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Degree theory and BMO;
Part I: Compact Manifolds without Boundaries

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§I.0. Introduction

In this paper we consider degree theory for mappings u from one compact smooth n-dimensional manifold X to a connected compact smooth manifold Y of the same dimension. These are manifolds without boundary and which are oriented.

The classical degree counts the "number of times" Y is covered by $u(X)$, taking into account algebraic multiplicity. For instance, if $u \in C^1$ and $y \in Y$ is a regular value of the map u, i.e., $u^{-1}(y)$ consists of a finite number of points $x_1, \ldots, x_k$ at each of which the Jacobian of the map, J_u, in terms of local coordinates (with the given orientation), is nonsingular, then

$$\deg(u, X, y) = \sum_j \operatorname{sgn} \det J_u(x_j).$$

A basic fact is that this degree is independent of the choice of the regular value y, and we then denote this degree by $\deg(u, X, Y)$.

Degree extends to continuous maps u from X to Y because of the fundamental fact that if $u, v \in C^1(X, Y)$, and are close in the C^0 topology, then they have the same degree. Degree theory is often defined directly for continuous maps via the action of the map on nth degree homology.

One of the important properties of degree is that if it is not zero then the map is onto Y. Another basic fact is that the degree is invariant under continuous deformation of the map (homotopy).

For a C^1-map there is an integral formula for the degree. Namely, if μ is a smooth n-form on Y then

$$\deg(u, X, Y) \int_Y \mu = \int_X \mu \circ u. \tag{0.1}$$

(see e.g., L. Nirenberg [1]). This may be expressed using local coordinates by

$$\int_X f(u) \det J_u(x) dx_1 \wedge \ldots \wedge dx_n,$$

if $\mu = f(y)dy_1 \wedge \ldots \wedge dy_n$. In particular, if X and Y are Riemannian manifolds, then

$$\deg(u, X, Y) = \frac{1}{\operatorname{vol}(Y)} \int_X \det J_u(x) d\sigma(x) \tag{0.2}$$

where $d\sigma$ is the volume element on X, and J_u is computed using geodesic normal coordinates at x and geodesic normal coordinates at $u(x)$.

Specializing further, consider $X = \partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^{n+1}$, and $Y = S^n$. Consider $u \in C^1(X, Y)$ and let $\tilde{u}$ be any C^1-extension inside Ω with values in $\mathbb{R}^{n+1}$. There is another integral formula for the degree of u:

$$\deg(u, \partial\Omega, S^n) = \frac{1}{|B|} \int_\Omega \det J_{\tilde{u}} dx_1 \ldots dx_{n+1}, \tag{0.3}$$

where $|B|$ is the volume of the unit ball B in $\mathbb{R}^{n+1}$. Since $\det J_{\tilde{u}}$ is a divergence expression, using Green's theorem, one easily obtains the equality of the two integral formulas.

Formulas (0.2), (0.3) suggest the possibility of extending degree theory to another class of maps — which need not be continuous — namely, maps in appropriate Sobolev spaces. This was done in the 80's:

(a) In connection with their proof of the existence of "large" harmonic maps, H. Brezis and J. M. Coron [1] (see also H. Brezis [2]) were led to consider degree for H^1 maps from S^2 to S^2. This degree is given by the integral on the right hand side of (0.2). To prove that this integral is an integer relies on the fact that smooth maps from S^2 to S^2 are dense in $H^1(S^2, S^2)$ (see R. Schoen and K. Uhlenbeck [1]).

(b) Motivated by a question concerning the Ginzburg-Landau equation (see Boutet de Monvel-Berthier, Georgescu and Purice [1]), L. Boutet de Monvel and O. Gabber introduced a degree for maps $u \in H^{1/2}(S^1, S^1)$. It is the familiar case of (0.2), namely the "change in argument"

$$\deg(u, S^1, S^1) = \frac{1}{2\pi i} \int_{S^1} \frac{du}{u} = \frac{1}{2\pi i} \int_{S^1} \bar{u} du. \tag{0.4}$$

Using the duality between $H^{1/2}$ and $H^{-1/2}$ one sees that this is well defined. (The degree may also be expressed in terms of the Fourier coefficients of u; see Section I.5. It is then transparent that degree makes sense for $u \in H^{1/2}$.) That the expression (0.4) (or the analogue in terms of the Fourier coefficients) is an integer for $u \in H^{1/2}$ is proved by approximation, as above. Alternatively, one may extend $u \in H^{1/2}(S^1, S^1)$ to $\tilde{u} \in H^1(B, \mathbb{R}^2)$, and then use formula (0.3). This $H^{1/2}$ degree is also used in Bethuel, Brezis and Hélein [1].

The natural generalization of (a) is to maps in the Sobolev space $W^{1,n}(X, Y)$, while (b) extends degree to maps in $W^{\frac{n}{n+1}, n+1}(\partial\Omega, S^n)$ — a space slightly larger than $W^{1,n}$. These are borderline spaces: embedding into continuous functions just fails.

In connection with degree for $H^{1/2}(S^1, S^1)$, L. Boutet de Monvel and O. Gabber made the interesting observation that the notion of degree for maps from S^1 to S^1 makes sense for maps in the class VMO: the closure in the BMO(= bounded mean oscillation) topology of smooth maps. However they did not establish the basic properties of VMO degree, such as stability under homotopy within VMO, surjectivity if deg $\neq 0$, etc. The VMO degree is not defined by an integral formula; it is defined via approximation. More precisely, they pointed out that if $u \in \mathrm{VMO}(S^1, S^1)$ and

$$\bar{u}_\varepsilon(\theta) = \frac{1}{2\varepsilon}\int_{\theta-\varepsilon}^{\theta+\varepsilon} u(s)ds,$$

then $|\bar{u}_\varepsilon(\theta)| \to 1$ uniformly in θ — despite the fact that u need not be continuous. Then, for ε small,

$$u_\varepsilon(\theta) = \frac{\bar{u}_\varepsilon(\theta)}{|\bar{u}_\varepsilon(\theta)|}$$

has a well defined degree, which is independent of ε. This is their definition of the degree.

In this paper we develop this concept for maps between n-dimensional manifolds X, Y, and establish its basic properties. The degree is defined via approximation, in the BMO topology, by smooth maps from X to Y.

A natural related question is: Are smooth maps from X to Y dense in $W^{1,p}(X, Y)$, with $1 \leq p < \infty$? Here X and Y might have different dimensions. For $p \geq \dim X$ the answer is always yes. For $p < \dim X$ the answer was given by F. Bethuel [1]: a necessary and sufficient condition for density is that $\Pi_{[p]}(Y) = 0$, where Π denotes the homotopy class and $[p]$ is the integral part of p.

We now describe the organization of the paper.

In Section I.1 we recall the notion of BMO maps in Euclidean spaces and describe its extension to maps between manifolds. For this purpose it is convenient to put a Riemannian metric on X and to embed Y smoothly into some $\mathbb{R}^N$. However, the notion of $\mathrm{BMO}(X, Y)$ is independent of the particular metric or embedding — as will be the degree.

The BMO (semi) norm of a map u from X to $\mathbb{R}^N$ is

$$\|u\|_{\mathrm{BMO}} = \sup_{\substack{x\in X\\ \varepsilon<r_0}} \fint_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)|d\sigma(y) \tag{0.5}$$

where

$$\bar{u}_\varepsilon(x) = \fint_{B_\varepsilon(x)} u d\sigma. \tag{0.6}$$

Here, for any $x \in X$, $B_\varepsilon(x)$ is the geodesic ball centered at x with radius $\varepsilon < r_0$, the injectivity radius of X and $⨍_A u$ denotes the average of u in a set A. A very convenient equivalent (semi) norm is

$$\|u\|_\star = \sup_{\substack{x\in X\\ \varepsilon<r_0}} ⨍_{B_\varepsilon(x)} ⨍_{B_\varepsilon(x)} |u(y)-u(z)|d\sigma(y)d\sigma(z). \tag{0.7}$$

(Incidentally, (0.7) suggests a notion of BMO for maps into a general metric space Y. Such maps enjoy some of the basic properties of BMO maps, e.g. the John-Nirenberg [1] inequality holds — via the usual proof.)

The space VMO(= vanishing mean oscillation) is the completion of smooth maps in the BMO norm. This space was introduced by Sarason [1] who established a useful characterization (see Lemma 3). In the same section we present some of the properties of VMO maps, such as the effect of left composition by a Lipschitz map F (see Lemma 2′ and the more general Lemma A.7 in Appendix A). The map $\Psi : u \mapsto F\circ u$, for $u \in \mathrm{BMO}(X,\mathbb{R}^N)$, *need not* be continuous in its dependence on u — as a map from BMO to BMO — but it is continuous at every $u \in$ VMO (see Lemma A.8 and Remark A.1). Lemma 4 gives a characterization of compact sets in VMO — an adaptation of Arzelà-Ascoli to VMO.

The proofs of many technical statements are given in Appendix A.

Section I.2 takes up various examples of BMO and VMO maps. In addition to continuous maps, VMO contains all the "borderline" Sobolev spaces $W^{s,p}$ for $1<p$, $sp=n$.

The degree for VMO maps is defined in Section I.3. The first main result, Theorem 1, deals with its stability under perturbation in VMO: given $u \in \mathrm{VMO}(X,Y)$, there exists δ depending on u such that, for $v \in \mathrm{VMO}(X,Y)$ with $\|u-v\|_{\mathrm{BMO}} < \delta$, it has the same degree as u; this implies in turn the invariance of degree under homotopy within VMO.

Surprisingly the δ really depends on u (see Lemma 6). This is in contrast to the standard perturbation of continuous maps; there the δ is uniform. We point out in Remark 7 that the degree can also be defined for u in $\mathrm{BMO}(X,Y)$ provided u is "close" to VMO.

In Section I.4 we carry over standard properties of degree to VMO. For example, we prove that if $\deg u \neq 0$, then u is "onto" Y. This is more subtle than for the continuous case because u may be changed on a set of measure zero. We are led to a notion of "essential range" of u which is independent of the choice of representatives in the class of equivalent maps.

The formulas (0.2), (0.3) extend when u is in some appropriate "borderline" Sobolev space (see Properties 4 and 5).

In Section I.5 we take up a natural question concerning maps from X to Y, not necessarily of the same dimension. $\mathrm{BMO}(X,Y)$ — as well as $L^p(X,Y)$, $1\le p\le\infty$

— is arcwise connected, while $\mathrm{VMO}(X,Y)$ has components which are simply the closures of the components of C^0 maps.

Section I.6 deals with a question first considered by R. Coifman and Y. Meyer [1], namely the possibility of lifting a map $u \in \mathrm{BMO}(X, S^1)$ to $\mathrm{BMO}(X, \mathbb{R})$. Theorem 3 asserts that this can be done with VMO if and only if u is homotopic to a constant within VMO. In Theorem 4, which is directly related to a result in Coifman and Meyer [1], we show that any $u \in \mathrm{BMO}(X, S^1)$ with small BMO norm may be written as $u = e^{i\varphi}$ with $\varphi \in \mathrm{BMO}(X, \mathbb{R})$ and $\|\varphi\|_{\mathrm{BMO}} \leq 4\|u\|_{\mathrm{BMO}}$. The proofs are quite technical. They make use of the John-Nirenberg inequality; various forms of this inequality for manifolds are presented in Appendix B.

Of course degree theory extends to maps on domains or manifolds with boundary. In Part II we will consider this situation for VMO maps. A new feature is that VMO maps in a domain need not have a trace on the boundary. This makes the theory more delicate.

The plan of the paper is the following:

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§I.1. BMO and VMO

Let X be a smooth n-dimensional compact manifold without boundary. In this section we recall the notion of BMO and VMO functions and maps defined on X and we state some of their properties. There is much literature on BMO, but mainly defined in Euclidean space; e.g., E. Stein [1] where many references may be found. People have worked with BMO on some manifolds, but the subject is mainly folklore to people in the field.

DEFINITION OF BMO FOR REAL FUNCTIONS ON X. We first put a smooth Riemannian metric on X. (Later we shall show that the notion of BMO is independent of the choice of metric.) Consider a real function f in $L^1(X)$, using the measure associated to the metric. Set

$$\|f\|_{\mathrm{BMO}} = \sup_{\substack{\varepsilon<r_0 \\ x\in X}} \fint_{B_\varepsilon(x)} |f(y) - \bar{f}_\varepsilon(x)| d\sigma(y), \tag{1}$$

where $r_0 = r_0(X)$, the injectivity radius of X (see e.g., M. P. do Carmo [1]), σ is the element of volume on X, $B_\varepsilon(x)$ denotes the geodesic ball in X of radius $\varepsilon < r_0$, centred at x, and

$$\bar{f}_\varepsilon(x) = \fint_{B_\varepsilon(x)} f(z) d\sigma(z).$$

As usual, $\fint_A f = \frac{1}{|A|}\int_A f$ denotes the average of f on A. BMO$(X,\mathbb{R})$ — often denoted by BMO — consists of those functions with $\|f\|_{\mathrm{BMO}} < \infty$. For these, (1) defines a norm on BMO — modulo constants, (see E. Stein [1]) — and BMO is complete under this norm.

Clearly

$$\begin{aligned}\fint_{B_\varepsilon(x)} |f(y) - \bar{f}_\varepsilon(x)| d\sigma(y) &\leq \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| d\sigma(z) d\sigma(y) \\ &= \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - \bar{f}_\varepsilon(x) + \bar{f}_\varepsilon(x) - f(z)| d\sigma(z) d\sigma(y) \\ &\leq 2 \fint_{B_\varepsilon(x)} |f(y) - \bar{f}_\varepsilon(x)| d\sigma(y).\end{aligned}$$

Consequently, the following is an equivalent norm on BMO:

$$\|f\|_* = \sup_{\substack{\varepsilon<r_0 \\ x\in X}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| d\sigma(y) d\sigma(z); \tag{1'}$$

in fact,

$$\|f\|_{\mathrm{BMO}} \leq \|f\|_* \leq 2\|f\|_{\mathrm{BMO}}. \tag{1''}$$

A first simple but useful property is

LEMMA 1. *There exists a constant C, depending on X (and the metric) such that for every $f \in$ BMO,*

$$\|f\|_{L^1} \leq C\|f\|_{\mathrm{BMO}} + \left|\int_X f\right|.$$

This is proved in the Appendix; see Lemma A.1.

REMARK 1. If we replace r_0 by any positive $r_1 < r_0$ we get a new norm $\|f\|_1$. The two norms are equivalent.

Indeed, $\|f\|_1 \le \|f\|_{\mathrm{BMO}} =: \|f\|_0$. Conversely, if $r_1 \le \varepsilon < r_0$ then

$$\fint_{B_\varepsilon(x)} |f - \bar{f}_\varepsilon(x)| \le 2 \fint_{B_\varepsilon(x)} |f|$$
$$\le \frac{2}{|B_{r_1}(x)|} \int_X |f| \le C \int_X |f|.$$

We now use Lemma 1. Since we may suppose $\int_X f = 0$, we obtain the desired conclusion

$$\|f\|_1 \le C\|f\|_0.$$

If one changes the Riemannian metric on X one obtains an equivalent BMO norm. More generally, if X_1 and X_2 are two smooth compact Riemannian manifolds of dimension n, without boundary, and $\varphi : X_1 \to X_2$ is a C^1 diffeomorphism, then $f \in \mathrm{BMO}(X_2)$ implies that $f \circ \varphi \in \mathrm{BMO}(X_1)$ and

$$\|f \circ \varphi\|_{\mathrm{BMO}(X_1)} \le C\|f\|_{\mathrm{BMO}(X_2)}$$

(see Lemma A.10 in the Appendix; a more general form, where φ is only quasi-conformal and $X_1 = X_2 = \mathbb{R}^n$, was proved by H. M. Reimann [1]).

If Ω is an open subset of X, we set

$$\|f\|_{\mathrm{BMO}(\Omega)} = \text{ the sup in (1) taken over all balls } B_\varepsilon(x) \text{ in } \Omega, \text{ with } \varepsilon < r_0.$$

An L^1 map $u : X \to \mathbb{R}^N$ belongs to $\mathrm{BMO}(X, \mathbb{R}^N)$ provided each component of u is in BMO. As norm, we use the definition (1), except that the absolute value refers to the Euclidean norm in $\mathbb{R}^N$.

DEFINITION OF BMO MAPS INTO A MANIFOLD. Let Y be a compact manifold without boundary which we always take to be smoothly embedded in some $\mathbb{R}^N$. We say that a map u belongs to $\mathrm{BMO}(X, Y)$, if $u \in \mathrm{BMO}(X, \mathbb{R}^N)$ and $u(x) \in Y$ a.e.

CLAIM. *The notion of* $\mathrm{BMO}(X, Y)$ *is independent of the metric on* X *and of the embedding of* Y.

That it does not depend on the metric in X follows from a previous consideration. To verify its independence of the embedding of Y, we use the following

LEMMA 2. *Let* F *be a Lipschitz map from* $\mathbb{R}^N$ *into* $\mathbb{R}^D$ *and let* $u \in \mathrm{BMO}(X, \mathbb{R}^N)$. *Then* $F \circ u$ *is in* $\mathrm{BMO}(X, \mathbb{R}^D)$ *and*

$$\|F \circ u\|_{\mathrm{BMO}} \le 2\|F\|_{Lip}\|u\|_{\mathrm{BMO}}.$$

This follows immediately from $(1'')$; see also the more general Lemma A.2.

Proof of Claim. Suppose φ_1 and φ_2 are smooth embeddings of Y into $\mathbb{R}^N$ and $\mathbb{R}^D$. Then $\eta = \varphi_2 \circ \varphi_1^{-1}$ is a smooth map of $\varphi_1(Y)$ onto $\varphi_2(Y)$. Let F be a Lipschitz continuous extension of η as a map from $\mathbb{R}^N$ to $\mathbb{R}^D$. Using Lemma 2 we see that the BMO norm using φ_2, is bounded by a constant times the BMO norm using φ_1. Thus the norms are equivalent. $\square$

Having chosen a Riemannian metric on X, and a smooth embedding of Y in some $\mathbb{R}^N$, $\mathrm{BMO}(X,Y)$ is equipped with a metric

$$d(u,v) = \|u - v\|_{\mathrm{BMO}(X,\mathbb{R}^N)}.$$

A different choice of the Riemannian metric on X and of the embedding of Y in some $\mathbb{R}^D$ yields an equivalent metric. Thus it makes sense to say that a sequence of maps $u_j : X \to Y$ converges to u in $\mathrm{BMO}(X,Y)$, independently of the choice of metric on X and embedding of Y.

In view of $(1')$ there is a natural notion of BMO of a map from X into any metric space Y, namely

$$\|u\|_* = \sup_{\substack{\varepsilon < r_0 \\ x \in X}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} \operatorname{dist}(u(y), u(z))\, d\sigma(y)\, d\sigma(z).$$

It is clear that $C^0(X) \subset \mathrm{BMO}(X)$; other examples of BMO functions will be given later (see Section I.2). In particular, the examples show that smooth functions are not dense in BMO. It is therefore natural to introduce the following definition (see D. Sarason [1]).

DEFINITION OF VMO FUNCTIONS AND MAPS. VMO is the completion of smooth functions in the BMO norm, namely, a real function f on X belongs to $\mathrm{VMO}(X,\mathbb{R})$ if $f \in \mathrm{BMO}(X,\mathbb{R})$, and there is a sequence (f_j) of smooth functions such that $\|f_j - f\|_{\mathrm{BMO}} \to 0$. In view of Lemma 1 we may also suppose that $\|f_j - f\|_{L^1} \to 0$.

VMO is equiped with the BMO norm.

REMARK 2. In the definition of $\mathrm{VMO}(X,\mathbb{R})$ one could use continuous f_j instead of smooth f_j. This follows easily from two facts:

(i) $C^\infty(X,\mathbb{R})$ is dense in $C^0(X,\mathbb{R})$,

(ii) $C^0(X,\mathbb{R}) \subset \mathrm{BMO}(X,\mathbb{R})$ and

$$\|f\|_{\mathrm{BMO}} \le 2|f|_{C^0}.$$

Similarly, one defines $\mathrm{VMO}(X, \mathbb{R}^N)$. Furthermore, a map $u : X \to \mathbb{R}^N$ belongs to $\mathrm{VMO}(X, Y)$ if $u \in \mathrm{VMO}(X, \mathbb{R}^N)$ and $u(x) \in Y$ a.e. $x \in X$.

As above, $\mathrm{VMO}(X, Y)$ is independent of the Riemannian metric on X. The fact that it is also independent of the choice of embedding of Y in some Euclidean space follows from a variant of Lemma 2. In view of this fact, unless we say otherwise, from now on we fix a Riemannian metric on X, and an embedding of Y in $\mathbb{R}^N$. The variant of Lemma 2 is:

LEMMA 2$'$. *Let F be a Lipschitz map of $\mathbb{R}^N$ into $\mathbb{R}^D$ and let $u \in \mathrm{VMO}(X, \mathbb{R}^N)$. Then $F \circ u \in \mathrm{VMO}(X, \mathbb{R}^D)$.*

For proof, see the more general Lemma A.7. A natural way to prove the lemma would be to take a sequence of smooth maps u_j tending to u in $\mathrm{BMO} \cap L^1$ and to show that $F \circ u_j \to F \circ u$ in BMO. Indeed, this method of proof works, but it is more delicate than it would appear. In fact $u \mapsto F \circ u$ is not continuous in $\mathrm{BMO} \cap L^1$; it is however continuous at points u in VMO. See Lemma A.8 and Remark A.1. The proof of Lemma 2$'$ that we present is different; it relies on Sarason's characterization of VMO:

LEMMA 3 (D. Sarason [1]). *$u \in \mathrm{VMO}(X, \mathbb{R}^N)$ iff $u \in \mathrm{BMO}(X, \mathbb{R}^N)$ and*

$$\lim_{\varepsilon \to 0} ⨍_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| = 0 \textit{ uniformly in } x \in X. \tag{2}$$

Again, in view of (1$'$), property (2) is equivalent to

$$\lim_{\varepsilon \to 0} ⨍_{B_\varepsilon(x)} ⨍_{B_\varepsilon(x)} |u(y) - u(z)| = 0 \quad \text{uniformly in } x \in X. \tag{2$'$}$$

The implication: VMO $\Longrightarrow$ (2) is easy. Indeed, given $\delta > 0$, there is a $v \in C^0(X, \mathbb{R}^N)$ such that

$$\|u - v\|_{\mathrm{BMO}} < \delta/2.$$

Write

$$\begin{aligned} ⨍_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| &\le ⨍_{B_\varepsilon(x)} |(u - v) - (\bar{u}_\varepsilon(x) - \bar{v}_\varepsilon(x))| + ⨍_{B_\varepsilon(x)} |v - \bar{v}_\varepsilon(x)| \\ &\le \frac{\delta}{2} + \frac{\delta}{2} \end{aligned}$$

provided ε is sufficiently small (depending on v). Property (2) follows easily.

The converse implication is more delicate. It is in fact a consequence of a more general form of Lemma 3, Lemma 3$'$ below. First some notation:

For $u \in \mathrm{BMO}(X, \mathbb{R}^N)$ and $0 < a < r_0(X)$, set

$$M_a = M_a(u) = \sup_{\substack{\varepsilon \le a \\ x \in X}} \fint_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| \le \|u\|_{\mathrm{BMO}}$$
$$M_0 = M_0(u) = \lim_{a \searrow 0} M_a(u).$$

LEMMA 3′ (D. Sarason [1]). *There is a constant A depending only on X (and choice of Riemannian metric), such that, if $u \in \mathrm{BMO}(X, \mathbb{R}^N)$, then*

$$M_0(u) \le \operatorname{dist}(u, \mathrm{VMO}(X, \mathbb{R}^N)) \le A\, M_0(u). \tag{3}$$

Here distance is measured using the BMO norm. More precisely,

$$\|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} \le A\, M_\varepsilon(u) \quad \forall \varepsilon < r_0, \quad \forall u \in \mathrm{BMO}(X, \mathbb{R}^N). \tag{4}$$

This is Lemma A.5.

Note that (2) says that $M_0(u) = 0$; hence (3) yields the implication $\Longleftarrow$ in Lemma 3. In addition we have

COROLLARY 1. *For any $u \in \mathrm{VMO}(X, \mathbb{R}^N)$,*

$$\|\bar{u}_\varepsilon - u\|_{\mathrm{BMO}} \longrightarrow 0, \ \bar{u}_\varepsilon \longrightarrow u \ \textit{in } L^1 \ \textit{as } \varepsilon \longrightarrow 0. \tag{5}$$

The first assertion follows from (3) and (4); the last assertion is well known. Another consequence of (4) is:

COROLLARY 2. *There is a constant $\widetilde{A}$ depending only on X (and choice of metric), such that*

$$\|\bar{u}_\varepsilon\|_{\mathrm{BMO}} \le \widetilde{A} \|u\|_{\mathrm{BMO}} \ \forall \varepsilon < r_0, \quad \forall u \in \mathrm{BMO}(X, \mathbb{R}^N). \tag{6}$$

If $u \in C^0(X, Y)$, the map $x \mapsto \bar{u}_\varepsilon(x)$ maps X into $\mathbb{R}^N$, but not into Y. However, for ε small it lies close to Y. (This is clear since $\bar{u}_\varepsilon \to u$ uniformly.) Surprisingly the same is true for $u \in \mathrm{VMO}(X, Y)$ — even though u need not be continuous. Indeed we have

$$\operatorname{dist}(\bar{u}_\varepsilon(x), Y) \le \fint_{B_\varepsilon(x)} |u(y) - \bar{u}_\varepsilon(x)| \le M_\varepsilon(u). \tag{7}$$

and $M_\varepsilon(u)$ tends to 0, by (2).

REMARK 3. Note that (7) holds if Y is any closed set in $\mathbb{R}^N$ — not just a smooth manifold.

This fact is at the heart of the paper, because it allows us to project $\bar{u}_\varepsilon$, for ε small, onto its nearest point in Y.

As pointed out in the introduction, the role of VMO maps in conjunction with (7) was first observed by L. Boutet de Monvel and O. Gabber.

Denote by P the projection operator in $\mathbb{R}^N$ to the nearest point on Y (this is well defined in a tubular neighbourhood of Y). For ε less than some ε_0,

$$u_\varepsilon(x) = P\bar{u}_\varepsilon(x) \text{ is well defined.} \tag{8}$$

Here is one more

COROLLARY 3. *There is a constant C depending only on X such that for any $a < r_0, \varepsilon \leq \varepsilon_0$,*

$$M_a(u_\varepsilon) \leq C\,(M_a(u) + M_\varepsilon(u)) \quad \forall u \in \mathrm{VMO}(X, Y).$$

Proof. We have

$$M_a(u_\varepsilon) \leq CM_a(\bar{u}_\varepsilon)$$

since P is Lipschitz continuous. On the other hand,

$$\begin{aligned} M_a(\bar{u}_\varepsilon) &\leq M_a(\bar{u}_\varepsilon - u) + M_a(u) \\ &\leq AM_\varepsilon(u) + M_a(u) \end{aligned}$$

by (4). □

Presumably, in the assertion of the corollary, the term $CM_\varepsilon(u)$ could be omitted. This is clear in Euclidean space.

We will be considering families of maps in $\mathrm{VMO}(X, Y)$. Let $\mathcal{F} \subset \mathrm{VMO}(X, Y)$ be a collection of maps. For each individual map $u \in \mathcal{F}$ we have

$$\lim_{\varepsilon \to 0} \mathrm{dist}(\bar{u}_\varepsilon(x), Y) = 0 \tag{9}$$

uniformly in $x \in X$, but this does not hold uniformly with respect to the map u. However, if $\mathcal{F}$ is a compact subset of $\mathrm{VMO}(X, Y)$, then (9) holds uniformly in $x \in X$ and $u \in \mathcal{F}$. This is an immediate consequence of (7) and the following:

LEMMA 4 (Characterization of compact sets in VMO). *Assume $\mathcal{F}$ is a compact subset of* $\mathrm{VMO}(X, \mathbb{R}^N)$. *Then*

$$\lim_{\varepsilon \to 0} M_\varepsilon(u) = 0 \quad \textit{holds uniformly in } u \in \mathcal{F}. \tag{10}$$

Conversely, if $\mathcal{F}$ is any collection of maps in VMO$(X, \mathbb{R}^N)$ *such that* (10) *holds, then $\mathcal{F}$ is contained in a compact subset of VMO$(X, \mathbb{R}^N)$.*

Proof of the first assertion (the second, which is more delicate, is proved in the Appendix; see Lemma A.16):

Given any $\delta > 0$ we may cover $\mathcal{F}$ by a finite number of balls

$$\mathcal{F} \subset \bigcup_{i=1}^{k} B_{\delta/2}(v_i)$$

(where B refers to balls for the BMO norm).

For each i there is some $\varepsilon_i > 0$ such that $\forall \varepsilon < \varepsilon_i$,

$$M_\varepsilon(v_i) < \delta/2. \tag{11}$$

Set $\varepsilon_0 = \min\limits_{1 \le i \le k} \varepsilon_i$. Given $v \in \mathcal{F}$, there is some i such that

$$\|v - v_i\|_{\mathrm{BMO}} < \delta/2.$$

Then $\forall \varepsilon < \varepsilon_0$

$$\begin{aligned} M_\varepsilon(v) \le M_\varepsilon(v - v_i) + M_\varepsilon(v_i) &\le \|v - v_i\|_{\mathrm{BMO}} + M_\varepsilon(v_i) \\ &\le (\delta/2) + (\delta/2) \quad \text{by}(11). \end{aligned}$$

□

Some further consequences of Lemma 3′ and Corollary 1 are:

COROLLARY 4. *Given $u \in$ VMO(X, Y),*

$$\|u_\varepsilon - u\|_{\mathrm{BMO}} \longrightarrow 0, \ u_\varepsilon \longrightarrow u \ \text{a.e. as } \varepsilon \to 0.$$

Proof. We have

$$\begin{aligned} \|u - P\bar{u}_\varepsilon\|_{\mathrm{BMO}} &\le \|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} + \|\bar{u}_\varepsilon - P\bar{u}_\varepsilon\|_{\mathrm{BMO}} \\ &\le \|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} + 2|\bar{u}_\varepsilon - P\bar{u}_\varepsilon|_{C^0} \\ &\le \|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} + 2\sup_x \operatorname{dist}(\bar{u}_\varepsilon(x), Y) \\ &\longrightarrow 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

by (5) and (9). □

COROLLARY 5. *Given $u \in \mathrm{VMO}(X,Y)$, there exists a sequence $u_j \in C^\infty(X,Y)$ such that $u_j \to u$ in* BMO *and a.e.*

Proof. Since $C^\infty(X,Y)$ is dense in $C^0(X,Y)$ the result follows with the aid of Corollary 4. □

§I.2. Some examples of BMO and VMO functions

As we have said in Remark 2, continuous functions f on X belong to VMO and

$$\|f\|_{\mathrm{BMO}} \leq 2|f|_{C^0}. \tag{12}$$

A less obvious class consists of functions in Sobolev spaces corresponding to limiting cases — where the embedding is into L^p for every $p < \infty$, but not into L^∞.

EXAMPLE 1. $W^{1,n}(X) \subset \mathrm{VMO}(X)$ with continuous embedding.

Proof. We first prove that $W^{1,n}(X) \subset \mathrm{BMO}(X)$, with continuous embedding. By Poincaré's inequality — which even holds on a manifold — we have, for $\varepsilon < r_0$,

$$\int_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| \leq C\varepsilon \int_{B_\varepsilon(x)} |\nabla u|.$$

Hence

$$\int_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| \leq C\varepsilon^n \left(\int_{B_\varepsilon(x)} |\nabla u|^n \right)^{1/n}$$

and thus

$$⨍_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| \leq C \left(\int_{B_\varepsilon(x)} |\nabla u|^n \right)^{1/n}, \tag{13}$$

which implies the desired conclusion. The embedding in VMO now follows easily from (13) and Lemma 3. □

More generally, we have:

EXAMPLE 2. $W^{s,p}(X) \subset \mathrm{VMO}(X)$ in the limiting case of the Sobolev embedding, i.e., $sp = n$, $0 < s < n$ (s may or may not be an integer). [1]

Proof. We distinguish two cases:

Case 1: $s \geq 1$. Then, by the Sobolev embedding, (see e.g., R. A. Adams [1], Theorem 7.57),

$$W^{s,p}(X) \subset W^{1,n}(X)$$

and the conclusion follows from Example 1.

Case 2: $0 < s < 1$. Recall that

$$W^{s,p}(X) = \left\{ u;\ \int_X \int_X \frac{|u(x) - u(y)|^p}{|\operatorname{dist}(x,y)|^{sp+n}} < \infty \right\}$$

and in our case $sp = n$, so that

$$W^{s,p}(X) = \left\{ u;\ \int_X \int_X \frac{|u(x) - u(y)|^p}{|\operatorname{dist}(x,y)|^{2n}} < \infty \right\}.$$

As before, we first prove that

$$W^{s,p} \subset \mathrm{BMO} \quad \text{with continuous injection.} \tag{14}$$

To prove (14), we compute for $\varepsilon \leq r_0$,

$$\begin{aligned} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)| &\leq \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} \frac{|u(y) - u(z)|}{|\operatorname{dist}(y,z)|^{\frac{2n}{p}}} C\varepsilon^{2n/p} \\ &\leq C \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} \frac{|u(y) - u(z)|^p}{|\operatorname{dist}(y,z)|^{2n}}, \end{aligned}$$

which yields (14). It then follows, as above, that $W^{s,p} \subset \mathrm{VMO}$. $\square$

---

1) For the definition and general properties of fractional Sobolev spaces, see e.g., R. A. Adams [1], Chapter VII.

Next we present some specific functions in BMO or VMO. The functions are defined on some bounded domain Ω in $\mathbb{R}^n$ containing the origin; Ω may be considered as an open subset of a compact manifold X.

EXAMPLE 3. The function $\log|x|$ belongs to BMO(Ω), for any n (see F. John and L. Nirenberg [1] or E. Stein[1], Chapter IV, Section I.1.2). However, $\log|x|$ is not in VMO. To see this, observe that

$$I = \fint_{B_\varepsilon(0)} \left| \log|y| - \fint_{B_\varepsilon(0)} \log|x| \right| = \fint_{B_1(0)} \left| \log|y| - \fint_{B_1(0)} \log|x| \right|,$$

and thus I does not tend to zero as $\varepsilon \to 0$.

EXAMPLE 4. The function $f(x) = \log\big|\log|x|\big|$ is in VMO(Ω). An easy way to verify this is to observe that f belongs to $W^{1,n}(\Omega)$, when $n \geq 2$, because

$$|\nabla f| \leq \frac{C}{|x|\big|\log|x|\big|}.$$

For $n = 1$, f is the trace on $\mathbb{R}$ of the function $\log\big|\log|x|\big|$ in $\mathbb{R}^2$ — which belongs to $W^{1,2} = H^1$ in any bounded region on $\mathbb{R}^2$. Consequently its trace belongs to $H^{1/2}(\Omega) = W^{\frac{1}{2},2}(\Omega)$. By Example 2, this function is then in VMO.

Applying Lemma 2′ we see that the functions $\exp(i\log|\log|x||)$ or $\sin(\log|\log|x||)$ also belong to VMO.

EXAMPLE 5. The function $f(x) = \big|\log|x|\big|^\alpha$, for $0 < \alpha < 1$ is in VMO(Ω).

Proof. Observe first that $f \in W^{1,n}_{\mathrm{loc}}$ in case $n > 1/(1-\alpha)$. If $n \leq 1/(1-\alpha)$, fix an integer $m > 1/(1-\alpha)$. Then the function $f(x)$ belongs to $W^{1,m}_{\mathrm{loc}}(\mathbb{R}^m)$. Consequently its trace on $\mathbb{R}^{m-1}$ belongs to $W^{1-\frac{1}{m},m}_{\mathrm{loc}}(\mathbb{R}^{m-1})$. Continuing to take traces, we find that $f \in W^{\frac{n}{m},m}_{\mathrm{loc}}(\mathbb{R}^n)$. Again by Example 2, $f \in$ VMO(Ω). $\square$

We conclude this section with a particular, but useful sequence of VMO functions in $\mathbb{R}$.

EXAMPLE 6. In $\mathbb{R}$, consider the sequence

$$f_j(x) = \begin{cases} 1 & \text{if } |x| \leq \dfrac{1}{j^2} \\ -1 - \dfrac{\log|x|}{\log j} & \text{if } \frac{1}{j^2} \leq |x| \leq \dfrac{1}{j} \\ 0 & \text{if } |x| \geq \dfrac{1}{j}. \end{cases}$$

Then

$$\|f_j\|_{H^{1/2}} \longrightarrow 0 \quad \text{as } j \to \infty.$$

In particular, $\|f_j\|_{\mathrm{BMO}} \to 0$.

Proof. Consider the sequence for $x \in \mathbb{R}^2$. One easily verifies that

$$\|f_j\|_{H^1(\mathbb{R}^2)} \longrightarrow 0.$$

The desired result then follows by taking trace. □

The reader may prefer a direct argument showing that $\|f_j\|_{\mathrm{BMO}} \to 0$, an argument which does not rely on trace. Here is one: f_j may be written as

$$f_j(x) = \min\left\{1, \max\left\{0, -1 - \frac{\log|x|}{\log j}\right\}\right\}.$$

Since $F(t) = \max\{0, t\}$ is Lipschitz with Lipschitz constant 1, it follows from Lemma 2 that

$$\begin{aligned}\left\|\max\{0, -1 - \frac{\log|x|}{\log j}\}\right\|_{\mathrm{BMO}} &\le \frac{2}{\log j}\|\log|x|\|_{\mathrm{BMO}} \\ &\le \frac{C}{\log j}.\end{aligned}$$

Similarly,

$$\|f_j\|_{\mathrm{BMO}} \le \frac{C}{\log j}.$$

This argument shows that for any n, f_j as defined above, in $\mathbb{R}^n$, satisfies

$$\|f_j\|_{\mathrm{BMO}} \le \frac{C}{\log j}.$$

□

REMARK 4. In Example 6 it seems natural to replace f_j by a simpler sequence of functions, in which f_j is linear on $(\frac{1}{j^2}, \frac{1}{j})$. However, the reader may verify that, then $\|f_j\|_{\mathrm{BMO}}$ does not tend to zero. Our sequence (f_j) is the kind of sequence which is commonly used to prove that in two dimensions, a point has zero capacity.

§I.3. Degree for VMO maps

This is our main topic. We consider VMO maps from X to Y. Here, X and Y are smooth n-dimensional compact manifolds without boundaries — *which we now assume to be oriented manifolds.* We shall define a degree for such maps and show that it has some of the usual properties of a degree.

We first put a Riemannian metric on X and consider Y as smoothly embedded in some $\mathbb{R}^N$. Recall that for a C^1 mapping $u : X \to Y$,

$$\deg u = \frac{1}{\operatorname{vol} Y} \int_X \det J_u(x),$$

where $J_u(x)$ is the Jacobian at x of the map u computed in terms of geodesic normal coordinates at x and at $u(x)$. This integral clearly makes sense for a map in $W^{1,n}(X, Y)$. We shall prove later that for such a map, this expression is indeed an integer. This fact suggests that degree theory, which extends to continuous maps, extends also to maps in $W^{1,n}(X, Y)$. In an attempt to find a general class of maps including both of these, the natural candidate seems to be the class VMO.

We now proceed to define the degree for a VMO map $u : X \to Y$.

DEFINITION. Let $u \in \mathrm{VMO}(X, Y)$. For $0 < \varepsilon$ small, recall

$$\bar{u}_\varepsilon(x) = \fint_{B_\varepsilon(x)} u \quad \text{and } u_\varepsilon(x) = P\bar{u}_\varepsilon(x).$$

Define

$$\deg(u, X, Y) = \deg(u_\varepsilon, X, Y)$$

for ε small. We claim that this is independent of ε. Indeed for ε small, since u_ε is continuous, $\deg(u_\varepsilon, X, Y)$ is defined. Furthermore, using the deformation $u_{t\varepsilon+(1-t)\varepsilon'}$, for $\varepsilon, \varepsilon'$ small, $0 \le t \le 1$, we see that $\deg u_\varepsilon = \deg u_{\varepsilon'}$.

In principle, $\deg(u, X, Y)$ depends on the choices of metric on X and of the embedding of Y. We shall see soon that it is independent of these choices. We first establish an important fact about this degree, namely, that it is stable under perturbation in VMO:

THEOREM 1. *Let $u \in \mathrm{VMO}(X, Y)$. Then there exists $\delta > 0$ depending on u, such that if $v \in \mathrm{VMO}(X, Y)$ and*

$$d(u, v) < \delta,$$

then

$$\deg v = \deg u. \tag{15}$$

Recall that $d(u,v)$ refers to the metric induced by the norm of $\mathrm{BMO}(X,\mathbb{R}^N)$ once an embedding of Y has been chosen. Easy consequences of Theorem 1 are

COROLLARY 6. *Let $H_t(\cdot)$ be a one-parameter family of VMO maps from X to Y, depending continuously in the BMO topology, on the parameter t. Then*

$$\deg H_t(\cdot) \quad \textit{is independent of } t.$$

COROLLARY 7. $\deg(u,X,Y)$ *is independent of the choices of Riemannian metric on X and of the embedding of Y.*

Proof. Suppose we have another metric on X and a smooth embedding of Y in some $\mathbb{R}^D$. We then obtain another family $\tilde{u}_\varepsilon$ mapping $X \to Y$. There is a corresponding degree $\tilde{d}$. By Corollary 4, $\tilde{u}_\varepsilon \to u$ in $\mathrm{BMO}(X,Y)$ as $\varepsilon \to 0$. Applying Theorem 1 we obtain the desired conclusion. □

In the proof of Theorem 1 we shall use

LEMMA 5. *Consider $u \in \mathrm{VMO}(X,Y)$. Suppose that for some constant vector $\xi \in \mathbb{R}^N$,*

$$u(x) + \xi \in Y \quad a.e.$$

Then

$$\deg(u+\xi) = \deg u.$$

Proof of Lemma 5. We need only consider $\xi \neq 0$, and, in fact, in this case we will also prove that both degrees are zero. We may suppose $\xi = (\xi_1, 0, 0)$, $\xi_1 > 0$ and also that $0 \in Y$, and that $y_1 \leq 0 \ \ \forall y \in Y$. Then

$$u_1(x) \leq -\xi_1, \quad \text{a.e. on } X.$$

Consequently for $\varepsilon < r_0$, the first component of $\bar{u}_\varepsilon(x) \leq -\xi_1$. It follows that for ε small, the first component of $P\bar{u}_\varepsilon(x) = u_\varepsilon(x)$ is less than $-\xi_1/2$. This implies that the image of u_ε does not cover Y, and so $\deg u_\varepsilon = \deg u = 0$.

Reversing the roles of u and $u+\xi$, we conclude that $\deg(u+\xi) = 0$. □

Proof of Theorem 1. Suppose the assertion of the theorem is false. Then there exists a sequence (v_j) such that

$$\|v_j - u\|_{\mathrm{BMO}} \longrightarrow 0 \quad \text{and } |\deg v_j - \deg u| \geq 1.$$

Since the (v_j) are compact in VMO, we know by Lemma 4 and (7) that

$$\operatorname{dist}(\bar{v}_{j,\varepsilon}(x), Y) \to 0 \quad \text{as } \varepsilon \to 0$$

uniformly in j and in $x \in X$. Hence there exists $\varepsilon_0 > 0$, such that

$$v_{j,\varepsilon} = P\bar{v}_{j,\varepsilon}$$

is well defined for all j and all $\varepsilon \le \varepsilon_0$. By definition,

$$\deg v_j = \deg v_{j,\varepsilon} \quad \forall j, \quad \forall \varepsilon \le \varepsilon_0. \tag{16}$$

Fix some $\varepsilon < \varepsilon_0$. Set

$$\xi_j = \fint_X (v_j - u).$$

By Lemma 1,

$$\int_X |v_j - u - \xi_j| \le C\|v_j - u\|_{\mathrm{BMO}} \to 0.$$

For a subsequence, we may assume ξ_j converges to some vector ξ, since Y is bounded. Hence $v_j \to u + \xi$ in L^1 and a.e. Therefore $u + \xi \in Y$ a.e. and also

$$\bar{v}_{j,\varepsilon} \to \bar{u}_\varepsilon + \xi \quad \text{uniformly on } X \text{ as } j \to \infty,$$

— recall that ε is fixed. Hence $v_{j,\varepsilon} \to (u+\xi)_\varepsilon$ as $j \to \infty$ uniformly on X. For j large it follows that $\deg v_{j,\varepsilon} = \deg(u+\xi)_\varepsilon$ and consequently

$$\deg v_j = \deg(u + \xi)$$

by (16) and by definition of $\deg(u+\xi)$. By Lemma 5 the proof is complete. □

REMARK 5. We have defined $\deg u$ with the aid of particular approximations of u by continuous functions tending to u in BMO, namely the u_ε (see Corollary 4), and we set $\deg u = \deg u_\varepsilon$. The preceding theorem shows that we could have used any approximation by continuous maps, tending to u in BMO. Theorem 1 is somewhat subtle for various reasons:

1) BMO convergence is weaker than uniform convergence but stronger than any L^p, $p < \infty$ (modulo constants). Degree is not preserved under small perturbations in L^p, $p < \infty$. For example, the following maps u_j of S^1 to S^1, have degree one, but their L^p limit is a constant — and thus has degree zero:

$$u_j(\theta) = e^{i\varphi_j(\theta)}$$

where $\varphi_j(\theta) = 0$ on $(0, 2\pi - \frac{1}{j})$ and φ_j goes linearly from 0 to 2π on $[2\pi - \frac{1}{j}, 2\pi]$.

2) Recall that when working with the C^0 norm, there is a *uniform* $\delta > 0$ such that
$$|u - v|_{C^0} < \delta \Longrightarrow \deg(u, X, Y) = \deg(v, X, Y).$$
For then, u is easily deformed to v via
$$P(tv + (1 - t)u).$$

Surprisingly, in Theorem 1, the δ really depends on u. Here is an example for $X = Y = S^1$ showing that if u and v are maps of S^1 to S^1 which are close even in $H^{1/2}$, they need not have the same degree.

LEMMA 6. *Given $\varepsilon > 0$ there are two smooth maps, u, v of S^1 to S^1 with*
$$\|u - v\|_{H^{1/2}} < \varepsilon \tag{17}$$
such that
$$\deg u = 0 \quad \textit{and} \deg v = 1. \tag{18}$$

Proof. **Step 1.** We first construct $u, v \in C^0(S^1, S^1)$ with $u - v$ in $H^{1/2}$, satisfying (17) and (18). Recall that there is a continuous function ρ defined on $\mathbb{R}$ with support, in $[\pi - \delta, \pi + \delta], \rho > 0$ in $(\pi - \delta, \pi + \delta)$, ρ symmetric about π nondecreasing on $(\pi - \delta, \pi), \rho(\pi) = 2$, and such that
$$\|\rho\|_{H^{1/2}} < \varepsilon.$$
Here δ depends on ε (see Example 6 in Section I.2).

Using ρ, we construct u and v of the form
$$u = e^{if}, \quad v = e^{i(f+g)}$$
on $[0, 2\pi]$ such that
$$f(0) = f(2\pi), \quad g(2\pi) - g(0) = 2\pi.$$
Thus we will have $\deg u = 0, \deg v = 1$.

We first define g as a continuous nondecreasing function on $(0, 2\pi)$, with
$$g(\theta) = \begin{cases} 0 & \text{on } [0, \pi - \delta] \\ 2\pi & \text{on } [\pi + \delta, 2\pi], \end{cases}$$
and such that
$$|e^{ig(\theta)} - 1| = \rho(\theta) \quad \text{on } [0, 2\pi].$$
This defines g in a unique manner.

Next we define f on $(\pi-\delta,\pi+\delta)$ as $f=-\arg(e^{ig}-1)$. Note that

$$f(\theta)\to-\frac{\pi}{2}\qquad \text{as } \theta\searrow(\pi-\delta),$$
$$f(\theta)\to-\frac{3\pi}{2}\qquad \text{as } \theta\nearrow(\pi+\delta).$$

We then extend f to $[0,2\pi]$ continuously so that $f(0)=f(2\pi)$. Any such extension will do.

The point is that

$$v-u=e^{if}(e^{ig}-1)\equiv\rho.$$

This is clear on $(\pi-\delta,\pi+\delta)$, and even clearer outside.

Step 2. We may approximate v in C^0 by smooth functions, and may approximate $u-v$ by smooth functions in the $C^0\cap H^{1/2}$ topology. The sum of these approximates is an approximation of u in C^0. □

REMARK 6. By a slight modification we may even construct two smooth maps u,v of S^1 to S^1 with

$$\|u-v\|_{H^{1/2}}<\varepsilon$$

such that

$$\deg u=0\quad\text{and }\deg v=k$$

(for any given integer k and any given $\varepsilon>0$).

REMARK 7. We have defined the degree for VMO maps of X to Y. The degree can, in fact, also be defined for $u\in \mathrm{BMO}(X,Y)$, with u "close" to VMO. More precisely, there is a $\delta>0$ such that if $u\in\mathrm{BMO}(X,Y)$, and

$$\operatorname{dist}(u,\mathrm{VMO}(X,Y)):=\inf_{v\in C^0(X,Y)}d(u,v)<\delta,$$

then u has a well defined degree. The distance d, and hence the number δ, depend on a particular choice of Riemannian metric on X and on the embedding of Y.

§I.4. Some properties of degree

The setting is the same as in the previous section. We consider VMO maps from X to Y and show that standard properties of degree carry over. Here are some:

Property 1. If $\deg u\neq 0$ then

$$\operatorname{ess} R(u)=Y.$$

Here, $\operatorname{ess} R(u)$, the essential range of u, has to be explained.

The notion of the range of a measurable map is not well defined, since the map may always be modified on a set of measure zero, keeping the new map in the same equivalence class. It is important to introduce a notion of the range of u which is independent of the choice of representative in the class of equivalent maps.

DEFINITION ($\operatorname{ess} R$). The essential range of a map u, $\operatorname{ess} R(u)$, is the smallest closed set Σ in Y such that

$$u(x) \in \Sigma \text{ a.e.}$$

CLAIM. *This is well defined.*

Proof. Let $(\Sigma_\alpha)_{\alpha\in\Lambda}$ be the family of all closed sets (Σ_α) in Y, such that $\forall\alpha$,

$$u(x) \in \Sigma_\alpha \text{ a.e.}$$

Set $\Sigma = \bigcap_{\alpha\in\Lambda} \Sigma_\alpha$. We assert that

$$u(x) \in \Sigma \text{ a.e.}$$

This follows easily from the general fact that there is a *countable* subset $J \subset \Lambda$ such that

$$\Sigma = \bigcap_{\alpha\in J} \Sigma_\alpha.$$

To see this, let $O_\alpha = \Sigma_\alpha^c$; $O = \bigcup_{\alpha\in\Lambda} O_\alpha = \Sigma^c$. The open set O may be written as a countable union of increasing compact subsets, K_i, $i = 1, 2, \dots$. Each K_i is covered by a finite number of the O_α. Hence O is the countable union of these. Σ is the intersection of their complements. □

The notion of essential range for a complex-valued measurable function f is commonly used in the theory of Banach algebras (see e.g. R. G. Douglas [1]). There it is defined as the set of all $\lambda \in \mathbb{C}$ for which $\{x \in X; |f(x) - \lambda| < \varepsilon\}$ has positive measure for every $\varepsilon > 0$. It is easy to see that this notion is equivalent to our definition when $\mathbb{C}$ is replaced by Y.

Proof of Property 1. We argue by contradiction. Suppose $\operatorname{ess} R(u)$ omits a point y_0. Then, for some $r > 0$,

$$\operatorname{ess} R(u) \cap B_r(y_0) = \phi.$$

Setting

$$\Sigma = Y \backslash B_r(y_0),$$

we clearly have $u(x) \in \Sigma$ a.e. Since $u \in \mathrm{VMO}$,

$$\lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| = 0 \quad \text{uniformly in } x.$$

Therefore

$$\operatorname{dist}(\bar{u}_\varepsilon(x), \Sigma) \to 0 \quad \text{uniformly in } x,$$

which implies

$$\operatorname{dist}(u_\varepsilon(x), \Sigma) \to 0 \text{ uniformly in } x.$$

Consequently, for ε small, $\deg u_\varepsilon = 0$, contradicting the assumption that $\deg u_\varepsilon = \deg u \neq 0$. □

Property 2 (Hopf). If $u, v \in \mathrm{VMO}(S^n, S^n)$ and have the same degree, then they are homotopic within $\mathrm{VMO}(S^n, S^n)$.

Proof. By our construction, $\deg u_\varepsilon = \deg u = \deg v = \deg v_\varepsilon$. The well known result of Hopf says that u_ε and v_ε are homotopic within $C^0(S^n, S^n)$ and therefore within $\mathrm{VMO}(S^n, S^n)$. On the other hand, u_ε is homotopic to u within $\mathrm{VMO}(S^n, S^n)$ (via $u_{t\varepsilon}$). □

Property 3 (Borsuk). Let U and V be symmetric open bounded neighbourhoods of the origin in $\mathbb{R}^n$, with smooth boundaries $\partial U, \partial V$ — each of which is connected. Let $u \in \mathrm{VMO}(\partial U, \partial V)$ be an odd map. Then $\deg u$ is odd.

Proof. One may simply apply Borsuk's theorem to u_ε which is also an odd map. □

Property 4. We return to the setting of Section I.3. Let $u \in W^{1,n}(X, Y)$ so that $u \in \mathrm{VMO}(X, Y)$ — see Example 1 in Section I.2. Then

$$\deg u \int_Y \mu = \int_X \mu \circ u \tag{19}$$

where μ is any smooth n-form on Y.

Observe that in local coordinates, the integrand involves the determinant of the Jacobian of the map, and hence is integrable. Formula (19) is well known for smooth maps (see e.g., L. Nirenberg [1]).

The proof relies on the following:

LEMMA 7. *Given $u \in W^{1,n}(X, Y)$, there is sequence (u_j) of smooth maps from X to Y, converging to u in $W^{1,n}$.*

Lemma 7 follows R. Schoen and K. Uhlenbeck [1], and is proved in the Appendix — see Lemma A.11.

REMARK 8. In general, if $p < n$, smooth maps from X to Y are not dense in $W^{1,p}(X,Y)$. However, F. Bethuel [1] has shown that they are dense iff the homotopy group $\pi_{[p]}(Y)$ is zero.

Assuming the lemma we give the

Proof of Property 4. Let (u_j) be the sequence of Lemma 7. Then, by convergence in $W^{1,n}$,

$$\int_X \mu \circ u_j \longrightarrow \int_X \mu \circ u.$$

On the other hand $u_j \to u$ in BMO, by Example 1 in Section I.2, and so, by Theorem 1, $\deg u_j = \deg u$ for j large. □

Property 5. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth connected boundary $\partial\Omega$. Suppose $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ (so its trace is defined on $\partial\Omega$) and suppose that

$$u : \partial\Omega \to S^{n-1}.$$

Then

$$\deg(u_{|_{\partial\Omega}}, \partial\Omega, S^{n-1}) = \frac{1}{|B_1|} \int_\Omega \det J_u. \tag{20}$$

Formula (20) is a well known formula for the degree in case u is smooth. In the more general situation, the right hand side makes sense because $u \in W^{1,n}$. The left hand side makes sense because $u_{|_{\partial\Omega}}$ belongs to $W^{1-\frac{1}{n},n}(\partial\Omega)$ which, by Example 2 in Section I.2, is contained in $\mathrm{VMO}(\partial\Omega)$.

COROLLARY 8. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth connected boundary $\partial\Omega$. If $u \in W^{1,n}(\Omega, S^{n-1})$ then*

$$\deg(u_{|_{\partial\Omega}}, \partial\Omega, S^{n-1}) = 0.$$

Returning to Property 5, we shall prove (20), as expected, via approximation; namely, using the following — which we formulate more generally (see Lemma A.13):

LEMMA 8. *Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth connected boundary $\partial\Omega$. Let Y be a compact manifold without boundary, smoothly embedded in $\mathbb{R}^N$. Let $u \in W^{1,n}(\Omega, \mathbb{R}^N)$ such that*

$$u(\partial\Omega) \subset Y.$$

Then there is a sequence (u_j) *of smooth maps of* $\overline{\Omega}$ *into* $\mathbb{R}^N$ *such that*

$$u_j(\partial\Omega) \subset Y, \quad \forall j, \text{ and } u_j \to u \text{ in } W^{1,n}.$$

Assertion (20) is a simple consequence of Lemma 8, for the $u_{j|\partial\Omega}$ converges to $u_{|\partial\Omega}$ in $W^{1-\frac{1}{n},n}(\partial\Omega)$ and hence in BMO. Thus

$$\deg(u_{j|\partial\Omega}, \partial\Omega, Y) \longrightarrow \deg(u_{|\partial\Omega}, \partial\Omega, Y)$$

by Theorem 1. Furthermore, the integrals on the right of (20) for the u_j tend to that for u.

We use Corollary 8 to prove a stronger result:

THEOREM 2. *Let* Ω *and* Z *be smooth bounded domains in* $\mathbb{R}^n$ *with* $\partial\Omega$ *and* ∂Z *connected. Let* $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ *be such that*

$$u(\partial\Omega) \subset \partial Z$$

and

$$\deg(u_{|\partial\Omega}, \partial\Omega, \partial Z) \neq 0. \tag{21}$$

Then

$$\operatorname{ess} R(u) \supset \overline{Z}.$$

Proof. Since $\operatorname{ess} R(u)$ is closed it suffices to show that $\operatorname{ess} R(u) \supset Z$. Suppose not, i.e., suppose that some $z_0 \in Z$ is not in $\operatorname{ess} R(u)$. Then a closed ball $\overline{B}_r(z_0)$ lies in Z and is disjoint from $\operatorname{ess} R(u)$. Let P be the nearest point projection onto $\overline{B}_r(z_0)$. Set

$$v = P \circ u.$$

Clearly $v \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $v(\Omega) \subset S_r(z_0)$, the sphere of radius r centred at z_0. Hence, by Corollary 8,

$$\deg(v_{|\partial\Omega}, \partial\Omega, S_r) = 0. \tag{22}$$

Next we claim that

$$\deg(u_{|\partial\Omega}, \partial\Omega, \partial Z) = \deg(v_{|\partial\Omega}, \partial\Omega, S_r). \tag{23}$$

The conclusion of the theorem is then an immediate consequence of (21–23).

Proof of (23). We reduce it to the smooth case. Namely, by Lemma 8, with $Y = \partial Z$, we know that there is a sequence (u_j) of smooth maps from $\overline{\Omega}$ into $\mathbb{R}^n$ such that

$u_j(\partial\Omega) \subset \partial Z$, and $u_j \to u$ in $W^{1,n}$. Since $u_{j|\partial\Omega} \to u_{|\partial\Omega}$ in $W^{1-\frac{1}{n},n}(\partial\Omega)$ it also converges in VMO$(\partial\Omega)$. Set $v_j = Pu_j$. Applying Lemma A.8 we find that $v_{j|\partial\Omega} \to v_{|\partial\Omega}$ in VMO$(\partial\Omega)$. By Theorem 1 and Example 2 of Section I.2, for j large,

$$\deg(u_{j|_{\partial\Omega}}, \partial\Omega, \partial Z) = \deg(u_{|_{\partial\Omega}}, \partial\Omega, \partial Z)$$

and

$$\deg(v_{j|_{\partial\Omega}}, \partial\Omega, S_r) = \deg(v_{|\partial\Omega}, \partial\Omega, S_r).$$

However, it is well known that

$$\deg(u_{j|\partial\Omega}, \partial\Omega, \partial Z) = \deg(u_j, \Omega, z_0)$$

and

$$\deg(v_{j|\partial\Omega}, \partial\Omega, S_r) = \deg(v_j, \Omega, z_0).$$

Finally, the two degrees on the right hand sides are the same by the following homotopy

$$H_t(x) = tu_j(x) + (1-t)v_j(x), \quad t \in [0,1].$$

This completes the proof of (23). □

REMARK 9. The assumption that u belongs to $W^{1,n}$ in Theorem 2 is sharp in that it may not be replaced by $u \in W^{1,p}$ with $p < n$ — even if u is smooth near $\partial\Omega$, so that the degree of $u_{|\partial\Omega}$ makes sense. Namely, for $n \geq 2$ and $\Omega = B_1(0)$ the map $u(x) = x/|x|$ is in $W^{1,p}(\Omega, \mathbb{R}^n)$ for any $p < n$; moreover $u_{|\partial\Omega} = \mathrm{Id}$ and so has degree one, but the conclusion of Theorem 2 does not hold since $\operatorname{ess} R(u) = S^{n-1}$.

The reader may ask if in Theorem 2, the condition $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ may be replaced by $u \in$ VMO$(\Omega, \mathbb{R}^n)$. This is a delicate issue, since maps in VMO(Ω) do not in general admit a trace on $\partial\Omega$. We will be led in Part II, Section 3, to the notion of a special class of maps in VMO(Ω) admitting a trace on $\partial\Omega$, which belongs again to VMO$(\partial\Omega)$. For such a class, which includes $W^{1,n}(\Omega)$, we will have a generalisation of Theorem 2 (in Part II, Section 4).

§I.5. Further comments

1. One may discuss BMO and VMO maps from compact X to compact Y even if their dimensions are different, say maps of S^n to S^k. The space of continuous maps from X to Y decomposes naturally into its components $\mathcal{C}_i$ namely, maps u and v are in the same component if there is a homotopy within $C^0(X,Y)$ of u to v. Similarly, the space of VMO maps from X to Y also decomposes into components via homotopy. There are two natural notions of homotopy for maps in VMO:

a) Two maps $u, v \in \mathrm{VMO}(X, Y)$ are said to be homotopic within $\mathrm{VMO} \cap L^1$ if there is some deformation $H \in C([0,1], \mathrm{VMO} \cap L^1)$ such that $H(0) = u$ and $H(1) = v$.

b) Two maps $u, v \in \mathrm{VMO}(X, Y)$ are said to be homotopic within VMO if there is some deformation $H \in C([0,1], \mathrm{VMO})$ such that $H(0) = u$ and $H(1) = v$.

Clearly, the first notion is stronger and, in general, it is *strictly stronger* (see Remark A.6). Surprisingly, the two notions are *equivalent* in the special case where $Y = S^k, k \geq 1$, (see Lemma A.23).

Homotopy classes of $\mathrm{VMO}(X, Y)$ in the sense of definition a) — homotopy within $\mathrm{VMO} \cap L^1$ — are in one-to-one correspondence with the homotopy classes of $C^0(X, Y)$. They are simply the closures of the above $\mathcal{C}_i$ in $\mathrm{VMO} \cap L^1$ (see Lemma A.21).

In contrast, the spaces $L^p(X, Y)$, $1 \leq p \leq \infty$, and $\mathrm{BMO}(X, Y)$ are arcwise connected. It suffices to prove this for $p = \infty$. We sketch a proof.

Step 1. Denote by $\mathrm{PC}(X, Y)$ the set of measurable maps from X to Y taking on only a finite number of values in Y. Given any measurable map $f : X \to Y$ and any $\varepsilon > 0$, there exists $g \in \mathrm{PC}(X, Y)$ such that $\|f - g\|_{L^\infty} < \varepsilon$.

Step 2. For $t \in [0,1]$, $tf(x) + (1-t)g(x)$ lies within ε of Y. Hence $h_t(x) = P(tf(x) + (1-t)g(x))$ lies in Y and connects g to f continuously in t, within $L^\infty(X, Y)$.

Step 3. Given $g_0, g_1 \in \mathrm{PC}(X, Y)$, they may be connected by a continuous arc within $L^\infty(X, Y)$. Namely, we may always assume that g_0 and g_1 have the form

$$g_0 = \Sigma \chi_{\omega_i} a_i, \quad g_1 = \Sigma \chi_{\omega_i} b_i$$

for some finite partition (ω_i) of X, with $a_i, b_i \in Y$. For each i let $\varphi_i(t), 0 \leq t \leq 1$, be a continuous arc in Y connecting a_i to b_i. Then the maps

$$g_t(x) = \Sigma \chi_{\omega_i}(x) \varphi_i(t) \quad 0 \leq t \leq 1$$

connect g_0 to g_1.

A continuous map from X to Y naturally induces a map from homology in X to homology in Y. The same is true for a VMO map u — via approximation by u_ε.

2. A. Granas pointed out that several authors have previously considered fixed point properties, and degree theory, for some classes of maps which are not continuous (see O. H. Hamilton [1], J. Stallings [1], H. A. De Kleine and J. E. Girolo [1]). A class which plays an essential role in their considerations is one introduced by J. Nash [1] called "connectivity maps". By this is meant that the graph of such a map f over every connected subset of X, is a connected set. H. A. De Kleine and J. E. Girolo [1] developed a degree theory for a somewhat more general class ("almost continuous" maps). However, their degree is a collection of integers. We do not see how our degree is related to theirs.

We point out, however, that if $u \in \mathrm{VMO}(X, Y)$ and A is a connected subset of X then $\Sigma = \operatorname{ess} R(u_{|A})$ is connected. This may be seen as follows: If Σ is not connected then $\Sigma = \Sigma_1 \cup \Sigma_2$ where Σ_1, Σ_2 are nonempty disjoint closed sets. Recall that since $u \in \mathrm{VMO}$,

$$\operatorname{dist}(u_\varepsilon(A), \Sigma) \longrightarrow \text{ as } \varepsilon \longrightarrow 0.$$

On the other hand, $u_\varepsilon(A)$ is connected. Therefore for ε small, $\operatorname{dist} u_\varepsilon(A)$ to either Σ_1 or Σ_2 is $< \frac{\delta}{2} = \frac{1}{2}\operatorname{dist}(\Sigma_1, \Sigma_2)$. Consequently, using a sequence $\varepsilon_i \to 0$, we conclude that $\operatorname{ess} R(u_{|A})$ is contained *either* in Σ_1 *or* in Σ_2. Impossible.

3. Our degree theory holds in particular for a map $u \in H^{1/2}(S^1, S^1)$. As remarked earlier, L. Boutet de Monvel and O. Gabber previously defined a degree for such maps, given by

$$\deg u = \frac{1}{2\pi i} \int_0^{2\pi} \bar{u}\, du. \tag{24}$$

This integral makes sense since $\bar{u} \in H^{1/2}$ and $\dot{u} \in H^{-1/2}$. That u is in $H^{1/2}$ may be expressed in terms of its Fourier coefficients. If

$$u = \sum_{-\infty}^{\infty} a_j e^{ij\theta}$$

then

$$\|u\|^2_{H^{1/2}} = \sum_{-\infty}^{\infty} |j|\ |a_j|^2.$$

I. M. Gelfand raised the question: what is $\deg u$ in terms of its Fourier coefficients? More generally, for maps $u : S^n \to S^n$, what is $\deg u$ in terms of its expansion coefficients in spherical harmonics? It is easily seen from (24) that

$$\deg u = \sum_{-\infty}^{\infty} j |a_j|^2. \tag{25}$$

From the fact that $|u(\theta)| = 1$ it is not a priori clear that the right hand side of (25) is an integer. The condition that $|u(\theta)| = 1$ on S^1 is equivalent to

$$\sum |a_j|^2 = 1$$

$$\sum_{j=-\infty}^{\infty} \bar{a}_j a_{j+k} = 0 \quad \text{for all integers } k \neq 0.$$

For a continuous map (or VMO map) $u : S^1 \to S^1$, its Fourier coefficients are defined. But the series in (25) need not be absolutely convergent.

Open Problem: What summation process makes it summable so that (25) holds?

When working with maps from S^n to S^n one would use the formula that

$$\deg u = \frac{1}{|S^n|} \int_{S^n} \det(u, u_{x_1}, \ldots, u_{x_n}) d\sigma$$

computed using normal geodesic coordinates $(x_1, \ldots, x_n)$. If one represents u via spherical harmonics, this leads to some complicated expressions.

4. Recently, M. Giaquinta, G. Modica and J. Soucek [1] introduced a notion of degree for rectifiable currents and for approximately differentiable maps with Jacobian determinant in L^1 (see also a related work by M. Esteban and S. Müller [1]). We do not know if it is related to our degree.

5. To every function $\varphi \in L^\infty(S^1, \mathbb{C})$ corresponds a Toeplitz operator T_φ in the Hardy space H^2; see e.g. R. G. Douglas [1], Chapter 7. When $\varphi \in C^0(S^1, \mathbb{C})$, T_φ is Fredholm if and only if $|\varphi| \geq \alpha > 0$; moreover

$$\operatorname{ind}(T_\varphi) = -\deg\left(\frac{\varphi}{|\varphi|}, S^1, S^1\right).$$

A similar result holds assuming only $\varphi \in L^\infty(S^1, \mathbb{C}) \cap \mathrm{VMO}(S^1, \mathbb{C})$; see Theorem 7.36 in Douglas [1] (where it is stated in different terms). We will return to this topic in Part II.

§I.6. Lifting of BMO maps

This section is largely inspired by an interesting result in R. Coifman and Y. Meyer [1].

One form of their result asserts that there exist constants $\delta, C > 0$ such that every $u \in \mathrm{BMO}((0,1), S^1)$ with

$$\|u\|_{\mathrm{BMO}} < \delta \tag{26}$$

may be lifted as

$$\begin{cases} u = e^{i\varphi}, \ \varphi \in \mathrm{BMO}((0,1), \mathbb{R}) \\ \|\varphi\|_{\mathrm{BMO}} \leq C\|u\|_{\mathrm{BMO}}. \end{cases} \tag{27}$$

We present variants of this result, in that we replace the interval $(0,1)$ by our compact n-dimensional manifold X without boundary. In addition, we will see that (26) is not needed when working with VMO, as opposed to BMO.

Here is a first result:

THEOREM 3. *Any $u \in \mathrm{VMO}(X, S^1)$ which is homotopic within VMO(X, S^1) to a constant map may be uniquely written as*

$$u = e^{i\varphi} \text{ with } \varphi \in \mathrm{VMO}\ (X, \mathbb{R}),\ 0 \leq \fint_X \varphi < 2\pi. \tag{28}$$

Furthermore, the map $u \mapsto \varphi$ is continuous from VMO$\cap L^1$ (respectively VMO) into VMO$\cap L^1$ (respectively VMO).

REMARK 10. (i) Note that we give no estimate of $\|\varphi\|_{\mathrm{BMO}}$. (ii) The converse of Theorem 3 also holds, namely, any u of the form in (28) is homotopic to a constant within VMO — because of Lemma A.8, via the homotopy $e^{it\varphi}$, $0 \leq t \leq 1$. (iii) As a consequence of Theorem 3, and Property 2 in Section I.4, we may assert that any map $u \in \mathrm{VMO}(S^1, S^1)$ with degree zero, may be written as $e^{i\varphi}$ with $\varphi \in \mathrm{VMO}(S^1, \mathbb{R})$, and conversely. More generally, any $u \in \mathrm{VMO}(S^1, S^1)$ may be written as

$$u(\theta) = e^{ik\theta + i\varphi(\theta)}$$

where $k = \deg u$, and $\varphi \in \mathrm{VMO}(S^1, \mathbb{R})$. This is easily seen by considering $e^{-ik\theta}u(\theta)$. (iv) If $\pi_1(X) = 0$, then *every* map $u \in \mathrm{VMO}(X, S^1)$ may be written as in (28). This is a consequence of the corresponding fact for continuous maps: one repeats the argument used in proving Theorem 3.

A variant which is closer to the result above of Coifman-Meyer is

THEOREM 4. *There exists $\delta > 0$ (depending only on X), such that if*

$$u \in \mathrm{BMO}\ (X, S^1)$$

and

$$\|u\|_{\mathrm{BMO}} \leq \delta, \tag{29}$$

then

$$u = e^{i\varphi} \text{ with } \varphi \in \mathrm{BMO}(X, \mathbb{R}),$$

and

$$\|\varphi\|_{\mathrm{BMO}} \leq 4\|u\|_{\mathrm{BMO}}. \tag{30}$$

The central idea in the proof of Theorem 3 and 4 is the same. We proceed as follows: (i) We approximate u by our u_ε (using averaging, $\bar{u}_\varepsilon$, and projection on S^1);

(ii) We lift u_ε as $u_\varepsilon = e^{i\varphi_\varepsilon}$, and derive estimates for φ_ε — in case of Theorem 4 we prove (30) for φ_ε; (iii) Finally, we show that φ_ε converges as $\varepsilon \to 0$ to the desired function $\varphi \in \mathrm{BMO}(X, \mathbb{R})$.

Our proof of Step (ii) is very different from that in R. Coifman and Y. Meyer [1]. It relies on the John-Nirenberg inequality, in the form described in Appendix B, while they used estimates of commutators. We have been informed that L. Carleson, in a personal communication to Y. Meyer in 1979, also proved Step (ii) using the John-Nirenberg inequality rather than commutators.

There is a result which includes Theorem 3 and part of Theorem 4; it involves $u \in \mathrm{BMO}(X, S^1)$ with small distance to $\mathrm{VMO}(X, S^1)$ — see Theorem 5.

Proof of Theorem 3. Since $u \in \mathrm{VMO}(X, S^1)$, there exists some $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$, $u_\varepsilon = P\bar{u}_\varepsilon$ is well defined, and converges to u in $\mathrm{BMO} \cap L^1$ as $\varepsilon \to 0$. In what follows, we always take $\varepsilon < \varepsilon_0$, and sometimes restrict ε further.

Step 1. There is some $\varepsilon_1 \le \varepsilon_0$ such that for every $\varepsilon \le \varepsilon_1$, u_ε may be written as

$$u_\varepsilon = e^{i\varphi_\varepsilon} \tag{31}$$

with $\varphi_\varepsilon \in C^0(X, \mathbb{R})$ and

$$0 \le \fint_X \varphi_\varepsilon < 2\pi. \tag{32}$$

Proof. We rely on Lemma A.23 according to which u is homotopic to a constant within $\mathrm{VMO} \cap L^1$. Denote the homotopy by $H(t), 0 \le t \le 1$, with $H(0) = u$, $H(1) =$ a constant. Since $H([0,1])$ is a compact set in VMO, by Lemma 4, there is some $\varepsilon_1 \le \varepsilon_0$ such that for every $\varepsilon \le \varepsilon_1, H(t)_\varepsilon$ is well defined. It yields a homotopy of u_ε to a constant within $C^0(X, S^1)$. Here we use the fact that $t \mapsto H(t)$ is continuous in L^1. We are therefore reduced to the classical continuous case, yielding (31). By adding an appropriate integral multiple of 2π to φ_ε, we may achieve (32).

From now on, $\varepsilon \le \varepsilon_1$.

Step 2. There is a constant $\overline{C}$ depending only on X such that

$$M_t(\varphi_\varepsilon) \le 2M_t(u_\varepsilon) + \overline{C} M_t^2(\varphi_\varepsilon) \quad \forall t < r_0, \quad \forall \varepsilon \le \varepsilon_1. \tag{33}$$

Proof. Since

$$|e^{it} - 1 - it| \le \frac{1}{2} t^2 \quad \forall t \in \mathbb{R}, \tag{34}$$

one easily finds that, for $\alpha, \beta \in \mathbb{R}$,

$$|\alpha - \beta| \le |e^{i\alpha} - e^{i\beta}| + \frac{1}{2}(\alpha - \beta)^2. \tag{35}$$

To verify (33) we recall that

$$M_t(\varphi_\varepsilon) \le \sup_{\substack{x\in X \\ r\le t}} \fint_{B_r(x)} \fint_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|.$$

Applying (35), with $\alpha = \varphi_\varepsilon(y)$, $\beta = \varphi_\varepsilon(z)$ we find, as in ($1''$), that

$$M_t(\varphi_\varepsilon) \le 2M_t(u_\varepsilon) + \frac{1}{2} \sup_{\substack{x\in X \\ r\le t}} \fint_{B_r(x)} \fint_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^2. \tag{36}$$

We now use Lemma B.4 to see that the last term in (36) is

$$\le \overline{C} M_t^2(\varphi_\varepsilon);$$

(33) is proved.

Step 3. There is some $a > 0$ such that

$$M_t(\varphi_\varepsilon) \le 4M_t(u_\varepsilon) \quad \forall t \le a, \quad \forall \varepsilon \le \varepsilon_1. \tag{37}$$

Proof. Since for $\varepsilon \le \varepsilon_1$, the family (u_ε) is compact in VMO, there is some $a > 0$, by Lemma 4, such that

$$M_a(u_\varepsilon) \le \frac{1}{9\overline{C}} \quad \forall \varepsilon \le \varepsilon_1.$$

We now claim that for such a, $M_t(\varphi_\varepsilon) < \frac{1}{2\overline{C}}$ $\forall t \le a$ — which yields (37) via (33).

Indeed, since $M_t(\varphi_\varepsilon)$ tends to zero as $t \to 0$, we see by (33), that $M_t(\varphi_\varepsilon) < \frac{1}{2\overline{C}}$ in some neighbourhood of $t = 0$. If the claim were false, then since $M_t(\varphi_\varepsilon)$ is continuous in t — see Lemma A.15 — there would be a first value of $t \le a$ such that

$$M_t(\varphi_\varepsilon) = \frac{1}{2\overline{C}}.$$

But then by (33), for that t,

$$\frac{1}{2\overline{C}} = M_t(\varphi_\varepsilon) \le 4M_t(u_\varepsilon) \le \frac{4}{9\overline{C}}.$$

Impossible.

Step 4. Existence of φ. We make use of an elementary, but very useful, observation of G. David (which simplifies our original presentation).

LEMMA 9. *Let $f \in C^0((0,\alpha),\mathbb{R})$, for some $\alpha > 0$, and assume that $\lim_{\varepsilon\to 0} e^{if(\varepsilon)}$ exists. Then $\lim_{\varepsilon\to 0} f(\varepsilon)$ exists.*

The proof of Lemma 9 is obvious and relies on the connectedness of the range of f.

Since $u_\varepsilon \in C^0((0,\varepsilon_1)\times X, S^1)$ we may lift it as $u_\varepsilon = e^{i\varphi_\varepsilon}$ with $\varphi_\varepsilon \in C^0((0,\varepsilon_1)\times X,\mathbb{R})$. Recall that $\lim_{\varepsilon\to 0} u_\varepsilon(x)$ exists at every Lebesgue point x of u. By Lemma 9, $\varphi(x) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(x)$ exists at every such x, hence a.e. on X. Moreover $u = e^{i\varphi}$.

We now prove that $\varphi \in \mathrm{VMO}$. From (37) we deduce that, for every $x \in X$,

$$⨍_{B_r(x)} ⨍_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)| \le 8M_r(u_\varepsilon) \quad \forall r \le a, \quad \forall \varepsilon \le \varepsilon_1.$$

Recall (see Corollary 3) that

$$M_r(u_\varepsilon) \le C(M_r(u) + M_\varepsilon(u)).$$

We may then pass to limit as $\varepsilon \to 0$, using Fatou's lemma, and conclude that

$$⨍_{B_r(x)} ⨍_{B_r(x)} |\varphi(y) - \varphi(z)| \le 8CM_r(u) \quad \forall r \le a.$$

It follows, by Sarason's Lemma 3, that $\varphi \in \mathrm{VMO}$.

Step 5. Uniqueness. Suppose φ_1 and φ_2 are solutions of (28). Then

$$\eta = \frac{1}{2\pi}(\varphi_1 - \varphi_2) \in \mathbb{Z} \text{ a.e.}$$

On the other hand, since $\eta \in \mathrm{VMO}$, $\operatorname{ess} R(\eta)$ is connected, see Item 2 in Section I.5. Hence $\operatorname{ess} R(\eta)$ is reduced to a point, i.e., η is constant; by (28), $\eta = 0$.

Step 6. Claim: There exists $\alpha > 0$ depending only on X such if $a < r_0$ and

$$M_a(u) \le \alpha,$$

then

$$M_a(\varphi) \le 4M_a(u).$$

Here u is as in the theorem, and φ is the unique solution of (28).

Proof. We may take $\alpha = 1/(9\overline{C})$ of Step 3 and repeat the arguments of Steps 2 and 3, deleting ε everywhere.

Step 7. Continuous dependence of $\mathbf{u} \mapsto \boldsymbol{\varphi}$. At first, let (u_j) be a sequence converging in VMO to u, each homotopic to a constant. We have the corresponding φ_j. By Lemma 4, we know that there exists some a_0 such that — for α in Step 6 —

$$M_{a_0}(u_j) \le \alpha \quad \forall j.$$

Hence for $a \le a_0$,

$$M_a(\varphi_j) \le 4M_a(u_j) \quad \forall j. \tag{38}$$

Since $M_a(u_j) \to 0$ as $a \to 0$, uniformly in j, the same is true for $M_a(\varphi_j)$. Consequently by Lemma 4, again, the (φ_j) lie in a compact set in VMO. A subsequence, still called (φ_j), converges in VMO to ψ.

In view of the fact that $0 \le ⨍_X \varphi_j \le 2\pi$ we may assume that

$$⨍_X (\varphi_j - \psi) \to \ell.$$

By Lemma 1, $\varphi_j \to \psi + \ell =: \varphi$ in L^1. For a further subsequence, still denoted φ_j, $\varphi_j \to \varphi$ a.e.

1. Continuity from VMO$\cap L^1$ into VMO$\cap L^1$. We suppose, then, in addition that $u_j \to u$ in L^1. Consequently

$$u = e^{i\varphi}.$$

Convergence of the full sequence (φ_j) follows from the uniqueness of φ.

2. Continuity from VMO to VMO. We established above that for a subsequence

$$\varphi_{j_k} \to \varphi \quad \text{in VMO} \cap L^1.$$

We can no longer infer that $u = e^{i\varphi}$; we can only say that

$$u = e^{i\varphi} + c$$

for some constant c. We need only consider the case $c \ne 0$. In this case, u takes its values in $S^1 \cap (S^1 + c)$, which consists of one or two points. Since $\operatorname{ess} R(u)$ is connected (see paragraph 2 in Section I.5), u must be a constant, thus also φ. Since $\varphi_j \to \varphi$ in BMO it follows that $\|\varphi_j\|_{\mathrm{BMO}} \to 0$. Continuity is proved. □

We turn now to the

Proof of Theorem 4. Given $u \in \mathrm{BMO}(X,Y)$, recall that $\bar{u}_\varepsilon$ is defined for every $\varepsilon < r_0$; by (7),

$$\operatorname{dist}(\bar{u}_\varepsilon(x), Y) \leq M_\varepsilon(u) \leq \|u\|_{\mathrm{BMO}}, \quad \forall x \in X.$$

Thus if $\|u\|_{\mathrm{BMO}} \leq$ some small δ (depending only on X), we may define $u_\varepsilon = P\bar{u}_\varepsilon$ for every $\varepsilon < r_0$.

Now consider $u \in \mathrm{BMO}(X, S^1)$ with $\|u\|_{\mathrm{BMO}} \leq \delta$. We will show that for δ sufficiently small, $u = e^{i\varphi}$, with $\|\varphi\|_{\mathrm{BMO}} \leq 4\|u\|_{\mathrm{BMO}}$.

Step 1. According to Lemma A.18 there is a δ depending only on X such that if $\|u\|_{\mathrm{BMO}} \leq \delta$, then for each $\varepsilon < r_0$, u_ε is homotopic (within $C^0(X, S^1)$) to a constant. Hence we lift u_ε and write it as

$$u_\varepsilon = e^{i\varphi_\varepsilon}, \quad \varphi_\varepsilon \in C^0(X, \mathbb{R}), \tag{39}$$

with

$$0 \leq \fint_X \varphi_\varepsilon < 2\pi. \tag{40}$$

By Step 2 in the proof of Theorem 3, we have

$$M_t(\varphi_\varepsilon) \leq 2M_t(u_\varepsilon) + \overline{C} M_t^2(\varphi_\varepsilon) \quad \forall t < r_0, \quad \forall \varepsilon < r_0. \tag{41}$$

Now

$$M_t(u_\varepsilon) \leq \|u_\varepsilon\|_{\mathrm{BMO}} \leq C\|\bar{u}_\varepsilon\|_{\mathrm{BMO}}$$

by Lemma 2, for some constant C.

Using Corollary 2 we conclude that

$$M_t(u_\varepsilon) \leq C^\star \|u\|_{\mathrm{BMO}} \leq C^\star \delta \quad \forall t, \varepsilon < r_0. \tag{42}$$

Next, arguing as in Step 3, we find that if

$$C^\star \delta < \frac{1}{9\overline{C}}, \tag{43}$$

then

$$M_t(\varphi_\varepsilon) \leq 4M_t(u_\varepsilon) \quad \forall t, \varepsilon < r_0. \tag{44}$$

In particular, by (42),

$$\|\varphi_\varepsilon\|_{\mathrm{BMO}} \leq 4C^\star \|u\|_{\mathrm{BMO}} \leq 4C^\star \delta. \tag{45}$$

Step 2. Existence of φ. We see, as in the proof of Theorem 3 (Step 4), that $\varphi(x) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x)$ exists a.e., and that $u = e^{i\varphi}$. From (45) we deduce that, for every $B_r(x)$,

$$\fint_{B_r(x)}\fint_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)| \le 8C^*\|u\|_{\mathrm{BMO}}. \tag{46}$$

Using Fatou's lemma we conclude that $\varphi \in \mathrm{BMO}$ and

$$\|\varphi\|_{\mathrm{BMO}} \le 8C^*\|u\|_{\mathrm{BMO}}. \tag{47}$$

Finally, we get rid of the factor C^* in (47). Namely, as in Step 2 of the proof of Theorem 3, applied to u and φ, we find

$$\begin{aligned}\|\varphi\|_{\mathrm{BMO}} &\le 2\|u\|_{\mathrm{BMO}} + \overline{C}\|\varphi\|^2_{\mathrm{BMO}}\\ &\le 2\|u\|_{\mathrm{BMO}} + 8C^*\overline{C}\delta\|\varphi\|_{\mathrm{BMO}}\end{aligned}$$

by (47). Assuming $16C^*\overline{C}\delta \le 1$ we obtain the desired conclusion. □

REMARK 11. The reader may think that the space S^1 plays a special role in Theorems 3 and 4. However this is not the case. One considers, in addition to the target space Y, a covering space Z. For $Y = S^1, Z$ is $\mathbb{R}$.

We denote by F the covering map of Z to Y, i.e., F is onto, and every point in Z has a neighbourhood U such that F is a diffeomorphism of U onto $F(U)$.

The proof of Theorem 3 extends to give the following:

THEOREM 3′. *Any u in $VMO(X,Y)$ which is homotopic within $VMO(X,Y) \cap L^1$ to a constant map may be written as*

$$u = F\circ\varphi \quad \textit{for some } \varphi \in \mathrm{VMO}(X,Z).$$

In the proof the following inequality replaces (35); here we use a Riemannian metric on Y (and its lift to Z): For $\alpha,\beta \in Z$,

$$\mathrm{dist}(\alpha,\beta) \le \mathrm{dist}(F(\alpha),F(\beta)) + C\,\overset{2}{\mathrm{dist}}(\alpha,\beta).$$

Similarly, one has an extension of Theorem 4:

THEOREM 4′. *There exists δ depending on X, Y and Z such that if $u \in \mathrm{BMO}(X,Y)$ and $\|u\|_{\mathrm{BMO}} \le \delta$, then u may be lifted to a map $\varphi \subset \mathrm{BMO}(X,Z)$, i.e., $u = F\circ\varphi$ such that*

$$\|\varphi\|_{\mathrm{BMO}} \le 4\|u\|_{\mathrm{BMO}}.$$

REMARK 12. We have carried out lifting for VMO or BMO maps. Can one do the same for Sobolev maps, say $u \in W^{s,p}(X, S^1)$? Some partial results are known (see F. Bethuel and X. Zheng [1], F. Demengel [1], P. Mironescu in H. Brezis [1]), also for X with boundary. If X is a bounded domain in $\mathbb{R}^n$ then the answer is positive in the following cases: (a) $sp > n$ (by Sobolev embedding) and (b) $s = 1, p = 2$, in any dimension. However if $s = 1$ and $p < 2$, the answer is sometimes negative.

REMARK 13. We now present a lifting result related to both Theorems 3 and 4. In doing so we consider the class $\mathcal{C}_\delta$ of maps $u \in \mathrm{BMO}(X, S^1)$ such that

$$d_0(u) = \mathrm{dist}(u, \mathrm{VMO}(X, \mathbb{R}^2)) \leq \delta \tag{48}$$

and such that there is continuous deformation $h(t, x)$,

$$\begin{cases} h \in C([0,1], \mathrm{BMO}(X, S^1) \cap L^1) \text{ satisfying} \\ h(0) = u, h(1) = \text{constant and} \\ d_0(h(t)) \leq \delta \quad \forall t \in [0,1]. \end{cases} \tag{49}$$

Recall that d_0 is equivalent to M_0 (see Lemma 3′); in fact $M_0 \leq d_0 \leq AM_0$.

THEOREM 5. *There exist $\delta, C > 0$ depending only on X (and a Riemannian metric on it) such that for every $u \in \mathcal{C}_\delta$, there is a lifting $\varphi \in \mathrm{BMO}(X, \mathbb{R})$ of u, i.e.,*

$$u = e^{i\varphi}, \tag{50}$$

satisfying

$$d_0(\varphi) \leq C d_0(u). \tag{51}$$

Here, $d_0(\varphi) = \mathrm{dist}(\varphi, \mathrm{VMO}(X, \mathbb{R}))$.

We do not include the proof here. It follows the lines of that of Theorem 4, but there are some additional technical points which the reader is spared.

Appendix A. Some useful estimates on BMO, et al.

We present some simple facts about BMO and Sobolev maps on manifolds. These are well known to people working on the subject, but are not all easily found in the literature. Unless stated otherwise, X is always assumed to be a connected compact Riemannian manifold without boundary.

LEMMA A.1. *There exists a constant C depending on X, such that for every $u \in \mathrm{BMO}(X,\mathbb{R})$,*

$$\|u\|_{L^1(X)} \leq C\|u\|_{\mathrm{BMO}} + \left|\int_X u\right|. \tag{A.1}$$

Proof. We may suppose that $\displaystyle\int_X u = 0$. We argue by contradiction; suppose there is no such C. Then there is a sequence u_j with $\displaystyle\int_X u_j = 0$, such that

$$\|u_j\|_{\mathrm{BMO}} \to 0 \quad \text{and} \quad \|u_j\|_{L^1} = 1. \tag{A.2}$$

Cover X by a finite number of balls $B_i = B_{r_0/2}(x_i)$. It follows from the definition of BMO that a subsequence of the u_j converges to a constant a.e., and in L^1, on each B_i. Necessarily, the constant is the same for all B_i. Since $\displaystyle\int_X u_j = 0$, the constant must be zero. This contradicts the second assertion in (A.2). □

LEMMA A.2. *Let $u \in \mathrm{BMO}(X,\mathbb{R}^N)$ and let F be a uniformly continuous mapping from $\mathbb{R}^N$ to $\mathbb{R}^D$. Then $F \circ u \in \mathrm{BMO}(X,\mathbb{R}^D)$.*

We shall make use of

LEMMA A.3. *Let F be a uniformly continuous map of $\mathbb{R}^N$ to $\mathbb{R}^D$. Then F has a concave modulus of continuity ω.*

Proof. For $t > 0$, set

$$\alpha(t) = \sup_{|a-b|<t} |F(a) - F(b)|.$$

It is easy to verify that $\forall t_0 > 0,\ \exists A = A(t_0)$ such that

$$\alpha(t) \leq At \quad \text{for } t \geq t_0.$$

(A depends on the function F). Hence $\alpha(t) \leq At + \alpha(t_0)\ \ \forall t \geq 0$. Thus we may introduce the concave hull $\omega(t)$ of $\alpha(t)$, namely the least concave function $\geq \alpha$. Since $\omega(t) \leq A(t_0)t + \alpha(t_0)$ it follows that

$$\lim_{t \searrow 0} \omega(t) \leq \alpha(t_0),$$

and consequently this limit must be zero. □

Proof of Lemma A.2. We use $(1')$ and $(1'')$. For any $B_\varepsilon(x)$ in X, $\varepsilon < r_0$,

$$\begin{aligned}\fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} |F(u(y)) - F(u(z))| &\le \fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} \omega(|u(y) - u(z)|) \\ &\le \omega\left(\fint_{B_\varepsilon(x)}\fint_{B_\varepsilon(x)} |u(y) - u_(z)|\right)\end{aligned}$$

by the concavity of ω. Hence, by $(1'')$,

$$\|F\circ u\|_{\mathrm{BMO}} \le \|F\circ u\|_\star \le \omega(\|u\|_\star) \le \omega\left(2\|u\|_{\mathrm{BMO}}\right).$$

$\square$

We shall often make use of the following simple

LEMMA A.4. *Given two measurable sets $A \subset B$ in a measure space, for any integrable function f*

$$\left|\fint_A f - \fint_B f\right| \le \fint_A \left|f - \fint_B f\right| \le \frac{|B|}{|A|}\fint_B \left|f - \fint_B f\right| \tag{A.3}$$

$$\fint_A \left|f - \fint_A f\right|,\ \fint_B \left|f - \fint_A f\right| \le 2\frac{|B|}{|A|}\fint_B \left|f - \fint_B f\right|. \tag{A.4}$$

We often refer to this as Lemma A–B.

Proof. Inequality (A.3) is obvious. The first term in (A.4) is bounded by

$$\fint_A \left|f - \fint_B f\right| + \left|\fint_B f - \fint_A f\right| \le 2\frac{|B|}{|A|}\fint_B \left|f - \fint_B f\right|,$$

by (A.3). Finally, the second term in (A.4) is bounded by

$$\fint_B \left|f - \fint_B f\right| + \left|\fint_B f - \fint_A f\right| \le \left(1 + \frac{|B|}{|A|}\right)\fint_B \left|f - \fint_B f\right|.$$

$\square$

We now restate and prove Sarason's Lemma 3'. Recall that for $u \in$ BMO $(X, \mathbb{R}^N)$ and $0 < a < r_0(X)$

$$M_a = M_a(u) = \sup_{\substack{\varepsilon \leq a \\ x \in X}} \fint_{B_\varepsilon(x)} |u - \bar{u}_\varepsilon(x)| \leq \|u\|_{\mathrm{BMO}},$$

$$M_0 = M_0(u) = \lim_{a \searrow 0} M_a(u).$$

LEMMA A.5. *There is a constant A depending on X (and its metric) such that if $u \in \mathrm{BMO}(X, \mathbb{R}^N)$, then*

$$M_0(u) \leq \mathrm{dist}(u, \mathrm{VMO}(X, \mathbb{R}^N)) \leq A M_0(u). \tag{A.5}$$

In fact

$$\|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} \leq A M_\varepsilon(u) \quad \forall \varepsilon < r_0. \tag{A.6}$$

Proof. To prove the first inequality in (A.5), note that $\forall u, v \in \mathrm{BMO}(X, \mathbb{R}^N)$,

$$M_a(u) \leq M_a(v) + M_a(u - v) \quad \forall a \in [0, r_0).$$

For $v \in C^0(X, \mathbb{R}^N)$, $M_0(v) = 0$ and thus

$$M_0(u) \leq M_0(u - v) \leq \|u - v\|_{\mathrm{BMO}}.$$

This yields the first inequality in (A.5). □

The other inequality in (A.5) is an obvious consequence of (A.6). The proof of (A.6) relies on the following simple

LEMMA A.6. *There is a number B depending only on X such that for any given numbers ε, δ, $0 < \varepsilon \leq \delta < r_0$, any ball $B_\delta(x)$ in X may be covered by a finite number of balls $B_\varepsilon(x_i)$, $x_i \in B_\delta(x), i = 1, \dots, K$, such that $\mathrm{dist}(x_i, x_j) \geq \varepsilon$ for $i \neq j$, and*

$$\sum_1^K |B_\varepsilon(x_i)| \leq B |B_\delta(x)|. \tag{A.7}$$

Proof of Lemma A.6. Let $B_{\varepsilon/2}(x_i), i = 1, \dots, K$, be a maximal collection of disjoint balls with centres x_i in $B_\delta(x)$. From the maximality, it follows easily that

$$\cup B_\varepsilon(x_i) \supset B_\delta(x).$$

Since

$$B_{\varepsilon/2}(x_i) \subset B_{\delta+\varepsilon/2}(x) \subset B_{2\delta}(x),$$
$$\sum |B_{\varepsilon/2}(x_i)| \leq |B_{2\delta}(x)|.$$

Therefore

$$\sum |B_\varepsilon(x_i)| \leq C \sum_i |B_{\varepsilon/2}(x_i)|$$
$$\leq C|B_{2\delta}(x)| \leq C|B_\delta(x)|.$$

□

We now return to the

Proof of (A.6). We first claim that

$$|\bar{u}_\varepsilon(y) - \bar{u}_\varepsilon(z)| \le CM_\varepsilon(u) \;\; \forall \varepsilon < r_0, \;\; \forall y, z \in X \;\text{ with } d(y,z) < \varepsilon/2, \tag{A.8}$$

where C depends only on X.

Indeed, by Lemma A–B (i.e., Lemma A.4), the following inequalities hold:

$$|\bar{u}_{\varepsilon/2}(y) - \bar{u}_\varepsilon(z)| \le CM_\varepsilon(u), \tag{A.9}$$

$$|\bar{u}_{\varepsilon/2}(x) - \bar{u}_\varepsilon(x)| \le CM_\varepsilon(u) \quad \forall \varepsilon < r_0, \quad \forall x \in X. \tag{A.10}$$

Their combination yields (A.8).

For $\varepsilon < r_0$ we have to estimate $\|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}}$; more precisely we want to show that for any $B_\delta(x) \subset X$, $\delta < r_0$,

$$I = \fint_{B_\delta(x)} \left| (u - \bar{u}_\varepsilon) - \fint_{B_\delta(x)} (u - \bar{u}_\varepsilon) \right| \le AM_\varepsilon(u). \tag{A.11}$$

We distinguish two cases.

Case (i): $\delta < \varepsilon/4$. We have

$$\begin{aligned} I &\le \fint_{B_\delta(x)} |u - \fint_{B_\delta(x)} u| + \fint_{B_\delta(x)} \fint_{B_\delta(x)} |\bar{u}_\varepsilon(y) - \bar{u}_\varepsilon(z)| \\ &\le M_\delta(u) + CM_\varepsilon(u) \quad \text{by (A.8)}, \\ &\le CM_\varepsilon(u) \qquad \text{since } \delta < \varepsilon. \end{aligned}$$

Case (ii): $\delta \ge \varepsilon/4$. We now use the covering of $B_\delta(x)$ by $B_{\varepsilon/2}(x_i)$, $i = 1, \dots, K$ given by Lemma A.6. Then

$$I \le 2 \sum_1^K \frac{1}{|B_\delta(x)|} \int_{B_{\varepsilon/2}(x_i)} |u - \bar{u}_\varepsilon|,$$

so that

$$\begin{aligned} I &\le \frac{2}{|B_\delta(x)|} \sum_1^K \int_{B_{\varepsilon/2}(x_i)} \left[|u(y) - \bar{u}_{\varepsilon/2}(x_i)| + |\bar{u}_{\varepsilon/2}(x_i) - \bar{u}_\varepsilon(x_i)| + |\bar{u}_\varepsilon(x_i) - \bar{u}_\varepsilon(y)| \right] \\ &\le \frac{C}{|B_\delta(x)|} M_\varepsilon(u) \sum_1^K |B_{\varepsilon/2}(x_i)| \qquad \text{by (A.10) and (A.8)}. \end{aligned}$$

Using Lemma A.6 we conclude that

$$I \le CM_\varepsilon(u).$$

$\square$

A simple consequence of (A.8) is the following: For any two points y, z in X, with $d = \operatorname{dist}(y,z) \geq \varepsilon/2$,

$$|\bar{u}_\varepsilon(y) - \bar{u}_\varepsilon(z)| \leq C\frac{\operatorname{dist}(y,z)}{\varepsilon} M_\varepsilon(u), \tag{A.12}$$

where C depends only on X. Indeed there is a chain of points $y, y_1, y_2, \ldots, y_{k-1}, z$, with $k = 1 + [\frac{2d}{\varepsilon}]$, such that the distance of any two successive ones is bounded by $\varepsilon/2$. Adding the corresponding inequalities (A.8) we obtain (A.12).

Returning to Lemma A.2 we know that $F \circ u$ is in BMO whenever $u \in$ BMO and F is uniformly continuous. The same holds in VMO:

LEMMA A.7. *Let $u \in \mathrm{VMO}(X, \mathbb{R}^N)$ and let F be a uniformly continuous map from $\mathbb{R}^N$ to $\mathbb{R}^D$. Then*

$$F \circ u \in \mathrm{VMO}(X, \mathbb{R}^D).$$

Proof. Recall — see the proof of Lemma A.2 — that

$$\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |F(u(y) - F(u(z))| \leq \omega \left(\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(y) - u(z)| \right).$$

Since the right hand side goes to zero as $\varepsilon \to 0$, we conclude by Sarason's characterization of VMO that $F \circ u \in \mathrm{VMO}(X, \mathbb{R}^D)$. □

We now take up a more delicate property, namely the continuity of the map $u \mapsto F(u)$. The map F induces a map Ψ from BMO(VMO) into BMO(VMO). We show that Ψ is continuous at every point $u \in$ VMO. We show also that Ψ need *not* be continuous outside VMO.

LEMMA A.8. *Let F be a uniformly continuous map from $\mathbb{R}^N$ to $\mathbb{R}^D$. Then Ψ is continuous in the BMO $\cap$ L^1 topology at every point u in VMO$(X, \mathbb{R}^N)$.*

Proof. Let u be in $\mathrm{VMO}(X, \mathbb{R}^N)$. Given ε we shall show that there exists $\tau > 0$, depending on ε and u, such that if $v \in \mathrm{BMO}(X, \mathbb{R}^N)$ and $\|v\|_{\mathrm{BMO}} + \|v\|_{L^1} < \tau$, then

$$\|F(u+v) - F(u)\|_{\mathrm{BMO}} < \varepsilon,$$

i.e., we show that for any ball $B_\delta(x)$ in X, $\delta < r_0$,

$$\begin{aligned} J = \fint_{B_\delta(x)} \fint_{B_\delta(x)} &|F(u(y) + v(y)) - F(u(y)) \\ &- (F(u(z) + v(z)) - F(u(z)))| < \varepsilon. \end{aligned} \tag{A.13}$$

For $\xi \in X$, $p \in \mathbb{R}^N$, set

$$G(\xi, p) = F(u(\xi) + p) - F(u(\xi)).$$

Given a measurable set A in X, we wish to estimate

$$I := \fint_A \fint_A |G(y, v(y)) - G(z, v(z))| \, dz dy. \tag{A.14}$$

We make use of the concave modulus of continuity ω of F of Lemma A.3 and establish two estimates for I:

$$I \leq 2\omega\left(\fint_A |v|\right) \tag{A.15}$$

and

$$I \leq 2\omega\left(2\fint_A \left|u - \fint_A u\right|\right) + \omega\left(2\fint_A \left|v - \fint_A v\right|\right). \tag{A.16}$$

Since $G(\xi, p) \leq \omega(|p|)$, we have

$$\begin{aligned} I \leq 2\fint_A |G(y, v(y))| &\leq 2\fint_A \omega(|v(y)|) \\ &\leq 2\omega\left(\fint_A |v|\right) \end{aligned}$$

by concavity of ω. We have proved (A.15).

To prove (A.16), we rely on an obvious inequality:

$$|G(\xi, p) - G(\eta, q)| \leq 2\omega(|u(\xi) - u(\eta)|) + \omega(|p - q|). \tag{A.17}$$

Using (A.17) in (A.14) we find

$$\begin{aligned} I &\leq \fint_A \fint_A [2\omega(|u(y) - u(z)|) + \omega(|v(y) - v(z)|)] \, dz dy \\ &\leq 2\omega\left(\fint_A \fint_A |u(y) - u(z)|\right) + \omega\left(\fint_A \fint_A |v(y) - v(z)|\right). \end{aligned}$$

This yields (A.16) (as in (1″)).

Recall now that ε is fixed and we wish to find τ. Since $u \in \mathrm{VMO}$, there exists $\delta_0 < r_0$, such that for every $\delta \leq \delta_0$ and every $x \in X$,

$$2\omega\left(2\fint_{B_\delta(x)} \left|u - \fint_{B_\delta(x)} u\right|\right) < \frac{\varepsilon}{2}.$$

Hence, taking $A = B_\delta(x)$, we see from (A.16) that

$$I \le \frac{\varepsilon}{2} + \omega\left(2\fint_{B_\delta(x)}\left|v - \fint_{B_\delta(x)} v\right|\right) \le \frac{\varepsilon}{2} + \omega(2\|v\|_{\mathrm{BMO}}). \tag{A.18}$$

Now we explain how to choose τ. We first require that

$$\omega(2\tau) \le \varepsilon/2, \tag{A.19}$$

so that, by (A.18), $I \le \varepsilon$ whenever $\delta \le \delta_0$. It remains to consider the case where $\delta > \delta_0$. Here we use (A.15) to conclude that

$$I \le 2\omega\left(\frac{\tau}{|B_{\delta_0}(x)|}\right) \le 2\omega(\alpha\tau)$$

where $\alpha = \operatorname*{Sup}_{x\in X} \frac{1}{|B_{\delta_0}(x)|}$. Choosing $\tau > 0$ such that $2\omega(\alpha\tau) < \varepsilon$, we obtain the desired conclusion, $I \le \varepsilon$, in both cases ($\delta \le \delta_0$ and $\delta > \delta_0$). $\square$

REMARK A.1. The following example shows that the map Ψ of Lemma A.8 need *not* be continuous at a point u in $\mathrm{BMO}(X, \mathbb{R}^N)$. Here we take $X = \mathbb{R}$, — it is not compact, but our functions will all have support in $[-1, 1]$.

Let σ be a positive function in VMO with support in $[-1, 1]$, even in x, decreasing and continuous on $(0, 1)$ and

$$\lim_{x\to 0} \sigma(x) = \infty.$$

Consider

$$u(x) = \begin{cases} 0 & \text{for } x < 0 \\ -1 & \text{for } 0 < x < \frac{1}{2} \\ 0 & \text{for } x > \frac{1}{2}. \end{cases}$$

Set $F(s) = s^+$, so that $u^+ = F(u) \equiv 0$. F is clearly Lipschitz but we claim that for $v_j = \sigma/j$,

$$\|F(u + v_j) - F(u)\|_{\mathrm{BMO}} \ge \frac{1}{4} \quad \text{for } j \text{ large}. \tag{A.20}$$

To see this, observe that there is a unique $\delta_j, 0 < \delta_j < 1$, such that $\sigma(\delta_j) = j$. Then

$$F(u + v_j) = \begin{cases} v_j & \text{for } x < 0, \\ v_j - 1 & \text{for } 0 < x < \delta_j, \\ 0 & \text{for } x > \delta_j. \end{cases}$$

Set

$$J = \fint_{-\delta_j}^{\delta_j} \left| F(u+v_j) - \fint_{-\delta_j}^{\delta_j} F(u+v_j) \right| .$$

One checks that

$$\fint_{-\delta_j}^{\delta_j} F(u+v_j) = \frac{1}{j} \fint_0^{\delta_j} \sigma - \frac{1}{2} = \fint_{-\delta_j}^{\delta_j} v_j - \frac{1}{2}.$$

Hence

$$\begin{aligned} J &= \fint_{-\delta_j}^{\delta_j} \left| v_j + u - \fint_{-\delta_j}^{\delta_j} v_j + \frac{1}{2} \right| \\ &\geq \fint_{-\delta_j}^{\delta_j} |u + \frac{1}{2}| - \|v_j\|_{\mathrm{BMO}} \\ &= \frac{1}{2} - \|v_j\|_{\mathrm{BMO}} \end{aligned}$$

which yields (A.20).

By a small modification of the F above, we may prove that

$$\|F(u+v_j) - F(u)\|_{\mathrm{BMO}} \geq \frac{1}{8}$$

for j large — for a *smooth* function F for which the Lipschitz constant is 1. Namely, take F to be any smooth nondecreasing function on $\mathbb{R}$ with $\dot{F} \leq 1$ such that

$$F(s) = \begin{cases} s & \text{for } s > 0 \\ -\frac{1}{8} & \text{for } s < -1. \end{cases}$$

Later we shall have need of an extension of Lemma A–B — where some averages are taken with respect to a positive weight function w satisfying

$$\frac{1}{C_0} \leq w \leq C_0,$$

for some constant $C_0 > 0$. For any measurable set Σ, and integrable function f on Σ we denote

$$\fint_{\Sigma, w} f := \left(\int_\Sigma w \right)^{-1} \int_\Sigma f w. \tag{A.21}$$

LEMMA A.9. *Let $A \subset B$ be measurable sets in X. Then*

$$\fint_B \left| f - \fint_{A,w} f \right| \leq 4C_0^2 \frac{|B|}{|A|} \fint_B \left| f - \fint_B f \right| , \tag{A.22}$$

and

$$\fint_{A,w}\left|f-\fint_{A,w}f\right| \le 4C_0^2\frac{|B|}{|A|}\fint_B\left|f-\fint_B f\right|. \tag{A.23}$$

Proof. Using Lemma A–B repeatedly we find

$$\begin{aligned}
\fint_B\left|f-\fint_{A,w}f\right| &\le \fint_B\left|f-\fint_A f\right| + \left|\fint_A f-\fint_{A,w}f\right| \\
&\le 2\frac{|B|}{|A|}\fint_B\left|f-\fint_B f\right| + \left|\left(\int_A w\right)^{-1}\int_A\left(f-\fint_A f\right)w\right| \\
&\le 2\frac{|B|}{|A|}\fint_B\left|f-\fint_B f\right| + C_0^2\fint_A\left|f-\fint_A f\right| \\
&\le 2\frac{|B|}{|A|}(1+C_0^2)\fint_B\left|f-\fint_B f\right|.
\end{aligned}$$

This proves (A.22).

Turning to (A.23), we have

$$\begin{aligned}
\fint_{A,w}\left|f-\fint_{A,w}f\right| &\le \fint_{A,w}\left|f-\fint_A f\right| + \left|\fint_A f-\fint_{A,w}f\right| \\
&\le C_0^2\fint_A\left|f-\fint_A f\right| + C_0^2\fint_A\left|f-\fint_A f\right|
\end{aligned}$$

as above,

$$\le 4C_0^2\frac{|B|}{|A|}\fint_B\left|f-\fint_B f\right|.$$

□

Next we establish the fact that the BMO notion is invariant under C^1 diffeomorphism. It is a simple but essential fact.

LEMMA A.10. *Let $\varphi : X_1 \to X_2$ be a C^1 diffeomorphism of a smooth compact n-dimensional Riemannian manifold without boundary onto another X_2. If $f \in \mathrm{BMO}(X_2)$ then $f\circ\varphi \in \mathrm{BMO}(X_1)$ and*

$$\|f\circ\varphi\|_{\mathrm{BMO}(X_1)} \le C\|f\|_{\mathrm{BMO}(X_2)}. \tag{A.24}$$

Here C depends only on the Riemannian manifolds X_1, X_2.

Proof. It is not difficult to verify that there are constants $\varepsilon_0, K > 0$ depending only on X_1 and X_2 such that for every $x \in X_1$, and every $\varepsilon < \varepsilon_0$

$$B_{\varepsilon/K}(\varphi(x)) \subset \varphi(B_\varepsilon(x)) \subset B_{\varepsilon K}(\varphi(x)). \tag{A.25}$$

Here ε_0 is less than the injectivity radius of X_1 and $\varepsilon_0 K$ is less than that of X_2.

For $f \in \mathrm{BMO}(X_2)$, set $g = f \circ \varphi$. To prove that $g \in \mathrm{BMO}(X_1)$ it suffices, by Remark 1 in Section I.1, to show that $\forall \xi \in X_1, \quad \forall \varepsilon < \varepsilon_0$

$$\sup_{\substack{\varepsilon<\varepsilon_0 \\ \xi \in X_1}} \fint_{B_\varepsilon(\xi)} |g - \bar{g}_\varepsilon(\xi)| \leq C\|f\|_{\mathrm{BMO}}. \tag{A.26}$$

We proceed to estimate

$$J = \fint_{B_\varepsilon(\xi)} |g - \bar{g}_\varepsilon(\xi)|$$

by changing variables; we require $\varepsilon < \varepsilon_0$. Setting $A_\varepsilon = \varphi(B_\varepsilon(\xi))$ we see easily that

$$J = \frac{1}{|B_\varepsilon(\xi)|} \int_{A_\varepsilon} |f - \bar{g}_\varepsilon(\xi)| w = \fint_{A_{\varepsilon,w}} |f - \bar{g}_\varepsilon(\xi)|,$$

where w is a smooth positive function obtained by the change of variables.

On the other hand

$$\bar{g}_\varepsilon(\xi) = \fint_{B_\varepsilon(\xi)} g = \frac{1}{|B_\varepsilon(\xi)|} \int_{A_\varepsilon} f w d\eta = \fint_{A_{\varepsilon,w}} f$$

since

$$\int_{A_\varepsilon} w = |B_\varepsilon(\xi)|.$$

Using Lemma A.9 and (A.25) we obtain the desired conclusion. □

Here is another proof of Lemma A.10. In view of (1′) and (1″) we estimate

$$\begin{aligned} I &= \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(\varphi(y)) - f(\varphi(z))| \\ &\leq \frac{C}{|B_\varepsilon(x)|^2} \int_{\varphi(B_\varepsilon(x))} \int_{\varphi(B_\varepsilon(x))} |f(\eta) - f(\xi)| \\ &\leq C \fint_{B_{\varepsilon K}(\varphi(x))} \fint_{B_{\varepsilon K}(\varphi(x))} |f(\eta) - f(\xi)|, \qquad \text{by (A.25)}, \\ &\leq C\|f\|_{\mathrm{BMO}}. \end{aligned}$$

□

Next we take up the proofs of two approximation lemmas of Section I.1: Lemmas 7 and 8. We restate them.

Let X, Y be our usual compact connected manifolds without boundaries. X has a Riemannian metric, $\dim X = n$, and Y is smoothly embedded in $\mathbb{R}^N$. (Here, Y need not have the same dimension as X.)

LEMMA A.11. *Given $u \in W^{1,n}(X,Y)$ i.e., $u \in W^{1,n}(X,\mathbb{R}^N)$ and $u(x) \in Y$ a.e., there exists a sequence (u^j) of smooth maps from X to Y tending to u in $W^{1,n}$.*

Proof. We first construct a sequence $(u^j) \subset W^{1,n}(X,Y) \cap C^0(X,Y)$ tending to u in $W^{1,n}(X,\mathbb{R}^N)$. Cover X by a finite number of balls $B_{r_0/4}(a_i) = B_i$. Let ζ_i be a subordinate partition of unity on X. Set $u_i = \zeta_i u$; clearly $u_i \in W^{1,n}(X,\mathbb{R}^N)$. Let φ be the smooth diffeomorphism (given by geodesic normal coordinates) from the Euclidean ball $B_{r_0}(0)$ onto $B_{r_0}(a_i), \varphi(0) = a_i$. Denote by v_i the transplant of u_i to $B_{r_0}(0)$, i.e.,

$$v_i(\xi) = u_i(\varphi(\xi)) \qquad \text{for } \xi \in B_{r_0}(0).$$

v_i has its support in $B_{r_0/4}(0)$ and belongs to $W^{1,n}$. Denote by $\bar{v}_{i,\varepsilon}(\xi)$ the Euclidean average of v_i in $B_\varepsilon(\xi)$

$$\bar{v}_{i,\varepsilon}(\xi) = \fint_{B_\varepsilon(\xi)} v_i.$$

It has support in $B_{r_0/2}(0)$ if $\varepsilon < r_0/4$, which we always assume. It is well known that $\bar{v}_{i,\varepsilon} \to v_i$ in $W^{1,n}(\mathbb{R}^n)$. Carrying back $\bar{v}_{i,\varepsilon}$ to X, we set

$$u_{i,\varepsilon}(x) = \bar{v}_{i,\varepsilon}(\varphi^{-1}(x)).$$

It has support in $B_{r_0/2}(a_i)$, and converges to u_i in $W^{1,n}(X,\mathbb{R}^N)$. Set

$$u_\varepsilon = \sum_i u_{i,\varepsilon}. \tag{A.27}$$

Clearly $u_\varepsilon \in C^0(X,\mathbb{R}^N) \cap W^{1,n}$ and $u_\varepsilon \to u$ in $W^{1,n}$ as $\varepsilon \to 0$.

CLAIM.

$$\operatorname{dist}(u_\varepsilon(x), Y) \to 0 \quad \textit{as } \varepsilon \to 0, \textit{ uniformly in } x \in X. \tag{A.28}$$

The main ingredient in proving the claim is the assertion that for every i,

$$J_\varepsilon(x) := \fint_{B_\varepsilon(x)} |u_i(y) - u_{i,\varepsilon}(x)| d\sigma(y) \to 0 \quad \text{uniformly in } x \in X. \tag{A.29}$$

Assuming (A.29), the claim follows easily because

$$\sum_i \fint_{B_\varepsilon(x)} |u_i(y) - u_{i,\varepsilon}(x)| d\sigma(y) \to 0.$$

The left hand side majorizes

$$\fint_{B_\varepsilon(x)} |u(y) - u_\varepsilon(x)| d\sigma(y) \geq \operatorname{dist}(u_\varepsilon(x), Y).$$

Then (A.28) holds.

Proof of (A.29). Note that $J_\varepsilon(x) = 0$ if $x \notin B_{r_0/2}(a_i)$.

We may write

$$u_{i,\varepsilon}(x) = \frac{1}{|B_\varepsilon(\xi)|} \int_{A_\varepsilon(x)} u_i w$$

where $A_\varepsilon(x) = \varphi(B_\varepsilon(\xi))$ and $\xi = \varphi^{-1}(x)$.

Here w is a smooth function coming from the change of variables, and

$$\frac{1}{C_0} \le w \le C_0,$$

with C_0 depending only on X. Recall that

$$\int_{A_\varepsilon(x)} w = |B_\varepsilon(\xi)|,$$

so that, using the notation (A.21),

$$u_{i,\varepsilon}(x) = \fint_{A_\varepsilon, w} u_i. \tag{A.30}$$

As in the proof of Lemma A.10, there are constants $0 < \varepsilon_1$ and $K > 1$, such that

$$B_{\varepsilon/K}(x) \subset A_\varepsilon(x) \subset B_{\varepsilon K}(x) \quad \forall x \in B_{r_0/2}(a_i), \quad \forall \varepsilon < \varepsilon_1,$$

and we require that $\varepsilon_1 K < r_0/4$.

Returning to J_ε, we have

$$\begin{aligned} J_\varepsilon(x) &= \fint_{B_\varepsilon(x)} \left| u_i(y) - \fint_{A_\varepsilon(x), w} u_i \right| \\ &\le C \fint_{B_{\varepsilon K}(x)} \left| u_i - \fint_{A_\varepsilon(x), w} u_i \right|. \end{aligned}$$

Applying (A.22) of Lemma A.9, we find

$$J_\varepsilon(x) \le C \fint_{B_{\varepsilon K}(x)} \left| u_i - \fint_{B_{\varepsilon K}(x)} u_i \right|.$$

By Example 1 of Section I.2, $J_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ uniformly in $x \in X$.

To summarize, we have a family $u_\varepsilon \in W^{1,n}(X,\mathbb{R}^N) \cap C^0(X,\mathbb{R}^N)$ such that as $\varepsilon \to 0$, $u_\varepsilon \to u$ in $W^{1,n}$ and $\operatorname{dist}(u_\varepsilon(x), Y) \to 0$ uniformly.

Using the projection P onto closest point in Y, the functions

$$\widetilde{u}_\varepsilon = Pu_\varepsilon$$

take values in Y and tend to u in $W^{1,n}$, since in general, as is well known, $u \mapsto F(u)$ is continuous in $W^{1,p}$ if F is Lipschitz and smooth. Letting $\varepsilon \to 0$ through a sequence ε_j, the functions

$$\widetilde{u}^j = \widetilde{u}_{\varepsilon_j}$$

belong to $C^0(X,Y) \cap W^{1,n}(X,Y)$ and tend to u in $W^{1,n}$.

For each fixed j there is a smooth map u^j from X to Y with

$$\|u^j - \widetilde{u}^j\|_{W^{1,n}} + |u^j - \widetilde{u}^j|_{C^0} \leq \frac{1}{j}$$

— by standard smoothing and projection on Y. $\square$

REMARK A.2. R. Schoen and K. Uhlenbeck [1] use a slightly different, but natural, approach in their proof (for $n = 2$). Namely, they embed X in some $\mathbb{R}^M$, and extend u to a tubular neighbourhood of X as constant on normals to X. They then mollify the above extension $\widetilde{u}$.

REMARK A.3. In Lemma A.11 if u is merely in $W^{1,p}(X,Y)$, $1 \leq p < n$ and also in $\mathrm{VMO}(X,Y)$, then there is a sequence (u^j) of smooth maps of X into Y, tending to u in $W^{1,p}$ and in BMO. This is proved in essentially the same way as the lemma, using in addition Lemma B.8 below. The latter is used to ensure that $u_i = \zeta_i u \in \mathrm{VMO}$.

Essentially the same argument as in the proof of Lemma A.11 yields the following more general form:

LEMMA A.12. *Assume $u \in W^{s,p}(X,Y)$ with $sp = n$, $0 < s < n$ (s may or may not be an integer). Then there exists a sequence (u^j) of smooth maps from X to Y tending to u in $W^{s,p}$.*

We use here Example 2 of Section I.2 instead of Example 1, and standard properties of $W^{s,p}$. A special case of Lemma A.12 occurs in F. Bethuel [2].

LEMMA A.13. *Assume $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with connected boundary $\partial\Omega$. Let $u \in W^{1,n}(\Omega,\mathbb{R}^N)$ be such that*

$$u(\partial\Omega) \subset Y$$

where Y is, as usual, a compact connected manifold smoothly embedded in $\mathbb{R}^N$. Then there exists a sequence (u_j) of smooth maps from $\overline{\Omega}$ to $\mathbb{R}^N$ such that

$$u_j(\partial\Omega) \subset Y \quad \forall j$$

and

$$u_j \to u \quad \text{in } W^{1,n}(\Omega, \mathbb{R}^N).$$

Proof. Set $\varphi = u_{|\partial\Omega}$, so that $\varphi \in W^{1-\frac{1}{n},n}(\partial\Omega, Y)$. Applying Lemma A.12 with $X = \partial\Omega, s = 1 - \frac{1}{n}, p = n$ (note that $\dim X = n-1$), we obtain a sequence (φ^j) of smooth maps from $\partial\Omega$ to Y such that

$$\varphi^j \to \varphi \quad \text{in } W^{1-\frac{1}{n},n}(\partial\Omega, \mathbb{R}^N).$$

Let v_j be the harmonic extension of φ^j in Ω. It is well known that

$$v_j \to v \quad \text{in } W^{1,n}(\Omega, \mathbb{R}^N),$$

where v is the harmonic extension of φ in Ω.

Since $u - v \in W_0^{1,n}(\Omega, \mathbb{R}^N)$ there is a sequence (w_j) in $C_c^\infty(\Omega, \mathbb{R}^N)$ such that

$$w_j \to (u - v) \quad \text{in } W^{1,n}(\Omega, \mathbb{R}^N).$$

The sequence $u_j = v_j + w_j$ has the required properties. $\square$

REMARK A.4. The same argument as in the proof of Lemma A.13 shows that if $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $1 \le p < n$ and

$$u(\partial\Omega) \subset Y$$

with $u_{|\partial\Omega} \in \mathrm{VMO}(\partial\Omega, Y)$, then there exists a sequence (u_j) of smooth maps from $\overline{\Omega}$ to $\mathbb{R}^N$ such that $u_j(\partial\Omega) \subset Y \quad \forall j$ and $u_j \to u$ in $W^{1,p}$.

If we do not make the assumption that $u_{|\partial\Omega} \in \mathrm{VMO}(\partial\Omega, Y)$ then the conclusion may fail. Here is an example. Let $\Omega = B_1 \subset \mathbb{R}^3$, let $Y = S^1$ and take $p = 2$. We use coordinates $x = (x_1, x_2, x_3) = (x', x_3)$. Consider the map

$$\varphi(x) = \frac{x'}{|x'|} \text{ defined on } \partial\Omega, \text{ with values in } S^1.$$

It is smooth except at the north and south poles. Near there, it belongs to $W^{1,q}(\partial\Omega)$ for any $q < 2$; in particular, it belongs to $H^{1/2}(\partial\Omega, \mathbb{R}^2)$ — by Sobolev. Hence φ admits an extension u into Ω, belonging to $H^1(\Omega, \mathbb{R}^2)$. Suppose there is a sequence of smooth maps $u_j \to u$ in $H^1(\Omega, \mathbb{R}^2)$ and $u_j : \partial\Omega \to S^1$. Then on some disc $D = \{x_3 = \text{constant}\} \cap \Omega$, $u_j \to u$ in $H^1(D)$, and therefore $u \to u$ in $H^{1/2}(\partial D)$.

By Theorem 1,

$$\deg(u_j, \partial D, S^1) = \deg(u, \partial D, S^1) \quad \text{for } j \text{ large.}$$

Clearly the right hand side equals 1. On the other hand the left hand side is zero because ∂D is the boundary of a spherical cap on $\partial\Omega$, which is mapped by u_j into S^1.

The argument is related to one of R. Schoen and K. Uhlenbeck [1], in which they prove that the map $x/|x|$ of Ω above into S^2, cannot be H^1–approximated by smooth maps into S^2.

Returning to the function M_t defined before Lemma 3′, we prove a useful fact:

LEMMA A.14. *There is a constant A depending only on X, such that for $u \in \mathrm{BMO}(X, \mathbb{R}^N)$,*

$$M_{2t}(u) \le A\, M_t(u) \quad \forall t < r_0/2.$$

Proof. In view of (1′), (1″) we estimate, for $t \le \delta \le 2t$,

$$I = \fint_{B_\delta(x)} \fint_{B_\delta(x)} |u(y) - u(z)|.$$

Using Lemma A.6, with $\varepsilon = t$, we cover $B_\delta(x)$ by balls $B_t(x_i)$, with $\operatorname{dist}(x_i, x_j) \ge t$ for $i \ne j$, and

$$\sum |B_t(x_i)| \le C |B_\delta(x)|.$$

Thus

$$I \le \frac{1}{|B_\delta(x)|^2} \left[\sum_{i \ne j} \int_{B_t(x_i)} \int_{B_t(x_j)} |u(y) - u(z)| + \sum_i \int_{B_t(x_i)} \int_{B_t(x_i)} |u(y) - u(z)| \right].$$

The last sum is bounded by

$$2M_t(u) \sum_i |B_t(x_i)|^2 \le C M_t(u) |B_\delta(x)|^2$$

while the first sum is bounded by

$$\sum_{i \ne j} \int_{B_t(x_i)} \int_{B_t(x_j)} \left[|u(y) - \bar{u}_t(x_i)| + |\bar{u}_t(x_i) - \bar{u}_t(x_j)| + |\bar{u}_t(x_j) - u(z)| \right]$$
$$\le 2M_t(u) \sum_{i \ne j} |B_t(x_i)|\, |B_t(x_j)| + \sum_{i \ne j} |\bar{u}_t(x_i) - \bar{u}_t(x_j)|\, |B_t(x_i)|\, |B_t(x_j)|.$$

Since $\operatorname{dist}(x_i, x_j) \leq 2\delta \leq 4t$, we find using $(A.12)$ that the expression above is bounded by

$$CM_t(u) \sum_{i \neq j} |B_t(x_i)|\, |B_t(x_j)| \leq CM_t(u) |B_\delta(x)|^2.$$

Thus we conclude that

$$I \leq C\, M_t(u).$$

□

LEMMA A.15. *For any $u \in \mathrm{BMO}(X, \mathbb{R}^N)$ the function $t \mapsto M_t(u)$ is continuous in $[0, r_0)$.*

The proof is left to the reader.

We now turn to the proof of Lemma 4 (the characterization of compact sets in VMO), which we restate

LEMMA A.16. *A set $\mathcal{F}$ in $VMO(X, \mathbb{R}^N)$ is relatively compact if and only if*

$$\lim_{\varepsilon \to 0} M_\varepsilon(u) \quad \textit{holds uniformly in } u \in \mathcal{F}.$$

After the statement of Lemma 4 we proved $\Rightarrow$. Now, we prove $\Leftarrow$.

It suffices to show that for any given $\delta > 0$, $\mathcal{F}$ may be covered by a finite number of balls in BMO of radius δ. We may assume that

$$\int_X u = 0 \quad \forall u \in \mathcal{F}.$$

We denote by $\bar{\bar{u}}_\varepsilon$ the ε-averaging iterated twice. Applying Lemma 3′ repeatedly we find

$$\begin{aligned}
\|u - \bar{\bar{u}}_\varepsilon\|_{\mathrm{BMO}} &\leq \|u - \bar{u}_\varepsilon\|_{\mathrm{BMO}} + \|\bar{u}_\varepsilon - \bar{\bar{u}}_\varepsilon\|_{\mathrm{BMO}} \\
&\leq A\, M_\varepsilon(u) + A\, M_\varepsilon(\bar{u}_\varepsilon) \\
&\leq A\, M_\varepsilon(u) + A\, M_\varepsilon(\bar{u}_\varepsilon - u) + A\, M_\varepsilon(u) \\
&\leq 2A\, M_\varepsilon(u) + A\|\bar{u}_\varepsilon - u\|_{\mathrm{BMO}} \leq (2A + A^2)\, M_\varepsilon(u).
\end{aligned}$$

By our hypothesis there exists ε, $0 < \varepsilon < r_0$ such that

$$\|u - \bar{\bar{u}}_\varepsilon\|_{\mathrm{BMO}} \leq 3A\, M_\varepsilon(u) \leq \delta/2 \quad \forall u \in \mathcal{F}.$$

Fix this ε. In view of Remark 1,

$$\|u\|_{\mathrm{BMO}} \leq C \quad \forall u \in \mathcal{F}$$

and then by Lemma 1,

$$\|u\|_{L^1} \le C \quad \forall u \in \mathcal{F}.$$

(Here, and in what follows, all the constants C depend on ε, which has been fixed.) Consequently,

$$\|\bar{u}_\varepsilon\|_{L^\infty} + \|\bar{\bar{u}}_\varepsilon\|_{L^\infty} \le C \quad \forall u \in \mathcal{F}. \tag{A.31}$$

We now claim that the family $(\bar{\bar{u}}_\varepsilon)$, $u \in \mathcal{F}$, satisfies the conditions of the Arzela-Ascoli theorem; more precisely, there is a constant C such that

$$J = |\bar{\bar{u}}_\varepsilon(x) - \bar{\bar{u}}_\varepsilon(y)| \le C \ \operatorname{dist}(x,y) \quad \forall x, y \in X, \quad \forall u \in \mathcal{F}. \tag{A.32}$$

Proof of (A.32). We distinguish two cases:

Case 1: $\operatorname{dist}(x,y) \ge \varepsilon/100$. In this case

$$J \le 2\|\bar{\bar{u}}_\varepsilon\|_{L^\infty} \le C.$$

Case 2: $\operatorname{dist}(x,y) < \varepsilon/100$. Then

$$\left| \frac{1}{|B_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(y)|} \right| \le C \ \operatorname{dist}(x,y).$$

Thus

$$J \le \int_{B_\varepsilon(x)} |\bar{u}_\varepsilon(z)| \left| \frac{1}{|B_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(y)|} \right| + \frac{1}{|B_\varepsilon(y)|} \left| \int_{B_\varepsilon(x)} \bar{u}_\varepsilon(z) - \int_{B_\varepsilon(y)} \bar{u}_\varepsilon(z) \right|.$$

We have only to estimate the last term, K. Let S be the symmetric difference of $B_\varepsilon(x)$ and $B_\varepsilon(y)$, i.e., $S = (B_\varepsilon(x) \cup B_\varepsilon(y)) \backslash (B_\varepsilon(x) \cap B_\varepsilon(y))$. Clearly,

$$S \subset \left(B_{\varepsilon+\operatorname{dist}(x,y)}(x) \backslash B_\varepsilon(x)\right) \cup \left(B_{\varepsilon+\operatorname{dist}(x,y)}(y) \backslash B_\varepsilon(y)\right).$$

Hence

$$|S| \le C \operatorname{dist}(x,y)$$

and therefore

$$K \le C|S| \, \|\bar{u}_\varepsilon\|_{L^\infty} \le C \operatorname{dist}(x,y) \qquad \text{by (A.31)}.$$

The desired inequality (A.32) follows by combining this with the earlier estimate.

Returning to the proof of Lemma A.16, we may now assert that the family $(\bar{\bar{u}}_\varepsilon)$, $u \in \mathcal{F}$, is relatively compact in $C^0(X, \mathbb{R}^N)$ and thus in $\mathrm{BMO}(X, \mathbb{R}^N)$. We may cover the family $(\bar{\bar{u}}_\varepsilon)$ by a finite number of balls in $\mathrm{BMO}(X, \mathbb{R}^N)$ of radius $\delta/2$. The concentric balls of radius δ then cover $\mathcal{F}$. $\square$

We mention a simple application of Lemma 3 to the sequence of truncates. Given a real-valued function f on X and an integer k, set

$$f^k(x) = \begin{cases} k & \text{if } f(x) \geq k \\ f(x) & \text{if } -k < f(x) < k \\ -k & \text{if } f(x) \leq -k. \end{cases}$$

LEMMA A.17. *For every $f \in \mathrm{VMO}(X, \mathbb{R})$ the sequence (f^k) converges to f in BMO.*

Proof. Given any $\delta > 0$ we will show that there exists k_0 such that

$$\|f - f^k\|_{\mathrm{BMO}} < \delta \quad \text{for } k > k_0.$$

Consider any $B_\varepsilon(x)$ in X. We have

$$⨍_{B_\varepsilon(x)} ⨍_{B_\varepsilon(x)} |f^k(y) - f^k(z)| \leq ⨍_{B_\varepsilon(x)} ⨍_{B_\varepsilon(x)} |f(y) - f(z)| \qquad \forall k.$$

It follows, with the aid of Lemma 3, that there exists $\varepsilon_0 > 0$, such that

$$⨍_{B_\varepsilon(x)} |(f - f^k) - ⨍_{B_\varepsilon(x)} (f - f^k)| < \delta \quad \text{for } \varepsilon < \varepsilon_0, \quad \forall k.$$

Since $f^k \to f$ in L^1, the same inequality holds for $\varepsilon \geq \varepsilon_0$ provided $k \geq k_0$, for some k_0 depending on ε_0. $\square$

REMARK A.5. If $f \in \mathrm{BMO}(X, \mathbb{R})$ then (f^k) need *not* converge to f in BMO. (It is easy to construct an example using $f(x) = \log |x|$; recall that this f belongs to BMO, but not VMO — see Example 3 in Section I.2).

We now present various results concerning homotopy properties for BMO and VMO maps. They are used in the proofs of Theorems 3 and 4, as well as in paragraph 1 of Section I.5. Let X, Y be our usual compact connected manifolds with X Riemannian (X and Y need not have the same dimension.)

LEMMA A.18. *There exists $\delta > 0$ (depending on X, Y) such that for every $u \in BMO(X, Y)$ with $\|u\|_{\mathrm{BMO}} < \delta$ and every $\varepsilon \in (0, r_0), u_\varepsilon$ is homotopic within $C^0(X, Y)$ to a constant.*

Proof. First observe that

$$\operatorname{dist}(\bar{u}_\varepsilon(x), Y) \leq \|u\|_{\mathrm{BMO}} \quad \forall x \in X, \quad \forall \varepsilon \in (0, r_0) \tag{A.33}$$

and thus $u_\varepsilon = P\bar{u}_\varepsilon$ is well defined for every $\varepsilon \in (0, r_0)$, provided $\|u\|_{\mathrm{BMO}} \le \delta < \delta_0$, with δ_0 sufficiently small. Moreover, u_ε is homotopic to $u_{\varepsilon'}$ within $C^0(X,Y)$ for every $\varepsilon, \varepsilon' \in (0, r_0)$, using the deformation $u_{t\varepsilon+(1-t)\varepsilon'}, 0 \le t \le 1$.

Fix any $\varepsilon \in (0, r_0)$ — for example $\varepsilon = r_0/2$. We have, by Lemma A.1,

$$\left|\bar{u}_\varepsilon(x) - \fint_X u\right| \le \frac{1}{|B_\varepsilon(x)|}\int_X \left|u - \fint_X u\right| \le C\|u\|_{\mathrm{BMO}}; \tag{A.34}$$

here C also depends on ε, but ε has been fixed and we do not stress the ε-dependence. Combining (A.33) and (A.34) we obtain

$$\operatorname{dist}\left((1-t)\bar{u}_\varepsilon(x) + t\fint_X u,\ Y\right) \le (C+1)\,\|u\|_{\mathrm{BMO}} \quad \forall x \in X, \quad \forall t \in [0,1].$$

Thus

$$P\left((1-t)\bar{u}_\varepsilon + t\fint_X u\right),\ 0 \le t \le 1,$$

is well defined provided $\|u\|_{\mathrm{BMO}} \le \delta < \delta_1$ with δ_1 sufficiently small. Hence u_ε is homotopic within $C^0(X,Y)$ to a constant, via $t \in [0,1]$. □

LEMMA A.19. *Given $u \in \mathrm{VMO}(X,Y)$ there exists $\delta = \delta(u) > 0$ such that every $v \in \mathrm{VMO}(X,Y)$ satisfying*

$$\|v-u\|_{\mathrm{BMO}} + \|v-u\|_{L^1} < \delta$$

is homotopic to u within VMO$\cap L^1$. Moreover δ is uniform when u lies in a compact subset $\mathcal{F}$ of VMO$\cap L^1$.

Proof. We argue by contradiction and assume that there is a sequence (u_j) in $\mathrm{VMO}(X,Y)$ such that $u_j \to u$ in $\mathrm{VMO} \cap L^1$ and each u_j is not homotopic to u within $\mathrm{VMO} \cap L^1$.

In view of Lemma 4 and (7) we know that there is some $\varepsilon_0 > 0$ such that $u_{j,\varepsilon}$ is well defined for every j and every $\varepsilon < \varepsilon_0$. *Fix* any $\varepsilon < \varepsilon_0$. Since $u_j \to u$ in L^1, we deduce that $u_{j,\varepsilon} \to u_\varepsilon$ uniformly, as $j \to \infty$. In particular, for j large, $u_{j,\varepsilon}$ is homotopic to u_ε within $C^0(X,Y)$ — and thus within $\mathrm{VMO} \cap L^1$.

On the other hand, u_ε is homotopic to u within $\mathrm{VMO} \cap L^1$ (through $u_{t\varepsilon}$, by Corollary 4), and similarly $u_{j,\varepsilon}$ is homotopic to u_j within $\mathrm{VMO} \cap L^1$. Therefore u_j is homotopic to u within $\mathrm{VMO} \cap L^1$. A contradiction.

The fact that δ is uniform when $u \in \mathcal{F}$ is easy to establish by contradiction. If not, there would exist equences (u_j) in $\mathcal{F}$ and (v_j) in VMO such that

$$\|v_j - u_j\|_{\mathrm{BMO}} + \|v_j - u_j\|_{L^1} \to 0$$

and, for each j, u_j is not homotopic to v_j within $\mathrm{VMO} \cap L^1$.

Since $\mathcal{F}$ is compact we may assume, for a subsequence, that $u_j \to u$ and $v_j \to u$ in $\mathrm{VMO} \cap L^1$. From the first assertion in the lemma we deduce that u_j and v_j are homotopic to u within $\mathrm{VMO} \cap L^1$, for j large. A contradiction. □

LEMMA A.20. *Assume $u, v \in C^0(X,Y)$ are homotopic within VMO$\cap L^1$. Then they are homotopic within $C^0(X,Y)$.*

Proof. Let $H(t)$ be a homotopy connecting u and v within VMO$\cap L^1$. Since $\mathcal{F} = H([0,1])$ is compact in VMO, we know by Lemma 4 that $H(t)_\varepsilon$ is well defined for all $\varepsilon < \varepsilon_0$ and all $t \in [0,1]$. Fix any $\varepsilon < \varepsilon_0$. Since $H \in C([0,1]), L^1)$ we deduce that $H(t)_\varepsilon$ is a homotopy connecting u_ε to v_ε within $C^0(X,Y)$. On the other hand, u_ε is homotopic to u within $C^0(X,Y)$ (via $u_{t\varepsilon}$), and similarly for v and v_ε. Thus, u and v are homotopic within $C^0(X,Y)$. □

Given a homotopy class $\mathcal{C}$ in $C^0(X,Y)$ we denote by $\overline{\mathcal{C}}$ its closure in the VMO$\cap L^1$ topology.

LEMMA A.21. *If $u, v \in \overline{\mathcal{C}}$, then u is homotopic to v within VMO$\cap L^1$. Conversely, if $u, v \in$ VMO(X,Y) are homotopic within VMO$\cap L^1$, then there exists a unique homotopy class $\mathcal{C}$ in $C^0(X,Y)$ such that $u, v \in \overline{\mathcal{C}}$.*

Proof. The first assertion is clear from Lemma A.19. We turn to the proof of the converse. Let (u_j) be a sequence in $C^0(X,Y)$ such that $u_j \to u$ in VMO$\cap L^1$ (we may for example take u_ε with $\varepsilon = 1/j$). Similarly, let (v_j) be a sequence in $C^0(X,Y)$ such that $v_j \to v$ in VMO$\cap L^1$. Applying Lemma A.19 we see that u_j is homotopic to u within VMO$\cap L^1$ for all $j \geq N$. Similarly, v_k is homotopic to v within VMO$\cap L^1$ for all $k \geq N$. Hence u_j is homotopic to v_k for all $j, k \geq N$, within VMO$\cap L^1$. We deduce from Lemma A.20 that u_j and v_k are also homotopic within $C^0(X,Y)$. Consequently there is a homotopy class $\mathcal{C}$ in $C^0(X,Y)$ such that $u_j, v_j \in \mathcal{C} \quad \forall j \geq N$. Thus $u, v \in \bar{\mathcal{C}}$.

Finally we prove the uniqueness of $\mathcal{C}$. It suffices to show that if $\mathcal{C}_1$ and C_2 are two homotopy classes in $C^0(X,Y)$ such that $\overline{\mathcal{C}}_1 \cap \overline{\mathcal{C}}_2 \neq \phi$, then $\mathcal{C}_1 = \mathcal{C}_2$. Let $u \in \overline{\mathcal{C}}_1 \cap \overline{\mathcal{C}}_2$ and let $(u_j) \subset \mathcal{C}_1, (v_j) \subset \mathcal{C}_2$ be sequences such that $u_j \to u$ in VMO$\cap L^1, v_j \to u$ in VMO$\cap L^1$. In view of Lemma A.19, we may assume that u_j and v_j are homotopic to u within VMO$\cap L^1$ for all j. By Lemma A.20 we know that u_j and v_j are homotopic within $C^0(X,Y)$, i.e., $\mathcal{C}_1 = \mathcal{C}_2$. □

In what follows we consider the special case where $Y = S^k, k \geq 1$, and we show that some of the properties concerning homotopy can be improved. One suprising fact is that the notion of "homotopy within VMO" is *equivalent* to the notion of "homotopy within VMO$\cap L^1$" (see Lemma A.23).

Throughout the rest of Appendix A we take $Y = S^k, k \geq 1$. A basic ingredient is

LEMMA A.22. *Let $u \in$ VMO(X,Y) be such that for some constant $c \neq 0$, $u + c \in Y$ a.e. Then u is homotopic to a constant within VMO$\cap L^1$. In particular, u is homotopic to $(u + c)$ within VMO$\cap L^1$.*

Proof. Set $\Sigma = Y \cap (Y - c)$; since $\Sigma \neq Y$, it is contractible to a point in Y, i.e., there is a continuous map $h(t, \sigma) : [0,1] \times \Sigma \to Y$ such that $h(0, \sigma) = \sigma \ \ \forall \sigma \in \Sigma$ and $h(1, \sigma)$ is a constant.

Set

$$H(t,x) = h(t, u(x)).$$

It is easy to verify (using Lemma A.7) that

$$H \in C([0,1], \mathrm{VMO} \cap L^1);$$

moreover $H(0,x) = u(x)$ and $H(1,x)$ is a constant. □

Next, an improvement of Lemma A.19.

LEMMA A.19′. *Given $u \in \mathrm{VMO}(X,Y)$ there exists $\delta = \delta(u) > 0$ such that every $v \in$ VMO (X,Y) satisfying*

$$\|v - u\|_{\mathrm{BMO}} < \delta$$

is homotopic to u within VMO$\cap L^1$. Moreover δ is uniform when u lies in a compact subset $\mathcal{F}$ of VMO.

Proof. We argue by contradiction and assume that there is a sequence (u_j) in $\mathrm{VMO}(X,Y)$ such that $u_j \to u$ in VMO and each u_j is not homotopic to u within VMO$\cap L^1$.

Set

$$c_j = \fint_X (u_j - u).$$

Passing to a subsequence we may assume that $c_j \to c$. Then, by Lemma 1, $u_j \to u+c$ in L^1.

In view of Lemma 4 and (7) we know that there is some $\varepsilon_0 > 0$ such that $u_{j,\varepsilon}$ is well defined for every j and every $\varepsilon < \varepsilon_0$. *Fix* any $\varepsilon < \varepsilon_0$. Since $u_j \to (u+c)$ in L^1 we deduce that $u_{j,\varepsilon} \to (u+c)_\varepsilon$ uniformly as $j \to \infty$.

In particular, for j large, $u_{j,\varepsilon}$ is homotopic to $(u+c)_\varepsilon$ within $C^0(X,Y)$ — and thus within VMO$\cap L^1$.

On the other hand, $(u+c)_\varepsilon$ is homotopic to $(u+c)$ within VMO$\cap L^1$ and similarly $u_{j,\varepsilon}$ is homotopic to u_j within VMO$\cap L^1$. Therefore u_j is homotopic to $(u+c)$ within VMO$\cap L^1$ for j large.

Finally, we apply Lemma A.22 to assert that $(u+c)$ is homotopic to u within VMO$\cap L^1$ (this is also true when $c = 0$!). Hence u_j is homotopic to u within VMO$\cap L^1$ for j large. A contradiction.

The fact that δ is uniform when $u \in \mathcal{F}$, a compact subset of VMO, is derived as in the proof of Lemma A.19. □

LEMMA A.23. *Assume $u, v \in$ VMO(X,Y) are homotopic within VMO. Then u, v are homotopic within VMO$\cap L^1$.*

Proof. Let $H(t)$ be a homotopy connecting u to v within VMO. Let $\mathcal{F} = H([0,1])$, so that $\mathcal{F}$ is compact subset of VMO. Let δ be as in Lemma A.19′ (relative to $\mathcal{F}$). There is a chain $u = u_0, u_1, \ldots, u_k, u_{k+1} = v$ in $\mathcal{F}$ such that $\|u_{i+1} - u_i\|_{\mathrm{BMO}} < \delta$ for $i = 0, 1, \ldots, k$. Thus u_{i+1} is homotopic to u_i within VMO$\cap L^1$ for $i = 0, 1, \ldots, k$. Consequently v is homotopic to u within VMO$\cap L^1$. □

REMARK A.6. The conclusion of Lemma A.23 may fail for a general manifold Y. Consider, for example, a manifold Y lying in $\mathbb{R}^3$ diffeomorphic to a 2-*d* torus T^2. Assume that Y contains a circle Σ contractible to a point in Y and another circle $\Sigma' = \Sigma + c$ (for some constant c) such that Σ' in *not* contractible to a point in Y. Let $X = \Sigma$; the map $u(x) = x$ is homotopic to a constant within VMO$\cap L^1$ and the map $v(x) = x + c$ (viewed as a map from X into Y) is not homotopic to a constant within VMO$\cap L^1$ (by Lemma A.20). On the other hand u and v are clearly homotopic — in fact they are the same — within VMO.

LEMMA A.24. *Let $\mathcal{C}$ be a homotopy class in $C^0(X,Y)$. Then its closure $\overline{\mathcal{C}}$ in VMO$\cap L^1$ coincides with its closure $\widetilde{\mathcal{C}}$ in VMO.*

Proof. Clearly $\overline{\mathcal{C}} \subset \widetilde{\mathcal{C}}$. To prove the reverse inclusion, consider some $u \in \widetilde{\mathcal{C}}$. There is a sequence (u_j) in $\mathcal{C}$ such that $u_j \to u$ in VMO. By Lemma A.19′, u_j is homotopic to u within VMO$\cap L^1$ for j sufficiently large. Applying Lemma A.21 we conclude that $u \in \bar{\mathcal{C}}$. □

Appendix B. John-Nirenberg inequality on manifolds, et al.

We begin by stating of the John-Nirenberg inequality on a cube Q_0 in $\mathbb{R}^n$ (with edges parallel to the axes). It is inequality (3)″ in John-Nirenberg [1]:

There exist $\beta, A > 0$ depending only on n such that if $u \in$ BMO$(Q_0, \mathbb{R}^N)$ and $\|u\|_{\mathrm{BMO}(Q_0)} \le 1$, then

$$\int_{Q_0} \left(e^{\beta|u - \bar{u}_{Q_0}|} - 1\right) \le A \int_{Q_0} |u - \bar{u}_{Q_0}| \tag{B.1}$$

where $\bar{u}_{Q_0} = ⨍_{Q_0} u$.

Here, $\| \ \|_{\mathrm{BMO}(Q_0)}$ refers to the sup in (1) taken over all parallel subcubes rather than balls.

An immediate consequence is: for p an integer ≥ 2,

$$\int_{Q_0} |u - \bar{u}_{Q_0}|^p \leq \frac{A}{\beta^p} p! \int_{Q_0} |u - \bar{u}_{Q_0}|.$$

The scaled version of this is

$$\begin{aligned} \int_{Q_0} |u - \bar{u}_{Q_0}|^p &\leq \frac{Ap!}{\beta^p} \|u\|^{p-1}_{\mathrm{BMO}(Q_0)} \int_{Q_0} |u - \bar{u}_{Q_0}| \\ &\leq C^p p! \|u\|^{p-1}_{\mathrm{BMO}(Q_0)} \int_{Q_0} |u - \bar{u}_{Q_0}|, \end{aligned} \tag{B.2}$$

with C depending only on n. It follows directly that, for a different C, depending only on n,

$$\int_{Q_0} \int_{Q_0} |u(y) - u(z)|^p \leq C^p p! \|u\|^{p-1}_{\mathrm{BMO}(Q_0)} \int_{Q_0} \int_{Q_0} |u(y) - u(z)|. \tag{B.3}$$

We wish next to present corresponding inequalities on a compact manifold X, without boundary. Here is a form of (B.3) on X.

LEMMA B.1. *There exists a constant A depending only on X, such that $\forall t < r_0/\sqrt{n}$, $\forall x \in X$,*

$$\int_{B_t(x)} \int_{B_t(x)} |u(y) - u(z)|^p \leq A^p p! \, M_t^{p-1}(u) \int_{B_{kt}(x)} \int_{B_{kt}(x)} |u(y) - u(z)| \tag{B.4}$$

where $k = \sqrt{n}$.

Proof. We use geodesic normal coordinates in $B_t(x)$. Then one easily sees that

$$I = \int_{B_t(x)} \int_{B_t(x)} |u(y) - u(z)|^p \leq C \int_Q \int_Q |\tilde{u}(y) - \tilde{u}(z)|^p$$

where in the right hand side Q is a cube centred at the origin (in our coordinate patch) with side length $2t$, and $\tilde{u}(y)$ represents the transplanted function. Here C comes from the change of variables, and depends only on X. By (B.3),

$$I \leq C^p p! \|\tilde{u}\|^{p-1}_{\mathrm{BMO}(Q)} \int_Q \int_Q |\tilde{u}(y) - \tilde{u}(z)|. \tag{B.5}$$

CLAIM. *There is a constant C depending only on X such that*

$$\|\tilde{u}\|_{\mathrm{BMO}(Q)} \leq C \, M_t(u). \tag{B.6}$$

Assuming that (B.6) holds, the proof of (B.4) is easily completed, since Q is contained in the transplant of $B_{kt}(x)$ so that

$$\int_Q \int_Q |\tilde{u}(y) - \tilde{u}(z)| \leq C \int_{B_{kt}(x)} \int_{B_{kt}(x)} |u(y) - u(z)|.$$

Proof of (B.6). We have to estimate for any parallel subcube $\widetilde{Q}$ of Q, centred at ξ, of side length 2τ,

$$J = \fint_{\widetilde{Q}} \fint_{\widetilde{Q}} |\tilde{u}(y) - \tilde{u}(z)|$$

in terms of $M_t(u)$. $\widetilde{Q}$ is contained in a ball $B_{k\tau}(\xi)$. Its transplant to X is contained in a ball $B_{K\tau}(\tilde{x})$ and contains $B_{\tau/K}(\tilde{x})$ where $\tilde{x}$ is the transplant of ξ and K is a constant depending only on X (see Proof of Lemma A.10). Hence

$$J \leq C \fint_{B_{K\tau}(\tilde{x})} \fint_{B_{K\tau}(\tilde{x})} |u(y) - u(z)| \leq 2C\, M_{K\tau}(u). \tag{B.7}$$

where C depends only on X.

By Lemma A.14 applied a number of times, we find that

$$J \leq C\, M_\tau(u).$$

Inserting this in (B.7) we obtain (B.6).

Lemma B.1 is proved. □

The next result is a more global form of Lemma B.1.

LEMMA B.2. *For every $t \leq r_0/\sqrt{n}$, and for every integer $p > 1$, there is a finite number of balls $B_t(x_i)$ in X, $i = 1, \dots, m$, depending on t, such that*

$$\int_X |u|^p \leq A^p p! M_t^{p-1}(u) \int_X |u| + 2^p \sum_i \frac{1}{|B_t(x_i)|^{p-1}} \left| \int_{B_t(x_i)} u \right|^p \tag{B.8}$$

where A depends only on X.

Before proving Lemma B.2 we present some more civilized corollaries.

LEMMA B.3. *There is a constant A depending only on X such that for $p > 1$, an integer,*

$$\int_X \Big|u - \fint_X u\Big|^p \leq A^p p! \|u\|_{\mathrm{BMO}}^{p-1} \Big\|u - \fint_X u\Big\|_{L^1}.$$

Proof. We may assume $\int_X u = 0$. In (B.8) we choose $t = r_0/\sqrt{n}$. Then we have

$$\int_X |u|^p \leq A^p p! \|u\|_{\mathrm{BMO}}^{p-1} \|u\|_{L^1} + A^p \|u\|_{L^1}^p.$$

The desired conclusion follows with the aid of Lemma 1. □

Another consequence is

LEMMA B.4.
$$\sup_{x\in X} \fint_{B_t(x)} \fint_{B_t(x)} |u(y)-u(z)|^p \le A^p p! M_t^p(u), \quad \forall t < r_0.$$

Proof. For $t \le r_0/\sqrt{n}$, the claim follows immediately from Lemma B.1 with the aid of Lemma A.14. Suppose $r_0/\sqrt{n} < t < r_0$; we may assume that $\int_X u = 0$. Then the desired result follows easily from Lemma B.3. □

Another simple consequence of Lemma B.3 is

LEMMA B.5. *Given $\theta > 0$, there is a number β depending on θ and on X such that if $\|u\|_{\mathrm{BMO}} \le 1$, then*
$$\int_X \left(e^{\beta|u-\bar{u}|} - 1\right) \le \theta \int_X |u-\bar{u}|$$
where $\bar{u} = \fint_X u$.

We point out that Lemmas B.3 and B.5 may be proved in a more direct fashion, not via Lemma B.2. Namely, one starts by proving, directly, Lemma B.5 using (B.1) locally and summing, as we did in the proof of Lemma B.1. Lemma B.3 follows from Lemma B.5.

We now turn to the proof of Lemma B.2. We shall make use of the following covering lemma:

LEMMA B.6. *Given $t, 0 < t < r_0$ and $k > 1$, there is a covering of X by a finite number of balls $B_t(x_i)$, $i = 1, \dots, m = m(t)$ with the property that every $y \in X$ belongs to at most μ balls $B_{kt}(x_i)$, where μ depends only on k and X.* **μ is independent of t**.

Proof. Consider a maximal family of disjoint balls in X of radius $t/2 : B_{t/2}(x_i)$, $i = 1, \dots, m$. Clearly the $B_t(x_i)$ cover X. Suppose $y \in X$ belongs to μ of the $B_{kt}(x_i)$, say for $i = 1, \dots, \mu$. Since $\mathrm{dist}(y, x_i) \le kt$, $i = 1, \dots, \mu$, it follows that
$$B_{t/2}(x_i) \subset B_{(k+\frac{1}{2})t}(y), \quad i = 1, \dots, \mu.$$
Since the balls $B_{t/2}(x_i)$ are disjoint, we find on adding their measures, that
$$\sum_{i=1}^{\mu} |B_{t/2}(x_i)| \le |B_{(k+\frac{1}{2})t}(y)|.$$
Using the fact that
$$\alpha r^n \le |B_r(x)| \le \beta r^n \quad \forall r,$$
for some positive constants α, β depending only on X, we deduce a bound for μ which depends only on k and X. □

Now the

Proof of Lemma B.2. Observe first that for any ball $B = B_t(x)$,

$$\left(\fint_B |u|^p\right)^{1/p} \le \left(\fint_B |u - \bar{u}_t(x)|^p\right)^{1/p} + |\bar{u}_t(x)|$$
$$\le \left[\fint_B \fint_B |u(y) - u(z)|^p\right]^{1/p} + |\bar{u}_t(x)|$$

by the triangle inequality. Hence

$$\int_B |u|^p \le \frac{2^p}{|B|} \int_B \int_B |u(y) - u(z)|^p + \frac{2^p}{|B|^{p-1}} \left| \int_B u \right|^p.$$

By Lemma B.1, we find, with a different A:

$$\int_B |u|^p \le \frac{A^p p!}{|B|} M_t^{p-1}(u) \int_{B_{kt}(x)} \int_{B_{kt}(x)} |u(y) - u(z)| + \frac{2^p}{|B|^{p-1}} \left| \int_B u \right|^p$$
$$\le A^p p! M_t^{p-1}(u) \int_{B_{kt}(x)} |u| + \frac{2^p}{|B|^{p-1}} \left| \int_B u \right|^p.$$

Using the preceding covering lemma, and this last inequality, with $B = B_t(x_i)$, and summing, we obtain the desired conclusion. □

LEMMA B.7. *There are constant β, C depending only on X such that for any measurable set $A \subset X$, and every $u \in \mathrm{BMO}(X, \mathbb{R}^N)$*

$$\beta \fint_A |u| \le \|u\|_{\mathrm{BMO}} \left(C + \log \frac{|X|}{|A|}\right) + \beta \left| \fint_X u \right|.$$

Proof. We may assume $\fint_X u = 0$ and $\|u\|_{\mathrm{BMO}} \le 1$. Recall Young's inequality: For $t > 0, \alpha \ge 1$,

$$\alpha t \le e^t + \alpha \log \alpha - \alpha. \tag{B.9}$$

We apply this with

$$\alpha = \frac{|X|}{|A|} \text{ and } t = \beta |u(x)|$$

with β as in Lemma B.5, where we take $\theta = 1$. Integrating the inequality over A we find

$$\beta |X| \fint_A |u| \le \int_A e^{\beta |u|} + |X| \log \frac{|X|}{|A|} - |X|.$$

Hence

$$\beta \fint_A |u| \le \fint_X (e^{\beta |u|} - 1) + \log \frac{|X|}{|A|}.$$

The desired conclusion follows with the aid of Lemmas B.5 and 1. □

Next, a lemma on the effect of multiplying a BMO function by some function.

LEMMA B.8. *Let a be a Lipschitz function on X and let f be in $BMO(X)$ (respectively VMO). Then af is in BMO (respectively VMO), and*

$$\|af\|_{\mathrm{BMO}} \le C\left(|a|_{C^0} + \|a\|_{\mathrm{Lip}}\right)\|f\|_{\mathrm{BMO}} + \|a\|_{\mathrm{BMO}}\left|\fint_X f\right|.$$

where C depends only on X.

Proof. We may assume that $\int_X f = 0$. We then have to evaluate

$$\begin{aligned} J &= \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |a(y)f(y) - a(z)f(z)| \\ &\le \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |(a(y)-a(z))f(y) + a(z)(f(y)-f(z))| \\ &\le 2\varepsilon\|a\|_{\mathrm{Lip}} \fint_{B_\varepsilon(x)} |f| + 2|a|_{C^0}\|f\|_{\mathrm{BMO}}. \end{aligned}$$

Using Lemma B.7, with $A = B_\varepsilon(x)$, the desired estimate follows. The VMO assertion follows easily from Lemma 3. ☐

REMARK B.1. Note that in Lemma B.8, instead of assuming that a is Lipschitz continuous, we could have assumed that a is Hölder continuous or even merely that

$$|a(x) - a(y)| \le \frac{C}{1 + |\log \mathrm{dist}(x,y)|} \quad \forall x \ne y.$$

D. Stegenga [1] has obtained necessary and sufficient conditions for a function a to be a multiplier preserving BMO.

We conclude this Appendix with a lemma asserting that if $u \in \mathrm{BMO}$, then $\bar{u}_\varepsilon$ is "almost" Lipschitz.

LEMMA B.9. *For $u \in \mathrm{BMO}(X, \mathbb{R}^N)$ and $\varepsilon < r_0$, there is a constant C_ε such that*

$$J = |\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(y)| \le C_\varepsilon \|u\|_{\mathrm{BMO}}\, \mathrm{dist}(x,y)\left(1 + \log\frac{\mathrm{diam} X}{\mathrm{dist}(x,y)}\right).$$

Proof. We may suppose that $\int_X u = 0$ and $\|u\|_{\mathrm{BMO}} = 1$. Then by Lemma 1, $\int_X |u| \le C$.

Case 1. $\operatorname{dist}(x,y) \geq \varepsilon/100$. In this case

$$J \leq \frac{1}{|B_\varepsilon(x)|}\int_X |u| + \frac{1}{|B_\varepsilon(y)|}\int_X |u| \leq C_\varepsilon.$$

Case 2. $\operatorname{dist}(x,y) < \varepsilon/100$. Then

$$\left|\frac{1}{|B_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(y)|}\right| \leq C_\varepsilon \operatorname{dist}(x,y).$$

Thus

$$J \leq \int_{B_\varepsilon(x)} |u(z)| \left|\frac{1}{|B_\varepsilon(x)|} - \frac{1}{|B_\varepsilon(y)|}\right| + \frac{1}{|B_\varepsilon(y)|}\left|\int_{B_\varepsilon(x)} u(z) - \int_{B_\varepsilon(y)} u(z)\right|.$$

As in the proof of Lemma A.16 we introduce the symmetric difference S of $B_\varepsilon(x)$ and $B_\varepsilon(y)$ and we have

$$|S| \leq C_\varepsilon \operatorname{dist}(x,y).$$

Applying Lemma B.7 we see that

$$\left|\int_S u(z)\right| \leq C|S|\left(C + \log\frac{|X|}{|S|}\right).$$

The desired inequality follows by combining this with the earlier estimate. □

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