On the Novikov conjecture

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Introduction

Signatures of quadratic forms play a central role in the classification theory of manifolds. The Hirzebruch theorem expresses the signature $\sigma(N) \in \mathbb{Z}$ of a $4k$-dimensional manifold N^{4k} in terms of the $\mathcal{L}$-genus $\mathcal{L}(N) \in H^{4*}(N;\mathbb{Q})$. The 'higher signatures' of a manifold M with fundamental group $\pi_1(M) = \pi$ are the signatures of the submanifolds $N^{4k} \subset M$ which are determined by the cohomology $H^*(B\pi;\mathbb{Q})$. The Novikov conjecture on the homotopy invariance of the higher signatures is of great importance in understanding the connection between the algebraic and geometric topology of high-dimensional manifolds. Progress in the field is measured by the class of groups π for which the conjecture has been verified. A wide variety of methods has been used to attack the conjecture, such as surgery theory, elliptic operators, C^*-algebras, differential geometry, hyperbolic geometry, bounded/controlled topology, and algebra.

The diffeomorphism class of a closed differentiable m-dimensional manifold M^m is distinguished in its homotopy type up to a finite number of possibilities by the rational Pontrjagin classes $p_*(M) \in H^{4*}(M;\mathbb{Q})$. Thom and Rochlin-Shvarc proved that the rational Pontrjagin classes $p_*(M)$ are

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combinatorial invariants by showing that they determine and are determined by the signatures of closed $4k$-dimensional submanifolds $N^{4k} \subset M \times \mathbb{R}^j$ (j large) with trivial normal bundle. A homotopy equivalence of manifolds only preserves the global algebraic topology, and so need not preserve the local algebraic topology given by the Pontrjagin classes. The Browder-Novikov-Sullivan-Wall surgery theory shows that modulo torsion invariants for $m \geq 5$ a homotopy equivalence of closed differentiable m-dimensional manifolds is homotopic to a diffeomorphism if and only if it preserves the signatures of submanifolds and the non-simply-connected surgery obstruction is in the image of the assembly map; this map is onto in the simply-connected case. (Here, torsion means both Whitehead groups and finite groups). Novikov proved the topological invariance of the rational Pontrjagin classes by showing that a homeomorphism preserves signatures of submanifolds with trivial normal bundles, using the fundamental group and non-compact manifold topology.

The object of this largely expository paper is to outline the relationship between the Novikov conjecture, the exotic spheres, the topological invariance of the rational Pontrjagin classes, surgery theory, codimension 1 splitting obstructions, the bounded/controlled topology of non-compact manifolds, the algebraic theory of Ranicki [45], [48], [49], and the method used by by Carlsson and Pedersen [14] to prove the conjecture for a geometrically defined class of infinite torsion-free groups π with $B\pi$ a finite complex and $E\pi$ a non-compact space with a sufficiently nice compactification. See Ferry, Ranicki and Rosenberg [19] for a wider historical survey of the Novikov conjecture.

The surgery obstruction groups $L_m(\mathbb{Z}[\pi])$ of Wall [56] are defined for any group π and $m(\bmod 4)$, to be the Witt group of $(-)^k$-quadratic forms over the group ring $\mathbb{Z}[\pi]$ for $m = 2k$, and a stable automorphism group of such forms for $m = 2k+1$. In [56] the groups $L_*(\mathbb{Z}[\pi])$ were understood to be the simple quadratic L-groups $L^s_*(\mathbb{Z}[\pi])$, the obstruction groups for surgery to simple homotopy equivalence, involving based f.g. free $\mathbb{Z}[\pi]$-modules and simple isomorphisms. Here, $L_*(\mathbb{Z}[\pi])$ are understood to be the free quadratic L-groups $L^h_*(\mathbb{Z}[\pi])$, the obstruction groups for surgery to homotopy equivalence, involving unbased f.g. free $\mathbb{Z}[\pi]$-modules and all isomorphisms. The simple and free L-groups differ in 2-torsion only, being related by the Rothenberg exact sequence

$$\ldots \longrightarrow L^s_m(\mathbb{Z}[\pi]) \longrightarrow L_m(\mathbb{Z}[\pi]) \longrightarrow \widehat{H}^m(\mathbb{Z}_2; Wh(\pi)) \longrightarrow L^s_{m-1}(\mathbb{Z}[\pi]) \longrightarrow \ldots$$

with $\widehat{H}^*(\mathbb{Z}_2; Wh(\pi))$ the (2-torsion) Tate $\mathbb{Z}_2$-cohomology groups of the duality involution on the Whitehead group $Wh(\pi)$. A normal map $(f,b) : M \longrightarrow N$ from an m-dimensional manifold M to an m-dimensional geometric Poincaré complex N with $\pi_1(N) = \pi$ has a surgery obstruction $\sigma_*(f,b) \in L_m(\mathbb{Z}[\pi])$ such that $\sigma_*(f,b) = 0$ if (and for $m \geq 5$ only if) (f,b) is normal bordant to a homotopy equivalence. The original treatment in [56] using forms and automorphisms was extended in Ranicki [45] to quadratic Poincaré complexes (= chain complexes with Poincaré duality). The surgery obstruction groups $L_*(\mathbb{Z}[\pi])$ were expressed in [45] as the cobordism groups of quadratic Poincaré complexes over $\mathbb{Z}[\pi]$.

The assembly maps in quadratic L-theory

$$A \ : \ H_*(X;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi_1(X)])$$

are defined in Ranicki [49] for any topological space X, abstracting a geometric construction of Quinn. The generalized homology groups $H_*(X;\mathbb{L}.(\mathbb{Z}))$ with coefficients in the simply-connected surgery spectrum $\mathbb{L}.(\mathbb{Z})$ are the cobordism groups of sheaves Γ over X of quadratic Poincaré complexes over $\mathbb{Z}$. For the purposes of this paper X can be taken to be a simplicial complex, and Γ can be taken to be a quadratic Poincaré cycle in the sense of [49]*. The assembly map A sends a quadratic Poincaré cycle Γ over X to the quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$

$$A(\Gamma) \ = \ q_!p^!\Gamma$$

with $p^!$ the pullback along the universal covering projection $p : \widetilde{X}\longrightarrow X$ and $q_!$ the pushforward along the unique map $q : \widetilde{X}\longrightarrow\{\text{pt.}\}$.

Novikov conjecture for a group π
The assembly maps for the classifying space $B\pi$

$$A \ : \ H_*(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$$

are rational split injections.

This will be called the **rational Novikov conjecture**, to distinguish it from:

Integral Novikov conjecture for a group π
The assembly maps $A : H_*(B\pi;\mathbb{L}.(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[\pi])$ are split injections.

The rational Novikov conjecture is trivially true for finite groups π; it has been verified for groups which have strong geometric properties.

The integral Novikov conjecture is known to be false for finite groups π; it has been verified for torsion-free groups which have strong geometric properties.

The verification of the integral Novikov conjecture π requires the construction of a 'disassembly' map

$$B \ : \ L_m(\mathbb{Z}[\pi]) \longrightarrow H_m(B\pi;\mathbb{L}.(\mathbb{Z})) \ ; \ C \longrightarrow B(C)$$

* The simplicial method applies to an arbitrary space X by considering algebraic Poincaré cycles over the simplicial complexes defined by the nerves of open covers of X. Hutt [25] has worked out the theory of sheaves of algebraic Poincaré complexes over an arbitrary space X.

such that $BA = 1$. Such a map B has to send a quadratic Poincaré complex C over $\mathbb{Z}[\pi]$ to a sheaf $B(C)$ over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$, with $BA(\Gamma)$ cobordant to Γ for any sheaf Γ over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$. It is possible to construct such B for any group π which has sufficient geometry that manifolds with fundamental group π have rigidity, meaning that homotopy equivalences can be deformed to homeomorphisms. Novikov [37] constructed B algebraically in the case of a free abelian group $\pi = \mathbb{Z}^n$, when $B\pi = T^n$ and A is an isomorphism. See Farrell and Jones [17] for a geometric construction of B in the case when $B\pi$ is realized by a compact aspherical Riemannian manifold all of whose sectional curvatures are nonpositive (when A is also an isomorphism), and the connection with the original Mostow rigidity theorem for hyperbolic manifolds.

The locally finite assembly maps in quadratic L-theory

$$A^{lf} \ : \ H^{lf}_*(X;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{C}_X(\mathbb{Z}))$$

are defined in Ranicki [49] for any metric space X, using the X-graded $\mathbb{Z}$-module category $\mathbb{C}_X(\mathbb{Z})$ of Pedersen and Weibel [41]. The locally finite generalized homology groups $H^{lf}_*(X;\mathbb{L}.(\mathbb{Z}))$ are the cobordism groups of locally finite sheaves Γ over X of quadratic Poincaré complexes over $\mathbb{Z}$. It was shown in Ranicki [48] that A^{lf} is an isomorphism for $X = O(K) \subseteq \mathbb{R}^{N+1}$ the open cone of a compact polyhedron $K \subseteq S^N$, which can be used to prove the topological invariance of the rational Pontrjagin classes (see 9.13 below). It is easier to establish that the locally finite assembly maps A^{lf} are isomorphisms than the ordinary assembly maps A. This is an algebraic reflection of the observed fact that rigidity theorems deforming homotopy equivalences to homeomorphisms are easier to prove for non-compact manifolds than for compact manifolds.

Carlsson and Pedersen [14] prove the integral Novikov conjecture for groups π with $B\pi$ a finite complex realized by a compact metric space such that the universal cover $E = E\pi$ admits a contractible π-equivariant compactification $\overline{E}$ with a metric such that compact sets in E become small when translated under π near the boundary $\partial E = \overline{E} \backslash E$. Bounded/controlled algebra is used to prove that A^{lf} is an isomorphism for $X = E$, and equivariant topology is used to construct an algebraic disassembly map B by means of $(A^{lf})^{-1}$. The conditions on the compactification allow E-bounded algebra/topology to be deformed to ∂E-controlled algebra/topology, i.e. to pass from homotopy equivalences to homeomorphisms. See Ferry and Weinberger [20] for a more geometric approach. The computation $Wh_{-*}(\{1\}) = 0$ of Bass, Heller and Swan [4] is an essential ingredient of both [14] and [20], since the lower K-groups of $\mathbb{Z}$ are potential obstructions to the disassembly of quadratic Poincaré complexes over $\mathbb{Z}$ in bounded algebra, or equivalently to compactifying simply-connected open manifolds in bounded topology.

In dealing with vector bundles, manifolds, homotopy equivalences, etc., only the oriented and orientation-preserving cases are considered. Manifolds are understood to be compact and differentiable, unless specified otherwise. Also, except for classifying spaces, only topological spaces which are finite-dimensional locally finite polyhedra or topological manifolds are considered.

§1. Pontrjagin classes and the $\mathcal{L}$-genus

The **Pontrjagin classes** of an m-plane bundle $\eta : X \longrightarrow BO(m)$ over a space X are integral characteristic classes

$$p_*(\eta) \in H^{4*}(X) .$$

The rational Pontrjagin character defines an isomorphism

$$\mathrm{ph} \ : \ KO(X)\otimes\mathbb{Q} \ = \ [X,\mathbb{Z}\times BO]\otimes\mathbb{Q} \xrightarrow{\simeq} H^{4*}(X;\mathbb{Q}) .$$

The $\mathcal{L}$-**genus** of an m-plane bundle $\eta : X \longrightarrow BO(m)$ is a rational cohomology class

$$\mathcal{L}(\eta) \in H^{4*}(X;\mathbb{Q})$$

whose components $\mathcal{L}_k(\eta) \in H^{4k}(X;\mathbb{Q})$ can be expressed as polynomials in the Pontrjagin classes $p_1, p_2, \dots$ with rational coefficients. The $\mathcal{L}$-genus determines and is determined by the rational Pontrjagin classes $p_k(\eta) \in H^{4k}(X;\mathbb{Q})$. The first two $\mathcal{L}$-polynomials are given by

$$\mathcal{L}_1 \ = \ \frac{1}{3}p_1 \ \ , \ \ \mathcal{L}_2 \ = \ \frac{1}{45}(7p_2-(p_1)^2) .$$

See Hirzebruch [23] and Milnor and Stasheff [32] for the textbook accounts of the Pontrjagin classes and the $\mathcal{L}$-genus.

The **Pontrjagin classes** and the $\mathcal{L}$-**genus** of an m-dimensional differentiable manifold M are the Pontrjagin classes and the $\mathcal{L}$-genus of the tangent m-plane bundle $\tau_M : M \longrightarrow BO(m)$

$$\begin{aligned} p_*(M) \ &= \ p_*(\tau_M) \in H^{4*}(M) , \\ \mathcal{L}(M) \ &= \ \mathcal{L}(\tau_M) \in H^{4*}(M;\mathbb{Q}) . \end{aligned}$$

By construction, the Pontrjagin classes and $\mathcal{L}$-genus are invariants of the differentiable structure of M: if $h : M' \longrightarrow M$ is a diffeomorphism then

$$\begin{aligned} \tau_{M'} \ &= \ h^*\tau_M \ : \ M' \longrightarrow BO(m) , \\ p_*(M') \ &= \ h^*p_*(M) \in H^{4*}(M') , \\ \mathcal{L}(M') \ &= \ h^*\mathcal{L}(M) \in H^{4*}(M';\mathbb{Q}) . \end{aligned}$$

§2. Signature

Definition 2.1 The **intersection form** of a closed $4k$-dimensional manifold N^{4k} is the nondegenerate symmetric form

$$\phi \ : \ H^{2k}(N;\mathbb{Q}) \times H^{2k}(N;\mathbb{Q}) \longrightarrow \mathbb{Q} \ ; \ (x,y) \longrightarrow \langle x \cup y, [N]\rangle$$

on the finite-dimensional $\mathbb{Q}$-vector space $H^{2k}(N;\mathbb{Q})$. The **signature** of N^{4k} is

$$\sigma(N) \ = \ \mathrm{signature}(H^{2k}(N;\mathbb{Q}),\phi) \in \mathbb{Z} \ .$$

□

Remarks 2.2 (i) An m-dimensional geometric Poincaré complex X is a finite CW complex with a fundamental class $[X] \in H_m(X)$ inducing isomorphisms

$$[X] \cap - \ : \ H^*(X) \xrightarrow{\simeq} H_{m-*}(X) \ .$$

Closed topological manifolds are the prime examples of geometric Poincaré complexes. The intersection form $(H^{2k}(X;\mathbb{Q}),\phi)$ and the signature $\sigma(X) \in \mathbb{Z}$ are defined for any $4k$-dimensional geometric Poincaré complex X, and are homotopy invariants of X.
(ii) The intersection form and signature are also defined for any $4k$-dimensional geometric Poincaré pair $(X,\partial X)$, such as a manifold with boundary $(M,\partial M)$.

□

Signature Theorem 2.3 (Hirzebruch) *The signature of a closed differentiable manifold N^{4k} is the evaluation of the $\mathcal{L}$-genus $\mathcal{L}(N) \in H^{4*}(N;\mathbb{Q})$ on $[N] \in H_{4k}(N;\mathbb{Q})$*

$$\sigma(N) \ = \ \langle \mathcal{L}(N), [N]\rangle \in \mathbb{Z} \ .$$

□

Transversality Theorem 2.4 *A continuous map $h : M'^m \longrightarrow M^m$ of differentiable m-dimensional manifolds is homotopic to a differentiable map. Given an n-dimensional submanifold $N^n \subset M^m$ it is possible to choose the homotopy in such a way that the differentiable map (also denoted by h) is transverse regular at N, with the restriction*

$$f \ = \ h| \ : \ N'^n \ = \ h^{-1}(N) \longrightarrow N^n$$

a degree 1 map of n-dimensional manifolds which is covered by a map of the normal $(m-n)$-plane bundles $b : \nu_{N' \subset M'} \longrightarrow \nu_{N \subset M}$.

□

Definition 2.5 A submanifold $N^n \subset M^m \times \mathbb{R}^j$ is **special** if it is closed, $n = 4k$ and the normal bundle is trivial

$$\nu_{N \subset M} \; = \; \epsilon^i \; : \; N \longrightarrow BSO(i) \quad (i = m + j - 4k) \; .$$

$\square$

Proposition 2.6 (Thom) *The rational Pontrjagin classes and the $\mathcal{L}$-genus of a manifold M are determined by the signatures of the special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$.*

Proof The Pontrjagin classes and the $\mathcal{L}$-genus of a special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$ are the images in $H^{4*}(N; \mathbb{Q})$ of the Pontrjagin classes and the $\mathcal{L}$-genus of M, that is

$$p_*(N) \; = \; e^* p_*(M) \quad , \quad \mathcal{L}(N) \; = \; e^* \mathcal{L}(M)$$

with

$$e \; : \; N \longrightarrow M \times \mathbb{R}^j \longrightarrow M \; .$$

The signature of N thus depends only on the homology class $e_*[N] \in H_{4k}(M; \mathbb{Q})$ represented by N

$$\begin{aligned} \sigma(N) \; &= \; \langle \mathcal{L}(N), [N] \rangle \\ &= \; \langle e^* \mathcal{L}(M), [N] \rangle \; = \; \langle \mathcal{L}(M), e_*[N] \rangle \in \mathbb{Z} \; . \end{aligned}$$

From now on, we shall write $e_*[N] \in H_{4k}(M; \mathbb{Q})$ as $[N]$. The cobordism classes of special submanifolds $N^{4k} \subset M^m \times \mathbb{R}^j$ are in one-one correspondence with the proper homotopy classes of proper maps

$$f \; : \; M \times \mathbb{R}^j \longrightarrow \mathbb{R}^i \quad (i \; = \; m + j - 4k)$$

with $N = f^{-1}(0)$ (assuming transverse regularity at $0 \in \mathbb{R}^i$). The set of proper homotopy classes is in one-one correspondence with the cohomotopy group $\pi^i(\Sigma^j M_+)$ of homotopy classes of maps $\Sigma^j M_+ \longrightarrow S^i$, with $\Sigma^j M_+$ the j-fold suspension of $M_+ = M \cup \{\text{pt.}\}$. By the Serre finiteness of the stable homotopy groups of spheres and Poincaré duality

$$\pi^i(\Sigma^j M_+) \otimes \mathbb{Q} \; = \; H^{m-4k}(M; \mathbb{Q}) \; = \; H_{4k}(M; \mathbb{Q}) \; .$$

The $\mathbb{Q}$-vector space $H_{4k}(M; \mathbb{Q})$ is thus spanned by the homology classes $[N]$ of special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$, and

$$\mathcal{L}(M) \in H^{4k}(M; \mathbb{Q}) \; = \; \mathrm{Hom}_{\mathbb{Q}}(H_{4k}(M; \mathbb{Q}), \mathbb{Q})$$

is given by

$$\begin{aligned} \mathcal{L}(M) \; &: H_{4k}(M; \mathbb{Q}) \longrightarrow \mathbb{Q} \; ; \\ & [N] \longrightarrow \langle \mathcal{L}(M), [N] \rangle \; = \; \langle \mathcal{L}(N), [N] \rangle \; = \; \sigma(N) \; . \end{aligned}$$

□

A PL homeomorphism of differentiable manifolds cannot in general be approximated by a diffeomorphism, by virtue of the exotic spheres.

Theorem 2.7 (Thom, Rochlin-Shvarc) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are combinatorial invariants.*

Proof Transversality also works in the PL category, so that the characterization (2.6) of the $\mathcal{L}$-genus in terms of signatures of special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$ can be carried out in the PL category. In particular, if $h : M' \longrightarrow M$ is a PL homeomorphism then

$$p_*(M') \;=\; h^* p_*(M) \;\;,\;\; \mathcal{L}(M') = h^*\mathcal{L}(M) \;.$$

□

Remark 2.8 Thom used PL transversality and the Hirzebruch signature theorem to define rational Pontrjagin classes $p_*(M)$ and the $\mathcal{L}$-genus $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$ for a PL manifold M. It is not possible to prove the topological invariance of the rational Pontrjagin classes by a mimicry of Thom's PL transversality argument: on the contrary, topological invariance is required for topological transversality.

□

Proposition 2.9 (Dold, Milnor) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are not homotopy invariants.*

Proof The stable classifying space G/O for fibre homotopy trivialized vector bundles is such that there is defined a fibration

$$G/O \longrightarrow BO \longrightarrow BG$$

with an exact sequence

$$\dots \longrightarrow \pi_{n+1}(BG) \longrightarrow \pi_n(G/O) \longrightarrow \pi_n(BO) \longrightarrow \pi_n(BG) \longrightarrow \dots \;.$$

The homotopy groups of the stable classifying space BG for spherical fibrations are the stable homotopy groups of spheres

$$\pi_*(BG) \;=\; \pi^S_{*-1} \;,$$

so that by Serre's finiteness theorem

$$\pi_*(BG)\otimes\mathbb{Q} \;=\; \pi^S_{*-1}\otimes\mathbb{Q} \;=\; 0 \;\;(*>1) \;.$$

By Bott periodicity $\pi_{4k}(BO) = \mathbb{Z}$, detected by the kth Pontrjagin class p_k. For any $k \geq 1$ there exists a fibre homotopy trivial $(j+1)$-plane bundle $\eta : S^{4k} \longrightarrow BO(j+1)$ (j large) over S^{4k} with

$$p_k(\eta) \;\neq\; 0 \in H^{4k}(S^{4k}) \;=\; \mathbb{Z} \;.$$

The sphere bundle $S(\eta)$ is a closed $(4k+j)$-dimensional manifold which is homotopy equivalent to $S(\epsilon^{j+1}) = S^{4k} \times S^j$, such that

$$\begin{aligned} p_k(S(\eta)) &= -p_k(\eta) \neq p_k(\epsilon^{j+1}) = 0 , \\ \mathcal{L}_k(S(\eta)) &= s_k p_k(S(\eta)) \neq \mathcal{L}_k(S(\epsilon^{j+1})) = 0 \\ &\in H^{4k}(S^{4k} \times S^j) = \mathbb{Z} \end{aligned}$$

with $s_k \neq 0 \in \mathbb{Z}$ the coefficient of p_k in $\mathcal{L}_k$. See 2.10 for a more detailed account.

□

Remark 2.10 Let Θ^m be the group of m-dimensional exotic differentiable spheres, and let $bP_{m+1} \subseteq \Theta^m$ be the subgroup of the exotic spheres Σ^m which occur as the boundary ∂W of a framed $(m+1)$-dimensional manifold W, as in Kervaire and Milnor [26]. For $m \geq 5$

$$\Theta^m = \pi_m(PL/O)$$

is a finite group. The classifying space PL/O for PL trivialized vector bundles fits into a fibration

$$PL/O \longrightarrow TOP/O \longrightarrow TOP/PL \simeq K(\mathbb{Z}_2, 3)$$

so that for $m \geq 5$

$$\Theta^m = \pi_m(PL/O) = \pi_m(TOP/O) ,$$

and the subgroup

$$\begin{aligned} bP_{m+1} &= \mathrm{im}(\pi_{m+1}(G/TOP) \longrightarrow \pi_m(TOP/O)) \\ &= \mathrm{im}(L_{m+1}(\mathbb{Z}) \longrightarrow \Theta^m) \subseteq \Theta^m \end{aligned}$$

is cyclic if m is odd, and is zero if m is even. The class $[\Sigma^m] \in bP_{m+1}$ of an exotic sphere Σ^m such that $\Sigma^m = \partial W$ for a framed $(m+1)$-dimensional manifold W is the image of the surgery obstruction

$$\sigma_*(f, b) \in \pi_{m+1}(G/TOP) = L_{m+1}(\mathbb{Z})$$

of the corresponding normal map $(f, b) : (W, \partial W) \longrightarrow (D^{m+1}, S^m)$ with $\partial f : \partial W \longrightarrow S^m$ a homotopy equivalence. We only consider the case $m = 4k-1$ here, with $k \geq 2$; the subgroup $bP_{4k} \subseteq \Theta^{4k-1}$ is cyclic of order

$$t_k = a_k 2^{2k-2}(2^{2k-1} - 1)\,\mathrm{num}(B_k/4k)$$

with B_k the kth Bernoulli number and $a_k = 1$ (resp. 2) if k is even (resp. odd). Let (W^{4k}, Σ^{4k-1}) be the framed $(2k-1)$-connected $4k$-dimensional manifold with homotopy $(4k-1)$-sphere boundary obtained by the E_8-plumbing of 8 copies of $\tau_{S^{2k}} : S^{2k} \longrightarrow BSO(2k)$, so that $[\Sigma^{4k-1}] \in bP_{4k}$ is a generator. Let

$$Q^{4k} \ = \ W^{4k} \cup c\Sigma^{4k-1}$$

be the framed $(2k-1)$-connected $4k$-dimensional PL manifold with signature $\sigma(Q) = 8$ obtained from (W^{4k}, Σ^{4k-1}) by coning off the boundary. The t_k-fold connected sum $\#_{t_k}\Sigma^{4k-1}$ is diffeomorphic to the standard $(4k-1)$-sphere S^{4k-1}, so that $\#_{t_k}Q^{4k}$ has a differentiable structure. The topological K-group of isomorphism classes of stable vector bundles over S^{4k}

$$\widetilde{KO}(S^{4k}) \ = \ \pi_{4k}(BO) \ = \ \pi_{4k}(BO(j+1)) \ \ (j \text{ large})$$

is such that there is defined an isomorphism

$$\pi_{4k}(BO) \xrightarrow{\simeq} \mathbb{Z} \ ; \ \eta \longrightarrow \langle p_k(\eta), [S^{4k}]\rangle / a_k(2k-1)! \ ,$$

by the Bott integrality theorem. The subgroup of fibre homotopy trivial bundles

$$\mathrm{im}(\pi_{4k}(G/O) \longrightarrow \pi_{4k}(BO)) \ = \ \ker(J : \pi_{4k}(BO) \longrightarrow \pi_{4k}(BG)) \subseteq \pi_{4k}(BO)$$

is the infinite cyclic subgroup of index

$$j_k \ = \ \mathrm{den}(B_k/4k)$$

with the generator $\eta : S^{4k} \longrightarrow BO(j+1)$ such that

$$p_k(\eta) \ = \ a_k j_k (2k-1)! \in H^{4k}(S^{4k}) \ = \ \mathbb{Z} \ .$$

For any fibre homotopy trivialization

$$h \ : \ J\eta \ \simeq \ J\epsilon^{j+1} \ : \ S^{4k} \longrightarrow BG(j+1)$$

the corresponding homotopy equivalence

$$S(h) \ : \ S(\eta) \xrightarrow{\simeq} S(\epsilon^{j+1}) \ = \ S^{4k} \times S^j$$

is such that the inverse image of $S^{4k} \times \{*\} \subset S^{4k} \times S^j$ is a submanifold of the type

$$N^{4k} \ = \ \#_{t_k} Q^{4k} \subset S(\eta) \ ,$$

and $S(h)$ restricts to a normal map

$$(f, b) \ = \ S(h)| \ : \ N^{4k} \longrightarrow S^{4k}$$

with $b:\nu_N \longrightarrow -\eta$. Moreover,

$$\begin{aligned}
\tau_N &= f^*(\eta) : N \longrightarrow BO(4k) , \\
p_k(N) &= f^*p_k(\eta) = a_k j_k (2k-1)! \in H^{4k}(N) = \mathbb{Z} , \\
\sigma(N) &= s_k p_k(N) = s_k a_k j_k (2k-1)! = 8t_k \in \mathbb{Z} ,
\end{aligned}$$

with

$$s_k = \frac{8t_k}{a_k j_k (2k-1)!} = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!}$$

the coefficient of p_k in $\mathcal{L}_k$. The homotopy equivalence $S(h): S(\eta) \longrightarrow S^{4k} \times S^j$ does not preserve the $\mathcal{L}$-genus, since

$$\begin{aligned}
\langle \mathcal{L}_k(S(\eta)), [N] \rangle &= \sigma(N) = 8t_k \\
&\neq \langle \mathcal{L}_k(S^{4k} \times S^j), [S^{4k}] \rangle = \sigma(S^{4k}) = 0 \in \mathbb{Z} .
\end{aligned}$$

(See 3.3 for more details in the special case $k=2$.) The homotopy equivalence $S(h): S(\eta) \longrightarrow S^{4k} \times S^j$ is not homotopic to a diffeomorphism since the surgery obstruction of (f,b) is

$$\begin{aligned}
\sigma_*(f,b) &= \frac{1}{8}(\sigma(N) - \sigma(S^{4k})) \\
&= t_k \neq 0 \in L_{4k}(\mathbb{Z}) = \mathbb{Z} .
\end{aligned}$$

These were the original examples due to Novikov [34] of homotopy equivalences of high-dimensional simply-connected manifolds which are not homotopic to diffeomorphisms. By the topological invariance of the rational Pontrjagin classes these homotopy equivalences are not homotopic to homeomorphisms.

□

§3. Splitting homotopy equivalences

Let M^m be an m-dimensional manifold, and let $N^n \subset M^m$ be an n-dimensional submanifold. Every map of m-dimensional manifolds $h: M' \longrightarrow M$ is homotopic to a map (also denoted by h) which is transverse regular at $N \subset M$, with the restriction

$$f = h| : N' = h^{-1}(N) \longrightarrow N$$

a degree 1 map of n-dimensional manifolds such that the normal $(m-n)$-plane bundle of N' in M' is the pullback along f of the normal $(m-n)$-plane bundle of N in M

$$\nu_{N' \subset M'} : N' \xrightarrow{f} N \xrightarrow{\nu_{N \subset M}} BSO(m-n) .$$

Let $i : N \longrightarrow M$, $i' : N' \longrightarrow M'$ be the inclusions. For any embedding $M' \subset S^{m+k}$ (k large) define a map of $(m-n+k)$-plane bundles covering f

$$\begin{aligned} b \;:\; \nu_{N'\subset S^{m+k}} \;&=\; \nu_{N'\subset M'} \oplus i'^*(\nu_{M'\subset S^{m+k}}) \\ &\longrightarrow \eta \;=\; \nu_{N\subset M} \oplus (h^{-1}i)^*(\nu_{M'\subset S^{m+k}})\ , \end{aligned}$$

so that $(f,b) : N' \longrightarrow N$ is a normal map. If $h : M' \longrightarrow M$ is a homotopy equivalence it need not be the case that (f,b) is a homotopy equivalence.

Definition 3.1 (i) A homotopy equivalence $h : M' \longrightarrow M$ of manifolds **splits** along a submanifold $N \subset M$ if h is homotopic to a map (also denoted h) which is transverse regular along $N \subset M$, and such that the restriction $h| : N' = h^{-1}(N) \longrightarrow N$ is a homotopy equivalence.
(ii) A homotopy equivalence $h : M' \longrightarrow M$ of manifolds ***h*-splits** along a submanifold $N \subset M$ if there exists an extension of $h : M' \longrightarrow M$ to a homotopy equivalence

$$(g;h,h') \;:\; (W;M',M'') \longrightarrow M \times ([0,1];\{0\},\{1\})$$

with $(W;M',M'')$ an h-cobordism and $h' : M'' \longrightarrow M$ split along $N \subset M$.
□

For $m \geq 5$ a homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds splits along a submanifold $N^n \subset M^m$ if and only if $h : M' \longrightarrow M$ h-splits with $\tau(M' \longrightarrow W) = 0 \in Wh(\pi_1(M))$, by the s-cobordism theorem.

See Chapter 23 of Ranicki [49] for an account of the Browder-Wall surgery obstruction theory for splitting homotopy equivalences along submanifolds. Here is a brief summary:

Proposition 3.2 (i) *If a homotopy equivalence of manifolds $h : M' \longrightarrow M$ is homotopic to a diffeomorphism then h splits along every submanifold $N \subset M$ and $\tau(h) = 0 \in Wh(\pi_1(M))$. A homotopy equivalence which does not split along a submanifold or is such that $\tau(h) \neq 0$ cannot be homotopic to a diffeomorphism.*
(ii) *The (free) LS-groups LS_* of Wall [56, §11] are defined for a manifold M^m and a submanifold $N^n \subset M^m$ with normal bundle*

$$\xi \;=\; \nu_{N\subset M} \;:\; N \longrightarrow BO(q) \;\; (q \;=\; m-n)$$

to fit into a commutative braid of exact sequences

$$
\begin{tikzcd}
H_i(M;\mathbb{L}.) \arrow[rr, bend left, "A"] \arrow[dr] & & L_i(\mathbb{Z}[\pi_1(M)]) \arrow[rr, bend left] \arrow[dr] & & LS_{i-q-1} \\
& L_i(\xi^{\,!}) \arrow[ur] \arrow[dr] & & \mathbb{S}_i(M) \arrow[ur] \arrow[dr] & \\
LS_{i-q} \arrow[ur] \arrow[rr, bend right] & & \mathbb{S}_i(\xi^{\,!}) \arrow[ur] \arrow[rr, bend right] & & H_{i-1}(M;\mathbb{L}.)
\end{tikzcd}
$$

with A the algebraic L-theory assembly map (§7), $L_*(\xi^!)$ the relative L-groups in the transfer exact sequence

$$
\begin{aligned}
\ldots \longrightarrow L_i(\mathbb{Z}[\pi_1(M\backslash N)]) \longrightarrow L_i(\xi^!) \longrightarrow L_{i-q}(\mathbb{Z}[\pi_1(N)]) \\
\longrightarrow L_{i-1}(\mathbb{Z}[\pi_1(M\backslash N)]) \longrightarrow L_{i-1}(\xi^!) \longrightarrow
\end{aligned}
$$

and similarly for $\mathbb{S}_*(\xi^!)$. The structure invariant $s(h) \in \mathbb{S}_{m+1}(M)$ (7.1) of a homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds has image $[s(h)] \in LS_n$, which has image $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(N)])$ the surgery obstruction of the n-dimensional normal map given by transversality

$$(f,b) \;=\; h| \;:\; N' \;=\; h^{-1}(N) \longrightarrow N \ .$$

For $n \geq 5$, $q \geq 1$ $h : M' \longrightarrow M$ h-splits along $N \subset M$ if and only if $[s(h)] = 0 \in LS_n$. For $q \geq 3$

$$
\begin{aligned}
\pi_1(M) \;&=\; \pi_1(N) \;=\; \pi_1(M\backslash N) \ \ , \\
L_*(\xi^!) \;&=\; L_*(\mathbb{Z}[\pi_1(M)]) \oplus L_{*-q}(\mathbb{Z}[\pi_1(M)]) \ , \\
LS_* \;&=\; L_*(\mathbb{Z}[\pi_1(M)])
\end{aligned}
$$

and

$$[s(h)] \;=\; \sigma_*(f,b) \in LS_n \;=\; L_n(\mathbb{Z}[\pi_1(M)])$$

so that for $n \geq 5$ $h : M' \longrightarrow M$ h-splits if and only if $\sigma_*(f,b) = 0 \in L_n(\mathbb{Z}[\pi_1(M)])$.

$\square$

The lens spaces give rise to homotopy equivalences $h : M' \longrightarrow M$ of manifolds in dimensions ≥ 3 with $\tau(h) \neq 0 \in Wh(\pi_1(M))$.

The exotic spheres give rise to homotopy equivalences $h : M' \longrightarrow M$ of manifolds which do not split along submanifolds. The following example gives an explicit homotopy equivalence $h : M'^m \longrightarrow M^m$ which does not split along a special submanifold $N^{4k} \subset M^m$ in the simply-connected case $\pi_1(N) = \pi_1(M) = \{1\}$.

Example 3.3 Take $k = 2$ in 2.10, with

$$a_2 \;=\; 1 \;\;,\;\; B_2 \;=\; \frac{1}{30} \;\;,\;\; j_2 \;=\; 240 \;\;,\;\; s_2 \;=\; \frac{7}{45} \;\;,\;\; t_2 \;=\; 56 \;.$$

Let (W^8, Σ^7) be the framed 3-connected 8-dimensional differentiable manifold with signature $\sigma(W) = 8$ obtained by the E_8-plumbing of 8 copies of $\tau_{S^4} : S^4 \longrightarrow BO(4)$, with boundary $\partial W = \Sigma^7$ the homotopy 7-sphere generating the exotic sphere group $\Theta^7 = \mathbb{Z}_{28}$. The 28-fold connected sum $\#_{28}\Sigma^7$ is diffeomorphic to the standard 7-sphere S^7. Let $\eta : S^8 \longrightarrow BO(q+1)$ (q large) be a fibre homotopy trivial $(q+1)$-plane bundle over S^8 such that

$$\eta \in \ker(J : \pi_8(BO) \longrightarrow \pi_S^7) \;=\; 240\mathbb{Z} \subset \pi_8(BO) \;=\; \mathbb{Z}$$

is the generator with

$$p_2(\eta) \;=\; -1440 \in H^8(S^8) \;=\; \mathbb{Z} \;.$$

The sphere bundle is a closed $(8+q)$-dimensional differentiable manifold $M' = S(\eta)$ with a homotopy equivalence

$$h \;:\; M' \;=\; S(\eta) \longrightarrow M \;=\; S(\epsilon^{q+1}) \;=\; S^8 \times S^q$$

which does not split along the special submanifold

$$N^8 \;=\; S^8 \times \{\mathrm{pt.}\} \subset M^{8+q} \;=\; S^8 \times S^q \;.$$

The inverse image of N is the special submanifold

$$N'^{\,8} \;=\; h^{-1}(N) \;=\; \#_{28} W \cup D^8 \subset M'^{\,8+j}$$

with

$$\begin{aligned}
\mathcal{L}_2(M') \;=\; \sigma(N') \;=\; \frac{7}{45}\langle -p_2(\eta), [S^8]\rangle \;=\; 28\cdot\sigma(W) \;=\; 224 \\
\neq\; h^*\mathcal{L}_2(M) \;=\; \sigma(N) \;=\; 0 \;\in H^8(M';\mathbb{Q}) \;=\; \mathbb{Q} \;, \\
p_2(M') \;=\; 1440 \neq h^*p_2(M) \;=\; 0 \in H^8(M') \;=\; \mathbb{Z} \;.
\end{aligned}$$

The codimension q splitting obstruction of h along $N \subset M$ is the surgery obstruction of the 8-dimensional normal map

$$(f,b) \;=\; h| \;:\; N' \longrightarrow N \;,$$

which is

$$\begin{aligned}[s(h)] &= \sigma_*(f,b) = \frac{1}{8}(\sigma(N')-\sigma(N)) \\ &= 28 \in LS_8 = L_8(\mathbb{Z}) = \mathbb{Z}\ .\end{aligned}$$

□

Example 3.4 If $m-4k \geq 3$ and $k \geq 2$ a homotopy equivalence $h : M' \longrightarrow M$ of simply-connected m-dimensional manifolds splits along a simply-connected $4k$-dimensional submanifold $N^{4k} \subset M$ if and only if the surgery obstruction

$$\sigma_*(f,b) = \frac{1}{8}(\sigma(N')-\sigma(N)) \in LS_{4k} = L_{4k}(\mathbb{Z}) = \mathbb{Z}$$

is 0, which for special $N \subset M$ is equivalent to

$$\langle (h^{-1})^*\mathcal{L}_k(M') - \mathcal{L}_k(M), [N]\rangle = 0 \in \mathbb{Q}\ .$$

□

Codimension 1 splitting obstruction theory is particularly significant for the topological invariance of the rational Pontrjagin classes and the Novikov conjectures. See §8 below for an account of the codimension 1 theory for homotopy equivalences of compact manifolds. In §10 there is a corresponding account for proper homotopy equivalences of open manifolds, making use of the evident modification of Definition 3.1 :

Definition 3.5 (i) A proper homotopy equivalence $h : W' \longrightarrow W$ of open manifolds **splits** along a closed submanifold $N \subset W$ if h is proper homotopic to a map (also denoted h) which is transverse regular along $N \subset W$, and such that the restriction $h| : N' = h^{-1}(N) \longrightarrow N$ is a homotopy equivalence.
(ii) A proper homotopy equivalence $h : W' \longrightarrow W$ of open manifolds ***h*-splits** along a closed submanifold $N \subset W$ if there exists an extension of $h : W' \longrightarrow W$ to a proper homotopy equivalence

$$(g; h, h') : (V; W', W'') \longrightarrow W \times ([0,1]; \{0\}, \{1\})$$

with $(V; W', W'')$ a proper h-cobordism and $h' : W'' \longrightarrow W$ split along $N \subset W$.

□

See Ranicki [42] for an algebraic development of the projective L-groups $L^p_*(\mathbb{Z}[\pi])$, which are related to the free L-groups $L_*(\mathbb{Z}[\pi])$ by a Rothenberg-type exact sequence

$$\begin{aligned}\dots \longrightarrow L_m(\mathbb{Z}[\pi]) \longrightarrow L^p_m(\mathbb{Z}[\pi]) &\longrightarrow \widehat{H}^m(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \\ &\longrightarrow L_{m-1}(\mathbb{Z}[\pi]) \longrightarrow \dots\ .\end{aligned}$$

See Pedersen and Ranicki [40] for a geometric interpretation of projective L-theory in terms of normal maps from compact manifolds to finitely dominated geometric Poincaré complexes.

Proposition 3.6 *Let $h : W' \longrightarrow W = N \times \mathbb{R}$ be a proper homotopy equivalence of open m-dimensional manifolds, with N a closed $(m-1)$-dimensional manifold. Let*

$$(f, b) \ = \ h| \ : \ N' \ = \ h^{-1}(N \times \{0\}) \longrightarrow N$$

be the normal map of closed $(m-1)$-dimensional manifolds obtained by transversality, with

$$W'^{+} \ = \ h^{-1}(N \times \mathbb{R}^{+}) \ , \quad W'^{-} \ = \ h^{-1}(N \times \mathbb{R}^{-}) \subset W'$$

such that

$$h \ = \ h^{+} \cup_f h^{-} \ : \ W' \ = \ W'^{+} \cup_{N'} W'^{-} \longrightarrow W \ = \ (N \times \mathbb{R}^{+}) \cup_{N \times \{0\}} (N \times \mathbb{R}^{-})$$

and

$$\pi_1(N) \ = \ \pi_1(N') \ = \ \pi_1(W'^{+}) \ = \ \pi_1(W'^{-}) \ (= \ \pi \ \textit{say}) \ .$$

(i) *The spaces W'^{+}, W'^{-} are finitely dominated, and the Wall finiteness obstruction*

$$[W'^{+}] \ = \ -[W'^{-}] \ = \ (-)^m [W'^{+}]^* \in \widetilde{K}_0(\mathbb{Z}[\pi])$$

is the splitting obstruction, such that $[W'^{+}] = 0$ if (and for $m \geq 6$) only if h splits along $N \times \{0\} \subset W = N \times \mathbb{R}$.
(ii) *The Tate $\mathbb{Z}_2$-cohomology class*

$$[W'^{+}] \in \widehat{H}^m(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$$

is the proper h-splitting obstruction, such that $[W'^{+}] = 0$ if (and for $m \geq 6$) only if $h : W' \longrightarrow W$ h-splits along $N \times \{0\} \subset W = N \times \mathbb{R}$.
(iii) *The surgery obstruction of (f, b) is the image of the Tate $\mathbb{Z}_2$-cohomology class of $[W'^{+}]$*

$$\begin{aligned} \sigma_*(f, b) \ &= \ [W'^{+}] \in \mathrm{im}(\widehat{H}^m(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \longrightarrow L_{m-1}(\mathbb{Z}[\pi])) \\ &= \ \ker(L_{m-1}(\mathbb{Z}[\pi]) \longrightarrow L^p_{m-1}(\mathbb{Z}[\pi])) \ . \end{aligned}$$

Proof (i)+(ii) The finiteness obstruction for arbitrary $\pi_1(N)$ is just the end invariant of Siebenmann [53], and is the obstruction to killing $\pi_*(W'^{+}, N')$ by handle exchanges (= ambient surgeries).

(iii) Let $\widetilde{W}'^{+}, \widetilde{W}'^{-}, \widetilde{N}, \widetilde{N}'$ be the universal covers of W'^{+}, W'^{-}, N, N' respectively. The homology $\mathbb{Z}[\pi]$-modules are such that

$$H_*(\widetilde{N}') \;=\; H_*(\widetilde{N}) \oplus H_{*+1}(\widetilde{W}'^{+}, \widetilde{N}') \oplus H_{*+1}(\widetilde{W}'^{-}, \widetilde{N}')$$

and the quadratic Poincaré kernel of (f, b) (Ranicki [45]) is the hyperbolic $(m-1)$-dimensional quadratic Poincaré complex on

$$C(\widetilde{W}'^{+}, \widetilde{N}')_{*+1} \oplus C(\widetilde{W}'^{-}, \widetilde{N}')_{*+1} \;\simeq\; C(\widetilde{W}'^{+}, \widetilde{N}')_{*+1} \oplus C(\widetilde{W}'^{+}, \widetilde{N}')^{m-*}$$

which is equipped with a projective null-cobordism.

□

Remarks 3.7 (i) 3.6 (i) is a special case of the codimension 1 bounded splitting Theorem 10.1. The unobstructed case $\pi = \{1\}$ is the splitting result of Browder [6].
(ii) The projective L-groups are such that

$$L_m(\mathbb{Z}[\pi][z, z^{-1}]) \;=\; L_m(\mathbb{Z}[\pi]) \oplus L^p_{m-1}(\mathbb{Z}[\pi])$$

with

$$\begin{aligned}\sigma_*((f, b) \times 1_{S^1}) \;&=\; (0, \sigma^p_*(f, b)) \\ &\in L_m(\mathbb{Z}[\pi][z, z^{-1}]) \;=\; L_m(\mathbb{Z}[\pi]) \oplus L^p_{m-1}(\mathbb{Z}[\pi])\end{aligned}$$

for any normal map (f, b) of finitely dominated $(m-1)$-dimensional geometric Poincaré complexes with fundamental group π (Ranicki [43]).
(iii) The vanishing of the projective surgery obstruction in 3.6 (ii)

$$\sigma^p_*(f, b) \;=\; 0 \in L^p_{m-1}(\mathbb{Z}[\pi])$$

corresponds to the vanishing of the free surgery obstruction

$$\sigma_*((f, b) \times 1_{S^1}) \;=\; 0 \in L_m(\mathbb{Z}[\pi][z, z^{-1}]) \;.$$

For $m \geq 5$ this is realized by the geometric wrapping up construction (Hughes and Ranicki [24]) of an $(m+1)$-dimensional normal bordism

$$(F, B) \;:\; (L; N' \times S^1, \partial_+ L) \longrightarrow N \times S^1 \times ([0, 1]; \{0\}, \{1\})$$

with $\partial_+ F = F| : \partial_+ L \longrightarrow N \times S^1$ a homotopy equivalence and

$$\begin{aligned}(F, B)| \;&=\; (f, b) \times 1_{S^1} \;:\; N' \times S^1 \longrightarrow N \times S^1 \;, \\ (L \backslash \partial_+ L, N' \times S^1) \;&=\; (W'^{+}, N') \times S^1 \;, \\ \tau(\partial_+ F) \;&=\; [W'^{+}] \in \mathrm{im}(\widetilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z}))\end{aligned}$$

with

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \longrightarrow Wh(\pi \times \mathbb{Z}) \ ; \ [P] \longrightarrow \tau(-z : P[z, z^{-1}] \longrightarrow P[z, z^{-1}])$$

the geometrically significant variant of the injection of Bass [3, XII]. The infinite cyclic covering of (F, B) induced from the universal covering $\mathbb{R} \longrightarrow S^1$

$$(\overline{F}, \overline{B}) \ : \ (\overline{L}; N' \times \mathbb{R}, \overline{\partial_+ L}) \longrightarrow N \times \mathbb{R} \times ([0,1]; \{0\}, \{1\})$$

is homotopy equivalent to an extension of (f, b) to a finitely dominated m-dimensional geometric Poincaré bordism

$$(F_1, B_1) \ : \ (W'^+; N', N) \longrightarrow N \times ([0,1]; \{0\}, \{1\})$$

with $(F_1, B_1)| = 1 : N \longrightarrow N$.

(iv) By the codimension 1 splitting theorem of Farrell and Hsiang [15] (8.1) the Whitehead torsion $\tau(h) \in Wh(\pi \times \mathbb{Z})$ of a homotopy equivalence $h : M' \longrightarrow M = N \times S^1$ of closed m-dimensional manifolds is such that

$$\tau(h) \in \mathrm{im}(Wh(\pi) \longrightarrow Wh(\pi \times \mathbb{Z})) \ \ (\pi \ = \ \pi_1(N))$$

if (and for $m \geq 6$ only if) h splits along $N \times \{*\} \subset N \times S^1$. The projection $Wh(\pi \times \mathbb{Z}) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi])$ of Bass [3, XII] sends the Whitehead torsion $\tau(h) \in Wh(\pi \times \mathbb{Z})$ to the splitting obstruction of 3.6

$$[\tau(h)] \ = \ [\overline{M}'^{+}] \in \widetilde{K}_0(\mathbb{Z}[\pi])$$

for the proper homotopy equivalence $\overline{h} : \overline{M}' \longrightarrow \overline{M} = N \times \mathbb{R}$ obtained from h by pullback from the universal cover $\mathbb{R} \longrightarrow S^1$. The h-splitting obstruction of $h : M' \longrightarrow M$ is the Tate $\mathbb{Z}_2$-cohomology class

$$[\tau(h)] \ = \ [\overline{M}'^{+}] \in LS_{m-1} \ = \ \widehat{H}^m(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi])) \ .$$

(The identification $LS_{m-1} = \widehat{H}^m(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi]))$ is the h-version of the identification $LS^s_{m-1} = \widehat{H}^m(\mathbb{Z}_2; Wh(\pi))$ obtained by Wall [56, Thm. 12.5] for the corresponding codimension 1 s-splitting obstruction group).

□

§4. Topological invariance

A homeomorphism of differentiable manifolds cannot in general be approximated by a diffeomorphism, by virtue of the exotic spheres. Thus it is not at all obvious that the $\mathcal{L}$-genus $\mathcal{L}(M)$ and the rational Pontrjagin classes $p_*(M)$ are topological invariants of a differentiable manifold M. Surgery theory for simply-connected compact manifolds is adequate for the construction and classification of exotic spheres. The topological invariance of the rational Pontrjagin classes requires surgery on non-simply-connected compact manifolds and/or simply-connected non-compact manifolds. The original proof due to Novikov [24] made use of the torus, as subsequently formalized in Ranicki [37, Appendix C16] using bounded L-theory: if $h : M' \longrightarrow M$ is a homeomorphism of manifolds then for any $j \geq 1$ the homeomorphism $h \times 1 : M' \times \mathbb{R}^j \longrightarrow M \times \mathbb{R}^j$ can be approximated by a differentiable $\mathbb{R}^j$-bounded homotopy equivalence, and the signatures of special submanifolds are $\mathbb{R}^j$-bounded homotopy invariants. (See §9 for a brief account of bounded surgery theory). Recently, Gromov [21] obtained a new proof of the topological invariance using the non-multiplicativity of the signature on surface bundles instead of torus geometry and the algebraic K- and L-theory of the group rings of the free abelian groups. I am grateful to Gromov for sending me a copy of [21]. In 4.1 the two methods are related to each other using algebraic surgery theory.

Theorem 4.1 (Novikov [36]) *The rational Pontrjagin classes and the $\mathcal{L}$-genus are topological invariants.*

Proof By 2.6 it suffices to prove that the signatures of special submanifolds are homeomorphism invariant, i.e. that if $h : M'^m \longrightarrow M^m$ is a homeomorphism of differentiable (or PL) manifolds then

$$\sigma(N) \;=\; \sigma(N') \in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}$$

for any special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$, with

$$N' \;=\; h'^{-1}(N) \subset M' \times \mathbb{R}^j$$

the transverse inverse image of any differentiable (or PL) approximation $h' : M' \times \mathbb{R}^j \longrightarrow M \times \mathbb{R}^j$ to $h \times 1_{\mathbb{R}^j}$. Every special submanifold is (ambient) cobordant to a simply-connected one, so it may be assumed that N is simply-connected, $\pi_1(N) = \{1\}$. The surgery obstruction of the $4k$-dimensional normal map

$$(f, b) \;=\; h'| \;:\; N' \longrightarrow N$$

is

$$\sigma_*(f, b) \;=\; \frac{1}{8}(\sigma(N') - \sigma(N)) \in L_{4k}(\mathbb{Z}) \;=\; \mathbb{Z}$$

which is a codimension i splitting obstruction, with $i = m + j - 4k$. There are at least four distinct ways of showing that $\sigma_*(f, b) = 0$:

1. use $\mathbb{R}^i$-bounded L-theory as in Ranicki [48], [49], and the computation $L_*(\mathbb{C}_{\mathbb{R}^i}(\mathbb{Z})) = L_{*-i}(\mathbb{Z})$,
2. as in the original proof of Novikov [36] use $T^{i-1} \subset \mathbb{R}^i$ and the computation $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{i-1}]) = 0$,

3. use $T^{i-1}\subset\mathbb{R}^i$ and the computation $L_*(\mathbb{Z}[\mathbb{Z}^{i-1}])=H_*(T^{i-1};\mathbb{L}_\bullet(\mathbb{Z}))$ of Novikov [37] and Ranicki [43]*,
4. follow Gromov [21] and use a hypersurface $B^{i-1}\subset\mathbb{R}^i$ (assuming i is odd) with a fibre bundle $F^{i-1}\longrightarrow E\longrightarrow B^{i-1}$ such that the total space has signature $\sigma(E)\neq 0$.

For 1. note that the homeomorphism

$$h_0 \;=\; (h\times 1_{\mathbb{R}^j})| \;:\; (h\times 1_{\mathbb{R}^j})^{-1}(N\times\mathbb{R}^i)\longrightarrow N\times\mathbb{R}^i$$

can be approximated by a differentiable $\mathbb{R}^i$-bounded homotopy equivalence which is normal bordant to the $\mathbb{R}^i$-bounded normal map

$$(f,b)\times 1_{\mathbb{R}^i} \;:\; N'\times\mathbb{R}^i\longrightarrow N\times\mathbb{R}^i\ ,$$

so that

$$\begin{aligned}\sigma_*(f,b) \;=\; \sigma_*((f,b)\times 1_{\mathbb{R}^i}) \;&=\; \sigma_*(h_0) \;=\; 0\\ &\in L_{4k}(\mathbb{Z}) \;=\; L_{4k+i}(\mathbb{C}_{\mathbb{R}^i}(\mathbb{Z})) \;=\; \mathbb{Z}\end{aligned}$$

– see 9.14 for a (somewhat) more detailed account.

For 2. proceed as in [36], making repeated use of codimension 1 splitting (3.6). To start with, approximate the homeomorphism

$$h_1 \;=\; (h\times 1_{\mathbb{R}^j})| \;:\; W_1 \;=\; (h\times 1_{\mathbb{R}^j})^{-1}(N\times T^{i-1}\times\mathbb{R})\longrightarrow N\times T^{i-1}\times\mathbb{R}$$

by a differentiable proper homotopy equivalence $h_1' : W_1\longrightarrow N\times T^{i-1}\times\mathbb{R}$. Since $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{i-1}])=0$ it is possible to split h_1' along $N\times T^{i-1}\times\{0\}\subset N\times T^{i-1}\times\mathbb{R}$, with the restriction

$$f_1 \;=\; h_1'| \;:\; N_1 \;=\; h_1'^{-1}(N\times T^{i-1}\times\{0\})\longrightarrow N\times T^{i-1}$$

a homotopy equivalence normal bordant to

$$(f,b)\times 1_{T^{i-1}} \;:\; N'\times T^{i-1}\longrightarrow N\times T^{i-1}\ .$$

* The full L-theoretic computation $L_*(\mathbb{Z}[\mathbb{Z}^{i-1}])=H_*(T^{i-1};\mathbb{L}_\bullet(\mathbb{Z}))$ requires the K-theoretic computation $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{i-1}])=0$, but for the topological invariance of the rational Pontrjagin classes it suffices to know that the map $-\otimes\sigma^*(T^{i-1}) : L_{4*}(\mathbb{Z})=\mathbb{Z}\longrightarrow L_{4*+i-1}(\mathbb{Z}[\mathbb{Z}^{i-1}])$ is a rational injection – this is a formal consequence of the splitting theorem $L^h_*(A[z,z^{-1}])=L^h_*(A)\oplus L^p_{*-1}(A)$ of [37] and [43] applied inductively to $\mathbb{Z}[\mathbb{Z}^{i-1}]=\mathbb{Z}[\mathbb{Z}^{i-2}][z,z^{-1}]$, and the identity $L^h_*(A)[1/2]=L^p_*(A)[1/2]$ given by the Rothenberg-type sequence relating the projective and free L-groups of a ring with involution A.

Pass to the infinite cyclic cover $\overline{T}^{i-1} = T^{i-2} \times \mathbb{R}$ of $T^{i-1} = T^{i-2} \times S^1$ and apply the same procedure to the proper homotopy equivalence

$$h_2 \ = \ \overline{f}_1 \ : \ W_2 \ = \ \overline{N}_1 \longrightarrow N \times \overline{T}^{i-1} \ = \ N \times T^{i-2} \times \mathbb{R} \ .$$

After $i-1$ applications of 3.6 there is obtained a homotopy equivalence of $4k$-dimensional manifolds $f_i : N_i \longrightarrow N$ normal bordant to $(f,b) : N' \longrightarrow N$, so that

$$\sigma_*(f,b) \ = \ \sigma_*(f_i) \ = \ 0 \in L_{4k}(\mathbb{Z}) \ .$$

(Alternatively, apply the splitting theorem of Farrell and Hsiang [15] i times – cf. 3.7 (iii)).

For 3. and 4. suppose given a closed hypersurface $U^{i-1} \subset \mathbb{R}^i$ with a neighbourhood $U \times \mathbb{R} \subset \mathbb{R}^i$, regard $N \times U \times \mathbb{R}$ as a codimension 0 submanifold of $M \times \mathbb{R}^j$ by

$$N \times U \times \mathbb{R} \subset N \times \mathbb{R}^i \subset M \times \mathbb{R}^j \ ,$$

and define the codimension 0 submanifold of $M' \times \mathbb{R}^j$

$$W^{4k+i} \ = \ (h \times 1_{\mathbb{R}^j})^{-1}(N \times U \times \mathbb{R}) \subset M' \times \mathbb{R}^j \ .$$

The restriction

$$(h \times 1_{\mathbb{R}^j})| \ : \ W \longrightarrow N \times U \times \mathbb{R}$$

is a homeomorphism. Let

$$V^{4k+i-1} \ = \ h''^{-1}(N \times U \times \{0\}) \subset W^{4k+i}$$

be the codimension 1 transverse inverse image of any differentiable (or PL) approximation $h'' : W \longrightarrow N \times U \times \mathbb{R}$ to $(h \times 1_{\mathbb{R}^j})|$. The $(4k+i-1)$-dimensional normal map

$$(g,c) \ = \ h''| \ : \ V \longrightarrow N \times U$$

is normal bordant to $(f,b) \times 1_U : N' \times U \longrightarrow N \times U$. The surgery obstruction of (g,c) is thus given by the surgery product formula of Ranicki [45]

$$\begin{aligned}\sigma_*(g,c) \ &= \ \sigma_*((f,b) \times 1_U) \ = \ \sigma_*(f,b) \otimes \sigma^*(U) \\ &\in \mathrm{im}(L_{4k}(\mathbb{Z}) \otimes L^{i-1}(\mathbb{Z}[\pi_1(U)]) \longrightarrow L_{4k+i-1}(\mathbb{Z}[\pi_1(U)])) \ ,\end{aligned}$$

with $\sigma^*(U) \in L^{i-1}(\mathbb{Z}[\pi_1(U)])$ the symmetric signature of U – see §6 for a brief account of the symmetric signature. Also, by 3.6 (ii)

$$\begin{aligned}\sigma_*(g,c) \ &= \ [W^+] \in \mathrm{im}(\widehat{H}^{4k+i}(\mathbb{Z}_2; \widetilde{K}_0(\mathbb{Z}[\pi_1(U)])) \longrightarrow L_{4k+i-1}(\mathbb{Z}[\pi_1(U)])) \\ &= \ \ker(L_{4k+i-1}(\mathbb{Z}[\pi_1(U)]) \longrightarrow L^p_{4k+i-1}(\mathbb{Z}[\pi_1(U)])) \ .\end{aligned}$$

For 3. take $U = T^{i-1} \subset \mathbb{R}^i$. It follows from $\widetilde{K}_0(\mathbb{Z}[\mathbb{Z}^{i-1}]) = 0$ that $\sigma_*(g,c) = 0$. The map

$$-\otimes \sigma^*(T^{i-1}) \ : \ L_{4k}(\mathbb{Z}) \longrightarrow L_{4k+i-1}(\mathbb{Z}[\mathbb{Z}^{i-1}])$$

is a (split) injection which sends $\sigma_*(f,b)$ to

$$\sigma_*(f,b)\otimes\sigma^*(T^{i-1}) \ = \ \sigma_*(g,c) \ = \ 0 \in L_{4k+i-1}(\mathbb{Z}[\mathbb{Z}^{i-1}])$$

so that $\sigma_*(f,b) = 0$.
For 4. assume that i is odd, say $i = 2n+1$, and let $U = B^{2n} \subset \mathbb{R}^{2n+1}$ be a hypersurface for which there exists a fibre bundle

$$F^{2n} \longrightarrow E^{4n} \xrightarrow{p} B^{2n}$$

such that the total space E is a $4n$-dimensional manifold with signature

$$\sigma(E) \neq 0 \in L^{4n}(\mathbb{Z}) \ = \ \mathbb{Z} \ .$$

(Any such B bounds a simply-connected manifold, so that the simply-connected symmetric signature is $\sigma(B) = 0 \in L^{2n}(\mathbb{Z})$, but $\sigma^*(B) \neq 0 \in L^{2n}(\mathbb{Z}[\pi_1(B)])$.) For example, take the n-fold cartesian product

$$F \ = \ F_1^{(n)} \longrightarrow E \ = \ E_1^{(n)} \xrightarrow{p=p_1^{(n)}} B \ = \ B_1^{(n)}$$

of one of the surface bundles over a surface

$$F_1^2 \longrightarrow E_1^4 \xrightarrow{p_1} B_1^2$$

with $\sigma(E_1) \neq 0 \in L^4(\mathbb{Z}) = \mathbb{Z}$ constructed by Atiyah [2], using an embedding $B_1 \times \mathbb{R} \subset \mathbb{R}^3$ to define an embedding

$$B_1^{(n)} \ = \ \prod_1^n B_1 \subset \mathbb{R}^{2n+1}$$

by

$$B_1^{(n)} \subset B_1^{(n-1)} \times \mathbb{R}^3 \ = \ B_1^{(n-2)} \times (B_1 \times \mathbb{R}) \times \mathbb{R}^2 \subset B_1^{(n-2)} \times \mathbb{R}^5 \subset \ldots \subset \mathbb{R}^{2n+1} \ .$$

Let

$$F^{2n} \longrightarrow Q^{4k+4n+1} \longrightarrow W^{4k+2n+1}$$

be the induced fibre bundle over W. The algebraic surgery transfer map induced by p

$$p^! \ : \ L_{4k+2n}(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{4k+4n}(\mathbb{Z}[\pi_1(E)])$$

sends $\sigma_*(g,c)$ to

$$\begin{aligned} p^!\sigma_*(g,c) &= [Q^+] \\ &\in \mathrm{im}(\widehat{H}^{4k+4n+1}(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi_1(E)]))\longrightarrow L_{4k+4n}(\mathbb{Z}[\pi_1(E)])) \\ &= \ker(L_{4k+4n}(\mathbb{Z}[\pi_1(E)])\longrightarrow L^p_{4k+4n}(\mathbb{Z}[\pi_1(E)])) \end{aligned}$$

with signature

$$\sigma_*(f,b)\sigma(E) \;=\; 0 \in L_{4k+4n}(\mathbb{Z}) \;=\; \mathbb{Z}$$

(Lück and Ranicki [28]), so that $\sigma_*(f,b)=0$.

□

Remarks 4.2 (i) Novikov's proof of the topological invariance of the rational Pontrjagin classes was in the differentiable category, but it applies equally well in the PL category. In fact, the proof led to the disproof of the manifold Hauptvermutung by Casson and Sullivan — see Ranicki [50]. A homeomorphism of PL manifolds cannot in general be approximated by a PL homeomorphism. The proof also led to the subsequent development by Kirby and Siebenmann [27] of the classification theory of high-dimensional topological manifolds. It is now possible to define the $\mathcal{L}$-genus and the rational Pontrjagin classes for a topological manifold, and the Hirzebruch signature theorem $\sigma(N)=\langle\mathcal{L}(N),[N]\rangle$ also holds for topological manifolds N^{4k}.
(ii) Let W be an open $(4k+1)$-dimensional manifold with a proper map $g: W\longrightarrow\mathbb{R}$ transverse regular at $0\in\mathbb{R}$, so that

$$V^{4k} \;=\; g^{-1}(0)\subset W^{4k+1}$$

is a closed $4k$-dimensional submanifold. Novikov [35] defined the signature of (W,g) by

$$\sigma(W,g) \;=\; \mathrm{signature}(H^{2k}(W)/H^{2k}(W)^{\perp},[\phi])\in\mathbb{Z}$$

with

$$\begin{aligned} &\phi \;:\; H^{2k}(W)\times H^{2k}(W) \longrightarrow \mathbb{Z}\;;\; (x,y)\longrightarrow \langle x\cup y,[V]\rangle\;, \\ &H^{2k}(W)^{\perp} \;=\; \{x\in H^{2k}(W)\,|\,\phi(x,y)=0 \text{ for all } y\in H^{2k}(W)\}\;, \end{aligned}$$

and identified

$$\sigma(W,g) \;=\; \sigma(V)\in L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}\;,$$

thus proving that $\sigma(V)$ is a proper homotopy invariant of (W,g). (In the context of the bounded L-theory of Ranicki [48] this is immediate from the computation $L^{4k+1}(\mathbb{C}_{\mathbb{R}}(\mathbb{Z}))=L^{4k}(\mathbb{Z})$.) This signature invariant was used in [35] to prove that $\mathcal{L}_k(M) \in H^{4k}(M;\mathbb{Q})$ is a homotopy invariant for any closed $(4k+1)$-dimensional manifold M, as follows. $\mathcal{L}_k(M)$ is detected by the signatures of special $4k$-dimensional submanifolds $N^{4k} \subset M^{4k+1} \times \mathbb{R}^j$ with

$$\begin{aligned} \sigma(N) &= \langle \mathcal{L}(N), [N] \rangle \\ &= \langle \mathcal{L}_k(M), [N] \rangle \in L^{4k}(\mathbb{Z}) = \mathbb{Z} . \end{aligned}$$

The Poincaré dual $[N]^* \in H^1(M)$ of $[N] \in H_{4k}(M)$ is represented by a map $f : M \longrightarrow S^1$ with a lift to a proper map $\overline{f} : \overline{M} \longrightarrow \mathbb{R}$ such that the transverse inverse image

$$V_N^{4k} = f^{-1}(1) \subset M$$

is diffeomorphic to $\overline{f}^{-1}(0) \subset \overline{M}$ and cobordant to N, so that

$$\sigma(N) = \sigma(V_N) = \sigma(\overline{M}, \overline{f}) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$$

is a homotopy invariant of (M, f). A homotopy equivalence $h : M' \longrightarrow M$ induces a proper homotopy equivalence $\overline{h} : \overline{M}' \longrightarrow \overline{M}$, so that

$$\mathcal{L}_k(M') = h^* \mathcal{L}_k(M) \in H^{4k}(M';\mathbb{Q}) .$$

For any map $f : M \longrightarrow S^1$ with transverse inverse image

$$V^{4k} = f^{-1}(1) \subset M^{4k+1}$$

the 'higher signature' of (M, f)

$$\begin{aligned} f_*(\mathcal{L}_k(M) \cap [M]) &= \langle \mathcal{L}(V), [V] \rangle \\ &= \sigma(V) \in H_1(S^1) = \mathbb{Z} \subset H_1(S^1;\mathbb{Q}) = \mathbb{Q} \end{aligned}$$

is thus a homotopy invariant of (M, f), verifying the Novikov conjecture for $\pi = \mathbb{Z}$ (5.2). The proof of topological invariance of the rational Pontrjagin classes in Novikov [36] grew out of this, leading on to the formulation of the general conjecture and the verification for free abelian π in Novikov [37].
(iii) Gromov's proof of topological invariance does not use surgery theory: the actual method of [21] extends the symmetric form defined by Novikov [35] for open $(4k+1)$-dimensional manifolds to the context of cohomology with coefficients in a flat hermitian bundle (as used by Lusztig [29] and Meyer [30]).
(iv) The topological invariance of Whitehead torsion (originally proved by Chapman) was proved in Ranicki and Yamasaki [52] using controlled K-theory. The parallel development of controlled L-theory will give yet another proof of the topological invariance of the rational Pontrjagin classes.

$\square$

§5. Homotopy invariance

Definition 5.1 The **higher $\mathcal{L}$-genus** of an m-dimensional manifold M with fundamental group $\pi_1(M) = \pi$ is

$$\mathcal{L}_\pi(M) \ = \ f_*(\mathcal{L}(M) \cap [M]) \in H_{m-4*}(B\pi;\mathbb{Q}) \ ,$$

with $f : M \longrightarrow B\pi$ classifying the universal cover $\widetilde{M}$ and $\mathcal{L}(M) \cap [M] \in H_{m-4*}(M;\mathbb{Q})$ the Poincaré dual of the $\mathcal{L}$-genus $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$.

□

Conjecture 5.2 (Novikov [37, §11]) *The higher $\mathcal{L}$-genus is a homotopy invariant: if $h : M'^m \longrightarrow M^m$ is a homotopy equivalence of m-dimensional manifolds then*

$$\mathcal{L}_\pi(M) \ = \ \mathcal{L}_\pi(M') \in H_{m-4*}(B\pi;\mathbb{Q}) \ .$$

□

Definition 5.3 A submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$ is **π-special** if it is special and the Poincaré dual $[N]^* \in H^{m-4k}(M;\mathbb{Q})$ of $[N] \in H_{4k}(M;\mathbb{Q})$ is such that

$$[N]^* \in \mathrm{im}(f^* : H^{m-4k}(B\pi;\mathbb{Q}) \longrightarrow H^{m-4k}(M;\mathbb{Q})) \ .$$

The **higher signatures** of M are the signatures $\sigma(N) \in \mathbb{Z}$ of the π-special manifolds $N \subset M \times \mathbb{R}^j$.

□

Remarks 5.4 (i) The higher $\mathcal{L}$-genus of an m-dimensional manifold M with $\pi_1(M) = \pi$ is detected by the higher signatures. As before, let $f : M \longrightarrow B\pi$ classify the universal cover $\widetilde{M}$ of M. The $\mathbb{Q}$-vector space $H^{m-4k}(B\pi;\mathbb{Q})$ is spanned by the elements of type $x = e^*(1)$ for a proper map

$$e \ : \ B\pi \times \mathbb{R}^j \longrightarrow \mathbb{R}^i \ \ (i = m + j - 4k)$$

with large j. (It is convenient to assume here that $B\pi$ is compact). For any such x, e the composite

$$e(f \times 1) \ : \ M \times \mathbb{R}^j \xrightarrow{f \times 1} B\pi \times \mathbb{R}^j \xrightarrow{e} \mathbb{R}^i$$

can be made transverse regular at $0 \in \mathbb{R}^i$, with

$$N^{4k} \ = \ (e(f \times 1))^{-1}(0) \subset M \times \mathbb{R}^j$$

a π-special submanifold. The higher $\mathcal{L}$-genus of M is such that

$$\begin{aligned} &\mathcal{L}_\pi(M) \ : \ H^{m-4k}(B\pi;\mathbb{Q}) \longrightarrow \mathbb{Q} \ ; \\ &x \longrightarrow \langle x, \mathcal{L}_\pi(M)\rangle \ = \ \langle \mathcal{L}(M) \cup f^*(x), [M]\rangle \ = \ \langle \mathcal{L}(N), [N]\rangle \ = \ \sigma(N) \ . \end{aligned}$$

(ii) The Novikov conjecture is equivalent to the homotopy invariance of the higher signatures: if $h : M'^m \longrightarrow M^m$ is a homotopy equivalence then

$$\sigma(N) \;=\; \sigma(N') \in \mathbb{Z}$$

for any π-special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$, with the inverse image

$$N' \;=\; (h \times 1)^{-1}(N) \subset M' \times \mathbb{R}^j$$

also a π-special submanifold.

□

See Chapter 24 of Ranicki [49] for a more detailed account of the higher signatures.

§6. Cobordism invariance

Very early on in the history of the Novikov conjecture (essentially already in [37]) it was recognized that the conjecture is equivalent to the algebraic Poincaré cobordism invariance of the higher $\mathcal{L}$-genus, and also to the injectivity of the rational assembly map $A_\pi : H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.

See Ranicki [45] for the **symmetric** (resp. **quadratic**) **L-groups** $L^m(R)$ (resp. $L_m(R)$) of a ring with involution R, which are the cobordism groups of m-dimensional symmetric (resp. quadratic) Poincaré complexes (C,ϕ) consisting of an m-dimensional f.g. free R-module chain complex C with a symmetric (resp. quadratic) Poincaré duality $\phi : C^{m-*} \simeq C$. The symmetrization maps $1+T : L_m(R) \longrightarrow L^m(R)$ are isomorphisms modulo 8-torsion. The quadratic L-groups $L_*(R)$ are the Wall surgery obstruction groups, and depend only on $m(\mathrm{mod}\, 4)$. The symmetric L-groups are not 4-periodic in general. The L-groups of $\mathbb{Z}$ are given by

$$L^m(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } m \equiv 0(\mathrm{mod}\, 4) \\ \mathbb{Z}_2 & \text{if } m \equiv 1(\mathrm{mod}\, 4) \\ 0 & \text{otherwise} \end{cases} \quad , \quad L_m(\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z} & \text{if } m \equiv 0(\mathrm{mod}\, 4) \\ \mathbb{Z}_2 & \text{if } m \equiv 2(\mathrm{mod}\, 4) \\ 0 & \text{otherwise .} \end{cases}$$

The symmetric L-groups $L^*(R)$ and the symmetric signature were introduced by Mishchenko [33].

Definition 6.1 The **symmetric signature** of an m-dimensional geometric Poincaré complex X with universal cover $\widetilde{X}$ is the symmetric Poincaré cobordism class

$$\sigma^*(X) \;=\; (C(\widetilde{X}),\phi) \in L^m(\mathbb{Z}[\pi_1(X)])$$

with ϕ the m-dimensional symmetric structure of the Poincaré duality chain equivalence $[X] \cap - : C(\widetilde{X})^{m-*} \longrightarrow C(\widetilde{X})$.

□

The standard algebraic mapping cylinder argument shows:

Proposition 6.2 *The symmetric signature is both a cobordism and a homotopy invariant of a geometric Poincaré complex.*

□

The symmetric signature is a non-simply-connected generalization of the signature; for $m = 4k$ the natural map $L^m(\mathbb{Z}[\pi_1(X)]) \longrightarrow L^m(\mathbb{Z}) = \mathbb{Z}$ sends $\sigma^*(X)$ to the signature $\sigma(X)$.

Definition 6.3 The **quadratic signature** of a normal map of m-dimensional manifolds with boundary $(f, b) : (M', \partial M') \longrightarrow (M, \partial M)$ and with $\partial f : \partial M' \longrightarrow \partial M$ a homotopy equivalence is the cobordism class of the quadratic Poincaré complex kernel

$$\sigma_*(f, b) \;=\; (C(f^!), \psi) \in L_m(\mathbb{Z}[\pi_1(M)]) \;,$$

with ψ the quadratic structure on the algebraic mapping cone $C(f^!)$ of the Umkehr $\mathbb{Z}[\pi_1(M)]$-module chain map

$$f^! \;:\; C(\widetilde{M}) \;\simeq\; C(\widetilde{M}, \partial\widetilde{M})^{m-*} \xrightarrow{\widetilde{f}^*} C(\widetilde{M}', \partial\widetilde{M}')^{m-*} \;\simeq\; C(\widetilde{M}') \;.$$

□

Proposition 6.4 *The quadratic signature of a normal map (f, b) is the surgery obstruction of Wall [56], such that $\sigma_*(f, b) = 0$ if (and for $m \geq 5$ only if) (f, b) is normal bordant to a homotopy equivalence.*

□

The symmetrization of the quadratic signature is the symmetric signature

$$(1 + T)\sigma_*(f, b) \;=\; \sigma^*(M' \cup_{\partial f} -M) \in L^m(\mathbb{Z}[\pi_1(M)])$$

where $-M$ refers to M with the opposite orientation $[-M] = -[M]$.

The rational surgery obstruction of a normal map $(f, b) : M' \longrightarrow M$ of closed m-dimensional manifolds with fundamental group π

$$\sigma_*(f, b) \otimes \mathbb{Q} \in L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$$

depends only on the difference of the higher $\mathcal{L}$-genera

$$\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M) \in H_{m-4*}(B\pi; \mathbb{Q}) \;.$$

For any finitely presented group π the $\mathbb{Q}$-vector space $H_{m-4*}(B\pi; \mathbb{Q})$ is spanned by the differences $\mathcal{L}_\pi(M') - \mathcal{L}_\pi(M)$ for normal maps $(f, b) : M' \longrightarrow M$ of closed m-dimensional manifolds with fundamental group π.

Definition 6.5 The **rational assembly map** in quadratic L-theory is

$$\begin{aligned} A_\pi \ &: H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q} \ ; \\ &\quad \mathcal{L}_\pi(M') - \mathcal{L}_\pi(M) \longrightarrow \sigma_*(f,b)\otimes\mathbb{Q} \ = \ \frac{1}{8}(\sigma^*(M')-\sigma^*(M)) \ , \end{aligned}$$

with

$$A_\pi\mathcal{L}_\pi(M) \ = \ \frac{1}{8}\sigma^*(M)\in L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q} \ = \ L^m(\mathbb{Z}[\pi])\otimes\mathbb{Q} \ .$$

□

The $\mathcal{L}$-genus $\mathcal{L}(M)\in H^{4*}(M;\mathbb{Q})$ is not in general a homotopy invariant of an m-dimensional manifold M, except for the $4k$-dimensional component $\mathcal{L}_k(M)\in H^{4k}(M;\mathbb{Q})$ in the case $m=4k$ – this is a homotopy invariant by virtue of the signature theorem

$$\sigma(M) \ = \ \langle \mathcal{L}_k(M),[M]\rangle \in \mathbb{Z} \ .$$

The simply-connected surgery exact sequence (§7) shows that if M is a simply-connected m-dimensional manifold and $m-4k\geq 1$, $m\geq 5$ then the $\mathbb{Q}$-vector space $H^{4k}(M;\mathbb{Q})$ is spanned by the differences

$$(h^{-1})^*\mathcal{L}_k(M')-\mathcal{L}_k(M)\in H^{4k}(M;\mathbb{Q})$$

for homotopy equivalences $h: M'\longrightarrow M$.

Proposition 6.6 *The Novikov conjecture holds for π if and only if the rational assembly map*

$$A_\pi \ : \ H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q}$$

is injective for each $m(\mathrm{mod}\,4)$.

Proof The rational assembly map

$$A \ : \ H^{4*}(M;\mathbb{Q}) \ = \ H_{m-4*}(M;\mathbb{Q}) \xrightarrow{f_*} H_{m-4*}(B\pi;\mathbb{Q}) \xrightarrow{A_\pi} L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q}$$

is such that

$$A((h^{-1})^*\mathcal{L}(M')-\mathcal{L}(M)) \ = \ A_\pi(\mathcal{L}_\pi(M')-\mathcal{L}_\pi(M))\in L_m(\mathbb{Z}[\pi])\otimes\mathbb{Q}$$

for any homotopy equivalence $h: M'\longrightarrow M$ of m-dimensional manifolds with $\pi_1(M)=\pi$, and $f: M\longrightarrow B\pi$ the classifying map. The $\mathbb{Q}$-vector space $H_{m-4*}(B\pi;\mathbb{Q})$ is spanned by the differences $\mathcal{L}_\pi(M')-\mathcal{L}_\pi(M)$. The non-simply-connected surgery exact sequence (§7) identifies the subspace of $H^{4*}(M;\mathbb{Q})$ spanned by the differences

$$(h^{-1})^*\mathcal{L}(M')-\mathcal{L}(M)\in H^{4*}(M;\mathbb{Q})$$

with the kernel of A, and

$$\ker(f_* : H_{m-4*}(M;\mathbb{Q}) \longrightarrow H_{m-4*}(B\pi;\mathbb{Q})) \subseteq \ker(A) .$$

The Novikov conjecture predicts that for all m-dimensional manifolds M with $\pi_1(M) = \pi$

$$f_*(\ker(A)) \; = \; \{0\} \subseteq H_{m-4*}(B\pi;\mathbb{Q}) ,$$

or equivalently that $\ker(f_*) = \ker(A)$. In turn, this is equivalent to the injectivity of $A_\pi : H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q}$.

□

§7. The algebraic L-theory assembly map

The integral versions of the topological invariance of the rational Pontrjagin classes and of the Novikov conjecture on the homotopy invariance of the higher signatures involve the algebraic L-spectra and the algebraic L-theory assembly map defined in Ranicki [49].

The **symmetric L-spectrum** $\mathbb{L}^\bullet(R)$ of a ring with involution R is defined in [49] using n-ads of symmetric forms over R, with homotopy groups

$$\pi_*(\mathbb{L}^\bullet(R)) \; = \; L^*(R) .$$

The **generalized homology spectrum** $\mathbb{H}_\bullet(X;\mathbb{L}^\bullet(R))$ of a topological space X is defined in [49] using sheaves over X of symmetric Poincaré complexes over R, with homotopy groups

$$\pi_*(\mathbb{H}_\bullet(X;\mathbb{L}^\bullet(R))) \; = \; H_*(X;\mathbb{L}^\bullet(R))$$

the cobordism groups of such sheaves. The assembly map

$$A \; : \; \mathbb{H}_\bullet(X;\mathbb{L}^\bullet(R)) \longrightarrow \mathbb{L}^\bullet(R[\pi_1(X)])$$

is defined by pulling back a symmetric Poincaré sheaf over X to the universal cover $\widetilde{X}$, and then assembling the stalks to obtain a symmetric Poincaré complex over $R[\pi_1(X)]$. Similarly for the **quadratic L-spectrum** $\mathbb{L}_\bullet(R)$.

The 0th space of the quadratic L-spectrum $\mathbb{L}_\bullet(\mathbb{Z})$ is such that

$$\mathbb{L}_0(\mathbb{Z}) \; \simeq \; L_0(\mathbb{Z}) \times G/TOP .$$

As usual, G/TOP is the classifying space for fibre homotopy trivialized topological bundles, with a fibration sequence

$$G/TOP \longrightarrow BTOP \longrightarrow BG .$$

Let $\mathbb{L}_\bullet = \mathbb{L}_\bullet\langle 1\rangle(\mathbb{Z})$ be the 1-connective cover of $\mathbb{L}_\bullet(\mathbb{Z})$, with 0th space such that

$$\mathbb{L}_0 \simeq G/TOP .$$

For any space X define the **structure spectrum**

$$\mathbb{S}_\bullet(X) = \text{homotopy cofibre}(A : \mathbb{H}_\bullet(X;\mathbb{L}_\bullet) \longrightarrow \mathbb{L}_\bullet(\mathbb{Z}[\pi_1(X)])) ,$$

to fit into a cofibration sequence of spectra

$$\mathbb{H}_\bullet(X;\mathbb{L}_\bullet) \xrightarrow{A} \mathbb{L}_\bullet(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathbb{S}_\bullet(X) ,$$

with A the spectrum level assembly map. The **structure groups**

$$\mathbb{S}_*(X) = \pi_*(\mathbb{S}_\bullet(X))$$

are the cobordism groups of sheaves over X of quadratic Poincaré complexes over $\mathbb{Z}$ such that the assembly quadratic Poincaré complex over $\mathbb{Z}[\pi_1(X)]$ is contractible. The structure groups are the relative groups in the **algebraic surgery exact sequence**

$$\dots \longrightarrow \mathbb{S}_{m+1}(X) \longrightarrow H_m(X;\mathbb{L}_\bullet) \xrightarrow{A} L_m(\mathbb{Z}[\pi_1(X)]) \longrightarrow \mathbb{S}_m(X) \longrightarrow \dots .$$

If X is an m-dimensional CW complex then $H_*(X;\mathbb{L}_\bullet) = H_*(X;\mathbb{L}_\bullet(\mathbb{Z}))$ for $* > m$ and $\mathbb{S}_*(X) = \mathbb{S}_{*+4}(X)$ for $* > m+1$.

Proposition 7.1 (Ranicki [49]) (i) *An m-dimensional geometric Poincaré complex X has a* **total surgery obstruction**

$$s(X) \in \mathbb{S}_m(X)$$

such that $s(X) = 0$ if (and for $m \geq 5$ only if) X is homotopy equivalent to a closed m-dimensional topological manifold.
(ii) *A closed m-dimensional topological manifold M has a* **symmetric L-theory orientation**

$$[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^\bullet(\mathbb{Z}))$$

which is represented by the symmetric Poincaré orientation sheaf, with assembly the symmetric signature

$$A([M]_{\mathbb{L}}) = \sigma^*(M) \in L^m(\mathbb{Z}[\pi_1(M)]) .$$

(iii) *A normal map $(f,b) : M' \longrightarrow M$ of closed m-dimensional topological manifolds has a* **normal invariant**

$$[f,b]_{\mathbb{L}} \in H_m(M;\mathbb{L}_\bullet) = [M, G/TOP]$$

which is represented by the sheaf over M of the quadratic Poincaré complex kernels over $\mathbb{Z}$ of the normal maps

$$(f,b) \;=\; h| \;:\; N' \;=\; h^{-1}(N) \longrightarrow N \quad (N^n \subset M^m) \;,$$

with assembly the surgery obstruction

$$A([f,b]_{\mathbb{L}}) \;=\; \sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(M)]) \;.$$

The symmetrization of the normal invariant is the difference of the symmetric L-theory orientations

$$(1+T)[f,b]_{\mathbb{L}} \;=\; f_*[M']_{\mathbb{L}} - [M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^\bullet(\mathbb{Z})) \;.$$

(iv) *A homotopy equivalence $h : M' \longrightarrow M$ of closed m-dimensional topological manifolds has a* **structure invariant**

$$s(h) \in \mathbb{S}_{m+1}(M)$$

which is represented by the $\mathbb{Z}[\pi_1(M)]$-contractible quadratic Poincaré kernel sheaf of (iii) *and is such that $s(h) = 0$ if (and for $m \geq 5$ only if) h is h-cobordant to a homeomorphism. Moreover, for $m \geq 5$ every element $x \in \mathbb{S}_{m+1}(M)$ is the structure invariant $x = s(h)$ of a homotopy equivalence $h : M' \longrightarrow M$. The structure group $\mathbb{S}_{m+1}(M)$ is thus the topological manifold structure set of the Browder-Novikov-Sullivan-Wall surgery theory*

$$\mathbb{S}_{m+1}(M) \;=\; \mathcal{S}^{TOP}(M) \;,$$

with a surgery exact sequence

$$\begin{aligned} \dots \longrightarrow L_{m+1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathcal{S}^{TOP}(M) \longrightarrow{} & [M, G/TOP] \\ \longrightarrow{} & L_m(\mathbb{Z}[\pi_1(M)]) \;. \end{aligned}$$

□

Remarks 7.2 (i) The symmetric and quadratic L-spectra of $\mathbb{Z}$ are given rationally by

$$\mathbb{L}^\bullet(\mathbb{Z}) \otimes \mathbb{Q} \;=\; \mathbb{L}_\bullet(\mathbb{Z}) \otimes \mathbb{Q} \;=\; \bigvee_k K(\mathbb{Q}, 4k) \;,$$

so that for any space X

$$H_m(X;\mathbb{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_m(X;\mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(X;\mathbb{Q}) \;.$$

(ii) The symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^\bullet(\mathbb{Z}))$ of a closed m-dimensional topological manifold M is an integral refinement of the $\mathcal{L}$-genus, with

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; [M] \cap \mathcal{L}(M) \in H_m(M;\mathbb{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(M;\mathbb{Q})$$

detected by the signatures $\sigma(N)$ of special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$. As before, let $\pi_1(M) = \pi$ and let $f : M \longrightarrow B\pi$ be the classifying map of the universal cover $\widetilde{M}$. The image $f_*[M]_{\mathbb{L}} \in H_m(B\pi;\mathbb{L}^\bullet(\mathbb{Z}))$ is an integral refinement of the higher $\mathcal{L}$-genus, with

$$f_*[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; \mathcal{L}_\pi(M) \in H_m(B\pi;\mathbb{L}^\bullet(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(B\pi;\mathbb{Q})$$

detected by the signatures $\sigma(N)$ of π-special submanifolds $N^{4k} \subset M \times \mathbb{R}^j$.
(iii) The normal invariant $[f,b]_{\mathbb{L}} \in H_m(M;\mathbb{L}_\bullet)$ of an m-dimensional normal map $(f,b) : M' \longrightarrow M$ is given rationally by the difference of the Poincaré duals of the $\mathcal{L}$-genera

$$\begin{aligned}[f,b]_{\mathbb{L}} \otimes \mathbb{Q} \;&=\; f_*(\mathcal{L}(M') \cap [M']) - (\mathcal{L}(M) \cap [M]) \\ &\in H_m(M;\mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \;=\; H_{m-4*}(M;\mathbb{Q})\ .\end{aligned}$$

(iv) The construction and the verification of the combinatorial invariance of the symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^\bullet(\mathbb{Z}))$ is quite straightforward for a PL manifold M. The construction and topological invariance of $[M]_{\mathbb{L}}$ for a topological manifold M is much more complicated — see §9 below.
(v) For $m \geq 5$ an m-dimensional geometric Poincaré complex X is homotopy equivalent to a closed m-dimensional manifold M if and only if there exists a symmetric L-theory orientation $[X]_{\mathbb{L}} \in H_m(X;\mathbb{L}^\bullet(\mathbb{Z}))$ such that $A([X]_{\mathbb{L}}) = \sigma^*(X) \in L^m(\mathbb{Z}[\pi_1(X)])$, modulo 2-primary torsion invariants.

□

Integral Novikov conjecture 7.3 *The assembly map in quadratic L-theory*

$$A \;:\; H_*(B\pi;\mathbb{L}_\bullet(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$$

is a split injection.

□

The algebraic surgery exact sequence for the classifying space $B\pi$ of a group π

$$\dots \longrightarrow \mathbb{S}_{m+1}(B\pi) \longrightarrow H_m(B\pi;\mathbb{L}_\bullet) \xrightarrow{A} L_m(\mathbb{Z}[\pi]) \longrightarrow \mathbb{S}_m(B\pi) \longrightarrow \dots$$

is such that

$$\begin{aligned}&\mathrm{im}(\mathbb{S}_{m+1}(B\pi) \longrightarrow H_m(B\pi;\mathbb{L}_\bullet)) \\ &\qquad = \ker(A : H_m(B\pi;\mathbb{L}_\bullet) \longrightarrow L_m(\mathbb{Z}[\pi])) \subseteq L_m(\mathbb{Z}[\pi])\end{aligned}$$

consists of the images of the normal invariants

$$f_*[s(h)] = f_*[h,b]_{\mathbb{L}} \in H_m(B\pi;\mathbb{L}_\bullet)$$

of all homotopy equivalences $h : M' \longrightarrow M$ of m-dimensional topological manifolds with $\pi_1(M) = \pi$, and with $f : M \longrightarrow B\pi$ classifying the universal cover.

Remarks 7.4 (i) The integral Novikov conjecture for π implies the original Novikov conjecture for π, since the integral assembly map A induces the rational assembly map

$$A \otimes 1 \ : \ H_m(B\pi;\mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \ = \ H_{m-4*}(B\pi;\mathbb{Q}) \longrightarrow L_m(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \ .$$

(ii) The integral Novikov conjecture is generally false if π has torsion, e.g. if $\pi = \mathbb{Z}_2$.
(iii) The integral Novikov conjecture has been verified for many torsion-free groups π using codimension 1 splitting methods, starting with the free abelian case $\pi = \mathbb{Z}^i$ (when A is an isomorphism) — see §§8,10 below for further discussion.
(iv) The integral Novikov conjecture has been verified geometrically for many groups π such that the classifying space $B\pi$ is realized by an aspherical Riemannian manifold with sufficient symmetry to ensure geometric rigidity, so that homotopy equivalences of manifolds with fundamental group π can be deformed to homeomorphisms — see Farrell and Hsiang [16], Farrell and Jones [17] for example.

□

Here is how algebraic surgery theory translates rigidity results in geometry into verifications of the integral Novikov conjecture:

Proposition 7.5 *If π is a finitely presented group such that (at least for $m \geq 5$) there is a systematic procedure for deforming every homotopy equivalence $h_0 : M_0 \longrightarrow N$ of closed m-dimensional manifolds with $\pi_1(N) = \pi$ to a homeomorphism $h_1 : M_1 \longrightarrow N$, via an $(m+1)$-dimensional normal bordism*

$$(g,c) \ : \ (W;M_0,M_1) \longrightarrow N \times ([0,1];\{0\},\{1\})$$

with $g|_{M_i} = h_i : M_i \longrightarrow N$ $(i = 0,1)$, then the integral Novikov conjecture holds for π.

Proof The realization theorem of Wall [56] identifies $L_{m+1}(\mathbb{Z}[\pi])$ with the bordism group of normal maps $(f,b) : (K,\partial K) \longrightarrow (L,\partial L)$ of compact $(m+1)$-dimensional manifolds with boundary which restrict to a homotopy equivalence $\partial f = f| : \partial K \longrightarrow \partial L$ on the boundary, with $\pi_1(L) = \pi$. The generalized homology group $H_{m+1}(B\pi;\mathbb{L}_\bullet)$ has a similar description, with the added condition that $\partial f : \partial K \longrightarrow \partial L$ be a homeomorphism (including $\partial K = \partial L = \emptyset$ as a special case). Given systematic deformations of homotopy equivalences to homeomorphisms as in the statement there is defined a direct sum system

$$H_{m+1}(B\pi;\mathbb{L}_\bullet) \underset{B}{\overset{A}{\rightleftarrows}} L_{m+1}(\mathbb{Z}[\pi]) \underset{D}{\overset{C}{\rightleftarrows}} \mathbb{S}_{m+1}(B\pi)$$

verifying the integral Novikov conjecture for π, with

$$\begin{aligned}
B \ &: \ L_{m+1}(\mathbb{Z}[\pi]) \longrightarrow H_{m+1}(B\pi;\mathbb{L}_\bullet) \ ; \ (f,b) \longrightarrow (f,b)\cup(g,c) \\
&\qquad\qquad\qquad (h_0 = \partial f : M = \partial K \longrightarrow N = \partial L) \ , \\
C \ &: \ L_{m+1}(\mathbb{Z}[\pi]) \longrightarrow \mathbb{S}_{m+1}(B\pi) \ ; \\
&\qquad \sigma_*((f,b) : (K,\partial K) \longrightarrow (L,\partial L)) \longrightarrow s(\partial f : \partial K \longrightarrow \partial L) \ , \\
D \ &: \ \mathbb{S}_{m+1}(B\pi) \longrightarrow L_{m+1}(\mathbb{Z}[\pi]) \ ; \ s(h) \longrightarrow \sigma_*(g,c) \ .
\end{aligned}$$

□

The chain complex treatment in Ranicki [45], [49] of the surgery obstruction of Wall [56] associates a quadratic Poincaré complex $\sigma_*(f,b)$ over $\mathbb{Z}[\pi]$ (resp. a sheaf over $B\pi$ of quadratic Poincaré complexes over $\mathbb{Z}$) to any normal map $(f,b) : (M,\partial M) \longrightarrow (N,\partial N)$ with ∂f a homotopy equivalence (resp. a homeomorphism) and $\pi_1(N) = \pi$. In principle, this allows the translation into algebra of any geometric construction of a disassembly map B.

§8. Codimension 1 splitting for compact manifolds

The primary obstructions to deforming a homotopy equivalence of high-dimensional manifolds $h : M' \longrightarrow M$ with $\pi_1(M)$ torsion-free to a homeomorphism are the splitting obstructions along codimension 1 submanifolds $N \subset M$. The method was initiated by Browder [7], where manifolds with fundamental group $\pi_1 = \mathbb{Z}$ were studied by considering surgery on codimension 1 simply-connected manifolds.

Codimension 1 Splitting Theorem 8.1 (Farrell and Hsiang [15]) *Let M^m be an m-dimensional manifold, and let $N^{m-1} \subset M^m$ be a codimension 1 submanifold with trivial normal bundle, such that*

$$\pi_1(M) \ = \ \pi \times \mathbb{Z} \ \ , \ \ \pi_1(N) \ = \ \pi \ .$$

The Whitehead torsion of a homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds is such that $\tau(h) \in \mathrm{im}(Wh(\pi) \longrightarrow Wh(\pi \times \mathbb{Z}))$ if (and for $m \geq 6$ only if) h splits along $N \subset M$.

K-theoretic proof. This was the original proof in [15]. Let $\widetilde{M}$ be the universal cover of M. The infinite cyclic cover $\overline{M} = \widetilde{M}/\pi$ of M can be constructed from M by cutting along N, with

$$\overline{M} \ = \ \overline{M}^+ \cup_N \overline{M}^-$$

for two ends $\overline{M}^+, \overline{M}^-$ with

$$\pi_1(\overline{M}^+) \ = \ \pi_1(\overline{M}^-) \ = \ \pi_1(N) \ = \ \pi$$

and similarly for $M', N' = h^{-1}(N) \subset M'$. The $\mathbb{Z}$-equivariant homotopy equivalence $\overline{h} : \overline{M}' \longrightarrow \overline{M}$ has a decomposition

$$\overline{h} \ = \ \overline{h}^+ \cup_g \overline{h}^- \ : \ \overline{M}' \ = \ \overline{M}'^+ \cup_{N'} \overline{M}'^- \longrightarrow \overline{M} \ = \ \overline{M}^+ \cup_N \overline{M}^-$$

with the restriction

$$(g, c) \ = \ h| \ = \ \overline{h}| \ : \ N' \ = \ h^{-1}(N) \ = \ \overline{h}^{-1}(N) \longrightarrow N$$

a normal map. Since h is a homotopy equivalence the natural $\mathbb{Z}[\pi]$-module chain map of the relative cellular $\mathbb{Z}[\pi]$-module chain complexes

$$C(\widetilde{N}', \widetilde{N}) \ \simeq \ C(\widetilde{M}'^+, \widetilde{M}^+) \oplus C(\widetilde{M}'^-, \widetilde{M}^-)$$

is a chain equivalence. Now $C(\widetilde{N}', \widetilde{N})$ is a finite f.g. free $\mathbb{Z}[\pi]$-module chain complex, so that $C(\widetilde{M}'^+, \widetilde{M}^+)$ and $C(\widetilde{M}'^-, \widetilde{M}^-)$ are finitely dominated (i.e. chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi]$-module chain complex). The reduced projective class

$$[C(\widetilde{M}'^+, \widetilde{M}^+)] \ = \ -[C(\widetilde{M}'^-, \widetilde{M}^-)] \in \widetilde{K}_0(\mathbb{Z}[\pi])$$

is the $\widetilde{K}_0$-component of $\tau(h) \in Wh(\pi \times \mathbb{Z})$ in the decomposition

$$Wh(\pi \times \mathbb{Z}) \ = \ Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi]) \oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])$$

of Bass [3, XII]. The $\mathbb{Z}[\pi]$-module Poincaré duality chain equivalence

$$C(\widetilde{N}', \widetilde{N})^{m-1-*} \ \simeq \ C(\widetilde{N}', \widetilde{N})$$

on the chain complex kernel

$$C(\widetilde{N}', \widetilde{N}) \ = \ C(g^! : C(\widetilde{N}) \longrightarrow C(\widetilde{N}'))$$

restricts to a chain equivalence

$$C(\widetilde{M}'^+, \widetilde{M}^+)^{m-1-*} \ \simeq \ C(\widetilde{M}'^-, \widetilde{M}^-) \ .$$

Thus $h : M' \longrightarrow M$ splits along $N \subset M$ (i.e. $(g, c) : N' \longrightarrow N$ is a homotopy equivalence) if and only if $C(\widetilde{M}'^+, \widetilde{M}^+)$ is chain contractible. For $m \geq 6$ the $\widetilde{K}_0$-component is 0 if and only if it is possible to modify N' by handle exchanges inside $\overline{M}'$ in the style of Browder [6] and Siebenmann [53] until

$(g,c) : N' \longrightarrow N$ is a homotopy equivalence, if and only if the $\mathbb{R}$-bounded homotopy equivalence $\overline{f} : \overline{M}' \longrightarrow \overline{M}$ splits along $N \subset \overline{M}$. The $\widetilde{\mathrm{Nil}}_0$-components (which are Poincaré dual to each other) are the obstructions to such modifications inside a fundamental domain of the infinite cyclic cover $\overline{M}'$ of M'.

L-theoretic proof. The surgery obstruction theory of Wall [56] can be used to give an L-theoretic proof of the splitting theorem, at least in the unobstructed case $\tau(h) \in \mathrm{im}(Wh(\pi))$. The surgery exact sequence for the appropriately decorated topological manifold structure set of M

$$\begin{aligned} \dots \longrightarrow L^{\mathrm{im}(Wh(\pi))}_{m+1}(\mathbb{Z}[\pi\times\mathbb{Z}]) \longrightarrow \mathbb{S}^{\mathrm{im}(Wh(\pi))}_{m+1}(M) & \\ \longrightarrow H_m(M;\mathbb{L}_\bullet) \longrightarrow L^{\mathrm{im}(Wh(\pi))}_{m}(\mathbb{Z}[\pi\times\mathbb{Z}]) \longrightarrow \dots \end{aligned}$$

combined with the algebraic computation of Ranicki [43]

$$L^{\mathrm{im}(Wh(\pi))}_{m}(\mathbb{Z}[\pi\times\mathbb{Z}]) \;=\; L_m(\mathbb{Z}[\pi]) \oplus L_{m-1}(\mathbb{Z}[\pi])$$

(or the geometric winding tricks of [56, 12.9] or Cappell [10]) give an exact sequence

$$\begin{aligned} \dots \longrightarrow \mathbb{S}_{m+1}(M\backslash N) \longrightarrow \mathbb{S}^{\mathrm{im}(Wh(\pi))}_{m+1}(M) \longrightarrow \mathbb{S}_m(N) & \\ \longrightarrow \mathbb{S}_m(M\backslash N) \longrightarrow \dots . \end{aligned}$$

The codimension 1 h-splitting obstruction of [56, §11] is the Tate $\mathbb{Z}_2$-cohomology class

$$[s(h)] \;=\; [C(\widetilde{M}'^{+},\widetilde{M})] \in LS_{m-1} \;=\; \widehat{H}^m(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi])) \ .$$

The structure set $\mathbb{S}^{\mathrm{im}(Wh(\pi))}_{m+1}(M)$ of homotopy equivalences of m-dimensional manifolds $h : M' \longrightarrow M$ such that $\tau(h) \in \mathrm{im}(Wh(\pi))$ is thus identified with the structure set $\mathbb{S}_{m+1}(N \longrightarrow M\backslash N)$ of homotopy equivalences $h : M' \longrightarrow M$ which split along $N \subset M$

$$\mathbb{S}^{\mathrm{im}(Wh(\pi))}_{m+1}(M) \;=\; \mathbb{S}_{m+1}(N \longrightarrow M\backslash N) \ .$$

(The relative $\mathbb{S}$-group $\mathbb{S}_{m+1}(N \longrightarrow M\backslash N)$ is denoted $\mathbb{S}_{m+1}(\xi^!)$ in the terminology of 3.2).

□

Example 8.2 For $m \geq 6$ a homotopy equivalence of m-dimensional manifolds of the type

$$h \;:\; M'^m \longrightarrow M^m \;=\; N^{m-1}\times S^1$$

is homotopic to

$$g\times \mathrm{id}_{S^1} \;:\; M' \;=\; N'\times S^1 \longrightarrow M \;=\; N\times S^1$$

for a homotopy equivalence of $(m-1)$-dimensional manifolds $g : N' \longrightarrow N$ if and only if $\tau(h) \in \mathrm{im}(Wh(\pi) \longrightarrow Wh(\pi \times \mathbb{Z}))$ $(\pi = \pi_1(N))$. The structure set of homotopy equivalences $h : M' \longrightarrow M$ which split along $N \subset M$ is given in this case by

$$\mathbb{S}^{\mathrm{im}(Wh(\pi))}_{m+1}(M) \;=\; \mathbb{S}_{m+1}(N) \oplus \mathbb{S}_m(N)\ .$$

□

For the remainder of §8 we shall assume that M is a connected manifold and that $N \subset M$ is a connected codimension 1 submanifold with trivial normal bundle

$$\nu_{N \subset M} \;=\; \epsilon \;:\; N \longrightarrow BO(1)$$

and such that $\pi_1(N) \longrightarrow \pi_1(M)$ is injective. As in the general theory of Wall [56, §12] there are two cases to consider:
(A) $N \subset M$ separates M, so that $M \backslash N$ has two components M_1, M_2, with

$$\pi_1(M) \;=\; \pi_1(M_1) *_{\pi_1(N)} \pi_1(M_2)$$

the amalgamated free product determined by the injections $\pi_1(N) \longrightarrow \pi_1(M_1)$, $\pi_1(N) \longrightarrow \pi_1(M_2)$, by the Seifert-Van Kampen theorem,
(B) $N \subset M$ does not separate M, so that $M_1 = M \backslash N$ is connected, with

$$\pi_1(M) \;=\; \pi_1(M_1) *_{\pi_1(N)} \{z\}$$

the HNN extension determined by the two injections $\pi_1(N) \longrightarrow \pi_1(M_1)$.
For example, if M is a genus 2 surface and $N = S^1 \subset M$ separates M with $M \backslash N = M_1 \sqcup M_2$ the disjoint union of punctured tori then (N, M) is of type (A), with $\pi_1(M_1) = \pi_1(M_2) = \mathbb{Z} * \mathbb{Z}$, $\pi_1(N) = \mathbb{Z}$, while $(M', N') = (S^1, \{\mathrm{pt.}\})$ is of type (B).

Waldhausen [55] obtained a splitting theorem for the algebraic K-theory of amalgamated free products and HNN extensions along injections, involving the K-groups $\widetilde{\mathrm{Nil}}_*$ of nilpotent objects, generalizing the splitting theorem of Bass [3, XII] for the Whitehead group of a polynomial extension. The Mayer-Vietoris exact sequence of [55] is

$$\begin{aligned}
\dots \longrightarrow\ & Wh(\pi_1(N)) \oplus \widetilde{\mathrm{Nil}}_1 \longrightarrow Wh(\pi_1(M_1)) \oplus Wh(\pi_1(M_2)) \\
\longrightarrow\ & Wh(\pi_1(M)) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(N)]) \oplus \widetilde{\mathrm{Nil}}_0 \\
\longrightarrow\ & \widetilde{K}_0(\mathbb{Z}[\pi_1(M_1)]) \oplus \widetilde{K}_0(\mathbb{Z}[\pi_1(M_2)]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(M)]) \longrightarrow \dots
\end{aligned}$$

with $Wh(\pi_1(M)) \longrightarrow \widetilde{\mathrm{Nil}}_0$ a split surjection, setting $M_2 = \emptyset$ in case (B). See Remark 8.7 below for a brief account of the algebraic transversality used in [55], and its extension to algebraic L-theory.

Codimension 1 Splitting Theorem 8.3 (Cappell [9]) *Let M^m be an m-dimensional manifold, and let $N^{m-1} \subset M^m$ be a codimension 1 submanifold with trivial normal bundle, such that $\pi_1(N) \longrightarrow \pi_1(M)$ is injective. The algebraic L-theory of $\mathbb{Z}[\pi_1(M)]$ is such that there is defined a Mayer-Vietoris exact sequence*

$$\begin{aligned} \dots \longrightarrow\ & L^I_i(\mathbb{Z}[\pi_1(N)]) \oplus \mathrm{UNil}_{i+1} \longrightarrow L_i(\mathbb{Z}[\pi_1(M_1)]) \oplus L_i(\mathbb{Z}[\pi_1(M_2)]) \\ \longrightarrow\ & L_i(\mathbb{Z}[\pi_1(M)]) \longrightarrow L^I_{i-1}(\mathbb{Z}[\pi_1(N)]) \oplus \mathrm{UNil}_i \\ \longrightarrow\ & L_{i-1}(\mathbb{Z}[\pi_1(M_1)]) \oplus L_{i-1}(\mathbb{Z}[\pi_1(M_2)]) \longrightarrow L_{i-1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \dots \end{aligned}$$

with

$$\begin{aligned} I &= \mathrm{im}(Wh(\pi_1(M)) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(N)])) \\ &= \ker(\widetilde{K}_0(\mathbb{Z}[\pi_1(N)]) \longrightarrow \widetilde{K}_0(\mathbb{Z}[\pi_1(M_1)]) \oplus \widetilde{K}_0(\mathbb{Z}[\pi_1(M_2)])) \\ &\subseteq \widetilde{K}_0(\mathbb{Z}[\pi_1(N)]) \end{aligned}$$

and $L_i(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathrm{UNil}_i$ a split surjection onto an L-group of unitary nilpotent objects, setting $M_2 = \emptyset$ in case (B). The codimension 1 h-splitting obstruction (3.2) of a homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional manifolds along $N \subset M$ is given by

$$[s(h)] = ([\tau(h)], \sigma_*(f,b)) \in LS_{m-1} = \widehat{H}^m(\mathbb{Z}_2; I) \oplus \mathrm{UNil}_{m+1} .$$

The first component is the obstruction to the existence of a normal bordism to a split homotopy equivalence, the image $[\tau(h)] \in \widehat{H}^m(\mathbb{Z}_2; I)$ of the Tate $\mathbb{Z}_2$-cohomology class of the Whitehead torsion

$$\tau(h) = (-)^{m+1}\tau(h)^* \in \widehat{H}^{m+1}(\mathbb{Z}_2; Wh(\pi_1(M))) .$$

The second component is the surgery obstruction

$$\sigma_*(f,b) \in \mathrm{UNil}_{m+1} \subseteq L_{m+1}(\mathbb{Z}[\pi_1(M)])$$

of a normal bordism

$$(f,b) : (W; M', M'') \longrightarrow M \times ([0,1]; \{0\}, \{1\})$$

from $h : M' \longrightarrow M$ to a split homotopy equivalence $h' : M'' \longrightarrow M$ given by the nilpotent normal cobordism construction of Cappell [10] in the case $[\tau(h)] = 0 \in \widehat{H}^m(\mathbb{Z}_2; I)$.

□

The I-intermediate quadratic L-groups $L^I_*(\mathbb{Z}[\rho])$ in 8.3 are such that there is defined a Rothenberg-type exact sequence

$$\dots \longrightarrow L_m(\mathbb{Z}[\rho]) \longrightarrow L^I_m(\mathbb{Z}[\rho]) \longrightarrow \widehat{H}^m(\mathbb{Z}_2; I) \longrightarrow L_{m-1}(\mathbb{Z}[\rho]) \longrightarrow \dots$$

with $\widehat{H}^*(\mathbb{Z}_2;I)$ the Tate $\mathbb{Z}_2$-cohomology groups of the duality involution $*:I\longrightarrow I$.

See Ranicki [46, §7.6] for a chain complex interpretation of the nilpotent normal cobordism construction and the identification

$$LS_{m-1} \;=\; \widehat{H}^m(\mathbb{Z}_2;I)\oplus \mathrm{UNil}_{m+1}\ .$$

Example 8.4 In the situation of 8.1

$$\pi_1(M) \;=\; \pi\times\mathbb{Z}\ \ ,\ \ \pi_1(N) \;=\; \pi$$

it is the case that

$$\begin{array}{l} Wh(\pi_1(M)) \;=\; Wh(\pi)\oplus \widetilde{K}_0(\mathbb{Z}[\pi])\oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])\oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])\ , \\ \mathrm{UNil}_* \;=\; 0\ \ ,\ \ \widehat{H}^*(\mathbb{Z}_2;\widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])\oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])) \;=\; 0\ \ ,\ \ I \;=\; \widetilde{K}_0(\mathbb{Z}[\pi]) \end{array}$$

and the codimension 1 h-splitting obstruction of 8.3

$$[s(h)] \;=\; [\tau(h)] \;=\; [C(\widetilde{M}'^{+},\widetilde{M}^{+})]\in LS_{m-1} \;=\; \widehat{H}^m(\mathbb{Z}_2;\widetilde{K}_0(\mathbb{Z}[\pi]))$$

is the Tate $\mathbb{Z}_2$-cohomology class of the codimension 1 splitting obstruction

$$\begin{array}{rl} [\tau(h)] \;=\; (-)^{m+1}[\tau(h)]^*\in & \mathrm{coker}(Wh(\pi)\longrightarrow Wh(\pi\times\mathbb{Z})) \\ & =\; \widetilde{K}_0(\mathbb{Z}[\pi])\oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])\oplus \widetilde{\mathrm{Nil}}_0(\mathbb{Z}[\pi])\ . \end{array}$$

□

Corollary 8.5 *Let π be a finitely presented group such that:*

either (A) $\pi=\pi_1*_\rho\pi_2$ is an amalgamated free product, with π_1,π_2,ρ finitely presented,

or (B) $\pi=\pi_1*_\rho\{z\}$ is an HNN extension, with π_1,ρ finitely presented, and let

$$I \;=\; \mathrm{im}(Wh(\pi)\longrightarrow\widetilde{K}_0(\mathbb{Z}[\rho]))\subseteq \widetilde{K}_0(\mathbb{Z}[\rho])\ .$$

The algebraic L-theory Mayer-Vietoris exact sequence

$$\begin{array}{l} \dots\longrightarrow L^I_i(\mathbb{Z}[\rho])\oplus\mathrm{UNil}_{i+1}\longrightarrow L_i(\mathbb{Z}[\pi_1])\oplus L_i(\mathbb{Z}[\pi_2])\longrightarrow L_i(\mathbb{Z}[\pi]) \\ \qquad\longrightarrow L^I_{i-1}(\mathbb{Z}[\rho])\oplus\mathrm{UNil}_i\longrightarrow L_{i-1}(\mathbb{Z}[\pi_1])\oplus L_{i-1}(\mathbb{Z}[\pi_2])\longrightarrow\ \dots \end{array}$$

extends to a Mayer-Vietoris exact sequence of $\mathbb{S}$-groups

$$\begin{array}{l} \dots\longrightarrow \mathbb{S}^I_i(B\rho)\oplus\mathrm{UNil}_{i+1}\longrightarrow \mathbb{S}_i(B\pi_1)\oplus\mathbb{S}_i(B\pi_2)\longrightarrow\mathbb{S}_i(B\pi) \\ \qquad\longrightarrow \mathbb{S}^I_{i-1}(B\rho)\oplus\mathrm{UNil}_i\longrightarrow \mathbb{S}_{i-1}(B\pi_1)\oplus\mathbb{S}_{i-1}(B\pi_2)\longrightarrow\ \dots \end{array}$$

interpreting $\pi_2 = \emptyset$ in case (B).

Proof This a formal consequence of 8.4 and the Mayer-Vietoris exact sequence of generalized homology theory

$$\begin{aligned} \dots \longrightarrow h_i(B\rho) \longrightarrow h_i(B\pi_1) \oplus h_i(B\pi_2) \longrightarrow h_i(B\pi) \\ \longrightarrow h_{i-1}(B\rho) \longrightarrow h_{i-1}(B\pi_1) \oplus h_{i-1}(B\pi_2) \longrightarrow \ \dots\ , \end{aligned}$$

with $h_*(\ ) = H_*(\ ;\mathbb{L}_\bullet)$.

□

Theorem 8.6 (Cappell [11]) *The Novikov conjecture holds for the class of finitely presented groups π obtained from $\{1\}$ by amalgamated free products and HNN extensions along injections.*

Proof If the groups $G = \pi_1, \pi_2, \rho$ in 8.5 are such that the algebraic L-theory assembly maps $A : H_*(BG;\mathbb{L}_\bullet(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[G])$ are rational isomorphisms then so is the algebraic L-theory assembly map $A : H_*(B\pi;\mathbb{L}_\bullet(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[\pi])$, and the Novikov conjecture holds for π. Apply the 5-lemma to the $\mathbb{S}$-group Mayer-Vietoris exact sequence of 8.5, noting that the UNil-groups have exponent ≤ 8 and so make no rational contribution.

□

(The actual inductively defined class of groups for which the Novikov conjecture was verified in [11] is somewhat larger.)

Remark 8.7 The finite presentation conditions in 8.5, 8.6 are necessary because the L-theory Mayer-Vietoris exact sequence of Cappell [9] was only stated for the group rings of finitely presented groups, since the proof used geometric methods. In fact, it is possible to state and prove an L-theory Mayer-Vietoris exact sequence for amalgamated free products and HNN extensions along injections of any rings with involution, allowing the hypothesis of finite presentation to be dropped. Pending the definitive account of Ranicki [51], here is a brief account of the algebraic proof of the L-theory sequence, extending the method used by Waldhausen [55] to prove the algebraic K-theory Mayer-Vietoris exact sequence.

A ring morphism $f : R\longrightarrow R'$ determines induction and restriction functors

$$\begin{aligned} f_! \ &:\ \{R\text{-modules}\} \longrightarrow \{R'\text{-modules}\}\ ; \\ &\qquad M \longrightarrow f_!M \ =\ R' \otimes_R M \text{ with } r'(1\otimes x) = r'\otimes x\ , \\ f^! \ &:\ \{R'\text{-modules}\} \longrightarrow \{R\text{-modules}\}\ ; \\ &\qquad M' \longrightarrow f^!M' \ =\ M' \text{ with } rx' = f(r)x'\ . \end{aligned}$$

Let R be a ring such that:

either (A) $R = R_1 *_S R_2$ is the amalgamated free product determined by injections of rings $i_1 : S\longrightarrow R_1$, $i_2 : S\longrightarrow R_2$ such that R_1, R_2 are free as

(S,S)-bimodules,

or (B) $R = R_1 *_S [z,z^{-1}]$ is the HNN extension determined by two injections $i_1, i_1' : S \longrightarrow R_1$, with respect to both of which R_1 is a free (S,S)-bimodule.

As in the Serre-Bass theory there is an infinite tree T with augmented simplicial R-module chain complex

$$\Delta(T;R) \ : \ 0 \longrightarrow k_! k^! R \longrightarrow (j_1)_! j_1^! R \oplus (j_2)_! j_2^! R \longrightarrow R \longrightarrow 0$$

with

$$j_1 \ : \ R_1 \longrightarrow R \ \ , \ \ j_2 \ : \ R_2 \longrightarrow R \ \ , \ \ k \ = \ j_1 i_1 \ = \ j_2 i_2 \ : \ S \longrightarrow R$$

the inclusions, and

$$j_1^! R \ = \ \sum_{T_1^{(0)}} R_1 \ \ , \ \ j_2^! R \ = \ \sum_{T_2^{(0)}} R_2 \ \ , \ \ k^! R \ = \ \sum_{T^{(1)}} S \ \ ,$$

setting $R_2 = 0$ in case (B). Thus for any finite f.g. free R-module chain complex C there is defined a Mayer-Vietoris presentation

$$(*) \ \ C \otimes_R \Delta(T;R) \ : \ 0 \longrightarrow k_! k^! C \longrightarrow (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \longrightarrow C \longrightarrow 0$$

with $j_1^! C$ an infinitely generated free R_1-module chain complex, $j_2^! C$ an infinitely generated free R_2-module chain complex, and $k^! C$ an infinitely generated free S-module chain complex. For any subtree $U \subset T$ the augmented simplicial R-module chain complex $\Delta(U;R)$ defines a Mayer-Vietoris presentation of R

$$C(U;R) \ : \ 0 \longrightarrow k_! \sum_{U^{(1)}} S \longrightarrow (j_1)_! \sum_{U_1^{(0)}} R_1 \oplus (j_2)_! \sum_{U_2^{(0)}} R_2 \longrightarrow R \longrightarrow 0 \ ,$$

such that if U is finite then $\sum\limits_{U_1^{(0)}} R_1$ is a f.g. free R_1-module, $\sum\limits_{U_2^{(0)}} R_2$ is a f.g. free R_2-module, and $\sum\limits_{U^{(1)}} S$ is a f.g. free S-module. Let C be n-dimensional, with

$$C_r \ = \ R^{c_r} \ \ (0 \leq r \leq n)$$

a f.g. free R-module of rank c_r. There exist finite subtrees

$$U_r \subset T \ \ (0 \leq r \leq n)$$

such that the f.g. free submodules

$$(D_1)_r \ = \ \sum_{U_{r,1}^{(0)}} R_1^{c_r} \subset j_1^! C_r \ = \ \sum_{T_1^{(0)}} R_1^{c_r} \ ,$$

$$(D_2)_r \ = \ \sum_{U_{r,2}^{(0)}} R_2^{c_r} \subset j_2^! C_r \ = \ \sum_{T_2^{(0)}} R_2^{c_r} \ ,$$

$$E_r \ = \ \sum_{U_r^{(1)}} S^{c_r} \subset k^! C_r \ = \ \sum_{T^{(1)}} S^{c_r}$$

define f.g. free subcomplexes

$$D_1 \subset j_1^! C \quad , \quad D_2 \subset j_2^! C \quad , \quad E \;=\; D_1 \cap D_2 \subset k^! C$$

with a Mayer-Vietoris presentation

$$0 \longrightarrow k_! E \longrightarrow (j_1)_! D_1 \oplus (j_2)_! D_2 \longrightarrow C \longrightarrow 0 \ .$$

This type of algebraic transversality (a generalization of the linearization trick of Higman [22] for matrices over a Laurent polynomial extension) was used in [55] to obtain the Mayer-Vietoris exact sequence in algebraic K-theory

$$\begin{aligned} \dots \longrightarrow K_i(S) \oplus \widetilde{\mathrm{Nil}}_{i+1} &\longrightarrow K_i(R_1) \oplus K_i(R_2) \longrightarrow K_i(R) \\ &\longrightarrow K_{i-1}(S) \oplus \widetilde{\mathrm{Nil}}_i \longrightarrow K_{i-1}(R_1) \oplus K_{i-1}(R_2) \longrightarrow K_{i-1}(R) \longrightarrow \dots \end{aligned}$$

with $K_i(R) {\longrightarrow} \widetilde{\mathrm{Nil}}_i$ split surjections.

Now suppose that R, R_1, R_2, S are rings with involution. Given a finite f.g. free R-module chain complex C apply $C \otimes_R -$ to $(*)$ above, to obtain an exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$(**)\ \ 0 \longrightarrow k^! C \otimes_S k^! C \longrightarrow (j_1^! C \otimes_{R_1} j_1^! C) \oplus (j_2^! C \otimes_{R_2} j_2^! C) \longrightarrow C \otimes_R C \longrightarrow 0$$

with $\mathbb{Z}[\mathbb{Z}_2]$ acting by $x \otimes y {\longrightarrow} \pm\, y \otimes x$. In the terminology of Ranicki [45] the following algebraic transversality holds : for any n-dimensional quadratic complex over R

$$(C, \psi \in Q_n(C) = H_n(\mathbb{Z}_2; C \otimes_R C))$$

there exist an $(n-1)$-dimensional quadratic complex (E, θ) over S and n-dimensional quadratic pairs

$$\Gamma_1 \;=\; ((i_1)_! E {\longrightarrow} D_1, (\delta_1 \theta, (i_1)_! \theta)) \quad , \quad \Gamma_2 \;=\; ((i_2)_! E {\longrightarrow} D_2, (\delta_2 \theta, (i_2)_! \theta))$$

over R_1, R_2 such that the union n-dimensional quadratic complex over R is homotopy equivalent to (C, ψ)

$$(j_1)_! \Gamma_1 \cup (j_2)_! \Gamma_2 \;\simeq\; (C, \psi) \ .$$

The algebraic Poincaré splitting method of Ranicki [46, §§7.5,7.6] gives a Mayer-Vietoris exact sequence in quadratic L-theory

$$\begin{aligned} \dots \longrightarrow L_i^I(S) \oplus \mathrm{UNil}_{i+1} &\longrightarrow L_i(R_1) \oplus L_i(R_2) \longrightarrow L_i(R) \\ &\longrightarrow L_{i-1}^I(S) \oplus \mathrm{UNil}_i \longrightarrow L_{i-1}(R_1) \oplus L_{i-1}(R_2) \longrightarrow L_{i-1}(R) \longrightarrow \dots \end{aligned}$$

with $L_i(R) \longrightarrow \mathrm{UNil}_i$ split surjections and

$$I \;=\; \mathrm{im}(K_1(R) \longrightarrow K_0(S)) \;=\; \ker(K_0(S) \longrightarrow K_0(R_1) \oplus K_0(R_2)) \subseteq K_0(S)\ ,$$

using the algebraic transversality given by $(**)$ to replace the geometric transversality of [46, 7.5.1]. There is a corresponding Mayer-Vietoris exact sequence in symmetric L-theory. This type of algebraic Poincaré transversality was already used in Milgram and Ranicki [31] and Ranicki [48] for the L-theory of Laurent polynomial extensions and the associated lower L-theory.

□

§9. With one bound

The applications of bounded and controlled algebra to splitting theorems in topology and the Novikov conjectures depend on the development of an algebraic theory of transversality : algebraic Poincaré complexes in categories associated to topological spaces are shown to have enough transversality properties of manifolds mapping to the spaces to construct a 'disassembly' map. For the sake of simplicity we shall restrict attention to the bounded algebra of Pedersen and Weibel [41] and Ranicki [48], even though it is the continuously controlled algebra of Anderson, Connolly, Ferry and Pedersen [1] which is actually used by Carlsson and Pedersen [14].

Given a metric space X and a ring A let $\mathbb{C}_X(A)$ be the **X-bounded free A-module** additive category, with objects the direct sum of f.g. free A-modules graded by X

$$M \;=\; \sum_{x \in X} M(x)$$

such that $M(K) = \sum\limits_{x \in K} M(x)$ is a f.g. free A-module for every bounded subspace $K \subseteq X$, and with morphisms the A-module morphisms

$$f \;=\; \{f(y,x)\} \;:\; M \;=\; \sum_{x \in X} M(x) \longrightarrow N \;=\; \sum_{y \in X} N(y)$$

for which there exists a number $b > 0$ with $f(y,x) = 0 : M(x) \longrightarrow N(y)$ for all $x, y \in X$ with $d(x,y) > b$.

A **proper eventually Lipschitz map** $f : X \longrightarrow Y$ of metric spaces is a function (not necessarily continuous) such that the inverse image of a bounded set is a bounded set, and there exist numbers $r, k > 0$ depending only on f such that for all $s > r$ and all $x, y \in X$ with $d(x,y) < s$ it is the case that $d(f(x), f(y)) < ks$. Such a map induces a functor

$$\begin{aligned} f_! \;&:\; \mathbb{C}_X(A) \longrightarrow \mathbb{C}_Y(A)\ ;\\ M \;=\; \sum_{x \in X} M(x) \longrightarrow f_! M \;&=\; \sum_{y \in Y} \Bigg(\sum_{x \in f^{-1}(y)} M(x) \Bigg)\ . \end{aligned}$$

If $f : X \longrightarrow Y$ is a homotopy equivalence in the proper eventually Lipschitz category then $f_! : \mathbb{C}_X(A) \longrightarrow \mathbb{C}_Y(A)$ is an equivalence of additive categories, inducing isomorphisms in algebraic K-theory.

Let $\mathbb{P}_X(A)$ be the idempotent completion of $\mathbb{C}_X(A)$, the additive category in which an object (M,p) is an object M in $\mathbb{C}_X(A)$ together with a projection $p = p^2 : M \longrightarrow M$, and a morphism $f : (M,p) \longrightarrow (N,q)$ is a morphism $f : M \longrightarrow N$ in $\mathbb{C}_X(A)$ such that $qfp = f : M \longrightarrow N$. The reduced projective class group of $\mathbb{P}_X(A)$ is defined by

$$\widetilde{K}_0(\mathbb{P}_X(A)) \;=\; \mathrm{coker}(K_0(\mathbb{C}_X(A)) \longrightarrow K_0(\mathbb{P}_X(A))) \;.$$

Example 9.1 A bounded metric space X is contractible in the proper eventually Lipschitz category, so that $\mathbb{C}_X(A)$ is equivalent to the additive category of based f.g. free A-modules, $\mathbb{P}_X(A)$ is equivalent to the additive category of f.g. projective A-modules and

$$\begin{aligned} K_*(\mathbb{C}_X(A)) &= K_*(\mathbb{P}_X(A)) \;=\; K_*(A) \quad (* \neq 0) \;, \\ K_0(\mathbb{C}_X(A)) &= \mathrm{im}(K_0(\mathbb{Z}) \longrightarrow K_0(A)) \quad , \quad K_0(\mathbb{P}_X(A)) \;=\; K_0(A) \;, \\ \widetilde{K}_0(\mathbb{P}_X(A)) &= \mathrm{coker}(K_0(\mathbb{Z}) \longrightarrow K_0(A)) \;=\; \widetilde{K}_0(A) \;. \end{aligned}$$

□

Suppose given a metric space X with a decomposition

$$X \;=\; X^+ \cup X^- \;.$$

Define for any $b \geq 0$ the subspaces

$$\begin{aligned} X_b^+ &= \{x \in X \,|\, d(x,y) \leq b \text{ for some } y \in X^+\} \;, \\ X_b^- &= \{x \in X \,|\, d(x,z) \leq b \text{ for some } z \in X^-\} \;, \\ Y_b &= \{x \in X \,|\, d(x,y) \leq b \text{ and } d(x,z) \leq b \text{ for some } y \in X^+, z \in X^-\} \;. \end{aligned}$$

The inclusions $X^+ \longrightarrow X_b^+$, $X^- \longrightarrow X_b^-$ are homotopy equivalences in the proper eventually Lipschitz category, so that

$$K_*(\mathbb{C}_{X_b^+}(A)) \;=\; K_*(\mathbb{C}_{X^+}(A)) \quad , \quad K_*(\mathbb{C}_{X_b^-}(A)) \;=\; K_*(\mathbb{C}_{X^-}(A)) \;.$$

Proposition 9.2 (Pedersen and Weibel [41], Carlsson [12]) *For any metric space X and any decomposition $X = X^+ \cup X^-$ there is defined a Mayer-Vietoris exact sequence in bounded K-theory*

$$\begin{aligned} \dots \longrightarrow\; & K_n(\mathbb{P}_{X^+}(A)) \oplus K_n(\mathbb{P}_{X^-}(A)) \longrightarrow K_n(\mathbb{P}_X(A)) \\ & \xrightarrow{\partial} \varinjlim_b K_{n-1}(\mathbb{P}_{Y_b}(A)) \longrightarrow K_{n-1}(\mathbb{P}_{X^+}(A)) \oplus K_{n-1}(\mathbb{P}_{X^-}(A)) \longrightarrow \dots \end{aligned}$$

with

$$Y_b \;=\; \{x\in X\,|\, d(x,y)\leq b \text{ and } d(x,z)\leq b \text{ for some } y\in X^+, z\in X^-\}\ .$$

Proof The original proof in [41] (for open cones) and the generalization in [12] use the heavy machinery of the algebraic K-theory spectra. For $n=1$ there is a direct proof in Ranicki [48], as follows. Every finite chain complex C in $\mathbb{C}_X(A)$ is such that there exist subcomplexes $C^+,C^-\subseteq C$ with $C^\pm$ defined in $\mathbb{C}_{X^\pm_b}(A)$ and $C^+\cap C^-$ defined in $\mathbb{C}_{Y_b}(A)$ for some $b\geq 0$. Thus C admits a 'Mayer-Vietoris presentation'

$$0 \longrightarrow C^+\cap C^- \longrightarrow C^+\oplus C^- \longrightarrow C \longrightarrow 0\ .$$

If C is contractible then C^+ and C^- are $\mathbb{P}_{Y_b}(A)$-finitely dominated chain complexes. The reduced version $\widetilde{\partial}$ of the connecting map ∂ in the Mayer-Vietoris exact sequence

$$\begin{aligned}\dots \longrightarrow K_1(\mathbb{C}_{X^+}(A))\oplus K_1(\mathbb{C}_{X^-}(A)) \longrightarrow K_1(\mathbb{C}_X(A))\\ \xrightarrow{\partial} \varinjlim_b K_0(\mathbb{P}_{Y_b}(A)) \longrightarrow K_0(\mathbb{P}_{X^+}(A))\oplus K_0(\mathbb{P}_{X^-}(A)) \longrightarrow \dots\end{aligned}$$

sends the Whitehead torsion $\tau(C)\in K_1(\mathbb{C}_X(A))$ to the reduced projective class

$$\widetilde{\partial}\tau(C) \;=\; [C^+] \;=\; -[C^-]\in \varinjlim_b \widetilde{K}_0(\mathbb{P}_{Y_b}(A))\ ,$$

which is such that $\widetilde{\partial}\tau(C)=0$ if and only if there exists a presentation (C^+,C^-) with $C^+,C^-,C^+\cap C^-$ contractible. See [48] for further details. □

Example 9.3 For any metric space Y let

$$X \;=\; Y\times\mathbb{R}\ \ ,\ \ X^+ \;=\; Y\times\mathbb{R}^+\ \ ,\ \ X^- \;=\; Y\times\mathbb{R}^-\ ,$$

so that

$$X \;=\; X^+\cup X^-\ \ ,\ \ X^+\cap X^- \;=\; Y\times\{0\}\ .$$

In this case

$$\begin{aligned}K_*(\mathbb{P}_{X^+}(A)) &\;=\; K_*(\mathbb{P}_{X^-}(A)) \;=\; 0 \ \ \text{(Eilenberg swindle)}\ ,\\ K_{*+1}(\mathbb{P}_X(A)) &\;=\; \varinjlim_b K_*(\mathbb{P}_{Y_b}(A)) \;=\; K_*(\mathbb{P}_Y(A))\ .\end{aligned}$$

The connecting map

$$\partial\ :\ K_1(\mathbb{C}_X(A)) \;=\; K_1(\mathbb{P}_X(A)) \longrightarrow K_0(\mathbb{P}_Y(A))\ ;\ \tau(C)\longrightarrow [C^+] \;=\; -[C^-]$$

is an isomorphism, with $\tau(C)$ the torsion of a contractible finite chain complex C in $\mathbb{C}_X(A)$ and (C^+, C^-) any Mayer-Vietoris presentation of C.

$\square$

A CW complex M is **X-bounded** if it is equipped with a proper map $M \longrightarrow X$ such that the diameters of the images of the cells of M are uniformly bounded in X, so that the cellular chain complex $C(M)$ is defined in $\mathbb{C}_X(\mathbb{Z})$. We shall only be concerned with metric spaces X which are allowable in the sense of Ferry and Pedersen [18], and finite-dimensional X-bounded CW complexes M which are (-1)- and 0-connected in the sense of [18], with a bounded fundamental group π. The cellular chain complex $C(\widetilde{M})$ of the π-cover $\widetilde{M}$ of M is defined in $\mathbb{C}_X(\mathbb{Z}[\pi])$. Similarly for cellular maps, with induced chain maps in $\mathbb{C}_X(\mathbb{Z}[\pi])$.

If $f : M \longrightarrow N$ is an X-bounded homotopy equivalence of X-bounded CW complexes with bounded fundamental group π the X-bounded Whitehead torsion is given by

$$\begin{aligned} \tau(f) &= \tau(\widetilde{f} : C(\widetilde{M}) \longrightarrow C(\widetilde{N})) \\ \in Wh(\mathbb{C}_X(\mathbb{Z}[\pi])) &= \operatorname{coker}(K_1(\mathbb{C}_X(\mathbb{Z})) \oplus \{\pm\pi\} \longrightarrow K_1(\mathbb{C}_X(\mathbb{Z}[\pi]))) \end{aligned}$$

with $\widetilde{f} : C(\widetilde{M}) \longrightarrow C(\widetilde{N})$ the induced chain equivalence in $\mathbb{C}_X(\mathbb{Z}[\pi])$. If $X = X^+ \cup X^-$ the algebraic splitting obstruction

$$\partial\tau(f) \in \varinjlim_b K_0(\mathbb{P}_{Y_b}(\mathbb{Z}[\pi]))$$

is such that $\partial\tau(f) = 0$ if and only if f is X-bounded homotopic to an X-bounded homotopy equivalence (also denoted by f) such that the restrictions $f| : f^{-1}(Y) \longrightarrow Y$ are Y-bounded homotopy equivalences, with $Y = X^+, X^-, Y_b$ (for some $b \geq 0$).

The **lower K-groups** $K_{-*}(A)$ of Bass [3, XII] are defined for any ring A to be such that

$$K_1(A[\mathbb{Z}^i]) = \sum_{j=0}^{i} \binom{i}{j} K_{1-j}(A) \oplus \widetilde{\mathrm{Nil}}\text{-groups} .$$

For a group ring $A = \mathbb{Z}[\pi]$

$$Wh(\pi \times \mathbb{Z}^i) = \sum_{j=0}^{i} \binom{i}{j} Wh_{1-j}(\pi) \oplus \widetilde{\mathrm{Nil}}\text{-groups} ,$$

where the lower Whitehead group are defined by

$$Wh_{1-j}(\pi) = \begin{cases} Wh(\pi) & \text{if } j = 0 \\ \widetilde{K}_0(\mathbb{Z}[\pi]) & \text{if } j = 1 \\ K_{1-j}(\mathbb{Z}[\pi]) & \text{if } j \geq 2 . \end{cases}$$

Bass, Heller and Swan [4] proved that $Wh(\mathbb{Z}^i) = 0$ $(i \geq 1)$, so that

$$Wh_{1-*}(\{1\}) \ = \ 0 \ .$$

Example 9.4 (Pedersen [38]) The $\mathbb{R}^i$-bounded K-groups of a ring A are the lower K-groups of A

$$K_*(\mathbb{P}_{\mathbb{R}^i}(A)) \ = \ K_{*-i}(A) \ .$$

The $\mathbb{R}^i$-bounded Whitehead groups of a group π are the lower Whitehead groups

$$Wh(\mathbb{C}_{\mathbb{R}^i}(\pi)) \ = \ Wh_{1-i}(\pi) \ \ (i \geq 1) \ .$$

□

There is a corresponding development of bounded L-theory.

An involution on the ground ring A induces a duality involution on the X-bounded A-module category

$$* \ : \ \mathbb{C}_X(A) \longrightarrow \mathbb{C}_X(A) \ ; \ M \ = \ \sum_{x \in X} M(x) \longrightarrow M^* \ = \ \sum_{x \in X} M(x)^* \ ,$$

with $M(x)^* = \mathrm{Hom}_A(M(x), A)$.

Definition 9.5 (Ranicki [47], [48]) The **X-bounded symmetric L-groups** $L^*(\mathbb{C}_X(A))$ are the cobordism groups of symmetric Poincaré complexes in $\mathbb{C}_X(A)$. Similarly for the **X-bounded quadratic L-groups** $L_*(\mathbb{C}_X(A))$.

□

The symmetrization maps $1 + T \ : \ L_*(\mathbb{C}_X(A)) \longrightarrow L^*(\mathbb{C}_X(A))$ are isomorphisms modulo 8-torsion. For bounded X $\mathbb{C}_X(A)$ is equivalent to the category of f.g. free A-modules and

$$L^*(\mathbb{C}_X(A)) \ = \ L^*(A) \ \ , \ \ L_*(\mathbb{C}_X(A)) \ = \ L_*(A) \ .$$

The functor

$$\begin{aligned}&\{\text{metric spaces and proper eventually Lipschitz maps}\}\\ &\qquad \longrightarrow \{\mathbb{Z}\text{-graded abelian groups}\} \ ; \ X \longrightarrow L_*(\mathbb{C}_X(A))\end{aligned}$$

was shown in Ranicki [48] to be within a bounded distance (in the non-technical sense) of being a generalized homology theory. The functor is homotopy invariant, and has the following bounded excision property:

Proposition 9.6 (Ranicki [48, 14.2]) *For any metric space X and any decomposition $X = X^+ \cup X^-$ there is defined a Mayer-Vietoris exact sequence in bounded L-theory*

$$\begin{aligned}&\dots \longrightarrow L_n(\mathbb{C}_{X^+}(A)) \oplus L_n(\mathbb{C}_{X^-}(A)) \longrightarrow L_n(\mathbb{C}_X(A))\\ &\quad \xrightarrow{\partial} \varinjlim_b L^{J_b}_{n-1}(\mathbb{P}_{Y_b}(A)) \longrightarrow L_{n-1}(\mathbb{C}_{X^+}(A)) \oplus L_{n-1}(\mathbb{C}_{X^-}(A)) \longrightarrow \dots \ ,\end{aligned}$$

with

$$\begin{aligned} Y_b &= \{x \in X \mid d(x,y) \le b \text{ and } d(x,z) \le b \text{ for some } y \in X^+, z \in X^-\}\ , \\ J_b &= \ker(\widetilde{K}_0(\mathbb{P}_{Y_b}(A)) \longrightarrow \widetilde{K}_0(\mathbb{P}_X(A))) \\ &\qquad \subseteq \widetilde{K}_0(\mathbb{P}_{Y_b}(A)) \ = \ \mathrm{coker}(K_0(\mathbb{C}_{Y_b}(A)) \longrightarrow K_0(\mathbb{P}_{Y_b}(A)))\ . \end{aligned}$$

The J_b-intermediate quadratic L-groups $L_*^{J_b}(\mathbb{P}_{Y_b}(A))$ are such that there is defined a Rothenberg-type exact sequence

$$\begin{aligned} \dots \longrightarrow L_n(\mathbb{C}_{Y_b}(A)) \longrightarrow L_n^{J_b}(\mathbb{P}_{Y_b}(A)) & \\ \longrightarrow \widehat{H}^n(\mathbb{Z}_2; J_b) \longrightarrow L_{n-1}(\mathbb{C}_{Y_b}(A)) \longrightarrow \dots & \end{aligned}$$

with $\widehat{H}^*(\mathbb{Z}_2; J_b)$ the Tate $\mathbb{Z}_2$-cohomology groups of the duality involution $* : J_b \longrightarrow J_b$.

□

The **lower L-groups** $L_*^{\langle -j\rangle}(A)$ of Ranicki [43] are defined for any ring with involution A to be such that

$$L_n(A[\mathbb{Z}^i]) \ = \ \sum_{j=0}^{i} \binom{i}{j} L_{n-j}^{\langle 1-j\rangle}(A)\ ,$$

with $L_*^{\langle 1\rangle}(A) = L_*^h(A) = L_*(A)$ the free L-groups and $L_*^{\langle 0\rangle}(A) = L_*^p(A)$ the projective L-groups.

Example 9.7 The $\mathbb{R}^i$-bounded L-groups of a ring with involution A were identified in Ranicki [48] with the lower L-groups of A

$$L_*(\mathbb{C}_{\mathbb{R}^i}(A)) \ = \ L_{*-i}^{\langle 1-i\rangle}(A)\ .$$

□

Definition 9.8 The **X-bounded symmetric signature** of an m-dimensional X-bounded geometric Poincaré complex M with bounded fundamental group π is the cobordism class

$$\sigma^*(M) \ = \ (C(\widetilde{M}), \phi) \in L^m(\mathbb{C}_X(\mathbb{Z}[\pi]))\ ,$$

with ϕ the symmetric structure of the Poincaré duality chain equivalence $[M] \cap - : C(\widetilde{M})^{m-*} \longrightarrow C(\widetilde{M})$.

□

The standard algebraic mapping cylinder argument shows:

Proposition 9.9 *The X-bounded symmetric signature is an X-bounded homotopy invariant of an X-bounded geometric Poincaré complex.*

□

Definition 9.10 Let $(f, b) : (M', \partial M') \longrightarrow (M, \partial M)$ be a normal map from an X-bounded m-dimensional manifold with boundary $(M', \partial M')$ to an X-bounded m-dimensional geometric Poincaré pair $(M, \partial M)$, such that M has bounded fundamental group π, and $\partial f : \partial M' \longrightarrow \partial M$ is an X-bounded homotopy equivalence. The **X-bounded quadratic signature** of (f, b) is the quadratic Poincaré cobordism class

$$\sigma_*(f, b) \ = \ (C(f^!), \psi) \in L_m(\mathbb{C}_X(\mathbb{Z}[\pi])) \ ,$$

with ψ the quadratic structure on the algebraic mapping cone $C(f^!)$ of the Umkehr chain map in $\mathbb{C}_X(\mathbb{Z}[\pi])$

$$f^! \ : \ C(\widetilde{M}) \ \simeq \ C(\widetilde{M}, \partial\widetilde{M})^{m-*} \ \xrightarrow{f^*} \ C(\widetilde{M}', \partial\widetilde{M}')^{m-*} \ \simeq \ C(\widetilde{M}) \ .$$

□

The quadratic Poincaré complex $(C(f^!), \psi)$ in 9.10 can be obtained in two (equivalent) ways: either by the X-bounded version of Wall [56, §§5,6] by first performing geometric surgery below the middle dimension to obtain a quadratic form/formation in $\mathbb{C}_X(\mathbb{Z}[\pi])$ as in Ferry and Pedersen [18], or by the X-bounded version of Ranicki [45], using algebraic Poincaré complexes and the chain bundle theory of Weiss [57].

Proposition 9.11 *The X-bounded quadratic signature is the bounded surgery obstruction of Ferry and Pedersen [18], such that $\sigma_*(f, b) = 0$ if (and for $m \geq 5$) (f, b) is normal bordant to an X-bounded homotopy equivalence.*

□

The symmetrization of the X-bounded quadratic signature is the X-bounded symmetric signature

$$(1 + T)\sigma_*(f, b) \ = \ \sigma^*(M' \cup_{\partial f} -M) \in L^m(\mathbb{C}_X(\mathbb{Z}[\pi])) \ .$$

Let M be an X-bounded CW complex with bounded fundamental group π. See Ranicki [49, Appendix C5] for the construction of the **locally finite assembly maps**

$$A^{lf} \ : \ \mathbb{H}^{lf}_\bullet(M; \mathbb{L}_\bullet) \longrightarrow \mathbb{L}_\bullet(\mathbb{C}_X(\mathbb{Z}[\pi])) \ .$$

The locally finite homology spectrum $\mathbb{H}^{lf}_\bullet(M; \mathbb{L}_\bullet)$ is defined using locally finite sheaves over M of quadratic Poincaré complexes over $\mathbb{Z}$, and the L-spectrum $\mathbb{L}_\bullet(\mathbb{C}_X(\mathbb{Z}[\pi]))$ is defined using quadratic Poincaré complexes in $\mathbb{C}_X(\mathbb{Z}[\pi])$. The **$X$-bounded structure groups of M**

$$\mathbb{S}^b_*(M) \ = \ \pi_*(A^{lf} : \mathbb{H}^{lf}_\bullet(M; \mathbb{L}_\bullet) \longrightarrow \mathbb{L}_\bullet(\mathbb{C}_X(\mathbb{Z}[\pi])))$$

are the relative groups in the X**-bounded algebraic surgery exact sequence**

$$\dots \longrightarrow \mathbb{S}^b_{m+1}(M) \longrightarrow H^{lf}_m(M;\mathbb{L}_\bullet) \xrightarrow{A^{lf}} L_m(\mathbb{C}_X(\mathbb{Z}[\pi])) \\ \longrightarrow \mathbb{S}^b_m(M) \longrightarrow \dots .$$

Proposition 9.12 (Ranicki [48], [49])
(i) *An m-dimensional X-bounded manifold M with bounded fundamental group π has an $\mathbb{L}^\bullet(\mathbb{Z})$-coefficient fundamental class $[M]_{\mathbb{L}} \in H^{lf}_m(M;\mathbb{L}^\bullet(\mathbb{Z}))$ with locally finite assembly the X-bounded symmetric signature*

$$A^{lf}([M]_{\mathbb{L}}) \;=\; \sigma^*(M) \in L^m(\mathbb{C}_X(\mathbb{Z}[\pi])) .$$

A normal map $(f,b) : M' \longrightarrow M$ has a **normal invariant**

$$[f,b]_{\mathbb{L}} \in H^{lf}_m(M;\mathbb{L}_\bullet) \;=\; H^0(M;\mathbb{L}_\bullet) \;=\; [M, G/TOP] .$$

The surgery obstruction of (f,b) is the image of the normal invariant under the locally finite assembly map

$$\begin{aligned}\sigma_*(f,b) \;&=\; A^{lf}([f,b]_{\mathbb{L}}) \in \mathrm{im}(A^{lf} : H^{lf}_m(M;\mathbb{L}_\bullet) \longrightarrow L_m(\mathbb{C}_X(\mathbb{Z}[\pi]))) \\ &=\; \ker(L_m(\mathbb{C}_X(\mathbb{Z}[\pi])) \longrightarrow \mathbb{S}^b_m(M)) .\end{aligned}$$

(ii) *An m-dimensional X-bounded geometric Poincaré complex M has a* **total surgery obstruction**

$$s^b(M) \in \mathbb{S}^b_m(M)$$

such that $s^b(M) = 0$ if (and for $m \geq 5$ only if) M is X-bounded homotopy equivalent to an m-dimensional X-bounded topological manifold. The total surgery obstruction has image $[s^b(M)] = 0 \in H^{lf}_{m-1}(M;\mathbb{L}_\bullet)$ if and only if the Spivak normal fibration $\nu_M : M \longrightarrow BG$ admits a TOP reduction $\widetilde{\nu}_M : M \longrightarrow BTOP$, in which case $s^b(M) = [\sigma_*(f,b)]$ is the image of the X-bounded surgery obstruction $\sigma_*(f,b) \in L_m(\mathbb{C}_X(\mathbb{Z}[\pi]))$ for any normal map $(f,b) : M' \longrightarrow M$.
(iii) *An X-bounded homotopy equivalence $h : M' \longrightarrow M$ of m-dimensional X-bounded topological manifolds has a* **structure invariant**

$$s^b(h) \in \mathbb{S}^b_{m+1}(M)$$

such that $s^b(h) = 0$ if (and for $m \geq 5$ only if) h is X-bounded homotopic to a homeomorphism. Moreover, for $m \geq 5$ every element $s \in \mathbb{S}^b_{m+1}(M)$ is the *structure invariant $s = s^b(h)$ of such an X-bounded homotopy equivalence $h : M' \longrightarrow M$. Thus*

$$\mathbb{S}^b_{m+1}(M) \;=\; \mathcal{S}^{b,TOP}(M)$$

is the X-bounded topological manifold structure set of M, with a surgery exact sequence

$$\begin{aligned} \dots \longrightarrow L_{m+1}(\mathbb{C}_X(\mathbb{Z}[\pi])) \longrightarrow \mathcal{S}^{b,TOP}(M) &\longrightarrow [M, G/TOP] \\ &\longrightarrow L_m(\mathbb{C}_X(\mathbb{Z}[\pi])) \end{aligned}$$

as in Ferry and Pedersen [18, §11].

□

For any subspace $K \subseteq S^N$ define the **open cone** metric space

$$O(K) \;=\; \{tx \,|\, x \in K, t \geq 0\} \subseteq \mathbb{R}^{N+1} \ ,$$

such that for compact K

$$H^{lf}_{*+1}(O(K); \mathbb{L}_\bullet) \;=\; \widetilde{H}_*(K; \mathbb{L}_\bullet) \ .$$

In particular, $O(S^N) = \mathbb{R}^{N+1}$ and

$$H^{lf}_{*+1}(O(S^N); \mathbb{L}_\bullet) \;=\; \widetilde{H}_*(S^N; \mathbb{L}_\bullet) \;=\; L_{*-N}(\mathbb{Z}) \ .$$

Proposition 9.13 (Ranicki [48], [49]) (i) *The locally finite assembly maps*

$$A^{lf} \;:\; H^{lf}_*(O(K); \mathbb{L}_\bullet(\mathbb{Z})) \longrightarrow L_*(\mathbb{C}_{O(K)}(\mathbb{Z}))$$

are isomorphisms for any compact polyhedron $K \subseteq S^N$, with $\mathbb{S}^b_*(O(K)) = 0$. Similarly for symmetric L-theory.
(ii) *The symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M; \mathbb{L}^\bullet(\mathbb{Z}))$ of a closed m-dimensional manifold M is a topological invariant.*
Proof (i) For any ring with involution A every quadratic complex (C, ψ) in $\mathbb{C}_{O(K)}(A)$ is cobordant to the assembly $A(\Gamma)$ of a locally finite sheaf Γ over $O(K)$ of quadratic complexes over A. If (C, ψ) is a quadratic Poincaré complex it may not be possible to choose Γ such that each of the stalks is a quadratic Poincaré complex over A — the reduced lower K-groups $\widetilde{K}_{-*}(A)$ are the potential obstructions to such a quadratic Poincaré disassembly. This is an $O(K)$-bounded algebraic L-theory version of the lower Whitehead torsion obstruction (10.1 below) to codimension 1 splitting of $O(K)$-bounded homotopy equivalences of $O(K)$-bounded open manifolds. For $A = \mathbb{Z}$ the obstruction groups are $\widetilde{K}_{-*}(\mathbb{Z}) = Wh_{1-*}(\{1\}) = 0$ by Bass, Heller and Swan [4]. See [49, Appendix C14] and §10 below for further details.

(ii) Let $M_+ = M \cup \{\mathrm{pt.}\}$. Regard $M \times \mathbb{R}$ as an $(m+1)$-dimensional $O(M_+)$-bounded geometric Poincaré complex via the projection $M \times \mathbb{R} \longrightarrow O(M_+)$, with $O(M_+)$ defined using any embedding $M_+ \subset S^N$ (N large). The symmetric L-theory orientation of M is the $O(M_+)$-bounded symmetric signature of $M \times \mathbb{R}$

$$\begin{aligned} &\sigma^*(M \times \mathbb{R}) \;=\; [M]_{\mathbb{L}} \\ &\in L^{m+1}(\mathbb{C}_{O(M_+)}(\mathbb{Z})) \;=\; H^{lf}_{m+1}(O(M_+);\mathbb{L}^\bullet(\mathbb{Z})) \;=\; H_m(M;\mathbb{L}^\bullet(\mathbb{Z}))\ . \end{aligned}$$

A homeomorphism $h : M' \longrightarrow M$ determines an $O(M_+)$-bounded homotopy equivalence $h \times 1 : M' \times \mathbb{R} \longrightarrow M \times \mathbb{R}$, so that

$$\begin{aligned} [M]_{\mathbb{L}} \;&=\; \sigma^*(M \times \mathbb{R}) \;=\; (h \times 1)_*\sigma^*(M' \times \mathbb{R}) \;=\; h_*[M']_{\mathbb{L}} \\ &\in H_m(M;\mathbb{L}^\bullet(\mathbb{Z})) \;=\; L^{m+1}(\mathbb{C}_{O(M_+)}(\mathbb{Z}))\ . \end{aligned}$$

See [49, Appendix C16] for further details.

□

Remark 9.14 (i) As in the original proof of the topological invariance of the rational Pontrjagin classes due to Novikov [36] it suffices to prove the topological invariance of signatures of special submanifolds – cf. 2.6. As in the proof of 4.1 suppose given a homeomorphism $h : M'^m \longrightarrow M^m$ of m-dimensional (differentiable) manifolds and a special submanifold $N^{4k} \subset M^m \times \mathbb{R}^j$. Let

$$W \;=\; N \times \mathbb{R}^i \subset M \times \mathbb{R}^j \quad (i \;=\; m+j-4k)$$

be a regular neighbourhood of N in $M \times \mathbb{R}^j$, and let

$$W' \;=\; (h \times \mathrm{id}_{\mathbb{R}^j})^{-1}(W) \subset M' \times \mathbb{R}^j\ .$$

Now W' is an $(m+j)$-dimensional $\mathbb{R}^i$-bounded manifold which is $\mathbb{R}^i$-bounded homotopy equivalent to W, so that the $\mathbb{R}^i$-bounded symmetric signatures are such that

$$\sigma^*(W') \;=\; \sigma^*(W) \;=\; \sigma(N) \in L^{m+j}(\mathbb{C}_{\mathbb{R}^i}(\mathbb{Z})) \;=\; L^{4k}(\mathbb{Z}) \;=\; \mathbb{Z}\ .$$

Let $N'^{4k} \subset W'$ be the inverse image submanifold obtained by making the homeomorphism $(h \times \mathrm{id}_{\mathbb{R}^j})| : W' \longrightarrow W$ transverse regular at $N \subset W$, so that N' is the transverse inverse image of $0 \in \mathbb{R}^i$ under $W' \longrightarrow \mathbb{R}^i$. The algebraic isomorphism $L^{m+j}(\mathbb{C}_{\mathbb{R}^i}(\mathbb{Z})) \cong L^{4k}(\mathbb{Z})$ of Ranicki [48] sends $\sigma^*(W')$ to $\sigma(N')$. Thus

$$\sigma(N') \;=\; \sigma^*(W') \;=\; \sigma^*(W) \;=\; \sigma(N) \in \mathbb{Z}\ ,$$

giving (yet again) the topological invariance of the signatures of special submanifolds.
(ii) The topological invariance of signatures of special submanifolds is a formal consequence of the topological invariance of the symmetric L-theory orientation, as follows. If $N^{4k} \subset M^m \times \mathbb{R}^j$ is a special submanifold there exists a proper map

$$e \ : \ M \times \mathbb{R}^j \longrightarrow \mathbb{R}^i \ \ (i \ = \ m + j - 4k)$$

such that $N = e^{-1}(0)$, and there is defined a commutative diagram

$$\begin{tikzcd}
H_m(M;\mathbb{L}^{\cdot}(\mathbb{Z})) \ = \ H^{lf}_{m+j}(M \times \mathbb{R}^j;\mathbb{L}^{\cdot}(\mathbb{Z})) \arrow[dd] \arrow[dr, "e_*"] & \\
 & H^{lf}_{m+j}(\mathbb{R}^i;\mathbb{L}^{\cdot}(\mathbb{Z})) \ = \ L^{4k}(\mathbb{Z}) \\
H^{lf}_{m+j}(M \times \mathbb{R}^j, (M \times \mathbb{R}^j)\backslash N;\mathbb{L}^{\cdot}(\mathbb{Z})) \ = \ H_{4k}(N;\mathbb{L}^{\cdot}(\mathbb{Z})) \arrow[ur, "A"] &
\end{tikzcd}$$

with A the simply-connected symmetric L-theory assembly map. The symmetric L-theory orientation $[M]_{\mathbb{L}} \in H_m(M;\mathbb{L}^{\bullet}(\mathbb{Z}))$ has image the signature of N

$$e_*([M]_{\mathbb{L}}) \ = \ A([N]_{\mathbb{L}}) \ = \ \sigma(N) \in L^{4k}(\mathbb{Z}) \ = \ \mathbb{Z} \ .$$

The topological invariance of the symmetric L-theory orientation $[M]_{\mathbb{L}}$ thus implies the topological invariance of the signatures $\sigma(N)$ of special submanifolds, and hence the topological invariance of the $\mathcal{L}$-genus and the rational Pontrjagin classes $\mathcal{L}(M), p_*(M) \in H^{4*}(M;\mathbb{Q})$ (as in 4.1).

□

§10. Codimension 1 splitting for non-compact manifolds

The obstruction theory for splitting homotopy equivalences of compact manifolds along codimension 1 submanifolds involves both algebraic K- and L-theory, as recalled in §8. In fact, the approach to the (integral) Novikov conjecture of Carlsson and Pedersen [14] makes use of the obstruction theory for splitting bounded homotopy equivalences of non-compact manifolds along codimension 1 submanifolds, which only requires algebraic K-theory obstructions to be considered.

Bounded Codimension 1 Splitting Theorem 10.1 (Ferry and Pedersen [18, 7.2], Ranicki [48, 7.5]) *Let $h : M'^m \longrightarrow M^m$ be an $X \times \mathbb{R}$-bounded homotopy equivalence of m-dimensional $X \times \mathbb{R}$-bounded manifolds with bounded* *fundamental group π. Assume the given proper map $\rho : M \longrightarrow X \times \mathbb{R}$ is transverse regular at $X \times \{0\} \subset X \times \mathbb{R}$, so that*

$$N^{m-1} \;=\; \rho^{-1}(X \times \{0\}) \subset M^m$$

is a codimension 1 X-bounded submanifold with trivial normal bundle and bounded fundamental group π. The $X \times \mathbb{R}$-bounded Whitehead torsion

$$\tau(h) \in Wh(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \;=\; \widetilde{K}_0(\mathbb{P}_X(\mathbb{Z}[\pi]))$$

is such that $\tau(h) = 0$ if (and for $m \geq 6$ only if) h splits along $N \subset M$.
K-theoretic proof. Make $h : M' \longrightarrow M$ transverse regular at $N \subset M$, and let $N' = h^{-1}(N) \subset M'$, so that as in the K-theoretic proof of 8.1 we have

$$h \;=\; h^+ \cup_g h^- \;:\; M' \;=\; M'^+ \cup_{N'} M'^- \longrightarrow M \;=\; M^+ \cup_N M^- \;.$$

Since h is an $X \times \mathbb{R}$-bounded homotopy equivalence the natural chain map is a chain equivalence in $\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])$

$$C(\widetilde{N}', \widetilde{N}) \;\simeq\; C(\widetilde{M}'^+, \widetilde{M}^+) \oplus C(\widetilde{M}'^-, \widetilde{M}^-) \;,$$

and Poincaré duality defines a chain equivalence in $\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])$

$$C(\widetilde{M}'^+, \widetilde{M}^+)^{m-1-*} \;\simeq\; C(\widetilde{M}'^-, \widetilde{M}^-) \;.$$

The restriction X-bounded normal map

$$(g, c) \;=\; h| \;:\; N' \longrightarrow N$$

is an X-bounded homotopy equivalence if and only if the chain complex $C(\widetilde{M}'^+, \widetilde{M}^+)$ is chain contractible. The isomorphism given by 9.3

$$\partial \;:\; Wh(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \xrightarrow{\;\simeq\;} \widetilde{K}_0(\mathbb{P}_X(\mathbb{Z}[\pi]))$$

sends $\tau(h)$ to the reduced projective class of the $\mathbb{C}_X(\mathbb{Z}[\pi])$-finitely dominated cellular $\mathbb{Z}[\pi]$-module chain complex $C(\widetilde{M}'^+, \widetilde{M}^+)$. For $m \geq 6$ $\tau(h) = 0$ if and only if it is possible to modify N' by X-bounded handle exchanges inside M' until the X-bounded normal map $h| : N' \longrightarrow N$ is a homotopy equivalence, if and only if h splits along $N \subset M$.
L-theoretic proof. The unobstructed case $\tau(h) = 0 \in Wh(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi]))$ proceeds as in the L-theoretic proof of 10.1 to compute the simple $X \times \mathbb{R}$-bounded topological manifold structure set of M

$$\begin{aligned} \dots \longrightarrow L^s_{m+1}(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) &\longrightarrow \mathbb{S}^{b,s}_{m+1}(M) \\ &\longrightarrow H^{lf}_m(M; \mathbb{L}_\bullet) \longrightarrow L^s_m(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \longrightarrow \dots \;. \end{aligned}$$

It follows from the algebraic computation of Ranicki [48]

$$L^s_{m+1}(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \;=\; L_m(\mathbb{C}_X(\mathbb{Z}[\pi]))$$

that there is defined an exact sequence

$$\dots \longrightarrow \mathbb{S}^b_{m+1}(M\backslash N) \longrightarrow \mathbb{S}^{b,s}_{m+1}(M) \longrightarrow \mathbb{S}^b_m(N) \longrightarrow \mathbb{S}^b_m(M\backslash N) \longrightarrow \dots .$$

The structure set $\mathbb{S}^{b,s}_{m+1}(M)$ of simple $X\times\mathbb{R}$-bounded homotopy equivalences of m-dimensional manifolds $h: M'{\longrightarrow}M$ is thus identified with the structure set $\mathbb{S}^b_{m+1}(N{\longrightarrow}M\backslash N)$ of $X\times\mathbb{R}$-bounded homotopy equivalences $h: M'{\longrightarrow}M$ which split along $N\subset M$

$$\mathbb{S}^{b,s}_{m+1}(M) \;=\; \mathbb{S}^b_{m+1}(N{\longrightarrow}M\backslash N) .$$

□

Example 10.2 For $m\geq 6$ an $X\times\mathbb{R}$-bounded homotopy equivalence of m-dimensional $X\times\mathbb{R}$-bounded manifolds of the type

$$h \;:\; M'^m \longrightarrow M^m \;=\; N^{m-1}\times\mathbb{R}$$

is homotopic to

$$g\times \mathrm{id}_{\mathbb{R}} \;:\; M' \;=\; N'\times\mathbb{R} \longrightarrow M \;=\; N\times\mathbb{R}$$

for an X-bounded homotopy equivalence of $(m-1)$-dimensional X-bounded manifolds $g: N'{\longrightarrow}N$ if and only if

$$\tau(h) \;=\; 0\in Wh(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \;=\; \widetilde{K}_0(\mathbb{P}_X(\mathbb{Z}[\pi])) .$$

The algebraic surgery exact sequences for the structure set $\mathbb{S}^{b,s}_{m+1}(N\times\mathbb{R})$ of simple $X\times\mathbb{R}$-bounded homotopy equivalences $h: M'{\longrightarrow}M$ and the structure set $\mathbb{S}^b_m(N)$ of X-bounded homotopy equivalences $g: N'{\longrightarrow}N$ are related by an isomorphism

$$\begin{tikzcd}
\dots \arrow[r] & H^{lf}_m(N;\mathbb{L}.) \arrow[r, "A^{lf}"] \arrow[d, "\cong"'] & L_m(\mathbb{C}_X(\mathbb{Z}[\pi])) \arrow[r] \arrow[d, "\cong"'] & \mathbb{S}^b_m(N) \arrow[r] \arrow[d, "\cong"'] & \dots \\
\dots \arrow[r] & H^{lf}_{m+1}(N\times\mathbb{R};\mathbb{L}.) \arrow[r, "A^{lf}"] & L^s_{m+1}(\mathbb{C}_{X\times\mathbb{R}}(\mathbb{Z}[\pi])) \arrow[r] & \mathbb{S}^{b,s}_{m+1}(N\times\mathbb{R}) \arrow[r] & \dots
\end{tikzcd}$$

so that

$$\mathbb{S}^{b,s}_{m+1}(N\times\mathbb{R}) \;=\; \mathbb{S}^b_m(N) ,$$

and simple $X\times\mathbb{R}$-bounded homotopy equivalences $h : M'\longrightarrow M$ split along $N\times\{0\}\subset M = N\times\mathbb{R}$.

□

Proposition 10.3 *Let N be a compact n-dimensional manifold, and let W be an open $(n+i)$-dimensional $\mathbb{R}^i$-bounded manifold with an $\mathbb{R}^i$-bounded homotopy equivalence $h : W\longrightarrow N\times\mathbb{R}^i$ $(i\geq 1)$. The $\mathbb{R}^i$-bounded Whitehead torsion*

$$\tau(h)\in Wh(\mathbb{C}_{\mathbb{R}^i}(\mathbb{Z}[\pi]))\ =\ Wh_{1-i}(\mathbb{Z}[\pi])\ \ (\pi=\pi_1(N))$$

is such that $\tau(h)=0$ if (and for $n\geq 5$ only if) h is $\mathbb{R}^i$-bounded homotopic to

$$g\times \mathrm{id}_{\mathbb{R}^i}\ :\ W\ =\ N'\times\mathbb{R}^i\ \longrightarrow\ N\times\mathbb{R}^i$$

for some closed codimension i submanifold $N'\subset W$, with $g : N'\longrightarrow N$ a homotopy equivalence.

Proof See Bryant and Pacheco [8] for a proof based on the geometric twist-glueing technique of Siebenmann [54]. Alternatively, apply 10.2 i times.

□

§11. Splitting the assembly map

This section is an outline of the infinite transfer method used by Carlsson and Pedersen [14] to prove the integral Novikov conjecture by splitting the algebraic L-theory assembly map

$$A\ :\ H_*(B\pi;\mathbb{L}.(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[\pi])$$

for torsion-free groups π with finite classifying space $B\pi$, such that $E\pi$ has a sufficiently nice compactification. The method may be viewed as a particularly well-organized way of avoiding the algebraic K-theory codimension 1 splitting obstructions to deforming homotopy equivalences of manifolds with fundamental group π to homeomorphisms.

The **homotopy fixed set** of a pointed space X with π-action is

$$X^{h\pi}\ =\ \mathrm{map}_\pi(E\pi_+,X)\ ,$$

with $E\pi_+=E\pi\cup\{\mathrm{pt.}\}$.

Let K be a connected compact polyhedron, regarded as a metric space. The action of the fundamental group $\pi=\pi_1(K)$ on the universal cover $\widetilde{K}$ induces an action of π on the spectrum $\mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))$, with the fixed point spectrum such that

$$\mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^\pi\ \simeq\ \mathbb{L}.(\mathbb{C}_K(\mathbb{Z}[\pi]))\ \simeq\ \mathbb{L}.(\mathbb{Z}[\pi])\ .$$

The action of π on the cofibration sequence of spectra

$$\mathbb{H}^{lf}_{.}(\widetilde{K};\mathbb{L}.)\ \xrightarrow{A^{lf}}\ \mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))\longrightarrow\mathbb{S}^b(\widetilde{K})$$

determines a cofibration sequence of the homotopy fixed point spectra

$$\mathbb{H}^{lf}_{.}(\widetilde{K},\mathbb{L}.)^{h\pi} \xrightarrow{A^{lf}} \mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^{h\pi} \longrightarrow \mathbb{S}^b(\widetilde{K})^{h\pi}$$

with a homotopy equivalence

$$\mathbb{H}^{lf}_{.}(\widetilde{K},\mathbb{L}.)^{h\pi} \simeq \mathbb{H}.(K,\mathbb{L}.) .$$

The infinite transfer maps of Ranicki [49, p. 328]

$$\mathrm{trf} : L_*(\mathbb{Z}[\pi]) = L_*(\mathbb{C}_K(\mathbb{Z}[\pi])) = L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^{\pi}) \longrightarrow L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))$$

extend to define a natural transformation of algebraic surgery exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & \mathbb{S}_{m+1}(K) \arrow[r] \arrow[d, "\mathrm{trf}"'] & H_m(K;\mathbb{L}.) \arrow[r, "A"] \arrow[d, "\mathrm{trf}"', "\cong"] & L_m(\mathbb{Z}[\pi]) \arrow[r] \arrow[d, "\mathrm{trf}"'] & \mathbb{S}_m(K) \arrow[r] \arrow[d, "\mathrm{trf}"'] & \dots \\
\dots \arrow[r] & \mathbb{S}^{b,h\pi}_{m+1}(\widetilde{K}) \arrow[r] & H^{lf,h\pi}_m(\widetilde{K};\mathbb{L}.) \arrow[r, "A^{lf}"] & L_m(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^{h\pi}) \arrow[r] & \mathbb{S}^{b,h\pi}_m(\widetilde{K}) \arrow[r] & \dots
\end{tikzcd}$$

with

$$\mathbb{S}^{b,h\pi}_*(\widetilde{K}) = \pi_*(\mathbb{S}^b(\widetilde{K})^{h\pi}) , \quad L_*(\mathbb{C}_{\widetilde{K}}(\mathbb{Z})^{h\pi}) = \pi_*(\mathbb{L}.(\mathbb{C}_{\widetilde{K}}(\mathbb{Z}))^{h\pi}) .$$

The composite

$$\mathbb{S}_{m+1}(K) \xrightarrow{\mathrm{trf}} \mathbb{S}^{b,h\pi}_{m+1}(\widetilde{K}) \longrightarrow \mathbb{S}^b_{m+1}(\widetilde{K})$$

sends the structure invariant $s(h) \in \mathbb{S}_{m+1}(K)$ of a homotopy equivalence $h : M' \longrightarrow M$ of compact m-dimensional manifolds with a π_1-isomorphism reference map $M \longrightarrow K$ to the $\widetilde{K}$-bounded structure invariant $s^b(\widetilde{h}) \in \mathbb{S}^b_{m+1}(\widetilde{K})$ of the induced $\widetilde{K}$-bounded homotopy equivalence $\widetilde{h} : \widetilde{M'} \longrightarrow \widetilde{M}$ of the universal covers.

The method of infinite transfers first applied by Carlsson [13] to the algebraic K-theory version of the Novikov conjecture has the following application in algebraic L-theory to the integral Novikov conjecture :

Proposition 11.1 *Let π be a group such that the classifying space $B\pi$ has the homotopy type of a finite CW complex, so that π is torsion-free. If the universal cover $E\pi$ of $B\pi$ is realized by a contractible metric space E with a free π-action and such that the locally finite assembly maps are isomorphisms*

$$A^{lf} : H^{lf}_*(E;\mathbb{L}.(\mathbb{Z})) \xrightarrow{\simeq} L_*(\mathbb{C}_E(\mathbb{Z}))$$

then the integral Novikov conjecture holds for π, i.e. the assembly maps

$$A \ : \ H_*(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi])$$

are split injections.

Proof The $E\pi$-bounded structure spectrum $\mathbb{S}^b(E\pi)$ is contractible, and hence so is the homotopy fixed point spectrum $\mathbb{S}^b(E\pi)^{h\pi}$. The locally finite assembly map

$$A^{lf}_\pi \ : \ \mathbb{H}^{lf}_.(E\pi;\mathbb{L}.(\mathbb{Z}))^{h\pi} \longrightarrow \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi}$$

is a homotopy equivalence, so that there are defined homotopy equivalences

$$\mathbb{H}.(B\pi;\mathbb{L}.(\mathbb{Z})) \ \simeq \ \mathbb{H}^{lf}_.(E\pi;\mathbb{L}.(\mathbb{Z}))^{h\pi} \ \simeq \ \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi} \ .$$

The infinite transfer maps

$$\mathrm{trf} \ : \ \mathbb{L}.(\mathbb{Z}[\pi]) \ \simeq \ \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{\pi} \longrightarrow \mathbb{L}.(\mathbb{C}_{E\pi}(\mathbb{Z}))^{h\pi} \ \simeq \ \mathbb{H}.(B\pi;\mathbb{L}.(\mathbb{Z}))$$

induce splitting maps $\mathrm{trf} : L_*(\mathbb{Z}[\pi])\longrightarrow H_*(B\pi;\mathbb{L}.(\mathbb{Z}))$ for the assembly maps $A : H_*(B\pi;\mathbb{L}.(\mathbb{Z}))\longrightarrow L_*(\mathbb{Z}[\pi])$.

$\square$

Example 11.2 Let $\pi = \mathbb{Z}^n$, so that

$$B\pi \ = \ T^n \quad , \quad E \ = \ E\pi \ = \ \mathbb{R}^n \ .$$

Compactify E by adding the $(n-1)$-sphere at infinity

$$\overline{E} \ = \ \mathbb{R}^n \cup S^{n-1} \ = \ D^n \ ,$$

extending the free $\mathbb{Z}^n$-action on $\mathbb{R}^n$ by the identity on $\partial E = S^{n-1}$. In this case the locally finite assembly isomorphisms

$$\begin{aligned} A^{lf} \ : \ H_*(D^n,S^{n-1};\mathbb{L}.(\mathbb{Z})) \ &= \ H^{lf}_*(\mathbb{R}^n;\mathbb{L}.(\mathbb{Z})) \\ &= \ \widetilde{H}_{*-1}(S^{n-1};\mathbb{L}.(\mathbb{Z})) \ = \ L_{*-n}(\mathbb{Z}) \\ &\xrightarrow{\simeq} \ L_*(\mathbb{C}_{\mathbb{R}^n}(\mathbb{Z})) \end{aligned}$$

and the assembly isomorphisms

$$A \ : \ H_*(T^n;\mathbb{L}.(\mathbb{Z})) \xrightarrow{\simeq} L_*(\mathbb{Z}[\mathbb{Z}^n])$$

were already obtained in Ranicki [43], [48], using the identification of the $\mathbb{R}^n$-bounded L-groups of a ring with involution A with the lower L-groups

$$L_*(\mathbb{C}_{\mathbb{R}^n}(A)) \ = \ L^{\langle 1-n\rangle}_{*-n}(A)$$

and the splitting theorem

$$L_*(A[\mathbb{Z}^n]) \;=\; \sum_{k=0}^{n} \binom{n}{k} L^{\langle 1-k\rangle}_{*-k}(A) \;,$$

with $L^{\langle -*\rangle}_*(\mathbb{Z}) = L_*(\mathbb{Z})$ by virtue of $Wh_{-*}(\{1\}) = 0$.

□

Example 11.3 Let $\pi = \pi_1(M)$ be the fundamental group of a complete closed n-dimensional Riemannian manifold with non-positive sectional curvature M. The universal cover $E = \widetilde{M}$ is a complete simply-connected open Riemannian manifold such that the exponential map at any point $x \in E$ defines a diffeomorphism

$$\exp_x \;:\; \tau_x(E) \;=\; \mathbb{R}^n \longrightarrow E$$

by the Hadamard-Cartan theorem, so that $M = B\pi$ is aspherical. The locally finite assembly map

$$A^{lf} \;:\; H^{lf}_*(E;\mathbb{L}.(\mathbb{Z})) \;=\; L_{*-n}(\mathbb{Z}) \longrightarrow L_*(\mathbb{C}_E(\mathbb{Z}))$$

is an isomorphism, so that the integral Novikov conjecture holds for π by 11.1. See Farrell and Hsiang [16] for the original geometric proof, which is generalized by Carlsson and Pedersen [14] (cf. 11.5 below) by abstracting the properties of the π-action on the compactification $\overline{E} = D^n$ near the sphere at ∞ $\partial E = \overline{E}\backslash E = S^{n-1}$.

□

Example 11.4 For any integer $g \geq 1$ let

$$\pi_g \;=\; \{a_1, a_2, \ldots, a_{2g} \,|\, [a_1, a_2] \ldots [a_{2g-1}, a_{2g}]\}$$

be the fundamental group of the closed oriented surface M_g of genus g, so that

$$B\pi_g \;=\; M_g \;\;,\;\; E \;=\; E\pi_g \;=\; \mathbb{R}^2 \;.$$

For $g = 1$ $M_g = T^2$, as already considered in 11.2. For $g \geq 2$ M_g has a hyperbolic structure, and the free action of π_g on $E = \mathbb{R}^2 = \mathrm{int}(D^2)$ extends to a (non-free) action on $\overline{E} = D^2$, which is the identity on $\partial E = S^1$. The hypotheses of 11.1 are satisfied, so that the assembly maps

$$A \;:\; h_*(B\pi_g) \;=\; H_*(B\pi_g;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi_g])$$

are split injections, and the integral Novikov conjecture holds for π_g. In fact, these assembly maps are isomorphisms, which may be verified by the following argument (for which I am indebted to C.T.C.Wall). By the Freiheitssatz for one-relator groups the subgroup $\rho_g \subset \pi_g$ generated by $a_1, a_2, \dots, a_{2g-1}$ is free, so that π_g is the α-twisted extension of ρ_g by $\mathbb{Z} = \{a_{2g}\}$

$$\{1\} \longrightarrow \rho_g \longrightarrow \pi_g \longrightarrow \mathbb{Z} \longrightarrow \{1\}$$

and

$$\mathbb{Z}[\pi_g] \;=\; \mathbb{Z}[\rho_g]_\alpha[z, z^{-1}]$$

is the α-twisted Laurent polynomial extension of $\mathbb{Z}[\rho_g]$, with

$$z \;=\; a_{2g} \;\;,\;\; \alpha(a_i) \;=\; (a_{2g})^{-1} a_i a_{2g} \;\; (1 \le i \le 2g-1) \;.$$

The assembly maps $A : h_*(B\rho_g) \longrightarrow L_*(\mathbb{Z}[\rho_g])$ are isomorphisms by Cappell [9]. A 5-lemma argument applied to the assembly map

$$\begin{array}{ccccccccc}
\dots \longrightarrow & h_n(B\rho_g) & \xrightarrow{1-\alpha} & h_n(B\rho_g) & \longrightarrow & h_n(B\pi_g) & \longrightarrow & h_{n-1}(B\rho_g) & \longrightarrow \dots \\
& \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle A} & & \Big\downarrow{\scriptstyle A} & \\
\dots \longrightarrow & L_n(\mathbb{Z}[\rho_g]) & \xrightarrow{1-\alpha} & L_n(\mathbb{Z}[\rho_g]) & \longrightarrow & L_n(\mathbb{Z}[\pi_g]) & \longrightarrow & L_{n-1}(\mathbb{Z}[\rho_g]) & \longrightarrow \dots
\end{array}$$

from the Wang exact sequence in group homology to the exact sequence of Ranicki [44] for the L-theory of a twisted Laurent polynomial extension (using $Wh(\pi_g) = 0$) shows that the assembly maps

$$A \;:\; h_*(B\pi_g) \;=\; H_*(B\pi_g; \mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}[\pi_g])$$

are isomorphisms.

□

Theorem 11.5 (Carlsson and Pedersen [14]) *Let π be a group with finite classifying space $B\pi$ such that the universal cover $E\pi$ is realized by a contractible metric space E with a free π-action, and with a compactification $\overline{E}$ such that:*

(a) *the free π-action on E extends to a π-action on $\overline{E}$ (which need not be free),*
(b) *$\overline{E}$ is contractible,*
(c) *compact subsets of E become small near the boundary $\partial E = \overline{E} \backslash E$, i.e. for every point $y \in \partial E$, every compact subset $K \subseteq E$ and for every neighbourhood U of y in $\overline{E}$, there exists a neighbourhood V of y in $\overline{E}$ so that if $g \in \pi$ and $g(K) \cap V \neq \emptyset$ then $g(K) \subset U$.*

Then the integral Novikov conjecture holds for π.

□

The proof of 11.5 uses infinite transfer maps (as in 11.1), but with the continuously controlled category $\mathbb{B}_{X,Y}(\mathbb{Z})$ of Anderson, Connolly, Ferry and Pedersen [1] replacing the bounded category $\mathbb{C}_E(\mathbb{Z})$ of Pedersen and Weibel [41]. For a compact metrizable space X and a closed subspace $Y \subseteq X$ $\mathbb{B}_{X,Y}(\mathbb{Z})$ is the category with the same objects as $\mathbb{C}_E(\mathbb{Z})$, where $E = X\backslash Y$. A morphism in $\mathbb{B}_{X,Y}(\mathbb{Z})$

$$f \;=\; \{f(x',x)\} \;:\; A \;=\; \sum_{x\in E} A(x) \longrightarrow B \;=\; \sum_{x'\in E} B(x')$$

is a $\mathbb{Z}$-module morphism such that for every $y \in Y$ and every neighbourhood $U \subseteq X$ of y there is a neighbourhood $V \subseteq U$ such that

$$f(x',x) \;=\; 0 \;:\; A(x) \longrightarrow B(x') \;\; (x \in V, x' \in X\backslash U)$$

(or equivalently $f(A(V)) \subseteq B(U)$). If E is dense in X and compact subsets of E become small near the boundary $\partial E = Y$ in $\overline{E} = X$ there is defined a forgetful functor $\mathbb{C}_E(\mathbb{Z}) \longrightarrow \mathbb{B}_{X,Y}(\mathbb{Z})$. This functor induces isomorphisms in K- and L-theory in certain cases with X contractible (e.g. if $E = O(K)$ is the open cone on a compact subcomplex $K \subseteq S^N$ and $X = O(K) \cup K$ is the closed cone, with $Y = K \subset X$), but it is not known if it does so in general. See Pedersen [39] for the relationship between the bounded and continuously controlled categories.

The algebraic transversality of Ranicki [48], [49] is extended in Carlsson and Pedersen [14, 5.4] to prove that the continuously controlled L-theory assembly maps

$$A \;:\; H^{lf}_*(E;\mathbb{L}.(\mathbb{Z})) \;=\; H_*(X,Y;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_*(\mathbb{B}_{X,Y}(\mathbb{Z}))$$

are isomorphisms if $E = E\pi$ and $(X,Y) = (\overline{E}, \partial E)$ are as in 11.5 – this is the key step in the proof. As in 10.1 there are potential lower Whitehead torsion obstructions to splitting, which are avoided by the computation $Wh_{-*}(\{1\}) = 0$ of Bass, Heller and Swan [4]. The assembly map $A : H^{lf,h\pi}_m(E;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_m(\mathbb{B}_{\overline{E},\partial E}(\mathbb{Z})^{h\pi})$ in the commutative square

$$\begin{array}{ccc}
H_m(B\pi;\mathbb{L}.(\mathbb{Z})) & \xrightarrow{\quad A \quad} & L_m(\mathbb{Z}[\pi]) \\
\mathrm{trf}\Big\downarrow\cong & & \Big\downarrow\mathrm{trf} \\
H^{lf,h\pi}_m(E;\mathbb{L}.(\mathbb{Z})) & \xrightarrow{\quad A \quad} & L_m(\mathbb{B}_{\overline{E},\partial E}(\mathbb{Z})^{h\pi})
\end{array}$$

is an isomorphism, giving the splitting of the assembly map

$$A \;:\; H_m(B\pi;\mathbb{L}.(\mathbb{Z})) \longrightarrow L_m(\mathbb{Z}[\pi]) \;.$$

Example 11.6 As already noted by Carlsson and Pedersen [14], the work of Bestvina and Mess [5] shows that negatively curved groups in the sense of Gromov satisfy the conditions of Theorem 11.5, so that the integral Novikov conjecture holds for these groups. The fundamental groups π of complete Riemannian manifolds (of finite homotopy type) $B\pi$ with non-positive curvature are the main examples of such groups – cf. 11.3.

□

If π is in the class of groups satisfying the conditions of 11.5

$$L_*(\mathbb{Z}[\pi]) \ = \ H_*(B\pi;\mathbb{L}.)\oplus\mathbb{S}_*(B\pi)\ .$$

It is worth investigating the extent to which $\mathbb{S}_*(B\pi)$ is determined by the Cappell UNil-groups.

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