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Elliptic genera, modular forms over KO_*, and the Brown-Kervaire invariant

Serge Ochanine *
CNRS, Université de Paris-Sud, Mathématique, bât 425, F-91405 Orsay, France

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Let Ω_*^{SO} be the oriented cobordism ring and Λ any commutative $\mathbb{Q}$-algebra. An *elliptic genus* over Λ, as originally defined in [14], is a ring homomorphism

$$\varphi\colon\ \Omega_*^{SO} \to \Lambda$$

satisfying

$$\sum_{i \geq 0} \varphi[\mathbb{C}P_{2i}]\, u^{2i} = (1 - 2\delta u^2 + \varepsilon u^4)^{-1/2}$$

Here

$$\delta = \varphi[\mathbb{C}P_2] \quad \text{and} \quad \varepsilon = \varphi[\mathbb{H}P_2]$$

are two parameters in Λ which determine φ completely.

In the most interesting *universal* examples, Λ is the ring $\mathbb{Q}[[q]]$ of formal power series over $\mathbb{Q}$, and for any oriented manifold V, $\varphi[V]$ is the q-expansion of a level 2 modular form whose values at the two cusps are, up to an inessential factor, the $\hat{A}$-genus $\hat{A}[V]$ and the signature $\sigma(V)$ (cf. [9, 5, 10, 23, 8]).

Though defined for oriented manifolds, the elliptic genera reveal their most striking properties, such as rigidity (constancy) under compact Lie group actions ([3, 15]) or integrality [6], on spin manifolds. Both rigidity and integrality rely on the fact noticed by E. Witten [22] that in the universal examples, the coefficients of $\varphi[V]$ are indices of twisted Dirac operators, therefore KO-characteristic numbers.

In this paper we consider a refined elliptic genus

$$\beta_q\colon\ \Omega_*^{\text{spin}} \to KO_*[[q]]$$

whose values are q-expansions of level 2 modular forms over the coefficient ring KO_* of the real K-theory. In dimensions divisible by 4, $\beta_q[V]$ is essentially the above universal genus $\varphi[V]$. On the other hand, in dimensions $8m+1$,

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* *Present address:* Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, W-5300 Bonn 3, Federal Republic of Germany

$8m+2$, $\beta_q[V]$ is a modular form over $\mathbb{F}_2$ (in the sense of J.-P. Serre [18]), and can be expressed as a polynomial in the basic form $\bar{\varepsilon} = \sum_{n \geqq 1} q^{(2n-1)^2}$:

$$\beta_q[V] = a_0 + a_1 \bar{\varepsilon} + \ldots + a_m \bar{\varepsilon}^m.$$

It turns out that a_0 is the Atiyah invariant while a_m is the KO-part of the Brown-Kervaire invariant of V.

Being a refinement of an elliptic genus, β_q retains at least a few of the properties of the latter. For example, M. Bendersky ([2]) recently proved that $\beta_q[V] = 0$ for a spin manifold V admitting an odd type semi-free circle action, which implies the vanishing of both the Atiyah invariant and the KO-part of the Brown Kervaire invariant*.

1 Definition of β_q

Let E be a real vector bundle over X. Writing $\Lambda^i(E)$ and $S^i(E)$ respectively for the exterior and the symmetric powers of E, and

$$\Lambda_t(E) = \sum_{i \geqq 0} \Lambda^i(E)\, t^i,$$

$$S_t(E) = \sum_{i \geqq 0} S^i(E)\, t^i,$$

one defines the *Witten characteristic class* Θ_q ([22], cf. [10]) by

$$\Theta_q(E) = \bigotimes_{n \geqq 1} (\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)).$$

For any E, $\Theta_q(E)$ is a formal power series in q whose coefficients are virtual vector bundles over X. Moreover, one has

$$\Theta_q(E) = 1 - E \cdot q + \ldots$$

and

$$\Theta_q(E \oplus F) = \Theta_q(E) \cdot \Theta_q(F).$$

Therefore Θ_q canonically extends to $KO(X)$:

$$\Theta_q\colon KO(X) \to KO(X)[[q]].$$

Let $\beta_q(E)$ be defined by

$$\beta_q(E) = \Theta_q(E - \dim E).$$

Then

$$\beta_q(E) = b_0(E) + b_1(E)\, q + \ldots$$

where

$$b_0(E) = 1$$

$$b_i(E) \in \widetilde{KO}(X) \qquad (i > 0)$$

and

$$\beta_q(E \oplus F) = \beta_q(E) \cdot \beta_q(F).$$

* For a proof valid for all odd type actions see [16].

It is easy to see that $b_i (i \geqq 0)$ are stable KO-characteristic classes and can be expressed as polynomials in the Pontrjagin classes π_i defined by (cf. [21]):

$$\sum \pi_i(E)\, u^i = \Lambda_t(E - \dim E),$$

where

$$u = \frac{t}{(1+t)^2}.$$

For example

$$b_1 = -\pi_1$$
$$b_2 = \pi_2 - \pi_1$$
$$b_3 = -\pi_3 + 4\pi_2 - \pi_1^2 - 4\pi_1$$

and, more generally,

$$b_i = (-1)^i \pi_i + \text{lower terms}.$$

Let now V^n be a closed spin manifold, and $[V^n] \in KO_n(V^n)$ be the fundamental class of V^n in real K-theory.

Definition.

$$\beta_q[V^n] = \beta_q(TV)[V^n] = \sum_{i \geqq 0} b_i[V^n]\, q^i,$$

where TV is the tangent bundle of V^n and

$$b_i[V^n] = b_i(TV)[V^n] \in KO_n = KO_n(\text{point})$$

is the KO-characteristic number corresponding to b_i.

One can easily see that β_q defines a ring homomorphism (genus)

$$\beta_q\colon\ \Omega_*^{\text{spin}} \to KO_*[[q]].$$

Considered as $\mathbb{Z}/8$-graded, the ring KO_* is generated by two elements η and ω of degree 1 and 4 respectively subject to the relations

$$2\eta = \eta^3 = \eta\omega = 0, \qquad \omega^2 = 4.$$

Clearly, β_q preserves the degree mod 8.

Let

$$ph\colon\ KO^*(X) \to H^{**}(X; \mathbb{Q})$$

be the Pontrjagin character defined as the composition

$$KO^*(X) \xrightarrow{\otimes \mathbb{C}} K^*(X) \xrightarrow{\text{Chern char.}} H^{**}(X; \mathbb{Q}).$$

For $X = $ point one has $KO^*(X) \cong KO_*$ and $H^{**}(X; \mathbb{Q}) \cong \mathbb{Q}$, and ph is entirely determined by

$$ph(\eta) = 0, \qquad ph(\omega) = 2.$$

In particular, ph is integral:

$$ph\colon\ KO_* \to \mathbb{Z}.$$

Composing β_q with ph leads to a genus

$$\varphi_q = ph \circ \beta_q \colon \Omega_*^{\mathrm{spin}} \to \mathbb{Z}[[q]]$$

such that

$$\varphi_q[V^n] = \sum_{i \geqq 0} ph(b_i[V^n])\, q^i = \sum_{i \geqq 0} ph(b_i(TV))\, \hat{\mathfrak{A}}(TV)[V^n]\, q^i$$

where $\hat{\mathfrak{A}}(TV)$ is the total $\hat{\mathfrak{A}}$-class of V^n. In particular, the constant term of $\varphi_q[V^n]$ is the $\hat{A}$-genus $\hat{A}[V^n]$.

Theorem 1 ([10, 23]) φ_q *is the restriction to* Ω_*^{spin} *of an elliptic genus*

$$\varphi_q \colon \Omega_*^{SO} \to \mathbb{Q}[[q]]$$

with parameters

$$\delta = -\tfrac{1}{8} - 3 \sum_{n \geqq 1} \Big(\sum_{\substack{d \mid n \\ d \text{ odd}}} d \Big) q^n$$

$$\varepsilon = \sum_{n \geqq 1} \Big(\sum_{\substack{d \mid n \\ n/d \text{ odd}}} d^3 \Big) q^n \quad \square$$

2 Modular forms over graded rings

It turns out that $\beta_q[V^n]$ can be interpreted as a modular form of degree n over the graded ring KO_*.

If Γ is a subgroup of $SL_2(\mathbb{Z})$ of finite index, let $M_*^\Gamma(\mathbb{C})$ be the graded ring of modular forms over $\mathbb{C}$ for Γ. Thus $M_w^\Gamma(\mathbb{C})$ is the group of forms of weight w. We will always identify a modular form from $M_*^\Gamma(\mathbb{C})$ with its q-expansion. This way $M_*^\Gamma(\mathbb{C})$ becomes a subring in $\mathbb{C}[[q^{1/h}]]$, where h is the smallest positive integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ belongs to Γ.

Let now $M_*^\Gamma(\mathbb{Z})$ be the graded subring of $M_*^\Gamma(\mathbb{C})$ of forms having integral q-expansions

$$M_*^\Gamma(\mathbb{Z}) = \bigoplus_n M_n^\Gamma(\mathbb{Z})$$

$$M_n^\Gamma(\mathbb{Z}) = M_n^\Gamma(\mathbb{C}) \cap \mathbb{Z}[[q^{1/h}]].$$

For any graded commutative ring with unit

$$R_* = \bigoplus_n R_n,$$

the canonical injection

$$M_*^\Gamma(\mathbb{Z}) \to \mathbb{Z}[[q^{1/h}]]$$

extends to a ring homomorphism

$$R_* \otimes_{\mathbb{Z}} M_*^\Gamma(\mathbb{Z}) \to R_*[[q^{1/h}]].$$

We define $M^\Gamma(R_*)$ to be the image of this homomorphism, and will call its elements *modular forms over* R_* *for* Γ.

Notice that $M^\Gamma(R_*)$ is canonically a graded R_*-algebra:

$$M^\Gamma(R_*) = \bigoplus_n M^\Gamma(R_n),$$

where $M^\Gamma(R_n)$ is the image of $R_n \otimes M_*^\Gamma(\mathbb{Z})$. We refer to the elements of $M^\Gamma(R_n)$ as forms of *degree n*.

If for a certain n, R_n has no torsion, then

$$R_n \otimes M_*^\Gamma(\mathbb{Z}) \to M^\Gamma(R_n)$$

is an isomorphism. In this case,

$$M^\Gamma(R_n) = \bigoplus_w M_w^\Gamma(R_n),$$

where

$$M_w^\Gamma(R_n) \cong R_n \otimes M_w^\Gamma(\mathbb{Z}).$$

We will say that forms from $M_w^\Gamma(R_n)$ have *weight w*.

In the general situation, a form $F \in M^\Gamma(R_n)$ may come from integral forms of different weights, and the weight of F cannot be defined correctly. Instead, one defines an increasing *filtration* of $M^\Gamma(R_n)$ as follows: a form $F \in M^\Gamma(R_n)$ has filtration $\leqq f$ if F is the image of an element of

$$R_n \otimes \Big(\bigoplus_{w \leqq f} M_w^\Gamma(\mathbb{Z})\Big),$$

i.e. if

$$F = \sum r_j F_j,$$

where $F_j \in M_*^\Gamma(\mathbb{Z})$ are forms of weight $\leqq f$ and $r_j \in R_n$.

3 Modular forms over KO_*

From now on Γ will designate the group $\Gamma_0(2)$ of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that $c \equiv 0 \pmod 2$. The series δ and ε of theorem 1 are the basic examples of modular forms for $\Gamma_0(2)$. More precisely, let

$$\delta_0 = -8\delta = 1 + 24q + 24q^2 + 96q^3 + \ldots$$

Proposition 1 (cf. [8], Anhang I) (i) $\delta_0 \in M_2^\Gamma(\mathbb{Z})$, $\varepsilon \in M_4^\Gamma(\mathbb{Z})$;
(ii) $M_*^\Gamma(\mathbb{Z}) = \mathbb{Z}[\delta_0, \varepsilon]$.

Consider now $M^\Gamma(KO_*)$. For $n \equiv 0 \pmod 4$, one has $KO_n \cong \mathbb{Z}$. Thus

$$M^\Gamma(KO_n) \cong KO_n \otimes M_*^\Gamma(\mathbb{Z}).$$

It follows that:

(a) a modular form of degree $n=8m$ and weight w over KO_* can be written in a unique way as a polynomial $P(\delta_0, \varepsilon)$ of weight w with integer coefficients;
(b) a modular form of degree $n=8m+4$ and weight w over KO_* can be written in a unique way as $\omega P(\delta_0, \varepsilon)$, where $P(\delta_0, \varepsilon)$ is a polynomial of weight w with integer coefficients.

Notice now that one has $\delta_0 \equiv 1 \pmod 2$. Let $\bar{\varepsilon}$ be the reduction mod 2 of $\varepsilon \in \mathbb{Z}[[q]]$. It is easy to see that

$$\bar{\varepsilon} = \sum_{n \geqq 1} q^{(2n-1)^2} = q + q^9 + q^{25} + \dots$$

For $n=8m+r (r=1, 2)$, one has $KO_n = \mathbb{F}_2 \eta^r$ and the map

$$KO_n \otimes M_*^\Gamma(\mathbb{Z}) \to KO_n[[q]]$$

is essentially the reduction mod 2:

$$\eta^r \otimes P(\delta_0, \varepsilon) \mapsto \eta^r \bar{P}(1, \bar{\varepsilon}),$$

where $P(\delta_0, \varepsilon)$ is a polynomial with integer coefficients and $\bar{P}$ is its reduction mod 2. As $\bar{\varepsilon} = q + \dots$, the powers of $\bar{\varepsilon}$ are linearly independent over $\mathbb{F}_2$. Therefore:

(c) a modular form F of degree $n=8m+r (r=1, 2)$ and filtration $\leqq f$ over KO_* can be written in a unique way as $\eta^r Q(\bar{\varepsilon})$, where

$$Q(\bar{\varepsilon}) = a_0 + a_1 \bar{\varepsilon} + \dots + a_s \bar{\varepsilon}^s (a_i \in \mathbb{F}_2)$$

and $4s \leqq f$. The filtration of F is exactly $4s$ if and only if $a_s \neq 0$.

The additive structure of $M^\Gamma(KO_*)$ is completely described by (a), (b), and (c). The ring structure is given by the following theorem.

Theorem 2 (i) *The kernel of*

$$KO_* \otimes M_*^\Gamma(\mathbb{Z}) \to M^\Gamma(KO_*)$$

is the principal ideal generated by $\eta \otimes (\delta_0 - 1)$.
(ii) *The commutative* KO_**-algebra* $M^\Gamma(KO_*)$ *is generated by* δ_0 *and* ε *subject to the single relation* $\eta \delta_0 = \eta$.

The proof is immediate from the above description of

$$KO_* \otimes M_*^\Gamma(\mathbb{Z}) \to KO_*[[q]].$$

4 $\beta_q[V^n]$ as a modular form

We will now see that $\beta_q[V^n]$ is a modular form of degree n over KO_*.

Theorem 3 (i) *If* $n=4s$, *then* $\beta_q(\Omega_n^{\text{spin}})$ *is the set of all modular forms of degree* n *and weight* $2s$ *over* KO_*.
(ii) *If* $n=8m+r (r=1, 2)$, *then* $\beta_q(\Omega_n^{\text{spin}})$ *is the set of all modular forms of degree* n *and filtration* $\leqq 4m$ *over* KO_*.
(iii) $\beta_q(\Omega_*^{\text{spin}})$ *is the subring of* $M^\Gamma(KO_*)$ *generated by* η, $\omega\delta_0$, δ_0^2 *and* ε.

Proof. Part (iii) clearly follows from (i), (ii) and the above description of $M^\Gamma(KO_*)$.

Part (i) is a simple consequence of the definition of φ_q, the description of ph and the following theorem:

Theorem 4 ([6], cf. [10]) *For any spin manifold V^{4s}, $\varphi_q[V^{4s}]$ is a modular form from $M^\Gamma_{2s}(\mathbb{Z})$. More precisely,*

$$\varphi_q(\Omega^{\text{spin}}_{8m}) = M^\Gamma_{4m}(\mathbb{Z})$$
$$\varphi_q(\Omega^{\text{spin}}_{8m+4}) = 2M^\Gamma_{4m+2}(\mathbb{Z})$$

The proof of the remaining part (ii) relies on the following construction due to R.E. Stong (cf. [21, p. 341] for the details):

Let $\bar{S}^1$ be the circle equipped with its non-trivial spin structure. $\bar{S}^1$ represents the non-zero element of $\Omega^{\text{spin}}_1 \cong \mathbb{F}_2$. If V is an $(8m+2)$-dimensional spin manifold, then $\bar{S}^1 \times V$ is the boundary of a compact spin manifold U. On the other hand, $2\bar{S}^1$ is the boundary of a compact spin manifold M^2. Therefore one can form a closed $(8m+4)$-dimensional spin manifold $T(V)$ by glueing together two copies of U and $-M^2 \times V$ along

$$\partial(2U) = 2\bar{S}^1 \times V = \partial(M^2 \times V).$$

Though involving arbitrary choices of M^2 and U, this construction induces a well-defined homomorphism

$$T\colon\ \Omega^{\text{spin}}_{8m+2} \to \Omega^{\text{spin}}_{8m+4} \otimes \mathbb{F}_2.$$

Let

$$t\colon\ KO_2 \to KO_4 \otimes \mathbb{F}_2$$

be the isomorphism which sends η^2 to $\omega \otimes 1$.

Proposition 2 (cf. [21, p. 343]) *If ξ is a polynomial in the Pontrjagin classes π_i, then one has in $KO_4 \otimes \mathbb{F}_2$:*

$$\xi[T(V)] \otimes 1 = t(\xi[V]). \quad \square$$

Roughly speaking, $\xi[V]$ is the reduction mod 2 of $\xi[T(V)]$.

Let $I_* \subset \Omega^{\text{spin}}_*$ be the ideal of classes with vanishing Pontrjagin KO-characteristic numbers. Proposition 2 implies that T induces a homomorphism

$$\tilde{T}\colon\ \Omega^{\text{spin}}_{8m+2}/I_{8m+2} \to (\Omega^{\text{spin}}_{8m+4}/I_{8m+4}) \otimes \mathbb{F}_2.$$

Proposition 3 (cf. [21, p. 344]) *$\tilde{T}$ is an isomorphism.* $\square$

The coefficients of $\beta_q[V]$ are Pontrjagin KO-characteristic numbers. Therefore one has:

$$\beta_q[T(V)] \otimes 1 = t(\beta_q[V])$$

in $(KO_4 \otimes \mathbb{F}_2)[[q]]$. By Theorem 3 (i),

$$\beta_q[T(V)] = \omega P(\delta_0, \varepsilon),$$

where $P(\delta_0, \varepsilon)$ is a polynomial of weight $4m+2$ in δ_0, ε with integer coefficients. Therefore

$$\beta_q[V] = \eta^2 \bar{P}(1, \bar{\varepsilon})$$

is a modular form of degree $8m+2$ and filtration $\leqq 4m$ over KO_*. Proposition 3 implies that all such forms can be obtained from spin manifolds V, and this settles the case of manifolds of dimension $8m+2$.

The proof in the case of $(8m+1)$-dimensional manifolds is similar. Instead of T one considers the multiplication by $\bar{S}^1$ homomorphism

$$S\colon \Omega^{\text{spin}}_{8m} \to \Omega^{\text{spin}}_{8m+1}.$$

If ξ is a polynomial in the classes π_i, then

$$\xi[\bar{S}^1 \times M] = \eta \cdot \xi[M]$$

for any spin manifold M. Thus S induces a homomorphism

$$\tilde{S}\colon \Omega^{\text{spin}}_{8m}/I_{8m} \to \Omega^{\text{spin}}_{8m+1}/I_{8m+1}.$$

Proposition 4 (cf. [21, p. 344]) *$\tilde{S}$ is into.* □

It follows that

$$\beta_q(\Omega^{\text{spin}}_{8m+1}) = \eta \cdot \beta_q(\Omega^{\text{spin}}_{8m})$$

and the result follows from (i) and the description of $M^\Gamma(KO_*)$.

5 Characteristic classes a_i

Let $h(q) = q + \ldots$ be any series from $\mathbb{Z}[[q]]$ whose reduction mod 2 is

$$\sum_{n \geqq 1} q^{(2n-1)^2} = q + q^9 + q^{25} + \ldots$$

For example, one can take $h(q) = \varepsilon(q)$. Another possible choice for $h(q)$ is the Ramanujan series

$$\Delta(q) = q \prod_{n \geqq 1} (1-q^n)^{24} = q - 24q^2 + 252q^3 - \ldots\star$$

For any real vector bundle E over X define

$$\alpha_t(E) \in KO(X)[[t]]$$

by

$$\alpha_t(E) = \beta_q(E),$$

where

$$t = h(q).$$

Since the leading term of $h(q)$ is q, this series is invertible in $\mathbb{Z}[[q]]$, therefore $\alpha_t(E)$ is well-defined. Clearly, one has

$$\alpha_t(E \oplus F) = \alpha_t(E)\, \alpha_t(F).$$

⋆ It is an amusing exercise to show that $\Delta \equiv \varepsilon (\text{mod } 2)$ and even, as noticed by P. Landweber, $\Delta \equiv \varepsilon (\text{mod } 16)$.

If

$$\alpha_t(E) = a_0(E) + a_i(E)\, t + a_2(E)\, t^2 + \dots,$$

then $a_i(E)$ is a polynomial in the Pontrjagin classes $\pi_i(E)$ such that

$$a_0(E) = 1$$
$$a_i(E) \in \widetilde{KO}(X) \qquad (i > 0)$$

and

$$a_i(E) = (-1)^i\, \pi_i(E) + \text{lower terms}.$$

Notice that while $a_i(E)$ depends on the choice of $h(q)$, its reduction mod 2, that is its image in $KO(X) \otimes \mathbb{F}_2$ is independent of any choice.

By definition of a_i, for any spin manifold V^n one has:

$$\beta_q[V^n] = a_0[V^n] + a_1[V^n]\, t + a_2[V^n]\, t^2 + \dots,$$

where

$$a_i[V^n] = a_i(TV)[V^n].$$

On the other hand, the reduction mod 2 of $\beta_q[V^n]$ is of the form (cf. Sect. 3):

$$a_0 + a_1\, \bar{\varepsilon} + \dots + a_m\, \bar{\varepsilon}^m,$$

where $a_i \in KO_n \otimes \mathbb{F}_2$ and $m = [n/8]$. Comparing these two expressions leads to the following:

Theorem 5 (i) *For $i > [n/8]$, one has $a_i\, [V^n] \otimes 1 = 0$ in $KO_n \otimes \mathbb{F}_2$.*
(ii) *One has in $(KO_n \otimes \mathbb{F}_2)[[q]]$:*

$$\beta_q[V^n] \equiv a_0[V^n] + a_1[V^n]\, \bar{\varepsilon} + \dots + a_m[V^n]\, \bar{\varepsilon}^m,$$

where $m = [n/8]$.

6 The Brown-Kervaire invariant

Notice that for $n = 8m + 2$, the constant term $a_0[V^n] = 1[V^n]$ is the so-called Atiyah invariant ([1]). We will see now that $a_m[V^n]$ has an interpretation in terms of the Brown-Kervaire invariant of V^n.

Let V^n, $n = 8m + 2$, be a spin manifold. As mentioned earlier, $\bar{S}^1 \times V = \partial U$, where U is a compact spin manifold. It is shown in [13] that the signature $\sigma(U)$ is divisible by 8, and that

$$k(V) = \sigma(U)/8 \in \mathbb{F}_2$$

is a spin cobordism invariant satisfying

$$k(\bar{S}^1 \times \bar{S}^1 \times M) = \sigma(M) \bmod 2$$

for all $8m$-dimensional spin manifolds M. For a large class of manifolds, including all complex-spin manifolds [20], $k(V)$ agrees with the Brown-Kervaire invariant [4]. For a general spin manifold V, $k(V)$ can be thought of as the KO-part of the Brown-Kervaire invariant (cf. [13] for the details).

More generally, one defines an invariant $\kappa(V^n)\in KO_n\otimes \mathbb{F}_2$ by

$$\kappa(V^n)=\begin{cases}\sigma(V), & n\equiv 0 \pmod 8\\ k(\bar{S}^1\times V)\,\eta, & n\equiv 1 \pmod 8\\ k(V)\,\eta^2, & n\equiv 2 \pmod 8\\ (\sigma(V)/16)\omega, & n\equiv 4 \pmod 8\end{cases}$$

The multiplicative properties of k are summarized by saying that κ defines a ring homomorphism

$$\kappa\colon \Omega_*^{\text{spin}}\to KO_*\otimes \mathbb{F}_2.$$

A new proof of this will be given later.

Theorem 6 *Let V^n be a spin manifold. Then*

$$a_m[V^n]=\kappa(V^n)$$

in $KO_n\otimes\mathbb{F}_2$, where $m=[n/8]$.

Proof. Consider first the case when $n=8m+4$. According to Theorem 3,

$$\beta_q[V^n]=\omega(a_0\,\delta_0^{2m+1}+a_1\,\delta_0^{2m-1}\,\varepsilon+\ldots+a_m\,\delta_0\,\varepsilon^m),$$

where $a_i\in\mathbb{Z}$. Then

$$\varphi_q[V^n]=2(a_0\,\delta_0^{2m+1}+a_1\,\delta_0^{2m-1}\,\varepsilon+\ldots+a_m\,\delta_0\,\varepsilon^m).$$

If we consider φ_q as an elliptic genus over $\mathbb{Z}[\delta,\varepsilon]$, the signature $\sigma(V^n)$ is obtained by specializing $\delta=1$, $\varepsilon=1$, or $\delta_0=-8$, $\varepsilon=1$. Thus,

$$\begin{aligned}\sigma(V^n)&=2(a_0(-8)^{2m+1}+a_1(-8)^{2m-1}+\ldots+a_m(-8))\\ &\equiv 16a_m \pmod{32},\end{aligned}$$

and

$$\kappa(V^n)=a_m\,\omega \bmod 2.$$

On the other hand, by Theorem 5,

$$a_m\,\omega=a_m[V^n] \bmod 2,$$

therefore

$$\kappa(V^n)=a_m[V^n] \bmod 2.$$

If $n=8m+2$, Proposition 2 gives

$$\begin{aligned}t(a_m[V^n])&=a_m[T(V)] \bmod 2\\ &=(\sigma(T(V))/16)\,\omega \bmod 2\end{aligned}$$

by the previous case.

By definition,

$$T(V)=(2U)\cup(-M^2\times V),$$

where $\partial U=\bar{S}^1\times V$. Thus

$$\sigma(T(V))=2\sigma(U).$$

On the other hand,

$$k(V) = \frac{\sigma(U)}{8} = \frac{\sigma(T(V))}{16} \bmod 2.$$

Comparing with the above expression for $t(a_m[V^n])$, we obtain:

$$a_m[V^n] = k(V^n)\,\eta^2 = \kappa(V^n).$$

If $n = 8m+1$,

$$a_m[V^n]\,\eta = a_m[\bar{S}^1 \times V^n] = k(\bar{S}^1 \times V^n)\,\eta^2,$$

therefore

$$a_m[V^n] = k(\bar{S}^1 \times V^n)\,\eta = \kappa(V^n)$$

since the multiplication by η is an isomorphism $KO_1 \xrightarrow{\cong} KO_2$.

Finally, if $n = 8m$, then

$$a_m[V^n]\,\eta^2 = a_m[\bar{S}^1 \times \bar{S}^1 \times V^n] = k(\bar{S}^1 \times \bar{S}^1 \times V^n)\,\eta^2 = \sigma(V^n)\,\eta^2,$$

and

$$a_m[V^n] \equiv \sigma(V^n) (\mathrm{mod}\ 2). \quad \square$$

Corollary 1 $\kappa\colon \Omega_*^{\mathrm{spin}} \to KO_* \otimes \mathbb{F}_2$ *is a ring homomorphism.*

Proof. Let V_1 and V_2 be two spin manifolds of dimension n_1 and n_2 respectively, and let

$$m_1 = [n_1/8], \qquad m_2 = [n_2/8], \qquad m = [(n_1+n_2)/8].$$

By Theorem 6,

$$\kappa(V_1 \times V_2) = a_m[V_1 \times V_2] = \sum_{i_1+i_2=m} a_{i_1}[V_1]\, a_{i_2}[V_2].$$

Notice that $m \geqq m_1 + m_2$. If $m = m_1 + m_2$, then Theorem 5 (i) and theorem 6 imply:

$$\kappa(V_1 \times V_2) = a_{m_1}[V_1]\, a_{m_2}[V_2] = \kappa(V_1)\,\kappa(V_2).$$

If $m > m_1 + m_2$, then Theorem 5 (i) gives

$$\kappa(V_1 \times V_2) = 0$$

and one has to check that

$$\kappa(V_1)\,\kappa(V_2) = 0.$$

But $m > m_1 + m_2$ is possible only in one of the following cases:

(1) $$n_1 \equiv n_2 \equiv 4 (\mathrm{mod}\ 8).$$

In this case

$$\kappa(V_1)\,\kappa(V_2) = 0$$

since $\omega^2 \equiv 0 (\mathrm{mod}\ 2)$.

(2) $$n_1 \equiv 5, 6, 7 (\mathrm{mod}\ 8) \quad \text{or} \quad n_2 \equiv 5, 6, 7 (\mathrm{mod}\ 8).$$

In this case $\kappa(V_1)$ or $\kappa(V_2)$ is zero. $\square$

Corollary 2 *Let V^n, $n=8m+r(r=1,2)$ be a spin manifold. The filtration of $\beta_q[V^n]$ is exactly $4m$ if and only if $\kappa(V^n)\neq 0$.*

Proof. This follows from Theorem 6 and the description of $M^\Gamma(KO_{8m+r})$ in Sect. 3. □

7 The *SU*-case

Theorem 3 describes the subring $M_*=\beta_q(\Omega_*^{\text{spin}})\subset M^\Gamma(KO_*)$. Using the results of [6] one can easily determine the image of the special unitary cobordism ring Ω_*^{SU} under β_q. We will focus on the dimensions $8m+1$, $8m+2$, leaving the easier remaining cases to the reader.

Theorem 7 (i) *If $n=8m+1$, then $\beta_q(\Omega_n^{SU})\subset\beta_q(\Omega_n^{\text{spin}})$ is the subgroup of forms of the form $\eta P(\varepsilon^2)$ where P is a polynomial of degree $\leqq m/2$ over $\mathbb{F}_2$.*
(ii) *If $n=8m+2$, then $\beta_q(\Omega_n^{SU})=\beta_q(\Omega_n^{\text{spin}})$.*

Corollary. *If M^n, $n=8m+1$, is an SU-manifold, then*

$$a_i[M^n]=0$$

for all odd *i. For instance,*

$$\pi_1[M^n]=0,$$
$$(\pi_3+\pi_1^2)[M^n]=0.$$

Proof. (i) According to [6], an element from $\varphi_q(\Omega_{8m}^{SU})$ can be written as

$$2P(\delta_0^2,\varepsilon)+Q(\delta_0^2,\varepsilon^2),$$

where P, Q are two polynomials with integer coefficients. On the other hand, one has

$$\Omega_{8m+1}^{SU}=[\bar{S}^1]\cdot\Omega_{8m}^{SU}$$

where $\bar{S}^1$ is the circle S^1 equipped with its non-trivial SU-structure (cf. [21, Chap. X]). Therefore,

$$\beta_q(\Omega_{8m+1}^{SU})=\eta\cdot\beta_q(\Omega_{8m}^{SU})$$

and the result follows.

Part (ii) is an immediate consequence of the following proposition.

Proposition 5 *The canonical map*

$$\Omega_{8m+2}^{SU}\to\Omega_{8m+2}^{\text{spin}}/I_{8m+2}$$

is onto. In other words, any spin manifold of dimension $8m+2$ has the same KO-characteristic numbers as an SU-manifold.

Proof. Notice first that the homomorphism T used in the proof of Theorem 4 can be defined using SU-manifolds: there is a homomorphism

$$T^c\colon\ \Omega_{8m+2}^{SU}\to\Omega_{8m+4}^{SU}\otimes\mathbb{F}_2$$

with preserves the mod 2 KO-characteristic numbers. Let $I^c_* \subset \Omega^{SU}_*$ be the ideal of classes with vanishing KO-characteristic numbers. Then T^c induces a homomorphism

$$\tilde{T}^c\colon \Omega^{SU}_{8m+2}/I^c_{8m+2} \to (\Omega^{SU}_{8m+4}/I^c_{8m+4}) \otimes \mathbb{F}_2,$$

and there is a commutative diagram

$$\begin{array}{ccc} \Omega^{SU}_{8m+2}/I^c_{8m+2} & \xrightarrow{\tilde{T}^c} & (\Omega^{SU}_{8m+4}/I^c_{8m+4}) \otimes \mathbb{F}_2 \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\ \Omega^{\mathrm{spin}}_{8m+2}/I_{8m+2} & \xrightarrow{\tilde{T}} & (\Omega^{\mathrm{spin}}_{8m+4}/I_{8m+4}) \otimes \mathbb{F}_2 \end{array}$$

in which λ and μ are induced by the forgetful homomorphism. One has to show that λ is onto. It is well known (cf. [19]) that

$$\Omega^{SU}_{8m+4} \to \Omega^{\mathrm{spin}}_{8m+4}/\mathrm{Tors}$$

is onto. As $I_{8m+4} = \mathrm{Tors}\,\Omega^{\mathrm{spin}}_{8m+4}$, this implies that μ is onto. Thus to prove the proposition, it will suffice to show that $\tilde{T}^c$ is onto.

Let $B_* \subset \Omega^{SO}_*/\mathrm{Tors}$ be the subring of classes represented by U-manifolds with spherical determinant. According to Stong ([21, p. 282]), B_* is a polynomial algebra and $\Omega^{SU}_{8m+4}/I^c_{8m+4} \subset B_{8m+4}$ is exactly the subgroup $2B_{8m+4}$.

Let M^{8m+4} be an SU-manifold, and let W^{8m+4} be a U-manifold with spherical determinant such that $[M] = 2[W]$ in B_{8m+4}. Dualizing the determinant of W gives an SU-manifold V^{8m+2} and we have

$$W = U \cup (-D^2 \times V)$$

where U is an SU-manifold with boundary $\bar{S}^1 \times V$, namely the complement of a tubular neighbourhood of V in W (cf. [13]).

By definition, $T^c([V])$ is represented by the manifold $Z = (2U) \cup (-M^2 \times V)$, where M^2 is an SU-manifold such that $\partial M^2 = 2\bar{S}^1$. It is easy to see that Z is cobordant to $2W$ as a U-manifold. Therefore Z and $2W$ have the same rational Pontrjagin numbers. Hence Z and M have the same KO-characteristic numbers, that is represent the same element in $\Omega^{SU}_{8m+4}/I^c_{8m+4}$. □

8 Final remarks

(1) According to Theorem 6, the reduction mod 2 of the class a_m measures the KO-part of the Brown-Kervaire invariant in dimension $8m+2$. For instance,

$$k(V^{10}) = \pi_1[V^{10}]$$
$$k(V^{18}) = (\pi_2 + \pi_1)[V^{18}]$$
$$k(V^{26}) = (\pi_3 + \pi_1^2)[V^{26}].$$

Other sequences $a_0, a_1, \ldots$ having the same property have been constructed in [13]. For example,

$$a_m = L_{2m}(\pi_1, \ldots, \pi_{2m}) + (\pi_1^3 + \pi_1 \pi_2 + \pi_3) L_{2m-2}(\pi_1, \ldots, \pi_{2m-2}),$$

where L_{2m} is the reduced mod 2 Hirzebruch's polynomial, is such a sequence. A simple comparison of the first few terms shows that the new classes a_m have far fewer terms. Besides, they have better multiplicative properties. The classes a_m have been used in [17] to represent $k(V)$ as the index of a twisted Dirac operator on V.

Notice that the mod 2 reduction of $h(q)$ is of the form $q + o(q^8)$. Therefore on has

$$a_m \equiv b_m (\mathrm{mod}\ 2)$$

for $m \leqq 8$. Thus in dimensions $n \leqq 71$, $\kappa(V)$ is measured by the Witten class $b_{[n/8]}$

(2) The genus

$$\varphi\colon\ \Omega_*^{\mathrm{SO}} \to M^\Gamma(\mathbb{Z}[1/2])$$

was used by Landweber, Ravenel and Stong [12] to construct an elliptic (co)homology theory $E\ell\ell_*$([10, 11]). Namely they showed that

$$E\ell\ell_*(\,) = \Omega_*^{\mathrm{SO}}(\,) \otimes_\varphi M^\Gamma(\mathbb{Z}[1/2])[\varepsilon^{-1}]$$

is a homology theory. Here $M^\Gamma(\mathbb{Z}[1/2])$ is considered as an Ω_*^{SO}-module via φ.

By analogy with the Conner-Floyd isomorphism ([7])

$$KO_*(\,) \cong \Omega_*^{\mathrm{Sp}}(\,) \otimes KO_*$$

one can ask whether the functor

$$\Omega_*^{\mathrm{Sp}}(\,) \otimes_{\beta_q} M_*[\varepsilon^{-1}],$$

where $M_* \subset M^\Gamma(KO_*)$ is the image of β_q described in Theorem 3 (iii), is a homology theory. A positive answer to this question would provide a way of eliminating the undesirable 1/2 in the definition of $E\ell\ell_*(\,)$.

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Note added in proof

An elliptic homology theory with coefficient ring M_* has been recently defined by M. Kreck and S. Stolz.