alpha_j e_j$ where $w \in W$ and $t_j \in \mathbb{R}$. Since L_1 is a lagrangian subspace $\langle v, e_j \rangle = t_j \alpha_j \in \mathbb{R}$. The coefficients t_j must be zero otherwise $\alpha_j \in \mathbb{R} \cap S^1 = \{\pm 1\}$ would imply $\alpha_j^2 = 1$. Hence $W = L_1 \cap L_2$. □

We can now characterize $\mathscr{L}_P$ as a homogeneous space.

Theorem 1.11. *For any $(L_1, L_2) \in \mathscr{L}_P$ there is a natural isomorphism:*

$$U(V)/O_{L_1}(V) \cap O_{L_2}(V) \cong \mathscr{L}_P.$$

Proof. By Theorem 1.9 the unitary group $U(V)$ acts transitively on $\mathscr{L}_P$. The stabilizer of $(L_1, L_2) \in \mathscr{L}_P$ is $U(V)_{(L_1,L_2)} = O_{L_1}(V) \cap O_{L_2}(V)$. $\square$

Remark. Projection onto the first factor yields a map $\mathscr{L}_P \to \mathscr{L}$ which can be identified with the natural map $U(V)/O_{L_1}(V) \cap O_{L_2}(V) \to U(V)/O_{L_1}(V)$. This map is the projection of a smooth fiber bundle. By Corollary 1.8 its fiber, $O_{L_1}(V)/O_{L_1}(V) \cap O_{L_2}(V)$, is a real flag manifold.

2. Lagrangian pairs in a symplectic vector space

Let (E, ω) be a symplectic vector space, i.e., E is a real vector space and ω is a nondegenerate 2-form. $L \subset E$ is a *lagrangian subspace* if it is self-annihilating for ω and maximal with respect to this property.

Definition 2.1. A complex structure J on E is *compatible* if ω is J invariant and *positive* if in addition $\langle x, y\rangle_J = \omega(Jx, y) + \iota\omega(x, y)$ defines a (positive definite) hermitian form on (E, J).

The set of positive complex structures on (E, ω) will be denoted $C^+(E)$. Note that $L \subset E$ is a lagrangian subspace if and only if for some $J \in C^+(E)$ it is a maximal totally real subspace of $(E, \langle\cdot\,,\cdot\rangle_J)$. Since we will consider different positive complex structures on a fixed underlying symplectic vector space (E, ω), the notation of the previous section will be augmented as follows: Let L_1, L_2, L be lagrangian subspaces of (E, ω) and $J \in C^+(E)$. $U(E, J)$ is the unitary group of $(E, \langle\cdot\,,\cdot\rangle_J)$. $O_L(E, J)$ is the orthogonal group with respect to L of $(E. \langle\cdot\,,\cdot\rangle_J)$. $\sigma(L_1, L_2; J)$ is the characteristic polynomial of (L_1, L_2) computed in $(E, \langle\cdot\,,\cdot\rangle_J)$.

The *symplectic group*, $\mathrm{Sp}(E)$, is the group of all real linear automorphisms of E which preserve ω.

In what follows we will assume that (E, ω) is finite dimensional and that (L_1, L_2) is a fixed pair of lagrangian subspaces in E.

For any complex polynomial G and complex number λ define multi(G, λ) to be zero if λ is not a root of G and to be the multiplicity of λ if λ is a root of G.

Lemma 2.2. *If* $J \in C^+(E)$, *then* $\dim L_1 \cap L_2 = \mathrm{multi}(\sigma(L_1, L_2; J), 1)$.

Proof. We use the hermitian structure $\langle\cdot\,,\cdot\rangle_J$ on (E, J). Suppose $\dim L_1 \cap L_2 > 0$. If $e_1 \in L_1 \cap L_2$ is a unit vector, extend $\{e_1\}$ to an orthonormal basis $B_1 = \{e_1, \ldots, e_n\}$ for L_1 and also to an orthonormal basis $B_2 = \{e_1, f_2, \ldots, f_n\}$ for L_2. The Souriau matrix S of (L_1, L_2) with respect to (B_1, B_2) has $S_{11} = 1$ and $S_{1j} = S_{j1} = 0$ for $j > 1$. Hence 1 is a root of the characteristic polynomial of S which by definition is $\sigma(L_1, L_2; J)$.

Suppose $\beta_1^2 = 1, \beta_2^2, \ldots, \beta_m^2$ are the distinct roots of $\sigma(L_1, L_2; J)$. By Proposition 1.6 there is an orthogonal decomposition $L_1 = W_{\beta_1} \oplus \cdots \oplus W_{\beta_m}$ where $\dim W_{\beta_j} = \text{multi}(\sigma(L_1, L_2; J), \beta_j^2)$. $W_{\beta_1} = \{v \in L_1 \mid \sqrt{1}\, v \in L_2\} = L_1 \cap L_2$, completing the proof. □

For the remainder of this section P will be a monic complex polynomial of degree $\frac{1}{2}\dim_{\mathbb{R}} E$ whose roots lie on the unit circle in $\mathbb{C}$. Define for $J \in C^+(E)$

$$X_P(L_1, L_2; J) = \{A \in \text{Sp}(E) \mid \sigma(A^{-1}(L_1), A^{-1}(L_2); J) = P\}.$$

Lemma 2.3. *$X_P(L_1, L_2; J)$ is not empty if and only if* $\text{multi}(P, 1) = \dim L_1 \cap L_2$.

Proof. Suppose $X_P(L_1, L_2; J)$ is not empty, say $A \in X_P(L_1, L_2; J)$. Then $\dim L_1 \cap L_2 = \dim A^{-1}(L_1) \cap A^{-1}(L_2) = \text{multi}(P, 1)$, the last equality by Lemma 2.2.

Suppose $\text{multi}(P, 1) = \dim L_1 \cap L_2$. By Lemma 1.10 there exists a lagrangian subspace L_2' such that $L_1 \cap L_2' = L_1 \cap L_2$ and $\sigma(L_1, L_2'; J) = P$. By [3, Proposition 2.2.18] there is $A \in \text{Sp}(E)$ such that $A(L_1) = L_1$ and $A(L_2') = L_2$ (in fact $A|_{L_1}$ can taken to be the identity). Then $A \in X_P(L_1, L_2; J)$. □

Proposition 2.4. *For any $A_0 \in X_P = X_P(L_1, L_2; J)$, X_P is the double coset $\text{Sp}(E)_{(L_1,L_2)} A_0 U(E, J)$ where $\text{Sp}(E)_{(L_1,L_2)} = \{A \in \text{Sp}(E) \mid A(L_i) = L_i,\ i = 1, 2\}$.*

Proof. Suppose $B \in X_P$. Then $\sigma(B^{-1}(L_1), B^{-1}(L_2); J) = P$. By Theorem 1.9 there exists $\phi \in U(E, J)$ such that $\phi B^{-1}(L_i) = A_0^{-1}(L_i)$ for $i = 1, 2$. Thus $A_0 \phi B^{-1} \in \text{Sp}(E)_{(L_1,L_2)}$ and so $B \in \text{Sp}(E)_{(L_1,L_2)} A_0 U(E, J)$.

Conversely, suppose $B = \psi A_0 \phi$ where $\psi \in \text{Sp}(E)_{(L_1,L_2)}$ and $\phi \in U(E, J)$. Then $B^{-1}(L_i) = \phi^{-1} A_0^{-1} \psi^{-1}(L_i) = \phi^{-1} A_0^{-1}(L_i)$ for $i = 1, 2$. Again by Theorem 1.9 $\sigma(B^{-1}(L_1), B^{-1}(L_2); J) = \sigma(A_0^{-1}(L_1), A_0^{-1}(L_2); J)$ and so $B \in X_P$. □

We now determine the set $C_P^+(L_1, L_2)$ of all positive complex structures J such that $\sigma(L_1, L_2; J) = P$.

Theorem 2.5. *$C_P^+(L_1, L_2)$ is not empty if and only if* $\text{multi}(P, 1) = \dim L_1 \cap L_2$. *If $J \in C_P^+(L_1, L_2)$, then there is an isomorphism*:

$$\text{Sp}(E)_{(L_1,L_2)} / O_{L_1}(E, J) \cap O_{L_2}(E, J) \cong C_P^+(L_1, L_2)$$

where $\text{Sp}(E)_{(L_1,L_2)} = \{A \in \text{Sp}(E) \mid A(L_i) = L_i,\ i = 1, 2\}$.

Proof. If $C_P^+(L_1, L_2)$ is not empty, then Lemma 2.2 implies that $\text{multi}(P, 1) = \dim L_1 \cap L_2$.

For any $J \in C^+(E)$ and $A \in \text{Sp}(E)$ we have $AJA^{-1} \in C^+(E)$. Since $A^{-1} : (E, \langle\cdot,\cdot\rangle_{AJA^{-1}}) \to (E, \langle\cdot,\cdot\rangle_J)$ is an isometry of hermitian vector spaces it follows that $\sigma(A^{-1}(L_1), A^{-1}(L_2); J) = \sigma(L_1, L_2; AJA^{-1})$. Suppose $\text{multi}(P,1) = \dim L_1 \cap L_2$.

Choose any $J_0 \in C^+(E)$. By Lemma 2.3 there exists $A_0 \in \mathrm{Sp}(E)$ such that $\sigma(A_0^{-1}(L_1), A_0^{-1}(L_2); J_0) = P$. Then $J_1 = A_0 J_0 A_0^{-1} \in C_P^+(L_1, L_2)$ and so in particular $C_P^+(L_1, L_2)$ is not empty.

For $J \in C_P^+(L_1, L_2)$ define $\theta_J : X_P(L_1, L_2; J) \to C_P^+(L_1, L_2)$ by $\theta_J(A) = AJA^{-1}$. θ_J is surjective since any $J' \in C^+(E)$ is of the form AJA^{-1} for some $A \in \mathrm{Sp}(E)$. Note that $\theta_J(A_1) = \theta_J(A_2)$ if and only if $A_2^{-1}A_1 \in U(E, J)$. Thus θ_J induces a bijection $X_P(L_1, L_2; J)/U(E, J) \cong C_P^+(L_1, L_2)$. By Proposition 2.4 $X_P(L_1, L_2; J)$ is the double coset $\mathrm{Sp}(E)_{(L_1,L_2)}U(E, J)$ and thus

$$\begin{aligned} X_P(L_1, L_2; J)/U(E, J) &= [\mathrm{Sp}(E)_{(L_1,L_2)}U(E, J)]/U(E, J) \\ &= \mathrm{Sp}(E)_{(L_1,L_2)}/\mathrm{Sp}(E)_{(L_1,L_2)} \cap U(E, J) \\ &= \mathrm{Sp}(E)_{(L_1,L_2)}/O_{L_1}(E, J) \cap O_{L_2}(E, J). \qquad \square \end{aligned}$$

The special case of a pair (L_1, L_2) of *transverse* lagrangian subspaces, i.e., $L_1 \cap L_2 = (0)$, is of particular interest.

If (L_1, L_2) is transverse, then $\mathrm{Sp}(E)_{(L_1,L_2)}$ can be identified with $\mathrm{GL}(n, \mathbb{R})$, the group of $n \times n$ invertible real matrices where $2n = \dim_{\mathbb{R}} E$. By Corollary 1.8 the subgroup $O_{L_1}(E, J) \cap O_{L_2}(E, J)$ of $\mathrm{Sp}(E)_{(L_1,L_2)}$ can be identified with the block diagonal embedding of $O(k_1) \times \cdots \times O(k_m)$ in $\mathrm{GL}(n, \mathbb{R})$ where $k_1, \ldots, k_m$ are the multiplicities of $\sigma(L_1, L_2; J)$ and $O(k)$ is the group of $k \times k$ orthogonal matrices. Note that $k_1 + \cdots + k_m = n$.

We express these observations as follows. Suppose 1 is not a root of P and P has multiplicities $k_1, \ldots, k_m$.

Corollary 2.6. *If (L_1, L_2) is transverse, then there is a diffeomorphism*:

$$C_P^+(L_1, L_2) \cong \mathrm{GL}(n, \mathbb{R})/O(k_1) \times \cdots \times O(k_m).$$

Since the inclusion of $O(n)$ in $\mathrm{GL}(n, \mathbb{R})$ is a homotopy equivalence we conclude:

Corollary 2.7. *If (L_1, L_2) is transverse, then $C_P^+(L_1, L_2)$ is homotopy equivalent to the real flag manifold $O(n)/O(k_1) \times \cdots \times O(k_m)$.*

In particular $C_P^+(L_1, L_2)$ is connected if (L_1, L_2) is transverse because the real flag manifolds are connected.

3. Applications to vector bundles

A *symplectic vector bundle* $(\mathscr{E}, \omega)$ over a smooth manifold M consists of a smooth real vector bundle $\mathscr{E}$ over M together with a smooth field ω of symplectic forms on the fibres of $\mathscr{E}$.

A vector subbundle $\eta \subset \mathscr{E}$ is a *lagrangian subbundle* if for each $x \in M$ the fiber η_x is a lagrangian subspace of $(\mathscr{E}_x, \omega_x)$. $(\mathscr{E}, \omega)$ admits a lagrangian subbundle if and only if the structure group of $\mathscr{E}$ can be reduced from $\mathrm{Sp}(2n, \mathbb{R})$ to $O(n)$ where the fiber dimension of $\mathscr{E}$ is $2n$.

Given a symplectic vector bundle $(\mathscr{E}, \omega)$ there is an associated convex cone bundle $\Sigma(\mathscr{E})$ called the *Siegel bundle* of $\mathscr{E}$ whose fiber over $x \in M$ is the space $C^+(\mathscr{E}_x)$ of positive complex structures on $(\mathscr{E}_x, \omega_x)$ (see [3, p. 59]). A positive complex structure on $(\mathscr{E}, \omega)$ is equivalent to a smooth cross section of $\Sigma(\mathscr{E})$.

A pair (η_1, η_2) of lagrangian subbundles of the symplectic vector bundle $(\mathscr{E}, \omega)$ will be called *regular* if the intersection $\eta_1 \cap \eta_2$ is a subbundle of $\mathscr{E}$, equivalently, if the intersection $(\eta_1)_x \cap (\eta_2)_x$ has constant dimension.

In what follows P will be a monic complex polynomial of degree equal to one-half the fiber dimension of $\mathscr{E}$ and whose roots lie on the unit circle in $\mathbb{C}$. Given a regular pair (η_1, η_2) of lagrangian subbundles of $\mathscr{E}$ such that $\mathrm{multi}(P, 1) = \dim(\eta_1)_x \cap (\eta_2)_x$, we can define $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ to be the subbundle of $\Sigma(\mathscr{E})$ whose fiber over $x \in M$ is $C_P^+((\eta_1)_x, (\eta_2)_x)$. A cross section of $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ is equivalent to a positive complex structure J on $(\mathscr{E}, \omega)$ such that $\sigma((\eta_1)_x, (\eta_2)_x; J) = P$ for all $x \in M$. Since the fiber $C_P^+((\eta_1)_x, (\eta_2)_x)$ is typically not contractible, in general there may be obstructions to the existence of such a J.

Remark. If M is contractible, then $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ is a trivial bundle and thus admits a cross section.

The case when (η_1, η_2) is *transverse*, i.e., $\eta_1 \cap \eta_2$ is the zero bundle, is of particular interest.

Proposition 3.1. *Suppose (η_1, η_2) is transverse. If M has the homotopy type of a 1-complex, then $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ admits a cross section.*

Proof. Let $X \subset M$ be a 1-complex such that the inclusion is a homotopy equivalence. By Corollary 2.7 the fiber $C_P^+((\eta_1)_x, (\eta_2)_x)$ is homotopy equivalent to a real flag manifold and thus connected. It follows that any section of $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ over the 0-skeleton of X can be extended to all of X. Since $X \hookrightarrow M$ is a homotopy equivalence any section over X can be extended to M. $\square$

For an arbitrary manifold M the following result is an easy consequence of the preceding theory.

Proposition 3.2. *Suppose (η_1, η_2) is transverse. Let $\lambda \in S^1 - \{\pm 1\}$. Then there exists a positive complex structure J on $(\mathscr{E}, \omega)$ for which $\lambda \eta_1 = \eta_2$.*

Proof. Let $P(X) = (X - \lambda^2)^n$. By Corollary 2.6 the fiber $C_P^+((\eta_1)_x, (\eta_2)_x)$ of $\Sigma_P(\mathscr{E}; \eta_1, \eta_2)$ is diffeomorphic to $\mathrm{GL}(n, \mathbb{R})/O(n) \cong \mathbb{R}^n$ which is contractible. Hence for any M the bundle $\Sigma_P(\mathscr{E};\, \eta_1,\, \eta_2)$ admits a cross section. Any such cross section defines a positive complex structure on $(\mathscr{E}, \omega)$ with the property $\lambda\eta_1 = \eta_2$. □

The results of this section are, of course, equally valid for continuous symplectic vector bundles over CW complexes.

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