SMOOTH FIXED-POINT FREE ACTIONS OF COMPACT LIE GROUPS ON DISKS

Robert Alan Oliver

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I. BACKGROUND AND SUMMARY

The question of whether compact groups can have actions on acyclic spaces, especially on disks and Euclidean spaces, without fixed points, has a long history. The Brauer and Lefschetz fixed-point theorems proved that no cyclic group has a fixed-point free action on a finite polyhedron which has no rational homology. P. A. Smith, in the 1930's, proved theorems about $\mathbb{Z}_p$-actions which show that groups of prime power order are unable to have fixed-point free actions, not only on acyclic spaces, such as disks, but also on non-compact spaces such as Euclidean space.

In the 1940's, Smith [19] stated the question as his "proposition [G,n]": does G have a fixed-point free action on the Euclidean space $\mathbb{R}^n$?, and proved some nonexistence results for small n. The question came up twice in Eilenberg's paper "On the Problems of Topology" [6]: posed by Smith for cyclic groups and by Montgomery for compact groups in general.

The first counterexamples to these questions came around 1960. Floyd and Richardson [9] constructed a fixed-point free action of the finite group A_5 on a disk; Conner and Floyd [4] constructed such actions of finite cyclic groups, not of prime power order, on non-compact contractible manifolds, actions which were modified by Kister [16] to give fixed-point free actions on Euclidean spaces. Later, work by Conner and Montgomery [5] and by Hsiang and Hsiang [13] showed that <u>any</u> non-Abelian connected Lie group has a smooth fixed-point free action on Euclidean space.

At about the same time as the first counterexamples were constructed, Greever [10] found a large family of finite solvable groups which could not have fixed-point free actions on integrally acyclic spaces when certain finiteness conditions held; in particular his results applied to smooth or simplicial actions on disks and other compact manifolds with the right homology, but not to actions on Euclidean spaces. This, and the large number of examples which began to hint that the only compact Lie groups with no fixed-point free actions on Euclidean spaces were tori extended by p-groups, showed that the problem, when limited to smooth or simplicial actions on compact acyclic manifolds, is far different and more complicated than the problem with actions on Euclidean spaces.

This thesis is centered around the question of which compact Lie groups have fixed-point free actions on (sufficiently high dimensional) disks or other compact acyclic manifolds. In the case of finite solvable groups, the question is answered completely for disks, or for compact R-acyclic manifolds, where $R = \mathbb{Z}$, $\mathbb{Z}_p$, or $\mathbb{Q}$.

The following collections of finite solvable groups are defined. Let $\mathcal{G}$ be the collection of all groups G, with a cyclic normal subgroup $\mathbb{Z}_n \triangleleft G$, such that $|G/\mathbb{Z}_n| = q^k$, some prime power. For p prime, let $\mathcal{G}_p$ be the collection of all groups G with a normal subgroup $P \triangleleft G$, $|P| = p^k$, and such that G/P is in $\mathcal{G}$. The following will be proven:

<u>Theorem</u>: Let G be a finite solvable group. G has a smooth fixed-point free action on a disk if and only if $G \notin \mathcal{G}_p$, for all p. G has a smooth fixed-point free action on some compact $\mathbb{Z}_p$-acyclic manifold if and only if $G \notin \mathcal{G}_p$. G has such an action on some compact $\mathbb{Q}$-acyclic manifold if and

only if $G \notin \mathcal{G}$. In particular, a finite abelian group has a smooth fixed-point free action on a disk if and only if it has three or more non-cyclic Sylow subgroups.

Thus, these conditions show that the smallest abelian group with a smooth fixed-point free action on a disk is $\mathbb{Z}_{30} \oplus \mathbb{Z}_{30}$, of order 900. The smallest solvable groups with such actions have order 72: two such groups are $S_3 \oplus A_4$ and $\mathbb{Z}_3 \oplus S_4$. The Floyd-Richardson example mentioned above then shows that A_5, of order 60, is the smallest finite group with such an action.

The necessity of the above conditions for the existence of actions is simply an extension of the methods of Greever, which involves checking the Euler characteristics possible for the fixed-point sets. The key extra step is an extension of a theorem of Floyd, showing that under certain finiteness conditions (including the case of a smooth action on a compact manifold) the Euler characteristic of the fixed-point set of a $\mathbb{Z}_n$-action is equal to the Lefschetz number of the action of a generator.

To show the existence of the actions described above, it is first shown that it suffices to construct simplicial fixed-point free actions on finite complexes of the right homotopy type: the method used is more complicated than that of Floyd and Richardson in [9] so as to apply also to positive-dimensional Lie groups. The problem is then reduced to one of constructing actions on $\mathbb{Z}_p$-acyclic complexes: if G has a fixed-point free action on such a space for all p, then some finite join of such spaces will be a contractible complex on which G acts, also without fixed points. Specific actions of certain groups are constructed, and then all other desired actions obtained by manipulating the fixed-point sets, or by passing to larger groups which contain the original groups as subgroups or quotient groups.

Results for finite non-solvable groups, or for positive-dimensional compact Lie groups, are more scattered. Smooth fixed-point free actions on disks have been constructed for some of the groups $PSL(2,q)$, including an example for A_5 independent of the Floyd-Richardson example. Similar examples have been found for two compact connected simple Lie groups, $SU(2)$ and $SU(3)$.

Chapter II is devoted to several general constructions and techniques which will be used to simplify the problem. The necessity of the conditions described above for a finite solvable group to have fixed-point free actions on certain spaces will be proven in Chapter III. In Chapter IV specific actions will be constructed, which will be used in Chapter V to prove the sufficiency of the above conditions. Results on non-solvable finite groups and positive-dimensional groups will be discussed in Chapter VI. All transformation groups considered will be compact Lie groups.

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II. GENERAL TECHNIQUES

The first step is to reduce the problem to one on finite complexes. The basic type of space used will be the finite equivariant CW complex (see, e.g. Illman [15]).

<u>Definition:</u> Let G be a compact Lie group. A zero-dimensional G-equivariant CW complex is a disjoint union of homogeneous G-spaces G/H_i. An n-dimensional G-equivariant CW complex is a space X, obtained from an (n-1)-dimensional G-equivariant CW complex Y, by attaching spaces $G/H_i \times D_i^n$ (where G acts trivially on the n-disk D_i^n), via equivariant maps $\varphi_i : G/H_i \times S_i^{n-1} \longrightarrow Y$.

When G is a finite group, this condition is equivalent to specifying that X have a CW complex structure, such that any $g \in G$ sends open n-cells to open n-cells, and sends an n-cell to itself only by the identity map. Note that for any compact Lie group G, any finite G-equivariant CW complex has finitely generated homology (because $(G/H \times D^n, G/H \times S^{n-1})$ does). It has been proven by Illman [15] that any smooth manifold with a smooth G-action has the structure of an equivariant CW complex; if the manifold is compact, it will be a finite complex. The following theorem makes it possible to go in the other direction, from finite complexes to smooth compact manifolds.

<u>Theorem 1.</u> Let K be a finite G-equivariant CW complex. Then there is a smooth, compact manifold M with a smooth G-action, and an equivariant embedding $i: K \longrightarrow M$ into its interior, such that $\pi_1(i)$ and $H_*(i; \mathbb{Z})$ are

isomorphisms. Furthermore, M can be chosen such that the isotropy subgroup of any point in M is contained in the isotropy subgroup of some point of K, and such that M^G is a regular neighborhood of K^G.

Thus, if K is simply connected, M will have the same homotopy type, and if the action on K is fixed-point free, the same will be true of the action on M.

The equivariant tubular neighborhood theorem (see, e.g. Bredon [2, p. 306]) shows that the boundary of any smooth G-manifold M has an invariant neighborhood equivariantly diffeomorphic to $\partial M \times [0,1)$ (with the fixed action on $[0,1)$). Thus, the corners which occur when taking the product of two manifolds, a disk bundle over a manifold, or a manifold with a handle attached, all can be smoothed equivariantly.

The following lemma will be needed to prove the theorem:

Lemma 1. Given smooth manifolds M^n and N^p, where M is compact and $p \geq 2n + 3$, and a continuous function $f_0: M \longrightarrow N$, there is a homotopy $F: M \times I \longrightarrow N$ of f_0, such that f_i is a smooth embedding for all $i > 0$.

Proof: Define $\tilde{F}: M \times (0,1] \longrightarrow N$, by $\tilde{F}(m,t) = f_0(m)$. Define the positive function $\delta: M \times (0,1] \longrightarrow \mathbb{R}$ by $\delta(m,t) = t$. Fix a metric on N.

$\operatorname{Dim}(N) \geq 2 \cdot \dim (M \times (0,1]) + 1$, so $\tilde{F}$ can be δ-approximated by a smooth one-to-one immersion $F': M \times (0,1] \longrightarrow N$. Set $f_t(m) = F'(m,t)$, then f_t is a smooth embedding $(t > 0)$ since M is compact. Set $F = f_0 \cup F': M \times [0,1] \longrightarrow N$. Then F is continuous and is the required homotopy. $||$

Theorem 1 will be proven by induction, starting with the subcomplex $K^0 \cup K^G$, where K^0 is the union of the cells $(G/H \times D^0)$. The theorem is true for this subcomplex: embed K^G in some regular neighborhood, and leave alone the components of K^0 not in K^G. Theorem 1 now follows from the following lemma:

Lemma 2. Assume K is a finite G-equivariant CW complex, with subcomplex L, where $K = L \cup_f (G/H \times D^n)$ for some equivariant $f: G/H \times S^{n-1} \longrightarrow L$, some $H \subsetneq G$. Assume M_0 is a compact manifold with smooth G action, with the embedding $i: L \longrightarrow M_0$ fulfilling the conclusion of the theorem. Then the theorem holds for K.

Proof: Let $j: S^{n-1} \longrightarrow G/H \times S^{n-1}$ be the inclusion map $j(x) = (eH, x)$. One may assume $\dim M_0^H \geq 2n+1$ (if not replace M_0 by $M_0 \times D^k \supseteq M_0$). Apply Lemma 1 to the map $ifj: S^{n-1} \longrightarrow M_0^H$, obtaining the map

$$\alpha: S^{n-1} \times I \longrightarrow \text{int } M_0^H$$

where $\alpha_0 = ifj$ and α_t is a smooth embedding for $t > 0$.

Choose D a disk with a linear action of G, such that some $x \in \partial D$ has isotropy subgroup H. Set $M_1 = M_0 \times D$, let $i_0: M_0 \longrightarrow M_1$ be the embedding of the zero section, and define $\hat{\alpha}: G/H \times S^{n-1} \times I \longrightarrow M_1$ by

$$\hat{\alpha}(gH, x, t) = (g \cdot \alpha(x, t), t \cdot gx).$$

Then $(\hat{\alpha} | G/H \times S^{n-1} \times 0) = i_0 if$, and the restriction of $\hat{\alpha}$ to $G/H \times S^{n-1} \times (0,1]$ is embedded in $M_1 - M_0$, with $G/H \times S^{n-1} \times 1$ the inverse image of ∂M_1, and smoothly embedded. Denote that embedding by $\beta: G/H \times S^{n-1} \longrightarrow \partial M_1$.

Let $i_1 : M_1 \longrightarrow W$ be a smooth, equivariant embedding of M_1 in a linear representation of G; let M_2 be an equivariant tubular neighborhood of M_1 in W. This induces a smooth embedding $\beta' = i_1\beta: G/H \times S^{n-1} \longrightarrow \partial M_2$, which restricts to $\beta' j: S^{n-1} \longrightarrow \partial M_2$.

As H-bundles:

$$W \times S^{n-1} = \tau_W | S^{n-1} = \tau(S^{n-1}) \oplus \mathbb{R} \times S^{n-1} \oplus \tau_{eH}(G/H) \times S^{n-1} \oplus \nu_{\partial M_2}(G/H \times S^{n-1})\big|_{S^{n-1}}$$

Set $V = \mathbb{R}^n \oplus \tau_{eH}(G/H)$ (an H-representation where $\mathbb{R}^n$ has the trivial action); then $W \times S^{n-1} = V \times S^{n-1} \oplus \nu_{\partial M_2}(G/H \times S^{n-1})\big|_{S^{n-1}}$.

Let $\widetilde{V}$ be a real G-representation whose restriction to H contains V as a direct summand: $\widetilde{V} = V \oplus V_1$ as H-representations. Set $M_3 = D(\widetilde{V}) \times M$; $i_2 : M_2 \longrightarrow M_3$ the embedding $x \longrightarrow (0, x)$.

As an H-bundle over S^{n-1},

$$\nu_{\partial M_3}(G/H \times S^{n-1})\big|_{S^{n-1}} = \nu_{\partial M_2}(G/H \times S^{n-1})\big|_{S^{n-1}} \oplus (V \oplus V_1) \times S^{n-1} = (W \oplus V_1) \times S^{n-1},$$

and so $\nu_{\partial M_3}(G/H \times S^{n-1}) \cong (G \times_H (W \oplus V_1)) \times S^{n-1}$. Let $D(W \oplus V_1)$ be the disk representation associated to $W \oplus V_1$, and attach $(G \times_H D(W \oplus V_1)) \times D^n$ via this isomorphism to ∂M_3 to get the smooth G-manifold M_4. The embeddings

$$G/H \times D^n \longrightarrow (G \times_H D(W \oplus V_1)) \times D^n \qquad \text{(zero-section)}$$

$$i_2 i_1 \hat{\alpha}: G/H \times S^{n-1} \times I \longrightarrow M_3$$

$$i_2 i_1 i_0 i : L \longrightarrow M_3$$

define an embedding of K into M_4. By the Van Kampen and Meyer Vietoris theorems, this embedding still induces an isomorphism of fundamental groups and integral homology.

The manifold M_3 is a vector bundle over M_0, and therefore any isotropy subgroup of M_3 is contained in one of M_0. The handle $G \times_H D(W \oplus V_1)) \times D^n$ is a bundle over $G/H \times D^n$, and so any of its isotropy subgroups is contained in a conjugate of H.

Since $G \neq H$, $(M_4)^G$ is a disk bundle over $(M_0)^G$, which was assumed to be a regular neighborhood of $L^G = K^G$, so $(M_4)^G$ is a regular neighborhood of K^G. ||

Corollary: Assume K a contractible finite G-equivariant CW complex. Then G has a smooth action on a disk, any of whose isotropy subgroups is contained in an isotropy subgroup of K.

Proof: By the theorem, G has a smooth action on some compact contractible manifold M_0, where all isotropy subgroups of M_0 are contained in isotropy subgroups of K. Embed M_0 smoothly in some linear representation of G; let M_1 be the disk bundle of an equivariant tubular neighborhood of M_0. By a theorem of Whitehead [22, p. 298], M_1 is a disk if M_0 was embedded with sufficiently high codimension. Isotropy subgroups of M_1 are contained inside those of M_0. ||

Thus, a compact Lie group G has a smooth fixed-point free action on a compact R-acyclic manifold, for a given ring R, if and only if G acts without fixed points on an R-acyclic finite G-equivariant CW complex; G has such an action on a disk if and only if it has a fixed-point free action on a contractible finite G-equivariant CW complex. The next proposition shows that it will suffice to consider the rings $\mathbb{Z}_p$, for prime p. First, two lemmas about joins of spaces will be needed.

Lemma 3. Assume X and Y are Hausdorff spaces. If X is $\mathbb{Z}_p$-acyclic, then so is $X*Y$. If X and Y are connected, then $X*Y$ is simply connected.

Proof: $X*Y = (X \times Y \times I)/\sim$, where $\sim$ is defined by

$$(x_1, y, 0) \sim (x_2, y, 0) \quad \text{for } x_i \in X,\ y \in Y$$

$$(x, y_1, 1) \sim (x, y_2, 1) \quad \text{for } x \in X,\ y_i \in Y$$

Let $U \subseteq X*Y$ be the image of $X \times Y \times [0,1)$, V the image of $X \times Y \times (0,1]$. Then $X \subseteq V$, $Y \subseteq U$, and $X \times Y \subseteq U \cap V$ (the level $\frac{1}{2}$) are deformation retracts. Thus, there is a Meyer-Vietoris exact sequence

$$\longrightarrow H_n(X \times Y;\mathbb{Z}_p) \longrightarrow H_n(X;\mathbb{Z}_p) \oplus H_n(Y;\mathbb{Z}_p) \longrightarrow H_n(X*Y;\mathbb{Z}_p) \longrightarrow$$

If X is $\mathbb{Z}_p$-acyclic, then $H_*(X \times Y;\mathbb{Z}_p) \longrightarrow H_*(Y;\mathbb{Z}_p)$ is an isomorphism, and $H_*(X*Y;\mathbb{Z}_p) \cong H_*(X;\mathbb{Z}_p)$.

If X and Y are connected, then so are U, V, and $U \cap V$. The Van Kampen theorem applies, and $\pi_1(X*Y) = 0$. ||

Lemma 4. Assume X and Y are finite G- and H-equivariant CW complexes, respectively. Then $X*Y$ is a finite $G \times H$-equivariant CW complex.

Proof: $X*Y$ can be constructed by attaching, to the disjoint union $X \cup Y$, one cell $(G \times H/G_0 \times H_0) \times (D^{m+n+1})$ for each pair $(G/G_0 \times D^m, H/H_0 \times D^n)$ of cells of X and Y. ||

Proposition 1. G has a fixed-point free action on a $\mathbb{Q}$-acyclic finite equivariant complex if and only if it has such an action on a $\mathbb{Z}_p$-acyclic complex for some prime p. G has a smooth fixed-point free action on a disk if and only if it has such an action on a $\mathbb{Z}_p$-acyclic complex for all primes p.

Proof: A space with finitely generated integral homology is $\mathbb{Q}$-acyclic if and only if it is $\mathbb{Z}_p$-acyclic for some prime p. If G has a smooth fixed-point free action on a disk, that action has the structure of a finite G-equivariant CW complex, which is $\mathbb{Z}_p$-acylic for all p.

Assume G acts without fixed point on the finite complexes $X_p : X_p$ Z_p-acyclic. Since X_2 has finitely generated integral homology, it has p-torsion for only a finite set $\mathscr{P}$ of primes. Set $X = X_2 * (\underset{p\in\mathscr{P}}{*} X_p)$ if $\mathscr{P} \neq \phi$; $X = X_2 * X_3$ if $\mathscr{P} = \phi$. Then X is $\mathbb{Z}$-acyclic and simply connected; thus contractible. If X is the join of n spaces, then it is a G^n-equivariant CW complex, with no isotropy subgroup containing the diagonal $G \subseteq G^n$. By the corollary, G^n has a smooth action on a disk, with no isotropy subgroup containing G, so restricting the action to G gives a fixed-point free smooth action of G on a disk. $||$

Notice that, if effective actions are desired, the spaces may be crossed by any effective action of the group on a disk, and the acyclic or contractibility conditions will still hold on the new complex. Thus, if $H \lhd G$ are compact Lie groups, the existence of a smooth fixed-point free action of G/H on a compact acyclic manifold or disk implies the existence of an effective fixed-point free action on the same type of space.

It is also sometimes possible to go from actions of a subgroup to actions of the whole group, as described in the following proposition.

Proposition 2. Assume $H \subseteq G$, as subgroup with finite index n. If H acts on X with fixed-point set F, then G has an action on X^n with fixed-point set F', the image of F under the diagonal map $\Delta: X \longrightarrow X^n$. If X is a smooth manifold, with the action of H smooth, then G acts smoothly on X^n.

Proof: Let G/H be the finite set of right cosets; choose some splitting map $t: G/H \longrightarrow G$ with $t(He) = e$. Define $p: G \longrightarrow H$ by $p(g) = g \cdot t(Hg)^{-1}$. The function p is continuous, and

$$p(h) = h \quad \text{for } h \in H$$

$$p(hg) = h \cdot p(g) \quad \text{for } h \in H, g \in G.$$

The space X^n can be described as $X^{G/H}$: the space of functions from $G/H \longrightarrow X$. Define the action $\pi: G \times X^{G/H} \longrightarrow X^{G/H}$ by

$$\pi(g, \xi)(Ha) = p(a)^{-1} p(ag) \cdot \xi(Ha\, g)$$

π is well-defined, since $p(ha)^{-1} p(ha\, g) = (h \cdot p(a))^{-1}(h \cdot p(ag))$. It is an action of G, since

$$\begin{aligned}\pi(g_1, \pi(g_2, \xi))(Ha) &= p(a)^{-1} p(ag_1) \cdot \pi(g_2, \xi)(Ha\, g_1)\\ &= p(a)^{-1} p(ag_1) \cdot p(ag_1)^{-1} p(ag_1 g_2) \xi(Ha\, g_1 g_2)\\ &= \pi(g_1 g_2, \xi)(Ha)\end{aligned}$$

The action is continuous; since for fixed g, the action on each coordinate is the action of some $h \in H$ (with the coordinates permuted), π is smooth if the original action was smooth.

Clearly, every point of $F' = \Delta(F)$ is fixed by π. For any $\xi \in X^n$, fixed by π: for any $a \in G$, $h \in H$,

$$\xi(Ha) = \pi(a^{-1}ha, \xi)(Ha) = p(a)^{-1}p(ha)\xi(Ha) = [p(a)^{-1} \cdot h \cdot p(a)]\xi(Ha)$$

so $\xi(Ha) \in F$ for all $Ha \in G/H$. Then

$$\xi(He) = \pi(a, \xi)(He) = \xi(Ha)$$

for all $a \in G$, and $\xi \in F'$. $||$

It should be noted that some of the constructions in this chapter would be much simpler if one were only interested in finite groups. Theorem 1 could be proven as done by Floyd and Richardson [9], by embedding the complex simplicially in a sphere and taking the second derived neighborhood, and then smoothing using the methods of Hirsch [12]. The join constructions are also less awkward when finite groups are involved.

III. GROUPS WITHOUT FIXED-POINT FREE ACTIONS

This chapter will be devoted to proving Theorem 1 stated below. Again, $\mathcal{G}$ is the collection of finite groups having a cyclic subgroup of prime power index, and $\mathcal{G}_p$ is the collection of finite groups G, with normal subgroup P of order p^n, such that $G/P \in \mathcal{G}$. The following notation will be used throughout this chapter:

Notation. An action of a group G on a space X will be said to have condition (F) if the Čech cohomology group $\check{H}^*(X^H; \mathbb{Z})$ of the fixed-point set of any subgroup $H \subseteq G$ is finitely generated.

Theorem 1. Assume the finite group G acts on the finitistic space X with condition (F). If $G \in \mathcal{G}$ and X is $\mathbb{Q}$-acyclic, then $X^G \neq \phi$. If $G \in \mathcal{G}_p$ and X is $\mathbb{Z}_p$-acyclic, then $X^G \neq \phi$. Here, the term finitistic is used as defined by Bredon [2, p. 133]: a paracompact Hausdorff space X is finitistic if every open covering of X has a finite-dimensional refinement. Note that a closed subspace of a finitistic space X is finitistic; also, if the finite group G acts on X, then X/G is still finitistic.

The following standard theorems will be used in this chapter (see, e.g., Bredon [2, Chapter III, §7]).

T 1. Assume $\mathbb{Z}_p$ acts on the finitistic space X, where $\check{H}^*(X; \mathbb{Z}_p)$ is finitely generated. Then $\check{H}^*(X^{\mathbb{Z}_p}; \mathbb{Z}_p)$ and $\check{H}^*(X/\mathbb{Z}_p; \mathbb{Z}_p)$ are finitely generated.

T 2. Assume $\mathbb{Z}_p$ acts on the finitistic $\mathbb{Z}_p$-acyclic space X

$\mathbb{Z}_p$-acyclic in terms of Čech cohomology). Then $X^{\mathbb{Z}_p}$ and $X/\mathbb{Z}_p$ are $\mathbb{Z}_p$-acyclic.

<u>T 3.</u> Assume $\mathbb{Z}_p$ acts on the finitistic space X, where $\check{H}^*(X;\mathbb{Z}_p)$ is finitely generated. Then the Euler characteristics $\chi(X)$, $\chi(X^{\mathbb{Z}_p})$ and $\chi(X/\mathbb{Z}_p)$ are well-defined in terms of $\mathbb{Z}_p$ Čech cohomology (by T 1), and

$$\chi(X/\mathbb{Z}_p) = \frac{1}{p}[\chi(X) + (p-1)\chi(X^{\mathbb{Z}_p})]$$

In particular, $\chi(X) \equiv \chi(X^{\mathbb{Z}_p}) \pmod{p}$.

The following lemma follows from T 1:

<u>Lemma 1.</u> Let G be a finite solvable group acting continuously on the space X, where $\check{H}^*(X;\mathbb{Z})$ is finitely generated. Then $\check{H}^*(X/G;\mathbb{Z})$ is finitely generated.

<u>Proof:</u> Since, for $K \lhd H \subseteq G$, $X/H = (X/K)/(H/K)$, it will suffice to prove the theorem for $G = \mathbb{Z}_p$, p prime. In this case, $\check{H}^*(X/G;\mathbb{Z}_p)$ is finitely generated, by T 1, and so (from the universal coefficient theorem) $\check{H}^*(X/G;\mathbb{Z}) \otimes \mathbb{Z}_p$ is a finitely generated group.

This group is the cokernel of the map

$$\cdot p\colon \check{H}^*(X/G;\mathbb{Z}) \longrightarrow \check{H}^*(X/G;\mathbb{Z})$$

which is multiplication by p. That map is equal to the composition

$$\check{H}^*(X/G;\mathbb{Z}) \longrightarrow \check{H}^*(X;\mathbb{Z}) \longrightarrow \check{H}^*(X/G;\mathbb{Z})$$

of the transfer map and the map induced by the projection. Thus $(\cdot p)$ has finitely generated image and cokernel, and $\check{H}^*(X/G;\mathbb{Z})$ is finitely generated. ||

During the rest of the chapter, all Euler characteristics used will be defined in terms of rational Čech cohomology. They will be used only on spaces with finitely generated integral cohomology, and so T 3 will still hold.

Theorem 1 will be proven by computing the Euler characteristics of fixed-point sets of the group action. The first step is to do this for cyclic group actions; the following lemma is needed.

Lemma 2. Assume $\mathbb{Z}_n$ acts linearly on the rational vector space $\mathbb{Q}^k$. If $s, t \in \mathbb{Z}_n$ generate the same subgroup, then their characters under the representation, $\chi(s)$ and $\chi(t)$, are equal.

Proof: Let $M \in GL(\mathbb{Q}, k)$ be the matrix determined by the action of $s \in \mathbb{Z}_n$; $t \equiv rs \pmod n$ for some r relatively prime to n, and M^r is the matrix corresponding to the action of $t \in \mathbb{Z}_n$. Let $P_M(x)$, $P_{M^r}(x)$ be the characteristic polynomials of these matrices; they are polynomials in $\mathbb{Q}[x]$ all of whose roots in $\mathbb{C}$ are n^{th} roots of unity. Furthermore, since M and M^r are diagonalizable over $\mathbb{C}$, the roots of $P_{M^r}(x)$ are the r^{th} powers of the roots of $P_M(x)$.

It is a standard result of Galois theory (see, e.g. Lang[17, p. 206]) that for any positive integer m, $\prod_{\xi \in S_m} (x-\xi)$, the product taken over the set S_m of all primitive m^{th} roots of unity, is an irreducible polynomial over $\mathbb{Q}$. Thus, for any $m|n$, the elements of S_m occur with equal multiplicity as roots of $P_M(x)$. The function $\xi \longrightarrow \xi^r$ permutes the elements of S_m, for any $m|n$, and so the characteristic polynomials $P_M(x)$ and $P_{M^r}(x)$ are equal. The trace of a matrix is determined by one coefficient in its characteristic polynomial, so $\mathrm{tr}(M) = \mathrm{tr}(M^r)$, and $\chi(s) = \chi(t)$. $\quad ||$

The following proposition was proven by Floyd [8] in the case when n is a prime power.

Proposition 1. Assume the cyclic group $\mathbb{Z}_n$ acts on the finitistic space X with condition (F). Let $g: X \longrightarrow X$ be the action of $1 \in \mathbb{Z}_n$. Then

$$\chi(X^{\mathbb{Z}_n}) = \Lambda(g)$$

where $\Lambda(g)$ is the Lefschetz number of g.

Proof: For some choice of basis of $\check{H}^k(X;\mathbb{Q})$, let M_k denote the matrix of g^*. Then

$$\dim \check{H}^k(X/\mathbb{Z}_n;\mathbb{Q}) = \dim\,[\check{H}^k(X;\mathbb{Q})^{\mathbb{Z}_n}] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{tr}(M_k^i)$$

(since for V a rational representation of a finite group G, $V^G \otimes \mathbb{C} = (V \otimes \mathbb{C})^G$). Thus, taking the alternating sum over integers k,

$$\chi(X/\mathbb{Z}_n) = \frac{1}{n}\sum_{i=1}^{n} \Lambda(g^i).$$

Collecting terms on the right which are equal under Lemma 2, one gets

$$\chi(X/\mathbb{Z}_n) = \frac{1}{n}\sum_{d|n} \varphi\left(\frac{n}{d}\right) \Lambda(g^d)$$

where $\varphi(k)$ is the Euler φ-function. Now, by induction on n, it will suffice to prove

$$\chi(X/\mathbb{Z}_n) = \frac{1}{n}\sum_{d|n} \varphi\left(\frac{n}{d}\right) \chi(X^{\mathbb{Z}_{n/d}})$$

or $$\chi(X/\mathbb{Z}_n) = \frac{1}{n} \sum_{d|n} \varphi(d)\, \chi(X^{\mathbb{Z}_d}). \tag{1}$$

If $n = p^\alpha$, a prime power, then the induced action of $\mathbb{Z}_p = (\mathbb{Z}_{p^\alpha})/\mathbb{Z}_{p^{\alpha-1}})$ on $X/\mathbb{Z}_{p^{\alpha-1}}$ has fixed-point set $X^{\mathbb{Z}_{p^\alpha}}$. T 3 applies ($\check{H}^*(X/\mathbb{Z}_{p^{\alpha-1}};\mathbb{Z}_p)$ is finitely generated by T 1), and

$$p^\alpha \chi(X/\mathbb{Z}_{p^\alpha}) = [p^{\alpha-1}\chi(X/\mathbb{Z}_{p^{\alpha-1}}) + \varphi(p^\alpha)\cdot\chi(X^{\mathbb{Z}_{p^\alpha}})].$$

Iterating this, one gets

$$p^\alpha \chi(X/\mathbb{Z}_{p^\alpha}) = \sum_{i=0}^{\alpha} \varphi(p^i)\, \chi(X^{\mathbb{Z}_{p^i}}),$$

so (1) is proven in this case.

It will now suffice to show that (1) holds for $n = ab$, when it is assumed to hold for the relatively prime integers a and b. Note that, if $(|G|, |H|) = 1$ and $G \times H$ acts on the space Y, then $(Y^G)/H = (Y/H)^G$ (any subgroup is of the form $K \times L$). Then

$$\begin{aligned}
\chi(X/\mathbb{Z}_n) &= \frac{1}{a} \sum_{e|a} \varphi(e)\, \chi\left((X/\mathbb{Z}_b)^{\mathbb{Z}_e}\right) \\
&= \frac{1}{a} \sum_{e|a} \varphi(e)\, \chi(X^{\mathbb{Z}_e}/\mathbb{Z}_b) \\
&= \frac{1}{a} \sum_{e|a} \varphi(e)\cdot\left[\frac{1}{b} \sum_{f|b} \varphi(f)\, \chi(X^{\mathbb{Z}_e \times \mathbb{Z}_f})\right] \\
&= \frac{1}{n} \sum_{ef|ab} \varphi(ef)\, \chi(X^{\mathbb{Z}_{ef}}) = \frac{1}{n} \sum_{d|n} \varphi(d)\, \chi(X^{\mathbb{Z}_d}). \qquad ||
\end{aligned}$$

<u>Corollary</u>. Assume $\mathbb{Z}_n$ acts on the finitistic $\mathbb{Q}$-acyclic space X with condition (F). Then $\chi(X^{\mathbb{Z}_n}) = 1$. ||

Proof of Theorem. If $G \in \mathcal{G}$, then there is $\mathbb{Z}_n \lhd G$ with $|G/\mathbb{Z}_n| = q^k$, q a prime. If G acts on the finitistic $\mathbb{Q}$-acyclic space X with condition (F), then so does $\mathbb{Z}_n$, and $\chi(X^{\mathbb{Z}_n}) = 1$. $G/\mathbb{Z}_n$ acts on $X^{\mathbb{Z}_n}$ with fixed-point set X^G, and by T 3, $\chi(X^G) \equiv \chi(X^{\mathbb{Z}_n}) = 1 \pmod q$. Thus, $X^G \neq \phi$.

If $G \in \mathcal{G}_p$, then there is $P \lhd G$ with $G/P \in \mathcal{G}$. If G acts on the finitistic $\mathbb{Z}_p$-acyclic space X with condition (F), then X^P is $\mathbb{Z}_p$-acyclic by T 2. Now G/P acts on the $\mathbb{Z}_p$-acyclic (and thus $\mathbb{Q}$-acyclic) space X^P with condition (F), and with fixed-point set X^G. Thus, $X^G \neq \phi$. ||

It should be noted that the restrictions that X be finitistic and the action have condition (F) include the case of a simplicial action of a finite group on a finite simplicial complex; in particular Theorem 1 applies to the case of a smooth action on a compact manifold. Thus, the following corollary, which is half of Theorem 1 of Chapter I, follows:

Corollary. If $G \in \mathcal{G}_p$, G has no smooth fixed-point free action on any compact $\mathbb{Z}_p$-acyclic manifold; in particular, G has no fixed-point free smooth action on any disk. If $G \in \mathcal{G}$, then G has no smooth fixed-point free action on any compact $\mathbb{Q}$-acyclic manifold. ||

IV. EXPLICIT CONSTRUCTIONS OF ACTIONS

In this chapter, fixed-point free actions of specific groups G on $\mathbb{Z}_t$-acyclic complexes will be constructed, for various pairs (G,t). All transformation groups considered will be finite, and all actions will have the structure of finite equivariant CW complexes. The results of this chapter will be summarized in Chapter V, and used to prove the other half of the theorem stated in Chapter I.

First some specific constructions will be made.

Proposition 1. Let G be a finite group, with subgroup $\mathbb{Z}_p^n \triangleleft G$ of index q (p,q distinct primes). Let $\psi: \mathbb{Z}_q \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_p^n)$ be the action defined by conjugation in G; assume $\mathbb{Z}_q$ acts freely on $\mathcal{H} = \{H \subseteq \mathbb{Z}_p^n : |H| = p^{n-1}\}$. Then G has a fixed-point free action on a $\mathbb{Z}_t$-acyclic space for any prime $t \neq p$.

Proposition 2. $\mathbb{Z}_p^2$ acts on some $\mathbb{Z}_t$-acyclic space with exactly $(p+1)$ fixed-points, for any prime $t \neq p$.

Proof (of both propositions): Let $h = |\mathcal{H}|$ (where $\mathcal{H} = \{H \subseteq \mathbb{Z}_p^2 : |H| = p\}$ in the case of Proposition 2). Give D^2 a cellular structure with one 2-cell, h 1-cells $\{a_1, \ldots, a_h\}$ and h vertices $\{v_1, \ldots, v_h\}$. Choose $f: \{a_1, \ldots, a_h\} \longrightarrow \mathcal{H}$ some bijection.

Define $X = (\mathbb{Z}_p^n \times D^2)/\sim$, with the action of $\mathbb{Z}_p^n$ induced by multiplication on the first factor, and the relation $\sim$ defined by:

$$(g_1, v_i) \sim (g_2, v_i) \quad \text{for} \quad g_1, g_2 \in \mathbb{Z}_p^n, \qquad i=1,\ldots,h$$

$$(g,x) \sim (gh,x) \quad \text{for} \quad g \in \mathbb{Z}_p^n,\ x \in a_i,\ h \in H_i = f(a_i)$$

For $H_i \in \mathcal{N}$, X/H_i consists of p copies of D^2, glued together along $\bigcup_{j \neq i} a_j$; it is thus contractible. It follows that

$$\tilde{H}_*(X;F)^{H_i} \cong \tilde{H}_*(X/H_i;F) = 0$$

for any field F of characteristic $t \neq p$. Assume that F is a field of characteristic t containing the p^{th} roots of unity; then representations of $\mathbb{Z}_p^n$ over F split as sums of one-dimensional representations, each of which fixes some $H \in \mathcal{N}$. Thus, $\tilde{H}_*(X;F)^H = 0$ for all $H \in \mathcal{N}$ implies that $\tilde{H}_*(X;F) = 0$, so $\tilde{H}_*(X;\mathbb{Z}_t) = 0$.

Thus, X is $\mathbb{Z}_t$-acyclic for all primes $t \neq p$. In the case $n=2$, $\mathbb{Z}_p^2$ fixes exactly $h=p+1$ points of X (the vertices $\{v_i\}$), this proves Proposition 2.

To prove Proposition 1, note that q divides h (since $\mathbb{Z}_q$ acts freely on $\mathcal{N}$). Fix $\mathbb{Z}_q \subseteq G$; give D^2 a $\mathbb{Z}_q$-action which permutes the 1-cells freely, and now require that $f\colon \{a_i\} \longrightarrow \mathcal{N}$ be $\mathbb{Z}_q$-equivariant. Define an action of $\mathbb{Z}_q$ on X by

$$\alpha(g,x) = (\alpha g \alpha^{-1}, \alpha x) \qquad \text{for } \alpha \in \mathbb{Z}_q,\ g \in \mathbb{Z}_p^n,\ x \in D^2.$$

The action is clearly well-defined on the vertices and the interiors of the 2-cells of X, and is well-defined on the 1-cells since:

$$\alpha(gh,x) = ((\alpha g \alpha^{-1})(\alpha h \alpha^{-1}), \alpha x) \sim (\alpha g \alpha^{-1}, \alpha x) = \alpha(g,x)$$

for $\alpha \in \mathbb{Z}_q$, $g \in \mathbb{Z}_p^n$, $x \in a_i$, $h \in H_i$ (so $\alpha h \alpha^{-1} \in \alpha H_i$).

Furthermore, for $\alpha \in \mathbb{Z}_q$, $h \in \mathbb{Z}_p^n$, $(g,x) \in X$:

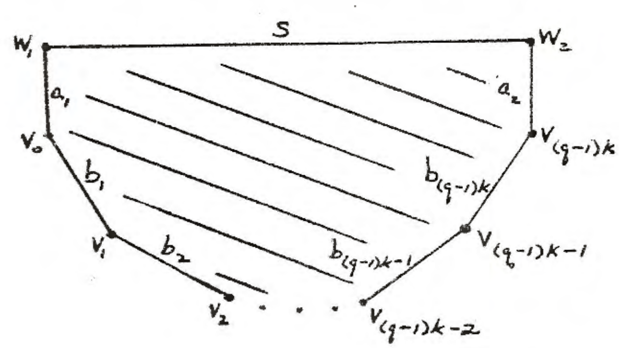
$$\alpha(h(\alpha^{-1}(g,x))) = \alpha(h(\alpha^{-1}g\alpha, \alpha^{-1}x)) = \alpha(h\alpha^{-1}g\alpha, \alpha^{-1}x)$$

$$= (\alpha h \alpha^{-1} g, x) = (\alpha h \alpha^{-1})(g,x)$$

and so the actions of $\mathbb{Z}_p^n$ and $\mathbb{Z}_q$ combine to give a well-defined action of G on X. The action is fixed-point free by construction, and X has been shown to be $\mathbb{Z}_t$-acyclic for all primes $t \neq p$. $||$

Proposition 3. For p,q distinct primes, assume $\mathbb{Z}_p^2 \lhd G$ with index q. Let $\psi: \mathbb{Z}_q \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_p^2)$ be the action defined by conjugation in G; assume $\mathbb{Z}_q$ acts on the set $\mathcal{X} = \{H \subseteq \mathbb{Z}_p^2 : |H| = p\}$ fixing exactly two elements. Then for any $t \neq p$, G has a fixed-point free action on a $\mathbb{Z}_t$-acyclic space.

Proof: Let $k = \frac{p-1}{q}$, the number of free orbits in the action of $\mathbb{Z}_q$ on $\mathcal{X}$. Define the space $Y \cong D^2$ with the following cellular structure:



Y has vertices $\{w_1, w_2, v_0, v_1, \ldots, v_{(q-1)k}\}$, 1-cells $\{s, a_1, a_2, b_1, \ldots, b_{(q-1)k}\}$ and one 2-cell.

Denote by H_1, H_2 the elements of $\mathcal{X}$ fixed by $\mathbb{Z}_q$. Define a one-

-one map

$$f\{b_1, b_2, \ldots, b_{(q-1)k}\} \longrightarrow \mathcal{H} - \{H_1, H_2\}$$

whose image contains exactly (q-1) subgroups from each free orbit of $\mathbb{Z}_q$.

Define $Y' = (\mathbb{Z}_q \times Y)/\sim$, where

$$(g_1, y) \sim (g_2, y) \qquad \text{for all } g_1, g_2 \in \mathbb{Z}_q,\ y \in s.$$

Left multiplication in $\mathbb{Z}_q$ induces an action of $\mathbb{Z}_q$ on Y'. Let $p: Y' \longrightarrow Y$ be the orbit map; let $A_i = p^{-1}(a_i \cup w_i)$, $V = p^{-1}(\{v_0, \ldots, v_{(q-1)k}\})$, $B = p^{-1}(\{b_1, \ldots, b_{(q-1)k}\})$. Define a $\mathbb{Z}_q$-equivariant map $f': B \longrightarrow \mathcal{H} - \{H_1, H_2\}$ which upon restriction to the elements of B on one leaf of Y', is equal to f.

Define $X = (\mathbb{Z}_p^2 \times Y')/\sim$, where the relation is given by:

$$(g_1, y) \sim (g_2, y) \qquad \text{if } g_1, g_2 \in \mathbb{Z}_p^2,\ y \in V$$

$$(g, y) \sim (gh, y) \qquad \text{if } g \in \mathbb{Z}_p^2, h \in H_i, y \in A_i \quad (i=1,2)$$

$$(g, y) \sim (gh, y) \qquad \text{if } g \in \mathbb{Z}_p^2, y \in b \in B,\ h \in f'(b)$$

Actions of $\mathbb{Z}_p^2$ and $\mathbb{Z}_q$ on X are induced by

$$h(g, y) = (hg, y) \qquad \text{for } h \in \mathbb{Z}_p^2,\ (g, y) \in \mathbb{Z}_p^2 \times Y'$$

$$k(g, y) = (\psi k(g), ky) \qquad \text{for } k \in \mathbb{Z}_q,\ (g, y) \in \mathbb{Z}_p^2 \times Y'$$

These actions are well-defined under the relation $(\sim)$. Furthermore, fixing $\mathbb{Z}_q \subseteq G$, these two actions induce a well-defined fixed-point free action of G on X. Calculations exactly parallel those in Proposition 1.

As in Proposition 1, to prove that $\widetilde{H}_*(X; \mathbb{Z}_t) = 0$ for any prime $t \neq p$, it will suffice to prove that X/H is contractible, for any $H \in \mathcal{H}$.

In all of these cases, this orbit space consists of p copies of Y' identified along a contractible subspace. The proposition follows. $||$

Let $\mathbb{Z}_{p,q}(q \mid p-1)$ be the non-abelian group of order pq $(p,q$ prime). The fourth space needed will be a $\mathbb{Z}_t$-acyclic space, for $t \neq p$, with a $\mathbb{Z}_{p,q}$ action having exactly $q+1$ fixed points. The space will be constructed by exhibiting a connected one-dimensional complex Y (with the $q+1$ fixed points), and showing that $H_1(Y;\mathbb{Z}_t)$, with the induced $\mathbb{Z}_{p,q}$ action, is a free $\mathbb{Z}_t[\mathbb{Z}_{p,q}]$-module. Then 2-cells can be added to produce a $\mathbb{Z}_t$-acyclic space with $\mathbb{Z}_{p,q}$ action.

In order to do this, it will be necessary to study some modular representations of $\mathbb{Z}_{p,q}$. In general, for $H \subseteq G$ finite groups, $\mathbb{Z}_t[G/H]$ will be used to designate the $\mathbb{Z}_t$ vector space with basis the elements of G/H; the action of G on G/H makes this into a $\mathbb{Z}_t[G]$ module. The natural projection $\mathbb{Z}_t[G/H] \longrightarrow \mathbb{Z}_t[G/K]$, for $H \subseteq K \subseteq G$, will be denoted $p(H,K)$. When $t \nmid |K/H|$, the transfer map $\mathbb{Z}_t[G/K] \longrightarrow \mathbb{Z}_t[G/H]$ will be denoted $t(K,H)$, and when convenient $\mathbb{Z}_t[G/K]$ will be identified with its image in $\mathbb{Z}_t[G/H]$.

The first two lemmas about projective $\mathbb{Z}_t[G]$-modules follow from standard theorems about modular representations and characters. (See, e.g. Serre [18, §14.3, 18.1, 18.2]).

Lemma 1. Let M_1 and M_2 be finitely generated projective $\mathbb{Z}_t[G]$-modules. Assume that for at least one cyclic subgroup $\mathbb{Z}_n \subseteq G$ from every conjugacy class such that $t \nmid n$, $M_1 \cong M_2$ as $\mathbb{Z}_t[\mathbb{Z}_n]$-modules. Then

$M_1 \cong M_2$ as $\mathbb{Z}_t[G]$-modules.

Lemma 2. Let M_1, M_2, M_3 be finitely generated projective $\mathbb{Z}_t[G]$-modules. If $M_1 \oplus M_2 \cong M_1 \oplus M_3$, then $M_2 \cong M_3$.

Proof: From the theory of modular characters, $\chi_{M_i}(g)$, defined for $g \in G$ of order prime to t, depends only on the action of g on M_i; then, if $\langle g \rangle$ is the cyclic group generated by g, $\chi_{M_1}(g) = \chi_{M_2}(g)$ if the modules are isomorphic as $\mathbb{Z}_t[\langle g \rangle]$ modules. Characters are constant on conjugacy classes, so the hypothesis of Lemma 1 implies that $\chi_{M_1}(g) = \chi_{M_2}(g)$ for all g of order prime to t. Since the M_i are projective, $M_1 \cong M_2$.

Lemma 2 results from the fact that finitely generated projective $\mathbb{Z}_t[G]$-modules split uniquely into sums of indecomposable projectives. (This second lemma will not be used until later in the chapter). ||

Lemma 3. Let $0 \to H \to G \to \mathbb{Z}_t \to 0$ be a short exact sequence of finite groups, where $t \nmid |H|$. For M a $\mathbb{Z}_t[G]$-module, if M is projective as a $\mathbb{Z}_t[\mathbb{Z}_t]$-module, for some fixed $\mathbb{Z}_t \subseteq G$, then M is a projective $\mathbb{Z}_t[G]$-module.

Proof: For L some free $\mathbb{Z}_t[G]$-module, there is a $\mathbb{Z}_t[G]$-linear surjection $p: L \to M$. M is projective as a $\mathbb{Z}_t[\mathbb{Z}_t]$-module, so there is a map $f_0: M \to L$ linear over this subring, such that $pf_0 = 1$. Set

$$f(m) = \frac{1}{|H|} \sum_{h \in H} h \cdot f_0(h^{-1}m).$$

Then for $h_0 \in H$: $f(h_0 m) = \frac{1}{|H|} \sum_{h \in H} h \cdot f_0((h_0^{-1}h)^{-1}m) = h_0 : f(m)$. For $g \in \mathbb{Z}_t$,

$$f(gm) = \frac{1}{|H|} \sum_{h \in H} h \cdot f_0(h^{-1}gm) = \frac{1}{|H|} \sum_{h \in H} g \cdot (g^{-1}hg) \cdot f_0((g^{-1}hg)^{-1}m)$$

$$= g \cdot f(m).$$

So f is $\mathbb{Z}_t[G]$-linear, $pf = 1$, and it follows that M is projective. $||$

<u>Lemma 4</u>. M, N R-modules, $p: M \oplus N \longrightarrow N$ an R-linear surjection, with $p|N$ an isomorphism. Then $\operatorname{Ker} p \cong M$.

<u>Proof:</u> The composite $\operatorname{Ker} p \longrightarrow M \oplus N \xrightarrow{\text{proj}} M$ is an isomorphism. $||$

<u>Lemma 5</u>. Let $G = \mathbb{Z}_{p,q}$, fixing some subgroup $\mathbb{Z}_q \subseteq G$. Let

$M = \operatorname{Ker} p(1, \mathbb{Z}_p): \mathbb{Z}_t[G] \longrightarrow \mathbb{Z}_t[G/\mathbb{Z}_p]$.

$N = \operatorname{Ker} p(\mathbb{Z}_q, G): \mathbb{Z}_t[G/\mathbb{Z}_q] \longrightarrow \mathbb{Z}_t$

where t is some prime not equal to p.

Then M and N are projective $\mathbb{Z}_t[G]$-modules, and $M \cong N^q$.

<u>Proof:</u> $\mathbb{Z}_q$ acts freely on G and $G/\mathbb{Z}_p$ (under left translation) and so, as $\mathbb{Z}_t[\mathbb{Z}_q]$-modules, the first projection becomes

$$p(1, \mathbb{Z}_p) = \overset{p}{\oplus} \mathrm{Id}: \overset{p}{\oplus} \mathbb{Z}_t[\mathbb{Z}_q] \longrightarrow \mathbb{Z}_t[\mathbb{Z}_q]$$

and by Lemma 4, $M \cong \overset{p-1}{\oplus} \mathbb{Z}_t[\mathbb{Z}_q]$. $\mathbb{Z}_q$ acts on $G/\mathbb{Z}_q$ fixing one coset and acting freely on the others, and so as $\mathbb{Z}_t[\mathbb{Z}_q]$-modules, the second projection becomes

$$p(\mathbb{Z}_q, G) = \mathrm{Id} \oplus \Big(\overset{\underline{p-1}}{\oplus^q} p(1, \mathbb{Z}_q) \Big) : \mathbb{Z}_t \oplus \Big(\overset{\underline{p-1}}{\oplus^q} \mathbb{Z}_t[\mathbb{Z}_q] \Big) \longrightarrow \mathbb{Z}_t$$

with kernel $N \cong \overset{\frac{p-1}{q}}{\oplus} \mathbb{Z}_t[\mathbb{Z}_q]$ by Lemma 4.

Thus, M and N are both free as $\mathbb{Z}_t[\mathbb{Z}_q]$-modules, so if $t = q$, M and N are projective $\mathbb{Z}_t[G]$-modules, by Lemma 3. If $t \neq q$, then $t \nmid |G|$, $\mathbb{Z}_t[G]$ is semi-simple, and so M and N are trivially projective, with $M \cong N^q$ as $\mathbb{Z}_t[\mathbb{Z}_q]$-modules.

Upon restriction to $\mathbb{Z}_t[\mathbb{Z}_p]$, the second projection becomes

$$p(1, \mathbb{Z}_p): \mathbb{Z}_t[\mathbb{Z}_p] \longrightarrow \mathbb{Z}_t$$

The first projection becomes

$$\overset{q}{\oplus} p(1, \mathbb{Z}_p): \overset{q}{\oplus} \mathbb{Z}_t[\mathbb{Z}_p] \longrightarrow (\mathbb{Z}_t)^q$$

Thus, $M \cong N^q$ as $\mathbb{Z}_t[\mathbb{Z}_p]$-modules, and by Lemma 1, $M \cong N^q$ as $\mathbb{Z}_t[G]$-modules. ||

Proposition 4. Let $G = \mathbb{Z}_{p,q}$, t a prime not equal to p. Then $\mathbb{Z}_{p,q}$ has an action on a $\mathbb{Z}_t$-acyclic space with exactly (q+1) fixed points.

Proof: Given S^1 a cellular structure with vertices $\{v_0, \ldots, v_q\}$ and 1-cells $\{a_0, \ldots, a_q\}$. Let $Y = (G \times S^1)/\sim$, with the G action induced by left multiplication in the first factor, where the relation $\sim$ is given by:

$$(g_1, v_i) \sim (g_2, v_i) \text{ for all } g_1, g_2 \in G, \quad i=0, \ldots, q$$

$$(g, x) \sim (gh, x) \text{ for } x \in a_0, \ g \in G, \ h \in \mathbb{Z}_p \triangleleft G$$

$$(g, x) \sim (gh, x) \text{ for } x \in a_i \ (i=1, \ldots, q), \ g \in G, \ h \in \mathbb{Z}_q$$

$$\text{for some fixed } \mathbb{Z}_q \subseteq G.$$

Y is connected, so it remains to calculate $H_1(Y;\mathbb{Z}_t)$ as a G-representation. Let $K_* = \mathrm{Ker}\,(C_*(Y;\mathbb{Z}_t) \longrightarrow C_*(Y/\mathbb{Z}_p;\mathbb{Z}_t))$, since $t \nmid p$, the transfer map splits $C_*(Y;\mathbb{Z}_t)$ as the chain complex direct sum

$$C_*(Y;\mathbb{Z}_t) \cong C_*(Y/\mathbb{Z}_p;\mathbb{Z}_t) \oplus K_*$$

and so in homology

$$H_*(Y;\mathbb{Z}_t) \cong H_*(Y/\mathbb{Z}_p;\mathbb{Z}_t) \oplus H(K_*).$$

Since $\mathbb{Z}_p \lhd G$, $Y/\mathbb{Z}_p$ has an induced G action; the projection and transfer maps are equivariant, so the direct sum decompositions hold as G-representations.

The space $Y/\mathbb{Z}_p$ has the homotopy type of a fixed point with q loops attached, where $\mathbb{Z}_q = G/\mathbb{Z}_p$ permutes the loops freely. Thus, $H_1(Y/\mathbb{Z}_p;\mathbb{Z}_t) \cong \mathbb{Z}_t[G/\mathbb{Z}_p]$.

$H_0(K_*) = 0$, so $0 \longrightarrow H_1(K_*) \longrightarrow K_1 \longrightarrow K_0 \longrightarrow 0$ is exact.

$K_1 = \mathrm{Ker}\,(\mathbb{Z}_t[G/\mathbb{Z}_p] \longrightarrow \mathbb{Z}_t[G/\mathbb{Z}_p]) \oplus \left(\mathrm{Ker}\,(\mathbb{Z}_t[G/\mathbb{Z}_q] \longrightarrow \mathbb{Z}_q)\right)^q = N^q \cong M.$

$K_0 = 0$, so $H_1(K_*) \cong M$ (as in Lemma 5), and

$$H_1(Y;\mathbb{Z}_t) \cong \mathbb{Z}_t[G/\mathbb{Z}_p] \oplus M \cong \mathbb{Z}_t[G].$$

The composite $\pi_1(Y) \longrightarrow H_1(Y;\mathbb{Z}) \longrightarrow H_1(Y;\mathbb{Z}_t)$ is surjective, so let $\varphi: S^1 \longrightarrow Y \in \pi_1(Y)$ be in the inverse image of a generator of $\mathbb{Z}_t[G] \cong H_1(Y;\mathbb{Z}_t)$. Let $g\varphi$ be the composite with the action of g on Y; then $\{g\varphi : g \in G\}$ generates, under its image in $H_1(Y;\mathbb{Z}_t)$, a basis as a $\mathbb{Z}_t$-vector space, and the space $X = Y \cup_{g\varphi} (G \times D^2)$ is a $\mathbb{Z}_t$-acyclic G-space, still with $\{v_i\}$ the only fixed points. $||$

The next step is to show how the spaces constructed in Propositions 2 and 4 can be combined to give fixed-point free actions of larger groups (note that, by the results of Chapter III, neither $\mathbb{Z}_p^2$ nor $\mathbb{Z}_{p,q}$ have themselves fixed-point free actions on $\mathbb{Q}$-acyclic spaces). The technique will be to show that, under the right conditions, certain fixed-point sets may be removed.

<u>Lemma 6</u>. Assume G acts on the space X with fixed-point set F. Assume G also acts on some space E, and that there is an equivariant map $\varphi: E \longrightarrow F$ (i.e., φ factors through E/G) which is an R-homology equivalence (some ring R). Then there is a G-space Y, with fixed-point set homeomorphic to E^G, such that $H_*(Y;R) \cong H_*(X;R)$.

<u>Proof:</u> Let $i: X \longrightarrow M^n$ be an embedding of X into a smooth compact manifold, given by Theorem 1 in Chapter II ($X \longrightarrow M$ and $F \longrightarrow M^G$ are homology equivalences). Then $i\varphi: E \longrightarrow M^G$ is an equivariant R-homology equivalence. Let ξ be the equivariant normal bundle to M^G in M, and $\eta = (i\varphi)^* \xi$ its pullback, as a G-equivariant bundle over E. Letting D and S denote the associated disk and sphere bundles, respectively, one has

$$i\varphi: (D\eta, S\eta) \longrightarrow (D\xi, S\xi)$$

as an R-homology equivalence. Let $Y = D\eta \bigcup_{i\varphi | S\eta} \overline{(M-D\xi)}$. By the equivariant skeletal approximation theorem (see Illman [15]), the attaching map may be assumed to preserve skeleta, for some equivariant CW structure on $\overline{(M-D\xi)}$, and so Y is a finite G-equivariant CW complex. The induced map $\psi: Y \to M$ is an R-homology equivalence. Clearly $Y^G \cong E^G$. $||$

Lemma 7. Assume the finite group G has a normal subgroup H of index q, where $\frac{1}{q} \in R$. If G acts on the R-acyclic space X with $X^G \cong S^1$, then G acts on some R-acyclic space Y with $Y^G = \phi$.

Proof: Let $E = S^1$ with the G-action induced by a free $\mathbb{Z}_q$-action. Then $E/G \cong S^1$, and the resulting equivariant map $E \to S^1 \cong X^G$ is an R-homology equivalence. By Lemma 6, G acts on some R-acyclic space Y with $Y^G \cong E^G = \phi$. $||$

Denote by A_n a discrete set of n points, $B_n \cong \Sigma A_r$ the (unreduced) suspension. $A_m \cup B_n$ will refer to the disjoint union of the two spaces.

Lemma 8. G, R as above. Assume G acts on the R-acyclic space X with $X^G \cong A_m \cup B_n$ $(n \geq 2, m \geq 1)$. Then G acts on some R-acyclic space Y with $Y^G \cong A_{n-2} \cup B_m$.

Proof: G acts on ΣX with fixed-point set $F \cong B_m \underset{\psi}{\cup} (A_n * S^1)$, where for b_1, b_2 the suspension points of B_m, $\psi: \{b_1, b_2\} \to A_n * S^1$ is some map. Define $\psi': \{b_1, b_2\} \to A_n * S^1$ which sends the b_i to two distinct points of $A_n \subseteq A_n * S^1$. Thus $F' = B_m \underset{\psi'}{\cup} (A_n * S^1)$ has the homotopy type of F. The action of G on S^1 induced by the free action of $\mathbb{Z}_q \cong G/H$ induces an action of G on F', and an R-homology equivalence $F' \to F'/G \cong F' \simeq F$. Lemma 6 applies, and G has an action on some R-acyclic space Y, with

$F^G \cong (F')^G \cong A_{n-2} \cup B_m$. $\|$

Proposition 5. Assume G is a finite group with a normal subgroup H of index q, where $\frac{1}{q} \in R$. If G acts on the R-acyclic spaces X and Y with $(1+m)$ and $(1+n)$ fixed points, respectively, where $(m,n) = 1$, then G has a fixed-point free action on some R-acyclic space.

Proof: Choose positive integers a and b such that $bn-am = 1$. Let $\hat{X} = \overset{a}{\vee} X$, $\hat{Y} = \overset{b}{\vee} Y$: the one-point unions (at a fixed-point) of a and b copies of X, Y respectively. G acts on $\hat{X}$ and $\hat{Y}$ with $(1+am)$ and $(1+bn)$ fixed-points.

The one-point union $\hat{Y} \vee \Sigma\hat{X}$ is R-acyclic and has fixed-point set homeomorphic to $A_{bn} \cup B_{bn}$. If bn is odd, repeated application of Lemma 8 produces a G-space with fixed-point set $A_1 \cup B_1$, whose suspension has fixed-point set homotopically equivalent to S^1. If bn is even, the same produces a G-space with fixed-point set $B_2 \cong S^1$. In either case, Lemma 7 applies to produce an R-acyclic space with a fixed-point free G-action. $\|$

Proposition 5 can be immediately applied to the spaces constructed in Propositions 2 and 3 to give a wide variety of fixed-point free actions.

Corollary. For distinct primes p, q, t, $\mathbb{Z}_p^2 \oplus \mathbb{Z}_q^2$ acts on a $\mathbb{Z}_t$-acyclic space without fixed point.

Proof: $\mathbb{Z}_p^2$, $\mathbb{Z}_q^2$ act on $\mathbb{Z}_t$-acyclic spaces with $(p+1), (q+1)$ fixed points. $\|$

Corollary. For distinct primes p, q, r, where $r | q-1$, and $t \neq p, q$, $\mathbb{Z}_p^2 \oplus \mathbb{Z}_{q,r}$ acts on a $\mathbb{Z}_t$-acyclic space without fixed-point. $||$

Corollary. For distinct primes p, q, r, s, $q | p-1$, $s | r-1$, and $t \neq p, r$, $\mathbb{Z}_{p,q} \oplus \mathbb{Z}_{r,s}$ acts on a $\mathbb{Z}_t$-acyclic space without fixed point. $||$

For $n | p-1$, where p is prime but n need not be, let $\mathbb{Z}_{p,n}$ denote the extension of $\mathbb{Z}_p$ by $\mathbb{Z}_n$ under a monomorphism $\mathbb{Z}_n \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_p)$.

Corollary. For distinct primes p, q, r, where $qr | p-1$, and $t \neq p$, $\mathbb{Z}_{p,qr}$ acts without fixed points on a $\mathbb{Z}_t$-acyclic space. $||$

Proof: One may assume $t \neq q$ without loss of generality; then $\mathbb{Z}_{p,r} \lhd \mathbb{Z}_{p,qr}$ has index q. By Proposition 4, the subgroups $\mathbb{Z}_{p,q}$ and $\mathbb{Z}_{p,r}$ induce actions of $\mathbb{Z}_{p,qr}$ on $\mathbb{Z}_t$-acyclic spaces with $(q+1), (r+1)$ fixed points, so Proposition 4 applies. $||$

For $q | p-1$, let $G(p^2, q)$ be the extension of $\mathbb{Z}_p^2$ by $\mathbb{Z}_q$ defined by a monomorphism $\mathbb{Z}_q \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_p^2)$ into the center of $\mathrm{Aut}\,(\mathbb{Z}_p^2)$.

Corollary. For $q | p-1$, $t \neq p, q$, $G(p^2, q)$ has a fixed-point free action on a $\mathbb{Z}_t$-acyclic space. $||$

Proof: $G(p^2, q)$ has subgroups $\mathbb{Z}_p^2$ and $\mathbb{Z}_{p,q}$. $||$

The explicit construction of one further space with action will be needed in the following chapter. For p, q primes with $q | p-1$, let $G = G(p^2, q)$, the group of order $p^2 q$ as defined above; a fixed-point free action of G on

$\mathbb{Z}_q$-acyclic space will be constructed.

Let K be the normal subgroup of G of order p^2, $K \cong \mathbb{Z}_p^2$. Let $\mathcal{K} = \{H \subseteq K: |H| = p\}$. For $H \in \mathcal{K}$, set

$$M_H = \text{Ker } p(H,K): \mathbb{Z}_q[G/H] \longrightarrow \mathbb{Z}_q[G/K]$$

$$N_H = \text{Ker } p(H\mathbb{Z}_q, G): \mathbb{Z}_q[G/H\mathbb{Z}_q] \longrightarrow \mathbb{Z}_q$$

M_H and N_H are induced from the modules M and N of Lemma 5 (for $t = q$) under the homomorphism $G \longrightarrow G/H \cong \mathbb{Z}_{p,q}$. Thus, as $\mathbb{Z}_q[G]$-modules, M_H and N_H are projective ($\mathbb{Z}_q[G/H]$ is a direct summand of $\mathbb{Z}_q[G]$), and $M_H \cong N_H^q$.

The following lemma for decomposing $\mathbb{Z}_q[K]$ will be needed.

Lemma 9. For $H \in \mathcal{K}$, let $N'_H = \text{Ker}(\mathbb{Z}_q[K/H] \longrightarrow \mathbb{Z}_q)$. Then $N'_H \subseteq \mathbb{Z}_q[K/H] \subseteq \mathbb{Z}_q[K]$ under the transfer map, $\mathbb{Z}_q \subseteq \mathbb{Z}_q[K]$ under the transfer map $t(K,0)$, and

$$\mathbb{Z}_q[K] = \mathbb{Z}_q + \bigoplus_{H \in \mathcal{K}} N'_H.$$

Proof: For any $\mathbb{Z}_q[K]$-linear map $\varphi: \mathbb{Z}_q[K/H_1] \longrightarrow \mathbb{Z}_q[K/H_2]$, when $H_1 \neq H_2$, any $x \in \text{Im}\varphi$ is fixed by the actions of both H_1 and H_2; thus $\text{Im}\varphi \subseteq \mathbb{Z}_q[K/H_2]^K = \mathbb{Z}_q$. Since $N'_{H_2} \subseteq \mathbb{Z}_q[K/H_2]$, and $N'_{H_2} \cap \mathbb{Z}_q = 0$, one has

$$\text{Hom}_{\mathbb{Z}_q[K]}(\mathbb{Z}_q[K/H_1], N'_{H_2}) = 0, \quad \text{or}$$

$$\text{Hom}_{\mathbb{Z}_q[K]}(N'_{H_1}, N'_{H_2}) = \text{Hom}_{\mathbb{Z}_q[K]}(\mathbb{Z}_q, N'_{H_2}) = 0$$

Since $(N'_H)^K = 0$ for all H, $\text{Hom}_{\mathbb{Z}_q[K]}(N'_H, \mathbb{Z}_q) = 0$.

Since each of the submodules $\mathbb{Z}_q$ or N'_H is a direct summand of $\mathbb{Z}_q[K]$ $(q \nmid |K|)$, and the corresponding projections

$$\Sigma N'_H \longrightarrow \mathbb{Z}_q, \quad \mathbb{Z}_q \oplus \sum_{\substack{H \neq H_0 \\ H \in \mathcal{K}}} N'_H \longrightarrow N'_{H_0}$$

are zero, the submodules must be linearly independent. Counting dimensions (over the field $\mathbb{Z}_q$)

$$\dim(\mathbb{Z}_q) + \sum_{H \in \mathcal{K}} \dim N'_H = 1 + (p+1)(p-1) = p^2 = \dim \mathbb{Z}_q[K]$$

and so $\mathbb{Z}_q[K] = \mathbb{Z}_q + \bigoplus_{H \in \mathcal{K}} N'_H$. $||$

In a similar manner, one has the inclusions $\mathbb{Z}_q, N_H \subseteq \mathbb{Z}_q[G/\mathbb{Z}_q]$ and $\mathbb{Z}_q[G/K], M_H \subseteq \mathbb{Z}_q[G]$.

Regarded as representations of K, the first inclusions are isomorphic to those $\mathbb{Z}_q, N'_H \subseteq \mathbb{Z}_q[K]$, and the inclusions in $\mathbb{Z}_q[G]$ decompose into a direct sum of q copies of it. Thus

$$\mathbb{Z}_q[G/\mathbb{Z}_q] = \mathbb{Z}_q + \bigoplus_{H \in \mathcal{K}} N_H \quad \text{and}$$

$$\mathbb{Z}_q[G] = \mathbb{Z}_q[G/K] + \bigoplus_{H \in \mathcal{K}} M_H.$$

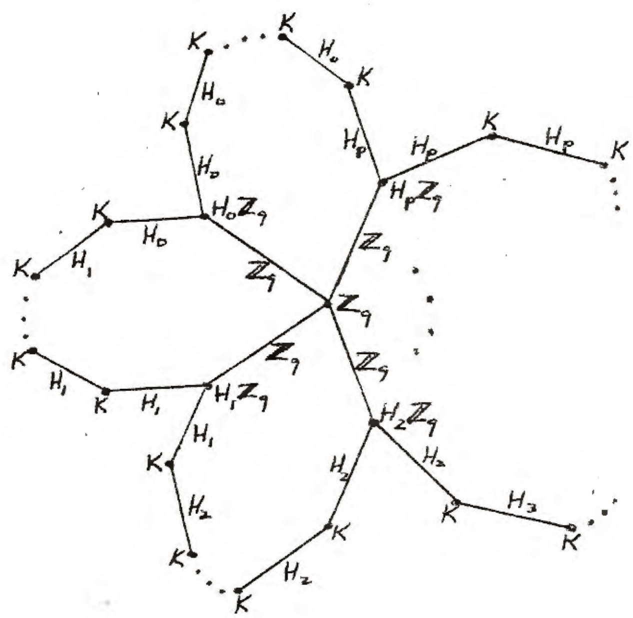
Furthermore, the kernel of $p(\mathbb{Z}_q, G) \colon \mathbb{Z}_q[G/\mathbb{Z}_q] \longrightarrow \mathbb{Z}_q$ is isomorphic to $\bigoplus_{H \in \mathcal{K}} N_H$.

<u>Proposition 6.</u> G has a fixed-point free action on a $\mathbb{Z}_q$-acyclic complex.

<u>Proof:</u> As in the construction of the $\mathbb{Z}_{p,q}$-action, a one-dimensional

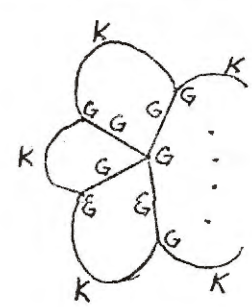
G-complex will be constructed (without fixed point) such that $H_1(Y; \mathbb{Z}_q)$ is a free $\mathbb{Z}_q[G]$-module. Then 2-cells can be attached to produce the G-space X, which is $\mathbb{Z}_q$-acyclic, and with the action on $(X-Y)$ free.

Define $Y = (G \times \tilde{Y})/\sim$, where $\tilde{Y}$ is the orbit space pictured below, and the relation $(\sim)$ is: $(g,x) \sim (gh,x)$ for $g \in G$, $x \in \tilde{Y}$ and $h \in G'$, the subgroup of G associated to the cell containing x.



Here, $\mathcal{H} = \{H_0, \ldots, H_p\}$, and each $H_i\mathbb{Z}_q$ is the isotropy subgroup at exactly one vertex in $\tilde{Y} = Y/G$. Each subgroup H_i is the isotropy subgroup over exactly $(p+1 - \frac{p-1}{q})$ one-cells of $\tilde{Y}$.

Consider Y/K. As a (G/K)-space, it has the same orbit space $\tilde{Y}$, with the following isotropy subgroups:



So Y/K has the homotopy type of a "many-leafed clover," and, as a G-representation,

$$H_1(Y/K;\mathbb{Z}_q) \cong \overset{p+1}{\oplus}\ \mathbb{Z}_q[G/K].$$

Let K_* be the kernel of the projection $C_*(Y;\mathbb{Z}_p) \longrightarrow C_*(Y/K;\mathbb{Z}_p)$. The projection and transfer maps are equivariant chain maps, so as G-representations

$$H_i(Y;\mathbb{Z}_q) \cong H_i(Y/K;\mathbb{Z}_q) \oplus H_i(K_*).$$

The spaces Y and Y/K are both connected, so $H_0(K_*) = 0$ and $H_1(K_*)$ is the kernel of the surjection $K_1 \longrightarrow K_0$.

From the construction of Y, and from previous lemmas,

$$K_1 \cong \Big[\bigoplus_{H\in\mathcal{N}} N_H\Big]^{p+1} \qquad \text{(from the } G/\mathbb{Z}_q \text{ orbits)}$$

$$\oplus \bigoplus_{H\in\mathcal{N}} \Big[M_H\Big]^{p+1-\frac{p-1}{q}} \qquad \text{(from the } G/\mathbb{Z}_q \text{ orbits)}$$

$$K_0 \cong \bigoplus_{H\in\mathcal{N}} N_H \qquad \text{(from the } G/\mathbb{Z}_q \text{ orbit)}$$

$$\oplus \bigoplus_{H\in\mathcal{N}} \Big[N_H\Big] \qquad \text{(from the } G/H\mathbb{Z}_q \text{ orbits)}$$

So $K_1 \cong \Big(\bigoplus_{H\in\mathcal{N}} N_H\Big)^{(p+1)(q+1)-(p-1)}$, $K_0 \cong \Big(\bigoplus_{H\in\mathcal{N}} N_H\Big)^{2}$. Both K_1 and K_0 are projective, so $H_1(K_*) \oplus K_0 \cong K_1$; and by Lemma 2,

$$H_1(K_*) \cong \Big(\bigoplus_{H\in\mathcal{N}} N_H\Big)^{(p+1)q} = \Big(\bigoplus_{H\in\mathcal{N}} M_H\Big)^{p+1}.$$

Thus, $H_1(Y;\mathbb{Z}_q) \cong \mathbb{Z}_q[G/K]^{p+1} \oplus \Big(\bigoplus_{H\in\mathcal{N}} M_H\Big)^{p+1} \cong \mathbb{Z}_q[G]^{p+1}$. $\quad ||$

V. PROOF OF THE MAIN THEOREM

In Chapter IV, fixed-point free actions of a group G on a finite $\mathbb{Z}_t$-acyclic G-equivariant CW complex were constructed for a wide variety of pairs (G,t). The results of that chapter are listed below (where p, q, r and s are always distinct primes):

	$\underline{G}$	$\underline{t}$
1)	$\mathbb{Z}_p^n \lhd G$ of index q, where the induced action of $\mathbb{Z}_q$ on $\mathbb{Z}_p^n$ leaves no index p subgroup invariant	$t \neq p$
2)	$\mathbb{Z}_p^2 \lhd G$ of index q, where the induced action of $\mathbb{Z}_q$ on $\mathbb{Z}_p^2$ leaves exactly two index p subgroups invariant	$t \neq p$
3)	$G(p^2, q)$	$t \neq p$
4)	$\mathbb{Z}_{p,qr}$	$t \neq p$
5)	$\mathbb{Z}_p^2 \oplus \mathbb{Z}_q^2$	$t \neq p, q$
6)	$\mathbb{Z}_p^2 \oplus \mathbb{Z}_{q,r}$	$t \neq p, q$
7)	$\mathbb{Z}_{p,q} \oplus \mathbb{Z}_{r,s}$	$t \neq p, r$

Cases 1) and 2) follow from Propositions 1 and 3 of Chapter IV. Part of case 3) follows from Proposition 5. The other cases were proven in corollaries to Proposition 4.

These seven cases will be extended to prove the remaining half of Theorem 1 in Chapter I. Roughly, it will be shown that for any finite solvable G not in $\mathcal{S}_t$, there are subgroups $H \lhd K \subseteq G$ such that K/H is listed above as having a fixed-point free action on a finite $\mathbb{Z}_t$-acyclic complex, thus on a compact $\mathbb{Z}_t$-acyclic manifold. It then follows, by Proposition 2 of Chapter II, that G also has such an action.

In order to simplify some of the proofs in this chapter, "$\mathscr{S}'_p$" will be used to denote the collection of finite solvable groups which do not have smooth fixed-point free actions on compact $\mathbb{Z}_p$-acyclic manifolds. The main theorem in this chapter will be that $\mathscr{S}'_p \subseteq \mathscr{S}_p$.

The first step is to combine and extend cases 1) to 3) to a larger class of groups. Three lemmas will be needed.

Lemma 1. If the finite group G is not abelian, then $G/Z(G)$ is not cyclic. (In particular, $[G: Z(G)]$ is not prime.)

Proof: Assume otherwise: if $G/Z(G) \cong \mathbb{Z}_n$, choose $x \in G$ which maps to a generator of $G/Z(G)$. Any element of G can be written in the form $x^k z$ for $z \in Z(G)$, and any two of these commute. This contradicts the assumption that G is not abelian. ||

Lemma 2. Let G be a finite group, q a prime not dividing the order of G. Let $\xi: G \to G$ be an automorphism fixing some subgroup $H \triangleleft G$; assume $\xi^q = 1$. If $\xi | H: H \to H$ and $\xi / H: G/H \to G/H$ are both the identity, then $\xi = 1$.

Proof: For any $x \in G$, $x^{-1}\xi(x) \in H$, so $\xi(x^{-1}\xi(x)) = x^{-1}\xi(x)$. This implies $\xi^2(x) = x(x^{-1}\xi(x))^2$; by induction

$$\xi^n(x) = x(x^{-1}\xi(x))^n.$$

Then $x = \xi^q(x) = x(x^{-1}\xi(x))^q$, so $(x^{-1}\xi(x))^q = 1$ and $x = \xi(x)$. ||

Lemma 3. For p prime, $p \nmid n$, if $a^n \equiv 1 \pmod{p^m}$ and $a \equiv 1 \pmod{p}$, then $a \equiv 1 \pmod{p^m}$.

Proof: $\left(\frac{a^n-1}{a-1}\right) = a^{n-1} + a^{n-2} + \ldots + 1 \equiv n \pmod{p}$; thus $p \nmid \left(\frac{a^n-1}{a-1}\right)$. Since $p^m \mid a^n-1$, it follows that $p^m \mid (a-1)$, or $a \equiv 1 \pmod{p^m}$. $||$

Proposition 1. Let $0 \longrightarrow P \longrightarrow G \longrightarrow \mathbb{Z}_q \longrightarrow 0$ be a short exact sequence, where P is a non-cyclic p-group, for p and q distinct primes. If $G \in \mathscr{S}'_t$, for any prime $t \neq p$, then $G \cong P \oplus \mathbb{Z}_q$.

Proof: Fix $\mathbb{Z}_q \subseteq G$, fix $x \in \mathbb{Z}_q$ a generator. Let $\xi: P \longrightarrow P$ be the map taking $g \longrightarrow xgx^{-1}$. It will suffice to show that $\xi = 1$.

First assume $P \cong \mathbb{Z}_p^2$. Let $\mathscr{H} = \{H \subseteq P: |H| = p\}$; ξ induces an action of $\mathbb{Z}_q$ on $\mathscr{H}$. Since $\mathscr{H}$ has order $(p+1)$, and $q \nmid p$, the number of invariant subgroups cannot be one. If the number were zero or two, case 1) or 2) would imply $G \notin \mathscr{S}'_t$. Thus, at least three subgroups are invariant; choose $a, b \in P$ such that a, b and (ab) generate these subgroups. Then $\xi(a) = a^\alpha$, $\xi(b) = b^\beta$, $\xi(ab) = a^\gamma b^\gamma = a^\alpha b^\beta$, so $\alpha \equiv \gamma \equiv \beta \pmod{p}$, and $\xi(g) = g^\alpha$ for any $g \in P$. If $\alpha \not\equiv 1 \pmod{p}$ then $G \cong G(p^2, q) \notin \mathscr{S}'_t$ (case 3). It follows that $\alpha \equiv 1 \pmod{p}$, and ξ is the identity.

Next, take the case $P \cong \mathbb{Z}_p^n$ $(n > 2)$, and assume the proposition has been proven for $\mathbb{Z}_p^k$ $(2 \leq k < n)$. Again, let $\mathscr{H} = \{H \subseteq P: |H| = p^{n-1}\}$. The map $\xi: P \longrightarrow P$ must leave some subgroup in $\mathscr{H}$ invariant, otherwise $G \notin \mathscr{S}'_t$ by case 1). Let $H \in \mathscr{H}$ be invariant; $H\mathbb{Z}_q \in \mathscr{S}'_t$, as a subgroup of G, and so by the induction hypothesis, $\xi | H = 1$. Let $K \subseteq H$ be a proper subgroup; $G/K \in \mathscr{G}'_t$, so again by induction, the induced map $(\xi/K): G/K \longrightarrow G/K$ is the identity. By Lemma 2, $\xi = 1$.

Now let P be any non-cyclic abelian p-group; again assume the proposition has been proven for abelian groups of smaller order. Let $\hat{P} = \{g \in P: g^p = 1\}$; $\hat{P} \cong \mathbb{Z}_p^n$, some $n > 1$, and is invariant under ξ. So $\hat{P}\mathbb{Z}_q \subseteq G$ is in $\mathscr{G}'_t$, and by the previous paragraph, $\xi | \hat{P} = 1$. Moreover, $G/\hat{P} \in \mathscr{G}'_t$, so if $P/\hat{P}$ is not cyclic, $\xi/\hat{P} \in \mathrm{Aut}\,(P/\hat{P})$ is the identity (by the induction hypothesis) and $\xi = 1$ by Lemma 2.

If $\hat{P} = P$ we are clearly done: so assume $P/\hat{P} \cong \mathbb{Z}_{p^k}$ for $k \geq 1$. Choose $y \in P$ such that $(y\hat{P})$ generates $P/\hat{P}$, then $y^{p^k} \neq 1$ is the smallest power of y in $\hat{P}$. Let $P' \subseteq \hat{P}$ be the subgroup generated by y^{p^k}; then P/P' is not cyclic (otherwise P would be cyclic or P' a direct summand). Now $\xi/P' = 1$ $(G/P' \in \mathscr{G}'_t)$, and it follows from Lemma 2 that $\xi = 1$.

It remains to consider the case when P is not abelian. Again, the proposition will be assumed for non-cyclic groups of smaller order. By Lemma 1, $P/Z(P)$ is not cyclic (and $Z(P) \neq 1$), so $\xi/Z(P) = 1$. Choose some normal subgroup $\hat{P} \triangleleft P$ such that $Z(P) \subseteq \hat{P}$ with index p (so $\hat{P} \subsetneq P$). If $\hat{P}$ is not cyclic, $\hat{P}\mathbb{Z}_q \in \mathscr{G}'_t$ and by the induction hypothesis, $\xi | \hat{P} = 1$ and $\xi = 1$. If $\hat{P} \cong \mathbb{Z}_{p^k}$ is generated by g, then $\xi(g) = g^s$, and $s^q \equiv 1 \pmod{p^k}$. On the other hand, $g^{-1}\xi(g) \in Z(P)$ $(\xi/Z(P) = 1)$ so $s \equiv 1 \pmod{p}$. It follows by Lemma 3 that $s \equiv 1 \pmod{p^k}$, so $\xi(g) = g$. Again, $\xi | \hat{P} = 1$ and so $\xi = 1$. $||$

We are now ready to prove the main theorem of this chapter. This will be done by simplifying a normal series of $G \in \mathcal{S}'_p$ until it fits the definition of $\mathcal{S}_p$. The following notation will be used: G will be said to be of type $\langle F_1, F_2, \ldots, F_n \rangle$ if it has a normal series

$$0 = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = G \quad (H_i \lhd G)$$

with $F_i = H_i/H_{i-1}$. Under this notation, $\mathcal{S}$ is the collection of all groups of type $\langle \mathbb{Z}_n, Q \rangle$, where Q is of prime power order, and $\mathcal{S}_p$ consists of all groups of type $\langle P, \mathbb{Z}_n, Q \rangle$, where P is of p-power order and Q of prime power order (for some prime possibly equal to p).

The following theorem of P. Hall [11, p.99] will be used: if G is a solvable group of order ab, for relatively prime a and b, then G has a subgroup of order a, and any two such are conjugate. In particular, a short exact sequence of solvable groups, with kernel and cokernel of relatively prime order, splits.

Theorem 1. For any solvable group $G \notin \mathcal{S}_p$, G has a smooth fixed-point free action on a $\mathbb{Z}_p$-acyclic manifold.

Proof: This is equivalent to showing $\mathcal{S}'_t \subseteq \mathcal{S}_t$. Choose any $G \in \mathcal{S}'_t$, G is solvable and is thus of the type $\langle \mathbb{Z}_{p_1}^{\alpha_1}, \ldots, \mathbb{Z}_{p_k}^{\alpha_k} \rangle$ for some collection of elementary abelian p-groups. The theorem will be proven in four reductions:

Step 1. G is of the form $\langle P_1, K, P_2 \rangle$, where $p \nmid |K|$ and the P_i are p-groups.

Step 2. G is of the form $\langle P_1, Q_1, \ldots, Q_k, P_2 \rangle$, where Q_i is a q_i-group for primes $q_1 > q_2 > \ldots > q_k$ (all distinct from p).

Step 3. G is of the form $\langle P_1, \mathbb{Z}_a, \oplus Q'_j \oplus P_2\rangle$, where Q'_j is a q'_j-group, all q'_j, p are distinct primes not dividing a.

Step 4. G is of the form $\langle P, \mathbb{Z}_n, Q\rangle$, where P is a p-group, Q a q-group (possibly $q=p$) and some n.

Step 1. It will suffice, by induction, to prove the following: If G is of the form $\langle H, \mathbb{Z}_q^n, P_1, K, P_2\rangle$, for $q \neq p$, $p \nmid |K|$ and P_i p-groups, then G is of the form $\langle H, \widetilde{P}_1, \mathbb{Z}_q^n, K, \widetilde{P}_2\rangle$, where $\widetilde{P}_i$ are p-groups.

Let $H \lhd H_1 \lhd H_2 \lhd H_3 \lhd G$ be the normal subgroups of G which determine the form described above. Set $S = [H_2/H, G|H] \lhd G$ (and contained in H_2/H). Let $\varphi\colon G/H \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_q^n)$ be the map induced by conjugation in $\mathbb{Z}_q^n \cong H_1/H$. Since $H_2/H \in \mathscr{S}'_p$, $\varphi|_{H_2/H}$ is the identity unless $n=1$; in either case, its image is in the center of $\mathrm{Aut}\,(\mathbb{Z}_q^n)$, and so $\varphi|S$ goes to the identity. Let $S' \lhd G/H$ be the subgroup generated by S and $\mathbb{Z}_q^n$, thus $\varphi|S'$ goes to the identity and $S' \cong \mathbb{Z}_q^n \oplus (S'/\mathbb{Z}_q^n)$ (it is already a semidirect product by the above theorem of Hall). Thus, G is of the form $\langle H, \widetilde{P}_1, \mathbb{Z}_q^n, P_1/\widetilde{P}_1, K, P_2\rangle$ where $\widetilde{P}_1 = S'/\mathbb{Z}_q^n$.

Let $\psi\colon G/S'H \longrightarrow \mathrm{Aut}\,(H_2/S'H)$ be the map determined by conjugation; for $g \in G$, $h \in H_2$, $(ghg^{-1})h^{-1} \in S'H$, so $\mathrm{im}\,\psi = 1 \in \mathrm{Aut}\,(H_2/S'H)$. In particular, $H_3/S'H \cong K \oplus (P_1/\widetilde{P}_1)$, and G is of the form $\langle H, \widetilde{P}_1, \mathbb{Z}_q, K, \widetilde{P}_2\rangle$ for $\widetilde{P}_2 \cong G/H_2K$, a p-group.

Step 2. It will suffice to show that if G is of the form $\langle H_1, \mathbb{Z}_{r_1}^{k_1}, \mathbb{Z}_{r_2}^{k_2}, H_2\rangle$, where $r_1 < r_2$ are primes distinct from p, then G is of the form

$\langle H_1, \mathbb{Z}_{r_2}^{k_2}, \mathbb{Z}_{r_1}^{k_1}, H_2 \rangle$.

Let $H_1 \lhd K_1 \lhd K_2 \lhd G$ be the normal subgroups of G determining the above form. Since $K_2/H_1 \in \mathscr{S}'_t$, then by Proposition 1, either $k_1 = 1$ or $K_2/H_1 \cong \mathbb{Z}_{r_1}^{k_1} \oplus \mathbb{Z}_{r_2}^{k_2}$. In the latter case, we are done; in the former case, r_2 does not divide the order of Aut $(\mathbb{Z}_{r_1})$ and so K_2/H_1 is a direct sum anyway.

<u>Step 3</u>. Here, it will suffice to show that if $G' = G/P_1$ is of the form $\langle K_1, Q, K_2 \rangle$, where $p \nmid |K_1|, |Q|$, the order of Q is relatively prime to the orders of K_1 and K_2, K_1 is cyclic, and Q is a q-group, then G' is of the form $\langle K_1 \oplus Q, K_2 \rangle$ or $\langle K_1, Q \oplus K_2 \rangle$. If Q is not cyclic, then G' has the second form.

Let $H \lhd G'$ be the normal subgroup of type $\langle K_1, Q \rangle$. The exact sequence $0 \longrightarrow H/K_1 \longrightarrow G'/K_1 \xrightarrow{\pi} G'/H \longrightarrow 0$ splits, determining a map $\varphi: K_2 = G'/H \longrightarrow \mathrm{Aut}\,(Q)$. If Q is not cyclic, then for any $g \in K_2$ of prime order, $\pi^{-1}(\langle g \rangle) \in \mathscr{S}'_p$ and so $\varphi(g) = 1$, by Proposition 1. The same holds true for g of prime power order: if $\varphi(g^t) = 1$ (where $|g|$ is a power of t), then $\langle g^t \rangle$ is normal in $\pi^{-1}(\langle g \rangle)$, and the application of Proposition 1 to $\pi^{-1}(\langle g \rangle)/\langle g^t \rangle$ implies $\varphi(g) = 1$. It follows that $\varphi(g) = 1$ for all $g \in K_2$, and so $G'/K_1 \cong Q \oplus K_2$.

If Q is cyclic, and φ is trivial, then it is still the case that $G'/K \cong Q \oplus K_2$. Assume $\varphi \neq 1$: then for some $g \in G'/K_1$, of order n relatively prime to q,

$gxg^{-1} = x^k$ for all $x \in Q$, where $k \not\equiv 1 \pmod{q^m}$ $(q^m = |Q|)$.

Since $k^n \equiv 1 \pmod{q^m}$, then $k \not\equiv 1 \pmod q$ (by Lemma 3). Thus, $gxg^{-1}x^{-1} = x^{k-1}$ generates Q, so $[H/K_1, G'/K_1] = H/K_1$. Conjugation also defines $\psi: G'/K_1 \longrightarrow \mathrm{Aut}\,(K_1)$; since $\mathrm{Aut}\,(K_1)$ is abelian, any commutator goes to the identity. Thus, $\psi(Q) = 1$, so $H \cong Q \oplus K_1$.

Step 4. We now have G in the form $\langle P_1, \mathbb{Z}_a, \oplus Q_j' \oplus P_2 \rangle$, where Q_j' is a q_j'-group, q_j' distinct primes differing from p, and $\underline{a}$ relatively prime to p and q_j'. Note that any non-cyclic q-group has a normal subgroup with quotient isomorphic to $\mathbb{Z}_q^2$: this is obvious for an abelian group, and for a non-abelian group it follows by repeatedly dividing out its center (using Lemma 1). Also observe that if $\mathbb{Z}_{q^n}$ is extended non-trivially by a q'-group $Q'(q' \neq q)$, then the extension contains $\mathbb{Z}_{q,q'}$ as a quotient group of some subgroup: the image of $Q \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_{q^n})$ lies in the subgroup of order $(q-1)$, and so Q acts non-trivially on $\mathbb{Z}_{q^n}/\mathbb{Z}_{q^{n-1}}$.

Thus, under the action of $\oplus Q_j' \oplus P_2$ on $\mathbb{Z}_a$, at most one of the summands can act non-trivially (applying cases 4) and 7)). Let Q be that summand; if $Q \neq P_2$, then P_2 can be absorbed into P_1; and G can now be expressed in the form $\langle P, \mathbb{Z}_a \oplus (\oplus Q_j''), Q \rangle$ (P a p-group), where $Q = 1$ or Q acts non-trivially on Z_a and trivially on the other factors. In the latter case, one must have $\oplus Q_j''$ cyclic, by case 6), and $G \in \mathscr{S}_p$. If $Q = 1$, at most one of the Q_j'' can be non-cyclic (case 5)) and so $G \in \mathscr{S}_p$ in this case. ||

Corollary. For any finite group $G \notin \mathcal{G}$, G has a smooth fixed-point free action on a compact $\mathbb{Q}$-acyclic manifold.

Proof: If G is solvable, then $G \notin \mathcal{G}_p$ for some prime p (take, e.g., $p \nmid |G|$) and G acts without fixed point on a compact $\mathbb{Z}_p$-acyclic manifold. If G is not solvable, then for some $H \triangleleft K \subseteq G$, K/H is simple and has a fixed-point free action on some even dimensional projective space (Proposition 2, Chapter VI). Then G has a fixed-point free action on some product of even dimensional projective spaces, which is $\mathbb{Z}_q$-acyclic for any odd prime q. ||

Theorem 1 can now be combined with Proposition 1 of Chapter II and with the results of Chapter III to give the main result:

Corollary. A finite solvable group G has a smooth fixed-point free action on a disk if and only if $G \notin \mathcal{G}_p$ for all primes p. ||

In particular, one gets for abelian groups:

Corollary. A finite abelian group G has a smooth fixed-point free action on a disk if and only if three or more of the Sylow subgroups of G are non-cyclic. ||

The smallest abelian group with such an action is $\mathbb{Z}_{30} \oplus \mathbb{Z}_{30}$, with order 900.

There are two solvable groups of order 72 which have smooth fixed-point free actions on disks: $A_4 \oplus S_3$ and $S_4 \oplus \mathbb{Z}_3$. Each of these has quotient groups of subgroups isomorphic to A_4 and $\mathbb{Z}_3 \oplus S_3$, where actions of these groups on $\mathbb{Z}_t$-acyclic complexes for $t \neq 2$, $t \neq 3$ respectively, were constructed directly in Chapter IV (Propositions 1 and 3, respectively). That these are the smallest solvable groups with smooth fixed-point free actions on disks is shown by the following:

Theorem 2. If G is a finite solvable group of order less than 72, then $G \in \mathcal{S}_p$ for some prime p.

Proof: Any group of order $p^\alpha q$ or pqr, for primes p, q, r, is in some $\mathcal{S}_p$ by examination of its composition series. This leaves the cases $|G| = 36$ or 60.

If $|G| = 36$, G has a normal series all of whose components are elementary p-groups; the only possibilities which do not immediately show $G \in \mathcal{S}_2$ or $\mathcal{S}_3$ are $\langle \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \rangle$, $\langle \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_3 \rangle$, $\langle \mathbb{Z}_2, \mathbb{Z}_3^2, \mathbb{Z}_2 \rangle$ and $\langle \mathbb{Z}_3, \mathbb{Z}_2^2, \mathbb{Z}_3 \rangle$. Since the only extension of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ is $\mathbb{Z}_6$, the first case reduces to $\langle \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_2 \rangle$, or $G \in \mathcal{S}_3$, and the second case to $\langle \mathbb{Z}_6, \mathbb{Z}_6 \rangle$, which has the same form as the first case. Similarly, the third case reduces to $\langle \mathbb{Z}_2 \oplus \mathbb{Z}_3^2, \mathbb{Z}_2 \rangle$ or $G \in \mathcal{S}_3$. In the fourth case, either G is of type $\langle \mathbb{Z}_3, \mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \rangle$ (and $G \in \mathcal{S}_3$) or G is of type $\langle \mathbb{Z}_3, A_4 \rangle$. Since A_4 has no subgroup of index 2, $\mathbb{Z}_3$ must be in the center of G. Thus, G is also of the form $\langle \mathbb{Z}_3 \oplus \mathbb{Z}_2^2, \mathbb{Z}_3 \rangle$, and $G \in \mathcal{S}_2$.

If $|G| = 60$, there are eight possibilities for the components of a normal series which do not immediately show $G \in \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_5$. Four of them, $\langle \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_2 \rangle$, $\langle \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_3, \mathbb{Z}_2 \rangle$, $\langle \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_2 \rangle$ and $\langle \mathbb{Z}_5, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2 \rangle$ imply $G \in \mathcal{S}_2, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_5$, respectively, since $\mathbb{Z}_{15}$ is the only group of type

$\langle \mathbb{Z}_3, \mathbb{Z}_5 \rangle$ or $\langle \mathbb{Z}_5, \mathbb{Z}_3 \rangle$, etc. If G is of type $\langle \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_5 \rangle$ or $\langle \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_2, \mathbb{Z}_3 \rangle$, then it is of type $\langle \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_{10} \rangle$ or $\langle \mathbb{Z}_2, \mathbb{Z}_5, \mathbb{Z}_6 \rangle$, cases which were considered above $(G \in \mathcal{S}_2)$. A group of type $\langle \mathbb{Z}_3, \mathbb{Z}_2^2, \mathbb{Z}_5 \rangle$ is in $\mathcal{S}_3$, since $\mathbb{Z}_2^2 \oplus \mathbb{Z}_5$ is the only group of type $\langle \mathbb{Z}_2^2, \mathbb{Z}_5 \rangle$. If G is of type $\langle \mathbb{Z}_5, \mathbb{Z}_2^2, \mathbb{Z}_3 \rangle$, then $G/\mathbb{Z}_5 \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$ (so $G \in \mathcal{S}_5$), or $G/\mathbb{Z}_5 \cong A_4$. In this last case G is a semi-direct product; the only homomorphism $A_4 \longrightarrow \mathrm{Aut}\,(\mathbb{Z}_5) \cong \mathbb{Z}_4$ is the trivial one, so $G \cong \mathbb{Z}_5 \oplus A_4 \in \mathcal{S}_2$. $\|$

VI. NON-SOLVABLE AND POSITIVE-DIMENSIONAL GROUPS

In the previous three chapters, a complete description has been given of which finite solvable groups have smooth fixed-point free actions on disks, and on certain other classes of compact acyclic manifolds. In this chapter, partial results are given towards extending this to all compact Lie groups.

The first step in extending the results of Chapter III is the following lemma:

Lemma 1. Assume a torus T acts on the compact space X. If X is ($\mathbb{Z}$-acyclic, $\mathbb{Z}_p$-acyclic) (under Čech cohomology), then so is the fixed-point set X^T (assuming $\check{H}^*(X^T)$ finitely generated in the $\mathbb{Z}$-acyclic case).

Proof: Assume first that X is $\mathbb{Z}_p$-acyclic. There is a sequence $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots \subseteq T$ of finite subgroups of T of p-power order such that $\bigcup_{i=1}^{\infty} P_i$ is dense in T: for instance, set $P_i = (\mathbb{Z}_{p^i})^n$, when T is the n-dimensional torus. By T2 (Chapter III), each fixed-point set X^{P_i} is $\mathbb{Z}_p$-acyclic. Since $X^T = \bigcap_{i=1}^{\infty} X^{P_i}$, it follows that $\check{H}^*(X^T; \mathbb{Z}_p) = \varinjlim \check{H}^*(X^{P_i}, \mathbb{Z}_p)$ $= \mathbb{Z}_p$ (all of the spaces are compact). Thus, X^T is $\mathbb{Z}_p$-acyclic.

If X is $\mathbb{Z}$-acyclic, it is $\mathbb{Z}_p$-acyclic for all primes p. By the above, X^T is $\mathbb{Z}_p$-acyclic for all p, and so is $\mathbb{Z}$-acyclic. $||$

With this lemma, the problem for compact Lie groups with abelian identity component is reduced to that for finite groups:

Proposition 1. Let G be a compact Lie group with abelian identity component T. Then G has a smooth fixed-point free action on a compact ($\mathbb{Z}$-, $\mathbb{Z}_p$-, Q-acyclic) manifold if and only if the finite group G/T does. $||$

Thus, it remains to see what happens with finite non-solvable groups, and with compact Lie groups with non-abelian identity component. It is conjectured that all of these have smooth fixed-point free actions on disks. By application of Proposition 2 of Chapter II (an action of H on X induces an action of G on X^n, where $H \subseteq G$ has finite index n), it would suffice to construct smooth fixed-point free actions of any finite non-abelian simple group, and any compact connected non-abelian Lie group, on disks. The following proposition further reduces the problem:

Proposition 2. Assume G is compact connected non-abelian, or finite non-abelian simple. Then G has a smooth fixed-point free action on a compact $\mathbb{Z}_p$-acyclic manifold for any odd prime p.

Proof: By a theorem of Hsiang and Hsiang [14, p. 366], every compact connected non-abelian Lie group has an irreducible representation on $\mathbb{R}^{2k+1}$ for some $k \geq 1$. If G is a finite non-abelian simple group, it has even order [7], and so the decomposition of the left regular representation on $\mathbb{R}[G]$ must yield at least one odd-dimensional irreducible representation other than the trivial one, which must therefore be of dimension at least three. In either case, the representation of G on $\mathbb{R}^{2k+1}$ induces a smooth fixed-point free action on P^{2k}, which is $\mathbb{Z}_p$-acyclic for all odd primes p. ||

Thus, by Proposition 1 of Chapter II, it would suffice to construct finite $\mathbb{Z}_2$-acyclic G-equivariant complexes without fixed-points for the groups G described above. This has been done for the groups $SU(2)$ and $SU(3)$.

The following notation will be used for subgroups of $U(n)$. Let $\hat{k} = (k_1, k_2, \ldots, k_s)$ be a partition of n $(k_i > 0, \Sigma k_i = n)$. Then set $U(\hat{k}) = U(k_1) \times \ldots \times U(k_s) \subseteq U(n)$. Set $T(\hat{k})$ to be the inverse image of $S_{k_1} \times \ldots \times S_{k_s}$ under the projection $N(T) \longrightarrow N(T)/T \cong S_n$ (T is the subgroup of diagonal matrices, a maximal torus).

<u>Lemma 2</u>. For any n, $\tilde{H}^*(U(n)/N(T);\mathbb{Q}) = 0$.

<u>Proof</u>: $U(1)/T'$ is a point. In general, the fibration $U(n-1) \longrightarrow U(n) \longrightarrow S^{2n-1}$ induces a fibration $U(n-1)/T^{n-1} \longrightarrow U(n)/T^n \longrightarrow \mathbb{C}P^{n-1}$. The Serre spectral sequence shows that if the base space of a fibration $F \longrightarrow E \longrightarrow B$ is simply connected, and $H^*(F, \mathbb{Z})$, $H^*(B;\mathbb{Z})$ are both zero in odd dimensions and free finitely generated groups in even dimensions, then $H^*(E;\mathbb{Z}) \cong H^*(B;\mathbb{Z}) \otimes H^*(F;\mathbb{Z})$. In particular, by induction on the above fibration

$$H^*(U(n)/T^n;\mathbb{Z}) \cong \bigotimes_{k=1}^{n-1} H^*(\mathbb{C}P^k;\mathbb{Z}).$$

Thus, $H^*(U(n)/T^n;\mathbb{Q})$ is zero in odd dimension, and $\chi(U(n)/T^n) = n!$ The finite group S_n acts freely on $U(n)/T^n$ with orbit space $U(n)/N(T)$;

Thus $\chi(U(n)/N(T)) = 1$ and

$$\mathrm{proj}^*: H^*(U(n)/N(T);\mathbb{Q}) \longrightarrow H^*(U(n)/T;\mathbb{Q})$$

is a monomorphism. Then $\dim[H^*(U(n)/N(T);\mathbb{Q})] = 1$ and the space is $\mathbb{Q}$-acyclic. $||$

<u>Proposition 3</u>. For $\hat{k} = (k_1, \ldots, k_s)$ a partition of n, consider the following maps induced by the natural projections:

$$\begin{tikzcd}
H^*(U(n)/T(\hat{k});\mathbb{Z}) \arrow[r, "f"] & H^*(U(n)/T;\mathbb{Z}) \\
H^*(U(n)/U(\hat{k});\mathbb{Z}) \arrow[u, "g"] \arrow[ur, "h=fg"'] &
\end{tikzcd}$$

Then $H^*(U(n)/T(\hat{k});\mathbb{Z}) = \mathrm{Ker}\,(f) \oplus \mathrm{Im}\,(g)$, and g is one-to-one.

<u>Proof:</u> It will suffice to show that h is one-to-one and that $\mathrm{Im}\,(f) = \mathrm{Im}\,(h)$. Two fibrations will be considered:

(1) $\mathop{\mathrm{X}}_{i=1}^{s} (U(k_i)/N(T^{k_i})) = U(\hat{k})/T(\hat{k}) \longrightarrow U(n)/T(\hat{k}) \longrightarrow U(n)/U(\hat{k})$

(2) $\mathop{\mathrm{X}}_{i=1}^{s} (U(k_i)/T^{k_i}) = U(\hat{k})/T^n \longrightarrow U(n)/T^n \longrightarrow U(n)/U(\hat{k})$

Since $U(\hat{k})/T(\hat{k})$ is $\mathbb{Q}$-acyclic, by Lemma 1, the map $g \otimes \mathbb{Q}: H^*(U(n)/U(\hat{k});\mathbb{Q}) \longrightarrow H^*(U(n)/T(\hat{k});\mathbb{Q})$ is an isomorphism, by (1), and $\mathrm{Im}\,(f)/\mathrm{Im}\,(h)$ is finite (all cohomology is finitely generated since all spaces are compact manifolds). Furthermore, $H^*(U(n)/U(\hat{k});\mathbb{Z})$ must be finite in odd dimensions, since $H^*(U(n)/T(\hat{k});\mathbb{Q}) \subseteq H^*(U(n)/T;\mathbb{Q})$ which is zero in odd dimensions. By a result of Borel [1, p. 202], $H^*(U(n)/U(\hat{k});\mathbb{Z})$ is free.

Thus, referring to the proof of Lemma 2, all spaces in the fibration (2) have integral cohomology zero in odd dimensions and free in even dimensions. The spectral sequence must collapse, and so $h: H^*(U(n)/U(\hat{k});\mathbb{Z}) \longrightarrow H^*(U(n)/T;\mathbb{Z})$ is one-to-one. Furthermore, $H^*(U(n)/T;\mathbb{Z})/\mathrm{Im}(h)$ is free. So $\mathrm{Im}(f)/\mathrm{Im}(h)$ is a finite subgroup of a free group, and thus zero. ||

The next step is to establish a formula for computing the cohomology of the orbit space of a finite group action in a certain situation: when G acts on X, and the cohomology of X/H is known, for some $H \subseteq G$, we want to compute $\check{H}^*(X/G;R)$ if $[G:H]$ is invertible in R. It will be assumed that X is finitistic, and Čech cohomology will be used. The derivation exactly parallels the calculations in [3, Ch. XII, §§8-9], where it is done for group cohomology.

For any subgroups $H \subseteq K \subseteq G$, the insertion $H \longrightarrow K$ induces a projection $X/H \longrightarrow X/K$; the induced map on cohomology will be denoted $i(K,H): \check{H}^*(X/K;R) \longrightarrow \check{H}^*(X/H;R)$. Using the techniques in Bredon [2, Ch. III, §6], the transfer map $t(H,K): \check{H}^*(X/H;R) \longrightarrow \check{H}^*(X/K;R)$ is defined, and $t(H,K)\,i(K,H) = [K:H]\cdot \mathrm{id}: \check{H}^*(K/H;R) \longrightarrow \check{H}^*(K/H;R)$. If $[K:H]$ is invertible in R, the composition is an isomorphism, and so $i(K,H)$ is a monomorphism. Thus, the problem is reduced to identifying a subspace of $\check{H}^*(X/H;R)$.

For any $a \in G$, let $r_a : X \longrightarrow X$ be the action of a: $r_a(x) = ax$. For any $H \subseteq G$, this induces a map $r_a : X/H \longrightarrow X/aHa^{-1}$; denote by $r_a^* : \check{H}^*(X/aHa^{-1};R) \longrightarrow \check{H}^*(X/H;R)$ the induced map in cohomology.

Lemma 3. For $H \subseteq G$, for $z \in \check{H}^*(X/H;R)$,

$$i(G,H)t(H,G)z = \sum_i r_{a_i}^* t(H \cap a_i H a_i^{-1}, a_i H a_i^{-1}) \, i(H, H \cap a_i H a_i^{-1}) z$$

where $\{a_i\}$ is a set of double coset representatives for H: $G = \bigcup_i Ha_iH$ is a disjoint union.

Proof: The corresponding theorem for homology can easily be proven in the case of a simplicial action on a complex: it can be proven on the chain level by regarding $C_*(X/K)$ as a subspace of $C_*(X)$ (under the transfer map) and doing all calculations in $C_*(X)$. The original equation will then hold on cochains, and thus in simplicial cohomology. One may then generalize this to an arbitrary continuous action on a finitistic space by using G-covering approximations on X, and taking direct limits to obtain Čech cohomology. ||

From this lemma, the desired condition follows immediately. First note that for any $a \in G$, $K_1 \subseteq K_2 \subseteq G$,

$$r_a^* t(K_1, K_2) r_{a^{-1}}^* = t(a^{-1}K_1 a, a^{-1}K_2 a)$$

and

$$r_a^* i(K_2, K_1) r_{a^{-1}}^* = i(a^{-1}K_2 a, a^{-1}K_1 a)$$

Proposition 4. Let $H \subseteq G$, and assume $[G:H]$ is invertible in R. Then for $z \in \check{H}^*(X/H;R)$, $z \in \operatorname{Im} i(G,H)$ if and only if

$$r_a^* i(H, H \cap aHa^{-1})z = i(H, a^{-1}Ha \cap H)\, z$$

for at least one a in every double coset $HaH \subseteq G$.

<u>Proof:</u> If $z = i(G,H)\,x$, then $r_a^* i(H, H \cap aHa^{-1})\, z$

$= [r_a^* i(G, H \cap aHa^{-1}) r_{a^{-1}}^*]\, r_a^* x = i(G, a^{-1}Ha \cap H)\, x = i(H, a^{-1}Ha \cap H)\, z.$

If the equation holds for all a_i, where $\{a_i\}$ is a set of double coset representatives, then

$$r_{a_i}^* t(H \cap a_i H a_i^{-1},\ a_i H a_i^{-1})\, i(H,\ H \cap a_i H a_i^{-1})\, z$$

$$= r_{a_i}^* t(H \cap a_i H a_i^{-1}, a_i H a_i^{-1})\, r_{a_i^{-1}}^* \ i(H, a_i^{-1} Ha_i \cap H)\, z$$

$$= t(a_i^{-1} Ha_i \cap H, H) i(H, a_i^{-1} Ha_i \cap H) z = [H: a_i^{-1} Ha_i \cap H]\, z$$

so $i(G,H)t(H,G)z = \left(\sum_i [H: a_i^{-1} Ha_i \cap H]\right)\cdot z = [G:H]\, z.$

(The identity $[G:H] = \sum_i [H: a_i^{-1} Ha_i \cap H]$ can be obtained by applying the lemma to a fixed action on some space.) Thus, $[G:H]z \in \operatorname{Im}(i(G,H))$, and so is z. $\|$

<u>Proposition 5.</u> The sequence

$$0 \longrightarrow H^*(U(3)/N(T^3); \mathbb{Z}_2) \xrightarrow{p_1^*} H^*(U(3)/T(2,1); \mathbb{Z}_2) \xrightarrow{p_2^*} H^*(U(3)/T^3; \mathbb{Z}_2)$$

is exact in positive dimensions.

<u>Proof:</u> Let $X = U(3)/T^3$, then S_3, the Weyl group of $U(3)$, acts

freely on X, with $U(3)/N(T^3) = X/S_3$, $U(3)/T(2,1) = X/\mathbb{Z}_2$, for some $\mathbb{Z}_2 \subseteq S_3$. In particular, since $[S_3 : \mathbb{Z}_2] = 3$, p_1^* is a monomorphism.

Choose $z \in H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ with $p_2^*(z) = 0$. If S_3 is the permutation group on $\{1,2,3\}$, and $\mathbb{Z}_2$ is generated by (12), then $\{e, (23)\}$ are a set of double coset representatives. The condition of Proposition 4 certainly holds for $a = e$. If $a = (2\,3)$, $H = \{e, (12)\}$, then $aHa^{-1} \cap H = \{e, (1\,3)\} \cap \{e, (1\,2)\}$
$= \{e\}$.

Thus $p_2^* = i(H, H \cap aHa^{-1}) = i(H, a^{-1}Ha \cap H)$, and since $p_2^*(z) = 0$, the hypothesis of Proposition 4 is fulfilled: $z \in \mathrm{Im}\, p_1^*$. Thus, $\ker(p_2^*) \subseteq \mathrm{im}(p_1^*)$.

In the proof of Lemma 2, it was shown that $H^*(U(3)/T^3; \mathbb{Z})$ is free; in particular, $\mathrm{im}(p_2^* p_1^*)$ is free. By Lemma 2, $\widetilde{H}^*(U(3)/N(T); \mathbb{Z})$ contains only torsion so in integral homology, the map
$H^*(U(3)/N(T); \mathbb{Z}) \longrightarrow H^*(U(3)/T^3; \mathbb{Z})$ is zero in positive dimensions. Thus, in $\mathbb{Z}_2$-cohomology, $p_2^* p_1^*$ is zero in positive dimensions. ||

Corollary. Let $p_1^* : H^*(U(3)/N(T); \mathbb{Z}_2) \longrightarrow H^*(U(3)/T(2,1); \mathbb{Z}_2)$ be as above. Let $p_3^* : H^*(U(3)/U(2,1); \mathbb{Z}_2) \longrightarrow H^*(U(3)/T(2,1); \mathbb{Z}_2)$ be as in Proposition 1. Then $p_1^* \oplus p_3^*$ is an isomorphism in positive dimensions, and onto with kernel $\mathbb{Z}_2$ in dimension zero.

Proof: By Proposition 3, p_3^* is 1-1 with

$$\mathrm{Im}\, p_3^* \oplus \mathrm{Ker}\, p_2^* = H^*(U(3)/T(2,1); \mathbb{Z}_2)$$

where p_2^* is as in Proposition 5. By Proposition 5, $\mathrm{Ker}\, p_2^* \cong \mathrm{Im}\, p_1^*$, in positive dimensions, and p_1^* is one-to-one. Thus,

$p_3^* \oplus p_1^*: H^*(U(3)/N(T);\mathbb{Z}_2) \oplus H^*(U(3)/U(2,1);\mathbb{Z}_2) \longrightarrow H^*(U(3)/T(2,1);\mathbb{Z}_2)$ is an isomorphism in positive dimensions. In dimension zero, this is just the map $\mathrm{id} \oplus \mathrm{id}: \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ which is surjective. $||$

Proposition 6. $U(3)$ has a fixed-point free action on a finite $\mathbb{Z}_2$-acyclic equivariant CW-complex, such that all isotropy subgroups contain a maximal torus.

Proof: Set $X = (U(3) \times I)/\sim$, where

$$(g,t) \sim (gh,t) \quad \text{for } t \in I,\ g \in U(3),\ h \in T(2,1)$$

$$(g,0) \sim (gh,0) \quad \text{for } g \in U(3),\ h \in U(2,1)$$

$$(g,1) \sim (gh,1) \quad \text{for } g \in U(3),\ h \in N(T^3)$$

Then $U(3)$ acts on X with the required isotropy subgroups.

$H^*(X;\mathbb{Z}_2)$ is acyclic: apply the Meyer-Vietoris sequence and use the preceding corollary. $||$

Corollary. Let G be a compact Lie group with an irreducible three-dimensional representation $G \longrightarrow U(3)$ not induced from any representation of a subgroup. Then G has a smooth fixed-point free action on a compact $\mathbb{Z}_2$-acyclic manifold.

Proof: It follows from Proposition 6, and from Theorem 1 of Chapter II, that $U(3)$ has a smooth action on a compact $\mathbb{Z}_2$-acyclic manifold, with all isotropy subgroups contained in a conjugate of $U(2) \times U(1)$ or $N(T^3)$.

Since the representation $G \longrightarrow U(3)$ is irreducible, the image of G is not contained in any conjugate of $U(2) \times U(1)$; since it is not induced, the image of G is not contained in any conjugate of $N(T^3)$. Thus, the image of G is not contained in any isotropy subgroup of the action of $U(3)$, and so the induced action of G on the same space is without fixed-point. $||$

Corollary. $SU(3)$, $SU(2)$, $U(2)$, $SO(3)$, and A_5 all have smooth fixed-point free actions on compact $\mathbb{Z}_2$-acyclic manifolds (and thus on disks). $||$

The above procedure suggests one possible way to attack the main problem: construct actions of unitary (or orthogonal) groups with restrictions on the isotropy subgroups which occur, so that fixed-point free actions of all other desired groups can be induced via representations. One possibility would be the following:

Proposition 7. Assume that for all $n \geq 1$, there exists an action of $U(2n+1)$ on a finite $\mathbb{Z}_2$-acyclic equivariant CW complex (and thus on some compact $\mathbb{Z}_2$-acyclic manifold M_n) such that every isotropy subgroup is conjugate to a subgroup of $N(U(\hat{k}))$ for some partition $\hat{k}$ of $2n+1$. Then every compact connected non-abelian Lie group, and every non-abelian finite simple group, would have smooth fixed-point free actions on disks.

Proof: The conditions on the isotropy subgroups would imply that any irreducible representation $G \longrightarrow U(2n+1)$, not induced from any representation of a subgroup, induces a fixed-point free action of G on M_n. Every compact connected non-abelian Lie group has an irreducible odd-dimensional complex representation: the complexification of the representation constructed by Hsiang and Hsiang [14, p. 366]; since the group has no closed subgroups of finite index, the representation is not induced.

In the case of finite simple groups, it suffices to construct actions of the minimal simple groups, as classified by Thompson [21]. These include:

1) $PSL(2,q)$ for $q = 2^p$ or 3^p (p prime) or $q \equiv \pm 2 \pmod 5$ and q prime, $q > 3$

2) $Sz(q)$ for $q = 2^p$, p an odd prime

3) $PSL(3,3)$

In case 1), when q is odd, $PSL(2,q)$ has an irreducible q-dimensional representation, and no subgroups of index a factor of q. When $q = 2^p$, $PSL(2,q)$ has an irreducible $(q-1)$-dimensional representation, and no subgroups of index a factor of $(q-1)$. Suzuki [20] has shown that $Sz(q)$ has irreducible representations of dimension $(q-1)(q-2r+1)$ (where $q = 2r^2$), and again no subgroups of index dividing that dimension.

The third case is even easier: $PSL(3,3)$ has a subgroup isomorphic to $\mathbb{Z}_3 \oplus S_3$, for which a fixed-point free action on a $\mathbb{Z}_2$-acyclic complex has already been constructed (Proposition 3 of Chapter IV). ||

It should also be noted that certain of the groups $PSL(2,q)$ have solvable subgroups which have fixed-point free actions on $\mathbb{Z}_2$-acyclic spaces from the results of Chapter IV. For instance, if q is prime, and $n = \frac{q-1}{2}$ is not a prime power, then $PSL(2,q)$ has subgroups isomorphic to $\mathbb{Z}_{q,n}$, and the fourth corollary to Proposition 5 of Chapter IV applies.

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Abstract

This thesis is centered around the problem of under what conditions a compact Lie group can act without fixed points on an acyclic space, especially on disks and other compact acyclic manifolds: the fact that cyclic groups of non-prime-power order can act smoothly without fixed points on Euclidean space, but not on any compact, rationally acyclic manifold (by the Lefschetz fixed-point theorem) illustrates immediately how significant the restriction imposed by compactness is.

In the case of finite solvable groups, I use the following notation. Let $\mathcal{S}$ be the collection of finite groups which have a cyclic normal subgroup of prime-power index. For any prime p, let $\mathcal{S}_p$ be the collection of finite groups G, with normal subgroup P, such that $|P| = p^n$ and $G/P \in \mathcal{S}$. The following theorem is then proven:

Let G be a finite solvable group. G has a fixed-point free action on a (sufficiently high-dimensional) disk if and only if $G \notin \mathcal{S}_p$, for all primes p. G has a smooth fixed-point free action on some compact $\mathbb{Z}_p$-acyclic manifold if and only if $G \notin \mathcal{S}_p$. G has such an action on some compact $\mathbb{Q}$-acyclic manifold if and only if $G \notin \mathcal{S}$. Thus, a finite abelian group has a smooth fixed-point free action on a disk if and only if it has three or more non-cyclic Sylow subgroups.

In particular, these conditions show that the smallest abelian group with a smooth fixed-point free action on a disk is $\mathbb{Z}_{30} \oplus \mathbb{Z}_{30}$, of order 900. The smallest solvable groups with such actions have order 72: two

such groups are $S_3 \oplus A_4$ and $\mathbb{Z}_3 \oplus S_4$.

Results for finite non-solvable groups, or for positive-dimensional compact Lie groups, are more scattered. Smooth fixed-point free actions on disks can be constructed for a number of simple groups using the same constructions as for solvable groups, but constructing such actions in general is more complicated. Examples of such actions are also constructed for two compact simple Lie groups, SU(2) and SU(3), and the general problem of classifying which compact Lie groups have smooth fixed-point free actions on disks is discussed briefly.