XXX.—*Non-Alternate* ± *Knots.* By Professor C. N. LITTLE, Ph.D. Communicated by Professor TAIT. (With Three Plates.)

(Read July 3, 1899.)

1. The following paper is a contribution to the theory of non-alternate ± knots, together with a census of these knots for Order Ten; that is, all the knots are given which have in *reduced* form just ten crossings, and in which the thread does not proceed alternately over and under.

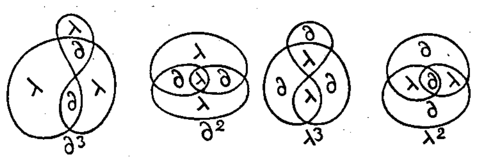
The census was begun in the fall of '93, and carried so far that the forms were drawn. The matter was then laid aside and taken up anew in the spring of '99.

2. Having postulated an endless one-dimensional continuum which may change its length and form in any way, subject only to the condition that it can never have a double point, and consequently no one portion can be made to break through another, I understand by a knot *a continuum which can not be brought to a circular form.*

3. The above definition makes of a knot a purely mathematical concept. As physical approximations for the continuum may be taken:—(*a*) a vortex filament of frictionless ether; (*b*) a flexible, extensible thread. It is convenient and can lead to no error, although keeping to the mathematical conception, nevertheless to speak of the continuum as a thread.

4. Through the work of LISTING,* TAIT,† KIRKMAN,‡ and the writer,§ but pre-

* "Vorstudien zur Topologie," *Göttinger Studien*, 1847, pp. 859–866. To the kind courtesy of Professors FELIX KLEIN and P. STÄCKEL, for which I here express my appreciation, I owe the opportunity to examine the topological *Nachlasse* of GAUSS and LISTING. The former will appear in the forthcoming Bd. VIII., *Gesammelte Werke*, and must not be commented upon in advance of publication. The latter contains among the drawings of reduced knots not figured in the "Vorstudien," a sheet bearing date March 18, 1849, on which are the following forms marked as equivalent:—



$\begin{cases} 2\partial^3 + \partial^2 \\ 2\lambda^3 + \lambda^2 \end{cases}$

The interest of this series lies in the fact that it shows that Professor LISTING fifty years ago recognised the amphicheiral character of this knot.

† These *Transactions*, vol. xxviii. pp. 145–190; vol. xxxii. pp. 327–342; *ibid.*, pp. 493–506. See also *Collected Scientific Papers*, vol. i. pp. 273–346.

‡ These *Transactions*, vol. xxxii. pp. 281–309; *ibid.*, pp. 483–491.

§ *Trans. Connecticut Academy*, vol. vii. pp. 1–17; these *Transactions*, vol. xxxvi. pp. 253–255.

eminently of Professor Tait, the theory of the *alternate* knot is well understood. It may be useful to recapitulate the main points of this theory:

(*a*) The knot is *reduced* (projected with fewest crossings) if no one of the compartments into which its projection divides the plane is opposite itself.

(*b*) Reduced forms of the same knot divide the plane into two sets of vertically opposite parts, with a *constant* number of parts in each set. This gives a convenient basis for classification.

(*c*) It is a simple problem in the circular arrangement of letters to determine from a given reduced form all reduced forms of the knot. Hence it is easy to say as to two given forms whether or not they are projections of the same knot.

(*d*) Simple methods are known by which *all* the knots of a given *order* (minimum number of crossings) can be found.

(*e*) The theory of amphicheiral knots.

5. It is quite the contrary with the non-alternates. They constitute an almost untouched field, bristling with difficulties. In these *Transactions*, vol. xxxv. part ii. p. 664, I published a census of these knots for Orders Eight and Nine; for brevity I shall refer to this paper under the letter A in brackets.

6. It is there stated, [A] § 8, *There is no reduced non-alternate* ± *knot of fewer crossings than eight.* I proceed to give formal proof*:—

In class II., order n, all knots are obviously alternate.

In class III., order n, the leading partition (set of compartments with smaller number of parts) has three parts, say A, B, C. If any one of the connections (AB), (BC), (CA) is null, the form is not a knot. If any one is a single crossing the knot is alternate, [A] § 3. If two of them have each two crossings there must be a link, that is, at least two threads. Hence there must be for these connections at least 3, 3, 2 crossings respectively, and a non-alternate knot of class III. must be at least an eightfold.

Consider class IV., order n. Here there are in the leading partition four parts, A, B, C, D, and these have six connections.

First, let any one, say (AB), be null. Then in order that the form may be the projection of a reduced knot, (AC), (AD), (BC), (BD) must exist. There are two cases:—
(*a*) Let (CD) also be null. If now any one of the existing connections be a single crossing, say λ, then all of the crossings of the other connections will also be λ, by [A] § 3, and the form alternate. But if not, the form is at least elevenfold, since only one connection can consist of an even number of crossings, else the form would be a link.
(*b*) Let (CD) exist. We now have D joined to C by B, by A, and by the one or more crossings of (CD). If either (DB) or (CB), and at the same time either (DA) or (CA), are single crossings, the form is alternate. If (DB) and (CB), or (DA) and (CA), are

* The classification of alternate knots according to the number of parts in the projection, in that set of vertically opposite compartments which has the smaller number, does not answer for non-alternates, since the same knot can be projected in forms belonging to different classes. Later, in § 9, a new basis of classification will be proposed. In this and the following sections, however, the term class has the old signification.

both even, the form is a link. Hence to have a non-alternate knot, A (or B) must furnish at least five crossings and the remaining three connections at least one each. The knot is at least an eightfold.

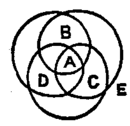
Second, let all six connections exist. If each is a single crossing the form is a link of three threads. If one connection has two crossings, and the other five one each, the form has two threads. In all other cases in this class the form is at least an eightfold.

Class V. and higher classes do not exist in orders lower than Order Eight. This completes the proof of the theorem that there exists no non-alternate ± knot with fewer than eight crossings.

7. *No reduced, non-alternate knot with three consecutive overs has fewer than eleven crossings.*

Since in classes II., III., and IV., order n, it is always possible to go from any part of the leading partition to any other part of the same partition with not more than two crossings, no *sequence of three* (three consecutive overs) can exist in these classes. Hence no reduced eightfold knot with a sequence of three can exist.

All other ninefolds of class V. can be derived from the ninefold 35, below, by erasing crossings at some of the existing connections of the leading partition and adding the crossings erased at other connections. Although it requires three crossings by every path to go from A to E, nevertheless the form as it stands cannot have three consecutive overs, for this would require that we should have three consecutive overs on a closed circuit of five crossings; but this is impossible, since, in order to have in a reduced knot three consecutive overs on a closed circuit, the circuit must have at least seven crossings. If (BC), (CD), or (DB) were erased it would be possible to go from A to E with two crossings. If an A (or E) crossing be erased, say (AB), we have a link, and for knot must add the crossing erased to (BC), (EC), (BD), or (ED). But this will leave D and C connected by the 2-gon A and the crossing (DC). By the transposition of these it becomes possible to go from A to E with two crossings. Hence no reduced ninefold knot can have a sequence of three.



All class V. tenfolds are to be had from the ninefold 35 by the process above described, except that the number of crossings added must exceed by one the number erased. If none be erased and one added, we have either a link or the tenfold 87, which can have no sequence of three, from A to E, except upon a five-crossing circuit. As above, we may not erase (BC), (CD), or (DB). If (BA) be erased and two crossing added in any way to (BC), (EC), (BD), (ED), we again have, after transposition of 2-gon A and the crossing (CD), only two portions of the thread between A and E. If one crossing be added to any one of the four mentioned connections and the other to (DA), (CA), or (DC), A can still be brought into the same position with respect to E. If the second crossing be added to (CE) or (DE) there will be two threads. Hence no class V. tenfold can have a three sequence.

Lastly, suppose a class VI. form to have the sequence of three of fig. 1. The parts A, B, C, D, a, b, c, d must all be distinct, and A, B, a, b can not be 2-gons, or the form would be reducible. Nor can the latter be 3-gons. For, suppose any one, as A, to be a 3-gon. If, now, the thread be shifted from the position fig. 2 to that of fig. 3, the part A is lost to the leading partition and the form comes under class V., which, as has just been shown, can have no form with three consecutive overs. Hence we have the portion of the class VI. tenfold shown in fig. 4. A moment's consideration shows that the parts marked are all distinct. Of the three pairs of adjacent parts C, c; D, d; F, f, only one in each can be a 2-gon. Hence to complete the form at least two additional crossings must be introduced, so that the form becomes at least an elevenfold. Still less is it possible for A, B, a, or b to have more sides than four. Hence no class VI. tenfold can have a three sequence.

C	a	B	d
c	A	b	D

Fig. 1.

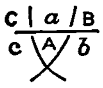


Fig. 2.

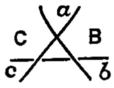


Fig. 3.

	F	f	
C	a	B	d
c	A	b	D

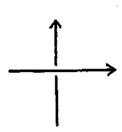
Fig 4.

This completes the proof of the theorem that no reduced non-alternate knot with three consecutive overs has fewer than eleven crossings.

It is easy to verify these theorems by inspection of the plates of alternate forms.

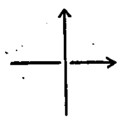
An irreducible elevenfold knot form with three consecutive overs is shown at B, the last in Pl. III. As alternate it is No. 39 of my census.*

8. Twist.—Let the direction of the moving point, continuously tracing the projection of a knot be recorded at each crossing by marking the thread with arrows. Crossings are of two kinds, as shown in the figs. 5 and 6. I call the first a *twist* of $+\pi$, the second a *twist* of $-\pi$.



$+\pi$

Fig. 5.



$-\pi$

Fig. 6.

Theorem.—*The total twist of a reduced knot is constant for all forms in which the knot can be projected.* The proof is very simple. The twist of the crossings is not altered by any of the transformations permissible to alternate forms, since these consist of rotations of a portion of the knot through an angle of π about an axis in the plane of the knot projection. This does not change the relation of the arrows to a crossing nor a λ crossing to a γ, or *vice versa.* In the changes peculiar to non-alternate forms the thread is shifted from one portion of the knot to another, so as to alter the position of two consecutive overs (or unders). Either the twist of the two crossings is unchanged, or else the twist of the two crossings was originally unlike, and in both crossings it is reversed.

9. Knots may be classified according to twist. The non-alternate tenfolds will be so classified in the following census.

10. An amphicheiral knot is one that can be distorted into its own perversion; and

* *Trans. R. S. E.*, vol. xxxvi. pl. I. (39. $D_1 . \mu$.)

the twist of the perversion of a form is the negative of the twist of the original; hence *the twist of an amphicheiral must be zero*, since it must at the same time change sign and be constant.

Census of Tenfold Non-Alternates.

11. The basis of a non-alternate census of any order is the corresponding census of alternates of the same order, since projections of the former, looked at as plane curves with double points, must be included in the projections of the latter, also so regarded.

12. It will be remembered that two censuses* of tenfold alternates were made independently. These agreed in classes III. and V. The one discrepancy in class IV. was due to an oversight of Professor TAIT, and by him corrected before his plates were printed. There were four discrepancies in class VI., for all of which I was responsible. The substantial agreement of these results, obtained by quite different methods, leaves no doubt that we have all the tenfold alternates without error or omission.

Because of their greater elegance, I have used Professor TAIT'S drawings upon which to mark the crossings; the numbers given in the following sections and on the plates are those of the knots of his census

13. The alternate tenfold knots which can in projection also be the projection of non-alternate knots are the following:—

(*a*) Those in which two parts have three necessarily alternate connections:—

2	39	57	75	115
5	42	63	100	116
18	47	64	103	118
23	50	69	109	123
36	56	70	114	

The two or three forms of each of these were at once drawn and marked. For example, 123 has two 3-gons connected by the alternate connections 2-gon, 2^2 coil and 2^4 coil. Any one of these connections may be γ (or λ) and the other two λ (or γ), giving six forms—the three found on the plates in knots: I, class I.; I, class III.; and I, class VI., and their perversions.

(*b*) More complicated:—

6	27	38	86	95
8	28	44	87	96
9	29	46	88	97
10	30	54	89	98
12	31	59	90	99
13	32	82	91	110
17	33	83	92	111
25	34	84	93	112
26	35	85	94	113

* *Trans. R. S. E.*, vol. xxxii.; *Trans. Conn. Acad.*, vol. vii.

14. For the latter, tables of all possible crossings were made out, as described in [A] § 6. To secure accuracy these tables were written out twice, once by myself and again later under my direction.* The limitation of § 7 keeps these tables in bounds. Nevertheless a considerable number of the forms given are reducible, and these must in the after work be detected and excluded. This was invariably done by so distorting the form that a sequence of three overs occurred.

15. Upon these tables also the *twist* of each form was computed.

16. From the tables the crossings were marked upon the tracings, giving all possible *forms* of non-alternate knots.

Forms are regarded as distinct only when compartments are dissimilar, or when the direct connections of the compartments are dissimilar in the number or character of the crossings. The tables lead to a certain number of forms equivalent to others already given, and differing only in the circumstance that the form had been rotated through an angle π about an axis of symmetry in the plane of the projection. These were excluded from the plates. In the case of forms which as alternates are amphicheiral, two equivalent forms occur which differ in that the amplexum partition of one is made the non-amplexum partition of the other. Such duplicates are also omitted.

17. The task of finding the knots from the knot forms was exceedingly laborious, and one that I should not have been able to accomplish except for the constant check given by the twist.

The process was as follows:—The twenty-four knots of which the forms obtained in (α) § 13 are projections were easily found. Every other form was examined to see if it could be distorted so as to have two parts with three alternate connections. If this could be done, its proper place in these twenty-four knots was at once known. Where this was found to be impossible, the form was distorted into a form of which the knot was known. In the end it was necessary to see that all permissible distortions of the forms of each knot gave only forms of the same knot. The conditions attendant upon this process are very numerous; I must repeat the caution previously given that until a simple test is found which shall distinguish the forms of one knot from those of every other in the same class, it may happen that non-alternate knots regarded as distinct may be in reality the same.

18. The identification of a given tenfold non-alternate by means of the census may, in a particular instance, require some labour. In the first place it must be ascertained that the given knot is reduced. For this, unfortunately, no simple test is as yet known. Next observe the twist, to determine the class in which the knot will be found. Then find the number of the corresponding alternate knot and compare with the list of § 19. If this number is to be found in more than one knot of the class, a closer comparison with the forms of the plate must be made. In case of apparent failure to find the form, the limitations of § 16 must be kept in mind.

* By Dr H. F. Blichfeldt, now instructor, at the time a student in Stanford University, for whose care my thanks are due.

The Tenfold Alternate ± Knots.

19. In the following list the forms of § 13 (*a*) are put at the beginning of the knots in which they occur. Other forms are given in numerical order; this is not the case upon the plates.

Class I.—Twist 0. Six knots, and forty-six numbered forms.

I.—70, 116, 123, 26, 33, 88, 94, 94, 97, 99, 112, 113, 113.
II.—18, 57, 75, 17, 17, 34, 44, 54, 93, 93, 95, 111.
III.—2, 63, 8, 10, 25, 25, 84, 85, 89, 90.
IV.—5, 50, 64, 6, 6, 8, 46.
V.—25, 28, 87.
VI.—59.

Class II.—Twist 2π. Nine knots, eighty-four numbered forms.

I.—56, 115, 26, 27, 38, 38, 88, 88, 96, 97, 112.
II.—2, 63, 114, 10, 25, 30, 44, 83, 83, 85, 90, 92, 92.
III.—23, 69, 34, 92, 95, 98.
IV.—39, 42, 47, 12, 12, 13, 13, 29, 82, 84, 84, 86, 86, 86, 86, 89.
V.—42, 47, 12, 12, 13, 28, 28, 29, 29, 30, 82, 82, 82, 84, 84, 85, 86, 86, 87, 89, 91.
VI.—39, 42, 12, 13, 84, 84.
VII.—31, 32, 38, 88, 96, 112.
VIII.—34, 89, 91.
IX.—91, 98, 110.

Class III.—Twist 4π. Ten knots, ninety-eight numbered forms.

I.—36, 70, 123, 33, 94, 99, 113.
II.—23, 69, 118, 25, 28, 34, 82, 85, 87, 91, 91, 93, 98, 110, 110, 111.
III.—56, 100, 115, 26, 31, 32, 33, 88, 94, 96, 112.
IV.—57, 75, 109, 54, 17, 93, 111.
V.—50, 64, 103, 6, 8, 25, 25, 28, 30, 44, 46, 85, 87, 90, 90, 92, 92, 93, 95, 98, 110, 110, 111.
VI.—28, 29, 93.
VII.—26, 27, 32, 33, 35, 59, 94, 96, 97, 97, 113.
VIII.—26, 27, 27, 88, 94, 96.
IX.—6, 8, 9, 10, 44, 59, 95.
X.—10, 25, 83, 83, 92, 92, 98.

Class IV.—Twist 6π. Eight knots, sixty-five numbered forms.

I.—36, 70, 116, 97, 99, 99, 113.
II.—18, 57, 109, 17, 54, 54, 95.
III.—5, 50, 103, 6, 8, 25, 30, 30, 44, 46, 46, 54, 85, 85, 90, 95, 98, 111.
IV.—28, 82, 85.
V.—10, 25, 28, 29, 85, 87, 89, 89, 91, 110.
VI.—8, 9, 10, 30, 44, 46, 59, 83, 85, 89, 111.
VII.—31, 32.
VIII.—33, 33, 35, 59, 96, 97, 99.

Class V.—Twist 8π. Two knots, ten numbered forms.

I.—63, 114, 10, 29, 83, 86, 89.
II.—27, 32, 96.

Class VI.—Twist 10π. Eight knots, fifty numbered forms.

I.—36, 116, 123, 26, 31, 33, 38, 88, 94, 97, 99, 112, 113.
II.—23, 118, 25, 87, 92, 98, 110.
III.—100, 115, 27, 32, 96.
IV.—18, 75, 109, 8, 17, 28, 34, 54, 85, 93, 95, 111.
V.—5, 64, 103, 6, 30, 44, 46.
VI.—10, 29, 83, 89.
VII.—9.
VIII.—35.

20. It is interesting to note that in order ten as in order eight no amphicheiral non-alternate knots have made their appearance. The numbers of knots must be doubled, therefore, to include perversions.

To the 253 alternate knots previously described this census adds 86 non-alternate, making 339 reduced knots of ten crossings.

These new knots are shown upon Plates I., II., III., the perversions being obtained as usual by holding the plates before a mirror and looking at the images. In knot II of class II., Plate I., § 16 should have excluded the second form of 83. The first 59 in knot IX of class III. on Plate II. is also manifestly superfluous.

PROF. LITTLE: NON-ALTERNATE ± KNOTS.

PLATE I.

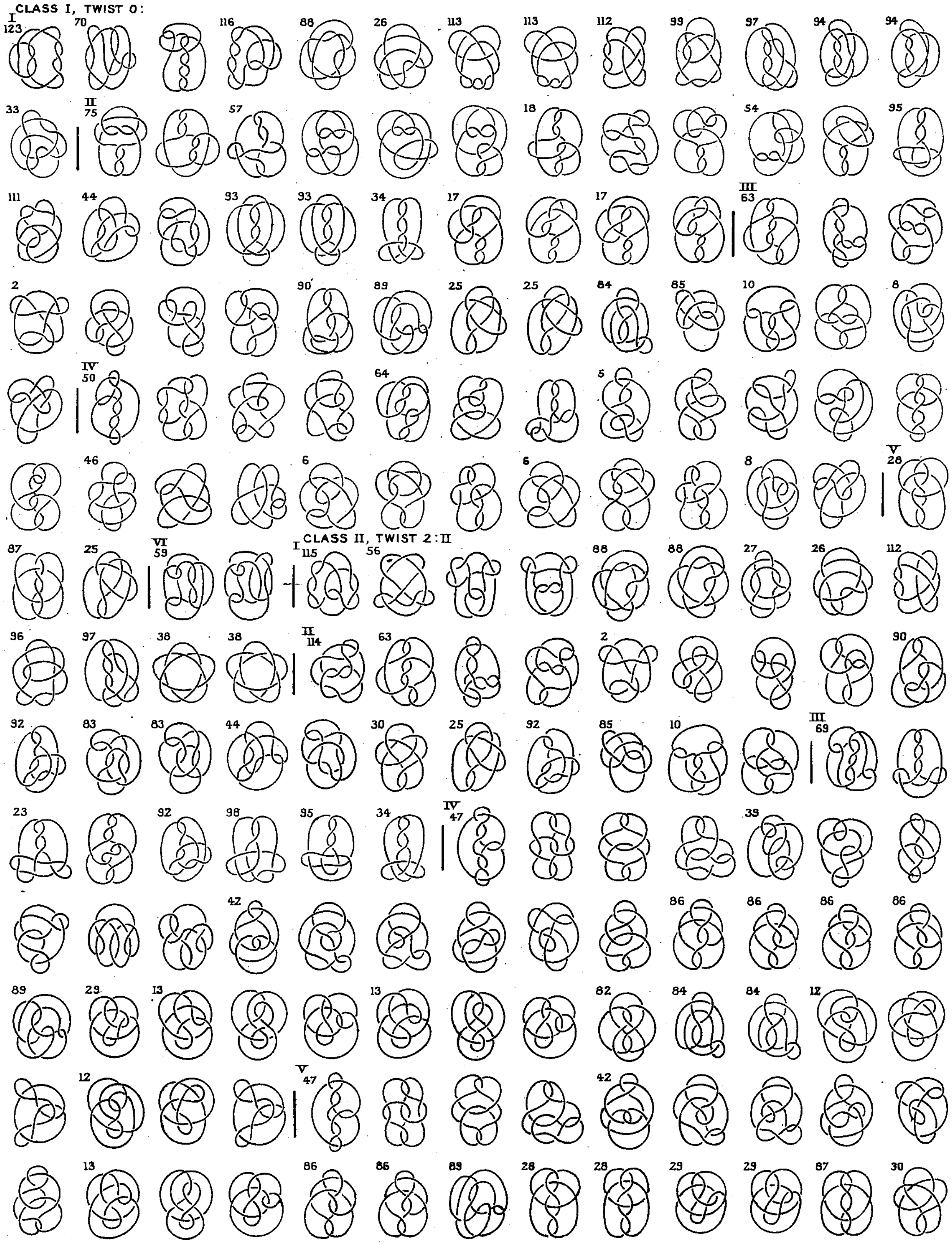


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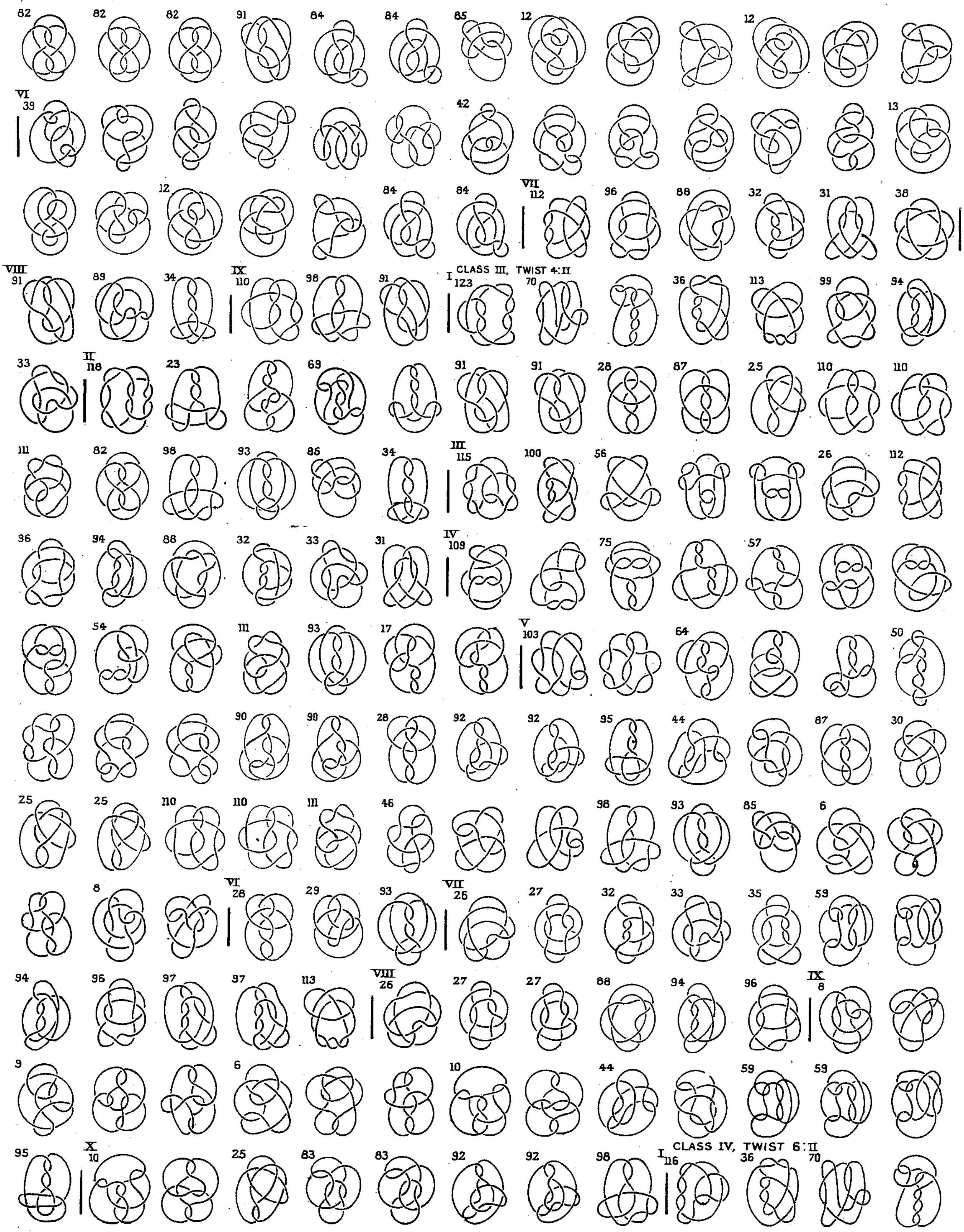
PLATE II.



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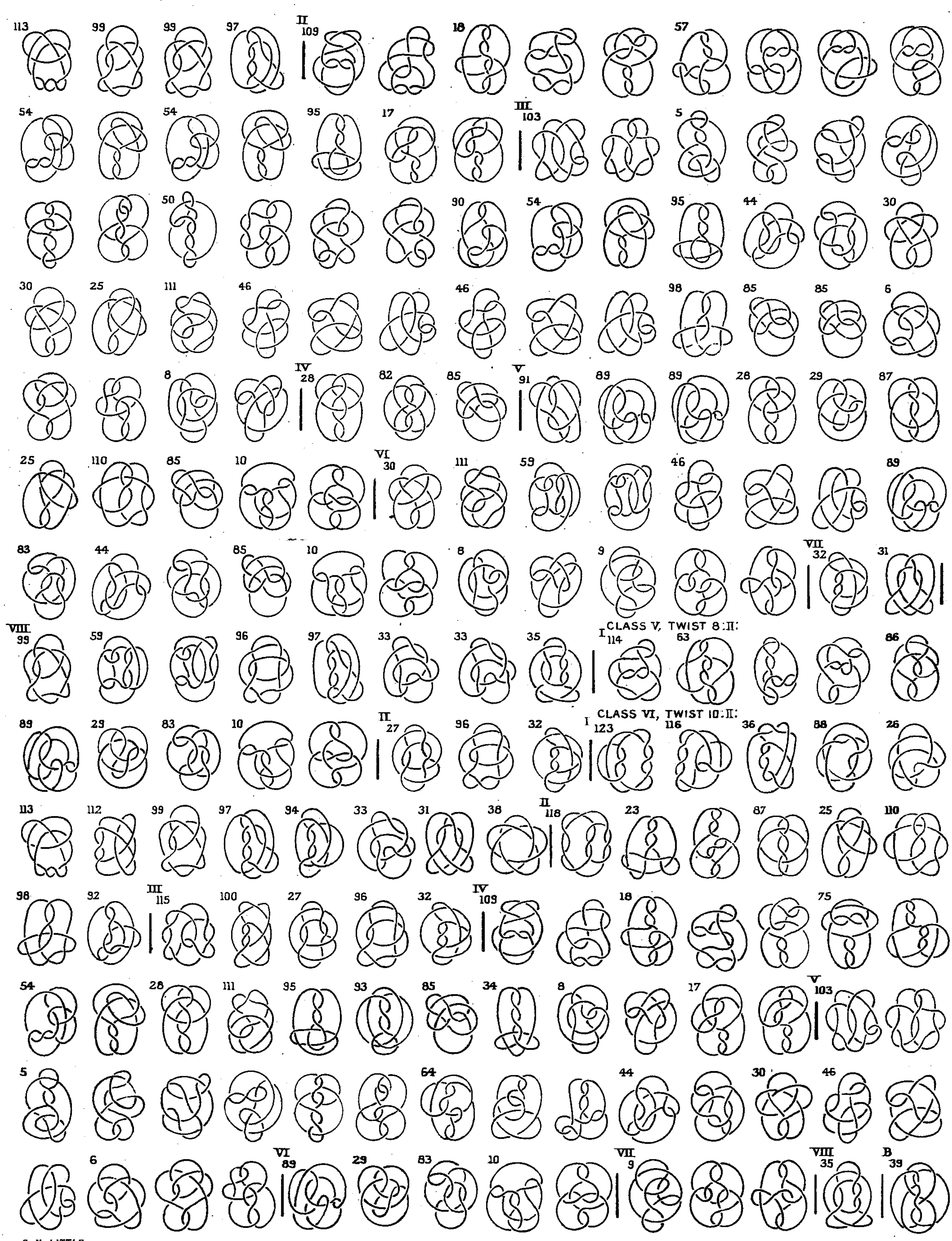
Prof. Little: Non-Alternate ± Knots.

Plate III.



C. N. LITTLE.

A. RITCHIE & SON, EDIN.