SIGNATURE DEFECTS AND ETA FUNCTIONS OF DEGENERATIONS OF ABELIAN VARIETIES

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Abstract. In this paper, we will calculate the eta functions for torus bundles over S^1 which arise from boundaries of degenerations of Abelian varieties when the local monodromies are unipotent or have finite orders. By using the special values of the eta functions, we obtain the signature defects for such degenerations.

§1 Introduction.

(1.1) In [H2], Hirzebruch defined a signature defect for a cusp singularity of a Hilbert modular variety associated to a totally real number field of degree d and calculated them for Hilbert modular surfaces ($d = 2$) by using his beautiful explicit resolution of the cusp singularities. Based on these computations, Hirzebruch showed that if $d = 2$ the signature defects coincide with special values of Shimizu's L-function [S] and he conjectured that this fact also holds even for the cusps in the higher degree cases. This conjecture was proved by Atiyah, Donnelly and Singer in [ADS] by using the index theorems for manifolds with boundary developed in [APS-I, II, III].

A framed manifold (Y, α) is a pair of a compact oriented smooth manifold Y of real dimension $4k - 1$ and a trivialization α of the tangent bundle of Y. Then Y bounds a smooth compact oriented manifold X. In [H2], Hirzebruch defined the signature defect $\sigma(Y, \alpha)$ for a general framed manifold (Y, α) as the difference between the evaluation of L-polynomial of relative Pontrjagin classes X in the fundamental class $[X, Y]$ and the signature on $H^{2k}(X, Y, \mathbb{R})$. (It should be noted that the signature defect depends only on the boundary (Y, α).) The signature defect for a cusp is defined as that of the framed manifold (Y, α) arising from the boundary of a small neighborhood X of the cusp. In case of $d = 2$, Hirzebruch calculated $\sigma(Y, \alpha)$ by using the explicit calculation of good minimal resolutions of the cusp singularities. However, in higher dimensional cases, though we have Hironaka resolutions of singularities in general, it is rather difficult to get enough information from them.

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On the other hand, for a framed manifold (Y, α) one can define the eta function $\eta(D, s)$ associated to a certain first order self-adjoint operator D induced by the flat connection (cf. §2). By using index theorems in [APS-I, II, III], Atiyah, Donnelly and Singer [ADS] proved that that the special value $\eta(D, 0)$ coincides with the signature defect $\sigma(Y, \alpha)$ for a cusp of a Hilbert modular variety. Furthermore they proved that $\eta(D, 0)$ coincides with a special value of Shimizu L-function, which established the conjecture in general.

In [APS-I], another type of signature defect was defined for a Riemannian manifold X with boundary $\partial X = Y$ under the assumption that the metric of X is product near boundary. From the definition (cf. (2.4)), we denote it by $\delta(X, Y)$, though it depends only on Y and its metric. They also defined the tangential signature operator B_Y and the eta function $\eta(B_Y, s)$ (cf. §2). Then a conclusion of the index theorem in [APS-I] can be stated as $\delta(X, Y) = \eta(B_Y, 0)$.

(1.2) The main purpose of this paper is to calculate these signature defects and the eta functions of some torus bundles over the unit circle S^1 arising from boundaries of one parameter degenerations of abelian varieties (of complex dimension n = 2k-1) in algebraic geometry. An one parameter degeneration of abelian varieties is a projective holomorphic map $f : X \longrightarrow D$ from a smooth complex manifold onto a unit disk such that for $t \in D^* := D - 0$ the fibre $X_t = f^{-1}(t)$ is an abelian variety (i.e. a complex torus with a projective embedding). If the relative complex dimension is $n = 2k - 1$, the boundary Y is a torus bundle over S^1 of real dimension $4k - 1$. Note that if the relative dimension n is even, then the signature defects of Y should be defined to be zero.

Let us recall the classification of one parameter degenerations of elliptic curves due to Kodaira [K] for a motivation. Let $f : X \longrightarrow D$ be a degeneration of abelian varieties of complex relative dimension 1 (i.e. elliptic curves) and let $g : Y \longrightarrow S^1$ be its boundary. We assume that f has a holomorphic section. Moreover since the total space X is of dimension 2, we may assume that X is the minimal model, that is, X has no exceptional curve of the first kind. This assumption is very important to determine the configuration of the special fibre $X_0 = f^{-1}(0)$ (as a scheme theoretic fibre). Under these assumptions, Kodaira [K1] classified degenerations of elliptic curves into 10 types (see figure 1). They are essentially classified by means of the monodromy translation T on $H_1(X_1, \mathbb{Z})$. Note that in our case the monodromy T is always a parabolic or an elliptic element of $SL_2(\mathbb{Z})$.

For an oriented smooth fibre bundle of torus over a punctured Riemannian surface, one can define a signature cocycle as in §2 of [A] and Meyer [M] introduced an invariant $\phi : SL_2(\mathbb{Z}) \longrightarrow \mathbb{Q}$ whose coboundary is minus the signature cocycle. (cf. [§5, A].) In [A], Atiyah studied the relations between many invariants defined for each element $T \in SL_2(\mathbb{Z})$ including Meyer's invariant $\phi(T)$, the eta invariant $\eta(B_Y, 0)$, its "integral adiabatic limit" $\eta^0(B_Y, 0)$ and the signature defect $\sigma(Y, \alpha)$ (if it admits a framing). First of all, one can show that $\eta^0(B_Y, 0) = -\phi(T)$ for every element $T \in SL_2(\mathbb{Z})$ ([Prop. 5.12, A]). (Note that we use the orientation of Y opposite to one in [A].) Then in our case, by using the same argument as in [A] and considering existence of suitable elliptic surfaces we can prove that

$$\eta^0(B_Y,0) = -\phi(T) = -\frac{2}{3}e(X_0) + (N-1), \tag{1.3}$$

where $e(X_0)$ and N denote the Euler number and the number of irreducible components of the singular fibre X_0 respectively.

Moreover Atiyah showed that if T is elliptic, the eta invariant $\eta(B_Y,0)$ does not depend on the metric and hence $\eta(B_Y,0) = \eta^0(B_Y,0) = -\phi(T)$ and in particular we can obtain the signature defect $\delta(X,Y)$.

In the cases of parabolic monodromies (cases $(I)_b$ and $(I)_b^*$), these equalities do not follow immediately, but in this paper we will show $\sigma(Y,\alpha) = \eta^0(B_Y,0) = -\phi(T)$ in the case $(I)_b$. (See Corollary 4.15.)

Now thanks to the formula (1.3) and the Kodaira classification we can calculate the invariants $\eta^0(B_Y,0) = -\phi(T)$ explicitly. (See figure 1 below.)

Figure 1.

Type	I_0	I_0^*	$I_b (b \geq 1)$	$I_b^* (b \geq 1)$	II
T	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
order of T	1	2	∞	∞	6
$e(X_0)$	0	6	b	$b+6$	2
N	1	5	b	$b+5$	1
$\eta^0(B_Y,0)$	0	0	$b/3-1$	$b/3$	$-4/3$

Type	II^*	III	III^*	IV	VI^*
T	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
order of T	6	4	4	3	3
$e(X_0)$	10	3	9	4	8
N	9	2	8	3	7
$\eta^0(B_Y,0)$	$4/3$	-1	1	$-2/3$	$2/3$

(1.4) In order to obtain a formula of the signature defects like (1.3) in the higher dimensional case, we have to know about good birational models of degenerations and its central fibres. However, since the relative dimension of f is greater than 2, we do not have a unique *smooth* minimal model of X in general, therefore the topology and geometry of the central fibre of X will be subtler than in the case of $n = 1$.

On the other hand, the eta invariants depends only on the boundary manifold Y and we do not have to take care of the central fibre of X. Therefore, in this paper, we will mainly calculate the eta functions and the eta invariants. Since the index theorems relate some eta invariants to the signature defects, which are the differences of some topological objects and geometrical objects coming from X, it will be an interesting problem to study some good birational smooth models of X and study their central fibres from the view point of the signature defects.

(1.5) Now we will explain our problem more explicitly. The most important property of the torus bundle Y arising from a degeneration of abelian varieties is the quasi-unipotency of the monodromy transformation T on $H_1(X_t, \mathbb{Z})$. When T is unipotent, Y has a structure of homogeneous manifold with respect to a Lie group and it defines a natural framing α on Y, and this allows us to consider the signature defect $\sigma(Y, \alpha)$ of this framed manifold and also the eta function $\eta(D, s)$.

For a general degeneration of abelian varieties, one can take a finite base change $\tilde{D} \to D$ to make the monodromy T unipotent. Then a given degeneration becomes a quotient of a degeneration with a unipotent monodromy by a finite cyclic group. In general, the action does not preserve the framing, hence one can not induce the natural framing of the boundary Y from the unipotent case.

By a technical reason, in this paper we will deal with the following two cases.

(I) Unipotent monodromy cases and **(II) Finite monodromy cases.**

In the case (I), we will show that the eta function $\eta(D, s)$ associated to the flat connection coincides with the Riemann Zeta function up to an easy factor when $n = 1$ and it vanishes identically when $n \geq 2$. By using these facts, we will calculate the signature defect $\sigma(Y, \alpha)$ explicitly and we show that if $n \geq 2$ the invariant $\sigma(Y, \alpha)$ is an integer depending only on the dimension n and rank $(T - I)$.

In the case (II), we may assume that $f : X \longrightarrow D$ is diffeomorphic to some smooth model of quotient variety $G_l \backslash (A \times S^1)$ where A is an abelian variety and G_l is generated by the product $\tilde{\gamma} = (\gamma, e_l)$ of an automorphism γ of A fixing the origin and polarization and $e_l = \exp(2\pi i/l)$. The polarization induces a metric on A and the natural product metric on $A \times S^1$ is invariant under the action of $\tilde{\gamma}$. Therefore this metric induces a metric on Y which extends naturally to a metric on X, which is product near the boundary. By using this metric, one can define the signature defect $\delta(X, Y)$ and the eta function $\eta(B_Y, s)$. Moreover one can introduce the equivariant version of the eta function. We will calculate these eta functions explicitly and obtain the signature defect $\delta(X, Y)$ by using equivariant version of index theorem for manifolds with boundary due to Donnelly [D2].

(1.6) Now let us state our main theorems. Let $f : X \longrightarrow D$ be a degeneration of principally polarized abelian varieties over the unit closed disk of relative complex dimension $n = 2k - 1$ (for precise definition, see (3.1)), and $g : Y \longrightarrow S^1$ the boundary fibration. We always assume that f has a section.

(1.7) (I) Results in the cases of unipotent monodromies.

Assume that the monodromy T is unipotent. Then we can define a framing α on Y and the tangential signature operator D by using flat connection associated to the framing α instead of the Levi-Civita connection (see (2.16)).

The eta function $\eta(D, s)$ can be calculated as follows.

Theorem 1.7. *(cf. Theorem 4.7) Let (Y, α) be as above and assume that $N = T - I \neq 0$.*

(i) If $k = 1$ (i.e. $n = 1$) and $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b > 0$, then we have

$$\eta(D, s) = -4b(2\pi)^{-s}\zeta(s-1) \tag{1.8}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann Zeta function.

(ii) If $k > 1$ (i.e. $n = 2k - 1 > 1$), then

$$\eta(D, s) \equiv 0.$$

Next let us set $r = \mathrm{rank}(\mathrm{Ker}(T - I_{2n}))$. Then since $(T - I)^2 = 0$ we see that $n \leq r \leq 2n$. As for the signature defect $\sigma(Y, \alpha)$ of the framed manifold (Y, α) we have the following theorem.

Theorem 1.9. *(cf. Corollary 4.15)*

(i) *If $n = 1$ and $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b > 0$, we have*

$$\sigma(Y, \alpha) = \frac{b}{3} - 1.$$

(ii) *If $n > 1$, and $r > n$, we have $\sigma(Y, \alpha) = 0$*

(iii) *If $n > 1$, and $r = n$, the signature defect $\sigma(Y, \alpha)$ depends only on the relative dimension n (hence k). If we write as $\sigma(Y, \alpha) = s(k)$, $s(k)$ is equal to the signature of a certain explicit matrix. For small k, we have $s(2) = -1$ and $s(3) = -2$.*

Note that the assertion (i) and the figure 1 show that $\sigma(Y, \alpha) = \eta^0(Y, 0) = -\phi(T)$ in the case $(I)_b$ of Kodaira's classification. Moreover we note that the assertion (ii) follows from the fact that the fibration $Y \longrightarrow S^1$ has a non-trivial constant torus factor.

(1.10) (II) Results in the cases of finite monodromies.

When T has a finite order l, there exists a principally polarized abelian varieties A and an automorphism γ of A of order l fixing the origin and the polarization such that Y is diffeomorphic to the quotient of $W = A \times S^1$ by the cyclic group generated by $\tilde{\gamma} = (\gamma, e_l)$ where $e_l = \exp(2\pi i/l)$. Write the action of γ on the tangent space at the origin of A in the diagonal form

$$\gamma = (e_l^{a_1}, e_l^{a_2}, \cdots, e_l^{a_n}).$$

The product metric of $W = A \times S^1$ is invariant under the action of $\tilde{\gamma}$. With respect to the induced metric of Y, we can define the tangential signature operator B_Y on Y and hence the eta function $\eta(B_Y, s)$. Moreover the G-equivariant version of eta functions $\eta_{\tilde{\gamma}^j}(B_W, s)$ can be also defined. (See 2.18).

Our main result in this case can be stated as follows.

Theorem 1.11. *(Cf. Theorem 5.7.) Under the notation as above, we have*

$$\eta_{\tilde{\gamma}^j}(B_W, s) = (2\pi)^{-s}(-1)^k \cdot 2^{n+1} \cdot Z(\frac{2\pi j}{l}, s) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l}) \tag{1.12}$$

Moreover the eta function of Y (with respect to the induced metric) is given by

$$\eta(B_Y, s) = (2\pi)^{-s}(-1)^k \cdot 2^{n+1} \cdot (\frac{1}{l}\sum_{j=1}^{l-1} Z(\frac{2\pi m j}{l}, s) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l})). \tag{1.13}$$

Here $Z(q, s)$ denotes the following Dirichlet series:

$$Z(q, s) := \sum_{m=1}^{\infty} \frac{\sin(mq)}{m^s}.$$

From this theorem, we can obtain the signature defect $\delta(X, Y)$ as follows.

Theorem 1.14. *(cf. Corollary 5.11) Under the notation and the assumption in Theorem 1.11, we have the special values*

$$\eta_{\tilde{\gamma}^j}(B_W, 0) = (-1)^k \cdot 2^n \cdot \cot(\frac{\pi j}{l}) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l}). \tag{1.15}$$

Moreover the signature defect of $Y = G_l \backslash W \subset X$ is given by

$$\delta(X, Y) = \eta(B_Y, 0) = (-1)^k \cdot 2^n \cdot (\frac{1}{l}\sum_{j=1}^{l-1}(\cot(\frac{\pi j}{l})) \cdot \prod_{i=1}^{2k-1} \sin(\frac{2\pi j a_i}{l})). \tag{1.16}$$

We note that Donnelly calculated $\eta_{\tilde{\gamma}^j}(B_W, 0)$ by using the right hand side of the index theorem. For the relation between our results and his, we may refer the reader to Remark 5.14.

(1.17). The organization of this paper is as follows. In §2, we recall some necessary definition and backgrounds about signature defects, eta functions and index theorems. In §3, we will explain about degenerations of abelian varieties. Most of the results in these two sections should be well-known to experts. In §4, we will deal with the cases of unipotent monodromies. In §5, we will deal with the cases of finite monodromies.

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§2 Signature defects, Eta functions and Index theorems.

(2.1) Signature defects.

In this paper, we will deal with two kinds of signature defects, one of which is defined for an oriented Riemannian manifold with boundary and the other is defined for a framed manifold.

(2.2) Signature defects for Riemannian manifolds with boundary.

Let X be a compact oriented Riemannian manifold of dimension $4k$. We suppose that X may have non-empty smooth boundary $\partial X = Y^{4k-1}$ and assume that *the metric of X is a product near its boundary Y*. We define the symmetric bilinear form $b_{X,Y}$ induced by the cup product

$$H^{2k}(X,Y,\mathbb{R}) \times H^{2k}(X,Y,\mathbb{R}) \longrightarrow H^{4k}(X,Y,\mathbb{R}) \tag{2.3}$$

with evaluating on the fundamental cycle $[X,Y] \in H_{4k}(X,Y,\mathbb{R})$. The signature of this bilinear form will be denoted by $\mathrm{Sign}(X,Y)$. By using the Levi-Civita connection on the tangent bundle on T_X with respect the Riemannian metric, we can obtain the curvature forms and define the Pontrjagin forms $p_i(\Omega), 1 \leq i \leq k$ of X. Then we obtain the Hirzebruch L-polynomial $L_k(p_1(\Omega), p_2(\Omega), \cdots, p_k(\Omega))$ of X in these Pontrjagin forms. Following [APS-I], we have the following proposition and definition.

Proposition-Definition (2.4). *Let (X,Y) be as above and assume that the metric on X is product near the boundary Y. Then the difference*

$$\delta(X,Y) = \int_X L_k(p_1(\Omega), p_2(\Omega), \cdots, p_k(\Omega)) - \mathrm{Sign}(X,Y) \in \mathbb{Q} \tag{2.5}$$

depends only on the structure of an oriented Riemannian manifold of Y and does not depend on X. We call this difference $\delta(X,Y)$ the signature defect *of (X,Y) or simply of Y.*

(2.6) Signature defects for framed manifolds.

Next, again, let Y be a compact oriented manifold without boundary of dimension $4k-1$. A framing α of Y is a trivialization of the tangent bundle of Y, and if a framing α of Y exists the pair (Y,α) is called a framed manifold. For such a manifold, Hirzebruch [H2] defined a rational number $\sigma(Y,\alpha)$ as follows. Since Y has the trivial tangent bundle, all of its Pontrjagin and Stiefel-Whitney numbers vanish. Therefore Y bounds a $4k$-dimensional compact oriented differentiable manifold X. Since Y is framed, the tangent bundle of X is pulled back from a bundle over the quotient space X/Y. Then one can define the Pontrjagin classes of X as relative classes $p_j \in H^{4j}(X,Y,\mathbb{R})$. Then again we have $L_k(p_1,\cdots,p_k) \in H^{4k}(X,Y,\mathbb{R})$, and evaluating this with the class $[X,Y]$ we obtain a rational number $L_k(p_1,\cdots,p_k)[X,Y]$).

From the Hirzebruch signature theorem ([H1]) and Novikov additivity of the signature ([AS], p.588) we also have the following proposition and definition.

Proposition-Definition (2.7). *Let (Y,α) be a framed manifold of dimension $4k-1$ and let X be as above. Then the difference*

$$\sigma(Y,\alpha) = L_k(p_1,\cdots,p_k)[X,Y] - \mathrm{Sign}(X,Y) \in \mathbb{Q} \tag{2.8}$$

depends only on (Y,α) (does not depend on X). We call this difference $\sigma(Y,\alpha)$ the signature defect *of the framed manifold (Y,α).*

(2.9) A framing α of Y induces a natural Riemannian metric on Y and it can be extended to a metric on X which is product near ∂Y. Hence we can define $\delta(X,Y)$ as in (2.5). But in general the Pontrjagin forms defined by the curvature of Levi-Civita connection do not represent the Pontrjagin classes in (2.6). Therefore $\delta(X,Y)$ may be different from $\sigma(Y,\alpha)$.

(2.10) The relative Pontrjagin classes $p_j \in H^{4j}(X,Y,\mathbb{R})$ can be represented by differential forms on X as follows (cf. [ADS], §13). Let ∇ denote the flat connection on T_Y given by the framing of Y. Then this connection does not coincide with the Levi-Civita connection in general, hence its torsion tensor T_0 may not vanish. We can assume that there is a neighborhood of the boundary Y in X isomorphic to the product $Y \times I$ where $I=[0,1)$ is the unit interval, and we consider T_0 as a tensor over $Y\times I$ by lifting from the projection $Y\times I \to Y$. Choose a nonnegative C^∞ function f on I satisfying

$$0 \le f \le 1, \quad f([0,1/4]) = 1 \quad \text{and} \quad f([3/4,1)) = 0. \tag{2.10.1}$$

Define a tensor field T on $Y\times I$ by $f(t)T_0$. Then this can be extended to a tensor T over X by setting $T=0$ on $X-(Y\times I)$. Moreover X admits a Riemannian metric such that whose restriction on the neighborhood $Y\times I$ coincides with the product metric on $Y\times I$. Then as shown in [Hel, p48], there is a unique connection ϕ on the tangent bundle T_X such that ϕ preserves the metric of X and ϕ has torsion tensor T (cf. [Lemma 13.3, ADS]). Let $p_j(\phi)$ denote the Pontrjagin form defined from the curvature form of the connection ϕ by the Chern-Weil theory. (Note that $p_j(\phi)$ is zero on a small neighborhood of Y because locally at those points the connection ϕ coincides with the flat connection on Y. In particular, $L_k(p_1(\phi), \cdots, p_k(\phi)) = 0$ on $Y \times [0, 1/4]$. Then the differential form $p_j(\phi)$ represents the relative Pontrjagin class $p_j \in H^{4j}(X, Y, \mathbb{R})$. In particular we have

$$(2.11) \qquad L_k(p_1, \cdots, p_k)[X, Y] = \int_X L_k(p_1(\phi), \cdots, p_k(\phi)).$$

(2.12) Eta functions and Eta invariants.

An eta function is a spectral function for a first order self-adjoint differential operator on a Riemannian manifold. According to the case of Riemannian manifolds and framed manifolds, one can define suitable eta functions.

First let us recall the general definition of eta functions. Let Y be a compact oriented Riemannian manifold, E a vector bundle over Y. Assume that E is endowed with a smooth inner product, which induces the natural metric on the space of its smooth sections $\Gamma(Y, E)$. We will consider a first order elliptic differential operator $A : \Gamma(Y, E) \to \Gamma(Y, E)$ which is self-adjoint with respect to this inner product. Then A has pure point spectrum with real eigenvalues λ. We define the eta function of this operator A by

$$(2.13) \qquad \eta(A, s) := \sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-s},$$

whose sum is taken over all non-zero eigenvalues of A counting with multiplicity. (Here we define $\text{sign}(\lambda)$ by $\text{sign}(\lambda) = 1$ if $\lambda > 0$ and $\text{sign}(\lambda) = -1$ if $\lambda < 0$.) It is known that this converges for $\text{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s-plane ([Proposition 2.8, APS-III]). The *eta invariant* of A is defined to be the special values $\eta(A, 0)$ if it is finite. (For the finiteness of $\eta(A, 0)$ in general case, see also [APS-III].)

(2.14) Riemannian case.

Let Y be a compact oriented Riemannian manifold of dimension $4k - 1$, and let $E := \wedge^{ev}(T_Y^*) = \oplus_{p=0}^{2k-1} \wedge^{2p} (T_Y^*)$ be the bundle of smooth forms on Y of even degrees. We can define the differential operator $B_Y : \Gamma(Y, E) \longrightarrow \Gamma(Y, E)$ by by the formula

$$(2.15) \qquad B_Y(\phi) = (-1)^{k+p+1}(*d - d*)\phi \quad \text{for} \quad \phi \in \Gamma(Y, \wedge^{2p}(T_Y^*)).$$

Here d is the usual exterior differential and $*$ denotes the Hodge star operator defined by the Riemannian metric on Y. This is a first order elliptic self-adjoint operator, therefore we can define the eta function $\eta(B_Y, s)$ as in (2.13). It is proved that $\eta(B_Y, 0)$ is finite ([APS-I]), hence we obtain the eta invariant $\eta(B_Y, 0)$.

It should be noted that the eta function $\eta(B_Y, s)$ and hence the eta invariant $\eta(B_Y, 0)$ depend on Y and the metric of Y. If one scales the metric $g_{ij} \mapsto k^2 g_{ij}$, the eta function becomes $k^{-s}\eta(s)$. Therefore the eta function really depends on the metric unless $\eta(s) \equiv 0$, but this also shows that the eta invariant $\eta(0)$ is invariant under the scaling of the metric.

(2.16) Framed manifold case.

Let (Y, α) be a framed manifold of dimension $4k-1$. Then the framing α defines a flat connection ∇ on the tangent bundle Y which also induces flat connections on $\wedge^p T_Y^*$. Moreover we can define the natural Riemannian metric induced by the framing. According to [ADS], let d_∇ denote the skew covariant differential associated to this framing, that is, d_∇ is defined by the composition:

$$\Gamma(\wedge^p T_Y^*) \xrightarrow{\nabla} \Gamma(\wedge^p T_Y^* \otimes T_Y^*) \longrightarrow \Gamma(\wedge^{p+1} T_Y^*)$$

where the second map is the exterior multiplication. Denote by $*$ the Hodge star operator obtained by the framing. Then we can define the operator D on $\Gamma(Y, E)$ as in (2.15) by the formula

$$D(\phi) = (-1)^{k+p+1}(*d_\nabla - d_\nabla *)\phi \quad \text{for} \quad \phi \in \Gamma(Y, \wedge^{2p}(T_Y^*)). \tag{2.17}$$

This is again a first order elliptic self-adjoint differential operator, therefore we can define the eta function $\eta(D, s)$ as in (2.13). It is proved that $\eta(D, 0)$ is finite, hence we obtain the eta invariant $\eta(D, 0)$ ([APS-III, see also [ADS]).

(2.18) G-equivariant version.

Next we recall the G-equivariant version of eta functions. Let Y, E, $A : \Gamma(Y, E) \to \Gamma(Y, E)$ be as before. Suppose that G is a subgroup of the group of isometries of Y and assume that the action of G is lifted to an action on E preserving the inner product and commuting with the given operator A.

Then each element $\gamma \in G$ induces linear maps γ_λ on each eigenspace of A, and one can define the eta function of $\gamma \in G$ by

$$\eta_\gamma(A, s) = \sum_\lambda (\text{sign}\lambda)\text{Tr}(\gamma_\lambda)|\lambda|^{-s}. \tag{2.19}$$

Again, this function converges for $\text{Re}(s)$ sufficiently large and has a meromorphic continuation to the entire complex s-plane.

For an isometry $\gamma : Y \to Y$, one can define $\eta_\gamma(B_Y, s)$ and it is known that $\eta_\gamma(B_Y, 0)$ is finite. Moreover if γ is an automorphism of a framed manifold (Y, α), one can also define $\eta_\gamma(D, s)$ and show that $\eta_\gamma(D, 0)$ is finite ([APS-III]).

(2.20) Index Theorems.

We recall some of results in [APS-I, II, III] and [D1, D2] which we will use later.

(2.21) Index theorem for Riemannian Case. *(cf. Theorem (4.14) in [APS-I].) Let X, Y be as in (2.2). Let $\delta(X, Y)$ denote the signature defect (2.5) and $\eta(B_Y, 0)$ the eta invariant of Y. Then we have*

$$\eta(B_Y, 0) = \delta(X, Y). \tag{2.22}$$

In [APS-I], this was proved as a consequence of a general index theorem for Dirac operators on X with a certain global boundary conditions ([Theorem (4.2), APS-I]). Note that this general index theorem can be also applied for the operator D for a framed manifold (Y, α) and as shown in [ADS] we have the following theorem.

(2.23) Index theorem for framed manifolds. *(cf. Theorem 13.1 in [ADS]). Let (Y, α) be a framed manifold of dimension $4k-1$ and take X such that $\partial Y = X$ as in (2.6). Let $\eta(D, s)$ be the eta function defined in (2.16). Then we can write*

$$\eta(D, 0) = \int_X \mathcal{D}_0 - l_0. \tag{2.24}$$

Here l_0 is an integer and the integrand $\mathcal{D}_0$ is locally defined and invariant under scaling of the metric on X.

We may represent $\mathcal{D}_0$ more precisely by using the connection ϕ introduced in (2.10). As shown in Lemma 13.2 in [ADS], one can see that $\mathcal{D}_0$ is an $O(4k)$-invariant polynomial in the components of the curvature R and torsion tensor T and their covariant derivatives. Moreover at a point of X where $T = 0$ we have $\mathcal{D}_0 = L_k(p_1(\phi), \cdots, p_k(\phi))$. This can be proved by the scaling invariant property of $\mathcal{D}_0$. Note that from the construction of the connection ϕ in (2.10), we have $T = 0$ outside a small neighborhood of Y in X, where we have the equality $\mathcal{D}_0 = L_k(p_1(\phi), \cdots, p_k(\phi))$.

(2.25) G-equivariant version.

Let X be as in (2.2) and $Y = \partial X$. Let G be a compact Lie group which acts isometrically on X. Then each $\gamma \in G$ preserves the boundary Y and the given G-action is a product near the boundary. Now we will define the analogue of the signature defect for each $\gamma \in G$. Each element $\gamma \in G$ defines an action γ^* on $V := H^{2k}(X, Y, \mathbb{R})$ and it preserves the symmetric bilinear form $b_{X,Y}$ on V. This form $b_{X,Y}$ defines a symmetric real linear endomorphism $A : V \longrightarrow V$ and let V^+ (resp. V^-) denote the direct sum of the eigenspaces of positive eigenvalues (resp. negative eigenvalues). Since $b_{X,Y}$ is invariant under γ, both of the spaces V^+ and V^- are preserved by γ^*. We define the G-signature of γ by

$$\mathrm{sign}(\gamma, X, Y) := \mathrm{Tr}(\gamma^*_{|V^+}) - \mathrm{Tr}(\gamma^*_{|V^-}). \tag{2.26}$$

For each element $\gamma \in G$, the fixed point set $\Omega(\gamma)$ of γ is the disjoint union of compact connected totally geodesic submanifolds F of X. The normal bundle $N_{X/F}$ of any component $F \subset X$ decomposes as

$$N_{X/F} = N(-1) \oplus N(\theta_1) \oplus \cdots \oplus N(\theta_s) \tag{2.27}$$

where the differential of γ acts on $N(-1)$ via multiplication by -1 and on $N(\theta_i)$ via the rotation through the angle $\theta_i, 0 < \theta_i < \pi$. For each F, one has $\partial F \subset Y$.

We set

$$\mathcal{L}(F) = \prod_j \frac{x_j/2}{\tanh(x_j/2)} \tag{2.28}$$

where the Pontrjagin classes of F are the elementary symmetric functions in the x_j^2. Let m be the rank of $N(-1)$ which must be even. Then one may write

$$2^{-m/2}\mathcal{L}(N(-1))^{-1}e(N(-1)) = \prod_j \tanh(y_j/2) \tag{2.29}$$

where the Pontrjagin forms of $N(-1)$ are the elementary functions in the y_j^2 and the Euler form is the product of the y_j's. Since $N(-1)$ may not be globally orientable, the form $e(N(-1))$ must be interpreted as the Euler form relative to some local choice of orientation of F. Moreover we set

$$\mathcal{M}^{\theta_i} = \prod_j \frac{\tanh(\sqrt{-1}\theta_i/2)}{\tanh(\frac{z_j+\sqrt{-1}\theta_i}{2})} \tag{2.30}$$

where the elementary symmetric functions of the z_j's are the Chern forms of $N(\theta_i)$.

Quoting the G-signature theorem of Atiyah-Singer [AS], we have the following proposition and definition.

Proposition-Definition (2.31). *Under the notation and the assumption as above, we define the signature defect for an isometry γ of X by the difference:*

$$\delta(\gamma, X, Y) = \sum_{F\in\Omega(\gamma)} \int_F \mathcal{D}_F - \operatorname{sign}(\gamma, X, Y) \tag{2.32}$$

where $\mathcal{D}_F$ is given by

$$2^{(n-m)/2} *_F[(\prod_i \sqrt{-1}\tan(\theta_i/2))^{-c(\theta_i)})\mathcal{L}(F)\mathcal{L}(N(-1))^{-1}e(N(-1))\prod_i \mathcal{M}^{\theta_i}(N(\theta_i))]dV.$$

Here $*_F$ is the Hodge star operator relative to the local choice of orientation for F. Then if Y is empty, this difference $\delta(\gamma, X, Y)$ vanishes.

Donnelly ([D2]) proved a G-index formula for manifolds with boundary which generalizes results in [APS-I]. Applying the result to the signature complex, he obtained the following theorem.

Theorem 2.33. *(Cf. Theorem 2.1 in [D2].) Let X, Y and $\gamma \in G$ be as above and let $\eta_\gamma(B_Y, s)$ be the eta function defined in (2.18). Then we have*

$$\eta_\gamma(B_Y, 0) = \delta(\gamma, X, Y). \tag{2.34}$$

(2.35) Free actions of finite groups.

Let X, Y and G be as above. We consider the case when G is a group of a finite order. Assume that G acts freely on the boundary Y so that the quotient map $Y \longrightarrow Y' = Y/G$ is a regular covering space. By using the theory of characters, we obtain the following formula

$$\eta(B_{Y/G}, s) = \frac{1}{|G|} \sum_{\gamma \in G} \eta_\gamma(B_Y, s), \tag{2.36}$$

or equivalently

$$\eta(B_Y, s) - |G|\eta(B_{Y/G}, s) = -\sum_{\gamma \neq 1} \eta_\gamma(B_Y, s). \tag{2.37}$$

From this formula, we obtain the corresponding formula for the special values $\eta_\gamma(B_Y, 0)$ and $\eta(B_{Y/G}, 0)$.

§3 Degenerations of Abelian varieties.

In order to calculate invariants defined in §2 for a degeneration of Abelian varieties, we need to introduce a framing or a metric. For that purpose, we will introduce a uniformization of the boundary Y of the given degeneration of abelian varieties.

First let us recall some basic notations and definitions. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disk and $D^* = D - \{0\}$ the punctured disk. For a positive real number $\epsilon > 0$, set $D_\epsilon = \{z \in \mathbb{C} \mid |z| < 1 + \epsilon\} \supset D$.

Definition 3.1. An *one-parameter degeneration of principal polarized Abelian varieties over* D_ϵ (abbreviated to a degeneration of PPAV) is a proper surjective holomorphic map $f_\epsilon : X_\epsilon \longrightarrow D_\epsilon$ such that

(i) X_ϵ is a complex manifold,

(ii) for each $s \in D_\epsilon - 0$ the fibre $X_s = f^{-1}(s)$ is an Abelian variety,

(iii) and there exists a line bundle F on X which induces a principal polarization on each fibre X_s for $s \in D^*_\epsilon$.

An *one parameter degeneration of PPAV over the close unit disk* D is the restriction $f : X \longrightarrow D$ of a degeneration of PPAV $f_\epsilon : X_\epsilon \longrightarrow D_\epsilon$ for some $\epsilon > 0$ to the closed unit disk $D \subset D_\epsilon$.

(3.2) Complex uniformizations.

Let $f : X \longrightarrow D$ be a degeneration of PPAV of complex relative dimension n and let $g : Y \longrightarrow S^1$ be its boundary fibration. In the rest of this paper, we assume that every degeneration f admits a section $\sigma : D \longrightarrow X$ which we will often identify with the zero section. Setting $X^0 = f^{-1}(D^*)$, we obtain a smooth fibration $f^0 : X^0 \longrightarrow D^*$ of Abelian varieties. In the category of differentiable manifolds, this is nothing but a torus bundle over the punctured disk D^* with the typical fibre $(S^1)^{2n}$, and the boundary fibration $g : Y \longrightarrow S^1$ is a deformation retract of $f^0 : X^0 \longrightarrow D^*$.

We will recall here a complex uniformization of $f^0 : X^0 \longrightarrow D^*$ which induces a uniformization of the boundary Y.

Let $X_1 = f^{-1}(1)$ denote the fibre of f over $1 \in S^1$, and let T be the monodromy transformation on $H_1(X_1, \mathbb{Z})$ induced by rounding on the circle in counterclockwise. Since T is an automorphism of $H_1(X_1, \mathbb{Z})$ preserving principal polarization, we have $T \in Sp(2n, \mathbb{Z})$. By a standard argument (cf. e.g. Ueno [U]), there exists a multi-valued $n \times n$-matrix function $\tau(s)$ of $s \in D^*$ satisfying

$$(3.3) \qquad {}^t\tau(s) = \tau(s), \quad \mathrm{Im}\tau(s) >> 0,$$

(Hodge-Riemannian bilinear relation) and

$$(3.4) \qquad \tau(\exp(2\pi i) \cdot s) = (A\tau(s) + B)(C\tau(s) + D)^{-1},$$

where

$$(3.5) \qquad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{Z}), \;\; A, B, C, D \in M(n, \mathbb{Z})$$

is the matrix representation with respect to a suitable symplectic basis of $H_1(X_1, \mathbb{Z})$.

Note that since $\tau(s)$ is symmetric we also have a relation

$$(3.6) \qquad (A\tau(s) + B)(C\tau(s) + D)^{-1} = {}^t(C\tau(s) + D)^{-1}(\tau(s)^t A +^t B).$$

Let us set $U = \{t \in \mathbb{C} \mid \mathrm{Im} t \geq 0\}$, and let $\pi : U \longrightarrow D^*$ be the universal covering map given by $s = \pi(t) = \exp(2\pi i t)$.

We use column vectors to denote elements in $\mathbb{C}^n$ or in $\mathbb{Z}^{2n}$. For each $\vec{m} = \begin{pmatrix} \vec{m}_1 \\ \vec{m}_2 \end{pmatrix} \in \mathbb{Z}^{2n}$, and $\mu \in \mathbb{Z}$, we define an analytic automorphism $g(\vec{m}, \mu)$ of $(\mathbb{C}^n \times U)$ by

$$(3.7) \qquad g(\vec{m}, \mu)(\vec{z}, t) = ({}^t(C_\mu \tau(t) + D_\mu)^{-1}(\vec{z} + \tau(t)\vec{m}_1 + \vec{m}_2), t + \mu).$$

Here we set $\tau(t) = \tau(\pi(t))$ by an abuse of notation and $A_\mu, B_\mu, C_\mu, D_\mu$ are defined by the relation

$$T^\mu = \begin{pmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{pmatrix}.$$

By an easy calculation with using (3.6), one has the relation

$$(3.8) \qquad g(\vec{n}, \mu_2) g(\vec{m}, \mu_1) = g({}^t T^{\mu_1} \vec{n} + \vec{m}, \mu_1 + \mu_2)$$

for arbitrary elements $\vec{n}, \vec{m} \in \mathbb{Z}^{2n}$ and $\mu_1, \mu_2 \in \mathbb{Z}$. Therefore it is easy to see that all of elements $\{g(\vec{m}, \mu)\}_{\vec{m} \in \mathbb{Z}^{2n}, \mu \in \mathbb{Z}}$ form a properly discontinuous subgroup S of the analytic automorphism group of $\mathbb{C}^n \times U$ and each element $g(\vec{m}, \mu) \neq 1$ acts on it without fixed points. Moreover if we set

$$L := \{g(\vec{m}, 0) \mid \vec{m} \in \mathbb{Z}^{2n}\}, \tag{3.9}$$

L becomes a normal abelian subgroup of S and yields an exact sequence

$$0 \longrightarrow L \longrightarrow S \xrightarrow{h} \mathbb{Z} \longrightarrow 0. \tag{3.10}$$

Therefore we have quotient manifolds $S \backslash (\mathbb{C}^n \times U)$ and $L \backslash \mathbb{C}^n$, and since the natural projection $\mathbb{C}^n \times U \longrightarrow U$ is equivariant with respect to h we obtain the natural fibration

$$\phi : S \backslash (\mathbb{C}^n \times U) \longrightarrow \mathbb{Z} \backslash U \simeq D^*. \tag{3.11}$$

Note that this fibration ϕ has a natural section O which corresponds to $(\vec{0}, t)$.

Now we can state the following proposition. For a proof, see Ueno [U].

Proposition 3.12. *Let $f : X \longrightarrow D$ be a degeneration of principally polarized Abelian varieties with a section $\sigma : D \longrightarrow X$. Let S be the properly discontinuous group of analytic automorphisms of $\mathbb{C}^n \times U$ defined as above. Then we have the unique analytic isomorphism $\phi : X^o \longrightarrow S \backslash (\mathbb{C}^n \times U)$ which makes the following diagram commutative*

$$\begin{array}{ccc} X^o & \xrightarrow{\phi} & S \backslash (\mathbb{C}^n \times U) \\ & \searrow \quad \swarrow & \\ & D^* & \end{array} \tag{3.13}$$

and such that $\phi \circ \sigma = O$.

(3.14) Real uniformizations.

In proposition 3.12, we have a complex analytic isomorphism $\phi : X^0 \longrightarrow S \backslash (\mathbb{C}^n \times U)$ and this induces a diffeomorphism $\phi_{|Y} : Y \longrightarrow S \backslash (\mathbb{C}^n \times \mathbb{R})$. By using this, we can introduce the new real coordinate $(x_1, \cdots, x_{2n})$ of $\mathbb{C}^n$ by the formula:

$$\begin{pmatrix} z_1 \\ \cdot \\ z_n \\ \bar{z}_1 \\ \cdot \\ \bar{z}_n \end{pmatrix} = \begin{pmatrix} \tau(t) & 1_n \\ \overline{\tau(t)} & 1_n \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ x_n \\ x_{n+1} \\ \cdot \\ x_{2n} \end{pmatrix}. \tag{3.15}$$

This induces a diffeomorphism between $\mathbb{C}^n \times U$ and $\mathbb{R}^{2n} \times U$ and the action of $g(\vec{m}, \mu) \in S$ on $\mathbb{C}^n \times U$ is converted to the action of $\mathbb{R}^{2n} \times U$ as

$$g(\vec{m}, \mu)(\vec{x}, t) = ({}^t(T^{-\mu})(\vec{x} + \vec{m}),\ t + \mu), \tag{3.16}$$

hence we have a diffeomorphism

$$S\backslash(\mathbb{C}^n \times U) \xrightarrow{\simeq} S\backslash(\mathbb{R}^{2n} \times U). \tag{3.17}$$

Moreover this induces diffeomorphisms

$$Y \overset{\phi}{\xrightarrow{\simeq}} S\backslash(\mathbb{C}^n \times \mathbb{R}) \xrightarrow{\simeq} S\backslash(\mathbb{R}^{2n} \times \mathbb{R}). \tag{3.18}$$

Now it is obvious that the differentiable structure of Y (or X^0) is determined by the monodromy matrix T. More precisely it depends only on the $GL(2n, \mathbb{Z})$-conjugate class of T.

Remark 3.19. Since for $T \in Sp(2n, \mathbb{Z})$, we have $J_n^{-1} T J_n = {}^tT^{-1}$, which implies that ${}^tT^{-1}$ is $Sp(2n, \mathbb{Z})$-conjugate to T. Hence the action of S on $\mathbb{R}^{2n} \times U$ given by

$$g(\vec{m}, \mu)(\vec{x}, t) = ((T^{\mu})(\vec{x} + \vec{m}),\ t + \mu), \tag{3.20}$$

gives a quotient space $S\backslash(\mathbb{R}^{2n} \times U)$, which is diffeomorphic to the original quotient manifold by the action defined in (3.16). In this case, the group multiplication of $S = \{g(\vec{m}, \mu)\}_{\vec{m} \in \mathbb{Z}^{2n}, \mu \in \mathbb{Z}}$ can be given by

$$g(\vec{n}, \mu_2) g(\vec{m}, \mu_1) = g(\vec{m} + T^{-\mu_1}\vec{n}, \mu_1 + \mu_2).$$

(Compare this with (3.8).)

(3.21) Framings and metrics for degenerations of PPAV.

Let $f : X \longrightarrow D$ and $g : Y = \partial X \longrightarrow \partial D = S^1$ be as in (3.2). In order to define the signature defect $\delta(X, Y)$ or the eta function $\eta(B_Y, s)$ of Y, we need to fix a Riemannian metric of X which is product near the neighborhood of its boundary Y. Moreover if Y has a framing α, one can also define another signature defect $\sigma(Y, \alpha)$ for a framed manifold (Y, α) in (2.7) and also another eta function $\eta(D, s)$ as in (2.16).

In general, the boundary manifold Y admits no natural framing. But if the monodromy transformation T of $g : Y \longrightarrow S^1$ is unipotent, Y becomes a homogeneous manifold of a Lie group, hence Y admits a natural framing induced by left invariant vector fields.

The most important property of a torus bundle $g : Y \longrightarrow S^1$ arising from a degeneration of abelian varieties is that it always admits a semistable reduction after a finite base extension. In terms of the monodromy T, it can be stated as the special case of the following quasi-unipotency of T arising from degenerations of Hodge structures. (For a general statement and a proof, see for example [G-S].)

Theorem 3.22. *For a degeneration of abelian varieties $f : X \longrightarrow D$, the monodromy transformation T (on $H_1(X_t, \mathbb{Z})$) is quasi-unipotent. More precisely, there is a positive integers l and m such that*

$$(T^l - I)^m = 0.$$

Moreover since the Hodge level of $H_1(X_t, \mathbb{C})$ is 2, we can take $m = 2$.

From this results, one can show the following proposition.

Theorem 3.23. *Let $f : X \longrightarrow D$ be a degeneration of PPAV over the closed unit disk and $g : Y \longrightarrow S^1$ its boundary fibration. Let $D \longrightarrow D$ be the map induced by $u \mapsto s = u^l$ for some positive integer l. Let $\tilde{f} : \tilde{X} \longrightarrow D$ be a degeneration of PPAV which is obtained by a Hironaka resolution of singularities of the fibre product $X \times_D D \longrightarrow D$ and let $\tilde{g} : \tilde{Y} \longrightarrow S^1$ its boundary fibration. Then there exists a positive integer l such that:*
i) the monodromy of the fibration $\tilde{g} : \tilde{Y} \longrightarrow S^1$ is unipotent, and
ii) the natural map $\tilde{Y} \longrightarrow Y$ is a regular covering map whose Galois group is the cyclic group G_l of order l. (Hence one has $Y = G_l \backslash \tilde{Y}$.)

(3.24) Framings for semistable degenerations.

Let $f : X \longrightarrow D$ be a degeneration of PPAV whose monodromy T is unipotent. Set $L = H_1(X_1, \mathbb{Z})$ which is a free $\mathbb{Z}$-module of rank $2n$, $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, and denote by $<,>: L^* \times L \longrightarrow \mathbb{Z}$ the natural pairing. We consider the monodromy transformation T as an automorphism of L. Then $N = T - I$ becomes a nilpotent endomorphism of L such that $N^2 = 0$. Let $L_0 = \mathrm{Ker} N = \{x \in L | Nx = 0\}$ and set $r = \mathrm{rank} L_0$. We may choose a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_{2n}\}$ of L such that with respect to this basis of L, the matrix representation of T and N can be written as:

$$T = \begin{pmatrix} I_n & B \\ 0_n & I_n \end{pmatrix}, \quad N = \begin{pmatrix} 0_n & B \\ 0_n & 0_n \end{pmatrix}. \tag{3.25}$$

Since T arises from a degeneration of PPAV, we may assume that $T \in Sp(2n, \mathbb{Z})$ and B is a symmetric positive definite matrix by [Lemma 2.3, Nak]. Therefore we can assume that B has a form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix}, \tag{3.26}$$

such that B' is a symmetric positive definite matrix of size $2n - r$. (Note that L_0 is generated by $\mathbf{e}_1, \cdots, \mathbf{e}_r$.)

Put $V := \{\exp lN; l \in \mathbb{Z}\} \simeq \mathbb{Z}$. Then V acts on L from left by regarding elements of L as column vectors. Hence we have the extension of V by L:

$$0 \to L \to S \to V \to 0,$$

which is essentially equivalent to the extension in (3.10). We use a coordinate system $(x_1, x_2, \ldots, x_{2n})$ in $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{2n}$ so that $\vec{x} = \sum_{i=1}^{2n} x_i \mathbf{e}_i$ and

$z = \exp(tN)$ in $V_{\mathbb{R}} := V \otimes_{\mathbb{Z}} \mathbb{R} = \{\exp(tN); t \in \mathbb{R}\} \cong \mathbb{R}$. We denote the action of $z \in V_{\mathbb{R}}$ on $L_{\mathbb{R}}$ as l_z. Then for $z = \exp(tN)$ and $\vec{x} \in L_{\mathbb{R}}$ we have $l_z(\vec{x}) = \vec{x} + tN\vec{x}$. We define a group structure on $S_{\mathbb{R}} := L_{\mathbb{R}} \times V_{\mathbb{R}} \cong \mathbb{R}^{2n} \times \mathbb{R}$ so that S becomes a uniform lattice of it. More explicitly, this can be defined by the multiplication law

$$(\vec{x}_1, t_1)(\vec{x}_2, t_2) = (\vec{x}_1 + \exp(t_1 N)\vec{x}_2, t_1 + t_2). \tag{3.27}$$

Then we get a compact homogeneous manifold $S \setminus S_{\mathbb{R}}$ which is a fibre space over $V \setminus V_{\mathbb{R}} \cong S^1$ with the typical fibre $A = L \setminus L_{\mathbb{R}}$. By the construction and (3.14), we have a natural diffeomorphism

$$Y \simeq S \backslash S_{\mathbb{R}}.$$

Let $\mathcal{G}$ be the Lie algebra of $S_{\mathbb{R}}$ generated by the left invariant vector fields

$$X_i := -l_z(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial x_i} - tN\frac{\partial}{\partial x_i} \ (i = 1, 2, \dots, 2n),$$

$$\text{and } Z := -\frac{\partial}{\partial t} \text{ for } z = \exp tN. \tag{3.28}$$

Note that the relations between these vector fields are only $[X_i, Z] = NX_i$ ($i = 1, 2, \dots, 2n$). These left invariant vector fields on $S_{\mathbb{R}}$ pushed down to $Y \simeq S \backslash S_{\mathbb{R}}$ and defines a trivialization of the tangent bundles of Y. Now the following proposition is obvious.

Proposition (3.29). *Let $f : X \longrightarrow D$ be a degeneration of PPAV with unipotent monodromy T and $g : Y \longrightarrow S^1$ its boundary. Then under the diffeomorphism $Y \simeq S \backslash S_{\mathbb{R}}$, we obtain a natural framing α of Y induced by left invariant vector fields on $S_{\mathbb{R}}$ in (3.28).*

(3.30) Finite Group Actions.

By theorem (3.23), for a general degeneration $f : X \longrightarrow D$, the boundary $g : Y \longrightarrow S^1$ is a free quotient of the boundary $\tilde{g} : \tilde{Y} \longrightarrow S^1$ of a semistable degeneration by a cyclic group. In general a framing of $\tilde{Y}$ defined in proposition (3.29) is not invariant under the group action, hence we can not define the natural framing on Y. On the other hand, the metric induced by the framing of $\tilde{Y}$ may be invariant. And if it is true, we may calculate the signature defect or eta invariant for Y as a Riemannian manifold by using the equivariant version of eta invariant in (2.18).

For a technical reason, we will not deal with the general case but consider the degeneration $f : X \longrightarrow D$ with the monodromy T of finite order l. Then after the finite base extension $u \mapsto s = u^l$ we obtain the degeneration $\tilde{f} : \tilde{X} \longrightarrow D$ with a trivial monodromy and in this case we can always choose a model $\tilde{f} : \tilde{X} \longrightarrow D$ which is diffeomorphic to the product $A \times D$ for a principal polarized abelian variety A. Moreover, we can show that there exists an analytic automorphism

$$\gamma : A \longrightarrow A \tag{3.31}$$

preserving the polarization such that $f : X \longrightarrow D$ is diffeomorphic to a Hironaka resolution of singularities of the quotient of $A \times D$ by the action of the cyclic group G_l generated by the automorphism of $A \times D$

$$\tilde{\gamma}(\vec{z}, u) = (\gamma(\vec{z}), e_l u). \tag{3.32}$$

From this, we obtain the following proposition.

Proposition 3.33. *Let $f : X \longrightarrow D$ and $g : Y \longrightarrow S^1$ be as above. Then Y is diffeomorphic to a free quotient of $W = A \times S^1$ by the induced action of the cyclic group G_l as above. Moreover there exists a framing of A which induces a product framing α of $A \times S^1$ such that $\tilde{\gamma}$ becomes an isometry with respect to the metric induced by the framing α.*

Proof. The framing of A should be chosen so that the metric induced by the framing coincides with the metric induced by the polarization of A. Other parts are now obvious.

§4 Calculations in Unipotent Monodromy Cases.

Let us fix a positive integer k and set $n = 2k - 1$, and let $f : X \longrightarrow D$ be a degeneration of PPAV of dimension n with a unipotent monodromy and $g : Y \longrightarrow S^1$ its boundary. Then from proposition (3.29), we have the diffeomorphism $Y \simeq S \backslash S_{\mathbb{R}}$, and under this identification Y admits a natural framing α induced by the left invariant vector fields of $S_{\mathbb{R}}$ in (3.28). In this section, first we will calculate the eta function $\eta(D, 0)$ of the operator D as (2.17) for this framed manifold. Then by using the index theorem (2.23) we will relate this to the signature defect $\sigma(Y, \alpha)$ of the framed manifold (Y, α) defined in (2.7).

(4.1) The operator D and B_Y.

We have fixed a diffeomorphism

$$Y \simeq S \backslash S_{\mathbb{R}} \tag{4.2}$$

and also the framing α given by the left invariant vector fields $\{X_i, Z\}$ on $S_{\mathbb{R}}$ in (3.28). Let $\mathcal{M} = \mathcal{M}^{ev} =: \oplus_{p=0}^{n} \wedge^{2p} \mathcal{G}^* \otimes_{\mathbb{R}} \mathbb{C}$ denote the space of constant forms of even degrees on Y. Since by using the framing α the space $L^2(Y)^{ev}$ of L^2 forms of even degrees on Y is isomorphic to $L^2(Y) \otimes \mathcal{M}$ where $L^2(Y)$ denotes the the space of L^2-functions of Y. On $L^2(Y)^{ev}$, we can define two kinds of signature operators B_Y and D defined in (2.15) and (2.17) respectively. Since both of framing and the metric are invariant under the action of $S_{\mathbb{R}}$ on Y, we see that operators D and B_Y are $S_{\mathbb{R}}$-invariant. Since D^2 and B_Y^2 have the same leading symbol as the Laplacian, we have the following proposition. (cf. Proposition 6.4 [ADS]).

Proposition 4.3. *We can write the operators D and B_Y for the framed manifold (Y, α) as:*

$$D = -\sqrt{-1}\sum_{i=1}^{2n} X_i \otimes E_i + Z \otimes F, \quad \text{and} \tag{4.4}$$

$$B_Y = D + B_0 \tag{4.5}$$

with constant endomorphisms $E_i, F, B_0 \in \mathrm{End}(\mathcal{M})$. Moreover, E_i and B_0 are hermitian and F is skew-hermitian.

(4.6) Results.

Let (Y, α) be the framed manifold of dimension $4k-1$ as above. As far as the eta function $\eta(D, s)$ is concerned we have the following theorem.

Theorem 4.7. *Let (Y, α) be as above and assume that $N = T - I \neq 0$.*
(i) If $k = 1$ (i.e. $n = 1$) and $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b > 0$, then we have

$$\eta(D, s) = -4b(2\pi)^{-s}\zeta(s-1) \tag{4.8}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann Zeta function.

(ii) If $k > 1$ (i.e. $n = 2k-1 > 1$), then $\eta(D, s) \equiv 0$

Noting that $\zeta(-1) = -1/12$, we obtain the following corollary.

Corollary 4.9. *We have*

$$\eta(D, 0) = \begin{cases} \frac{b}{3} & \textit{if} \quad k = 1, \quad T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \textit{with} \quad b > 0, \\ 0 & \textit{if} \quad k \geq 2. \end{cases}$$

In order to give the signature defects $\sigma(Y, \alpha)$ for framed manifold (Y, α) defined in (2.7), we need a definition. Let $\mathcal{M}^{2k-1} := \wedge^{2k-1}\mathcal{G}^* \otimes \mathbb{C}$ be the space of constant forms on $S \setminus S_{\mathbb{R}}$ of degree $2k-1$ and set $d\mathcal{M}^{2k-1} := \{d\omega; \omega \in \mathcal{M}^{2k-1}\}$. We define a hermitian endomorphism B_1 by

$$B_1 := d* : d\mathcal{M}^{2k-1} \longrightarrow d\mathcal{M}^{2k-1}. \tag{4.10}$$

Theorem 4.11. *For the framed manifold* (Y, α) *as above, let* B_0 *be the endomorphism of* $\mathcal{M}$ *defined in (4.5) and denote by* $\mathrm{sign}(B_0)$ *and* $\mathrm{sign}(B_1)$ *signatures of the hermitian endomorphisms* B_0 *and* B_1 *respectively. Then we have:*
(i) $\mathrm{sign}(B_0) = \mathrm{sign}(B_1)$, *and*
(ii)

$$\sigma(Y, \alpha) = \eta(D, 0) + \mathrm{sign}(B_0) = \eta(D, 0) + \mathrm{sign}(B_1). \tag{4.12}$$

In particular, if $n > 1$, *then* $\sigma(Y, \alpha) = \mathrm{sign}(B_1)$ *is an integer.*

As far as $\mathrm{sign}(B_1)$ is concerned, we have the following proposition.

Proposition 4.13. *Let* $Y = S \backslash S_{\mathbb{R}}$ *be as above and set* $N = T - I_{2n}$ *and* $r = \mathrm{rank}(\mathrm{Ker} N)$. *Then the* $\mathrm{sign}(B_1)$ *depends only on* n *and* r. *Moreover we have the following.*

(i) *If* $r > n$, *we have* $\mathrm{sign}(B_1) = 0$.

(ii) *If* $r = n$, *then* $\mathrm{sign}(B_1)$ *depends only on the dimension* $n = 2k - 1$ *so write* $s(k) = \mathrm{sign}(B_1)$ *as a function of* k. *Then* $s(k)$ *is equal to the signature of an explicit symmetric matrix (see Proposition (4.53)). For small* k, *we can calculate* $s(k)$ *explicitly as*

$$s(1) = -1, \quad s(2) = -1, \quad s(3) = -2. \tag{4.14}$$

From (4.9), (4.11) and (4.13), we obtain the following corollary which gives the signature defect $\sigma(Y, \alpha)$.

Corollary 4.15.
(i) *If* $n = 1$ *and the monodromy* $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *with* $b > 0$, *we have*

$$\sigma(Y, \alpha) = \frac{b}{3} - 1.$$

(ii) *If* $n > 1$, *and* $r > n$, *we have* $\sigma(Y, \alpha) = 0$
(iii) *If* $n = 2k - 1 > 1$, *and* $r = n$, *we have* $\sigma(Y, \alpha) = s(k) \in \mathbb{Z}$.

Remark 4.16. As mentioned in §1, Atiyah [A] dealt with many kinds of invariants of 2-torus fibration or of monodromy T. He established the equality $\eta^0(Y, 0) = -\phi(T)$ for all 2-torus fibration Y over the circle with monodromy $T \in SL(2, \mathbb{Z})$. Here $\eta^0(Y, 0)$ is the adiabatic limit of the eta invariant and $\phi(T)$ is Meyer's invariant defined in [M]. The assertion (i) of (4.15) and the formula (1.3) show that the signature defect $\sigma(Y, \alpha)$ coincides with $\eta^0(Y, 0)$ and $-\phi(T)$ for $T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. It is an interesting problem to determine whether $\eta(Y, 0)$ coincides with its adiabatic limit in this case.

(4.17) Calculation of $\eta(D, s)$.

In this subsection, we will calculate the eta function $\eta(D, s)$ for $Y = S\backslash S_{\mathbb{R}}$ with the natural framing α. We use the notations in (3.24) and moreover we assume that the monodromy T has a form as in (3.25) and $N = T - I \neq 0$. From this assumption, we have

$$n \leq r(:= \mathrm{rank}(\mathrm{Ker}N)) < 2n, \quad NL \cap \mathbb{Z}\mathbf{e}_n \neq \{0\}. \tag{4.18}$$

Let us first recall a decomposition of the function space on $Y = S\backslash S_{\mathbb{R}}$. Let $L_1^* := \{\nu \in L^*; \nu N = 0\}$, where we regard elements of L^* as row vectors. Then we have the following decomposition of the right quasi-regular representation of $S_{\mathbb{R}}$.

Proposition 4.19.

$$L^2(S \setminus S_{\mathbb{R}}) \cong \bigoplus_{\nu \in L_1^*} L^2(\mathbb{Z} \setminus \mathbb{R}) \oplus \bigoplus_{\nu \in (L^* - L_1^*)/V} L^2(\mathbb{R}).$$

For $\nu \in L^*$, the differential operators X_i and Z act as $\nu(X_i) = -2\pi\sqrt{-1} < \nu + t\nu N, \mathbf{e}_i > (i = 1, 2, \dots, 2n)$ and $\nu(Z) = -\frac{d}{dt}$.

Proof. We identify $C^\infty(S \setminus S_{\mathbb{R}})$ with the invariant subspace $C^\infty(S_{\mathbb{R}})^S$ under the left action of S. Let $f(\vec{x}, t) \in C^\infty(S \setminus S_{\mathbb{R}})$. Since f is invariant under the action of L, we can expand it as a Fourier series

$$f(\vec{x}, t) = \sum_{\nu \in L^*} f_\nu(t)\mathbf{e}(< \nu, \vec{x} >),$$

where we use the notation $\mathbf{e}(\cdot) := \exp(2\pi\sqrt{-1}(\cdot))$. Further $\exp(lN)$ acts on f. We have

$$\begin{aligned}(\exp lN)^* f\ (\vec{x}, t) &= f(\vec{x} + lN\vec{x}, \vec{x} + l) \\ &= \sum_{\nu \in L^*} f_\nu(t + l)\mathbf{e}(< \nu, \vec{x} + lN\vec{x} >) \\ &= \sum_{\nu \in L^*} f_\nu(t + l)\mathbf{e}(< \nu + l\nu N, \vec{x} >).\end{aligned}$$

Thus we have $f_{\nu + l\nu N}(t) = f_\nu(t + l)$ for $\nu \in L^*$ and $l \in \mathbb{Z}$. Hence we have

$$f(\vec{x}, t) = \sum_{\mu \in L_1^*} f_\mu(t)\mathbf{e}(< \mu, \vec{x} >) + \sum_{\nu \in (L^* - L_1^*)/V} \left(\sum_{l \in \mathbb{Z}} f_\nu(t + l)\mathbf{e}(< \nu + l\nu N, \vec{x} >) \right),$$

where $f_\mu \in C^\infty(V \setminus V_{\mathbb{R}})$ for $\mu \in L_1^*$ and $f_\nu \in C^\infty(V_{\mathbb{R}})$ for $\nu \in (L^* - L_1^*)/V$.

Conversely, given $f_\mu \in L^2(V \setminus V_{\mathbb{R}})$ for $\mu \in L_1^*$ and $f_\nu \in L^2(V_{\mathbb{R}})$ for $\nu \in (L^* - L_1^*)/V$ the sum

$$\sum_{\mu \in L_1^*} f_\mu(t)\mathbf{e}(<\mu, \vec{x}>) + \sum_{\nu \in (L^*-L_1^*)/V} \left(\sum_{l \in \mathbb{Z}} f_\nu(t+l)\mathbf{e}(<\nu + l\nu N, \vec{x}>) \right)$$

lies in $L^2(S \setminus S_{\mathbb{R}})$ if it absolutely converges.

Let us write the operator D as in (4.4). Since the second order term of D^2 should coincide with the Laplacian $\Delta := -(\sum_{i=1}^{2n} X_i^2 + Z^2)$, we have a relation in $\{E_i, F\}$: $E_i^2 = Id$, $F^2 = -Id$ and any two of them anti-commute. Then we have

$$D^2 = \Delta + \sqrt{-1}\sum_{i=1}^{2n} NX_i \otimes E_i F.$$

We denote the restriction of D to the invariant subspace corresponding to $\nu \in L^*$ as:

$$D_\nu = -2\pi \sum_{i=1}^{r} <\nu, \mathbf{e}_i> E_i - 2\pi \sum_{j=r+1}^{2n} <\nu + t\nu N, \mathbf{e}_j> E_j - \frac{d}{dt} \otimes F.$$

From Proposition 4.18 one can write the eta function of D for $\mathrm{Re}(s) >> 0$ as:

$$\eta(D, s) = \sum_{\nu \in L_1^*} \eta_{D_\nu}(s) + \sum_{\nu \in (L^*-L_1^*)/V} \eta_{D_\nu}(s). \tag{4.19}$$

Lemma 4.20. *For $\nu \in L_1^*$ we have $\eta_{D_\nu}(s) = 0$.*

Proof. Set $\nu_i =<\nu, \mathbf{e}_i>$. For $\nu \in L_0$ we have

$$D_\nu = -2\pi \sum_{i=1}^{2n} \nu_i E_i - \frac{d}{dt} \otimes F. \tag{4.21}$$

Since $a\mathbf{e}_n \in NL$, and since $L_1^* \subset (NL)^\perp$, the unitary matrix E_n does not appear in D_ν, hence, it anti-commutes with D_ν, that is, we have $E_n^* D_\nu E_n = -D_\nu$. Thus $\eta_{D_\nu}(s) = -\eta_{D_\nu}(s) = 0$.

From Lemma 4.20 and (4.19) we have

$$\eta(D, s) = \sum_{\nu \in (L^*-L_1^*)/V} \eta_{D_\nu}(s) \tag{4.22}$$

for $\mathrm{Re}(s) >> 0$. We must calculate only the case $\nu \in L^* - L_1^*$. For $\nu \in L^*$ we have

$$D_\nu^2 = \Delta_\nu - 2\pi \sum_{i=1}^{2n} <\nu N, \mathbf{e}_i> E_i F,$$

where

$$\Delta_\nu = 4\pi^2 \sum_{i=1}^{2n} < \nu + t\nu N, \mathbf{e}_i >^2 - \frac{d^2}{dt^2}.$$

Set $||\nu + t\nu N||^2 = \sum_{i=1}^{2n} < \nu + t\nu N, \mathbf{e}_i >^2$ and $\nu_i =< \nu, \mathbf{e}_i >$. Then we have

$$||\nu + t\nu N||^2 = ||\nu||^2 + 2t \sum_{i=1}^{2n} \nu_i < \nu N, \mathbf{e}_i > + t^2 ||\nu N||^2.$$

Set $C = \sum_{i=1}^{2n} \nu_i < \nu N, \mathbf{e}_i >= \sum_{j=r+1}^{2n} \nu_j < \nu, N\mathbf{e}_j >$. Then we may write as

$$||\nu + t\nu N||^2 = (t||\nu N|| + \frac{C}{||\nu N||})^2 + ||\nu||^2 - \frac{C^2}{||\nu N||^2}.$$

Since $\nu N \neq 0$ for $\nu \in L^* - L_1^*$, set $u = \sqrt{\frac{2\pi}{||\nu N||}}(t||\nu N|| + \frac{C}{||\nu N||})$. Then we have

$$\Delta_\nu = 4\pi^2(||\nu||^2 - \frac{C^2}{||\nu N||^2}) + 2\pi||\nu N||(u^2 - \frac{d^2}{du^2}).$$

For Hermite polynomial $h_m(t) = e^{-t^2/2} \frac{d^m}{dt^m} e^{t^2/2}$, set $f_m(t) = e^{-t^2/2} h_m(t)$ $(m = 0, 1, 2, \dots)$. Then $f_m(t)$ satisfies the differential equation

$$\left[t^2 - \frac{d^2}{dt^2}\right] f_m(t) = (2m+1) f_m(t),$$

and $\{f_m(t); m = 0, 1, 2, \dots\}$ is a complete orthogonal basis of $L^2(\mathbb{R})$. Thus the eigenvalues of Δ_ν are $\{2\pi||\nu N||(2m+1) + 4\pi^2(||\nu||^2 - \frac{C^2}{||\nu N||^2})\}$.

Now the matrices $E_i F$ are hermitian and unitary, and any two of them anti-commute. Thus $(\sum_{i=1}^{2n} < \nu N, \mathbf{e}_i > E_i F)^2 = ||\nu N||^2$. Set $M = \sum_{i=1}^{2n} < \nu N, \mathbf{e}_i > E_i F$. Since M is hermitian, it has eigenvalues $\pm||\nu N||$. We decompose $\mathcal{M}$ into the sum $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ of the corresponding eigenspaces of M. Thus we have the following lemma.

Lemma 4.23. *The eigenvalues of D_ν^2 for $\nu \in L^* - L_1^*$ are*

$$\left\{2\pi||\nu N||(2m + 1 \mp 1) + 4\pi^2(||\nu||^2 - \frac{C^2}{||\nu N||^2}); m = 0, 1, 2, \dots\right\}.$$

Lemma 4.24. *The matrices E_i for $i = 1, \dots, r$ preserve any eigenspace $\mathcal{E}$ of D_ν^2, hence, we have*

$$\mathrm{Tr}(D_\nu|_{\mathcal{E}}) = -2\pi \sum_{i=1}^{r} \nu_i \mathrm{Tr}(E_i|_{\mathcal{E}}) - \mathrm{Tr}((D_\nu + 2\pi \sum_{i=1}^{r} \nu_i E_i)|_{\mathcal{E}}).$$

Proof. Since $D_\nu^2 = \Delta_\nu - 2\pi M$, and since $M = \sum_{j=r+1}^{2n} < \nu, N e_j > E_j F$, the matrices E_i for $i = 1, \dots, r$ commute with M, hence with D_ν^2. Thus we have the desired decomposition.

Lemma 4.25. *For $\nu \in L^* - L_1^*$ we have*

$$\mathrm{Tr}(D_\nu|_{\mathcal{E}}) = -2\pi\nu_n \mathrm{Tr}(E_n|_{\mathcal{E}}).$$

In particular, when $n \geq 2$ we have $\mathrm{Tr}(D_\nu|_{\mathcal{E}}) = 0$.

Proof. Since E_n is hermitian and unitary on $\mathcal{E}$, and since $E_n^* E_i E_n = -E_n$ for $i \neq 1$ and $E_n^* F E_n = -F$, we have $E_n^*(D_\nu + 2\pi \sum_{i=1}^r \nu_i E_i)|_{\mathcal{E}} E_n = -(D_\nu + 2\pi \sum_{i=1}^r \nu_i E_i)|_{\mathcal{E}}$. When $n \geq 2$, we have another hermitian and unitary matrix E_1 on $\mathcal{E}$ since $r \geq n \geq 2$.

Since we can write the eta function of D_ν as

$$\eta_{D_\nu}(s) = \sum \frac{\mathrm{Tr}(D_\nu|_{\mathcal{E}(\lambda^2)})}{|\lambda|^{s+1}}, \tag{4.26}$$

where the summation is taken over all eigenspaces $\mathcal{E}(\lambda^2)$ of D_ν^2 with eigenvalues λ^2.

From Lemmas 4.23, 4.24, 4.25 and 4.26, we see that $\eta_{D_\nu}(s) = 0$ unless $n = 1$. Therefore we obtains the assertion (ii) of Theorem (4.7).

(4.27) Continuation: the case $n = 1$.

In view of (4.19), (4.25) and (4.26), we must calculate $\mathrm{Tr}(E_1|_{\mathcal{E}})$ for $n = 1$. Set $N = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ with $b > 0$. Then we see that $C = b\nu_1\nu_2$ and $M = \nu_1 E_2 F$. Thus Lemma 4.23 for the case of $n = 1$ may be written as follows.

Lemma 4.28. *When $n = 1$, D_ν^2 has eigenvalues*

$$\{2\pi b|\nu_1|(2m+1) \mp 2\pi b\nu_1 + 4\pi^2\nu_1^2; m = 0, 1, 2, \dots\}.$$

Lemma 4.29. *For $\nu \in L^* - L_1^*$, we have*

$$\begin{aligned} \mathrm{Tr}(D_\nu|_{\mathcal{E}}) = 0 \quad & \textit{unless } \nu_1 > 0 \textit{ and } \mathcal{E} = \mathbb{C}f_0 \otimes \mathcal{M}_+, \\ & \textit{or } \nu_1 < 0 \textit{ and } \mathcal{E} = \mathbb{C}f_0 \otimes \mathcal{M}_-. \end{aligned}$$

Hence we have to calculate $\mathrm{Tr}(E_1|_{\mathcal{M}_\pm})$.

Lemma 4.30. *We have*

$$FE_1E_2 = Id.$$

Proof. We get the identity from a direct calculation. See section 4.3 of [G] for general dimension.

Lemma 4.31. *We have*

$$E_1|_{\mathcal{M}_\pm} = \pm Id.$$

The following proposition shows the assertion (i) of Theorem 4.7.

Proposition 4.32. *When $n = 1$ we have*

$$\eta(D, s) = -4b|2\pi|^{-s}\zeta(s-1).$$

Proof. When $n = 1$ we have $\dim \mathcal{M} = 2^2 = 4$. Since F gives a bijection between $\mathcal{M}_+$ and $\mathcal{M}_-$, we have $\dim \mathcal{M}_\pm = 2$. Thus we have

$$\eta(D, s) = \sum_{\nu \in (L^* - L_1^*)/V} \frac{-2\pi\nu_1 \cdot 2}{|2\pi\nu_1|^{s+1}}.$$

For $\nu = (\nu_1, \nu_2) \in L^* - L_1^*$ we have $\nu V = \{(\nu_1, lb\nu_1 + \nu_2); l \in \mathbb{Z}\}$. Thus we have

$$\eta(D, s) = -2 \sum_{\nu_1 = 1}^{\infty} \frac{b\nu_1 \cdot 4\pi\nu_1}{(2\pi\nu_1)^{s+1}} = -4b(2\pi)^{-s}\zeta(s-1).$$

(4.33) Calculation of Signature Defects $\sigma(Y, \alpha)$.

Let X be an oriented compact Riemannian manifold of dimension $2n + 2 = 4k$ with $\partial X = Y = S \setminus S_\mathbb{R}$ which is defined as in (2.6). Then from the index theorem (2.23) for a framed manifold (Y, α), we have

$$\eta(D, 0) = \int_X \mathcal{D}_0 - l_0,$$

where l_0 is an integer and $\mathcal{D}_0$ is invariant under scaling of the metric on X.

Let H be the subspace consisting of constant forms in $L^2(S \setminus S_\mathbb{R}) \otimes \mathcal{M}$ and $H^\perp$ the orthogonal complement.

Lemma 4.34. $\mathrm{Ker}(D) = H$ *and* $D^2 - 4\pi^2 \geq 0$ *on* $H^\perp$.

Proof. For $\mu \in L_1^*$ we have $D_\mu^2 - 4\pi^2||\mu||^2 \geq 0$ from (4.21). Set $||\nu'||^2 := \sum_{i=1}^r \nu_i^2$. Then for $\nu \in L^* - L_1^*$ we have $D_\nu^2 - 4\pi^2||\nu'||^2 \geq 0$ from Lemma 4.23.

Let us deform linearly from B_Y to D (cf. Proposition 4.3), hence set

$$D_t := tB_Y + (1-t)D = D + tB_0 \quad \text{for } 0 \leq t \leq 1.$$

Lemma 4.35. *We can scale L so that*

$$\mathrm{Ker}(D_t) = \begin{cases} H & \textit{for } t = 0 \\ \mathrm{Ker}(B_0) & \textit{for } t > 0 \end{cases}.$$

Proof. Put $L(\lambda) = \{\lambda x \in L_\mathbb{R}; x \in L\}$ for any positive real number λ. Note that the action of V on $L_\mathbb{R}$ preserves $L(\lambda)$. Thus we have a lattice $S(\lambda)$ of $S_\mathbb{R}$;

$$0 \to L(\lambda) \to S(\lambda) \to V \to 0.$$

The frame on $S_{\mathbb{R}}$ induces also a frame on $S(\lambda) \setminus S_{\mathbb{R}}$, hence, the metric g_λ. With respect to this metric g_λ we can define a differential operator D, which we denote by $D(\lambda)$ as in (2.17). Consider a diffeomorphism $\Psi_\lambda : S(\lambda) \setminus S_{\mathbb{R}} \to S \setminus S_{\mathbb{R}}$ defined by $\Psi_\lambda(S(\lambda)(x,t)) = S(x/\lambda, t)$. For $\phi \in L^2(S \setminus S_{\mathbb{R}}) \otimes \mathcal{M}$ we define a differential operator D_λ by $D_\lambda(\phi)(S(x,t)) := D(\lambda)(\phi \circ \Psi_\lambda)(\Psi_\lambda^{-1}(S(x,t))$. Then we have

$$D_\lambda(f \otimes \omega) = -\sqrt{-1}\frac{1}{\lambda}\sum_{i=1}^{2n} X_i f \otimes E_i \omega + Zf \otimes F\omega.$$

The operator D_λ is defined by the metric $(\Psi_\lambda^{-1})^* g_\lambda$ on $S \setminus S_{\mathbb{R}}$, in other words, defined by the frame $\{X_i/\lambda, Z; i = 1, 2, \ldots, 2n\}$. From (4.21) we have

$$(D_\lambda)_\mu^2 = \frac{4\pi^2}{\lambda^2}||\mu||^2 - \frac{d^2}{dt^2} \qquad \text{for } \mu \in L_1^*.$$

From Lemma 4.23 we obtain

$$(D_\lambda)_\nu^2 \geq \frac{4\pi^2}{\lambda^2}||\nu'||^2 \qquad \text{for } \nu \in L^* - L_1^*.$$

If we take λ sufficiently small, then we can scale L so that $2\pi/\lambda \geq ||B_0||$, that is, $\mathrm{Ker}(D_t) \subset H$.

We denote the eta function of D_t by $\eta(D_t, s)$. Then one has

$$\eta(D_t, s) = \eta(D_t|_H, s) + \eta(D_t|_{H^\perp}, s).$$

Lemma 4.36. *$\eta(D_t|_{H^\perp}, s)$ is continuous with respect to t.*

Proof. See Lemma 14.7 in [ADS].

From the Theorem 3.10 in [APS-III] we have

$$l_t = \int_X \mathcal{D}_t - \eta(D_t, 0), \tag{4.37}$$

where l_t is an integer and $\mathcal{D}_t$ is continuous in t. We can write the equality (4.37) as

$$l_t + \eta(D_t|_H, 0) = \int_X \mathcal{D}_t - \eta(D_t|_{H^\perp}, 0),$$

whose right hand side is continuous in t by Lemma 4.36, while the left hand side has values in integers. Thus we have

$$l_0 + \eta(D_0|_H, 0) = l_1 + \eta(D_1|_H, 0).$$

Since $\eta(D_0|_H, 0) = 0$ from Lemma 4.35, and since $l_1 = \mathrm{Sign}(X, Y)$ and $\eta(D_1|_H, 0) = \mathrm{sign}(B_0)$, we have

$$l_0 = \mathrm{Sign}(X, Y) + \mathrm{sign}(B_0). \tag{4.38}$$

(Cf. Theorem 14.10 in [ADS]).

From the remark after theorem (2.23), we see that outside the neighborhood $I \times Y$, $\mathcal{D}_0$ coincides with $L_k(p_1(\phi), \cdots, p_k(\phi))$ where $p_i(\phi)$ is the Pontrjagin form introduced in (2.10). Then from (2.7), we can write

$$\sigma(Y, \alpha) = \int_X L_k(p_1(\phi), \cdots, p_k(\phi)) - \mathrm{Sign}(X, Y). \tag{4.39}$$

Then setting $\Omega(\phi) = L_k(p_1(\phi), \cdots, p_k(\phi))$, from the remark as above, we obtain

$$\int_X (\Omega(\phi) - \mathcal{D}_0) = \int_{Y \times I} (\Omega(\phi) - \mathcal{D}_0).$$

Therefore from (2.24), (4.38) and (4.39) one has

$$\begin{aligned} \sigma(Y, \alpha) &= \int_X \Omega(\phi) - \mathrm{Sign}(X, Y) \\ &= \int_X \Omega(\phi) - (l_0 - \mathrm{sign}(B_0)) \\ &= \int_X \Omega(\phi) - \int_X \mathcal{D}_0 + \eta(D, 0) + \mathrm{sign}(B_0) \\ &= \int_{Y \times I} (\Omega(\phi) - \mathcal{D}_0) + \eta(D, 0) + \mathrm{sign}(B_0). \end{aligned} \tag{4.40}$$

Lemma 4.41.

$$\int_{Y \times I} (\Omega(\phi) - \mathcal{D}_0) = 0.$$

Proof. Let us recall the construction $p_i(\phi)$ in (2.10) and $\mathcal{D}_0$ in (2.23). Then just as in Proposition 13.5 in [ADS] we have

$$\Omega(\phi) - \mathcal{D}_0 = \sum a_i(f) P_i(T_0),$$

where $a_i(f)$ is a polynomial in f of (2.10.1) and in the derivatives of f with values in 1-forms, and $P_i(T_0)$ is an $O(4k-1)$-invariant $(4k-1)$-form valued polynomial in the components of T_0 and in the covariant derivatives with respect to the flat connection ∇. Moreover P_i is a finite linear combination of elementary monomials in the torsion tensor T_0 defined in [ABP]. Thus the integral which we want to know is essentially a finite linear combination of the integrals of $a_i(f)$ on I depending only on a choice of f satisfying (2.10.1). On the other hand, the integral in question coincides with $\sigma(Y, \alpha) - \eta(D, 0) - \mathrm{sign}(B_0)$ independent of f. Hence it must be zero.

From (4.40) and Lemma 4.41 we have

$$\sigma(Y, \alpha) = \eta(D, 0) + \mathrm{sign}(B_0), \tag{4.42}$$

which implies the first assertion of (ii) of Theorem 4.11.

Moreover we have the following lemma which implies (i) of Theorem 4.11.

Lemma 4.44. $\mathrm{sign}(B_0)$ *is equal to* $\mathrm{sign}(B_1)$ *where* B_1 *is the operator* $d*$ *restricted to* $d\mathcal{M}^{2k-1}$.

Proof. Apply Theorem 4.20 in [APS-I] on the space $\mathcal{M}^{ev}$ of constant forms of even degrees.

(4.45) Proof of Proposition 4.13.

Lemma 4.46. *The signature* $\mathrm{sign}(B_1)$ *defined in (4.10) depends only on the orientation and Lie group structure of* $S_{\mathbb{R}}$ *defined in (3.27). That is, it does not depends on the framing and the embedding of the lattice* S *into* $S_{\mathbb{R}}$.

Proof. First we note that two framings on Y giving the same orientation on Y can be transformed into each other by an element of $GL(4k-2,\mathbb{R})^+$. So if we fix a differential structure and an orientation on Y, $\mathrm{sign}(B_1)$ is a continuous function of the set of all framings, and hence can be regarded as a continuous function on $GL(4k-2,\mathbb{R})^+$. Since $\mathrm{sign}(B_1)$ takes an integer value and $GL(4k-2,\mathbb{R})^+$ is connected, we see that $\mathrm{sign}(B_1)$ does not depends on the framing. Now, from the definition of B_1 in (4.10) one can see that $\mathrm{sign} B_1$ only depends on the orientation and Lie group structure of $S_{\mathbb{R}}$.

Let T and N be as in (3.25) and B as in (3.26). Since B' in (3.26) is symmetric positive-definite matrix of size $2n-r$, we can find an invertible real matrix H of size $2n$ such that

$$HNH^{-1} = \begin{pmatrix} 0_n & I_{n.r} \\ 0_n & 0_n \end{pmatrix}, \tag{4.47}$$

where

$$I_{n,r} = \begin{pmatrix} 0 & 0 \\ 0 & I_{2n-r} \end{pmatrix}. \tag{4.48}$$

Lemma 4.49. *The Lie group structure of* $S_{\mathbb{R}}$ *defined in (3.27) depends only on the dimension* n *and* $r = \mathrm{rank}$ *of* $\mathrm{Ker}(N)$.

Proof. Let us consider the coordinate change $\vec{y} = H\vec{x}$ of $\mathbb{R}^{2n}$ as in (4.47). Then this induces a diffeomorphism of $S_{\mathbb{R}}$ and the Lie group structure defined in (3.27) is converted into

$$(\vec{y}_1, t_1)(\vec{y}_2, t_2) = (\vec{y}_1 + \exp(t_1 HNH^{-1})\vec{y}_2, t_1 + t_2).$$

By (4.47), we see that the Lie group structure depends only on n and r.

Now from (4.46) and (4.49), we have the following proposition which implies the first assertion of Proposition (4.13).

Proposition 4.50. *The signature* $\mathrm{sign}(B_1)$ *in (4.10) depends only on the dimension* n *and the rank* r $(n \le r \le 2n)$*.*

Next we will prove the assertion (i) of Proposition 4.13. Let ξ_i and $\zeta \in \mathcal{G}^*$ be the dual of X_i and Z, respectively. Then we have $d\xi_i = -(\xi_i N) \wedge \zeta$ and $d\zeta = 0$. In the following we use the notation $N^*\xi_i$ instead of $\xi_i N$. In (3.24) we define NL to has rank $2n - r$. Set $s = 2n - r$. Since $NL \subset L_0$, we may choose a basis of L_0 so that $NL \subset \oplus_{i=1}^{s} \mathbb{Z}\mathbf{e}_i$. Then we have

$$\begin{array}{llr} & N^*\xi_i \in \oplus_{j=r+1}^{2n} \mathbb{Z}\xi_j & \text{for } i = 1, \ldots, s \\ \text{and} & N^*\xi_i = 0 & \text{for } i = s+1, \ldots, 2n. \end{array} \tag{4.51}$$

Lemma 4.52. *When* $r > n$*, we have* $\mathrm{sign}(B_1) = 0$*.*

Proof. For a nonempty subset $I = \{i_1, \ldots, i_l\} \subset \{1, 2, \ldots, 2n\}$, write $\xi_I := \xi_{i_1} \wedge \cdots \wedge \xi_{i_l}$. Then a constant form $\omega \in \mathcal{M}^{2k-1}$ with $d\omega \neq 0$ can be written as a linear combination of

$$\xi_{I_1} \wedge \xi_{I_2} \wedge \xi_{I_3} =: \omega_{I_1, I_2, I_3},$$

where $I_1 \subset \{1, 2, \ldots, s\}, I_2 \subset \{s+1, s+2, \ldots, r\}$ and $I_3 \subset \{r+1, r+2, \ldots, 2n\}$, and $|I_1| + |I_2| + |I_3| = 2k - 1 (= n)$. We call that ω_{I_1, I_2, I_3} has type $(|I_1|, |I_2|, |I_3|)$. Since $*d*d$ preserves types, $(B_1)^2$ transforms $d\omega_{I_1, I_2, I_3}$ to a linear combination of $d\omega_{J_1, J_2, J_3}$'s satisfying $|I_i| = |J_i|$ $(i = 1, 2, 3)$ and $J_2 = I_2$. On the other hand, B_1 transforms $d\omega_{I_1, I_2, I_3}$ to a linear combination of $d\omega_{J_1, J_2, J_3}$'s satisfying $|J_1| = s + 1 - |I_1|, |J_2| = r - s - |I_2|$ and $|J_3| = s - 1 - |I_3|$, in particular, $J_2 = \{s+1, \ldots, r\} - I_2$. Thus we have $\mathrm{sign}(B_1) (= \eta(B_1, 0)) = 0$ unless $r = s$, that is, $r = n$.

The following proposition and proposition (4.50) shows that $\mathrm{sign}(B_1)$ can be calculated by the signature of an explicit matrix B_2 defined below. It may be interesting to determine $s(k) = \mathrm{sign}(B_2)$ as a function of k.

Proposition 4.53. *Assume that* $r = n$ *and that* N *has the matrix form as*

$$N = \begin{pmatrix} O_n & I_n \\ O_n & O_n \end{pmatrix}.$$

Let $\mathcal{M}_0$ *be the real vector space generated by symbols* ω_I *for all subsets* $I \subset \{1, 2, \ldots, n\}$ *with* $|I| = k$*, and let* B_2 *be the operator on* $\mathcal{M}_0$ *acting as*

$$B_2 \omega_I = \sum_{i \in I} (-1)^i \omega_{I^c \cup \{i\}}.$$

Then we have

$$\mathrm{sign}(B_1) = \mathrm{sign}(B_2).$$

Proof. From the proof of Lemma 4.52, we see that only the forms of type $(k, 0, k-1)$ contribute to $\mathrm{sign}(B_1)$. We denote η_i instead of ξ_{n+i} for $i = 1, 2, \ldots, n$. By our assumption on N we have $d\xi_i = -\eta_i \wedge \zeta$. For $I, J \subset \{1, 2, \ldots, n\}$ with $|I| = k, |J| = k-1$ the form $*d(\xi_I \wedge \eta_J)$ is a linear combination of $\xi_{I'} \wedge \eta_{J'}$ with $I' = I^c \cup \{i\}, J' = J^c \setminus \{i\}$ for some $i \in I \setminus J$. Note that $I' \cap J' = (I \cup J)^c$. The operator $(B_1)^2$ preserves $I \cap J$ of $d(\xi_I \wedge \eta_J)$'s, while B_1 interchanges $I \cap J$ and $(I \cup J)^c$. Hence only the forms $d(\xi_I \wedge \eta_J) \in d\mathcal{M}^{2k-1}$ with $I \cap J = \emptyset$ contribute to $\mathrm{sign}(B_1)$.

Set $\omega_I = \xi_I \wedge \eta_{I^c}$ for $I \subset \{1, 2, \ldots, n\}$ with $|I| = k$. By an elementary calculation we have

$$*d\omega_I = \sum_{i \in I} (-1)^i \omega_{I^c \cup \{i\}}.$$

§5 Calculations in Finite Monodromy Cases.

Let $f : X \longrightarrow D$ be a degeneration of PPAV of complex dimension $n = 2k-1$ with a monodromy T of order $l < \infty$, and let $g : Y \longrightarrow S^1$ be the boundary of X. As we see in (3.30), we obtain a principally polarized Abelian variety A and a holomorphic automorphism γ of A of order l which preserves the principal polarization such that there exists a diffeomorphism

$$Y \simeq G_l \backslash W \tag{5.1}$$

where

$$W = A \times S^1 \tag{5.2}$$

and G_l is a cyclic group of order l generated by the automorphism

$$\tilde{\gamma}(\vec{z}, u) = (\gamma(\vec{z}), e_l u). \tag{5.3}$$

Recall that $W = A \times S^1$ has a natural framing coming from the framings on both factors such that the metric induced by the framing coincides with the product of the metric on A induced by the polarization and the natural metric on S^1. Though $\tilde{\gamma}$ does not preserve the framing, it preserves the metric of W and hence this metric induces a natural metric on $Y = G_l \backslash W$.

Moreover the action of G_l on W can be naturally extended to $A \times D$ and X is diffeomorphic to a Hironaka resolution of the quotient variety $G_l \backslash (A \times D)$. Then the product metric on $A \times D$ induces a metric on X whose restriction on the boundary Y coincides with the above metric and the metric on X is product near the boundary. Therefore we can define the signature defect $\delta(X, Y)$ as in (2.5). We will fix these metric for X and Y from now on.

In this section, we will calculate the G-equivariant version of the eta functions $\eta_{\tilde{\gamma}^j}(B_W, s)$ for the isometry $\tilde{\gamma}^j$, and then by (2.6) the eta function $\eta(B_Y, s)$ of Y can be expressed as

$$\eta(B_Y, s) = \frac{1}{l}\sum_{j=0}^{l-1} \eta_{\tilde{\gamma}^j}(B_W, s). \tag{5.4}$$

Note that $\eta(B_W, s) \equiv 0$ since there is an orientation-reversing isometry on W. Therefore the sum in (5.4) can be taken only over non-trivial elements of G_l.

(5.5) Results.

By proposition 3.33 the framing of A gives a coordinate $(z_1, z_2, \cdots, z_n)$ of universal covering $\mathbb{C}^n$ of A and the metric on A is induced by the natural Euclidean metric on $\mathbb{C}^n$. Moreover the action of γ becomes an isometry of A. Therefore, after a suitable isometric coordinate change, one can see that the action of γ can be written in the diagonal form

$$\gamma((z_1, z_2, \cdots, z_n)) = (e_l^{a_1} z_1, e_l^{a_2} z_2, \cdots, e_l^{a_n} z_n), \tag{5.6}$$

where $e_l = \exp(2\pi i/l)$, a primitive l-th-root of the unity and $a_i \in \mathbb{Z}, 0 \leq a_i \leq l-1$ for $1 \leq i \leq n$. Now the main theorems of this section can be stated as follows.

Theorem 5.7. *Let $Y \longrightarrow S^1$ and $\gamma \in \mathrm{Aut}(A)$ be as above. Fix an integer j such that $1 \leq j \leq l-1$. Then with respect to the metric on $W = A \times S^1$ as above, the G-equivariant eta function of $\tilde{\gamma}^j$ defined in (2.19) is given by*

$$\eta_{\tilde{\gamma}^j}(B_W, s) = (2\pi)^{-s}(-1)^k \cdot 2^{n+1} \cdot Z(\frac{2\pi j}{l}, s) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l}) \tag{5.8}$$

where $Z(q, s)$ denotes the following Dirichlet series:

$$Z(q, s) := \sum_{m=1}^{\infty} \frac{\sin(mq)}{m^s}. \tag{5.9}$$

Therefore the eta function of Y with respect to the metric is given by

$$\eta(B_Y, s) = (2\pi)^{-s}(-1)^k \cdot 2^{n+1} \cdot (\frac{1}{l}\sum_{j=1}^{l-1} Z(\frac{2\pi m j}{l}, s) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l})). \tag{5.10}$$

Since it is easy to see that

$$Z(q, 0) = (1/2)\cot(\frac{q}{2}),$$

we have the following corollary which gives the signature defect of Y.

Corollary 5.11. *Under the notation and assumption in Theorem 5.7, we have*

$$\eta_{\tilde{\gamma}^j}(B_W,0) = (-1)^k \cdot 2^n \cdot \cot(\frac{\pi j}{l}) \cdot \prod_{i=1}^{n} \sin(\frac{2\pi j a_i}{l}). \tag{5.12}$$

Therefore the signature defect of $Y = G_l \backslash W \subset X$ *is given by*

$$\delta(X,Y) = \eta(B_Y,0) = (-1)^k \cdot 2^n \cdot (\frac{1}{l}\sum_{j=1}^{l-1} \cot(\frac{\pi j}{l}) \cdot \prod_{i=1}^{2k-1} \sin(\frac{2\pi j a_i}{l})) \tag{5.13}$$

Remark 5.14. Donnelly [D2] obtained the special value $\eta_{\tilde{\gamma}^j}(B_W,0)$ by using the index theorem (Theorem 2.32) and calculating the right hand side of (2.32). It is easy to see that $\mathrm{sign}(\tilde{\gamma}^j, A\times D, W) = 0$ for all $\tilde{\gamma}^j$ and so it suffices to determine the first part of the right hand side of (2.32). Moreover he showed that if γ^j has 1 as its eigenvalue the integrand of the first term of (2.32) is zero hence $\eta_{\tilde{\gamma}^j}(B_W,0) = 0$. (Note that in this case we obtain a slightly stronger result $\eta_{\tilde{\gamma}^j}(B_W,s) \equiv 0$ from (5.8).)

Otherwise γ^j has only isolated fixed points and the contribution from the fixed points can be easily calculated. In fact Donnelly obtained the following formula (cf. [Proposition 4.7, D2]):

$$\eta_{\tilde{\gamma}^j}(B_W,0) = \nu(\gamma^j)(-1)^k \cot(\frac{\pi j}{l}) \prod_{i=1}^{2k-1} \cot(\frac{\pi j a_i}{l}), \tag{5.15}$$

where $\nu(\gamma^j)$ denotes the number of the isolated fixed points of γ^j on A. From this and (5.12) we obtain the equality

$$\nu(\gamma^j) = 2^{2k-1} \prod_{i=1}^{2k-1} \frac{\sin(\frac{2\pi j a_i}{l})}{\cot(\frac{\pi j a_i}{l})}, \tag{5.16}$$

which gives the number of fixed points of γ^j. We remark that the formula (5.16) can be explained as follows. If we regard γ^j as an automorphism of the lattice L it can be represented by the integral matrix of size $4k-2$ whose eigenvalues are given by $\exp(\pm\frac{2\pi j a_i}{l})$ for $1 \le j \le 2k-1$. On the other hand one can easily show that the number of fixed points of γ^j is given by the determinant

$$\nu(\gamma^j) = \det(\gamma^j - I_{4k-2}).$$

Thus we obtain the formula

$$\nu(\gamma^j) = \prod_{i=1}^{2k-1} (2-2\cos(\frac{2\pi j a_i}{l})) = \prod_{i=1}^{2k-1} (\exp(\frac{2\pi j a_i}{l})-1)(\exp(\frac{-2\pi j a_i}{l})-1), \tag{5.17}$$

which is now equivalent to (5.16).

(5.18) Proof of Theorem 5.7: A Reduction.

In this subsection and the next, we will give a proof of Theorem 5.7. First we will recall a basic Fourier analysis on $W = A \times S^1$.

We have fixed an isomorphism $A = L\backslash\mathbb{C}^n \simeq L\backslash\mathbb{R}^{2n}$, where L is regarded as a lattice in $\mathbb{R}^{2n}$ and γ is a linear isometry of $\mathbb{R}^{2n}$ such that $\gamma(L) = L$. Moreover as in (3.24) we will choose a basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_{2n}\}$ of L and use the real coordinate $(x_1, x_2, \cdots, x_{2n})$ for $\vec{x} = \sum_{i=1}^{2n} x_i \mathbf{e}_i$.

Let $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ be the dual lattice of L and $<,>: L^* \times L \to \mathbb{Z}$ the natural pairing. Then we have the decomposition of $L_2(A \times S^1)$

$$L_2(A \times S^1) = \oplus_{(\nu,m)\in L^*\times\mathbb{Z}} \mathbb{C} \cdot \mathbf{e}_{\nu,m}(\vec{x}, t), \tag{5.19}$$

where we set $\mathbf{e}_{\nu,m}(\vec{x}, t) = \exp(2\pi\sqrt{-1}(<\nu, \vec{x}> + mt))$. Let $\mathcal{M}^{ev}(W)$ denote the space of constant forms on W of even degrees, and let B_W be the operator on W defined in (2.15). Note that since the flat connection on W with respect to the framing is nothing but the exterior differential d on W we see that the operator B_W coincides with the operator D defined for the framed manifold W in (2.16).

We extend the domain of B_W to $L_2(W)^{ev} := \mathcal{M}^{ev}(W) \otimes L_2(W)$. Then according to the decomposition (5.19), we have a decomposition of $L_2(W)^{ev}$

$$L_2(W)^{ev} = \oplus_{(\nu,m)\in L^*\times\mathbb{Z}} \mathcal{H}_{\nu,m} \tag{5.20}$$

where $\mathcal{H}_{\nu,m} := \mathcal{M}^{ev}(W) \otimes \mathbf{e}_{\nu,m}(\vec{x}, t)$. Since this decomposition is invariant under translations, we can decompose B_W as

$$B_W = \oplus_{(\nu,m)\in L^*\times\mathbb{Z}} B_{\nu,m}, \tag{5.21}$$

where $B_{\nu,m}$ denotes the restriction of B_W to the space $\mathcal{H}_{\nu,m}$.

Lemma 5.22. *Let $\lambda(\nu, m)$ be a positive eigenvalue of the operator $B_{\nu,m}$. Then we have*

$$\lambda(\nu, m) = 2\pi\sqrt{||\nu||^2 + m^2}, \tag{5.23}$$

and the eigenvalues of $B_{\nu,m}$ are $\pm\lambda(\nu, m)$. We let $\mathcal{H}^+_{\nu,m}$ and $\mathcal{H}^-_{\nu,m}$ denote the eigenspace of $B_{\nu,m}$ with the eigenvalue $\lambda(\nu, m)$ and $-\lambda(\nu, m)$ respectively. Then we have the decomposition

$$\mathcal{H}_{\nu,m} = \mathcal{H}^+_{\nu,m} \oplus \mathcal{H}^-_{\nu,m}. \tag{5.24}$$

Proof. Since B_W^2 coincides with the usual Laplacian, $B_{\nu,m}^2$ coincides with the scalar operator $C_{\nu,m}$ where

$$C_{\nu,m} = 4\pi^2(||\nu||^2 + m^2). \tag{5.25}$$

Then since $B_{\nu,m}$ itself is not a scalar operator we obtain the assertions.

Let λ be an eigenvalue of B_W and let $\mathcal{H}(\lambda)$ denote the corresponding eigenspace. According to the decomposition (5.20), we have the decomposition

$$\mathcal{H}(\lambda) = \oplus_{(\nu,m)\in L^*\times\mathbb{Z}} \quad \mathcal{H}_{\nu,m}(\lambda), \tag{5.26}$$

where we set $\mathcal{H}_{\nu,m}(\lambda) = \mathcal{H}(\lambda) \cap \mathcal{H}_{\nu,m}$.

Now let us consider the trace of action of $\tilde{\gamma}^j$ on $\mathcal{H}(\lambda)$. Recall that we can write $\tilde{\gamma}^j = (\gamma^j, e_l^j)$ where $\gamma^j : A \longrightarrow A$ is a holomorphic automorphism. Moreover recalling that $A = L\backslash\mathbb{R}^{2n}$ we can regard γ^j as an automorphism of the $\mathbb{Z}$-lattice L. We denote by ${}^t\gamma^j$ the dual action of γ^j to L^*.

Lemma 5.27. *For an automorphism $\tilde{\gamma}^j = (\gamma^j, e_l^j)$, $0 \le j \le l-1$, let $(\tilde{\gamma}^j)^*$ denote the induced action of $\tilde{\gamma}^j$ on $L_2(W)^{ev}$. Then we have*

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}(\lambda)}) = \sum_{(\nu,m),\ \ \nu={}^t\gamma^j\nu} \mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}_{\nu,m}(\lambda)}). \tag{5.28}$$

Proof. This easily follows from the decomposition of (5.26) and the fact that $(\tilde{\gamma}^j)^*$ induces the isomorphism

$$(\tilde{\gamma}^j)^* : \mathcal{H}_{\nu,m}(\lambda) \longrightarrow \mathcal{H}_{{}^t\gamma^j\nu,m}(\lambda).$$

Let us fix a positive eigenvalue λ of B_W and consider the operator $B_{\nu,m}$ such that $\lambda = \lambda(\nu, m)$. Moreover assume that ${}^t\gamma^j\nu = \nu$.

Proposition 5.29. *Under the notation and the assumption as above, assume moreover that $\nu \neq 0$. Then we have*

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}_{\nu,m}(\lambda)}) = \mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}_{\nu,m}(-\lambda)}). \tag{5.30}$$

In particular we have

(5.31)

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}(\lambda)}) - \mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}(-\lambda)}) = \sum_{m\in\mathbb{Z}} (\mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}_{0,m}(\lambda)}) - \mathrm{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{H}_{0,m}(-\lambda)})).$$

Proof. Take (ν, m) such that $\lambda = \lambda(\nu, m)$. Then assuming that ${}^t\gamma^j\nu = \nu$ and $\nu \neq 0$, we claim the following.

Claim 5.32: There exists a $(\tilde{\gamma}^j)^*$-invariant linear subspace $V \subset \mathcal{H}_{\nu,m}$ such that

$$\mathcal{H}_{\nu,m} \simeq V \oplus B_{\nu,m}(V) \qquad \text{(a direct sum)}.$$

Here we can only show that this claim implies (5.30), and we prove the claim later. For a non-zero element $\phi \in V$, the elements

$$B_{\nu,m}\phi \pm \lambda\phi$$

become non-zero eigen vectors of $B_{\nu,m}$ with eigenvalues $\pm\lambda$ respectively. (Note that $B_{\nu,m}^2 = \lambda^2$.) On the other hand, the action of $(\tilde{\gamma}^j)^*$ commutes with $B_{\nu,m}$, hence the actions of $(\tilde{\gamma}^j)^*$ on these two eigenvectors are the same. Taking basis of V, we obtain basis for $\mathcal{H}_{\nu,m}(\lambda)$ and $\mathcal{H}_{\nu,m}(-\lambda)$ as in the above way and this proves (5.30).

From Proposition (5.29) and the definition of G-equivariant eta functions in (2.19), we have the following corollary.

Corollary 5.33. *Notation being as above, let B_0 be the restriction of the operator B_W on the subspace $L_2(W)_0^{ev} := \oplus_{m\in\mathbb{Z}}\mathcal{H}_{0,m} \subset L_2(W)^{ev}$. Then we have*

$$\eta_{\tilde{\gamma}^j}(B_W, s) = \eta_{\tilde{\gamma}^j}(B_0, s) \tag{5.34}$$

where $\eta_{\tilde{\gamma}^j}(B_0, s)$ is an eta function defined in a similar way as in (2.19) for the operator B_0.

Now we will prove the claim 5.32. Let $\mathcal{M}^p(W)$ (resp. $\mathcal{M}^p(A)$) denote the space of the complex valued constant p-forms on $W = A \times S^1$ (resp. on A).

For each $p, 0 \le p \le 2n$, we have the following natural isomorphism

$$\mathcal{M}^{2p}(W) \simeq \mathcal{M}^{2p}(A) \oplus \mathcal{M}^{2p-1}(A) \wedge dt, \tag{5.35}$$

which yields a linear isomorphism

$$\mathcal{M}^{ev}(W) \simeq \oplus_{p=0}^{4k-2}\mathcal{M}^p(A). \tag{5.36}$$

For $j, 0 \le j \le 4k-2$, we define subspaces of $\mathcal{M}^{ev}(W)$ by

$$\mathcal{M}^{>j} = \oplus_{p>j}\mathcal{M}^p(A), \qquad \mathcal{M}^{\ge j} = \oplus_{p\ge j}\mathcal{M}^p(A). \tag{5.37}$$

The subspaces $\mathcal{M}^{<j}$ and $\mathcal{M}^{\le j}$ can be defined similarly. Moreover we identify $\mathcal{H}_{\nu,m}$ with $\mathcal{M}^{ev}(W)\mathbf{e}_{\nu,m}(\vec{x}, t)$ and we will omit $\mathbf{e}_{\nu,m}(\vec{x}, t)$ unless it will cause confusions. From the definition of $B_{\nu,m}$, we obtain the following lemma.

Lemma 5.38. *Under the notation as above, we have*

$$B_{\nu,m}(\mathcal{M}^{<2k-1}) \subset \mathcal{M}^{\ge 2k-1}, \quad B_{\nu,m}(\mathcal{M}^{>2k-1}) \subset \mathcal{M}^{\le 2k-1}.$$

We set $\mathcal{M}^p = \mathcal{M}^p(A)$ for simplicity. Let $p_1 : \mathcal{M}^{\ge 2k-1} \to \mathcal{M}^{2k-1}$ and $p_2 : \mathcal{M}^{\le 2k-1} \to \mathcal{M}^{2k-1}$ be the natural projections. From this lemma, we can define subspaces V_1 and V_2 of $\mathcal{M}^{2k-1}$ by

$$V_1 = p_1 \circ B_{\nu,m}(\mathcal{M}^{<2k-1}) \subset \mathcal{M}^{2k-1},$$
$$V_2 = p_2 \circ B_{\nu,m}(\mathcal{M}^{>2k-1}) \subset \mathcal{M}^{2k-1}.$$

The following lemma shows the claim 5.32, hence completes the proof of Proposition 5.29.

Lemma 5.39. *Under the notation as above, set $V = \mathcal{M}^{<2k-1} \oplus V_2$. Then V is invariant under the action of $(\tilde{\gamma}^j)^*$. Assume moreover that $\nu \neq 0$. Then we have $B_{\nu,m}(V) = \mathcal{M}^{>2k-1} \oplus V_1$ and*

$$\mathcal{H}_{\nu,m} \simeq \mathcal{M}^{ev}(W) = V \oplus B_{\nu,m}(V).$$

Proof. The first assertion is obvious from definition. We define degrees of elements in $\mathcal{M}^{ev}$ by means of the decomposition (5.36). Take an element $\varphi \in \mathcal{M}^{2k-1}(A)$. Then we can obtain the element $\varphi \wedge dt \cdot \mathbf{e}_{\nu,m}(\vec{x}, t) \in \mathcal{M}^{ev}(W)\mathbf{e}_{\nu,m}(\vec{x}, t)$ of degree $2k-1$. We will omit $\mathbf{e}_{\nu,m}(\vec{x}, t)$ from now on. By an explicit calculation we can write

$$B_{\nu,m}(\varphi \wedge dt) = 2\pi\sqrt{-1}(\xi_{2k-2} + \xi_{2k-1} + \xi_{2k}) \tag{5.40}$$

where ξ_j denotes the elements of degree j and they are given as

$$\xi_{2k-2} = *_A(\tau_\nu \wedge \varphi), \quad \xi_{2k-1} = m *_A (\varphi) \wedge dt, \quad \xi_{2k} = \tau_\nu \wedge (*_A\varphi). \tag{5.41}$$

Here we set $\tau_\nu = \sum_{i=1}^{2n} \nu_i dx_i$, $\nu_i =< \nu, \mathbf{e}_i >$ and $*_A$ denotes the Hodge star operator on A.

Now we remark that if $\varphi \wedge dt \in V_2$ then:

1) the degree $(2k-2)$ part of $B_{\nu,m}(\varphi \wedge dt)$ vanishes and,

2) $B_{\nu,m}(\varphi \wedge dt) \in V_1 \oplus \mathcal{M}^{>2k-1}$.

In fact, from the definition of V_2, we can find an element $\phi \in \mathcal{M}^{<2k-1}$ such that $\varphi \wedge dt + \phi \in B_{\nu,m}(\mathcal{M}^{>2k-1})$. Then since $B^2_{\nu,m}$ is a scalar operator, $B_{\nu,m}(\varphi \wedge dt + \phi) \in \mathcal{M}^{>2k-1}$, while $B_{\nu,m}(\phi) \in B_{\nu,m}(\mathcal{M}^{<2k-1}) \subset \mathcal{M}^{\geq 2k-1}$ by lemma 5.38. Therefore the degree $2k-2$ part of $B_{\nu,m}(\varphi \wedge dt)$ must vanish and we have $B_{\nu,m}(\varphi \wedge dt) \equiv -B_{\nu,m}(\phi) \mod \mathcal{M}^{>2k-1}$ which implies the second assertion.

By the same argument, we see that if $\varphi \wedge dt \in V_1$ then the degree $2k$ part of $B_{\nu,m}(\varphi \wedge dt)$ vanishes and $B_{\nu,m}(\varphi \wedge dt) \in V_2 \oplus \mathcal{M}^{<2k-1}$.

From these remarks and (5.41) we can conclude that if $\varphi \wedge dt \in V_1 \cap V_2$ then $\tau_\nu \wedge \varphi = \tau_\nu \wedge *_A(\varphi) = 0$. On the other hand, since $\nu \neq 0$, τ_ν is a non-zero 1-form on A. Then we can easily show that these last two equations imply that $\varphi = 0$, and hence we have

$$(\mathcal{M}^{<2k-1} \oplus V_2) \cap (\mathcal{M}^{>2k-1} \oplus V_1) = (0).$$

Moreover from the second remark we can see that $B_{\nu,m}(V_2) \subset \mathcal{M}^{>2k-1} \oplus V_1$. This implies together with lemma 5.38 and the definition of V_1 that

$$B_{\nu,m}(\mathcal{M}^{<2k-1} \oplus V_2) \subset \mathcal{M}^{>2k-1} \oplus V_1.$$

Similarly we have

$$B_{\nu,m}(\mathcal{M}^{>2k-1} \oplus V_1) \subset \mathcal{M}^{<2k-1} \oplus V_2.$$

Then since $B^2_{\nu,m}$ is a scalar operator, two inclusions above must be equalities. Setting $V = M^{<2k-1} \oplus V_2$, we see that $B_{\nu,m}(V) = \mathcal{M}^{>2k-1} \oplus V_1$ as desired.

Now in order to finish the proof, we need to show that V and $B_{\nu,m}(V)$ generate $\mathcal{M}^{ev}(W)$. Note that all elements of $\mathcal{M}^{<2k-1} \oplus \mathcal{M}^{>2k-1}$ are in $V \oplus B_{\nu,m}(V)$. Hence let us take an element $\varphi \wedge dt$ of $\mathcal{M}^{2k-1}$. Applying $B_{\nu,m}$ to the equality (5.40) and using (5.41), we have

$$B^2_{\nu,m}(\varphi \wedge dt) = 2\pi\sqrt{-1}(B_{\nu,m}(\xi_{2k-2}) + B_{\nu,m}(\xi_{2k-1}) + B_{\nu,m}(\xi_{2k})).$$

Now let us set $2\pi\sqrt{-1}B_{\nu,m}(\xi_{2k-1}) = \phi_{2k-2} + \phi_{2k-1} + \phi_{2k}$ as in (5.40). Then an explicit calculation shows that $\phi_{2k-1} = 4\pi^2 m^2 \varphi \wedge dt$. Since $B^2_{\nu,m} = 4\pi^2(||\nu||^2 + m^2)$, we hereby obtain

$$4\pi^2||\nu||^2(\varphi \wedge dt) = (2\pi\sqrt{-1}B_{\nu,m}(\xi_{2k-2}) + \phi_{2k}) + (2\pi\sqrt{-1}B_{\nu,m}(\xi_{2k}) + \phi_{2k-2}).$$

Since again we have $||\nu||^2 \neq 0$, this implies that $\varphi \wedge dt \in V \oplus B_{\nu,m}(V)$, which completes the proof of Lemma 5.39.

(5.42) Proof of Theorem 5.7.

Now we can start to prove theorem 5.7. The main part of the proof is showing the formula (5.8), from which the rest of assertions easily follow.

Fix $j, 1 \leq j \leq l-1$. From corollary (5.33) we have the equality

$$\eta_{\tilde{\gamma}^j}(B_W, s) = \eta_{\tilde{\gamma}^j}(B_0, s) \tag{5.43}$$

where B_0 is the restriction of the signature operator B_W to the subspace $L_2^{ev}(0) = \mathcal{M}^{ev}(W) \otimes L_2(S^1)$. One can write B_0 as

$$B_0 := F \otimes \frac{\partial}{\partial t} \tag{5.44}$$

where F is a real endomorphism on $\mathcal{M}^{ev} := \mathcal{M}^{ev}(W)$. According to the Fourier decomposition $L_2^{ev}(0) = \oplus_{m\in\mathbb{Z}}\mathcal{H}_{0,m}$ as in (5.20) we obtain the decomposition $B_0 = \oplus_{m\in\mathbb{Z}}B_{0,m}$. Moreover one can easily see that $B_{0,m} = 2\pi\sqrt{-1}mF$. Moreover since $B^2_{0,m} = 4\pi^2m^2$, we have $F^2 = -1$. Then we have the eigenspace decomposition

$$\mathcal{M}^{ev} = \mathcal{M}^{ev}(\sqrt{-1}) \oplus \mathcal{M}^{ev}(-\sqrt{-1}),$$

such that $\overline{\mathcal{M}^{ev}(\pm\sqrt{-1})} = \mathcal{M}^{ev}(\mp\sqrt{-1})$, for F is a real endomorphism.

Since the action $(\tilde{\gamma}^j)^*$ on $\mathcal{M}^{ev}$ commutes with F, it preserves the eigen spaces of F, hence we can consider the action of $(\tilde{\gamma}^j)^*$ on the spaces $\mathcal{M}^{ev}(\pm\sqrt{-1})$.

Lemma (5.45). *Under the assumption as above, we have*
(5.46)
$$\eta_{\tilde\gamma^j}(B_W,s)=\eta_{\tilde\gamma^j}(B_0,s)=(-4)\cdot(2\pi)^{-s}\mathrm{Im}(\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})\cdot Z(\frac{2\pi j}{l},s),$$

where $Z(q,s)$ is a Dirichlet series defined in (5.9).

Proof. The set of all eigenvalues of B_0 is given by $\{2\pi m, m\in\mathbb{Z}\}$. Then denote by $H_{2\pi m}$ the eigenspace of B_0 with the eigenvalue $2\pi m$. It is easy to see that

$$H_{2\pi m}=\mathcal{M}^{ev}(-\sqrt{-1})\mathbf{e}(mt)\oplus\mathcal{M}^{ev}(\sqrt{-1})\mathbf{e}(-mt). \tag{5.47}$$

Now let us fix a positive integer m. From (5.47) and the definition of $\tilde\gamma^j$ in (5.6), we have

$$\begin{aligned}(\tilde\gamma^j)^*_{|H_{2\pi m}}&=\mathbf{e}(mj/l)\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})+\mathbf{e}(-mj/l)\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(\sqrt{-1})})\\(\tilde\gamma^j)^*_{|H_{-2\pi m}}&=\mathbf{e}(-mj/l)\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})+\mathbf{e}(mj/l)\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(\sqrt{-1})}).\end{aligned}$$

On the other hand, from the relation $\overline{\mathcal{M}^{ev}(-\sqrt{-1})}=\mathcal{M}^{ev}(\sqrt{-1})$, we obtain the equality
$$\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(\sqrt{-1})})=\overline{\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})}.$$
Therefore

$$\begin{aligned}&(\tilde\gamma^j)^*_{|H_{2\pi m}}-(\tilde\gamma^j)^*_{|H_{-2\pi m}}\\&\qquad=(\mathbf{e}(mj/l)-\mathbf{e}(-mj/l))(\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})-\overline{\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(\sqrt{-1})})})\\&\qquad=-4\sin(\frac{2\pi mj}{l})\cdot\mathrm{Im}(\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})).\end{aligned}$$

Now the formula (5.46) follows from the definition of eta function in (2.19) and (5.9).

(5.48) Calculation of the trace $\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})$.

Next we calculate the imaginary part of the trace $\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})$.

Lemma (5.49). *The action of $(\tilde\gamma^j)^*$ preserves the decomposition (5.36) of $\mathcal{M}^{ev}(W)$. Moreover the action of F can be given as follows.*

$$\begin{aligned}F_{|\mathcal{M}^{2p}(A)}&=(-1)^{k+p+1}\cdot *_A:\mathcal{M}^{2p}(A)\longrightarrow\mathcal{M}^{4k-2-2p}(A)\\F_{|\mathcal{M}^{2p-1}(A)\wedge dt}&=(-1)^{k+p+2}\cdot *_A:\mathcal{M}^{2p-1}(A)\wedge dt\longrightarrow\mathcal{M}^{4k-1-2p}(A)\wedge dt.\end{aligned}$$

Here again $*_A$ denotes the Hodge star operator on A with respect to the Euclidean metric.

Proof. The first assertion is trivial from the definition of the action of $\tilde\gamma^j$. In order to show the second assertion, write $\phi(t)\in\mathcal{M}^{2p}(W)\otimes C^\infty(S^1)$ as $\phi(t)=$

$f(t)\cdot\omega + g(t)\cdot\varphi\wedge dt$ where $\omega\in\mathcal{M}^{2p}(A)$ and $\varphi\in\mathcal{M}^{2p-1}(A)$. Since $B_W(\phi(t)) = B_0(\phi(t)) = (-1)^{k+p+1}(*_W d - d*_W)(\phi)$, we have

$$\begin{aligned} B_0(f(t)\cdot\omega) &= (-1)^{k+p+1}\partial_t(f(t)) *_W (dt\wedge\omega) \\ B_0(g(t)\cdot\varphi\wedge dt) &= -(-1)^{k+p+1}\partial_t(g(t))dt\wedge *_W(\varphi\wedge dt) \\ &= -(-1)^{k+p+1+4k-1-2p}\partial_t(g(t)) \\ &= (-1)^{k+p+1}\partial_t(g(t)) *_W (\varphi\wedge dt)\wedge dt \end{aligned}$$

One can show that $*_W(dt\wedge\omega) = *_A(\omega)$ and $*_W(\varphi\wedge dt)\wedge dt = (-1) *_A (\varphi)\wedge dt$. Therefore we have

$$B_0(\phi(t)) = (-1)^{k+p+1}(\frac{\partial f(t)}{\partial t}\cdot *_A(\omega) - \frac{\partial g(t)}{\partial t}(*_A\varphi)\wedge dt)$$

which proves the second assertion.

Consider the following decomposition (cf. (5.37)):

$$\mathcal{M}^{ev}(W) = \mathcal{M}^{<2k-1}\oplus\mathcal{M}^{2k-1}\oplus\mathcal{M}^{>2k-1}.$$

Set $\mathcal{M}^{\neq 2k-1} := \mathcal{M}^{<2k-1}\oplus\mathcal{M}^{>2k-1}$. Then from lemma 5.49 we see that the subspace $\mathcal{M}^{\neq 2k-1}$ and $\mathcal{M}^{2k-1}$ are stable under F and therefore we have the natural decomposition

$$\mathcal{M}^{ev}(-\sqrt{-1}) = \mathcal{M}^{\neq 2k-1}(-\sqrt{-1})\oplus\mathcal{M}^{2k-1}(-\sqrt{-1}). \tag{5.50}$$

(5.51) Lemma. *The trace of* $(\tilde\gamma^j)^*$ *on the subspace* $\mathcal{M}^{\neq 2k-1}(-\sqrt{-1})$ *is a real number. Therefore we have*

$$\mathrm{Im}(\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})}) = \mathrm{Im}(\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{2k-1}(A)(-\sqrt{-1})}). \tag{5.52}$$

Proof. Take a non-zero form $\omega\in\mathcal{M}^{<2k-1}$. Since $F^2 = -1$ and $F\omega\in\mathcal{M}^{>2k-1}(A)$, the element $\omega + \sqrt{-1}F\omega \in M^{\neq 2k-1}$ is a non-zero eigen vector of F with the eigenvalues $-\sqrt{-1}$, and it is easy to see that this map gives an isomorphism $\mathcal{M}^{<2k-1}\simeq\mathcal{M}^{\neq 2k-1}(-\sqrt{-1})$ which commutes with $(\tilde\gamma^j)^*$. Therefore we have

$$\mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{\neq 2k-1}(-\sqrt{-1})}) = \mathrm{Tr}((\tilde\gamma^j)^*_{|\mathcal{M}^{<2k-1}(A)}). \tag{5.53}$$

Recalling that the de Rham isomorphism $H^p(A,\mathbb{C})\simeq\mathcal{M}^p(A)\simeq\wedge^p\mathcal{M}^1(A)$, since the automorphism γ^j on A induced from a real endomorphism of $\mathcal{M}^1(A)$, we see that the right hand side of (5.53) is real and this implies that the assertion.

(5.54) Hodge decomposition of $\mathcal{M}^{2k-1}(A)$.

Let us recall the Hodge decomposition of $\mathcal{M}^1 = \mathcal{M}^1(A)$

$$\mathcal{M}^1 = \mathcal{M}^{1,0}(A) \oplus \mathcal{M}^{0,1}(A)$$

where $\mathcal{M}^{1,0}(A) = \oplus_{i=1}^{2k-1}\mathbb{C}dz_i$ and $\mathcal{M}^{0,1} = \overline{\mathcal{M}^{1,0}(A)}$. This induces the Hodge decomposition of $\mathcal{M}^{2k-1}(A)$ as

$$\mathcal{M}^{2k-1}(A) = \oplus_{a+b=2k-1}H^{a,b}$$

where $H^{a,2k-1-a}$ denote the space of constant forms of type (a,b). Set $I = \{1,2,\cdots,2k-1\}$. For multi-indices $K_1, K_2 \subset I$ such that $|K_1|+|K_2| = 2k-1$, we define $dz_{K_1} \wedge \overline{dz_{K_2}}$ as usual. For a multi-index $M \subset I$, we set

$$w_M = \prod_{i \in M} dz_i \wedge \overline{dz_i}.$$

Then if $M = K_1 \cap K_2$ is not empty, we can write

$$dz_{K_1} \wedge \overline{dz_{K_2}} = \pm dz_{K_1'} \wedge \overline{dz_{K_2'}} \wedge w_M.$$

such that K_1', K_2' and M are mutually disjoint increasing multi-indices.

We have the following lemma. (For a proof, see p.161, Lemma 1.2 of [W].)

Lemma(5.56). *Suppose that K_1, K_2 and M are mutually disjoint increasing multi-indices. Then*

$$*_A(dz_{K_1} \wedge \overline{dz_{K_2}} \wedge w_M) = \xi(a,b,m)(dz_{K_1} \wedge \overline{dz_{K_2}} \wedge w_{M'})$$

for a non-vanishing constant $\xi(a,b,m)$, where $a = |K_1|, b = |K_2|, m = |M|$, and $M' = I - (K_1 \cup K_2 \cup M)$. Moreover,

$$\xi(a,b,m) = (\sqrt{-1})^{a-b}(-1)^{p(p+1)/2+m}(-2\sqrt{-1})^{p-(2k-1)}$$

where $p = a+b+2m$ is the total degree of $dz_{K_1} \wedge \overline{dz_{K_2}} \wedge w_M$.

Let $P^{a,b} \subset H^{a,b}$ be the space of the form of type (a,b) generated by $dz_{K_1} \wedge \overline{dz_{K_2}}$ such that $K_1 \cap K_2$ is empty. We can naturally define its complement $Q^{a,b}$ in $H^{a,b}$,

$$H^{a,b} = P^{a,b} \oplus Q^{a,b}. \tag{5.57}$$

Setting

$$P^{2k-1} = \oplus_{a+b=2k-1}P^{a,b}, \quad Q^{2k-1} = \oplus_{a+b=2k-1}Q^{a,b},$$

we have the decomposition

$$\mathcal{M}^{2k-1}(A) = P^{2k-1} \oplus Q^{2k-1}.$$

Note that the action of F on $\mathcal{M}^{2k-1}(A)$ is nothing but that of $(-1)^{2k+2}*_A = *_A$ (cf. Lemma 5.37). All of the actions of F and γ^j respect the Hodge decomposition and the decomposition of (5.57). We can easily obtain the following lemma from Lemma 5.56.

Lemma (5.58). *For a non-zero element of $\phi \in P^{a,2k-1-a}$, we have*

$$F(\phi) = *_A(\phi) = (-1)^a\sqrt{-1}\phi.$$

Moreover for a non-zero element $\tau = dz_{K_1} \wedge \overline{dz_{K_2}} \wedge w_M \in Q^{a,2k-1-a}$, the forms $\tau \mp \sqrt{-1} *_A \tau$ become non-zero eigen vectors of $F = *_A$ with eigenvalues $\pm\sqrt{-1}$. In particular we have

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|Q^{2k-1}(-\sqrt{-1})}) \in \mathbb{R}.$$

Proof. The first and second assertions follow from Lemma 5.56. From the second assertion we see that the action of $(\tilde{\gamma}^j)^*$ on the spaces $Q^{a,n-a}(-\sqrt{-1})$ and $Q^{a,n-a}(\sqrt{-1})$ isomorphic to each other. Therefore we have

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|Q^{a,n-a}(-\sqrt{-1})}) = \mathrm{Tr}((\tilde{\gamma}^j)^*_{|Q^{a,n-a}(\sqrt{-1})}).$$

This implies that the traces of $(\tilde{\gamma}^j)^*$ on the spaces $Q^{2k-1}(\pm\sqrt{-1})$ are equal to each other. Then the relation $\overline{Q^{2k-1}(\sqrt{-1})} = Q^{2k-1}(-\sqrt{-1})$ shows the last assertion.

From this lemma, we can easily obtain the following corollary.

Corollary 5.59. *We have*

$$\mathrm{Im}(\mathrm{Tr}((\gamma^j)^*_{|\mathcal{M}^{2k-1}(-\sqrt{-1})})) = \mathrm{Im}(\mathrm{Tr}((\gamma^j)^*_{|P^{2k-1}(-\sqrt{-1})})),$$

and

$$P^{2k-1}(-\sqrt{-1}) = P^{2k-1,0} \oplus P^{2k-3,2} \oplus \cdots P^{1,2k-2}. \tag{5.60}$$

Now we calculate the trace of $(\tilde{\gamma}^j)^*$ on $P^{2k-1}(-\sqrt{-1})$. Let $K_1 = \{i_1, i_2, \cdots, i_a\}$, and $K_2 = \{j_1, j_2, \cdots, j_b\}$ the multi-indices then

$$(\tilde{\gamma}^j)^*(dz_{K_1} \wedge \overline{dz_{K_2}}) = (\xi_{i_1}\xi_{i_2}\cdots\xi_{i_a})\overline{(\xi_{j_1}\xi_{j_2}\cdots\xi_{j_b})}dz_{K_1} \wedge \overline{dz_{K_2}}$$

where we have set $\xi_i = (e_l)^{ja_i}$ (cf. (5.6)). Therefore if we set

$$\Phi_j(y) = \prod_{i=1}^{2k-1}(\xi_i + \overline{\xi_i}y) = \sum_{p=0}^{2k-1} c_j y^j,$$

we have

$$\begin{aligned}\mathrm{Tr}((\tilde{\gamma}^j)^*_{|P^{2k-1}(-\sqrt{-1})}) &= (\xi_1\xi_2\cdots\xi_{2k-1} + \sum \xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2k-3}}\overline{\xi_{j_1}\xi_{j_2}} + \cdots) \\ &= c_0 + c_2 + \cdots c_{2k-2}.\end{aligned}$$

Therefore we have

$$\mathrm{Tr}((\tilde{\gamma}^j)^*_{|P^{2k-1}(-\sqrt{-1})}) = (\Phi_j(1) + \Phi_j(-1))/2,$$

while we know that $\Phi_j(-1)$ is pure imaginary and $\Phi_j(1)$ is real. From this and corollary (5.59) we have

$$
\begin{aligned}
\operatorname{Im}(\operatorname{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{M}^{ev}(-\sqrt{-1})})) &= \operatorname{Im}(\operatorname{Tr}((\tilde{\gamma}^j)^*_{|\mathcal{M}^{2k-1}(-\sqrt{-1})}) \\
&= \operatorname{Im}(\operatorname{Tr}((\tilde{\gamma}^j)^*_{|P^{2k-1}(-\sqrt{-1})}) \\
&= \Phi_j(-1)/(2\sqrt{-1}) = (\prod_{i=1}^{2k-1}(\xi_i - \xi_i^{-1}))/(2\sqrt{-1}) \\
&= (\prod_{i=1}^{2k-1}(2\sqrt{-1}\sin(\frac{2\pi j a_i}{l})))/(2\sqrt{-1}) \\
&= (-1)^{k-1}2^{2k-2}\prod_{i=1}^{2k-1}\sin(\frac{2\pi j a_i}{l}). \qquad (5.61)
\end{aligned}
$$

Finally we can deduce the formula (5.8) from Lemma (5.45) and the equality (5.61).

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