THE ANALYTIC GENERALIZED HOPF INVARIANT.
MANY-VALUED FUNCTIONALS

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§ 1. FORMULATION OF THE PROBLEM. MINIMAL MODEL

We consider a simply-connected manifold M^n (possibly not closed) and its homotopy groups $\pi_{q+1}(M^n)$.

Problem. [1] Given a homotopy type, what are the universal expressions in terms of differential forms on the manifolds M^n and S^{q+1} depending on a map $F\colon S^{q+1} \to M^n$ that give rise to $(q+1)$-forms on S^{q+1} whose integrals over S^{q+1} are homotopically invariant? To classify such expressions and to specify those that have the "rigidity" property, if the integrals over tlie sphere are integers up to a universal set of normalization constants depending only on the homotopy type of the manifold.

The simplest example is the Whitehead realization of the Hopf invariant for $F\colon S^{4n-1} \to S^{2n}$, where

$$(1) \qquad H(F) = \int_{S^{4n-1}} v \wedge F^*(\omega), \qquad dv = F^*(\omega).$$

(An integral $\int v \operatorname{curl} v \, d^3 v$ of this kind occurs in hydrodynamics.)

Homotopy invariant integrals that cannot be reduced to the Hurewicz homomorphism, that is, to integrals over spherical cycles of closed forms in M^n, are called "analytic generalized Hopf invariants". The "rigid formulae" are of particular interest (see below).

We assume further that the rational homotopy type of M^n is different from S^n (where the problem in question is solved by (1)).

It is well known that the rational homotopy type (or $\mathbb{Q}$-type) of simply-connected objects can be conveniently described by Sullivan's so-called minimal model. To each complex K with $\pi_1(K) = 0$ there corresponds a minimal model of its $\mathbb{Q}$-type, a free graded skew-commutative differential algebra over $\mathbb{Q}$ denoted by $A = A(K)$, which has free multiplicative generators $x_{j\alpha}$ of dimensions $j \geqslant 2$ such that

$$(2) \qquad \partial x_{j\alpha} = P_{j\alpha}(\ldots, x_{q\beta}, \ldots), \qquad q < j,$$

that is, the polynomials $P_{j\alpha}$ depend on the generators of dimension less than j, although $\dim \partial x_{j\alpha} = j + 1$.

If the homotopy type of K is realized as a smooth (possibly open) manifold M^n, then by a standard induction on tlie dimension we can construct a realization, that is, a homomorphism ψ of $A = A(K)$ into the algebra $\Lambda^*(M^n)$ of infinitely smooth real forms (see the end of § 3):

$$(3) \qquad\qquad \psi \colon\ A \to \Lambda^*(M^n),$$

which induces the cohomology isomorphism

$$H^*(A) = H^*(M^n).$$

We require that $\psi(x) = \psi(\partial y)$ in the construction of ψ for any x of dimension n, $\dim x = n$.

§ 2. Algebraic constructions

We consider an arbitrary skew-commutative graded algebra

$$T = \sum_{j \geqslant 0} T^j, \qquad T^0 = k$$

over the field $k = \mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$ with a differential $\partial \colon T^j \to T^{j+1}$. Let $H^1(T) = 0$. We introduce the minimal free extension $\bar{C}^{(q)}(T) \supset T$ such that $H^j(\bar{C}^{(q)}(T)) = 0$, $j \leqslant q$. This can be constructed as follows. We consider a sequence of extensions

$$T_0 = T \to T_1 \to T_2 \to \ldots \to \bar{C}^{(q)}(T),$$

where the embedding $H^j(T_k) \to H^j(T_{k+1})$ is zero for $j = 1, 2, \ldots, q$. We construct T_{k+1} by adding new free generators whose ∂-operator lies in T_k, which yields a minimal set of multiplicative generators of $H^*(T_k)$ in the dimensions $\leqslant q$.

Definition. The *homotopy group* of the algebra $\pi_{q+1}(T)$ is

$$\operatorname{Hom}(H^{q+1}(\bar{C}^{(q)}(T)), k) = H_{q+1}(\bar{C}^{(q)}(T)) = \pi_{q+1}(T).$$

If $T = \Lambda^*(M^n)$ is an algebra of forms, a minimal model, or its image in $\Lambda^*(M^n)$, then

$$\pi_{q+1}(T) = \pi_{q+1}(M^n) \otimes k.$$

This construction is sufficient for all purposes except the variational calculus. Having in mind subsequent applications, we perform (for $q + 1 \geqslant n$) a two-stage construction to single out in the finite algebra $\bar{C}_n^{(q)} = \bar{C}^{(q)}(T)$, $T = \psi(A)$ an important part. The result $C_n^{(q)}$ of the first stage is essentially independent of q. For $q + 1 < n$ nothing more is needed than the construction above.

As will be clear from § 3, this construction can be realized easily in the algebra of forms $\Lambda^*(S^{q+1} \times \mathbb{R})$ for a map $S^{q+1} \times \mathbb{R} \to M^n$.

We fix a minimal model A, which is a ring with a boundary operator of the form (2), and numbers n and q, $q + 1 \geqslant n$. We denote by $A_n \subset A$ the subring generated by the symbols $x_{j\alpha}$ for $j \leqslant n - 1$.

We denote by $\mathcal{J}_n \subset A$ the ideal generated by the elements of dimension n that are basis elements of $A / \operatorname{Im} \partial$ and by all $x \in A$ of dimension $j \geqslant n + 1$.

We now define I_n as $I_n = \mathcal{J}_n \cap A_n$. Clearly,

$$(4) \qquad\qquad A/\mathcal{J}_n = A_n/I_n, \qquad H^*(A/\mathcal{J}_n) = H^*(M^n).$$

We consider a differential extension of A_n, that is, a free skew-commutative algebra $D_n \supset C_n \supset A_n$, where the embedding $A_n \to C_n$ is zero in the cohomology of dimensions $j \leqslant n$. If $y_{j\gamma}$ is a minimal multiplicative basis of $H^*(M^n)$ and $\tilde{y}_{j\alpha}$ are their representatives in A_n, $\partial \tilde{y}_{j\gamma} = 0$, then we introduce new symbols (which also yields C_n and D_n)

$$(5) \quad \begin{cases} v_{j-1,\gamma}^{(0)}, \ b_{j-1,\gamma}^{(0)}, \ c_{j-2,\gamma}^{(0)}, \ \partial v_{j-1,\gamma}^{(0)} = \tilde{y}_{j\gamma}, \ \partial c_{j-2,\gamma}^{(0)} = b_{j-1,\gamma}^{(0)}, \\ C_n = A_n[\ldots, v_{j-1,\gamma}^{(0)}, \ldots], \ D_n = C_n[\ldots, b_{j-1,\gamma}^{(0)}, \ c_{j-2,\gamma}^{(0)}, \ldots]. \end{cases}$$

Iterating the construction we construct new embeddings

$$C_n \to C_{n,1}^{(q)} \to C_{n,2}^{(q)} \to \ldots \to C_{n,\infty}^{(q)},$$

where $C_{n,p+1}^{(q)}$ is obtained from $C_{n,p}^{(q)}$ by the adjunction of new generators $v_{j-1,\gamma}^{(p)}$ with $\partial v_{j-1,\gamma}^{(p)} \in C_{n,p}^{(q)}$ and the embeddings $C_{n,p}^{(q)} \to C_{n,p+1}^{(q)}$ are zero in the homology of dimensions $j \leqslant n$, $C_{n,0} = C_n$.

The algebras $D_{n,p}^{(q)}$ are constructed similarly, by adding to $C_{n,p}^{(q)}$ symbols $b_{j-1,\gamma}^{(p)}$, $c_{j-2,\gamma}^{(p)}$, $\partial c_{j-2,\gamma}^{(p)} = b_{j-1,\gamma}^{(p)}$.

All the algebras $D_{n,p}^{(q)}$ and $C_{n,p}^{(q)}$ are free. By construction, $H^*(C_n^{(q)}) = H^*(D_n^{(q)})$.

We consider the free skew-commutative algebras $C_n^{(q)}$ and $D_n^{(q)}$ with the following generators:

$$(6) \quad \begin{cases} C_n^{(q)} = A_n[\ldots, v_{j-1,\gamma}^{(p)}, \ldots], \\ D_n^{(q)} = C_n^{(q)}[\ldots, b_{j-1,\gamma}^{(p)}, c_{j-2,\gamma}^{(p)}, \ldots], \quad p \geqslant 0. \end{cases}$$

It is obvious that $H^*(C_n^{(q)}) = H^*(D_n^{(q)})$. We also introduce the differential quotient algebras (no longer free):

$$(7) \quad \begin{cases} \bar{C}_{n,0}^{(q)} = C_n^{(q)}/I_n C_n^{(q)}, \qquad \bar{D}_{n,0}^{(q)} = D_n^{(q)}/I_n D_n^{(q)}, \\ H^*(\bar{C}_{n,0}^{(q)}) = H^*(\bar{D}_{n,0}^{(q)}), \qquad I_n \subset A_n. \end{cases}$$

We consider the following differential q-extension of $\bar{C}_{n,0}^{(q)}$ and $\bar{D}_{n,0}^{(q)}$. Let $w^{(0)}$ be a minimal multiplicative basis of $H^*(\bar{C}_{n,0}^{(q)}) = H^*(\bar{D}_{n,0}^{(q)})$ in dimensions $k \leqslant q$ and $\widetilde{w}_{k\beta}^{(0)}$ representatives of it, $\partial \widetilde{w}_{k\beta}^{(0)} = 0$. We introduce the symbols

$$(8) \quad x_{k-1,\beta}^{0}, \ d_{k-1.\beta}^{(0)}, \ e_{k-2,\beta}^{(0)}, \ \partial x_{k-1,\beta}^{(0)} = \widetilde{w}_{k\beta}^{(0)}, \ \partial e_{k-2,\beta}^{(0)} = d_{k-2,\beta}^{(0)}.$$

Next, we set

$$(9) \quad \begin{cases} \bar{C}_{n,1}^{(q)} = \bar{C}_{n,0}[\ldots, x_{k-1,\beta}^{(1)}, \ldots], \\ \bar{D}_{n,1}^{(q)} = \bar{D}_{n,0}[\ldots, x_{k-1,\beta}^{(0)}, d_{k-1,\beta}^{(0)}, e_{k-2,\beta}^{(0)}, \ldots], \\ H^*(\bar{C}_{n,1}^{(q)}) = H^*(\bar{D}_{n,1}^{(q)}). \end{cases}$$

Iterating the construction (9) we introduce a minimal multiplicative basis $w_{k\beta}^{(1)} \in H^*(\bar{C}_{n,1}^{(q)})$ in dimensions $k \leqslant q$, new symbols and rings

(10)
$$\begin{cases} x_{k-1,\beta}^{(1)}, \ d_{k-1,\beta}^{(1)}, \ e_{k-2,\beta}^{(1)}, \ \partial x_{k-1,\beta}^{(1)} = \widetilde{w}_{k\beta}^{(1)} \in C_{n,1}^{(q)}, \ \partial e_{k-2,\beta}^{(1)} = d_{k-1,\beta}^{(1)}, \\ \bar{C}_{n,2}^{(q)} = \bar{C}_{n,1}^{(q)}[\dots, x_{k-1,\beta}^{(1)}, \dots], \\ \bar{D}_{n,2}^{(q)} = \bar{D}_{n,1}^{(q)}[\dots, d_{k-1,\beta}^{(1)}, e_{k-1,\beta}^{(1)}, \dots], \end{cases}$$

Thus we obtain a sequence of extensions (10)

$$\bar{C}_{n,0}^{(q)} \subset \bar{C}_{n,1}^{(q)} \subset \dots \subset \bar{C}_n^{(q)},$$
$$\bar{D}_{n,0}^{(q)} \subset \bar{D}_{n,1}^{(q)} \subset \dots \subset \bar{D}_n^{(q)},$$

By the construction,

(11)
$$H^*(\bar{D}_n^{(q)}) = H^*(\bar{C}_n^{(q)}).$$

Definition. We define a λ-*deformation* of the natural embedding $P \colon \bar{C}_n^{(q)} \to \bar{D}_n^{(q)}$ as an automorphism that is the identy on the image of A_n and is of the form

(12)
$$\begin{cases} x_{k-1,\beta}^{(p)} \xrightarrow{P} x_{k-1,\beta}^{(p)} + \lambda d_{k-1,\beta}^{(p)}, \\ v_{j-1,\beta}^{(p)} \xrightarrow{P} v_{j-1,\beta}^{(p)} + \lambda b_{k-1,\beta}^{(p)} \end{cases}$$

for all j, β, and all dimensions $\leqslant q - 1$.

We now construct the required differential extension of A_n to A. We consider the homomorphic embedding (6)

$$\varphi \colon \ A_n \mapsto C_n^{(q)}$$

and choose a minimal multiplicative basis in $\operatorname{Ker} \varphi^*$ in dimensions $j \geqslant n + 1$, $z_{j\alpha}^{(0)} \in \operatorname{Ker} \varphi^*$. Let $\tilde{z}_{j\alpha}^{(0)}$ be representatives of them in A_n, $\partial \tilde{z}_{j\alpha}^{(0)} = 0$

Let $m_{j-1,\alpha}^{(0)} \in C_n^{(q)}$, $n_{j-1,\alpha}^{(0)} \in A$;

$$\partial m_{j-1,\alpha}^{(0)} = \tilde{z}_{j\alpha}^{(0)},$$
$$\partial n_{j-1,\alpha}^{(0)} = \tilde{z}_{j\alpha}^{(0)}.$$

The first extension $A_{n,1} \supset A_n$ consists of A_n with all the elements $n_{j-1,\alpha}^{(0)}$ adjoined.

We construct the map φ_1 by the formula

$$\varphi_1 \colon \ A_{n,1} \to C_n^{(q)},$$
$$\varphi_1^* \colon \ H^*(A_{n,1}) \to H^*(C_n^{(q)}),$$
$$\varphi_1 \colon \ n_{j-1,\alpha}^{(0)} \mapsto m_{j-1,\alpha}^{(0)}.$$

By going over to the kernel $\operatorname{Ker} \varphi_1^*$ and bearing in mind that $H^j(A) = 0$ for $j > n$, we iterate the construction and obtain a sequence of homomorphisms (13) and embeddings, introducing the elements

(13)
$$m_{j-1,\alpha}^{(0)} \in C_n^{(q)}, \quad n_{j-1,\alpha}^{(0)} \in A_{n,s} \subset A,$$
$$A_n = A_{n,0} \subset A_{n,1} \subset \dots \subset A_{n,\infty} \subset A,$$
$$\varphi_j \colon A_{n,j} \to C_n^{(q)}, \quad \varphi_{n,\infty} \colon A_{n,\infty} \to C_n^{(q)}.$$

We denote the composition $A_{n,\infty} \to C_n^{(q)} \to \bar{C}_n^{(q)}$ by $\varkappa$, that is,

$$\varkappa\colon\ A_{n,\infty} \to \bar{C}_n^{(q)}, \qquad \varkappa^*\colon\ H^*(A_{n,\infty}) \to H^*(\bar{C}_n^{(q)}).$$

By going over to the kernel $\mathrm{Ker}\,\varkappa^*$, we construct by analogy to (13) differential extensions of the ring $A_{n,\infty} = B_{n,0}^{(q)}$ and homomorphisms

(14)
$$\begin{cases} B_{n,0}^{(q)} = A_{n,\infty} \subset B_{n,1}^{(q)} \subset \ldots \subset B_n^{(q)} \subset A, \\ \varkappa = \varkappa_0, \quad \varkappa_s\colon\ B_{n,s}^{(q)} \to \bar{C}_n^{(q)}, \quad \varkappa_\infty\colon\ B_n^{(q)} \to \bar{C}_n^{(q)}. \end{cases}$$

Lemma 1. *The ring $B_{n,\infty}^{(q)}$ contains all the elements of A_{q+1}.*

Remark. Since the ring $\bar{C}_n^{(q)}$ is not free, it can happen that $\varkappa(x) = 0$. For example, this is so for all $x \in A_n$ with $\dim x > n$.

In this case, if $\varkappa_{s-1}(x) = 0$ and x is a multiplicative generator in cohomology, one has to start in the extension of $\varkappa_s$ to A from the rule $\varkappa_s(\partial^{-1}x) = 0$ and also in the construction of all the $\varkappa_i$.

Generally speaking, there arises a homomorphism that is only partially determined and is not unique (the "higher Hurewicz homomorphism")

(15)
$$H^{(q)}\colon\ \pi_{q+1}(M^n) \to H_{q+1}(\bar{C}_n^{(q)}),$$

where $H_j(\bar{C}_n^{(q)}) = 0$ for $j \leqslant q$ by the construction and H_* are the linear forms on H^*.

We recall that $\pi_{q+1}(M^n)$ is represented as linear forms on A that are non-trivial only in the dimension $q+1$ and vanish on all elements that can be factored as products of elements of positive dimensions. The construction of $H_n^{(q)}$ is as follows. We consider the set of $(q+1)$-dimensional elements $u \in A$ such that

(16)
$$\partial\varkappa_\infty(u) = 0, \qquad \varkappa_\infty \in \bar{C}_n^{(q)}.$$

This set forms a subgroup $\widetilde{\Gamma}_{n,q}$ and generates a subgroup of classes modulo $\mathrm{Im}\,\partial$

(17)
$$t\colon\ \widetilde{\Gamma}_{n,q} \to \Gamma_{n,q} \subset H^{q+1}(\bar{C}_n^{(q)}).$$

There are two homomorphisms:

$$s\colon\ \pi_{q+1}(M^n) \to \widetilde{\Gamma}_{n,q}^* = \mathrm{Hom}(\widetilde{\Gamma}_{n,q}, Q),$$
$$t^*\colon\ H_{q+1}(\bar{C}_n^{(q)}) \to \widetilde{\Gamma}_{n,q}^*.$$

We set

(18)
$$H_n^{(q)} = (t^*)^{-1} \circ s.$$

We can also define the dual homomorphism

(19)
$$s^* \circ t^{-1}\colon\ H^{q+1}(\bar{C}_n^{(q)}) \to \pi_{q+1}^* = \mathrm{Hom}(\pi_{q+1}, Q),$$
$$s^* \circ t^{-1} = H_n^{(q)*}.$$

Lemma 2. *The homomorphism $H_n^{(q)*}$ is uniquely determined and is an isomorphism.*

Each element $z \in H^{q+1}(\bar{C}_n^{(q)})$ can be represented by a cocycle $\tilde{z} \in \bar{C}_n^{(q)}$, $\partial\tilde{z} = 0$, of the form

(20)
$$\tilde{z} = R_z(\ldots, v_{j\gamma}^{(p)}, x_{k\beta}^{(s)}, \ldots).$$

Lemma 3. *Suppose that the embedding $\bar{C}_n^{(q)} \to \bar{D}_n^{(q)}$ is subjected to a λ-deformation (12), that $\tilde{z}$ is a cocycle of $C_n^{(q)}$ of dimension $q+1$, and $\tilde{z}_\lambda$ its image under the λ-deformation. Then*

$$\tilde{z}_\lambda - \tilde{z} = \partial w_\lambda \in D_n^{(q)}.$$

Example. For the homotopy type of the bouquet $M^n \sim S^2 \vee S^2$ we can take any $n \geqslant 3$.

For $n = 3$ the manifold M^3 is obtained by deleting two distinct points from $\mathbb{R}^3$. We have the homomorphisms:

(21)
$$\begin{cases} H_3^{(2)}: \ \pi_3(M^3) \to H_3(\bar{C}_3^{(2)}), \\ H_3^{(3)}: \ \pi_4(M^n) \to H_4(\bar{C}_3^{(3)}). \end{cases}$$

In the minimal model the generators are

$$x_{2\alpha} = (x, y), \quad x_{3\alpha} = (z, w, t), \quad x_{4\alpha} = (a, b),$$
$$\partial z = x^2, \quad \partial w = xy, \quad \partial t = y^2,$$
$$\partial a = zy - wx, \quad \partial b = wy - tx, \quad \dots$$

Let $n = 3$. Then $A_3 = Q[x, y]$,

$$C_3^{(q)} = A_3[v_x, v_y], \quad \partial v_x = x, \quad \partial v_y = y.$$

We now construct $A_{3,1}$ and the homomorphism $\varphi_1: z \to xv_x, w \to xv_y, t \to yv_y$, where the images $\varphi_1(z), \varphi_1(w), \varphi_2(t)$ are cocycles in $\bar{C}_3^{(q)}$.

Next, we construct $A_{3,2}$, $\varphi_2(a) = xv_xv_y$, $\varphi_2(b) = yv_xv_y$. Clearly, $\varphi_2(a)$ and $\varphi_2(b)$ are cocycles in $\bar{C}_3^{(3)}$. We obtain homomorphisms of $H_3^{(2)}$ and $H_3^{(3)}$ onto the groups $\pi_3 \otimes Q$ and $\pi_4 \otimes Q$, using the homomorphisms π_1 and φ_2. The formulae (20) in $\bar{C}_3^{(q)}$ for $(z, w, t, a, b) \in A_{3,2}$ assume the form:

(22)
$$\begin{cases} \varkappa_x(u) = \varphi_2(u), \quad n = 3, \\ R_z = xv_x, \quad R_w = yv_x, \quad R_t = yv_y \quad (q + 1 = 3), \\ R_a = xv_xv_y, \quad R_b = yv_xv_y \quad (q + 1 = 4; \ n = 3). \end{cases}$$

§ 3. GEOMETRIC REALIZATION. THE ANALYTIC GENERALIZED HOPF INVARIANT. THE RIGIDITY PROPERTY

We use the geometric realization of the minimal model A indicated in § 1,

$$\psi: \ A \to \Lambda^*(M^n).$$

This geometric realization can also be subjected to a deformation $\psi \to \psi_\lambda$, where tlie images of all the generators $\psi(x_{j\alpha}) = \bar{x}_{j\alpha}$ are enlarged by following additional terms (by induction on the dimension): to tlie closed generators and, consequently, to the generators of lowest dimension exact forms are added

$$\bar{\varkappa}_{j\alpha} \to \bar{x}_{j,\alpha}(\lambda) = \bar{x}_{j\alpha} + \lambda \, d\theta_{j-1,\alpha}, \quad d\bar{x}_{j\alpha} = 0,$$

Suppose tliat tlie construction is completed for $j < m$. Then $d\bar{x}_{m\alpha} = P_{m\alpha}$,

$$d\bar{x}_{m\alpha}(\lambda) = P_{m\alpha}(\dots, \bar{x}_{q\beta}(\lambda), \dots), \qquad q < m.$$

Starting from the condition

$$d\psi = \psi\partial$$

we find a universal polynomial $Q_{m\alpha}(\dots, \bar{x}_{q\beta}, \lambda\theta_{q-1,\beta}, \dots)$ in all the symbols $\bar{x}_{q\beta}$, $\lambda\theta_{q-1,\alpha}$, $q < m$ such that $dQ_{m\alpha} = P_{m\alpha}(\lambda) - P_{m\alpha}(0)$.

Next we put

(23) $$\bar{x}_{m\alpha}(\lambda) = \bar{x}_{m\alpha} + Q_{m\alpha} + \lambda\, d\theta_{m-1,\alpha}.$$

Definition. We say that (23) defines a deformation ψ_λ of the geometric realization ψ.

We fix a realization ψ and consider an arbitrary C^∞-map

$$F\colon\ S^{q+1} \times \mathbb{R} \to M^n,$$

which induces a homomorphism of the minimal model

$$F^*\psi\colon\ A \to \Lambda^*(S^{q+1} \times \mathbb{R}).$$

Lemma 4. a) *There is a homomorphism of the "geometric realization" of the differential algebras*

$$\psi_F\colon\ \bar{C}_n^{(q)} \to \Lambda^*(S^{q+1} \times \mathbb{R}).$$

b) *For two distinct geometric realizations*

$$\psi_F^{(s)}\colon \bar{C}_n^{(q)} \to \Lambda^*(S^{q+1} \times \mathbb{R})$$

there is a homomorphism of differential algebras

$$\Phi_F\colon\ \bar{D}_n^{(q)} \to \Lambda^*(S^{q+1} \times \mathbb{R})$$

such that

$$\Phi_F/\bar{C}_n^{(q)} = \psi_F^{(0)},$$

$$\Phi_F/P\bar{C}_n^{(q)} = \psi_F^{(1)} \qquad (\lambda = 1),$$

that is, $\psi_F^{(1)}$ is obtained from $\psi_F^{(0)}$ by a λ-deformation for $\lambda = 1$.

The construction of ψ_F and Φ_F makes use of the exactness of all closed forms of dimension not exceeding q in the sphere $S^{q+1} \times \mathbb{R}$ and begins naturally with the realization of $C_n^{(q)}$. We note that $\psi_F\colon C_n^{(q)} \to \Lambda^*(S^{q+1} \times \mathbb{R})$ annihilates all the elements of $I_n C_n^{(q)}$, since $\psi_F(I_n) = 0$, etc.

We denote the images of all the $x_{j\alpha}$, $x_{k\beta}^{(s)}$, and $v_{l\gamma}^{(p)}$ under ψ_F also by $x_{j\alpha}$, $x_{k\beta}^{(s)}$, $v_{l\gamma}^{(p)} \in \Lambda^*(S^{q+1} \times \mathbb{R})$.

Theorem 1. *For any $z \in H^{q+1}(\bar{C}_n^{(q)})$ the formulae (20) determine the "analytic generalized Hopf invariant" $\mathcal{H}_z^{\mathrm{anal}}$ means of the integral*

(24) $$\mathcal{H}_z^{\mathrm{anal}}\{F\} = \int_{S^{q+1}} R_z(\dots, x_{j\alpha}, x_{k\beta}^{(s)}, v_{l\gamma}^{(p)}, \dots).$$

The formula (24) is homotopy-invariant and determines a linear form

$$\mathcal{H}_z^{\mathrm{anal}}\colon\ \pi_{q+1}(M^n) \otimes Q \to \mathbb{R}.$$

Then

$$\mathcal{H}_z^{\mathrm{anal}}(x) = c_{n,q}^\psi H_n^{(q)*}(z, x),$$

$$x \in \pi_{q+1}(M^n), \quad z \in H^{q+1}(\bar{C}_n^{(q)}).$$

Definition. If the constants $c_{n,q}^\psi$ remain unchanged under a deformation of the geometric realization ψ of the minimal model $A = A(M^n)$ in $\Lambda^*(M^n)$, then the element is said to be "*rigid*", $z \in H_{\mathrm{st}}^{q+1}(\bar{C}_n^{(q)})$.

Example. For $M^3 \sim S^2 \vee S^2$ the R_u are given by (22). In the geometric realization we have two closed forms $x, y \in \Lambda(M^3)$ representing $x, y \in H^2(M^3)$, $x, y \in A_3$. Next, we have the forms

$$\bar{x} = F^*(x), \quad \bar{y} = F^*(y), \quad dv_x = \bar{x}, \quad dv_y = \bar{y}.$$

According to (22) we obtain the densities of the "generalized analytic Hopf invariants" for

$$(25) \qquad \begin{cases} M^3 = R^3 \setminus (\mathcal{P}_1 \cup \mathcal{P}_2), \quad q+1 = 3, 4, \\ R_z = \bar{x}v_x, \quad R_w = \bar{y}v_x, \quad R_t = \bar{y}v_y, \\ R_a = \bar{x}v_xv_y, \quad R_b = \bar{y}v_xv_y. \end{cases}$$

Proposition. *The cocycles R_u for $u = (z, w, t, a, b)$ are rigid, that is, their integrals over a sphere of dimension S^{q+1} do not change under deformations of the minimal model in M^3.*

Let us prove this by way of illustration. Let τ_1 and τ_2 be 1-forms in M^3 that specify a deformation of the minimal model $x \to x + d\tau_1$, $y \to y + d\tau_2$. Then $v_x \to v_x + \bar{\tau}_1$, $\bar{\tau}_1 = F^*(\tau_1)$, $\bar{\tau}_2 = F^*(\tau_2)$.

a) Let $q + 1 = 3$. After the deformation we have

$$R'_z = (v_x + \bar{\tau}_1) \wedge (\bar{x} + d\bar{\tau}_1) = v_x\bar{x} + v_x\, d\bar{\tau}_1 + \bar{\tau}_1\bar{x} + \bar{\tau}_1\, d\bar{\tau}_1.$$

All these forms are closed in $\Lambda^*(S^3 \times \mathbb{R})$, where $F: S^3 \times \mathbb{R} \to M^3$. Any 3-form in M^3 is closed and exact, therefore $\bar{\tau}_1\bar{x}$ and $\bar{\tau}_1\, d\bar{\tau}_1$ are exact in $S^3 \times \mathbb{R}$.

We integrate $v_x\, d\bar{\tau}_1$ by parts:

$$v_x\, d\bar{\tau}_1 = -d(v\tau_1) + \bar{x}\bar{\tau}_1.$$

The second term on the right lies in $\operatorname{Im} F^*$ and is thus exact in $S^3 \times \mathbb{R}$.

The proof for w and t is quite similar.

b) Let $q + 1 = 4$. After the deformation we have

$$R'_a = (\bar{x} + d\bar{\tau}_1)(v_x + \bar{\tau}_1)(v_y + \bar{\tau}_2) = R_a + d\bar{\tau}_1 v_x v_y + \dots$$

All the omitted terms have the form $v_x F^*(\omega)$ or $v_y F^*(\omega)$, where ω is a 3-form in M^3. Since $d\omega = 0$ and $\omega = dh$ in M^3, integration by parts yields

$$v_x F^*(\omega) = -d(v_x\bar{h}) + \bar{x}\bar{h},$$

where $\bar{x}\bar{h} \equiv 0$, since $\dim M^3 < 4$. Forms like $v_x F^*(\omega)$ and $v_y F^*(\omega)$ are exact in $S^4 \times \mathbb{R}$. Next we consider the term $d\bar{\tau}_1 v_x v_y$ and integrate by parts:

$$d\bar{\tau}_1 v_x v_y = d(\bar{\tau}_1 v_x v_y) - \bar{\tau}_1\bar{x}v_y - \bar{\tau}_1 v_x\bar{y},$$

both additions have the form $v_x F^*(\omega)$ or $v_y F^*(\omega)$ and are thus exact, as above.

This completes the proof.

If we do not insist on rigidity, then our constructions can be extended in a natural way. Let $T \subset \Lambda^*(M^n)$ be a subalgebra that is closed under d. We construct its minimal differential extension $\bar{T} \subset \Lambda^*(M^n)$ such that the embedding homomorphism has kernel zero (is a monomorphism):

$$H^*(\bar{T}) \subset H^*(M^n).$$

This can be done by induction on the dimension, choosing in T a basis a_j of the kernel of the embedding of the cohomology of least dimension and adding to it a set of forms $d^{-1}(a_j)$ in $\Lambda^*(M^n)$, etc. If we choose the initial T by taking a set of closed forms in "general position" for a minimal multiplicative basis of $H^*(M^n)$,

then the algebra $\bar{T}$ is identical with the image $\psi(A)$, where A is Sullivan's minimal model (see §1). However, in a number of important special cases we can select subalgebras T that are not in general position. For example, we can take the algebra of two-sided invariant forms for symmetric spaces or of harmonic forms for Kähler manifolds (as in research on the $\mathbb{Q}$-type by Sullivan, Deligne, Griffiths, and Morgan) or subalgebras of them and a number of others. As before, we construct $\bar{C}^{(q)}(\bar{T})$ by means of a sequence of differential extensions

$$\bar{C}_0^{(q)} = \bar{T} \to \bar{C}_1^{(q)} \to \bar{C}_2^{(q)} \to \ldots \to \bar{C}^{(q)}(\bar{T}),$$

where all the embeddings are zero in the cohomology in dimensions not exceeding q (see §2).

Next, for smooth maps $S^{q+1} \times \mathbb{R} \to M^n$ we construct a geometric realization of the algebra $\bar{C}^{(q)}(\bar{T}) \subset \Lambda^*(S^{q+1} \times \mathbb{R})$, realizing all $\bar{C}_p^{(q)}(\bar{T})$, starting from $F^*(\bar{C}_0^{(q)}) = F^*(\bar{T})$ rather than from the minimal model.

Proposition. *Each element $z \in H^{q+1}(\bar{C}^{(q)}(\bar{T}))$ generates a linear form on the homotopy groups*

$$\mathcal{H}^{\mathrm{anal}} \colon \ \pi_{q+1}(M^n) \to \mathbb{R}.$$

In this construction we can talk of rigidity relative to deformations of a subalgebra $T_\lambda \subset \Lambda(M^n)$ or a homomorphism of some algebra $T \to \Lambda^*(M^n)$. The standard Hurewicz homomorphism becomes a special case of this construction (it is a rigid case; here T is the minimal model or the algebra of all forms). Another simple case is the subalgebra T consisting of the two forms Ω_k and Ω_l, $d\Omega_k = d\Omega_l = 0$, where $\Omega_k \wedge \Omega_l = 0$. The formula

$$\int_{S^{k+l+1}} v_{k-1} \wedge F^*(\Omega_l), \qquad dv_{k-1} = F^*(\Omega_k)$$

is homotopy-invariant and non-trivial on the Whitehead products (it is non-rigid).

§ 4. Many-valued functionals

We consider the space $\mathcal{L}$ of smooth maps $f \colon S^q \to M^n$ that are null-homotopic, $f \sim 0$. The group $\pi_q(\mathcal{L})$ is the same as $\pi_{q+1}(M^n)$.

Problem. To construct many-valued functionals $S\{f\}$ on $\mathcal{L}$, that is, closed 1-forms δS on $\mathcal{L}$ by a natural analytic method. To classify all such many-valued S for which δS is a local form depending on f and finitely many derivatives of it (that is, the Euler–Lagrange equation is a differential equation).

A natural analytic construction of many-valued functional follows from out previous results.

Theorem 2. *A many-valued functional on $\mathcal{L}$ is well-defined by (26):*

$$(26) \qquad\qquad S_u\{f\} = \int_{D^{q+1}} R^u$$

for any $u \in H^{q+1}(\bar{C}_n^{(q)})$ and a geometric realization ψ of the minimal model A in $\Lambda^*(M^n)$. Here $f \colon D^{q+1} \to M^n$ is a functional depending only on f on the boundary $\partial D^{q+1} = S^q$. If $H_n^{(q)*}(u) = 0$, then (26) is single-valued. For rigid elements, this condition is also necessary. In the subgroup of rigid elements

$$H_{\mathrm{st}}^{q+1} \subset H^{q+1}(\bar{C}_n^{(q)})$$

there is an integral lattice $\mathbb{Z}^m \subset H_{\mathrm{st}}^{q+1}$ independent of ψ such that the amplitude $\exp\{iS_u(j)\}$ for $u \in \mathbb{Z}^m$ is a single-valued functional ("quantization condition").

Remark 1. We always assume in what follows that all the operations $d^{-1}(\omega)$ in the construction of $S_u(f)$ are uniquely determined, where $dv = \omega$, $\delta v = *d*(v) = 0$ for all exact forms ω that occur in the constructions.

Remark 2. The simplest example $q = 2$, $M^2 = S^2$ is the density of the standard Hopf invariant. This many-valued functional $S_u\{f\}$ was mentioned to the author by Polyakov and Wigman who have shown that δS_u is local.

An important problem is to clarify when the variational derivative δS_u is local. "Local" many-valued functionals such that $S\{f\}$ depends on f and finitely many derivatives of it were determined by the author in 1981 (see [1]), and were then fully classified in §4 of the survey [2]. They have the form

$$(27) \qquad\qquad S\{f\} = S_0\{f\} + \int_{(N^q, f)} d^{-1}(\Omega),$$

where $f\colon N^q \to M^n$, N^q is closed, and Ω is an arbitrary closed $(q+1)$-form in M^n (more generally, there is a fibration $E \to N^q$ with base N^q and fibre M^n); the form Ω is taken in E, $d\Omega = 0$, and we consider the space of sections $f\colon N^q \to E$. Here, S_0 is an arbitrary single-valued local functional. This situation, however, can occur for $q < n - 1$. Suppose that $N^q = S^q$ and that the $\mathbb{Q}$-type of M^n is different from S^n.

Conjecture. *For $n < q + 1$ there are no other many-valued functionals with a single-valued local δS than those described by* (26).

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