Pontrjagin Classes, the Fundamental Group and some Problems of Stable Algebra[1]

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This paper deals mainly with problems connected with topological and homotopic invariance of Pontrjagin classes and some closely related problems of algebraic topology and stable algebra. These questions have arisen from the authors' papers on topological and homotopic invariance of rational Pontrjagin classes, based on the discovery of deep connections between characteristic classes and the fundamental group. There are a number of new stable algebraic problems connected with the diffeomorphism problem and Pontrjagin classes of nonsimply connected manifold (especially when $\pi_1 = \mathbf{Z} \times \cdots \times \mathbf{Z}$).

This paper is divided into three parts.

1. Rational Pontrjagin classes and the fundamental group. Homotopy invariance problem. Some questions of stable algebra.

2. Topological problems in the theory of classes. Smoothing of topological knots. Rational homological manifolds.

3. Pontrjagin-Hirzebruch classes with finite coefficient groups for homological and smooth manifolds. Topological microbundles. Classical Lens manifolds. (Results of author and V. A. Rohlin).

1. Rational Pontrjagin Classes and the Fundamental Group

The signature formula $(L_k, [M^{4k}]) = \sigma(M^{4k})$ was one of the most important discoveries in the theory of characteristic classes. Before 1964, it was the only known homotopically invariant relation ("rational"). The theorem of W. Browder and the author shows that this relation is really unique in a simply connected category: let $\tau \in \widetilde{KO}(M^n)$ be a stable tangent bundle and let $J: \widetilde{KO}(X) \to J(X)$ a J-homomorphism. The set $J^{-1} J(\tau)$ coincides exactly with the set of stable tangent bundles

[1] This paper had been written originally for a lecture at the Moscow International Congress (1966), but the subject of my $\frac{1}{2}$ hour lecture was, in fact, changed (to cobordism theory). I include here some additional remarks on the latest developments of these ideas.

of manifolds homotopically equivalent to M^n (all manifolds are closed, $n \geqslant 5$, $n \geqslant 2l+1$ or $n=6,14$). For $n=4k$, the analogous set contains all elements $\tau_i \in J^{-1}J(\tau)$ such that $L_k(\tau_i)-\sigma(M^{4k})=0$ and for $n=4k+2$, the Arf-invariant must be zero (see Izv. A. N. 1964, 28 N. 2, Supplement 1).

This theorem is false in non-simply connected categories. We prove some results and show interesting connections with the fundamental group. For any element $x \in H_{4k}(X,\mathbf{Q})$, we introduce the cohomological invariant $\sigma(x)$—the algebraic signature, in the sense of the author's papers (Dokl. A.N. 162 N. 6, Izv. A.N. 29 N. 6). Let $z \in H_{4k}(M^{n+4k},\mathbf{Q})$ and $Dz=y_1 \circ \cdots \circ y_n$ (D denotes Poincaré duality). We define a covering $P_z: \hat{M}_z \to M^{n+4k}$ depending on the element z: the subgroup $\operatorname{Im} P_{z*} \subset \pi_1(M^{n+4k})$ contains all $\gamma \in \pi_1$ such that $(\gamma, y_i)=0$, $i=1,\ldots,n$. There is one "canonical" element $\hat{z} \in H_{4k}(\hat{M}_z)$ (see Izv. A.N. 29, N. 6). Its definition is purely homotopic.

Theorem 1. (Author, DAN. 162 N. 6, 163 N. 2; Izv. A. N. 29, N. 6, 30 N. 1). *If one of the conditions* 1, 2, 2′, 3 *holds, then the "non-simply connected signature formula"* $(L_k(M^{n+4k}),z)=\sigma(\hat{z})$, $\hat{z} \in H_{4k}(\hat{M}_z)$ *is true.*

1. $n=1$.

2. $n=2$ *and* $rk[H_{2k+1}(\hat{M}_z,\mathbf{Q})]<\infty$.

2′. $n=2$, *and the intersection index on the group* $H_{2k+1}(\hat{M}_z,\mathbf{Q})$ *is identically zero.*

3. $n \geqslant 1$ *and* M^{n+4k} *is homotopically equivalent to a fiber bundle with base* (torus) T^n *and fiber-closed manifold* M^{4k}.

Corollary 1. $L_k(M^{4k+1})$ *is a homotopy invariant.*

Corollary 2. *For all* $k, p_k(M^q)=0$ *if* M^q *has the homotopy type of the torus* T^q. (From K-theory and Adam's results, we may now deduce that all homotopy tori are stably parallelizable.)

Many examples show that for $n \geqslant 2$, this simple form of the non-simply connected signature formula is false in general (Izv. A.N. 29 N.6, Example 2).

We have, however, the following important

Theorem 2. (Rohlin, Izv. A. N. 30 N. 3). *The scalar product* $(L_k(M^{4k+2}),z)$ *is a homotopy invariant if* $Dz=y_1 \cdot y_2$.

Rohlin's proof is non-constructive (except the case of part 2 of my earlier theorem). The problem of calculating the number (L_k,z) is not solved in general for $n=2$, $Dz=y_1,\ldots,y_n$, except the special case of Th. 1, part 2. It leads to interesting problems of stable algebra. Let $\tilde{J}(T^n)$ be a reduced J-functor and let $\tau \in J^{-1}(0)$. We apply the construction of Browder and myself and obtain a manifold M^n and a map $f: M^n \to T^n$ of degree 1 such that $f^*\tau \in \widetilde{KO}(M^n)$ is the stable tangent bundle and $\ker f_*^{(\pi_i)}=0$, $i<l$, $n=2l$. Consider the $\mathbf{Z}(\pi)$-module $N=\ker f_*^{(\pi_l)}$, $\pi=\mathbf{Z}\times\cdots\times\mathbf{Z}$. It has a "unimodular, equivariant and compact" scalar product $h\colon N \xrightarrow{\approx} \mathrm{Hom}_{\mathbf{C}}(N,\mathbf{Z})$ which is symmetric for $l=2q$ and skew-symmetric for $l=2q+1$; N is stable free (see Izv. A.N. 30 N. 1). The pair (N,h) is trivial iff $N=F_1+F_2$, and $(F_1,hF_1)=(F_2,hF_2)=0$. We have a sum $(N_1,h_1)\oplus(N_2,h_2)$ and Grothendieck groups $A(\pi)$, $B(\pi)$ (symmetric and skew-symmetric cases). We may deduce non constructively from Corollary 2:

Theorem 3. *For $\pi=\mathbf{Z}\times\mathbf{Z}$, the group $B(\pi)$ is non-trivial. (The group $A(\pi)$ is non-trivial also for $\pi=0$ because the usual signature is the stable invariant.)*

Let π be any group and let $A(\pi)$, $B(\pi)$ be the analogous Grothendieck groups of free modules with scalar product (symmetric or skew-symmetric) with some "Poincaré duality" restrictions, $\pi^{**}=\mathrm{Hom}(\mathrm{Hom}(\pi,\mathbf{Z}),\mathbf{Z})$. We can deduce from Corollary 2 the following important fact:

Statement. There are nontrivial homomorphisms Δ_j^i, $j=1,2$, $i\geqslant 0$,

$$\Delta_1^i\colon A(\pi)\to\Lambda^{4i}\pi^{**},\qquad \Delta_2^i\colon B(\pi)\to\Lambda^{4i+2}\pi^{**}$$

($\Lambda^k\pi^{**}$ are exterior powers of $\mathbf{Z}$-modules). Δ_1^0 is the usual signature of the $\mathbf{Z}$-module $N \underset{\mathbf{Z}(\pi)}{\otimes} \mathbf{Z}$ with induced symmetric scalar product (Note: $0\to\mathbf{Z}_0(\pi)\to\mathbf{Z}(\pi)\xrightarrow{\varepsilon}\mathbf{Z}$).

Δ_j^i are "higher signatures" in some sense. They must be such that in the geometric situation $\Delta_j^k(N,h)$ is equal to the dual Pontrjagin-Hirzebruch class $DL_k(M^n)=DL_k(\tau)\in H_{n-4k}(M^n)$, $\tau\in J^{-1}(0)\subset\widetilde{KO}(T^n)$. It is another important case of "non-simly connected Hirzebruch formula".

Problem. Define algebraically the "higher signatures" Δ_j^i.

This algebraic problem is closely connected with the following general topological problem:

Problem. Let $Dz=y_1,\dots,y_n$, $y_i\in H^1(M)$. Is the scalar product (L_k,z) a homotopy invariant? How can it be calculated?

The functors $A(\pi)$, $B(\pi)$ are interesting also for other groups. They have an involution $\alpha\colon A(\pi)\to A(\pi)$, $\alpha\colon B(\pi)\to B(\pi)$, such that $\alpha(N,h)=(N,-h)$. Especially interesting in topology are the groups $A_\alpha(\pi)=A(\pi)/\{x+\alpha(x)\}$ and $B_\alpha(\pi)=B(\pi)/\{x+\alpha(x)\}$ and the torsion groups $\operatorname{tor}A(\pi)$, $\operatorname{tor}B(\pi)$ [for $\pi=0$, $\alpha(x)=-x$]. The groups $A_\alpha(\pi)$, $B_\alpha(\pi)$ are nontrivial for $\pi=\mathbf{Z}_p$ (e.g. $p=5$), as follows from simple homotopy theory together with Atiyah's theorem (classical lens manifolds are

h-cobordant iff they are diffeomorphic) and my constructions in the diffeomorphism problem. *This statement is also non constructive.*

There are also other algebraic and topological problems connected with these and analogous functors. For example, we have the important fact: a) let $f: M_1^{2n} \to M_2^{2n}$ be a map of degree 1 and let $\ker f_*^{(\pi_i)} = 0$, $i < n$. If the map f is "tangential" (such that $f^*\tau(M_2) = \tau(M_1)$; τ denotes the stable tangent bundle), then the $\mathbf{Z}(\pi_1)$-module $N = \ker f_*^{(\pi_n)}$ with natural scalar product represents the element $x(f) \in A(\pi_1)$ for $n = 2l$ and $x(f) \in B(\pi_1)$ for $n = 2l+1$ and is such that $x(f) \in \operatorname{tor} A(\pi_1)$ or $\operatorname{tor} B(\pi_1)$. The correspondence $f \mapsto x(f)$ determines canonical maps $\Pi(M_2^{2n})/\tilde{J}(M_2^{2n}) \xrightarrow{x} A(\pi_1)$ for $n = 2l$, $\Pi(M_2^{2n})/\tilde{J}(M_2^{2n}) \xrightarrow{x} B(\pi_1)$ for $n = 2l+1$. ($\Pi(M^n)$ is a finite part of $\pi_{N+n}(T_N)$ where T is the Thom complex of the N-dimensional normal SO-bundle) such that $x(a+b-c) = x(a) + x(b) + \alpha \cdot x(c)$.

b) Let W^{2n} be a manifold with two boundaries M_1, M_2 and tangential retractions $r_i: W^{2n} \to M_i$. Then the obstruction to the diffeomorphism between M_1 and M_2 lies in the groups $A_\alpha(\pi_i)$ $(n = 2l)$ and $B_\alpha(\pi_1)$ $(n = 2l+1)$ or some of their factor groups.

We have here the important conjecture:

Conjecture. If the higher signatures Δ_j^i are complete invariants for the groups $A(\pi_1)$ and $B(\pi_1)$ (modulo finite groups), then the number of manifolds $\{M_i\}$ as above, $M_i \cup M_j = \partial W_{ij}^{2n}$ is finite (up to diffeomorphism homotopic to $r_i^{-1} \circ r_j: M_i \to M_j$) if and only if the natural maps $\Lambda^{2n-4k} \pi_1^* = \Lambda^{2n-4k} H^1(M_i) \to H^{2n-4k}(M_i)$ are monomorphic for all k.

(I can prove this in the case $M_i \sim T^{2n-1}$, $\dim M_i = 2n-1$, $n \geqslant 3$.)

Remark. Very recently, Shaneson proved nonconstructively (he used different terminology) that "higher signatures" are complete invariants (modulo finite groups) for obstructions to surgeries, and therefore for Hermitian and skew Hermitian integral, even, unimodular quadratic forms over free abelian fundamental groups (Bull. A. M. S., May 1968). The algebraic definitions are unknown! Wall, Hsiang and Shaneson deduced from this the important theorem that the number of n-dimensional homotopy tori (PL or smooth) is finite, $n \neq 4$. This fact depends on the "non simply connected Hirzebruch formula" in the same way that Milnor's classical theorem ($\Theta^{4k-1}(\partial\pi)$ is a finite group) does on the usual Hirzebruch formula. R. Kirby found remarkable applications of homotopy tori to purely continuous topology (Annulus Conjecture), developed now by Kirby and Siebenmann, Lashof and Rothenberg. The base of his approach is also toral open subsets in the "etale topology".

2. Topological Problems in the Theory of Characteristic Classes

I will not formulate here the corollaries of the topological invariance of the rational Pontrjagin classes (they may be found for example in my paper Izv. A.N. 1966, 30 N. 1, Introduction). I had always considered modern (1966) "topological invariance problems" (Pontrjagin classes, simple homotopy type) from the general point of view: All earlier "topological invariance" theorems (homology, Stiefel-Whitney classes) state, in fact, that some invariants are homotopy invariants. All "negative" results state, in fact, that some invariants are not combinatorial invariants (most important are Milnor's results: differentiable structure, stable tangent bundle and also integral Pontrjagin classes are not combinatorially and topologically invariant).

It was well known that rational Pontrjagin classes (and also Reidemeister-Whitehead torsion) are combinatorial, but not homotopy, invariants. What properties of "continuous homeomorphisms" can be used for studying topological invariants algebraically?

My approach to this problem is based on the study of "toral" open nonsimply connected subsets and their coverings in an open manifold W homeomorphic to $M^q \times \mathbf{R}^n$. These open "toral" subsets in the "etale" topology are very important in the homeomorphism problems. It is a trivial corollary of previous papers that for topological invariance of all rational Pontrjagin classes it is sufficient to prove the formula $(L_k(W), [M^{4k}]) = \sigma(M^{4k})$ (W is homeomorphic to $M^{4k} \times \mathbf{R}^n$) for simply connected manifolds M^{4k} (or even just for $M^{4k} = S^{\ k}$).

The main idea of the proof is to find a sequence of non simply connected interpolating manifolds (open and closed) with special properties: we first consider a subset $W_1 \subset W$ homeomorphic to $M^{4k} \times T^{n-1} \times \mathbf{R}$ (because $T^{n-1} \times \mathbf{R} \subset \mathbf{R}^n$) and prove that W_1 is diffeomorphic to $V_1 \times \mathbf{R}$ if $\pi_1(M^{4k}) = 0$ or $\tilde{K}_0(\pi_1 \times \mathbf{Z} \times \cdots \times \mathbf{Z}) = \tilde{K}_0(\pi_1(W_1)) = 0$. The pair (W_1, V_1) is the first and main one in the sequence of our manifolds, because $L_k(V_1) = L_k(W_1)$ and we can apply to V_1 the purely homotopic "non simply connected signature formula" (Theorem 1, part 3). This gives us the results: $(L_k(W), [M^{4k}]) = \sigma(M^{4k})$. But our process gives, in fact, a stronger result.

Theorem 4. *Let V_1 be homotopically equivalent to $M^q \times T^{n-1}$, $q \geqslant 5$, and let $\tilde{K}_0(\pi_1 \times Z \times \cdots \times \mathbf{Z}) = 0$ ($n-2$ times), $\pi_1 = \pi_1(M^q)$. The covering $\hat{V}_1 \to V_1$ homotopically equivalent to M^q (with monodromy group $\mathbf{Z} \times \cdots \times \mathbf{Z}$) is diffeomorphic to $\tilde{V} \times R^{n-1}$* (see Izv. A.N. 30 N. 1, Theorem 5 and Supplement 2).

I do not know any proof of topological invariance of classes without "toral" interpolating manifolds. However, the problem seems very far (formally) from the fundamental group. I also do not know whether the condition $\tilde{K}_0(\pi_1 \times \mathbf{Z} \times \cdots \times \mathbf{Z})=0$ is only technical in the problems connected with $M^q \times \mathbf{R}^n$.

We use here essentially (for technical purposes) a non-simply connected generalization of Browder's recent theorem: if W^{n+1} is "like" $M^n \times \mathbf{R}$ (M^n is closed, $n \geqslant 5$) and $\tilde{K}_0(\pi_1)=0$, then W^{n+1} is diffeomorphic to $V \times \mathbf{R}$) (see W. Browder, Cambr. Phil. Soc. 1965 for $\pi_1=0$ and the author, Izv. A.N. 1966, 30 N. 1 for $\pi_1 \neq 0$). I remark here that the last generalization of Browder's theorem itself (and also the generalization of Browder, Levine and Livesay's theorem about the boundary for open manifolds) was found independently by L. C. Siebenman; namely, he found the condition $\tilde{K}_0(\pi_1)=0$ in these problems, on the basis of Wall's idea in another problem ("direct summands" of the homotopy-finite complexes, Annals of Math. 1965). These ideas were developed recently by V. L. Golo who found here a "Poincaré duality law", which gives very strong restrictions on the "geometrically realizable" elements in $\tilde{K}_0(\pi)$, and constructed nontrivial geometric examples for $\pi=\mathbf{Z}_p$ (Doklady 1966). He uses here some results of Kummer theory reformulated in terms of $\tilde{K}_0(\pi)$. Soon after my papers (Dokl. A.N. 1965, N. 6; 163 N.2) Siebenmann developed somewhat the construction of a sequence of non-simply connected interpolating manifold and proved the "splitting theorem": let $\tilde{K}_0(\pi_1 \times \mathbf{Z} \times \cdots \times \mathbf{Z})=0$, $q \geqslant 5$, $n \geqslant 1$, $\pi_1=\pi_1(M^q)$. Then, if W is homeomorphic to $M^q \times \mathbf{R}^n$, M^q closed, it is diffeomorphic to $V \times \mathbf{R}^n$.[1] Interesting results were obtained by Sullivan and Wagoner in application of this result to the Hauptvermutung (Bul., AMS, 1967). In 1967, Sullivan, Lashof and Rothenberg proved the Hauptvermutung for almost all manifolds with $\pi_1=0$.

Let me give another example to illustrate the usefulness of considering non-simply connected open and closed submanifolds in purely topological problems. Let $S^n \subset S^{n+2}$ be a locally flat topological imbedding and let $n \geqslant 5$.

Theorem 5. *All such imbeddinds $S^n \subset S^{n+2}$ are topologically equivalent to differentiable imbeddings in some differentiable structure on S^n (from $\Theta^n(\partial\pi)$;* see Izv. A.N. 30 N. 1, Theorem 6).

The proof of this theorem uses essentially open sets $U_i \subset S^n$ and $W_i=U_i-S^n$ with $\pi_1(W_i)=\mathbf{Z}$, the sequence of which is "like" $S^n \times S^1 \times \mathbf{R}$.

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[1] This theorem is a direct corollary of Theorem 4 applied to the "toral" subset $M \times T^{n-1} \times \mathbf{R} \subset W$ only for $n \geqslant q$ because we have the immersion $V \times \mathbf{R}^n \to W$ by Theorem 4.

My first approach to the topological problems connected with rational Pontrjagin classes contains another idea (from the technical point of view) in the study of open manifolds homeomorphic to $M^{4k} \times R^n$ and $M^{4k} \times T^{n-1} \times R$ (see Dokl. A.N. 162 N. 6, Izv. A.N. 29 N. 6). It had given me the first nontrivial results in these problems (for example, the first negative solution of the Hurewicz problem about the homeomorphism and homotopy type of closed simply connected manifolds). This idea is based also on the fundamental group and coverings, but does not use the specific differentially-topological technique of the author and Browder (and its generalizations). The method of these papers is purely homological (for manifolds over $\mathbf{Q}$). But there are many difficulties here (not only technical ones) in codimensions greater then two. It would be very interesting to develop this method. Perhaps this problem is connected closely with the homotopy problems of the first part of our paper, because the parts 1 and 2 of Theorem 1 were obtained by this method.

3. Pontrjagin-Hirzebruch Classes over the Finite Coefficient Groups

We consider now the coefficient group $\mathbf{Z}_m$ and classes $p_k \in H^{4k}(M, \mathbf{Z}_m)$. Many homotopic invariance relations are known here. All of them are connected with J-functors and cohomological operations in some sense (Thom, Wu, Atiyah, Hirzebruch). However the class p_1 is not homotopically invariant for 5-manifolds (the rational class $p_1(M^5)$ is homotopically invariant). This may be seen with the classical lens manifolds $M^{2l+1} = L_p^{2l+1}$; $(g_0, \ldots, g_l)$, $g_i \neq 0$ (modulo p). The complete homotopy invariant here is $q = \prod_i g_i$ up to multiplication on $\lambda^{l+1} \not\equiv 0 \,(\operatorname{mod} p)$, $p_k = \left(\sum_i q_i^{2k}\right) x^{2k}, x \in H^2(L, Z)$.

As previous ("rational") results had shown, it is useful for invariance problems to give a "signature" definition of classes (modulo p) like the Thom-Rohlin-Schwartz definition of rational classes.

The problem of finding such a definition was solved by the author and V. A. Rohlin. We shall give a definition of classes $\delta_k \cdot L_k (\operatorname{mod} m)$ for $\mathbf{Q}_m$-homological manifolds ($\mathbf{Q}_m$ is the ring of rationals with denominators relatively prime to m), $\delta_k = \prod_{i \leq 4k} |\pi_{N+i}(S^N)|^2$.

Remark. We can define also the classes $\mu_k \cdot L_k$ for Q_m-manifolds, $\mu_k = \prod_{p \geq 2} p^{[(2k-1)/(p-1)]}$, by using more complicated arguments, connected with the problem of realization of the homology class by a submanifold with normal SO-bundle: the universal multiple α_i in this problem (such that $\alpha_i z$ is realizable for all $z \in H_i(M^n, Z)$, $n > 2i$) is equal to $\prod_{p>2} p^{[(i-1)/(2p-2)]}$, and $\alpha_{4k} = \mu_k$) (Dokl. A.N., 1960, v. 132 N. 5).

The idea of the definition of Rohlin and myself is this: we realize the homology class $Z\,(\mathrm{mod}\, m)$ by a regular map $f\colon V^{4k} \to M$ of a manifold with boundary $\partial V^{4k} = mW = W \cup \ldots \cup W$ (m times), such that the image of ∂V^{4k} is diffeomorphic to W and f on each component of ∂V^{4k} is a diffeomorphism. The normal bundle of f may be trivial (first variant) or an SO-bundle (second variant). We consider the natural quadratic form (possibly degenerate) on $H_{2k}(V^{4k}, \mathbf{Q})$ or $H^{2k}(V^{4k}, \partial V; \mathbf{Q})$ and its signature $\sigma(V^{4k})$.

Our definition is: $(L_k, Z) = \sigma(V^{4k}) \bmod m$ (in the first variant). This definition is correct. We use here essentially results of cobordism theory (all torsions have order 2) and the next lemma: let V_1^{4k}, V_2^{4k} be manifolds having the same closed component in their boundaries $W \subset V_1^{4k}$, $W \subset V_2^{4k}$; then the signature is an additive invariant: $\sigma(V_1^{4k} \bigcup_W V_2^{4k}) = \sigma(V_1^{4k}) + \sigma(V_2^{4k})$. If $\partial V = mW$, then $2W \sim 0$ in Ω_{SO} (or $W \sim 0$ in Ω_{SO} if $m = 2l+1$).[1]

Thus $2W = \partial \bar{V}$ and $\sigma\left(V^{4k} \bigcup_{2W} \bar{V} \cup \ldots \bigcup_{2W} \bar{V}\right) = \sigma(V^{4k})$ modulo $m/2$ for any $\bar{V}$, and we can apply to the manifold $\left(V^{4k} \bigcup_{2W} \bar{V} \cup \ldots \bigcup_{2W} \bar{V}\right)$ ($m/2$ times) the usual Hirzebruch formula.

Rohlin and I have proved the following theorems A and B.

Theorem A. *Let M^{4k+1} be a $\mathbf{Q}_m$-homological manifold and $z \in H_{4k}(M^{4k+1}, \mathbf{Q})$ a homology class such that $(\beta z)\cdot(\beta z) = 0$, β = Bokšteĭn coboundary. Then the scalar product $(L_k, Z) \bmod m$ is topologically invariant.*

The proof of theorem A is the development of purely homological papers (Novikov, DAN 126 N. 6; Izv. 29 N. 6 and Rohlin, Izv. 30 N. 3). The proof of theorem B is differential-topological.

Problem. Is this scalar product homotopically invariant? This problem seems to be connected with "coverings with fixed points".

Theorem B. *All classes $\delta_k L_k$ of smooth manifolds are topological invariants.*

We have now a number of corollaries:

[1] The Cobordism theory ($\mathrm{mod}\, p$) and its topological applications were pushed very far by D. Sullivan in 1967 in his paper about the space F/PL and the Hauptvermutung. This theory is based on manifolds W with $\partial W = pV$, as is our definition of p_k modulo p.

Corollary A. *There are universal numbers* $\lambda_k \neq 0$ *such that for any manifold* M, *complex* X *and map* $f: X \to M$ $(\dim X \leqslant k)$ *the element* $\lambda_k f^* \tau \in \widetilde{KO}(X)$ *is a topological invariant of* M *(*τ *is the stable tangent bundle of* M*). For example,* $\lambda_n \tau \in \widetilde{KO}(M^n)$ *is a topological invariant* $(X = M, f = 1)$. *The number* λ_5 *is equal to* $2^h \cdot 3^k$ *(possibly* $h = k = 0$*). We consider the classical lens 5-manifolds with* $\pi_1 = Z_5$. *All of them are homotopically equivalent, because the equation* $\lambda = \mu^3 \pmod 5$ *always has a solution. But they may have different classes* p_1.

Corollary B. *There are topologically nonequivalent classical lens manifolds with* $\pi_1 = Z_5$ *in dimension* 5 *(or the unitary transformation groups* $A^5 = 1$*), which are homotopy equivalent.*

Corollary C. *For almost all prime* p *(except a finite number, depending upon the dimension) the tangential homotopy type of lens manifolds is a topological invariant.*

Corollary D. *The Milnor map* $j: \widetilde{KO}(X) \to \tilde{K}_{top}(X)$ *for any complex* X, $\dim X \leqslant k$, *is such that* $\operatorname{Ker} j$ *is a periodic group with period* λ_k.