# MANIFOLDS WITH FREE ABELIAN FUNDAMENTAL GROUPS AND THEIR APPLICATIONS

## (Pontrjagin Classes, Smoothnesses, Multidimensional Knots)

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In this paper we establish the topological invariance of rational Pontrjagin classes of smooth and piecewise-linear manifolds and we draw a number of corollaries from this result. The methods are also applicable to other problems.

### Introduction

As the author has shown in previous papers [10–13], the question of the topological invariance of rational Pontrjagin classes is very closely related to certain problems of homotopy and differential topology of non-simply-connected manifolds and of their coverings, namely, those in which the fundamental group is free abelian. The reduction of the problem of the invariance of classes to homotopy problems in this group of papers by the author is linked by one general idea. This idea consists in making judicious selections from the notion of "continuous homeomorphism" by using special, non-simply-connected, open subsets which can then be studied by the means of a purely smooth topology making use of the non-simply-connectedness, although the fundamental group has no relation whatsoever to the problems originally posed. Thus, in the very first reference [10] ([13]) special cases of this problem were solved with the aid of analogs of the Hirzebruch formula on coverings, which already led to distinguishing between homeomorphism and homotopy type. A direct development of this "signature" method led the author to the proof of the topological invariance of the Pontrjagin-Hirzebruch class $L_k(M^n)$ for $n \leq 4k+3$. This intermediate discussion is indicated in the appendix; it was found before the general result of reference [11] and soon thereafter lost its interest to a great degree since the author succeeded in giving a more general proof of the invariance of classes (published in brief in [11]) which did not contain "signature" arguments and analogs of the Hirzebruch formula.

In the present paper the problem of the classes is solved by generalizing to the non-simply-connected case the techniques of papers [3] and [14] for investigating smoothnesses on manifolds of the type $M^n \times R$, $\pi_1(M^n) = Z + \cdots + Z$, although the reduction of the problem to such a problem in differential topology is, of course, also to be found in the first paper [10] by the author on the topological invariance of classes. A manuscript of W. Browder (later published in [4]) in which the problem of smoothnesses on manifolds of type $M \times R$ was solved for the simply-connected case $\pi_1(M) = 0$, was submitted at the very same time as the present paper and proved to be very useful to the present author. Certain arguments from [4] aided him during his work and he takes the opportunity here to thank W. Browder.

The results of the work are formulated in §1. The central result is Theorem 1, which establishes the topological invariance of rational Pontrjagin classes of smooth and piecewise-linear manifolds.

§2 is very important in the paper; it contains the reduction of Theorem 1 to Theorem 3 and also interrelates the remaining results. It is here that use is made of the fact that the manifolds $M_1$ and $M_2$ from Theorem 1 are homeomorphic.

Theorem 3 is proved in §§3–8. Of separate interest in itself is §5 which can easily be extended to a wider class of groups.

Theorem 6 from the theory of knots is proved in §9.

§10 contains (without proof) a generalization of Theorem 5.

A number of corollaries follow from Theorem 1 of this paper together with previously-known results of algebraic and differential topology.

*Certain consequences of the invariance of classes:*

1. The number of smooth structures on a simply-connected topological manifold $M^n$, $n \neq 4$, is finite and does not exceed the constant $c(M^n)$, where

$$c(M^n) < e^{q_n + \sum_{i=2}^{n} b_{n-i} \ln c_i + \sum_{4k} d_{4k}};$$

moreover,

$$q_n = \ln |\theta^n(\partial\pi)|, \quad d_i = \ln |\operatorname{Tor} H_i(M^n)|,$$

$$b_j = \max_{p \geqslant 2} rkH_j(M^n, Z_p), \quad c_j = a_j |\pi_{N+j}(S^N)|,$$

$a_j = 1$ when $j \not\equiv 1, 2 \pmod 8$ and $a_j = 2$ when $j \equiv 1, 2 \pmod 8$. This consequence arises from a comparison of Theorem 1 with Bott periodicity and the author's results in the diffeomorphism problem (see [14]).

The finiteness and the bound (with other universal constants) hold, analogously, for a number of combinatorial structures on $M^n$ under the same restrictions. Here it is necessary to make use of the result of Cerf that $\pi_0(\operatorname{diff} S^3) = 0$. Thus we have the Hauptvermutung with accuracy up to a finite number of possible $PL$-structures under our restrictions. This follows from reference [14] (see Appendix 2 of [14]).

2. As has already been mentioned in [10], for dimensions of the form $4k+2$, the difference between homeomorphism and homotopy type of closed simply-connected manifolds follows from the invariance of Pontrjagin classes and from



the results of the author and of Browder (see [3] and Appendix 1 of [14]). It follows from Theorem 1 that for any simply-connected manifold $M^n$, $n \geq 6$, in which the homology group $H_{4k}(M^n)$ is infinite for at least one $k \neq 0, n/4$, there exist an infinite number of smooth, or smooth except at a point, manifolds $M_i^n$ not pairwise homeomorphic but having a common homotopy type with $M^n$. When the above homological condition is not satisfied the number of such manifolds is automatically finite, as follows from [14].

3. On the odd-dimensional spheres $S^{2n+1}$, $n \geq 3$, there exist an infinite number of smooth, or smooth except at a point, operations on the circle $S^1$ without fixed points, which are not topologically equivalent pairwise. This fact follows as a result of applying the preceding paragraph to the factor-space $S^{2n+1}/S^1$ of homotopy type $CP^n$, since topologically equivalent operations generate homeomorphic factor-spaces.

4. Since the Pontrjagin numbers are topologically invariant, two smooth manifolds belonging to different classes of oriented cobordisms $\Omega_{SO}$ are always nonhomeomorphic.

5. All piecewise-linear manifolds with fractional Pontrjagin classes are nonhomeomorphic to smooth ones. Many such manifolds are known in each dimension $n \geq 8$, and many of them (although not all) are homotopically equivalent to smooth ones.

6. The $SO_n$-fiber spaces with the sphere $S^{4k}$ as base and the euclidean space $R^n$, the disc $D^n$ or the sphere $S^{n-1}$ when $n > 4k+1$ as layer, are completely classified from the topological point of view by the Pontrjagin class of this fiber. This is also true of a number of other examples. It has been long known (Dold) that here there are only a finite number of different homotopy types.

7. If on a smooth closed manifold $M^n$ we are given an elliptic integro-differential operator $A$ which transforms a section of the fiber $F_1$ over $M^n$ into sections of the fiber $F_2$ over $M^n$, then, as usual, it defines the "symbol" $\sigma(A)$. This symbol is the isomorphism of fibers $F_1$ and $F_2$ extended onto $\tau(M^n)$ and then bounded on the subspace

$$\tau(M^n) \setminus M^n \subset \tau(M^n),$$

where $\tau(M^n)$ is the space of the tangent fiber over $M^n$ with layer $R^n$ and $M^n \subset \tau(M^n)$ is the zero section. Since the space $\tau(M^n)$ and the section $M^n \subset \tau(M^n)$ do not depend on the smoothness on the manifold $M^n$, the "symbol of the operator $\sigma$" is a topologically invariant concept; however, in different smoothnesses on $M^n$ one and the same symbol $\sigma$ defines the operators $A_1$, $A_2$ given in different spaces but such that $\sigma(A_1) = \sigma(A_2)$ (the operators are defined nonuniquely but with accuracy up to a completely continuous component in each of the smoothnesses).

The well-known Atiyah-Singer formula expresses the index of the operator in terms of invariance of the triple $(F_1, F_2, \sigma)$, not dependent on smoothness, and in terms of the Pontrjagin classes of the smooth manifold $M^n$. From Theorem 1 it follows that the index of the operator is determined only by the symbol, independently of smoothness on the manifold $M^n$; the indices are the same for operators with a common (homotopy) symbol, given in different smoothnesses.

8. The natural map $\pi_i(BSO) \to \pi_i(B\text{ Top})$ is monomorphic, while the map $H^*(B\text{ Top}, Q) \to H^*(BSO, Q)$ is epimorphic.

9. The group of homotopy classes of the diffeomorphisms of a closed simply-connected manifold of dimension not less than five, has a finite index in the analogous group for the homeomorphisms (see Theorems 6.9 and 6.10 of [14]).

In conclusion I would like to thank V. A. Rohlin for advice and numerous useful discussions. It should be noted that the proof found by the author for the invariance of the Pontrjagin classes is to a considerable extent an extension of the line of work by Rohlin and Thom [15, 19] on this problem. I also thank S. P. Demuškin, I. R. Šafarevič and Ju. I. Manin for help in the many algebraic questions which arose during the work, and A. V. Černavskiĭ for questions related to Theorem 6.

## §1. Formulation of the results

The following theorems are basic to this paper from the point of view of application.

Theorem 1. *Let $M_1$ and $M_2$ be two smooth (or PL-) manifolds and let $h: M_1 \to M_2$ be a continuous homeomorphism. Then*

$$h^* p_i(M_2) = p_i(M_1),$$

*where $p_i(M_q)$, $q = 1, 2$, are rational Pontrjagin classes of the manifolds $M_1$ and $M_2$.*

Theorem 2. *Let $M^{4k}$ be a closed manifold, $W^{m+4k}$ a smooth closed manifold of the homotopy type of $M^{4k} \times T^m$, where $T^m$ is an m-dimensional torus, let $\pi_1(M^{4k}) = Z + \cdots + Z$, and let $h: W^{m+4k} \to M^{4k} \times T^m$ be a certain homotopy equivalence. Then*

$$(L_k(W^{m+4k}), h^*[M^{4k}] \otimes 1) = \tau(M^{4k}),$$

*where $L_k$ are the Hirzebruch polynomials and $\tau$ is the signature of the manifold.*

The condition $\pi_1 = Z + \cdots + Z$ in Theorem 2 can undoubtedly be removed, but we shall not do this here.

Now let $W$ be an open smooth manifold of dimension $n+1$, having the homotopy type of a closed $n$-dimensional manifold, and let us further be given a (possibly nonsmooth) transformation $T: W \to W$ acting discretely and such that the factor $W/T$ is compact. The following theorem holds under these conditions.

Theorem 3. *If $n \geq 5$ and if the group $\pi_1(W)$ is isomorphic to a free abelian group, then we can find a closed manifold $V$ such that $W$ is diffeomorphic to $V \times R$.*



This theorem is proved in §§3–8 and Theorems 1 and 2 for the smooth case (see §2) are derived from it. The case of $PL$-manifolds is entirely analogous and requires only a combinatorial analog of Theorem 3, which is proved without any changes with due regard to the author's remarks in [14] (see [14], Appendix 2 on combinatorial Morse surgery).

Among other results which can be extracted from Theorem 3 and its analogs, we mention the following.

Theorem 4. *Let $M^n$ be a closed manifold such that $\pi_1(M^n)$ is a free abelian group of rank $k$. Then the smoothness on the direct product $M^n \times R^q$ with $q > n$ is completely determined by a stable tangent bundle which can take only a finite number of values.*

Theorem 5. *Let $M^n$ be a smooth closed manifold, $\pi_1(M^n) = \pi$ a free abelian group of rank $k$, and let $M^n$ have the homotopy type of a fiber bundle with torus $T^l$ as base and layer $M^{n-l}$, where $M^{n-l}$ is a closed topological manifold. If $l \leq n-5$, the covering $\hat{M}$ over $M^n$ having the homotopy type of $M^{n-l}$ is diffeomorphic to the direct product $M_1^{n-l} \times R^l$, where $M_1^{n-l}$ is a closed smooth manifold.*

Theorem 5 follows directly from Theorem 3.

The following theorem can be extracted in indirect fashion from Theorem 3 or from a direct analog of it.

Theorem 6. *Let $S^n \subset S^{n+2}$, $n \geq 5$, be a topological locally flat imbedding. Then this imbedding is topologically equivalent to a smooth imbedding $S^n \subset S^{n+2}$ in some smoothness on $S^n$. In particular, the imbedding is locally flat.*

The derivation of Theorem 6 from the preceding results will be given at the end of the paper. In contrast to Theorems 1, 2, 4, 5 we shall here require some additional discussion (see §9).

At the end of the paper (§10) we shall also state without proof one generalization of Theorem 5.

## §2. Plan of the proofs of the fundamental theorems

1. The proofs of the fundamental theorems of the paper will be carried out along the following plan:

1) We shall first prove Theorem 3 (see §§3–8).

2) From Theorem 3 will be derived Theorem 1 for the simply-connected case, and Theorems 2, 4, 5 (see §2). It is well known that in Theorem 1 the general case follows from the simply-connected one. Further, it suffices to prove Lemma 2.1 (below) only for the spheres $S^{4k}$.

3) At the end of the paper we shall give separately the proof of Theorem 6 on the basis of Theorem 3 and its generalizations (see §§9, 10).

2. The proof of Theorem 3 will take up the main part of this paper. Here we indicate the plan for deriving Theorem 1 for the simply-connected case and for deriving Theorem 2, both from Theorem 3.

The following lemma is in essence contained in references [15, 16, 19]. It was communicated to the author by V. A. Rohlin a rather long time ago.

Lemma 2.1. *Let $W$ be any smooth manifold homeomorphic to $M^{4k} \times R^m$, where $M^{4k}$ is a simply-connected closed manifold. If the formula*

$$(L_k(W), [M^{4k}]) = \tau(M^{4k}),$$

*always holds, then the rational Pontrjagin classes of smooth simply-connected manifolds are topologically invariant.*

Here $L_k$ are the Hirzebruch polynomials and $\tau$ is the signature of the manifold. We do not prove this lemma, considering it to be very well known from the papers by Thom, Rohlin, Švarc (see [15, 16, 17]), where it is used mainly, it is true, for piecewise linear homeomorphisms.

Our aim is to prove the following assertion.

Lemma 2.2. *The formula*

$$(L_k(W), [M^{4k}]) = \tau(M^{4k}).$$

*always holds under the hypotheses of Lemma 2.1. Moreover, this formula holds for piecewise linear manifolds and for "combinatorial" Pontrjagin classes.*

From the conceptual point of view the derivation of Lemma 2.2 takes a central place in the paper since it is precisely here that we use the fact that the two manifolds are homeomorphic. The fact of the matter is that Theorem 3 by itself has no relation whatever with the problem of invariance of Pontrjagin classes.

We here proceed with the derivation.

We use the topological structure of the manifold $W$ in the following way. The ordinary torus $T^{m-1}$ can be smoothly realized in the euclidean space $R^m \supset T^{m-1} \times R$; we consider the open submanifold $i: W_1 \subset W$, where $W_1 = M^{4k} \times T^{m-1} \times R$, where, moreover, the imbedding $i: W_1 \subset W$ is defined in accordance with the homeomorphism $W \approx M^{4k} \times R^m$ and the imbedding $T^{m-1} \times R \subset R^m$. It is obvious that $i^* L_k(W) = L_k(W_1)$ and that $i_*: H_{4k}(W_1) \to H_{4k}(W)$ is an epimorphism. Therefore we can study the class of $L_k(W_1)$ instead of the class of $L_k(W)$. Since $W_1$ is homeomorphic to $(M^{4k} \times T^{m-1}) \times R$ and since $\pi_1(M^{4k}) = 0$, Theorem 3 is applicable to $W_1$ if $k > 1$ or if $k = 1$ but $m > 1$.

Later on below our discussion will be of a periodic nature. We indicate the construction of the first period:



a) On the basis of Theorem 3 we can find a closed submanifold $V_1 \subset W_1$, such that $W_1$ is diffeomorphic to $V_1 \times R$; therefore $L_k(W_1) = L_k(V_1)$.

b) We consider the covering over the torus $T^{m-2} \times R \to T^{m-1}$ and from this covering we construct the covering over $V_1$, where $V_1$ has the homotopy type of $M^{4k} \times T^{m-1}$

$$\hat{V}_1 \underset{p_1}{\to} V_1,$$

and, moreover, $\hat{V}_1$ has the homotopy type of $M^{4k} \times T^{m-2}$ and $Z$ is the group of the motions of the covering. Obviously, $L_k(\hat{V}_1) = p_1^* L_k(V_1)$ and the map

$$p_{1*} : H_{4k}(\hat{V}_1) \to H_{4k}(V_1)$$

is such that $H_{4k}(V_1) = \operatorname{Im} p_{1*} + A$, where $L_k/A = 0$ for a suitable choice of $A$.

c) We now denote $\hat{V}_1$ by $W_2$ and we note that Theorem 3 is once again applicable to the manifold $W_2$ if $k > 1$ or if $m - 1 > 1$. Thus we have the "period":

$$\underbrace{W_1 \supset V_1 \underset{p_1}{\leftarrow} \hat{V}_1 = W_2}\, \underbrace{\supset V_2 \underset{p_2}{\leftarrow} \hat{V}_2 = W_3}.$$

It is significant that $\dim W_2 = \dim W_1 - 1$ while the class of $L_k$ is essentially unaltered.

Further, from the manifold $W_2$ we once again seek, as in the first period, the manifolds $V_2 \subset W_2$ and $W_3 = \hat{V}_2$ and we go on in this way until we reach the simply-connected manifold $W_m$ of dimension $4k+1$ of the homotopy type of $M^{4k}$

If $4k > 4$ we can once again apply Theorem 3 to $W_m = V_m \times R$ and note that

$$(L_k(V_m), [V_m]) = (L_k(W), [M^{4k}])$$

by construction, and that

$$(L_k(V_m), [V_m]) = \tau(M^{4k})$$

by the Hirzebruch formula, since by construction $V_m$ has the homotopy type of $M^{4k}$ and is closed. Hence we obtain Lemma 2.2 for the case $4k > 4$.

If, however, $4k = 4$, then here we note that the manifold $V_{m-1}$ has the homotopy type of $M^{4k} \times S^1$. Then from Theorem 1 of the author's paper [13] it follows that $(L_k(V_{m-1}), [M^{4k}]) = \tau(M^{4k})$, and once more we obtain Lemma 2.2 for $k = 1$.

Theorem 2 is derived analogously from Theorem 3.

3. Let us derive Theorem 4 from Theorem 3. We consider a smooth manifold $W$, homeomorphic to $M^n \times R^m$ for large $m$. We smoothly imbed $M^n \subset W$ (see [5]). The neighborhood of $M^n$ in $W$ is the space of the $SO$-bundle $\beta$ such that

$$\beta \oplus \alpha(M) = \alpha(W),$$

where $\alpha(X)$ is the tangent fiber of the smooth manifold $X$.

We denote by $V = V^{n+m-1}$ the space of the $SO$-fiber $\beta$ with the sphere $S^{m-1}$ over $M^n$ as layer. From the manifold $W$ we discard the closed neighborhood of the manifold $M^n$ in $W$, homeomorphic to $M^n \times D^m$. What remains will be homeomorphic to

$$M^n \times S^{m-1} \times R = W_1.$$

By Theorem 3 $W_1$ is diffeomorphic to $V_1 \times R$, where $V_1$ is a smooth closed manifold of the homotopy type of $M^n \times S^{m-1}$. However, $V_1$ is $h$-homologic to the manifold $V$, the bundle space of the spheres $\beta$. Since $\pi_1 = Z + \cdots + Z$, $V_1$ is diffeomorphic to $V$ and the whole manifold $W$ is diffeomorphic to the bundle space of $\beta$ with $R^m$ over $M^n$ as layer. The theorem is proved.

Note that for $M^n = S^1$ the tangent fibers $\alpha(S^1)$ and $\alpha(W)$ are always trivial. Therefore $W = S^1 \times R^m$

4. We note that Theorem 5 follows formally from Theorem 3 for the case when the dimension of the torus is 1; for this we must examine the manifold $W$, being the covering over $M^n$ with motion group $Z$. The general case is derived by applying Theorem 3 successively to this situation.

## §3. A geometric lemma

The purpose of this section is to prove a lemma of a type which is rather usual in the theory of smooth imbeddings. The single feature which distinguishes it from the ordinary case is that we need it for the non-simply-connected case, although this does not give rise to significant changes in the proof.

Lemma 3.1.* *Let $(W^{n+1}, V^n)$ be a manifold in $W^{n+1} = W$, one of the components of whose boundary is $V^n = V$; $W^{n+1}$ can be open. If the imbedding $\pi_1(V) \to \pi_1(W)$ is an isomorphism and if the group $\pi_1(V)$ does not have a 2-torsion and all the groups $\pi_i(W, V)$ are null when $i \leq s$, then every map of pairs, $f_i$: $(D^{l+1}, S^l) \to (W, V)$, is homotopic to a smooth imbedding if $3l + 3 < 2n$ and $2l - n + 1 < s$. Furthermore, under the same restrictions on the dimensions, every finite collection of maps $f_i$: $(D^{l+1}, S^l) \to (W, V)$, $i = 1, \cdots, q$, is homotopic to a system of pairwise nonintersecting smooth imbeddings.*

Proof. We begin by considering the first part of the lemma, on the mapping of one object.

Let $f$: $(D^{l+1}, S^l) \to (W, V)$ be an arbitrary mapping of pairs. We consider the universal covering $(\hat{W}, \hat{V})$ and the covering pair map $\hat{f}$: $(D^{l+1}, S^l) \to (\hat{W}, \hat{V})$. Since the pair $(\hat{W}, \hat{V})$ is simple-connected, we can take it that the map $\hat{f}$ is a smooth imbedding (see [23]). Furthermore, the map $f$, from generality considerations, has only two points of self-intersection. These points of self-intersection form a sub-

* The author is not certain that this lemma cannot be extracted directly from the work of Haefliger [5] or of J. Levine. The lemma will be applied only for $n = 2l + 1$ and $n = 2l$ (see §8), and therefore the reader should not pay too much attention to it.



manifold $M^t \subset D^{l+1}$, in general, with a boundary, where $t = 2l - n + 1$. The map $f/M^t \to W$ is a two-sheeted covering. Let us show that this covering is trivial, i.e.

$$M^t = M_1^t \cup M_2^t$$

and

$$f(M_1^t) = f(M_2^t).$$

Indeed, if there were a connected component $M_0^t \subset M^t$ on which the map $f$ were two-sheeted, then the image $\hat{f}(M_0^t) \subset \hat{W}$ would be such that there would exist an element $\alpha \in \pi_1(V) = \pi_1(W)$ such that

$$\alpha(M_0^t) = M_0^t,$$

where $\alpha$: $\hat{W} \to \hat{W}$ and $\alpha^2/M_0^t = 1$; therefore we would have $\alpha^2 = 1$, which contradicts the hypotheses of the lemma.

Thus $M^t = M_1^t \cup M_2^t$ and $f(M_1^t) = f(M_2^t)$.

On the manifold $M_1^t$ we construct the Morse function $g$, equal to zero on the boundary $\partial M_1^t \subset S^l$. After passing through the first critical point $g = x_0$ the topology of the "region of large values" is changed. Let us show, by analogy with Haefliger [5], that we can correspondingly change the map

$$f: (D^{l+1}, S^l) \to (W, V),$$

so that instead of

$$M_1^t = \{g \geqslant 0\} = \{g \geqslant x_0 - \varepsilon\}$$

we shall have the self-intersection manifold

$$\bar{M}_1^t = \{g \geqslant x_0 + \varepsilon\}, \quad \varepsilon > 0,$$

for the new map

$$\bar{f}: (D^{l+1}, S^l) \to (W, V),$$

homotopic to the map $f$.

Consider the region $G = \{g \leq x_0 + \epsilon\}$. Let the index of the point $(g_0 = x_0,$ grad $g = 0)$ be $k$. Then

$$G = \partial M_1^t \times I(0, 1) \bigcup_h D^k \times D^{t-k},$$

Let

$$h: \partial D^k \times D^{t-k} \to \partial M_1^t \times 1.$$

$$S^{k-1} = h(\partial D^k \times 0) \subset \partial M_1^t,$$

$$D_0^k = h(D^k \times 0) \subset D^{l+1}.$$

Consider the disc $D^{k+1} \subset D^{l+1}$, where $\partial D^{k+1} = D_0^k \cup D^k$ is such that $D^{k+1} \cap M_2^t = \emptyset$,

$$D^{k+1} \cap M_1^t = D_0^k$$

(in the general position) and

$$D^{k+1} \cap \partial D^{l+1} = D_1^k$$

(in the general position). Let $T$ be a neighborhood of the disc $f(D^{k+1})$ in $W$ and let Int $T$ be its interior. We set

$$W' = W \setminus \operatorname{Int} T.$$

Obviously $W'$ is diffeomorphic to $W$; we have "squeezed out" the interior of $T$ from the boundary $\partial W = V$. Retaining the former notation we denote $W'$ by $W$ and $\partial W'$ by $V$.

Consider the abstract disc $D^{l+1}$ and the submanifolds $M_1^t, M_2^t$ in it. From $D^{l+1}$ we remove the set $D^{k+1} \subset D^{l+1}$ together with its "envelope" $f^{-1}f(D^{k+1})$, and in so doing we also have removed the neighborhood of the disc

$$f^{-1}f(D_0^k) \cap M_2^t = \bar{D}_0^k$$

from $D^{l+1}$. The topological effect of this operation is that the neighborhood of the disc $\bar{D}_0^k$ is removed from the disc $D^{l+1}$ in such a way that

$$\partial \bar{D}_0^k = \bar{D}_0^k \cap \partial D^{l+1}.$$

Therefore the boundary of this new body is $S^k \times S^{l-k}$ and the body itself is $D^{k+1} \times S^{l-k}$. We have

$$D' = D^{l+1} \setminus f^{-1}f(D^{k+1}) = D^{k+1} \times S^{l-k},$$
$$D' \cap \partial W' = S^k \times S^{l-k}.$$

The disc $D^{k+1} \times 0 \subset D'$ defines an element of the group

$$\pi_{k+1}(W', \partial W') = \pi_{k+1}(W, \partial W) = 0, \quad k+1 \leqslant s.$$

We consider a disc $D^{k+2} \subset W' = W$ such that

$$\partial D^{k+2} = D_0^{k+1} \cup D_1^{k+1},$$

$$D^{k+2} \cap \partial W' = D_0^{k+2},$$

$$D^{k+2} \cap f(D') = D_1^{k+1} = f(D^{k+1} \times 0)$$

(all the intersections are transversal). We perform surgery on the manifold $D'$ along the disc $D^{k+2}$, whereby the boundary undergoes Morse surgery over the base cycle $S^k \times 0$. After the surgery we again obtain a map of the disc, $\bar{f}: D^{l+1} \to W = W'$, while the manifold of the singularities is "diminished" by one critical point of the function $g: M_1^t \to R$.

More precisely, we have the map $f': D' \to W'$ induced by the map $f: D^{l+1} \to W$, such that

$$f': \partial D' \to \partial W', \qquad \partial D' = S^k \times S^{n-k}, \qquad D' = D^{k+1} \times S^{n-k},$$

and the singularity manifold for $f'$ is diffeomorphic to the region $\{g \geq x_0 + \epsilon\}$ on



$M_1^t$. On the disc $D^{k+1} \times 0$ the map $f'$ is one-to-one and there exists a disc $D^{k+2} \subset W'$ such that

$$\partial D^{k+2} = f'(D^{k+1} \times 0) \cup D_0^{k+1}$$

and

$$D_0^{k+1} \subset \partial W', \quad D^{k+2} \cap f'(D') = f'(D^{k+1} \times 0).$$

We consider the abstract disc $D^{k+2} \times D_0^{l-k}$, where $\partial D^{k+2} = D_0^{k+1} \cup D_1^{k+1}$, and we paste it on to $D'$ in the following manner:

$$A = D' \cup_h D^{k+2} \times D^{l-k}, \quad h: D_1^{k+1} \times D^{l-k} \to D' = D^{k+1} \times S^{l-1},$$

where $h(D_1^{k+1} \times 0) = D^{k+1} \times 0 \subset D'$; let

$$B = A \setminus [D^{k+2} \times \operatorname{Int} D^{l-k}].$$

The result of the pasting is diffeomorphic to the disc $B = D^{l+1}$. In a natural way there arises the map $f: D^{l+1} \to W'$,

$$D^{l+1} = B = A \setminus [D^{k+2} \times \operatorname{Int} D^{l-k}], \quad A = D' \cup_h D^{k+2} \times D^{l-k},$$

constructed in accordance with the map $f': D' \to W'$ and with the imbedded disc $D^{k+2} \subset W'$.

It is easy to see that the pair map

$$\bar{f}: (D^{l+1}, S^l) \to (W', \partial W') = (W, \partial W)$$

is homotopic to the map

$$f: (D^{l+1}, S^l) \to (W, \partial W)$$

and has self-intersections "more simply" at one critical point of the function $g$. By iterating the process we arrive at a map without self-intersections, which proves the first part of the lemma.

In a completely analogous way we can kill the intersections of the pair of imbeddings

$$f_1, f_2: (D^{l+1}, S^l) \to (W, \partial W).$$

This proves the second part of the lemma. The lemma is proved.

## §4. Analog of the Hurewicz theorem

Let $f: X \to Y$ be a map of complexes such that

$$f_*: \pi_1(X) \to \pi_1(Y)$$

is an isomorphism; let $f$ itself and the corresponding covering map $\hat{f}: \hat{X} \to \hat{Y}$ on the universal coverings $\hat{X}, \hat{Y}$, induce epimorphisms in all the dimensions:

$$H_i(\hat{X}) \xrightarrow{\hat{f}_*} H_i(\hat{Y}) \to 0,$$
$$H_i(X) \xrightarrow{f_*} H_i(Y) \to 0.$$

The following lemma holds under these conditions.

Lemma 4.1. *If the map $f^*: \pi_j(X) \to \pi_j(Y)$ is a monomorphism in all dimensions $j < k$, then it is an isomorphism in dimensions $j < k$ and is an epimorphism* *in dimension k, and the "Hurewicz theorem" holds for the kernels:*

a) $\operatorname{Ker} f_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)} = M_k$,

b) $M_k/Z_0(\pi) M_k = \operatorname{Ker} f_*^{(H_k)}$,

*where* $\pi = \pi_1(X) = \pi_1(Y)$, $Z_0(\pi)$ *is the kernel of the augmentation* $\epsilon: Z(\pi) \to Z$ *of the numerical group ring, and the homologic kernel* $M_k$ *is interpreted as a* $Z(\pi)$*-module.*

Before proving this lemma we point out those situations in which it will be applied.

1. Let $f: M_1^n \to M_2^n$ be a mapping of closed manifolds of degree $+1$ and let $\pi_1(M_1^n) = \pi_1(M_2^n)$. Then the map $\hat{f}: \hat{M}_1 \to \hat{M}_2$ of the universal (and of any other) covering has degree $+1$ as the natural map does. Therefore $\hat{f}$ induces the map $\hat{f}_*$ of open homologies and of $\hat{f}^*$-compact cohomologies; moreover,

$$\hat{f}_* D \hat{f}^* D(x) = x, \quad x \in H_q(\hat{M}_2^n).$$

Consequently,

$$H_q(\hat{M}_1) = \operatorname{Ker} \hat{f}_*^{(H_q)} + D\hat{f}^* D H_q(\hat{M}_2).$$

It is evident that Lemma 4.1 is applicable here.

2. Let $W$ be the smooth manifold from Theorem 3 ( see §1) and let $i: V_1 \subset W$ be a connected submanifold separating $W$ into two parts and realizing the base cycle of the group $H_n(W) = Z$, and, moreover, let it be such that $\pi_1(V_1) = \pi_1(W)$. We denote the "right" and "left" sides of $W$ with respect to $V_1$ by $A$ and $B$, respectively, where

$$A \cup B = W, \quad A \cap B = V_1.$$

Then, the assertions a) and b) following below are valid.

a) The imbeddings $i_1: V_1 \subset A$, $i_2: V_1 \subset B$ and $i: V_1 \subset W$ satisfy the hypotheses of Lemma 4.1.

b) On all the coverings we have the direct expansion

$$\operatorname{Ker} \hat{i}_*^{(H_k)} = \operatorname{Ker} \hat{i}_{1*}^{(H_k)} + \operatorname{Ker} \hat{i}_{2*}^{(H_k)},$$

and the maps

$$\hat{i}_{2*}: \operatorname{Ker} \hat{i}_{1*}^{(H_k)} \to H_k(\hat{B}),$$
$$\hat{i}_{1*}: \operatorname{Ker} \hat{i}_{2*}^{(H_k)} \to H_k(\hat{A})$$

are monomorphic, while the images $\hat{i}_{2*}\operatorname{Ker} \hat{i}_{1*}^{(H_k)}$ and $\hat{i}_{1*}\operatorname{Ker} \hat{i}_{2*}^{(H_k)}$ coincide with the kernels of the imbeddings $H_k(\hat{A}) \to H_k(\hat{W})$ and $H_k(\hat{B}) \to H_k(\hat{W})$.

We prove assertion a). Since

$$\pi_1(W) = \pi_1(A) *_{\pi_1(V_1)} \pi_1(B)$$

and $\pi_1(W) = \pi_1(V_1)$, we get that $\pi_1(A) = \pi_1(V_1)$ and $\pi_1(B) = \pi_1(V_1)$.

We consider the basis $x_1, \cdots, x_s \in H_k(W)$, which we realize by the cycles $z_1, \cdots, z_s \in W$. Then we can find an $N$ so large that $T^N z_1, \cdots, T^N z_s$ all lie wholly in $B \subset W$. Since $T_*$ is an isomorphism, these cycles form the basis of the group $H_k(W)$. Let $x \in H_k(A)$ and let $z \subset A$ be a cycle representing it. Then $z$ is homologous in $W$ to the linear combination $\Sigma a_i T^N z_i$ by means of the membrane $c \subset W$. The intersection $c \cap V_1$ is the cycle $\bar{z} \subset V_1$ representing the class of homologies $\bar{x} \in H_k(V_1)$ such that $x = i_{1*}\bar{x}$. The arguments for $B$ and for the whole of $W$ are identical.



We now consider the coverings $\hat{A}$, $\hat{B}$, $\hat{V}_1$, $\hat{W}$ and the covering imbeddings $\hat{i}$, $\hat{i}_1$, $\hat{i}_2$. Note that the homologies $H_k(\hat{V}_1)$, $H_k(\hat{A})$, $H_k(\hat{B})$, $H_k(\hat{W})$ are finitely generated $Z(\pi_1)$-modules since $\pi_1$ is a Noetherian group ($\pi_1 = Z + \cdots + Z$). Further arguments are identical, but instead of the basis of the group we must choose the $\pi_1$-basis of the module.

The same is true of all the intermediate coverings. Therefore Lemma 4.1 is applicable here.

We prove assertion b). If the intersection

$$\operatorname{Ker} \hat{i}_{1*}^{(H_k)} \cap \operatorname{Ker} \hat{i}_{2*}^{(H_k)}$$

is nonempty, we can find a cycle $z \subset \hat{V}_1$ which is homologous to zero in $\hat{A}$ and $\hat{B}$. The membranes define a cycle $c$ in $\hat{W}$ of dimension $k+1$. This cycle $c$, according to the above arguments, is homologous in $\hat{W}$ to the cycle $\bar{c} \subset \hat{W}$ such that $\bar{c} \cap \hat{V}_1 = \emptyset$, by means of the membrane $d \subset \hat{W}$. The intersection $d \cap \hat{V}_1$ is such that

$$\partial(d \cap \hat{V}_1) = c \cap \hat{V}_1 = z,$$

and $z$ is homologous to zero. Therefore

$$\operatorname{Ker} \hat{i}_{1*}^{(H)} \cap \operatorname{Ker} \hat{i}_{2*}^{(H)} = 0$$

on all the coverings.

We now consider the kernel of the imbedding $H_k(\hat{A}) \to H_k(\hat{W})$. Let $z$ be a cycle in $\hat{A}$, homologous to zero in $\hat{W}$ by means of the membrane $c$. Then, $z_1 = c \cap \hat{V}_1$ is such that $z = \hat{i}_{1*} z_1$ and $z_1 \in \operatorname{Ker} \hat{i}_{2*}^{(H)}$. The assertion is proved.

Proof of Lemma 4.1. Let us first consider the "simply-connected" case of the map $\hat{f}: \hat{X} \to \hat{Y}$. Let $C$ denote the cylinder of the map $\hat{f}$, shrinking down to $\hat{Y}$. We write down the exact sequences

$$\begin{array}{ccccccc} H_i(\hat{X}) & \to & H_i(\hat{Y}) & \to & H_i(C, \hat{X}) & \xrightarrow{\partial} & H_{i-1}(\hat{X}) \\ \uparrow H & & \uparrow H & & \uparrow H & & \uparrow H \\ \pi_i(X) & \to & \pi_i(\hat{Y}) & \to & \pi_i(C, \hat{X}) & \to & \pi_{i-1}(\hat{X}). \end{array}$$

Since $\hat{f}_*: \pi_{i-1}(\hat{X}) \to \pi_{i-1}(\hat{X})$ are monomorphisms when $i \leq k$,

$$\partial: \pi_i(C, \hat{X}) \to \pi_{i-1}(\hat{X})$$

trivially. Since $\hat{f}_*: H_j(\hat{X}) \to H_j(\hat{Y})$ are epimorphisms, $\partial: H_i(C, \hat{X}) \to H_{i-1}(\hat{X})$ are monomorphisms on the kernel $\operatorname{Ker} \hat{f}_*^{(H_{i-1})}$. Because $H\partial = \partial H$, then for the first $i$, where $\pi_i(C, \hat{X}) \neq 0$, we have

$$\pi_i(C, \hat{X}) = H_i(C, \hat{X})$$

and $\partial H$ is an isomorphism:

$$\pi_i(C, \hat{X}) \approx \operatorname{Ker} \hat{f}_*^{(H_{i-1})}.$$

But this can be so only when $i \geq k+1$; otherwise, $H\partial = 0$. When $i = k+1$,

$$\operatorname{Ker} \hat{f}_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)}$$

and when $i \leq k+1$ the map $\hat{f}_*: \pi_{i-1}(\hat{X}) \to \pi_{i-1}(\hat{Y})$ is an epimorphism.

Following Serre, let us convert the map $\hat{f}: \hat{Y} \to \hat{Y}$ into the fiber $\tilde{f}: X_1 \xrightarrow{F} Y_1$, where $X_1$, $Y_1$ are, respectively, of the homotopy type of $X$, $Y$, while $\tilde{f}$ is of the homotopy type of $f$. From the exact sequence of this fiber in homotopies we see that

$$\pi_k(F) = H_k(F) = \operatorname{Ker} \hat{f}_*^{(\pi_k)},$$

on the basis of the preceding results.

We consider the map $f: X \to Y$ and convert it into a fiber; the layer $F'$ is of the same homotopy type as $F$, and

$$\pi_k(F) = \operatorname{Ker} f_*^{(\pi_k)} = \operatorname{Ker} \hat{f}_*^{(H_k)} = M_k;$$

moreover, $\pi_i(F) = 0$, $i < k$.

Consider the spectral sequence of this fiber. Obviously, $E_2^{0,k} = M_k/Z_0(\pi)M_k$ and $E_2^{q,i} = 0$ when $0 < i < k$.

Since $f_*: H_{k+1}(X) \to H_{k+1}(Y)$ is an epimorphism, the differential

$$d_{k+1}: E_2^{k+1,0} \to E_2^{0,k}, \quad E_2^{k+1,0} = H_{k+1}(Y),$$

is trivial. Therefore

$$E_\infty^{0,k} = M_k/Z_0(\pi)M_k.$$

Obviously,

$$E_\infty^{0,k} = \operatorname{Ker} f_*^{(H_k)} = M_k/Z_0(\pi)M_k.$$

All the assertions of the lemma have been proved.

## §5. The functor $P = \operatorname{Hom}_c$ and its application to the study of the homological properties of maps of degree 1

Let $\pi$ be a Noetherian group, $K$ a ring or a field, $K(\pi)$ a group ring with coefficients in $K$, $\epsilon: K(\pi) \to K$ an augmentation, $K_0(\pi) = \operatorname{Ker} \epsilon$. We shall take it that $K$ is either $Z$ or a field. Let $M$ be a finitely-generated $K(\pi)$-module.

Definition 5.1. By the module $PM = \operatorname{Hom}_c(M, K)$ we mean the submodule $PM \subset \operatorname{Hom}(M, K)$ consisting of linear forms $h: M \to K$ such that for any element $x \in M$ the function $f_{h,x}(a) = (h, ax)$, $a \in \pi$, on the group is finite.

We note several simple properties of the functor $P = \operatorname{Hom}_c$.



1. The module $PF$ is free for a free module $F$.
2. For a projective module there exists the natural isomorphism $P^2: M \to P^2M$.
3. There always exists a natural map $P^2: M \to P^2M$ which, in general, is not monomorphic and not epimorphic. We denote the kernel of this map by $M_\infty \subset M$.

Then

$$0 \to M_\infty \to M \to P^2M \to \operatorname{Coker} P^2 \to 0.$$

Example 1. Let $p: \hat{M} \to M^n$ be a regular covering with motion group $\pi: \hat{M} \to \hat{M}$. The homologies $H_i(\hat{M}, K) = N_i$ are $K(\pi)$-modules, finitely generated if the group $\pi$ is Noetherian and $\hat{M}^n$ is a compact manifold. There exists the homomorphism

$$N_i/N_{i\infty} \to PN_{n-i},$$

established by the intersection index.

Example 2. Let $f: M_1^n \to M_2^n$ be a map of degree $+1$ and let

$$\pi_1(M_1^n) = \pi_1(M_2^n).$$

By $\hat{f}: \hat{M}_1 \to \hat{M}_2$ we denote the map of the coverings $\hat{M}_1 \to M_1^n$ and $\hat{M}_2 \to M_2^n$ with motion group $\pi$. We set

$$M_i = \operatorname{Ker} \hat{f}_*^{(H_i)} \subset H_i(\hat{M}_1)$$

By analogy with Example 1 we have

$$M_i/M_{i\infty} \xrightarrow{h} PM_{n-i},$$

$$(hx, y) = x \circ y.$$

Let us now consider derived functors of the functor $P = \operatorname{Hom}_c$. We shall denote them by $\operatorname{Ext}_c^i$, $i \geq 0$. Note that in contrast to the ordinary Hom, the functor $P = \operatorname{Hom}_c$ is not exact even for a field $K$. Therefore it is possible that

$$\operatorname{Ext}_c^i(M, K) \neq 0, \quad i > 0.$$

Example 3. Let $M_0$ be a module with one generator $u$ and let $au = u$ for all $a \in \pi$. If $\pi = Z + \cdots + Z$ is a free abelian group with $n$ generators, then

$$\operatorname{Ext}_c^n(M_0, K) = M_0$$

and

$$\operatorname{Ext}_c^i(M_0, K) = 0, \quad i < n.$$

Let us prove this fact. Consider the triangulated torus $T^n$ and the covering $R^n \to T^n$ with the group $\pi = Z + \cdots + Z$. Let $F_i$ denote the free $Z(\pi)$-module of $i$-dimensional chains on $R^n$. We have

$$0 \to F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \cdots \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} M_0 \to 0,$$

and, moreover, the sequence is exact since

$$H_i(R^n) = 0, \quad i > 0, \quad H_0(R^n) = M_0.$$

We apply the functor $P$ to the resolvent:

$$PF_n \leftarrow PF_{n-1} \leftarrow \cdots \leftarrow PF_1 \leftarrow PF_0,$$

but $PM_0 = 0$ and the complex written down is a complex of compact cochains for

$R^n$. Therefore
$$H_c^n(R^n, K) = M_0$$
and
$$H_c^i(R^n, K) = 0, \quad i < n,$$
and, moreover,
$$H_c^k(R^n, K) = \mathrm{Ext}_c^k(M_0, K).$$

The following simple lemma holds.

Lemma 5.1. *If the module $M$ is such that $\mathrm{Ext}_c^i(M, K) = 0$, $i > 0$, and if $\pi = Z + \cdots + Z$, then the module $PM$ is stably free, i.e. one can find a free module $F$ such that $PM + F$ is a free module.*

Proof. Since $\pi = Z + \cdots + Z$, we can find a free acyclic resolvent of finite length
$$0 \to F_l \to F_{l-1} \to \ldots \to F_0 \to M \to 0.$$
But, by the hypothesis of the lemma, the sequence
$$0 \leftarrow PF_l \leftarrow PF_{l-1} \leftarrow \ldots \leftarrow PF_0 \leftarrow PM$$
is exact. The functor $P$ possesses the property that the modules $PF_i$ are free. Furthermore, the functor $P$ is "semi-exact from the right"; it takes an epimorphism into a monomorphism. Therefore the kernel of the mapping $PF_0 \to PF_1$ is precisely $PM$. By virtue of the properties of a free module we have the equality
$$PF_2 + PF_0 = PF_1 + PM,$$
provable by the usual means; moreover, the $PF_i$ are free. The lemma is proved.

Let $C$ be a complex of free or projective modules:
$$C = \{\ldots \to F_l \xrightarrow{\partial} F_{l-1} \xrightarrow{\partial} \ldots \to F_1 \xrightarrow{\partial} F_0\}.$$
Then the groups $H_i(C) = N_i$ are $\pi$-modules. Consider the complex $PC$:
$$\{\leftarrow PF_i \xleftarrow{\delta} PF_{i-1} \xleftarrow{\delta} \ldots \xleftarrow{\delta} PF_0\}, \quad \delta = P\partial$$
whose homologies we denote by $H_c^i(C)$ since they have the sense of "cohomologies with compact supports".

The following fact is known: the spectral sequence $\{E_r, d_r\}$,
$$E_r = \sum_{p \geq 0,\, q \geq 0} E_r^{p,q}, \quad E_r^{p,q} = \mathrm{Ext}_c^p(N_q, K)$$
exists and the module
$$\sum_{p+q=l} E_\infty^{p,q}$$
is associated with $H_c^l(C)$.

This fact is "the formula of universal coefficients."

As is shown by examples, the functor $P$ is such that the modules $H_i(C) = N_i$ influence the $H_i^{+k}(C)$ for very large $k$ (see Example 3). We shall be interested in complexes which in some sense are manifolds and admit of some geometric realization.



The following are the necessary and sufficient conditions for the realizability of the complex
$$C = \{F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} F_0\}$$
as a covering with motion group $\pi$ over a finite complex:

a) freeness: all the $f_i$ are free modules;

b) $H_0(C) = M_0$ (see Example 3).

The necessary "geometric" requirement on the morphisms of complexes $f: C_1 \to C_2$ is then that
$$f_*: H_0(C_1) \to H_0(C_2)$$
is an isomorphism.

Later on we shall need manifolds and maps of degree 1. For realizability as a homological manifold, of course, we must have that the complexes of modules
$$C = \{F_n \xrightarrow{\partial} F_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} F_0\}$$
and
$$PC = \{PF_n \xleftarrow{\delta} PF_{n-1} \leftarrow \ldots \xleftarrow{\delta} PF_0\},$$
where $\delta = P\partial$, are "homotopically equivalent" in the algebraic sense (what this means is very well known). This gives us the Poincaré duality laws:
$$D: H_i(C) \approx H_c^{n-i}(C), \quad i \geq 0.$$
Furthermore, if we wish to obtain the duality law in a form connected with cohomological multiplication and the cut operation, we should require that the complex $C$ be a coalgebra, etc. We shall not formalize exactly all concepts we need. Note that for algebraic complexes obtained from the triangulation of manifolds we have the following: for maps of degree $\lambda$, $f: C_1^n \to C_2^n$ we can define the operator $Df^*D$: $C_2^n \to C_1^n$ such that
$$f_* Df^* D: \quad C_2^n \to C_2^n$$
is a multiplication by $\lambda$; however, if $\lambda = 1$, then
$$C_1^n = \mathrm{Ker}\, f + Df^*DC_2^n$$
Therefore, here there arises the complex $\mathrm{Ker}\, f$ composed of projective modules and such that the complex $P(\mathrm{Ker}\, f)$ is algebraically homotopic to it. Consequently we have the duality law
$$D: \quad H_i(\mathrm{Ker}\, f) = H_c^{n-i}(\mathrm{Ker}\, f),$$
and, moreover,
$$H_i(\mathrm{Ker}\, f) = \mathrm{Ker}\, f_*^{(H_i)}$$
and
$$H_c^{n-i}(\mathrm{Ker}\, f) = \mathrm{Coker}\, f^{*(H_c^{n-i})},$$

where the kernels and the cokernels are taken for the map of the complexes themselves, $f: C_1^n \to C_2^n$. Therefore we can apply separately the Poincaré duality laws and the "formula of universal coefficients" to the kernels $\operatorname{Ker} f_*^{(H_i)}$ and to the cokernels $\operatorname{Coker} f^*(H_c^i)$, as was already clear, of course.

The following theorem holds.

Theorem 5.1. *If $f: M_1^n \to M_2^n$ is a map of closed manifolds of degree $+1$, $n = 2k$, while $\hat{f}: \hat{M}_1 \to \hat{M}_2$ is a covering map, where the $\hat{M}_j$ are regular coverings over $M_j^n$ with motion group $\pi = Z + \cdots + Z$, and if the kernels $M_s = \operatorname{Ker} \hat{f}_*^{(H_s)} = 0$, $s < k$, then the kernel $M_k = \operatorname{Ker} \hat{f}_*^{(H_k)}$ is a stably free $Z(\pi)$-module.*

**Proof.** Since all the $M_s = 0$ when $s < k$,

$$\operatorname{Ext}_c^i (M_s, Z) = 0, \quad s < k,$$

and therefore, by virtue of the "formula of universal coefficients" mentioned earlier in the form of a spectral sequence, we get that

$$\operatorname{Coker} \hat{f}^{*(H_c^s)} = 0, \quad s < k.$$

Since

$$\operatorname{Coker} \hat{f}^{*(H_c^s)} = \operatorname{Ker} \hat{f}_*^{(H_{n-s})} = 0, \quad s < k,$$

all the $M_{n-s} = 0$ when $s < k$, $n = 2k$, and all the $M_q = 0$ except for $q = k$.

Consequently, by virtue of the "formula of universal coefficients,"

$$\operatorname{Coker} \hat{f}^{*(H_c^{k+q})} = \operatorname{Ext}_c^q (M_k, K).$$

But

$$\operatorname{Coker} \hat{f}^{*(H_c^{k+q})} = M_{k-q} = 0, \quad q > 0.$$

Therefore $\operatorname{Ext}_c^q (M_k, Z) = 0$ for all $q > 0$. By Lemma 1 the module $PM_k$ is stably free, $PM_k = M_k$. The theorem is proved.

In the case of odd $n = 2k + 1$, we again have $f: M_1^n \to M_2^n$ of degree $+1$, and $\hat{f}: \hat{M}_1 \to \hat{M}_2$ is the map of regular coverings with Noetherian motion group $\pi$.

Theorem 5.2. *If $M_s = \operatorname{Ker} f_*^{(H_s)} = 0$, $s < k$, the following relations hold:*

a) $PM_k = M_{k+1}$.

b) $\operatorname{Ext}_c^i (PM_k, Z) = \operatorname{Ext}_c^{i+2} (M_k, Z), \quad i \geqslant 1$.

c) *The sequence*

$$0 \to \operatorname{Ext}_c^1 (M_k, Z) \to M_k \xrightarrow{P^2} P^2 M_k \to \operatorname{Ext}_c^2 (M_k, Z) \to 0$$

$$(M_{k\infty} = \operatorname{Ext}_c^1 (M_k, Z), \ \operatorname{Coker} P^2 = \operatorname{Ext}_c^2 (M_k, Z))$$

*is exact.*

*If $\operatorname{Ext}_c^i (M_k, Z) = 0$, $i \geq 3$, the module $PM_{k+1} = P^2 M_k$ is stably free $(\pi = Z + \cdots + Z)$.*

The proof of this theorem is obtained very simply from the Poincaré duality



law

$$D: M_k = \operatorname{Coker} \hat{f}^{*(H_c^{k+1})},$$

$$D: M_{k+1} = \operatorname{Coker} \hat{f}^{*(H_c^k)}$$

and from the formula of universal coefficients in the form of a spectral sequence. Indeed, since $M_j = 0$, $j < k$,

$$\operatorname{Coker} \hat{f}^{*(H_c^k)} = PM_k = M_{k+1},$$

and we get item a). The isomorphism in item b) is established by the differential $d_2$, where

$$\begin{array}{ccc} d_2 : E_2^{k+1,\, i} & \to & E_2^{k,\, i+2} \\ \| & & \| \\ \operatorname{Ext}_c^i (M_{k+1}, Z) & & \operatorname{Ext}_c^{i+2} (M_k, Z), \end{array}$$

since $M_{k+j} = 0$, $j \geq 2$, and $\operatorname{Coker} \hat{f}^{*(H_c^{k+j})} = 0$, $j \geq 2$. Item c) also is obtained from the spectral sequence of the formula of universal coefficients, since

$$P^2 M_k = PM_{k+1} = E_2^{k+1,\, 0},$$

since the map

$$P^2 M_k \to \operatorname{Ext}_c^2 (M_k, Z)$$

is $d_2$, and since the module

$$\operatorname{Ker} d_2 + \operatorname{Ext}_c^1 (M_k, Z)$$

is associated with the module

$$M_k = \operatorname{Coker} \hat{f}^{*(H_c^{k+1})}.$$

The stable freeness of the module $P^2 M_k = PM_{k+1}$ follows from items a), b), c) and from Lemma 5.1 if

$$\operatorname{Ext}_c^i (PM_k, Z) = \operatorname{Ext}_c^{i+2} (M_k, Z) = 0, \quad i \geqslant 1.$$

Theorem 5.2 is proved.

**Remark.** For maps $f: M_1^n \to M_2^n$ of degree $+1$ the formula

$$\operatorname{Coker} \hat{f}^{*(H_c^k)} = \operatorname{Hom}_c (M_k, Z),$$

always holds on the coverings $\hat{f}: \hat{M}_1 \to \hat{M}_2$ if $M_j = 0$, $j < k$, whatever be $n$ and $k$.

Corollary 5.1. *If in the hypotheses of Theorem 5.2, $\pi = Z + Z$, the module $PM_{k+1} = P^2 M_k$ is stably free. (This fact is true also for the case $\pi = Z$, but it is trivial.)*

**Proof.** If $\pi = Z + Z$, then $\operatorname{Ext}_c^i (M_k, Z) = 0$ when $i \geq 3$ for any module $M_k$. By virtue of Theorem 5.2, we then obtain our assertion.

## §6. Stable freeness of modules of kernels under the hypotheses of Theorem 3

Let $V_1 \overset{i}{\subset} W$ be a connected submanifold separating $W$ into two parts $A$, $B$, where

$$A \cap B = V_1, \quad A \cup B = W.$$

We denote the imbeddings $V_1 \subset A$ and $V_1 \subset B$, as we did in §4, by $i_1$, $i_2$, and the imbedding of universal coverings over $V_1$ of $W$, $A$, $B$ by $\hat{i}: \hat{V}_1 \subset \hat{W}$, $\hat{i}_1: \hat{V}_1 \subset \hat{A}$, $\hat{i}_2: \hat{V}_1 \subset \hat{B}$. Here $W$ is an $(n+1)$-dimensional manifold having the homotopy type of the closed manifold $M^n$, the group $\pi = \pi_1(W)$ is Noetherian and a discrete transformation $T$ is given on $W$; moreover,

$$\pi_1(V_1) = \pi_1(A) = \pi_1(B) = \pi_1(W)$$

and the factor $W/T$ is compact. The following lemma holds.

Lemma 6.1. *If $\pi = Z + \cdots + Z$ and if the kernels $M_j = \operatorname{Ker} i_*^{(\pi_j)}$ are trivial when $j < k$, then when $n = 2k$ the modules*

$$M_k' = \operatorname{Ker} i_{1*}^{(\pi_k)}, \quad M_k'' = \operatorname{Ker} i_{2*}^{(\pi_k)}$$

*are stably free. However, if $n = 2k+1$ and if*

$$M_j' = \operatorname{Ker} i_{1*}^{(\pi_j)} = 0, \quad j < k,$$
$$M_j'' = \operatorname{Ker} i_{2*}^{(\pi_j)} = 0, \quad j < k+1,$$

*then the kernels $M_k' = \operatorname{Ker} i_{1*}^{(\pi_k)}$, $M_{k+1}'' = \operatorname{Ker} i_{2*}^{(\pi_{k+1})}$ are stably free. In both cases, under the hypotheses of the lemma there holds a natural isomorphism established by the intersection index of the cycles $M_k' = PM_{n-k}''$.*

Proof. Let $n = 2k$. By Theorem 5.1, under the hypotheses of Lemma 6.1 the module $M_k = M_k' + M_k''$ (see §4) is stably free. Therefore both modules $M_k'$ and $M_k''$ are projective, and since $\pi = Z + \cdots + Z$, $M_k'$ and $M_k''$ are stably free. As we know, $M_k$ is the kernel

$$\operatorname{Ker} \hat{i}_*^{(H_k)} = \operatorname{Ker} i_*^{(\pi_k)}.$$

Since

$$\operatorname{Ker} \hat{i}_*^{(H_k)} = \operatorname{Coker} \hat{i}^{*(H_c^k)} = PM_k$$

(see §5) and since both modules $M_k'$ and $M_k''$ have a zero intersection index, each one by itself, $M_k' = PM_k''$ and $M_k'' = PM_k'$, whence the lemma follows for even $n = 2k$.

Now let $n = 2k+1$. We first prove that under the hypotheses of the lemma, the kernel

$$M_{k+1}' = \operatorname{Ker} \hat{i}_{1*}^{(H_{k+1})}$$



is trivial. Because

$$\operatorname{Ker} \hat{i}_*^{(H_{k+1})} = M_{k+1} = M_{k+1}' + M_{k+1}'' = \operatorname{Coker} \hat{i}^{*(H_c^k)} = PM_k = PM_k',$$

we have

$$PM_k' \overset{h}{\approx} M_{k+1}' + M_{k+1}'',$$

moreover, $(hx, y) = x \circ y$, where $x \in M_{k+1}' + M_{k+1}''$, $y \in M_k' = M_k$ and $x \circ y$ is the intersection index. But the intersection index $M_k' \circ M_{k+1}'$ is identically zero. Therefore $M_{k+1}' = 0$.

Let us take a sufficiently large integer $s$. Then the intersection $T^sV_1 \cap V_1$ is empty. We denote the region between $V_1$ and $T^sV_1$ by $Q$ and we denote $T^sV_1$ itself by $V'$; $\partial Q = V_1 \cup V'$. Here we have considered that $T^sV_1 \subset A$.

Consider the imbeddings $j: \hat{V}_1 \subset \hat{Q}$, $j': \hat{V}' \subset \hat{Q}$ on the universal covering $\hat{W}$. We have (for sufficiently large $s$):

$$\operatorname{Ker} j_*^{(H_q)} = \begin{cases} 0, & q \neq k, \\ M_k', & q = k, \end{cases}$$

$$\operatorname{Ker} j_*'^{(H_q)} = \begin{cases} 0, & q \neq k+1, \\ M_{k+1}'', & q = k+1, \end{cases}$$

$$\operatorname{Coker} j_*^{(H_q)} \approx \begin{cases} 0, & q \neq k, \\ i_{2*} \quad M_k' \approx M_k', & q = k, \end{cases}$$

$$\operatorname{Coker} j_*'^{(H_q)} \approx \begin{cases} 0, & q \neq k+1 \\ i_{1*} \quad M_{k+1}'' \approx M_{k+1}'', & q = k+1. \end{cases}$$

From the above equalities we easily get

$$H_q(\hat{Q}, \hat{V}_1) = 0, \quad q \neq k,\ k+1,$$
$$H_k(\hat{Q}, \hat{V}_1) \approx H_{k+1}(\hat{Q}, \hat{V}_1) \approx M_k',$$
$$H_q(\hat{Q}, \hat{V}') = 0, \quad q \neq k+1,\ k+2,$$
$$H_{k+1}(\hat{Q}, \hat{V}') \approx H_{k+2}(\hat{Q}, \hat{V}_1') \approx M_{k+1}''.$$

Therefore

$$H_c^q(\hat{Q}, \hat{V}_1) \approx \begin{cases} 0, & q \neq k,\ k+1, \\ M_{k+1}'', & q = k,\ k+1, \end{cases}$$

$$H_c^q(\hat{Q}, \hat{V}') \approx \begin{cases} 0, & q \neq k+1,\ k+2, \\ M_k', & q = k+1,\ k+2. \end{cases}$$

By the formulas of universal coefficients for $H_c^q(\hat{Q}, \hat{V}_1)$,

$$M_{k+1}'' = PM_k' = H_c^k(\hat{Q}, \hat{V}_1),$$

$$d_2: \operatorname{Ext}_c^i(M_k', Z) \to \operatorname{Ext}_c^{i+2}(M_k', Z)$$

is an epimorphism when $i = 0$ and an isomorphism when $i > 0$. Recall that

$$M'_k \approx H_k(\hat{Q}, \hat{V}_1) \approx H_{k+1}(\hat{Q}, \hat{V}_1)$$

and

$$E_2^{p,q} = \mathrm{Ext}_c^p(H_q, Z), \quad d_2: E_2^{p,q} \to E_2^{p+2,q-1}.$$

Since $\pi = Z + \cdots + Z$, $\mathrm{Ext}_c^p = 0$ when $p > rk\pi$. Therefore

$$\mathrm{Ext}_c^i(M'_k, Z) = 0, \quad i > 0.$$

By Lemma 5.1 the module $PM'_k$ is stably free. Since $M'_k = PM''_{k+1}$, the same is true also for $M'_k$. The lemma is proved.

**Remark.** In proving the acyclicity of the module $M'_k$ we made use of the fact that $\mathrm{Ext}_c^i = \mathrm{Ext}_c^{i+2}$ and that $\mathrm{Ext}_c^p = 0$ when $p > rk\pi$. In reality, the triviality of the modules $\mathrm{Ext}_c^i(M'_k, Z)$ when $i > 0$ can be proved differently for any Noetherian group $\pi$ under the hypotheses of Lemma 6.1.

## §7. Homological effect of Morse surgery

Let $W$ have the same meaning as in the formulation of Theorem 3 (§1), $V_1 \overset{i}{\subset} W$, $W = A \cup B$, $A \cap B = V_1$, and, moreover, let the imbeddings

$$i_1: V_1 \subset A, \quad i_2: V_1 \subset B$$

be such that

$$\pi_1(V_1) = \pi_1(A) = \pi_1(B) = \pi_1(W)$$

and

$$\mathrm{Ker}\, i_{1*}^{(\pi_k)} = 0, \quad k < p, \qquad \mathrm{Ker}\, i_{2*}^{(\pi_k)} = 0, \quad k < n-p.$$

We set

$$\mathrm{Ker}\, i_{1*}^{(\pi_p)} = M'_p, \quad \mathrm{Ker}\, i_{2*}^{(\pi_{n-p})} = M''_{n-p}.$$

Both modules $M''_{n-p}$ and $M'_p$ are $Z(\pi)$-modules. According to Lemma 4.1,

$$M'_p = \mathrm{Ker}\, \hat{i}_{1*}^{(H_p)}, \quad M''_{n-p} = \mathrm{Ker}\, \hat{i}_{2*}^{(H_{n-p})}$$

On the universal covering $\hat{V}_1$ between $M'_p$ and $M''_{n-p}$ there exists a scalar product, integral and $\pi$-invariant, generated by the intersection index of the cycles.

By virtue of Lemma 4.1,

$$\mathrm{Ker}\, i_{1*}^{(H_p)} = M'_p / Z_0(\pi) M'_p$$

and

$$\mathrm{Ker}\, i_{2*}^{(H_{n-p})} = M''_{n-p} / Z_0(\pi) M''_{n-p}.$$

We choose a $\pi$-basis $\alpha_1, \cdots, \alpha_q$ in $M'_p$. Let $p$ satisfy the hypotheses of Lemma 3.1. We find discs $D_1^{p+1}, \cdots, D_q^{p+1} \subset A$ such that their boundaries $\partial D_j^{p+1} \subset \partial A = V_1$ realize the elements $\alpha_1, \cdots, \alpha_q \in M'_p$, and we paste on the handles

$$B' = B \cup T_1 \cup \ldots \cup T_q,$$

$$A' = A \setminus (\mathrm{Int}\, T_1 \cup \ldots \cup \mathrm{Int}\, T_q),$$



where the $T_j$ are the neighborhoods of discs $D_j^{p+1}$ in $A$. Then it is easy to see that for $V'_1 = \partial B' = \partial A'$, the kernels

$$\mathrm{Ker}\, \hat{i}_{1*}'^{(H_j)} = \widetilde{M}'_j$$

and

$$\mathrm{Ker}\, \hat{i}_{2*}'^{(H_j)} = \widetilde{M}''_j$$

will be arranged thus:

$$\widetilde{M}'_j = 0, \quad j \leqslant p, \quad \widetilde{M}'_j = M'_j, \quad j > p+1,$$

$$\widetilde{M}''_j = 0, \quad j < n-p-1, \quad \widetilde{M}''_j = M''_j, \quad j > n-p.$$

We denote the scalar product between the modules $M'_p$ and $M''_{n-p}$ by $(\ ,\ )$. Let $\beta_1, \cdots, \beta_t$ be the $\pi$-generators of the module $M''_{n-p}$. The following lemma then holds.

**Lemma 7.1.** *The module $\widetilde{M}''_{n-p-1}$ is described in the following manner: its generators $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_q$ are found in one-to-one correspondence with the generators of the module $M'_p$, while the relations are given by the generators of the module $M''_{n-p}$ as follows:*

$$\sum_{\substack{a \in \pi \\ m=1,\ldots,q}} (a^{-1}\beta_j, \alpha_m)\, a\tilde{\alpha}_m = 0.$$

**Proof.** The geometric meaning of the generators $\tilde{\alpha}_m$ is that they are the spheres $S_m^{n-p-1} \subset V'_1$ linked with the spheres $\partial D_m^{p+1} \subset V_1$ removed from $V_1$. It is obvious that the elements $\tilde{\alpha}_m$ are $\pi$-generators in $\widetilde{M}''_{n-p-1}$ since $M''_{n-p-1} = 0$.

Let us consider the geometric situation on the universal covering $\hat{W}$. The geometric meaning of the relations we have written is obvious, since on $\hat{W} \supset \hat{V}_1$ the cycle $\beta_j$ has intersection indices with the cycles $a\alpha_m$, $a \in \pi$, and after removing the neighborhoods of the cycles $\alpha_m$ from $V_1$ the cycle $\beta_j$ determines the relation indicated.

That this is a complete system of relations in our case ensues from the fact that it is a complete system of relations in the module

$$\hat{i}'_{1*}\widetilde{M}''_{n-p-1} \subset H_{n-p-1}(\hat{A}').$$

The fact is that $A'$ is obtained from $A$ homotopically by a simple removal of the discs $D_m^{p+1}$. It is easy to see that

$$\pi_{p+1}(A, V_1) = H_{p+1}(\hat{A}, V_1).$$

Since the relation in $\hat{i}'_* \widetilde{M}''_{n-p-1}$ arises at the expense of intersections of cycles from $H_{n-p}(\hat{A})$ with covering discs $\hat{D}_m^{p+1} \subset \hat{A}$ and since the map $H_{n-p}(\hat{V}_1) \to H_{n-p}(\hat{A})$ is epimorphic, the system of relations written down in the lemma is complete. The lemma is proved.

## §8. Proof of Theorem 3

Let $n \geq 5$. We retain all the notations for $V_1 \subset W$, $A$, $B$, $i_1$, $i_2$, $i$, $\hat{i}_1, \hat{i}_2, \hat{i}$, $M_t'$, $M_t''$, etc.

The proof of the theorem is carried out in three stages.

Stage 1. We achieve that $V_1 \subset W$ is connected and that

$$\pi_1(V) = \pi_1(W).$$

Here no constraints are imposed on $\pi_1(W)$ except that it be finitely determined.

Stage 2. The homotopic kernels of the imbedding $V_1 \subset W$ in the dimensions $k < [n/2]$ are killed by surgery, while on the basis of Lemma 3 the kernels

$$\operatorname{Ker} i_{2*}^{(\pi_t)} = M_t''$$

also are killed for odd $n = 2t + 1$. Here we use the fact that the fundamental group is Noetherian.

Stage 3. By pasting the one-sided handles $V_1 \to V_1 \# S^t \times S^{n-t}$ onto the manifold $V_1 \subset W$ we "stabilize" the module $M_t' \to M_t' + F$ when $n = 2t$ or $n = 2t + 1$ and we achieve that the kernel $M_t'$ becomes a free module over $Z(\pi)$. Here we apply the results of Theorem 5.2. Next, applying Lemma 3.1, we remove $M_t'$ and $M_{t+1}''$ for $n = 2t + 1$ and $M_t'$ and $M_t''$ for $n = 2t$ by surgery on the $\pi$-free basis from $M_t'$. On the basis of Lemma 7.1 the kernels in the remaining dimensions (including $M_{n-t-1}''$) remain trivial. As a result of the surgery we obtain a closed submanifold $V \subset W$ which is a deformation retract. Hence at this point Theorem 3 follows trivially: we can find a number $k$ such that $T^kV \cap V = \emptyset$. The neighborhood of the manifold $T^kV$ in $W$ is homeomorphic to $V \times R$. According to the preceding discussion, in this neighborhood we can find a smooth $V' \subset W$ close to $T^kV$, of the homotopy type of $W$. A smooth cobordism lies between $V$ and $V'$. Therefore this region is $V \times I(0, 1)$ and $V' = V$ because $Wh(\pi) = 0$, $\pi = Z + \cdots + Z$ (see [1, 2, 9]). Setting up such regions for all $k$, we see that $W = V \times R$.

The theorem is proved.

Remark. If in stage 3 the surgery had been performed not on the free $\pi$-basis in $M_t'$ but on any other one in accordance with the projection $F \to M_t'$, then after the surgery we would have obtained the module of the relations $R$, $0 \to R \to F \to M_t' \to 0$, where $R = \tilde{M}_{t+1}'$ (see §7) for the reconstructed manifold. By virtue of Lemma 7.1, for this manifold we would have had that

$$M_{n-t-1}'' = PM_{t+1}' = PR.$$



## §9. Proof of Theorem 6

Let $S^n \subset S^{n+2}$ be a topological locally flat imbedding and let $n \geq 5$. Note that the difference $G = S^{n+2} \setminus S^n$ is an open smooth manifold in which the "homotopy type at infinity" is that of $S^n \times S^1$. We construct a smooth closed manifold $V \subset G$ of the homotopy type of $S^n \times S^1$, which bounds in $S^{n+2}$ the manifold $D$ of the homotopy type of $S^n$, containing the "knot" $S^n \subset D \subset S^{n+2}$.

In the case when we already know that the knot $S^n \subset S^{n+2}$ is globally flat, i.e. it has a neighborhood $U \supset S^n$ which is homeomorphic to $S^n \times R^2$, this problem is easily solved with the help of Theorem 3: namely, we set $W = U \setminus S^n$. Then $W$ is homeomorphic to $S^n \times S^1 \times R$ and is smooth. By Theorem 3 we can find a smooth $V \subset W$ such that $W$ is diffeomorphic to $V \times R$. Obviously $V$ bounds in $U \supset W \supset V$ the manifold $D$ of the homotopy type of $S^n \subset D$, $n \geq 5$.

However, if global flatness is not known a priori, then we consider a decreasing sequence of smooth manifolds with boundary,

$$U_1 \supset U_2 \supset \ldots \supset U_i \supset \ldots,$$

such that $U_j \supset S^n$ and $\bigcap_j U_j = S^n$.

We set $W_j = U_j \setminus S^n$. Obviously the group $H_{n+1}(W_j) \neq 0$, and for a number $j_1$ large in comparison with $j_0 \gg 1$ the image

$$H_{n+1}(W_{j_1}) \to H_{n+1}(W_{j_0})$$

is isomorphic to the group $Z$.

If the numbers $j_0$, $j_1$ are sufficiently large, we can realize the base cycle of this image inside $W_{j_1}$ by the submanifold $V_1 \subset W_{j_1}$; it is easy to see that for sufficiently large $j_1 \gg j_0 \gg 1$, the map of the imbedding $V_1 \subset W_{j_0}$ is "made up" in the same way as the map $V_1 \to S^n \times S^1$. More precisely, this signifies that for a large number $j$ we can find a natural map $W_j \xrightarrow{g_j} S^n \times S^1$ (which in the case of global flatness can be considered as a simple projection onto $S^n \times S^1$) which by the same token induces the map $g_{j_1}: W_{j_1} \to S^n \times S^1$ for $j_1 \geq j$. The composition of the imbedding $V_1 \subset W_j$ and of $g_j: W_j \to S^n \times S^1$ is a map $f_j: V_1 \to S^n \times S^1$ of degree $+1$.

It is easy to achieve, as we did before, that $V_1$ is connected and that $\pi_1(V_1) = Z$. Then $V_1$ separates $W_{j_1}$ into two parts $A$ and $B$ and the homotopy kernels of the imbeddings $i_1: V_1 \subset A$ and $i_2: V_1 \subset B$, have all the very same properties as the kernels featured in Theorem 3 (see §4–8), although here, in contrast to the proof of Theorem 3, one cannot make use of the epimorphicity of the homological homomorphism of the imbeddings $V_1 \subset A$ and $V_1 \subset B$ for the study of these kernels. Note, however, that this epimorphicity holds with respect to the "inner" part of $A \subset W_{j_1}$ such that its closure in $S^{n+2}$ contains $S^n$. As before we denote the imbeddings $V_1 \subset A$ and $V_1 \subset B$ respectively by $i_1$ and $i_2$.

In view of the local flatness of the knot $S^n \subset S^{n+2}$, the manifold $G$ possesses the following property: we can find an $\epsilon > 0$ such that any map $h: P \to G$ of any complex $P$ is homotopic in $G$ to the map $\bar{h}: P \to G$ whose image lies at a distance $> \epsilon$ from $S^n$ in $S^{n+2}$. We take it that all the $W_j$ being studied lie in the $\epsilon$-neighborhood of the knot $S^n \subset S^{n+2}$, i.e. that $j$ is sufficiently large. But this just implies that Lemma 4.1 is applicable to the inner part of $A$ with respect to $V_1$. It is evident that Lemma 4.1 is applicable also to the map $f_j: V_1 \to S^n \times S^1$.

Just as we did in the proof of Theorem 3, we first kill, by pasting handles onto $V_1$ inside $G$, the kernels $\operatorname{Ker} i_{1*}^{(\pi_q)}$ and $\operatorname{Ker} i_{2*}^{(\pi_q)}$ when $q \le [n/2]$, and for odd $n+1$ also $\operatorname{Ker} i_{1*}^{(\pi_q)}$, $2q+1 = n+1$ (the dimension of $V_1$ here is $n+1$).

Note further that

$$\operatorname{Ker} i_{1*}^{(\pi_q)} = \operatorname{Ker} \hat{i}_{1*}^{(H_q)}$$

and

$$\operatorname{Ker} f_{j*}^{(\pi_q)} = \operatorname{Ker} \hat{f}_{j*}^{(H_q)},$$

and also that

$$\operatorname{Ker} \hat{f}_{j*}^{(H_q)} = \operatorname{Ker} \hat{i}_{1*}^{(H_q)} + \operatorname{Ker} \hat{i}_{2*}^{(H_q)},$$

whence it follows that the "Hurewicz theorem" from §4 is applicable to $\operatorname{Ker} \hat{i}_{2*}^{(H_q)}$.

Now, just as in the proof of Theorem 3, we perform surgery on $\operatorname{Ker} i_{2*}^{(\pi_q)}$ and we apply Lemma 7.1 with $n = 2q+1$. The case $n = 2q$ is analogous to Theorem 3 also by virtue of the remark made on the applicability of the "Hurewicz theorem" (Lemma 4.1) to the kernel $\operatorname{Ker} i_{2*}^{(\pi_q)}$.

Thus we have proved the following theorem.

Theorem 9.1. *Under the hypotheses of Theorem 6 one can find a submanifold $V \subset S^{n+2} \setminus S^n$ of the homotopy type of $S^n \times S^1$ such that the region $\bar{A} \subset S^{n+2}$, bounded by $V$, has the homotopy type of $S^n$.*

This is also an analog of Theorem 3 for the case being considered.

Note that by virtue of the Browder-Levine theorem (see [20], §5) the manifold $V$ is a fiber bundle with layer $\tilde{S}^n \in \theta^n(\partial\pi)$ and basis $S^1$. For even $n$ the group $\theta^n(\partial\pi) = 0$. However, be that as it may, $\tilde{S}^n$ is $PL$-homeomorphic to $S^1$ while $V$ is $PL$-homeomorphic to $S^n \times S^1$, since the group of $PL$-automorphisms of the sphere $S^n$ is connected. Our subsequent discussions will be in terms of $PL$-manifolds.

For the region $\bar{A}$, $\partial\bar{A} = V$, we take the "dual region", $PL$-homeomorphic to $D^{n+1} \times S^1$, and we paste together $\bar{A} \cup_h D^{n+1} \times S^1$, where $h: \partial D^{n+1} \times S^1 \to V$ is a $PL$-homeomorphism. As we know, under our hypotheses $\bar{A} \cup_h D^{n+1} \times S^1$ is $PL$-



homeomorphic to $S^{n+2}$. The original sphere $S^n$ lies in $\bar{A}$ and the complement $\bar{A} \setminus S^n$ shrinks down to $V = \partial\bar{A}$. Therefore the pair $(\bar{A} \cup_h D^{n+1} \times S^1, S^n)$ satisfies the Stallings theorem [18]. Without loss of generality we can take it that the imbedding $S^n \subset S^{n+2}$ is linear on a small simplex. From the method of reference [18] we get at once a variant of the result, which we state below.

There exists a homeomorphism (a $PL$-homeomorphism everywhere except in a small neighborhood of $S^n$) transforming $S^n$ into the standard sphere. Consequently, on the manifold $A$ we can give a new $PL$-structure such that:

a) it coincides with the old one on $\partial\bar{A}$;

b) $\partial\bar{A} = S^n \times S^1$ is $h$-cobordant to the boundary of the tubular neighborhood $T(S^n) \subset \bar{A}$.

Therefore in the new $PL$-structure we see that $\bar{A}$ is $PL$-homeomorphic to $S^n \times D^2$ (see [17]).

Hence the inference on the global flatness of the knot $S^n \subset S^{n+2}$ is obvious at this point.

Let us prove the rest of Theorem 6.

Everywhere except in the neighborhood of $S^n \subset \bar{A}$ there exists the $PL$-homeomorphism

$$d: \bar{A} \to S^n \times D^2,$$
$$d(S^n) = S^n \times 0.$$

To $S^n \times D^2$ we paste the closed complement $Q = \overline{(S^{n+2} \setminus \bar{A})}$ in accordance with the identification $d/\partial\bar{A} = \partial Q$. Then

$$M = S^n \times D^2 \cup_d Q,$$

where $d: \partial Q \to S^n \times S^1$ and $d/\partial Q$ is a $PL$-homeomorphism. It is easy to see that $M$ is a homotopy sphere of dimension $n+2$. Therefore we also obtain the joint transformation $d': M \to S^{n+2}$, where $d' = d/A$ and $d' = 1/Q$, taking the "knot" into a $PL$-knot with the direct product $S^n \times D^2 \subset M$. In such a situation the $PL$-knot is smoothed out and a smoothness from $\theta^n(\partial\pi)$ $(= bP^{n+1}$; see [7]) arises on $S^n \subset M$.

Theorem 6 is proved.

## §10. One generalization of Theorem 5

Let $K$ be a finite "Browder complex." In the simply-connected case this means that there exists an $n$-dimensional "fundamental cycle" $\mu \in H_n(K)$ such that the map $D: Z \to Z \cap \mu$ is the isomorphism $H^j(K) \to H_{n-j}(K)$. If the complex $K$ is non-simply-connected and if $p: K' \to K$ is a finite-sheeted cover with $m$ sheets, then it is necessary to require that $H_n(K') = Z$ and that the element

$$\mu' \in H_n(K'), \quad p^*\mu' = m\mu,$$

be defined such that the map $D: Z \to Z \cap \mu'$ is an isomorphism. If the group $\pi_1(K)$ is finite this gives us the definition of a Browder complex. In case $\pi_1(K)$ is infinite this is clearly insufficient. Let $K' \to K$ be a cover with subgroup $\pi' \subset \pi = \pi_1(K)$ and with layer $F = \pi/\pi'$, on which $\pi$ acts as left displacements. Let there be defined $\alpha \cdot f$, $\alpha \in \pi$, $f \in F$, and the groups $H^0(F)$, $H_0(F)$, $H_c^0(F)$, $H_0^{(0)}(F)$ on which $\pi$ acts (here $H_c^0(F)$ are functions on $F$ with values in $Z$, having a finite carrier, $H_0^{(0)}(F)$ are the infinite linear combinations $\Sigma a_i f_i$, $a_i \in Z$, $f_i \in F$). Then we have

$$H^*(K') = H^*(K, H^0(F)), \quad H_c^*(K') = H^*(K, H_c^0(F)),$$

$$H_*(K') = H_*(K, H_0(F)), \quad H_*^{(0)}(K') = H_*(K, H_0^{(0)}(F)),$$

and all the homologies are assumed with local coefficients.

Consider the generating element

$$g = \sum_i f_i \in H_0^{(0)}(F).$$

Then the comparison $Z \to Z \otimes g$ takes $H_i(K)$ into

$$H_i(K, H_0^{(0)}(F)) = H_i^{(0)}(K').$$

If $F$ consists of $m$ elements, the composition $p_*(Z \otimes g)$ is a multiplication by $m$: $Z \to mZ$.

Let us require that the maps $D: Z \to Z \cap (\mu \otimes g)$, $\mu \in H_n(K)$, be isomorphisms:

$$D: \ H_c^i(K') \to H_{n-i}(K'), \quad Z \in H_c^i(K'),$$

$$D: \ H^i(K') \to H_{n-i}^{(0)}(K'), \quad Z \in H^i(K').$$

As before, the element $\mu \in H_n(K)$ is a fundamental cycle in $K$ and $\mu \otimes g$ is the fundamental open cycle in $K'$.

In this case the complex $K$ is called a "Browder complex."

The following lemma holds.

Lemma 10.1. *If $W$ is an open smooth $(n-1)$-dimensional manifold having the homotopy type of a finite complex and if on $W$ there acts a (possibly nonsmooth) discrete transformation $T: W \to W$ such that the factor space is compact and that the group $H_n(W) = Z$, then $W$ is a Browder complex relative to an $n$-dimensional fundamental cycle.*

We leave the lemma without proof.* Note that the condition on the existence of the transformation $T$ can be replaced by the simple condition on the "homotopy type at infinite" for $W$.

*We remark that the proof is carried out by means of homologies with special families of carriers, introduced by Rohlin in work as yet unpublished.



The following theorem is easily extracted from Lemma 10.1 on the basis of Theorem 3, in which the hypothesis on the homotopy type of the closed manifold is replaced by Lemma 10.1.

Theorem 10.1. *Let $M^n$ be a closed smooth manifold, let $\pi_1(M^n) = \pi = Z + \cdots + Z$, and let the decomposition $\pi = \pi' + \pi''$ be given. Then the cover $M$ with fundamental group $\pi' \subset \pi$ is diffeomorphic to $M^{n-l} \times R^l$, where $l = rk\pi''$ and $M^{n-l}$ is a closed smooth manifold, $n - l \geq 5$.*

This theorem has been established by Browder and Levine (see [20]) for the case $\pi' = Z$, $\pi'' = 0$.

## APPENDIX I

### The signature formula

As in [10,13] we consider a manifold $M^n$, $n = m + 4k$, and an indivisible element $z \in H_{4k}(M^n, Z)$ such that $Dz = y_1, \cdots, y_m$, $y_j \in H^1(M^n, Z)$, $j = 1, \cdots, m$. As has been shown in [10,13], there exists one canonical element $\hat{z} \in H_{4k}(\hat{M}, Z)$, where $\hat{M}$ is the cover over $M^n$ with group $Z + \cdots + Z$ ($m$ of them), under which those and only those paths $\gamma \subset M^n$, for which

$$(\gamma, y_1) = \ldots = (\gamma, y_m) = 0$$

are covered by closed manifolds. Here we do not recall the algebraic definition of the element $\hat{z} \in H_{4k}(\hat{M}, Z)$ from the element $z$. Geometrically it is represented thus: we realize the cycles $Dy_j$ by the submanifolds $M_j^{n-1} \subset M^n$ and the cycle $z$ by their intersection

$$M^{4k} = M_1^{n-1} \cap \ldots \cap M_m^{n-1}.$$

In this case the manifold $M^{4k}$ is covered by a manifold closed in $\hat{M}$ and defines the cycle $\hat{z}$.

The following theorem holds when $m = 2$.

Theorem. *The formula*

$$(L_k(M^n), Z) = \tau(\hat{z}).$$

*holds if the intersection index of the cycles on the group $H_{2k+1}(\hat{M})$ is identically zero.*

Note that if the group $H_{2k+1}(\hat{M}, R)$ is finite dimensional the conditions of our theorem are fulfilled. Consequently this theorem is a generalization of Theorem 2 of [10].

Proof of the theorem. We consider the covering $\hat{M}$, defined earlier, on which lie the complete pre-images of the manifolds $M_1^{n-1}$ and $M_2^{n-1}$ under the projection $p: \hat{M} \to M^{4k+2}$, $n = 4k + 2$. The base transformations of the group $Z + Z$ of motions of the manifold $\hat{M}$ are denoted by $T_1, T_2: \hat{M} \to \hat{M}$. Then, the complete pre-image of the manifold $M_1^{n-1}$ is decomposed into the union $\bigcup_j M_j^{(1)}$ and the complete pre-image $p^{-1}(M_2^{n-1})$ is decomposed into the union $\bigcup_q M_q^{(2)}$; moreover, $M_s^{(\epsilon)}$, where $\epsilon = 1, 2$, $-\infty < s < +\infty$, separates the manifold $\hat{M}$ into two parts: $A_s^{(\epsilon)}$ and $B_s^{(\epsilon)}$, where

$$A_s^{(\epsilon)} \cup B_s^{(\epsilon)} = \hat{M}, \quad A_s^{(\epsilon)} \cap B_s^{(\epsilon)} = M_s^{(\epsilon)}.$$

Furthermore, the notation chosen here is such that

$$T_1 M_s^{(1)} = M_s^{(1)}, \quad T_2 M_s^{(1)} = M_{s+1}^{(1)},$$

$$T_2 M_s^{(2)} = M_s^{(2)}, \quad T_1 M_s^{(2)} = M_{s+1}^{(2)}$$

and for any $s$, $M_s^{(\epsilon)}$ are $Z$-coverings over $M_\epsilon^{n-1}$. The complete pre-image of the manifold $M^{4k} = M_1^{n-1} \cap M_2^{n-1}$ can be represented in the form

$$p^{-1}(M^{4k}) = \bigcup_{j,q} (M_j^{(1)} \cap M_q^{(2)}) = \bigcup_{i,q} M_{j,q}^{4k},$$

and, moreover, all the $M_{j,q}^{4k}$ are diffeomorphic to the original $M^{4k} = M_1^{n-1} \cap M_2^{n-1}$. The cycle which $M_{j,q}^{4k}$ defines in $M_j^{(1)}$ is denoted by $t_j \in H_{4k}(M_j^{(1)})$, $T_{1*}t_j = t_j$, and the imbedding $M_j^{(1)} \subset \hat{M}$, by $\lambda_j$. Obviously

$$\lambda_{j*} t_j = \hat{z}.$$

The formula

$$\tau(t_j) = \tau(M^{4k})$$

holds on the basis of Theorem 1 of [10] (or of Theorem 2 of [13]).

Let us prove that

$$\tau(t_j) = \tau(\hat{z}).$$

Let $j = 0$, $t_0 \in H_{4k}(M_0^{(1)})$. We denote $M_0^{(1)}$ simply by $M$, $t_0$ by $t$, $A_0^{(1)}$ by $A$ and $B_0^{(1)}$ by $B$. Then

$$B \cap A = M, \quad B \cup A = \hat{M}.$$

We denote the manifold $M_0^{(2)}$ by $N$. Then

$$M \cap N = M_{0,0}^{4k} = M^{4k}.$$

We now recall the result of [13]. If the equality $(\alpha^2, t) = 0$ holds for any element $\alpha \in H^{2k}(M, R)$ such that the cycle $\beta = \alpha \cap t \in H_{2k}(M)$ is homologous to zero in $A$ and in $B$, then the required formula

$$\tau(t) = \tau(\hat{z})$$

is valid.

Note that geometrically the cycle $\beta = \alpha \cap t$ lies on $M^{4k} = M \cap N$ and that the self-intersection index $\beta \circ \beta$ (on $M^{4k}$) equals $(\alpha^2, t)$ in $M$. Furthermore, on $M^{4k}$ the cycle $\beta$ is intersected by the open disc $D\alpha \in H_{2k+1}^{(0)}(M)$.

The membranes $\delta_1 \subset A$ and $\delta_2 \subset B$ span the cycle $\beta$ such that $\partial\delta_1 = \partial\delta_2 = \beta$. Further, the pair $M$ and $N$ divides $\hat{M}$ into four parts: $W_1, W_2, W_3, W_4$, where



$$\bigcap_i W_i = M^{4k},$$

$$\bigcup_i W_i = \hat{M}, \quad (W_1 \cup W_2) \cap (W_3 \cup W_4) = M,$$

$$(W_1 \cup W_4) \cap (W_2 \cup W_3) = N.$$

By $J_i \subset H^{2k+1}(M, R) \cap t \subset H_{2k}(M, R)$ we denote the subgroups consisting of elements which have representatives homologous to zero in $W_i$, $i = 1, 2, 3, 4$. Analogously, we introduce the subgroups $J_{(\epsilon)} \subset H^{2k+1}(M, R) \cap t$, $\epsilon = 1, 2$, consisting of elements homologous to zero in $A$ for $\epsilon = 1$ or in $B$ for $\epsilon = 2$.

Obviously,

$$J_1 \cup J_2 = J_{(1)}, \quad J_3 \cup J_4 = J_{(2)}.$$

We denote the group $H^{2k+1}(\hat{M}, R) \cap t$ by $H$. We introduce the operator $P: H \to H$ by setting

$$P(\alpha \cap t) = (T_{1*}\alpha) \cap t.$$

Since $T_{1*}t = t$, $P$ is an isomorphism. Note that $H$ is a finite-dimensional space over $R$.

The following relations hold:

$$P^k J_{(1)} \subset J_2, \quad P^{-k} J_{(1)} \subset J_1,$$

$$P^k J_{(2)} \subset J_3, \quad P^{-k} J_{(2)} \subset J_4$$

for sufficiently large $k$ in view of the finite dimensionality of $H$, $J_i$, $J_{(\epsilon)}$. Therefore (again because of finite dimensionality) we have

$$J_{(1)} = J_1 = J_2, \quad J_{(2)} = J_3 = J_4.$$

We now return to the element $\beta = \alpha \cap t$, homologous to zero in $A$ and in $B$, lying on $M^{4k}$ and represented by the cycle $\bar{\beta} \subset M^{4k}$. Since $\beta \in J_{(1)} \cap J_{(2)}$, the cycle $\bar{\bar{\beta}}$ on $T^{-2k}M^{4k}$ ($k$ large), representing, naturally, $P^{-2k}\beta$, is homologous to zero in the manifolds $T^{-2k}W_1$ and $T^{-2k}W_4$ if to this cycle we add the cycle $h \subset T^{-2k}M^{4k}$ homologous to zero in $M$. Because the group $H_{2k}(M^{4k})$ is finite dimensional, the number $k$ can be chosen so large that the membrane $\partial^{-1}(h)$ can be selected so as not to intersect with $T^{-k}M^{4k}$. Then the cycle $\bar{\bar{\beta}} \subset T^{-2k}M^{4k}$ is homologous to zero in the regions $T^{-k}W_1$ and $T^{-k}W_4$.

We denote the corresponding membrances by $\delta_3$ and $\delta_4$:

$$\delta_3 \subset T^{-k} W_1, \quad \delta_4 \subset T^{-k} W_4, \quad \partial\delta_3 = \partial\delta_4 = \bar{\bar{\beta}}.$$

Since $\alpha \in H^{2k}(M, R)$, it follows that $D\alpha \in H_{2k+1}^{(0)}(M, R)$ and $D\alpha$ is represented by an open cycle in $M$, whose intersection with $M^{4k}$ is $\bar{\beta}$ and with $T^{-2k}M^{4k}$ is $\bar{\bar{\beta}}$. The segment of this open cycle from $\bar{\beta}$ to $\bar{\bar{\beta}}$ is denoted by $d$, $\partial\alpha = \bar{\beta} - \bar{\bar{\beta}}$. We set

$$g_1 = \delta_3 - d + \delta_1,$$

$$g_2 = \delta_4 - d + \delta_2,$$

where $g_1$ and $g_2$ are $(2k+1)$-dimensional cycles in $\hat{M}$. The cycle

$$\bar{\bar{\beta}} = d \cap T^{-k} M^{4k}$$

is such that it is homologous to zero in $T^{-k}W_1$, $T^{-k}W_2$, $T^{-k}W_3$, $T^{-k}W_4$ and its self-intersection index in $T^{-k}M^{4k}$ equals

$$\bar{\bar{\beta}} \circ \bar{\bar{\beta}} = (\alpha^2,\ t) = \bar{\beta} \circ \bar{\beta}.$$

However, it is easy to see that

$$g_1 \circ g_2 = \bar{\bar{\beta}} \circ \bar{\bar{\beta}}$$

and

$$g_1 \circ g_2 = 0$$

by the hypotheses of the theorem. Hence we conclude that from the condition

$$(\alpha J_{(1)}, t) = (\alpha J_{(2)}, t) = 0$$

it follows that

$$(\alpha^2, t) = 0.$$

By analogy with [10,13], the theorem is proved.

We now draw several conclusions from the theorem we have proved.

1. It is easy to show that if the condition $N/Z_0(\pi)N = 0$ is fulfilled, then

$$N_{2k+1} = N_\infty \supset N^\perp_{2k+1}.$$

The important fact here is that each element $\sigma \in N$ satisfies the polynomial relation

$$Q(T_1, T_2)\sigma = 0,$$

where $T_1$, $T_2$ are generators of $\pi$ and $\epsilon Q = 1$, $\epsilon: Z(\pi) \to Z$. Indeed, if $\sigma_1, \cdots, \sigma_s$ are generators of $N$ over $Z(\pi)$ and if $N/Z_0(\pi)N = 0$, then we can find a matrix $P = (P_{ij})$ with coefficients in $Z(\pi)$ such that $\epsilon P = E$ and $\Sigma_i P_{ij}\sigma_i = 0$. But then

$$(\det P)\sigma_j = 0$$

and

$$Q = \det P, \qquad \varepsilon Q = 1.$$

We can take it that

$$Q = [1 + P_0(T_2)] + T_1 P_1(T_2) + \ldots + T_1{}^n P_n(T_2),$$

where $P_0$ depends only on positive powers of $T_2$ and $P_0(0) = 0$. Therefore the polynomial $Q$ is invertible into formal series in $T_1^j$ and $T_2^s$, where $j \geq 0$, $s \geq f(j) > -\infty$. Consequently the element $\sigma$ is homologous to zero in open homologies and is orthogonal to $N$ in the sense of the intersection index.

The sufficient condition stated here for the applicability of the theorem $(N/Z_0(\pi)N = 0)$ is satisfied, for example, if the image

$$p_*: H_{2k+1}(\hat{M}, R) \to H_{2k+1}(M^n, R)$$

is trivial and if the differential

$$d_2: E_2^{2,2k} \to E_2^{0,2k+1} = N/Z_0(\pi)N,$$
$$E_2^{2,2k} = H_{2k}^{\text{invariant}} \subset H_{2k}(\hat{M}),$$

is trivial in the Cartan spectral sequence for the cover $p: \hat{M} \to M^n$.

2. Let us give another proof of the topological invariance of the class of $L_k(M^n)$ when $n \leq 4k+3$ and $\pi_1(M^n) = 0$. Indeed, if $M^n$ is homeomorphic to $M^{4k} \times R^3$, where $M^{4k}$ is simple-connected and closed, then, as in §2, we can pick out a submanifold $W = M^{4k} \times T^2 \times R$ and realize the cycle $[M^{4k} \times T^2]$ by a smooth $V \overset{i}{\subset} W$ such that the homomorphism of the imbedding $i_*: \pi_q(V) \to \pi_q(W)$ is an isomorphism when $q \leq 2k$, which is trivial. Then $V$ separates $W$ into two parts $A$ and $B$, $A \cap B = V$, and $i_1: V \subset A$, $i_2: V \subset B$. We set

$$M'_{2k+1} = \operatorname{Ker} i_{1*}^{(H_{2k+1})}.$$

Since the intersection index on $M'_{2k+1} = \operatorname{Ker} \hat{i}_{1*}^{(H_{2k+1})}$ is trivial, where $\hat{i}_1: \hat{V} \subset \hat{A}$ (universal coverings), following Whitney we can realize the $Z(\pi)$-basis in $M'_{2k+1}$ by imbedded spheres and we can perform Morse surgery on them (the possibility of the realization is proved identically to the Whitney proof; see [6] for details). The surgery can be performed so that the Pontrjagin classes do not change; after surgery we obtain a manifold $V_1$ to which we can now apply the theorem of this appendix. Under the surgery it is evident that the "signature of the cycle" on the coverings over $V$ and $V_1$ also does not change. By comparing what we have said with the fundamental lemma of [13] applied to the imbedding $\hat{V} \subset \hat{W}$, with the above-mentioned theorem of this appendix and with the equality of the "signatures of the cycle" on $\hat{V}$ and $\hat{V}_1$, we obtain our assertion in accordance with the scheme of [13, 10].

## APPENDIX 2

### Unsolved problems related with the theory of characteristic classes

We mention here several problems directly connected with the results of the work of the author [10,11,12] and of Rohlin, related mainly to Pontrjagin classes.

I. Topological problems.

1.* Does there exist a number $n = n(k)$, depending only on $k$, such that for all simple $p > n(k)$ the Pontrjagin classes $p_k$ of modulus $p^h$ are topologically invariant? This should follow from the fact that the groups $\pi_i(B\,\mathrm{Top})$ are finitely-generated for all $i \leq 4k$. However, it is apparent that some generalization of the method of this paper or of the author's paper [13] is more suitable for answering this question. Such a result would have a good application, for example, to the

*Added in proof. Problem 1 has been solved recently in an as yet unpublished paper by the author and V. A. Rohlin.

classical lenses of dimensions $\geq 5$. For example, $p \neq 7$ for $k = 2$ (see [8]).

2. Are the rational Pontrjagin classes of complexes of rational homology manifolds topological invariants? We have affirmative answers here only for $L_k(M^n)$, $n \leq 4k+2$ (see [10,12]).

3. For the topological microbundles of Milnor can we determine rational Pontrjagin classes $p_i \in H^{4i}(B\,\mathrm{Top}, Q)$ satisfying the following axioms:

a) they coincide with the ordinary ones for $O$ and $PL$-microbundles;

b) the Whitney formula for sums;

c) the Hirzebruch formula for $L_k(M^{4k})$ and the author's formulas for $L_k(M^{4k+1})$ and sometimes for $L_k(M^{4k+m})$, $m > 1$ (see [10,13] and Theorem 2 of this paper).

II. Homotopy problems.

1. Let $z \in H_{4k}(M^n)$ be an element such that $Dz = y_1 \cdots y_m$, $m = n - 4k$, $y_i \in H^1(M^n)$. Is the scalar product $(L_k(M^n), z)$ a homotopy invariant? The author has solved this problem for $m = 1$, partially for $m = 2$ (see [10,12] and Appendix 1 of this paper) and sometimes for $m > 2$ (see Theorem 2 of this paper). For $m = 2$ the final solution has been obtained by Rohlin.

2. In those cases in which the preceding question has been answered affirmatively, there arises the problem of computing the classes of $L_k$ in terms of homotopy invariants. This problem has not been solved even in the case of the Rohlin theorem for the codimension $m = 2$. Important special cases of this problem will also be taken up in a later section, dealing with differential-topological questions.

III. Stably-algebraic problems.

Before dicussing the problems we give an algebraic introduction. Let $\pi$ be a Noetherian group and let $M$ be a finitely-generated $Z(\pi)$-module.

The homomorphism of the modules $h: M \to PM$, where $PM = \mathrm{Hom}_c(M, Z)$, is called the scalar product $(x, y) = hx(y)$. Symmetric and skew-symmetric cases naturally arise.

We say that the scalar product is unimodular if $h$ is an isomorphism.

If $\pi' \subset \pi$, then on $N = M/Z_0(\pi')M$ there arises the bilinear form $(px, py) = \Sigma_{a \in \pi'}(x, ay)$, being a scalar product in the same sense if $\pi'$ is a normal divisor. Here $p: M \to N$ is a natural projection. We call this bilinear form the induced scalar product.

We call a symmetric scalar product even, if $(x, x)$ is divisible by 2 and $(x, ax)$ is divisible by 2 for all $a \in \pi$, $a^2 = 1$.

For subgroups $\pi'$ of finite index in $\pi$ and of symmetric case, it makes sense to speak of the signature of the (induced) scalar product on $N = M/Z_0(\pi')M$, and of the signature of the form on $N$ defined as a function of the subgroup $\pi' \subset \pi$,



$\tau = \tau(\pi')$, if the index of $\pi'$ in $\pi$ is finite. We set $\tau(M) = \tau(\pi)$; $I(\pi')$ is the index of $\pi'$. Then we require that $\tau(\pi') = \tau(M)I(\pi')$.

Let the scalar product be skew-symmetric. We designate as an Arf-invariant the map $\phi: M \to Z_2$ such that $\phi(ax) = \phi(x)$, $a \in \pi$, and

$$\varphi(x+y) = \varphi(x) + \varphi(y) + (x, y) \bmod 2.$$

Let $\pi' \subset \pi$ and $N = M/Z_0(\pi')M$, $p: M \to N$. We designate as an induced Arf-invariant the map $\phi_{\pi'}: N \to Z_2$ such that

$$\varphi_{\pi'}(px) = \varphi(x) + \sum_{a \in \frac{\pi'}{2}} (x, ax) \bmod 2,$$

where $\pi'/2 \subset \pi'$ denotes a subset in $\pi'$ which contains one and only one element from any pair of elements $a, a^{-1} \in \pi$. The case $a = a^{-1}$ is not essential since then $(x, ax) = (a^{-1}x, x) = -(x, ax) = 0$. For $\phi_{\pi'}$, the correctness and identity of Arf'a are easily verified. If $\pi'$ has a finite index $I(\pi')$ in $\pi$, then a "total" Arf-invariant $\Phi(\pi') \in Z_2$ is defined on $M/Z_0(\pi')M$. We set $\phi(M) = \Phi(\pi)$. Then we let $\Phi(\pi') = \Phi(M)I(\pi')$.

Now let $\pi$ be a finite or an abelian group. We say that the module $M$ with a symmetric or skew-symmetric scalar product possesses Poincaré duality if for all subgroups $\pi' \subset \pi$ the induced scalar products are unimodular.

Let $F_1$ be a free module with two generators $x, y \in F_1$; moreover, $(x, ax) = (y, ay) = 0$ for all $a \in \pi$, $(x, ay) = 0$ for $a \neq 1$ and $(x, y) = 1$. Here we take the scalar product to be symmetric or skew-symmetric. In the latter case we also require that $\phi(x) = \phi(y) = 0$, i.e., that there exists an Arf-invariant of a special form in the module. Such a module we call a one-dimensional free module.

The sum $F = F_1 + \cdots + F_1$ with due regard to scalar product and Arf-invariant (for the skew-symmetric case) is called a free module.

Our examination of the isomorphisms, direct sums, etc., preserves all the existing structures.

Admissible classes of modules:

$C_1$: projective modules with symmetric even scalar product and Poincaré duality;

$C_1^0 \subset C_1$: modules with zero signature $\tau(M) = 0$;

$C_2$: projective modules with skew-symmetric scalar product, Poincaré duality and Arf-invariant;

$C_2^0 \subset C_2$: modules with zero Arf-invariant;

$C_2'$: as in $C_2$ but without reckoning Arf-invariant;

$\overline{C}_i \subset C_i$, $i = 1, 2$: inverse modules $M \subset \overline{C}_i$ for which we can find a module $M'$

such that $M + M' = F$ with due regard to all existing structures, where the module $F$ has been defined above.

The class $\overline{C}_2' \subset C_2$ without reckoning Arf-invariant is defined analogously.

We denote the subclasses $C_i^0 \cap \overline{C}_i$ by $D_i$.

With each class $C_1, C_2, C_2', C_1^0, C_2^0$ there is related in a natural way the "Grothendieck group":

$$A(\pi) = \widetilde{K}^0(C_1), \qquad B(\pi) = \widetilde{K}^0(C_2),$$

$$C(\pi) = \widetilde{K}^0(C_2'), \qquad D(\pi) = \widetilde{K}^0(C_1^0),$$

$$E(\pi) = \widetilde{K}^0(C_2).$$

The homomorphism $B(\pi) \to C(\pi)$ has been defined. The subclasses $\overline{C}_1, \overline{C}_2, \overline{C}_2', D_1, D_2$ define subgroups of "substantively inverse" elements.

The algebraic problem is to compute the groups $A(\pi), B(\pi), C(\pi), D(\pi), E(\pi)$. It would be of special interest to find these groups for $\pi = Z + \cdots + Z$ and for $\pi = Z_p$. For the latter case this is connected with the arithmetic of the number $p$, since here even the ordinary functor $\widetilde{K}^0(Z(Z_p))$, without regard to scalar products, could be nontrivial for "bad" $p$.

For $\pi = Z + Z$, the ordinary $\widetilde{K}^0(\pi)$ is trivial, but $B(\pi)$ and $C(\pi)$ are nontrivial as shown by Example 2 in §3 of [13]. As will be seen from the subsequent topological problems, all the $A, B, C, D, E$ are nontrivial for $\pi = Z + \cdots + Z$.

In case $\pi = Z + \cdots + Z$, we can take it that we are always dealing with scalar products on algebraic free modules since projective modules are stably free.

## IV. Differential-topological problems.

Our questions will refer to the following two situations.

a) There is a commutative diagram of maps of degree $+1$ and of (regular) coverings

$$\begin{array}{ccc} M_1^{2n} & \xrightarrow{f} & M_2^{2n} \\ p_1 \uparrow & & p_2 \uparrow \\ \hat{M}_1 & \xrightarrow{\hat{f}} & \hat{M}_2, \end{array}$$

where the monodromy group of the coverings is $\pi$ and where the element $\alpha \in K_R^0(M_2^{2n})$ is given such that $f^*\alpha \in K_R^0(M_1^{2n})$ is a "stable tangent bundle." Let us assume that the homological kernels of the map $\hat{f}$ are trivial in dimensions $< n$. Then the kernel $M = \operatorname{Ker} \hat{f}_*^{(H_m)}$ is a $\pi$-module and determines an element of $A(\pi)$ when $n = 2k$ or of $B(\pi)$ when $n = 2k + 1$. When $n = 3$ or $7$, we need only the image $B(\pi) \to C(\pi)$.



b) There is a membrane $W^{2n}$ with two boundaries $M_1^{2n-1}, M_2^{2n-1}$ and with retractions $r_i: W^{2n} \to M_i^{2n-1}$ which are tangential maps. On $r_i$ we impose constraints analogous to those on $f$ in example a) for the coverings $\hat{W} \to W^{2n}$, $\hat{M}_i \to M_i^{2n-1}$. Then the kernel $M = \operatorname{Ker} \hat{r}_i^{(H_n)}$ determines an element of $A(\pi)$, $n = 2k$, or of $B(\pi)$, $n = 2k + 1$; moreover, here it is easy to reduce these elements to $D(\pi)$ for $n = 2k$ or to $E(\pi)$ for $n = 2k + 1$.

Problems.

1. The realizability of the elements $x \in A(\pi), B(\pi), C(\pi), D(\pi), E(\pi)$ in the situations of examples a) and b).

2. The case in the preceding problem when in example a) the element $\alpha \in K_R^0(M_2^{2n})$ is a "stable tangent bundle" to $M_2^{2n}$, is of special interest.

3. The rational Pontrjagin classes: if in example a) the manifold $M_2^{2n}$ is the torus $T^{2n}$, then $\alpha \in \operatorname{Ker} J$ and the Pontrjagin classes

$$f^* p_i(\alpha) = p_i(M_2\ ),$$

are defined; moreover, $\pi = Z + \cdots + Z$. As the author has proven, a stable tangent bundle of manifolds of the homotopy type of $T^q$ is always trivial (this follows easily from Theorem 2 of this paper, from the Bott periodicity for $BO$, from the result of Adams on $J \otimes Z_2$-homeomorphism and from the fact that the suspension over the torus $T^q$ has the homotopy type of a union of spheres). Therefore the classes $p_i(\alpha) \in H^*(M_2^{2n})$ are not trivial when $\alpha \neq 0$ and the invariant $x(\alpha) \in A(\pi)$ for $n = 2k$ and $x(\alpha) \in C(\pi)$ for $n = 2k + 1$ is defined (possibly nonuniquely). The equality $x(\alpha) = 0$ implies the equality $\alpha = 0$ by a theorem of the author. The classes $p_i$ are linear forms in the exterior powers:

$$p_i(\alpha): \Lambda^{4i}\pi \to Z,$$

$$\pi = Z + \ldots + Z, \quad \operatorname{Hom}(\Lambda^{4i}\pi, Z) = \Lambda^{2n-4i}\pi.$$

In general, it is necessary to consider that $p_i(\alpha) \in \Lambda^{2n-4i}\pi$ when $\pi = Z + \cdots + Z$ ($2n$ of them).

The problem is to compute $p_i(\alpha) \in \Lambda^{2n-4i}\pi$ as a function of the element $x(\alpha) \in A(\pi)$ or $C(\pi)$. What we have said above shows that a connection between $p_i(\alpha)$ and $x(\alpha)$ definitely exists.

Finally, in this problem, instead of the torus $T^{2n} = M_2^{2n}$ we can take the product $S^{4k} \times T^{2n-4k}$, and then the question will be that of number. This question is closely related to problem 2 ("homotopy problems").

4. The situation of non-Noetherian fundamental groups is vague to the author; there are many geometric examples of "finite-dimensional groups" here, and the corresponding theory would have a number of applications. Of course, the functor

$P = \mathrm{Hom}_c$ can be introduced with the help of "locally finite" classes of bases, which are always geometric. However, in applications it is necessary that the modules of the kernels be finite dimensional over $Z(\pi)$. These questions, it is true, are not connected with characteristic classes and the author has not investigated them.

5. To investigate the odd-dimensional case $q = 2k + 1$. The constraints on the module which yield Theorem 5.2 of this paper are clearly insufficient.

Everywhere further on we shall denote $\widetilde{K}^0(Z(\pi))$ by $\widetilde{K}^0(\pi)$.

Let us note in addition that the ordinary $K^0(\pi)$, consisting of stable classes of projective modules, is imbedded in $D(\pi)$ and $E(\pi)$ in the following way:

If $\alpha \in K^0(\pi)$, then $P\alpha \in K^0(\pi)$, and there is a natural scalar product on the module $\alpha + P\alpha$. We obtain the imbeddings

$$K^0(\pi) \subset D(\pi) \subset A(\pi),$$
$$K^0(\pi) \subset E(\pi) \subset B(\pi),$$

by taking in the case of $E(\pi)$ the Arf-invariant on $\alpha \subset \alpha + P\alpha$ and on $P\alpha \subset \alpha + P\alpha$ to be trivial.

It is easy to prove the following theorem by using Poincaré duality, the formula of universal coefficients and other factors besides $P = \mathrm{Hom}_c(\ ,Z)$.

Theorem.* *If in Theorem 3 and in §2 we replace the group $\pi_1 = Z + \cdots + Z$ by another (Noetherian) group $\pi = \pi_1$, then the obstruction to the existence of the submanifold $V^n \subset W^{n+1}$, being a deformation retract in $W^{n+1}$, lies in the Grothendieck group $K^0(\pi)$, and the equality to zero of this obstruction is sufficient for the existence of the deformation retract $V^n \subset W^{n+1}$.*

Remark. The question of the uniqueness of such a $V^n \subset W^{n+1}$ leads to the $h$-cobordism problem and by the same token to $K^1(\pi)$ or, more precisely, to the factor-group $Wh(\pi)$ (see [8]). Thus we have the following situation.

A. The problem of the type of Theorem 3 and of §2 is connected only with $K^0(\pi)$ (or with its image in $A(\pi)$ and $B(\pi)$) and with $K^1(\pi) \to Wh(\pi)$. As we see from the proof of Theorem 6 (see §9) and from the paper by Browder, Levine and Livesay [21], these questions are analogous to the question of finding the boundary of an open manifold.

B. The diffeomorphism problem divides into the following:

1) The $J$-functor, the $K_R$-functor and the normal bundles of smooth manifolds;

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**Added in proof.* This theorem has been found independently by L. C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Thesis, Princeton Univ., Princeton, N. J., 1965. (Dissertation Abstracts 27B (1966), p. 2044-B).



here, $A(\pi)$ and $B(\pi)$ play a role when $n = 2k$ (see [3,22,14] and Appendix 1).

2) The realizations of classes in a Thom complex for $n = 2k$ (see problem 2); here the torsions tor $A(\pi)$ and tor $B(\pi)$ play a role (see Theorem 1 of [22] for $\pi_1 = 0$).

3) The relations between $h$-cobordism and homotopy class in a Thom complex (see Theorem 2 of [22]); here the inverse elements from $D(\pi)$, $E(\pi)$ play a role when $n = 2k - 1$.

4) For $n = 2k - 1$ in 1) and 2) and for $n = 2k$ in 3), there appear $\mathrm{Ext}^i_c$, whose role is not clear. They generalize torsion for $\pi_1 = 0$.

5) The relation between $h$-cobordism and diffeomorphism for $n \geq 5$ has been well studied and is connected only with $Wh(\pi) = K^1(\pi)/(\pi \cup -\pi)$.

## APPENDIX 3

### Algebraic remarks on the functor $P = \mathrm{Hom}_c$

Here we consider the following questions.

1. The connection between $\mathrm{Ext}^i_c(M, Z)$ and $\mathrm{Ext}^i_c(PM, Z)$.

2. The concept of the "reflexivity" of a module: $P^2M = M$.

3. The functor $Q$ for open homologies.

We start by considering the first question. Let $M$ be an admissible $\pi$-module. Consider the acyclic free (projective) resolvent

$$C = \{\ldots \to F_n \to \ldots \to F_0 \xrightarrow{\varepsilon} M \to 0\}$$

and apply the functor $P$:

$$PC = \{0 \to PM \xrightarrow{P\varepsilon} PF_0 \to \ldots \to PF_n \to \ldots\}.$$

We obtain a sequence which is exact in the term $PF_0$.

Now consider the resolvent of the module $PM$,

$$C' = \{\ldots \to F'_n \to \ldots \to F'_0 \xrightarrow{\varepsilon'} PM \to 0\}.$$

Let us paste together the complexes $C$ and $C'$:

$$C'' = \{\ldots \to F'_n \to \ldots \to F'_0 \xrightarrow{\delta} PF_0 \to \ldots \to PF_n \to \ldots\},$$
$$F'_0 \xrightarrow{\varepsilon'} PM \xrightarrow{P\varepsilon} PF_0, \qquad 0 \to PM \to 0$$

such that $\delta = (P\epsilon) \circ \epsilon'$.

We set $F''_n = F'_n$, $F'_{-n-1} = PF_n$, $n \geq 0$. Obviously we have

$$H_i(C'') = 0, \quad i \geqslant -1, \quad H_i(C'') = \mathrm{Ext}_c^{-i-1}(M, Z), \quad i \leqslant -2.$$

Furthermore, for the complex $PC''$,

$$H_c^i(C'') = H_i(PC'') = \mathrm{Ext}_c^i(PM, Z), \quad i > 0,$$
$$H_c^0(C'') = H_0(PC'') = \mathrm{Coker}\, P^2 = P^2M/\mathrm{Im}\, P^2,$$
$$H_c^{-1}(C'') = H_{-1}(PC'') = \mathrm{Ker}\, P^2 \subset M,$$
$$H_c^{-i}(C'') = H_{-i}(PC'') = 0, \quad i \geqslant +2.$$

All these equalities follow from the fact that $P^2$ is a natural isomorphism for projective modules. Thus Ker $P^2$ and Coker $P^2$ acquire a homological meaning.

Since $H_i(C'')$ and $H_c^i(C'')$ are connected by the Cartan-Eilenberg-Grothendieck spectral sequence, we can draw certain conclusions:

A. Let the homological dimension of the group $\pi$ be $n$ (for example, $\pi = Z + \cdots + Z$). Then we always have

$$\mathrm{Ext}_c^n(PM, Z) = \mathrm{Ext}_c^{n-1}(PM, Z) = 0.$$

B. If $\mathrm{Ext}_c^i(\mathrm{Ext}_c^i(M, Z), Z) = 0$, $i > 0$, then $\mathrm{Ker}\, P^2 = 0$.

C. If $\mathrm{Ext}_c^{i+1}(\mathrm{Ext}_c^i(M, Z), Z) = 0$, $i > 0$, then $\mathrm{Coker}\, P^2 = 0$.

The modules $M$ such that $P^2M = M$ are called reflexive, and the modules $M'$ such that $PM' \approx M'$ are called selfadjoint. Every reflexive module is a direct sum of selfadjoint ones and vice versa, since in this case $P(M + PM) = M + PM$ and $P$ is an additive functor.

Corollaries.

1. *If* $\mathrm{Ext}_c^i(M, Z) = 0$, $i > 0$, *and if* $\pi = Z + \cdots + Z$, *then* $M$ *is stably free, since* $PM$ *is stably free by Lemma* 5.1 *and* $P^2M = M$.

2. *If* $\pi = Z + Z$, *then for any module* $M$ *the module* $PM$ *is stably free, since* $\mathrm{Ext}_c^1(PM, Z) = \mathrm{Ext}_c^2(PM, Z) = 0$.

Note that this is not true even for $\pi = Z + Z + Z$, since there exists a module $M \neq 0$ which is reflexive and such that

$$\mathrm{Ext}_c^2(M, Z) = \mathrm{Ext}_c^3(M, Z) = 0,$$
$$\mathrm{Ext}_c^1(\mathrm{Ext}_c^1(M, Z), Z) = \mathrm{Ext}_c^2(\mathrm{Ext}_c^1(M, Z), Z) = 0,$$
$$\mathrm{Ext}_c^1(M, Z) = \mathrm{Ext}_c^3(\mathrm{Ext}_c^1(M, Z), Z) \neq 0.$$

We shall find such a module. Let $M_0$ be a one-dimensional module with a generator $u \in M_0$ such that $Z_0(\pi) \circ u = 0$. The resolvent of $M_0$ (see §5, Example 1) is three-dimensional,

$$0 \to F_3 \xrightarrow{d} F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{\varepsilon} M_0 \to 0;$$

and, moreover,

$$\mathrm{Ext}_c^i(M_0, Z) = 0, \quad 0 \leqslant i \leqslant 2, \quad \mathrm{Ext}_c^3(M_0, Z) = M_0.$$



Let $M = F_2/\mathrm{Im}\, d$. We have

$$0 \to F_3 \xrightarrow{d} F_2 \xrightarrow{\varepsilon} M \to 0.$$

Therefore $\mathrm{Ext}_c^i(M, Z) = 0$, $i > 1$, and $\mathrm{Ext}_c^1(M, Z) = \mathrm{Ext}_c^3(M_0, Z) = M_0$. This module $M$ yields the required example of a module which is reflexive but not projective for $\pi = Z + Z + Z$.

Let us introduce a topology in $Z(\pi)$: namely, as a base system of neighborhoods of zero we take all linear spaces over $Z$, generated by the elements $\alpha \in \pi \backslash A_i$, where $A_i$ is any finite set in $\pi$.

In a finitely generated module the topology is introduced thus: if $x_1, \cdots, x_k \in M$ are $\pi$-generators and if $A_1, \cdots, A_k$ are any finite sets in $\pi$, then as a neighborhood of zero in $M$ we take all the $x \in M$ such that $\lambda x = \Sigma_{i,j} \lambda_{ij} \alpha_{ij} x_j$, $\lambda \neq 0$, where $\alpha_{ij} \in \pi \backslash A_j$, $\lambda$, $\lambda_{ij} \in Z$. Such neighborhoods generate a system of neighborhoods of zero in $M$. In this topology the points are nonseparable, in general.

We have that the $PM$ are continuous characters of the continuous group $M$ in $Z$ (in the discrete topology), and moreover, $PM$ is a topological $Z(\pi)$-module. The Ker $P^2$ are points in $M$, infinitely close to zero.

We define the complement $Q$: $M \to \hat{M}$, where $\hat{M}$ is the compactification of $M$ and Ker $P^2$ is equated to zero in $\hat{M}$. The derived functors of functor $Q$ correspond to open homologies, so that for a field $K$ we have

$$Q = \mathrm{Hom}\,(\mathrm{Hom}_c(M, K), K),$$
$$\mathrm{Tor}_Q^i(M, K) = \mathrm{Hom}\,(\mathrm{Ext}_c^i(M, K), K), \quad i \geqslant 0.$$

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# EQUIVALENCE OF SYSTEMS OF INTEGER MATRICES

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1°. **Statement of the problem.** Let $A_k$ and $B_k$, $k = 1, 2, \cdots, N$, be two sets of square integer matrices of the same order $m$. We will call these two systems equivalent if there exists a unimodular integer matrix $C$ such that

$$C^{-1}A_k C = B_k, \qquad k = 1,2,\ldots,N.$$

In this article we will show that the problem of the equivalence of two given systems of matrices can be reduced to the question of whether a certain ideal of a $Z$-ring contained in a semi-simple algebra is principal (by a $Z$-ring we mean an associative ring with identity whose additive group is a free abelian group of finite rank).

2°. **The group $\mathfrak{G}$.** We will write each system of $N$ square matrices $U_1, \cdots, U_N$ of the same order $m$ as a row $(U_1, \cdots, U_N)$, defining in a natural way the addition of such rows and the multiplication of a row by a scalar.

Furthermore, we define

$$C(U_1,\ldots,U_N) = (CU_1,\ldots,CU_N)$$

and

$$(U_1,\ldots,U_N)C = (U_1C,\ldots,U_NC),$$

where $C$ is any square matrix of order $m$.

Put

$$A = (A_1,\ldots,A_N), \quad B = (B_1,\ldots,B_N),$$

where $A_1, \cdots, A_N$ and $B_1, \cdots, B_N$ are systems of integer matrices for which the question of equivalence is posed. Consider the set of rows

$$S = \{AX - XB\}$$

where $X$ ranges over all rational matrices of order $m$. It is clear that $S$ is a linear space over the field $Q$ of rational numbers and a basis over $Q$ can be found in a finite number of steps. We denote by $M$ the set of rows $\{AX - XB\}$, where $X$ ranges over all integer matrices of order $m$. Obviously $M$ is a lattice and its dimension is equal to the dimension of the space $S$, since $S = MQ$.

Also, we denote by $N$ the set of all integer matrices contained in $S$. Clearly $N$ is a lattice, $S \supset N \supseteq M$, and hence the dimension of $N$ is also equal to the dimension of $S$. Consequently the group

$$\mathfrak{G}(A,B) = N/M$$

is finite. The exponent (i.e. the least common multiple of the orders of the elements) of this group is denoted by $i(A, B)$. The construction of bases of the lattices $M$ and $N$, and hence the determination of the structure of the group $\mathfrak{G}(A, B)$