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SYMPLECTIC GROUPS

BY

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TO JEAN
– my wild Irish rose

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CONTENTS

PREFACE

My goal in these lectures is the isomorphism theory of symplectic groups over integral domains as illustrated by the theorem

$$\mathrm{PSp}_n(\mathfrak{o}) \cong \mathrm{PSp}_{n_1}(\mathfrak{o}_1) \Leftrightarrow n = n_1 \text{ and } \mathfrak{o} \cong \mathfrak{o}_1$$

for dimensions $\geqslant 4$. This is a sequel to my *Lectures on Linear Groups* where there was a similar objective with the linear groups in mind. Once again I will start from scratch assuming only basic facts from a first course in algebra plus a minimal number of references to the *Linear Lectures*. The simplicity of $\mathrm{PSp}_n(F)$ will be proved. My approach to the isomorphism theory will be more geometric and more general than the *CDC* approach that has been in use for the last ten years and that I used in the *Linear Lectures*. This geometric approach will be instrumental in extending the theory from subgroups of PSp_n ($n \geqslant 6$), where it is known, to subgroups of $\mathrm{P\Gamma Sp}_n$ ($n \geqslant 4$), where it is new([1]). There will be an extensive investigation and several new results([1]) on the exceptional behavior of subgroups of $\mathrm{P\Gamma Sp}_4$ in characteristic 2.

These notes are taken from lectures given at the University of Notre Dame during the school year 1974–1975. I would like to express my thanks to Alex Hahn, Kok-Wee Phan and Warren Wong for several stimulating discussions.

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Notre Dame, Indiana
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PREREQUISITES AND NOTATION

We assume a knowledge of the basic facts about sets, groups, fields and vector spaces.

If X and Y are sets, then pow X will denote the set of all subsets of X; $X \subset Y$ will denote strict inclusion; $X - Y$ will denote the difference set; $X \twoheadrightarrow Y$ will denote a surjection, $X \rightarrowtail Y$ an injection, $X \rightarrowtail\!\!\!\rightarrow Y$ a bijection, and $X \to Y$ an arbitrary mapping. If f: $X \to Y$ is a mapping and Z is a subset of X, i.e., Z is an element or point in pow X, then fZ is the subset $\{fz | z \in Z\}$ of Y; this provides a natural extension of f: $X \to Y$ to f: pow $X \to$ pow Y, namely the one obtained by sending Z to fZ for all Z in pow X; if f is respectively injective, surjective, bijective, then so is its extension to the power sets.

If X is any additive group, in particular, if X is a field or a vector space, then $\dot{X}$ will denote the set of nonzero elements of X; if X is a field, then $\dot{X}$ is to be regarded as a multiplicative group. Use $\mathbf{F}_q$ for the finite field of q elements. By a line, plane, hyperplane, in a finite n-dimensional vector space we mean a subspace of dimension 1, 2, $n - 1$, respectively.

V will always denote an n-dimensional vector space over a (commutative) field F with $0 \leqslant n < \infty$. After the appropriate definitions have been made (in fact, starting with Chapter 2) it will be assumed that V *is also a nonzero regular alternating space, i.e., that* V *is provided with a regular alternating form* q: $V \times V \to F$ *with* $2 \leqslant n < \infty$. *And* V_1, F_1, n_1, q_1 *will denote a second such situation.*

These lectures on the symplectic group are a sequel to:

> O. T. O'MEARA, *Lectures on linear groups*, CBMS Regional Conf. Ser. in Math., no. 22, Amer. Math. Soc., Providence, R.I., 1974, 87 pp.

which will be referred to as the *Linear Lectures*. In general we will try to keep things self-contained. Our general policy will be to redevelop concepts and restate propositions needed from the *Linear Lectures*, but not to rework proofs.

1. INTRODUCTION

1.1. Alternating Spaces

We say that a vector space V over the underlying field F is an alternating space if it is a composite object consisting of the vector space V and an alternating bilinear form q, i.e., a mapping $q\colon V \times V \rightarrowtail F$ with the properties

$$q(x, y + z) = q(x, y) + q(x, z),$$
$$q(x + y, z) = q(x, z) + q(y, z),$$
$$q(\alpha x, y) = \alpha q(x, y) = q(x, \alpha y),$$
$$q(x, x) = 0$$

for all x, y, z in V and all α in F. Note the consequence

$$q(x, y) = -q(y, x).$$

If $q\colon V \times V \rightarrowtail F$ is alternating, and if α is any element of F, then the mapping $q^{\alpha}\colon V \times V \rightarrowtail F$ defined by $q^{\alpha}(x, y) = \alpha q(x, y)$ is also alternating and the composite object consisting of the original vector space V with this new form q^{α} is an alternating space which will be written V^{α}.

A representation of an alternating space V into an alternating space W (both over F, both with forms written q) is, by definition, a linear transformation σ of V into W such that

$$q(\sigma x, \sigma y) = q(x, y) \qquad \forall x, y \in V.$$

An injective representation is called an isometry of V into W. And V and W are said to be isometric if there exists an isometry of V onto W. We let $V \rightarrowtail W$ denote a representation, $V \rightarrowtail W$ an isometry into, and $V \rightarrowtail\!\!\!\rightarrow W$ or $V \cong W$ an isometry onto. It is clear that the composite of two isometries is an isometry, and the inverse of an isometry is also an isometry. In particular the set of isometries of V onto V is a subgroup of the general linear group $\mathrm{GL}_n(V)$ of the abstract vector space V. This subgroup is called the symplectic group of the alternating space V and is written $\mathrm{Sp}_n(V)$. For any nonzero field element α we have $\mathrm{Sp}_n(V) = \mathrm{Sp}_n(V^{\alpha})$.

1.1.1. *Let σ be a linear transformation of an alternating space V into an alternating space W. Suppose there is a base $x_1, \ldots, x_n$ for V such that $q(x_i, x_j)$*

$= q(\sigma x_i, \sigma x_j)$ *for all* i, j. *Then* σ *is a representation.*

PROOF. This is a trivial consequence of the definitions. Q.E.D.

To each alternating space V with underlying alternating form q we associate mappings l and r of V into the dual space V' (of V regarded as an abstract vector space over F). The mapping l is the one taking a typical x in V to the linear functional $l(x)$ defined by $l(x)\colon y \to q(y, x)$, while r is the one taking x to $r(x)\colon y \to q(x, y)$. It is easily verified that $l\colon V \to V'$ and $r\colon V \to V'$ are linear transformations.

Recall that an $n \times n$ matrix A over F is called skewsymmetric if ${}^tA = -A$, alternating if ${}^tA = -A$ with 0's on the main diagonal. So alternating matrices are skewsymmetric. And skewsymmetric matrices are alternating when the characteristic of F is not 2.

Consider an alternating space V. We can associate a matrix B with a base $x_1, \ldots, x_n$ for V by forming the matrix B whose i, j entry is equal to $q(x_i, x_j)$. We call this matrix B the matrix of V in the base $x_1, \ldots, x_n$ and write

$$V \cong B \quad \text{in } x_1, \ldots, x_n.$$

If there is at least one base in which V has matrix B we write $V \cong B$. The matrix B associated with the alternating space V in the above way is clearly alternating. What happens under a change of base? Suppose $V \cong B'$ in $x'_1, \ldots, x'_n$, and let $T = (t_{ij})$ be the matrix carrying the first base to the second, i.e., let

$$x'_j = \sum_\lambda t_{\lambda j} x_\lambda;$$

then

$$q(x'_i, x'_j) = q\Big(\sum_\nu t_{\nu i} x_\nu, \sum_\lambda t_{\lambda j} x_\lambda\Big) = \sum_{\nu, \lambda} t_{\nu i} q(x_\nu, x_\lambda) t_{\lambda j},$$

so the equation

$$B' = {}^tTBT$$

describes the change in the matrix of V as the base is varied.

If V is just an abstract vector space with base $x_1, \ldots, x_n$, and if B is any $n \times n$ alternating matrix over F, there is a unique way of making V into an alternating space such that $V \cong B$ in $x_1, \ldots, x_n$: just define

$$q\Big(\sum_\nu \alpha_\nu x_\nu, \sum_\lambda \beta_\lambda x_\lambda\Big) = \sum_{\nu,\lambda} \alpha_\nu b_{\nu\lambda} \beta_\lambda,$$

where $b_{\nu\lambda}$ denotes the ν, λ entry of B.

1.1.2. *Suppose* V *is alternating, let* $\mathfrak{X}$ *be a base for* V, *and let* $V \cong A$ *in* $\mathfrak{X}$. *Then the matrix isomorphism determined by the base* $\mathfrak{X}$ *carries* $\mathrm{Sp}_n(V)$ *onto the group of all invertible* $n \times n$ *matrices* X *over* F *which satisfy the equation*

$${}^tXAX = A.$$

By the discriminant $d(z_1, \ldots, z_m)$ of vectors $z_1, \ldots, z_m$ in the alternating space V we mean the determinant

$$\det(q(z_i, z_j)).$$

In particular, if $x_1, \ldots, x_n$ is a base for V and if $V \cong B$ in this base, then

$$d(x_1, \ldots, x_n) = \det B.$$

If $x'_1, \ldots, x'_n$ is another base, the equation ${}^tTBT = B'$ shows that

$$d(x'_1, \ldots, x'_n) = \alpha^2 d(x_1, \ldots, x_n)$$

for some nonzero α in F. Hence the canonical image of $d(x_1, \ldots, x_n)$ in $0 \cup (\dot{F}/\dot{F}^2)$ is independent of the base; it is called the discriminant of the alternating space V, and it is written dV. The above set $0 \cup (\dot{F}/\dot{F}^2)$ is formed in the obvious way: Take the quotient group $\dot{F}/\dot{F}^2$, adjoin 0 to it, and define 0 times anything to be 0. If we write $dV = \beta$ with β in F we really mean that dV is equal to the canonical image of β in $0 \cup (\dot{F}/\dot{F}^2)$; this is equivalent to saying that V has a base $x_1, \ldots, x_n$ for which $d(x_1, \ldots, x_n) = \beta$. If $V = 0$ we define $dV = 1$.

1.1.3. Example. Consider an alternating space V with alternating form q, a base $\mathfrak{X} = \{x_1, \ldots, x_n\}$ for V, and the dual base $\mathfrak{X}' = \{y_1, \ldots, y_n\}$ for the dual space V' of V. Let $V \cong A$ in $\mathfrak{X}$. So $A = (a_{ij}) = (q(x_i, x_j))$. Then it is easy to see that the matrix of the linear transformation $l\colon V \rightarrowtail V'$ defined earlier with respect to the bases $\mathfrak{X}$ and $\mathfrak{X}'$ is equal to A; for if we write $lx_j = \Sigma_\lambda l_{\lambda j} y_\lambda$, then

$$a_{ij} = q(x_i, x_j) = l(x_j)(x_i) = \Big(\sum_\lambda l_{\lambda j} y_\lambda\Big)(x_i) = l_{ij}.$$

Similarly, the matrix of $r\colon V \rightarrowtail V'$ with respect to the bases $\mathfrak{X}$ and $\mathfrak{X}'$ is equal to ${}^tA$.

1.1.4. *Any m vectors $x_1, \ldots, x_m$ in an alternating space V with $d(x_1, \ldots, x_m) \neq 0$ are independent.*

Proof. A dependence $\Sigma_i \alpha_i x_i = 0$ yields $\Sigma_i \alpha_i q(x_i, x_j) = 0$ for $1 \leqslant j \leqslant m$; this is a dependence among the rows of the matrix $(q(x_i, x_j))$; and this is impossible since the discriminant is not 0. Q.E.D.

1.1.5. *The following are equivalent for an alternating space V:*

(1) $q(x, V) = 0 \Leftrightarrow x = 0$.

(2) $q(V, y) = 0 \Leftrightarrow y = 0$.

(3) $dV \neq 0$.

(4) *r is bijective.*

(5) *l is bijective.*

Proof. We can assume that $V \neq 0$. Fix a base $\mathfrak{X}$ for V and let $\mathfrak{X}'$ be its dual. Write $V \cong A$ in $\mathfrak{X}$. By Example 1.1.3 we have

$$\begin{aligned} dV \neq 0 &\Leftrightarrow \det A \neq 0 \\ &\Leftrightarrow A \text{ is invertible} \\ &\Leftrightarrow l \text{ is bijective;} \end{aligned}$$

hence (3) is equivalent to (5). Similarly (3) is equivalent to (4). And

$$l \text{ is bijective} \Leftrightarrow l(y) \neq 0 \quad \forall y \in \dot{V}$$
$$\Leftrightarrow l(y)V \neq 0 \quad \forall y \in \dot{V}$$
$$\Leftrightarrow q(V, y) \neq 0 \quad \forall y \in \dot{V}$$
$$\Leftrightarrow (q(V, y) = 0 \Rightarrow y = 0)$$
$$\Leftrightarrow (q(V, y) = 0 \Leftrightarrow y = 0)$$

so that (5) is equivalent to (2). Clearly (2) is equivalent to (1). Q.E.D.

1.1.6. Definition. An alternating space V is said to be regular if it satisfies any one of the five equivalent conditions in 1.1.5. An alternating space V is said to be degenerate if it is not regular. It is said to be totally degenerate if $q(V, V) = 0$.

If $V = 0$, then V is regular. If $V \neq 0$, then

$$V \text{ totally degenerate} \Rightarrow V \text{ degenerate}$$

by 1.1.5 and 1.1.6.

1.1.7. *Let σ: $V \rightarrowtail W$ be a representation of alternating spaces. If V is regular, then σ is an isometry.*

Proof. Take x in the kernel of σ. Then $q(x, V) \subseteq q(\sigma x, W) = 0$. So $x = 0$ by regularity. Q.E.D.

1.1.8. *To each base $\mathfrak{X} = \{x_1, \ldots, x_n\}$ of a regular alternating space V there corresponds a unique base $\mathfrak{Y} = \{y_1, \ldots, y_n\}$ of V, called the dual of $\mathfrak{X}$ with respect to q, such that $q(x_i, y_j) = \delta_{ij}$ for all i, j. If $V \cong A$ in $\mathfrak{X}$ and $V \cong B$ in $\mathfrak{Y}$, then $A = {}^tB^{-1}$.*

Proof. (1) Define $y_j = l^{-1}z_j$ for $1 \leqslant j \leqslant n$ where $z_1, \ldots, z_n$ denotes the dual of $\mathfrak{X}$ in the dual space V'. Then $\mathfrak{Y} = \{y_1, \ldots, y_n\}$ is a base since l is bijective. And

$$q(x_i, y_j) = l(y_j)(x_i) = z_j(x_i) = \delta_{ij}.$$

This proves the existence of $\mathfrak{Y}$. Uniqueness follows immediately from regularity.

(2) Write $y_j = \Sigma_\lambda t_{\lambda j} x_\lambda$. So $B = {}^tTAT$. Then

$$\sum_\lambda q(x_i, x_\lambda) t_{\lambda j} = q(x_i, y_j) = \delta_{ij},$$

so $AT = I$, so $B = {}^tT = {}^tA^{-1}$, so $A = {}^tB^{-1}$. Q.E.D.

Consider an alternating V with its associated alternating form q. We say that V has the orthogonal splitting

$$V = V_1 \perp \cdots \perp V_r$$

into subspaces $V_1, \ldots, V_r$ if V is the direct sum $V = V_1 \oplus \cdots \oplus V_r$ with the V_i pairwise orthogonal, i.e., with $q(V_i, V_j) = 0$ whenever $i \neq j$. We call the V_i the components of the orthogonal splitting. We say that the subspace U splits V, or that it is a component of V, if there exists a subspace W of V such that $V = U \perp W$. We have

$$d(V_1 \perp \cdots \perp V_r) = dV_1 \cdots dV_r,$$

the multiplication being performed in $0 \cup (\dot{F}/\dot{F}^2)$.

Consider two alternating spaces V and W over the same F, suppose we have an orthogonal splitting for $V = V_1 \perp \cdots \perp V_r$, and suppose W is a sum of subspaces $W = W_1 + \cdots + W_r$ with $q(W_i, W_j) = 0$ whenever $i \neq j$. Let a representation $\sigma_i \colon V_i \rightarrowtail W_i$ be given for each i $(1 \leqslant i \leqslant r)$. Then we know from linear algebra that there is a unique linear transformation σ of V into W which agrees with each σ_i on V_i. In fact it can easily be verified that this σ is actually a representation $\sigma \colon V \rightarrowtail W$. We write this representation in the form

$$\sigma = \sigma_1 \perp \cdots \perp \sigma_r.$$

The important case is where $V = W$, all $V_i = W_i$, and all $\sigma_i \in \mathrm{Sp}(V_i)$; in this event

$$\sigma_1 \perp \cdots \perp \sigma_r \in \mathrm{Sp}_n(V);$$

if we take another such $\tau_1 \perp \cdots \perp \tau_r$ we obtain the following rules:

$$(\sigma_1 \perp \cdots \perp \sigma_r)(\tau_1 \perp \cdots \perp \tau_r) = \sigma_1\tau_1 \perp \cdots \perp \sigma_r\tau_r,$$

$$(\sigma_1 \perp \cdots \perp \sigma_r)^{-1} = \sigma_1^{-1} \perp \cdots \perp \sigma_r^{-1},$$

$$\det(\sigma_1 \perp \cdots \perp \sigma_r) = (\det \sigma_1) \cdot \cdots \cdot (\det \sigma_r).$$

Consider an alternating space V over F. By the orthogonal complement U^* of a subspace U of V in V is meant the subspace

$$U^* = \{x \in V \mid q(x, U) = 0\}$$

which is also equal to

$$U^* = \{x \in V \mid q(U, x) = 0\}.$$

Define the radical of V to be the subspace rad $V = V^*$. Clearly

$$V \text{ regular} \Leftrightarrow \operatorname{rad} V = 0.$$

1.1.9. *Let V be alternating and suppose V is the sum of pairwise orthogonal subspaces, i.e., $V = V_1 + \cdots + V_r$ with $q(V_i, V_j) = 0$ whenever $i \neq j$. Then*

(1) rad V = rad $V_1 + \cdots +$ rad V_r.

(2) *V regular $\Leftrightarrow$ each V_i is regular.*

(3) *V regular $\Rightarrow V = V_1 \perp \cdots \perp V_r$.*

PROOF. (1) Take a typical x in rad V and write it $x = \Sigma x_\lambda$ with each x_λ in V_λ. Then for each i $(1 \leqslant i \leqslant r)$ we have

$$q(x_i, V_i) = q\left(\sum x_\lambda, V_i\right) \subseteq q(x, V) = 0,$$

so $x_i \in \operatorname{rad} V_i$, so $x \in \Sigma \operatorname{rad} V_i$. Conversely, if we take $x = \Sigma x_\lambda$ with each x_λ in rad V_λ, we have

$$q(x, V) \subseteq \sum q(x_\lambda, V_\lambda) = 0,$$

so $x \in \operatorname{rad} V$.

(2) This follows from (1) and the fact that an alternating space is regular if and only if its radical is 0.

(3) If $0 = \Sigma x_\lambda$ with each x_λ in V_λ, then

$$0 = q\left(\sum x_\lambda, V_i\right) = q(x_i, V_i),$$

so $x_i = 0$, so $V = V_1 \oplus \cdots \oplus V_r$, so $V = V_1 \perp \cdots \perp V_r$. Q.E.D.

1.1.10. *If U is any subspace of an alternating space V, then U^* is the annihilator in V of $l(U)$, i.e., $U^* = (l(U))^0$. In particular,* $\dim U + \dim U^* \geqslant n$.

PROOF. The proof follows directly from the definitions involved. Q.E.D.

1.1.11. *Let U be a regular subspace of an alternating space V. Then U splits V, in fact $V = U \perp U^*$. If $V = U \perp W$ is any other splitting, then $W = U^*$.*

PROOF. By regularity, $U \cap U^* = 0$. Hence by 1.1.10 we have

$$n \geqslant \dim(U + U^*) = \dim U + \dim U^* \geqslant n.$$

So $V = U \oplus U^*$. So $V = U \perp U^*$. Now consider $V = U \perp W$. Then $q(W, U) = 0$, so $W \subseteq U^*$, so $W = U^*$ by dimensions. Q.E.D.

1.1.12. *If U and W are arbitrary subspaces of a regular alternating space V, then*

(1) $\dim U + \dim U^* = n$.

(2) $U^{**} = U$.

(3) $(U + W)^* = U^* \cap W^*$.

(4) $(U \cap W)^* = U^* + W^*$.

(5) $\operatorname{rad} U = U \cap U^*$.

PROOF. Since V is regular l is bijective by 1.1.5; hence $\dim l(U) = \dim U$; hence $\dim U^* + \dim U = n$ by 1.1.10. This proves (1). Clearly $U \subseteq U^{**}$, so $U = U^{**}$ by dimensions. This proves (2). To prove (3),

$$\begin{aligned}(U + W)^* &= (l(U + W))^0 \\ &= (l(U) + l(W))^0 \\ &= (l(U))^0 \cap (l(W))^0 \\ &= U^* \cap W^*.\end{aligned}$$

Similarly with (4). Finally, (5) is trivial. Q.E.D.

Consider the radical $\operatorname{rad} V$ of the alternating space V and let U be any subspace of V for which $V = U \oplus \operatorname{rad} V$. Then clearly $V = U \perp \operatorname{rad} V$. We call any such splitting a radical splitting of V. Obviously U is not unique unless V is regular or V is totally degenerate. The equations

$$\operatorname{rad} V = \operatorname{rad} U \perp \operatorname{rad}(\operatorname{rad} V) = \operatorname{rad} U \perp \operatorname{rad} V$$

imply that $\operatorname{rad} U = 0$, and so U is regular.

1.1.13. THEOREM. *If V is a regular alternating space with $n > 0$, then*

$$V \cong \begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 0 & 1 \\ -1 & 0 \end{array} \end{pmatrix}$$

In particular, regular alternating spaces have even dimension and discriminant 1. *And regular alternating spaces of the same dimension over the same* F *are isometric.*

PROOF. By regularity we have vectors x and y in V with $q(x, y) = 1$. Since $q(x, x) = 0$, these two vectors must be independent, so $U = Fx + Fy$ is a plane. Clearly

$$U \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In particular U is regular by discriminants. Hence $V = U \perp W$ by 1.1.11. But W is also a regular alternating space. The first result then follows by an inductive argument. The second result is now trivial. To prove the third result apply 1.1.1. Q.E.D.

A base $\mathfrak{X}$ for a regular alternating space V is called hyperbolic if

$$V \cong \begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ -1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 0 & 1 \\ -1 & 0 \end{array} \end{pmatrix} \text{ in } \mathfrak{X},$$

it is called symplectic if

$$V \cong \left(\begin{array}{c|c} 0 & I_{n/2} \\ \hline -I_{n/2} & 0 \end{array}\right) \text{ in } \mathfrak{X}.$$

If

$$\mathfrak{X} = \{x_1, y_1, \ldots, x_{n/2}, y_{n/2}\}$$

is a hyperbolic base for V, then the rearrangement

$$\mathfrak{X}' = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

is symplectic, and conversely. A nonzero regular alternating space has a hyperbolic base by Theorem 1.1.13, and it therefore has a symplectic base too.

1.1.14. *Let V be a regular alternating space, let D be a totally degenerate subspace, and let $x_1, \ldots, x_r$ be a base for D. Then there is a regular subspace U of V of the form $U = P_1 \perp \cdots \perp P_r$ with P_i a regular plane and $x_i \in P_i$ for $1 \leqslant i \leqslant r$.*

PROOF. The case $r = 1$ is immediate. Proceed by induction to $r > 1$. Put $D_{r-1} = Fx_1 + \cdots + Fx_{r-1}$ and $D_r = D$. Then $D_{r-1} \subset D_r$, so $D_r^* \subset D_{r-1}^*$ by 1.1.12. Pick $y_r \in D_{r-1}^* - D_r^*$ and put $P_r = Fx_r + Fy_r$. Then $q(x_i, y_r) = 0$ for $1 \leqslant i \leqslant r-1$; hence $q(x_r, y_r) \neq 0$. Hence P_r is a regular plane containing x_r. By 1.1.11 we can write $V = P_r \perp P_r^*$. Then $P_r \subseteq D_{r-1}^*$ since $x_r \in D_{r-1}^*$ and $y_r \in D_{r-1}^*$; hence $D_{r-1} \subseteq P_r^*$. Apply an inductive argument to D_{r-1} regarded as a subspace of the regular alternating space P_r^*. Q.E.D.

1.1.15. *If M is a maximal totally degenerate subspace of a regular alternating space V, then* $\dim M = \frac{1}{2}\dim V$.

PROOF. We have $M \subseteq M^*$ since M is totally degenerate, so $\dim M \leqslant \dim M^* = n - \dim M$ by 1.1.12, so $\dim M \leqslant \frac{1}{2}\dim V$. If we had $\dim M < \frac{1}{2}\dim V$, then an easy application of 1.1.14 and 1.1.11 would produce a totally degenerate subspace strictly containing M, thereby denying the maximality of M. So $\dim M = \frac{1}{2}\dim V$. Q.E.D.

1.1.16. *If M_1 and M_2 are maximal totally degenerate subspaces of a regular alternating space V with $M_1 \cap M_2 = 0$, then, given a base $x_1, \ldots, x_r$ for M_1, there is a base $y_1, \ldots, y_r$ for M_2, such that $\{x_1, \ldots, x_r | y_1, \ldots, y_r\}$ is a symplectic base for V.*

PROOF. Of course $V = M_1 \oplus M_2$, by 1.1.15. Let $z_1, \ldots, z_r$ be a base for M_2. So $x_1, \ldots, x_r, z_1, \ldots, z_r$ is a base for V. Let $y_1, \ldots, y_r, y_{r+1}, \ldots, y_{2r}$ be the dual of this base with respect to q, as in 1.1.8. We have $M_2^* = M_2$, so $y_1, \ldots, y_r$ are in M_2 ($= M_2^*$), so $y_1, \ldots, y_r$ is a base for M_2, and

$$\{x_1, \ldots, x_r | y_1, \ldots, y_r\}$$

is clearly a symplectic base for V. Q.E.D.

1.1.17. *Suppose V is a regular alternating space and let*

$$\mathfrak{X} = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

be a symplectic base for V. Let M be the maximal totally degenerate space $Fx_1 + \cdots + Fx_{n/2}$. Then the matrix isomorphism associated with $\mathfrak{X}$ carries the group of linear transformations

$$\{\sigma \in \mathrm{Sp}_n(V) | \sigma M = M\}$$

onto the group of matrices of the form

$$\left(\begin{array}{c|c} {}^tC & B \\ \hline & C^{-1} \end{array}\right)$$

with C $\frac{1}{2}n \times \frac{1}{2}n$ invertible, and B $\frac{1}{2}n \times \frac{1}{2}n$ satisfying ${}^t(BC) = BC$.

PROOF. This can be verified by suitably applying 1.1.2. Q.E.D.

1.1.18. WITT'S THEOREM. *Let V and V' be isometric regular alternating spaces over the same field F. If U is any subspace of V, and if σ is an isometry of U into V', then there is a prolongation of σ to an isometry of V onto V'.*

PROOF. Take a radical splitting $U = W \perp \text{rad } U$ and let $x_1, \ldots, x_r$ be a base for rad U, with the understanding that $r = 0$ when rad $U = 0$. By applying 1.1.14 to the regular alternating space W^* we see that there is a subspace P of W^* of the form

$$P = P_1 \perp \cdots \perp P_r$$

in which P_i is a regular plane and $x_i \in P_i$ for $1 \leqslant i \leqslant r$. Since P is regular it splits W^*; hence there is a regular subspace S of W^* such that

$$V = P \perp S \perp W.$$

Put $U' = \sigma U$, $W' = \sigma W$ and $x_i' = \sigma x_i$ for $1 \leqslant i \leqslant r$. So

$$\text{rad } U' = \sigma(\text{rad } U) = Fx_1' + \cdots + Fx_r'.$$

And

$$U' = W' \perp \text{rad } U'$$

is a radical splitting. We can repeat the preceding argument to obtain a splitting

$$V' = P' \perp S' \perp W'$$

in which

$$P' = P_1' \perp \cdots \perp P_r'$$

where P_i' is a regular plane and $x_i' \in P_i'$ for $1 \leqslant i \leqslant r$. By suitably applying 1.1.1 we can find an isometry of P onto P' which agrees with σ on each x_i, and hence on rad U. Also, the given σ carries W to W'. Hence there is a prolongation of σ to an isometry of $P \perp W$ onto $P' \perp W'$. Now dim V = dim V' since V is given isometric to V'; hence dim S = dim S'; hence there is an isometry of S onto S' by Theorem 1.1.13. Hence there is a prolongation of σ to an isometry of $V = (P \perp W) \perp S$ onto $V' = (P' \perp W') \perp S'$. Q.E.D.

1.2. Projective Transformations

A geometric transformation g of the abstract vector space V onto the abstract vector space V_1 is a bijection $g\colon V \rightarrowtail\!\!\!\rightarrow V_1$ which has the following property for all subsets X of V: X is a subspace of V if and only if gX is a subspace of V_1.

It is clear that a composition of geometric transformations is geometric, and that the inverse of a geometric transformation is also geometric. If $g\colon V \rightarrowtail\!\!\!\rightarrow V_1$ is a geometric transformation, then g preserves inclusion, join, meet, Jordan-Hölder chains, among subspaces. So we have the following proposition.

1.2.1. *If g is a geometric transformation of V onto V_1, then*

$$g(U \cap W) = gU \cap gW, \qquad g(U + W) = gU + gW,$$
$$\dim_{F_1} gU = \dim_F U,$$
$$g0 = 0, \qquad gV = V_1,$$

holds for all subspaces U and W of V.

By the projective space $P(V)$ of V we mean the set of all subspaces of V. Thus $P(V)$ consists of the elements of pow V which are subspaces of V; $P(V)$ is a partially ordered set, the order relation being provided by set inclusion in V; any two elements U and W of $P(V)$ have a join and a meet, namely the subspaces $U + W$ and $U \cap W$, so that $P(V)$ is a lattice; $P(V)$ has an absolutely largest element V, and an absolutely smallest element 0; to each element U of $P(V)$ we attach the number $\dim_F U$; each U in $P(V)$ has a Jordan-Hölder chain $0 \subset \cdots \subset U$, and all such Jordan-Hölder chains have length $1 + \dim_F U$. Define

$$P^i(V) = \{U \in P(V) | \dim_F U = i\}$$

and call $P^1(V)$, $P^2(V)$, $P^{n-1}(V)$, the set of lines, planes, hyperplanes, of V.

A projectivity π of V onto V_1 is a bijection π: $P(V) \rightarrowtail\!\!\!\rightarrow P(V_1)$ which has the following property for all U, W in $P(V)$: $U \subseteq W$ if and only if $\pi U \subseteq \pi W$.

It is clear that a composition of projectivities is a projectivity, and that the inverse of a projectivity is also a projectivity. If π: $P(V) \rightarrowtail\!\!\!\rightarrow P(V_1)$ is a projectivity of V onto V_1, then π preserves order, join, meet, Jordan-Hölder chains, among the elements of $P(V)$ and $P(V_1)$. So we have the following proposition.

1.2.2. *If* π: $P(V) \rightarrowtail\!\!\!\rightarrow P(V_1)$ *is a projectivity of V onto V_1, then*

$$\pi(U \cap W) = \pi U \cap \pi W, \qquad \pi(U + W) = \pi U + \pi W,$$
$$\dim_{F_1} \pi U = \dim_F U,$$
$$\pi 0 = 0, \qquad \pi V = V_1,$$

holds for all elements U and W of $P(V)$. *In particular* π *carries* $P^1(V)$ *onto* $P^1(V_1)$, *and* π *is determined by its values on* $P^1(V)$, *i.e.,* π *is determined by its values on lines.*

If g: $V \rightarrowtail\!\!\!\rightarrow V_1$ is geometric, then the mapping $\bar{g}$: $P(V) \rightarrowtail\!\!\!\rightarrow P(V_1)$ obtained from g: pow $V \rightarrowtail\!\!\!\rightarrow$ pow V_1 by restriction is a projectivity of V onto V_1. Any projectivity π: $P(V) \rightarrowtail\!\!\!\rightarrow P(V_1)$ which has the form $\pi = \bar{g}$ for such a g will be called a projective geometric transformation of V onto V_1. The bar symbol will always be used to denote the projective geometric transformation $\bar{g}$ obtained from a geometric transformation g in the above way. So $\bar{g}$ sends the subspace U of V, i.e., the point U in $P(V)$, to the subspace gU of V_1. We have

$$\overline{g_1 \cdots g_t} = \bar{g}_1 \cdots \bar{g}_t$$

under composition, and

$$\bar{g}^{-1} = \overline{g^{-1}}$$

for inverses. In particular, composites and inverses of projective geometric transformations are themselves projective geometric transformations.

A geometric transformation of V is, by definition, a geometric transformation of V onto V. The set of geometric transformations of V is a subgroup of the group of permutations of V. It will be written $\Xi L_n(V)$ and will be called the general geometric group of V. By a group of geometric transformations of V we mean any subgroup of $\Xi L_n(V)$. The general linear group $GL_n(V)$, and the special linear group $SL_n(V) = \{\sigma \in GL_n(V) | \det \sigma = 1\}$, are therefore groups of geometric transformations. By a group of linear transformations of V we mean any subgroup of $GL_n(V)$.

A projectivity of V is, by definition, a projectivity of V onto V. The set of projectivities of V is a subgroup of the group of permutations of $P(V)$ which will be called the group of projectivities of V. The bar mapping then provides a homomorphism

$$^{-}: \Xi L_n(V) \twoheadrightarrow \text{group of projectivities of } V.$$

We sometimes use P instead of $^-$ and put

$$PX = \bar{X}$$

for the image $\bar{X}$ of a subset X of $\Xi L_n(V)$ under P. In particular $PGL_n(V)$ and $PSL_n(V)$ are subgroups of the group of projectivities of V called, respectively, the projective general linear group and the projective special linear group of V. It was established in the *Linear Lectures* that $P\Xi L_n(V)$ is the entire group of projectivities of V and so we use this symbol for this group. By a group of projectivities of V we mean any subgroup of $P\Xi L_n(V)$. By a projective group of linear transformations of V we mean any subgroup of $PGL_n(V)$.

For any nonzero α in F define the linear transformation r_α by

$$r_\alpha x = \alpha x \qquad \forall\, x \in V.$$

Thus r_α is in $GL_n(V)$. Any σ in $GL_n(V)$ which has the form $\sigma = r_\alpha$ for some such α will be called a radiation of V. The set of radiations of V is a normal subgroup of $GL_n(V)$ which will be written $RL_n(V)$. The isomorphism $RL_n \rightarrowtail \dot{F}$ is obvious. The following two propositions were established in the *Linear Lectures*.

1.2.3. *Let σ be an element of $GL_n(V)$. Then σ is in $RL_n(V)$ if and only if $\sigma L = L$ for all lines L in V. In particular,*

$$\ker(P|GL_n) = RL_n, \qquad \ker(P|SL_n) = SL_n \cap RL_n$$

and

$$PGL_n \cong GL_n/RL_n, \qquad PSL_n \cong SL_n/(SL_n \cap RL_n).$$

1.2.4. *The centralizer in $GL_2(V)$ of a nonradiation in $GL_2(V)$ is abelian.*

All that we have said so far in §1.2 is for abstract vector spaces and is taken directly from the *Linear Lectures*. Now let V be, in addition, a regular alternating space. Then $\mathrm{Sp}_n(V)$ is, of course, a group of geometric transformations of V. By a group of symplectic transformations of the alternating space V we mean any subgroup of $\mathrm{Sp}_n(V)$. The group $\mathrm{PSp}_n(V)$ is the group obtained by applying the homomorphism P to $\mathrm{Sp}_n(V)$, and it is called the projective symplectic group of the alternating space V. By a projective group of symplectic transformations of V we mean any subgroup of $\mathrm{PSp}_n(V)$.

1.2.5. *If V is a nonzero regular alternating space, then*

$$\mathrm{Sp}_n(V) \cap \mathrm{RL}_n(V) = (\pm 1_V),$$

$$\ker(\mathrm{P}|\mathrm{Sp}_n) = (\pm 1_V),$$

$$\mathrm{PSp}_n \cong \mathrm{Sp}_n/(\pm 1_V).$$

PROOF. The proof is left as an easy exercise. Q.E.D.

1.2.6. *If V is a regular alternating space with* $\dim V = 2$, *then* $\mathrm{Sp}_2(V) = \mathrm{SL}_2(V)$.

PROOF. Take a symplectic base $\mathfrak{X}$ for V and use 1.1.2 to show that an element σ of $\mathrm{GL}_2(V)$ is in $\mathrm{Sp}_2(V)$ if and only if $\det \sigma = 1$. Q.E.D.

A polarity of an abstract vector space V over F is a bijection $\mathrm{P}(V) \rightarrowtail\!\!\!\!\rightarrow \mathrm{P}(V)$, denoted $U \leftrightarrow U^{\#}$, such that

(1) $U \subseteq W \Leftrightarrow U^{\#} \supseteq W^{\#}$,

(2) $U^{\#\#} = U$,

for all U, W in $\mathrm{P}(V)$. If V is actually a regular alternating space over F, then $U \leftrightarrow U^*$ defines a polarity, called the polarity determined by the underlying alternating form q.

1.2.7. *Let V be an abstract vector space over F with $n \geqslant 2$. Suppose V is a regular alternating space under each of two alternating forms q_1 and q_2. Then q_1 and q_2 determine the same polarity if and only if there is a nonzero α in F such that $q_1 = q_2^{\alpha}$.*

PROOF. If $q_1^{\alpha} = q_2^{\alpha}$ the result is clear. We must prove the converse. Since V is regular under q_1 and q_2, the associated linear mappings l_1 and l_2 are bijective by 1.1.5 and 1.1.6, i.e., $l_1\colon V \rightarrowtail\!\!\!\!\rightarrow V'$ and $l_2\colon V \rightarrowtail\!\!\!\!\rightarrow V'$. It follows from 1.1.10 and the hypothesis that q_1 and q_2 determine the same polarity that $l_1(U) = l_2(U)$ for all subspaces U of V. Hence $l_2^{-1}l_1$ is an element of $\mathrm{GL}_n(V)$ that stabilizes all subspaces of V. In particular, $l_2^{-1}l_1$ stabilizes all lines of V. Hence $l_2^{-1}l_1 \in \mathrm{RL}_n(V)$ by 1.2.3. In other words there is a nonzero α in F such that $l_1(x) = l_2(\alpha x)$ for all x in V. But then $q_1(y, x) = q_2(y, \alpha x)$ for all y in V. So $q_1 = q_2^{\alpha}$. Q.E.D.

1.3. Residues

Suppose that V is an abstract vector space over F. Consider σ in $\mathrm{GL}_n(V)$. The residual space R, the fixed space P, and the residue, $\operatorname{res} \sigma$, of σ are defined by the equations

$$R = (\sigma - 1_V)V, \qquad P = \ker(\sigma - 1_V),$$
$$\text{res}\,\sigma = \dim R.$$

The subspaces R and P of V are called the spaces of σ. We have

$$\dim R + \dim P = n,$$
$$\sigma R = R, \qquad \sigma P = P,$$
$$\text{res}\,\sigma = 0 \Leftrightarrow \sigma = 1_V.$$

Obviously σ and σ^{-1} have the same R, P, res; and $P = \{x \in V | \sigma x = x\}$. If R is a line, plane, hyperplane, etc., we also refer to it as the residual line, etc., of σ. Similarly with the fixed line, etc.

1.3.1. *Convention.* Whenever a σ in $\mathrm{GL}_n(V)$ is under discussion, the letter R will automatically refer to the residual space of σ, the letter P to the fixed space. In the same way R_i and P_i will be associated with σ_i in $\mathrm{GL}_n(V)$.

1.3.2. *Let σ_1 and σ_2 be elements of* $\mathrm{GL}_n(V)$ *and put $\sigma = \sigma_1\sigma_2$. Then*

$$R \subseteq R_1 + R_2, \qquad P \supseteq P_1 \cap P_2,$$
$$\text{res}\,\sigma_1\sigma_2 \leqslant \text{res}\,\sigma_1 + \text{res}\,\sigma_2.$$

1.3.3. *Let σ_1 and σ_2 be elements of* $\mathrm{GL}_n(V)$ *and put $\sigma = \sigma_1\sigma_2$. Then*
(1) $V = P_1 + P_2 \Rightarrow R = R_1 + R_2$.
(2) $R_1 \cap R_2 = 0 \Rightarrow P = P_1 \cap P_2$.

1.3.4. *Let σ and Σ be elements of* $\mathrm{GL}_n(V)$. *Then the residual and fixed spaces of $\Sigma\sigma\Sigma^{-1}$ are ΣR and ΣP respectively. In particular* $\text{res}\,\Sigma\sigma\Sigma^{-1} = \text{res}\,\sigma$; *and $\sigma\Sigma = \Sigma\sigma$ implies $\Sigma R = R$ and $\Sigma P = P$.*

1.3.5. *Let σ_1 and σ_2 be elements of* $\mathrm{GL}_n(V)$. *Then $R_1 \subseteq P_2$ and $R_2 \subseteq P_1$ makes $\sigma_1\sigma_2 = \sigma_2\sigma_1$.*

1.3.6. *Let σ_1 and σ_2 be elements of* $\mathrm{GL}_n(V)$ *with $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Then*

$$R_1 \subseteq P_2 \quad \textit{and} \quad R_2 \subseteq P_1$$

provided either $R_1 \cap R_2 = 0$ or $V = P_1 + P_2$.

1.3.7. *Let σ be any element of* $\mathrm{GL}_n(V)$. *Then $\sigma^2 = 1_V$ if and only if $(\sigma|R) = -1_R$.*

An element σ in an arbitrary group with $\sigma^2 = 1$ is called an involution.

1.3.8. *Let $\sigma \neq 1_V$ be any element of* $\mathrm{GL}_n(V)$. *Then* $\det(\sigma|R) = \det\sigma$.

1.3.9. *If $V = V_1 \oplus V_2$ and $\sigma = \sigma_1 \oplus \sigma_2$ with $\sigma_1 \in \mathrm{GL}_{n_1}(V_1)$ and $\sigma_2 \in \mathrm{GL}_{n_2}(V_2)$, then*

$$R = R_1 \oplus R_2, \qquad P = P_1 \oplus P_2.$$

All that we have said so far in §1.3 is for abstract vector spaces and is taken directly from §1.3 of the *Linear Lectures*–only the proofs of the propositions have been omitted and these can be found by referring to the *Linear Lectures*. Now assume that V is, in addition, a regular alternating space.

1.3.10. *If V is a regular alternating space and $\sigma \in \mathrm{Sp}_n(V)$, then $R = P^*$ and $q(R, P) = 0$.*

PROOF. For any $x \in V$ and $p \in P$ we have

$$q(\sigma x - x, p) = q(\sigma x, p) - q(x, p) = q(\sigma x, p) - q(\sigma x, p) = 0,$$

so $q(R, P) = 0$, so $R \subseteq P^*$. But

$$\dim R = n - \dim P = \dim P^*.$$

So $R = P^*$. Q.E.D.

1.3.11. *If V is a regular alternating space and $\sigma = \sigma_1\sigma_2$ with $\sigma, \sigma_1, \sigma_2$ in $\mathrm{Sp}_n(V)$, then the following are true:*

(1) $V = P_1 + P_2$ implies $R = R_1 + R_2$ and $P = P_1 \cap P_2$;

(2) $R_1 \cap R_2 = 0$ implies $P = P_1 \cap P_2$ and $R = R_1 + R_2$;

(3) if $R_1 \subseteq P_2$, or $R_2 \subseteq P_1$, or $q(R_1, R_2) = 0$, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

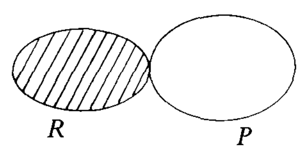
PROOF. The first part of (1) is contained in 1.3.3 as is the first part of (2). To prove the second part of (1) we note that by 1.3.10 and 1.1.12,

$$V = P_1 + P_2 \Rightarrow 0 = R_1 \cap R_2 \Rightarrow P = P_1 \cap P_2.$$

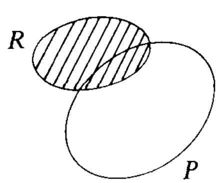
Similarly with the second part of (2). So consider (3). If $R_1 \subseteq P_2$, then $R_2 \subseteq P_1$ by 1.3.10, so $\sigma_1\sigma_2 = \sigma_2\sigma_1$ by 1.3.5. Similarly if $R_2 \subseteq P_1$. If $q(R_1, R_2) = 0$, then $R_1 \subseteq R_2^* = P_2$. Q.E.D.

1.3.12. DEFINITION. By a regular, degenerate, totally degenerate, element of $\mathrm{Sp}_n(V)$ (where V is a regular alternating space) we mean an element σ of $\mathrm{Sp}_n(V)$ whose R is, respectively, regular, degenerate, totally degenerate.

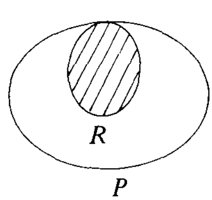
So σ regular is the same as $R \cap P = 0$, i.e., the action of σ is described pictorially by



while σ degenerate is equivalent to $R \cap P \neq 0$, i.e.,



and σ totally degenerate means $R \subseteq P$, i.e.,



1.3.13. *Let V be a regular alternating space. Then the following are true.*

(1) *If* char $F \neq 2$, *every involution in* $\mathrm{Sp}_n(V)$ *is regular.*

(2) *If* char $F = 2$, *the involutions in* $\mathrm{Sp}_n(V)$ *are the totally degenerate elements of* $\mathrm{Sp}_n(V)$.

PROOF. Apply 1.3.7. Q.E.D.

1.4. Transvections

Once again let us start out by assuming that V is just an abstract vector space.

An element σ in $\mathrm{GL}_n(V)$ is called a transvection if $\sigma = 1_V$ or if res $\sigma = 1$ with det $\sigma = 1$.

If σ is a transvection then so is $\Sigma\sigma\Sigma^{-1}$ for any Σ in $\mathrm{GL}_n(V)$.

1.4.1. *Let σ be an element of* $\mathrm{GL}_n(V)$ *with* res $\sigma = 1$ *and* $n \geqslant 2$. *Then*

(1) $R \subseteq P$ *if and only if σ is a transvection*;

(2) *If σ is a transvection, its characteristic vectors are* $\dot{P}$, *its characteristic roots are all* 1.

For any $a \in V$, $\rho \in V'$ with $\rho a = 0$, define the linear transformation $\tau_{a,\rho}$ of V into V by the equation

$$\tau_{a,\rho}x = x + (\rho x)a \qquad \forall\, x \in V.$$

One easily sees that

$$\tau_{a,\rho} \in \mathrm{GL}_n(V),$$

$$\tau_{a,\rho} = 1_V \Leftrightarrow a = 0 \text{ or } \rho = 0,$$

$$\tau_{\lambda a,\rho} = \tau_{a,\lambda\rho} \qquad \forall\, \lambda \in F,$$

and, if $\tau_{a,\rho} \neq 1_V$, that $\tau_{a,\rho}$ is a transvection with residual line Fa and fixed hyperplane ker ρ. In particular

$$\det \tau_{a,\rho} = 1.$$

1.4.2. *If $\tau_{a,\rho}$ and $\tau_{a',\rho'}$ are defined and not equal to 1_V, then $\tau_{a,\rho} = \tau_{a',\rho'}$ if and only if there is a λ in $\dot{F}$ such that $a' = \lambda a$ and $\rho' = \lambda^{-1}\rho$.*

We have

$$\tau_{a,\rho}\tau_{b,\varphi}x = \{x + (\rho x)a + (\varphi x)b\} + (\varphi x)(\rho b)a,$$

$$\tau_{a,\rho}\tau_{b,\rho} = \tau_{a+b,\rho},$$

$$\tau_{a,\rho}\tau_{a,\varphi} = \tau_{a,\rho+\varphi},$$

$$\tau_{a,\rho}^m = \tau_{ma,\rho} \qquad (m \in \mathbf{Z}),$$

$$\sigma\tau_{a,\rho}\sigma^{-1} = \tau_{\sigma a,\rho\sigma^{-1}} \qquad (\sigma \in \mathrm{GL}_n(V)),$$

provided all the τ's on the left are defined.

1.4.3. *Suppose* dim $V \geqslant 2$. *Let L be a line and H a hyperplane in V with $L \subseteq H$. Then there is a transvection σ in* $\mathrm{GL}_n(V)$ *with $R = L$ and $P = H$.*

1.4.4. *Let σ be any transvection in* $\mathrm{GL}_n(V)$ *with* $\sigma \neq 1_V$. *So* R *is a line and* P *is a hyperplane with* $R \subseteq P$. *If we take any nonzero* a *in* R *and any nonzero linear functional* ρ *annihilating* P, *then there is a* λ *in* F *such that* $\sigma = \tau_{\lambda a,\rho}$.

1.4.5. *Let* τ_1 *and* τ_2 *be transvections in* $\mathrm{GL}_n(V)$ *with* $n > 0$ *and let* $\alpha \in \dot{F}$. *Then* $r_\alpha \tau_1 = \tau_2$ *if and only if* $\alpha = 1$ *with* $\tau_1 = \tau_2$. *In particular,* $r_\alpha \tau_1$ *is not a transvection when* $\alpha \neq 1$.

1.4.6. *Let* σ_1 *and* σ_2 *be elements of* $\mathrm{GL}_n(V)$ *of residue* 1 *with* $\sigma_1\sigma_2 \neq 1_V$. *Then* $\operatorname{res} \sigma_1\sigma_2 = 1$ *if and only if* $R_1 = R_2$ *or* $P_1 = P_2$.

1.4.7. *Let* σ_1 *and* σ_2 *be nontrivial transvections in* $\mathrm{GL}_n(V)$. *Then* $\sigma_1\sigma_2$ *is a transvection if and only if* $R_1 = R_2$ *or* $P_1 = P_2$.

1.4.8. *Let* X *be a subgroup of* $\mathrm{GL}_n(V)$ *that consists entirely of transvections. Then all nontrivial elements of* X *either have the same residual line, or they all have the same fixed hyperplane.*

1.4.9. *Let* σ_1 *and* σ_2 *be nontrivial transvections in* $\mathrm{GL}_n(V)$. *Then* $\sigma_1\sigma_2 = \sigma_2\sigma_1$ *if and only if* $R_1 \subseteq P_2$ *and* $R_2 \subseteq P_1$.

All that we have said so far in §1.4 is for abstract vector spaces and is taken directly from §1.4 of the *Linear Lectures*–only the proofs of the propositions have been omitted. Now assume that V is, in addition, a regular alternating space with its alternating form q. For any $a \in V$, $\lambda \in F$, define the linear transformation $\tau_{a,\lambda}$ of V into V by the equation

$$\tau_{a,\lambda} x = x + \lambda q(x, a)a \qquad \forall\, x \in V.$$

One easily sees that

$$\tau_{a,\lambda} \in \mathrm{Sp}_n(V),$$
$$\tau_{a,\lambda} = 1_V \Leftrightarrow a = 0 \text{ or } \lambda = 0,$$
$$\tau_{\alpha a,\lambda} = \tau_{a,\alpha^2\lambda},$$

and, if $\tau_{a,\lambda} \neq 1_V$, that $\tau_{a,\lambda}$ is a transvection with residual line Fa. (The fixed hyperplane is $(Fa)^*$ but there is no need to emphasize this, by 1.3.10.)

1.4.10. *If* $\tau_{a,\lambda}$ *and* $\tau_{a',\lambda'}$ *are defined and not equal to* 1_V, *then* $\tau_{a,\lambda} = \tau_{a',\lambda'}$ *if and only if there is an* α *in* $\dot{F}$ *such that* $a' = \alpha a$ *and* $\alpha^2\lambda' = \lambda$. *In particular,*

$$\tau_{a,\lambda} = \tau_{a,\lambda'} \Leftrightarrow \lambda = \lambda',$$
$$\tau_{a,\lambda} = \tau_{a',\lambda} \Leftrightarrow a = \pm a'.$$

We have

$$\tau_{a,\lambda}\tau_{b,\nu} x = x + \lambda\big(q(x, a) + \nu q(x, b)q(b, a)\big)a + \nu q(x, b)b;$$
$$\tau_{a,\lambda}\tau_{a,\nu} = \tau_{a,\lambda+\nu},$$
$$\tau_{a,\lambda}^m = \tau_{a,m\lambda} \qquad (m \in \mathbf{Z}),$$
$$\sigma\tau_{a,\lambda}\sigma^{-1} = \tau_{\sigma a,\lambda} \qquad (\sigma \in \mathrm{Sp}_n(V)).$$

1.4.11. *If V is a regular alternating space with* $\dim V \geqslant 2$, *and if L is any line in V, then there is a transvection σ in* $\mathrm{Sp}_n(V)$ *with $R = L$.*

1.4.12. *Let V be a regular alternating space with* $\dim V \geqslant 2$ *and let σ be any transvection in* $\mathrm{Sp}_n(V)$ *with $\sigma \neq 1_V$. So R is a line. If we take any nonzero a in R, then there is a λ in F such that $\sigma = \tau_{a,\lambda}$.*

Proof. By 1.4.4 there is a linear functional ρ with $\rho(Fa)^* = 0$ such that $\sigma x = x + (\rho x)a$. By 1.1.5 we have $b \in V$ such that $l(b) = \rho$, i.e., $\rho x = q(x, b)$ for all x in V. Then $q((Fa)^*, b) = 0$, so $b \in (Fa)^{**}$, i.e., $b = \lambda a$ for some λ in F, i.e., σx has the desired form $x + \lambda q(x, a)a$. Q.E.D.

1.4.13. *Let V be a regular alternating space and let σ_1 and σ_2 be nontrivial transvections in* $\mathrm{Sp}_n(V)$. *Then $\sigma_1\sigma_2$ is a transvection if and only if $R_1 = R_2$.*

Proof. Apply 1.4.7 and 1.3.10. Q.E.D.

1.4.14. *Let V be a regular alternating space and let X be a subgroup of* $\mathrm{Sp}_n(V)$ *that consists entirely of transvections. Then all nontrivial elements of X have the same residual line.*

Proof. Apply 1.4.8 and 1.3.10. Q.E.D.

1.4.15. *Let V be a regular alternating space and let σ_1 and σ_2 be nontrivial transvections in* $\mathrm{Sp}_n(V)$. *Then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if and only if $q(R_1, R_2) = 0$.*

Proof. Apply 1.4.9 and 1.3.10. Q.E.D.

1.5. Matrices

We shall use $\mathrm{GL}_n(F)$ to denote the multiplicative group of invertible $n \times n$ matrices over F, and $\mathrm{SL}_n(F)$ for the subgroup consisting of those matrices of determinant 1. The group of scalar matrices, i.e., the group of matrices of the form $\mathrm{diag}(\alpha, \ldots, \alpha)$ with α in $\dot{F}$, will be written $\mathrm{RL}_n(F)$. For any even integer $n \geqslant 2$ define $\mathrm{Sp}_n(F)$ as the subgroup of $\mathrm{GL}_n(F)$ consisting of all $n \times n$ matrices X over F which satisfy the equation

$$ {}^tX \left[\begin{array}{c|c} 0 & I_{n/2} \\ \hline -I_{n/2} & 0 \end{array}\right] X = \left[\begin{array}{c|c} 0 & I_{n/2} \\ \hline -I_{n/2} & 0 \end{array}\right]. $$

If we fix a base $\mathfrak{X}$ for V, then the associated isomorphism of linear transformations to matrices induces

$$ \mathrm{GL}_n(V) \rightarrowtail \mathrm{GL}_n(F), $$
$$ \mathrm{SL}_n(V) \rightarrowtail \mathrm{SL}_n(F), $$
$$ \mathrm{RL}_n(V) \rightarrowtail \mathrm{RL}_n(F). $$

If V is, in addition, a regular alternating space and if $\mathfrak{X}$ is a symplectic base for V, then

$$ \mathrm{Sp}_n(V) \rightarrowtail \mathrm{Sp}_n(F) $$

by 1.1.2.

For matrices we define P as the natural homomorphism

$$\mathrm{P}\colon \mathrm{GL}_n(F) \to \mathrm{GL}_n(F)/\mathrm{RL}_n(F).$$

Thus $\mathrm{PSL}_n(F)$ is the image of $\mathrm{SL}_n(F)$ in $\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/\mathrm{RL}_n(F)$. The kernel of P restricted to $\mathrm{SL}_n(F)$ is, of course, $\mathrm{SL}_n(F) \cap \mathrm{RL}_n(F)$. If $\mathfrak{X}$ is a base for V, then $\mathrm{GL}_n(V) \rightarrowtail \mathrm{GL}_n(F)$ induces

$$\mathrm{PGL}_n(V) \rightarrowtail \mathrm{PGL}_n(F)$$

via cosets modulo RL_n; we call this the projective isomorphism associated with $\mathfrak{X}$. It induces

$$\mathrm{PSL}_n(V) \rightarrowtail \mathrm{PSL}_n(F).$$

If V is regular alternating and we fix a symplectic base $\mathfrak{X}$ for V it induces

$$\mathrm{PSp}_n(V) \rightarrowtail \mathrm{PSp}_n(F).$$

Given $n \geqslant 2$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$ and $\lambda \in F$, we use $t_{ij}(\lambda)$ to denote the elementary $n \times n$ matrix with 1's on the diagonal, λ in the (i, j) position, and 0's everywhere else.

Suppose $n \geqslant 2$. Fix a base $x_1, \ldots, x_n$ for the abstract vector space V, and let $\rho_1, \ldots, \rho_n$ be the corresponding dual base for V'. By an elementary transvection with respect to the base $x_1, \ldots, x_n$ we mean any transvection of the form $\tau_{\lambda x_i, \rho_j}$ $(i \neq j)$ with λ in F. It is easy to see that every transvection is an elementary transvection with respect to some base for V. The matrix correspondence associated with the base $x_1, \ldots, x_n$ establishes the correspondence $\tau_{\lambda x_i, \rho_j} \leftrightarrow t_{ij}(\lambda)$ between elementary transvections and elementary matrices.

1.6. Projective Transvections

Call an element k of the group of projectivities of V a projective transvection if it is of the form $k = \bar{\sigma}$ for some transvection σ in $\mathrm{GL}_n(V)$. It follows from 1.4.5 that the transvection σ representing a projective transvection $k = \bar{\sigma}$ is unique, and we accordingly call it the representative transvection of k. Define the residual and fixed spaces of a projective transvection as the corresponding spaces of its representative transvection. The R, P convention of §1.3 will be extended to projective transvections–if, for example, a projective transvection σ in $\mathrm{PGL}_n(V)$ (or $\bar{\sigma}$ with σ in $\mathrm{GL}_n(V)$) is under discussion, then R and P will automatically refer to its residual and fixed spaces respectively. Note that we make no attempt to define residual and fixed spaces for arbitrary elements of $\mathrm{PGL}_n(V)$. Of course, if σ is a projective transvection with $\sigma \neq 1$, then R is a line, P is a hyperplane, and $R \subseteq P$. If, on the other hand, we are given a line L, a hyperplane H, and $L \subseteq H$, then there always exists a projective transvection σ in $\mathrm{PGL}_n(V)$ having $R = L$ and $P = H$. If σ is a projective transvection and Σ is any element of $\mathrm{PGL}_n(V)$, then $\Sigma\sigma\Sigma^{-1}$ is also a projective transvection, and its spaces are ΣR and ΣP respectively. In particular

$$\sigma\Sigma = \Sigma\sigma \Rightarrow \Sigma R = R \text{ and } \Sigma P = P.$$

Note that we sometimes describe elements of $\mathrm{PGL}_n(V)$ in the form $\bar{\sigma}$ with σ in $\mathrm{GL}_n(V)$, and at other times in the form σ with σ in $\mathrm{PGL}_n(V)$.

1.6.1. *Suppose* dim $V \geqslant 3$. *Let* σ_1 *and* σ_2 *be nontrivial projective transvections in* $\mathrm{PGL}_n(V)$. *Then* $\sigma_1\sigma_2$ *is a projective transvection if and only if* $R_1 = R_2$ *or* $P_1 = P_2$.

1.6.2. *Let* X *be a subgroup of* $\mathrm{PGL}_n(V)$ *that consists entirely of projective transvections. Then all nontrivial elements of* X *either have the same residual line, or they all have the same fixed hyperplane.*

1.6.3. *Let* σ_1 *and* σ_2 *be nontrivial projective transvections in* $\mathrm{PGL}_n(V)$. *Then* $\sigma_1\sigma_2 = \sigma_2\sigma_1$ *if and only if* $R_1 \subseteq P_2$ *and* $R_2 \subseteq P_1$.

So far in §1.6, V has just been an abstract vector space over F–see the *Linear Lectures* for proofs. Now let V be, in addition, a regular alternating space.

1.6.4. *If* V *is a regular alternating space then the representative transvection of a projective transvection in* $\mathrm{PSp}_n(V)$ *belongs to* $\mathrm{Sp}_n(V)$.

PROOF. By 1.2.6 we can assume that $n \geqslant 4$. It is enough to show that if σ is a transvection in $\mathrm{SL}_n(V)$ with $\alpha\sigma$ in $\mathrm{Sp}_n(V)$, then $\alpha = \pm 1$. Since $\dim P \geqslant n - 1 > \frac{1}{2}n$, and since all totally degenerate subspaces of V have dimension $\leqslant \frac{1}{2}n$ by 1.1.15, there are vectors x, y in P with $q(x,y) = 1$. Then

$$1 = q(x,y) = q(\alpha\sigma x, \alpha\sigma y) = q(\alpha x, \alpha y) = \alpha^2 q(x,y) = \alpha^2,$$

so $\alpha = \pm 1$ as required. Q.E.D.

1.6.5. *If* V *is a regular alternating space and* σ *is a projective transvection in* $\mathrm{PSp}_n(V)$, *then* $R = P^*$.

PROOF. The representative transvection of σ is in $\mathrm{Sp}_n(V)$ by 1.6.4; apply 1.3.10. Q.E.D.

1.6.6. *Let* V *be a regular alternating space with* dim $V \geqslant 4$ *and let* σ_1 *and* σ_2 *be nontrivial projective transvections in* $\mathrm{PSp}_n(V)$. *Then* $\sigma_1\sigma_2$ *is a projective transvection if and only if* $R_1 = R_2$.

PROOF. Apply 1.6.1 and 1.6.5. Q.E.D.

1.6.7. *Let* V *be a regular alternating space and let* X *be a subgroup of* $\mathrm{PSp}_n(V)$ *that consists entirely of projective transvections. Then all nontrivial elements of* X *have the same residual line.*

PROOF. Apply 1.6.2 and 1.6.5. Q.E.D.

1.6.8. *Let* V *be a regular alternating space and let* σ_1 *and* σ_2 *be nontrivial projective transvections in* $\mathrm{PSp}_n(V)$. *Then* $\sigma_1\sigma_2 = \sigma_2\sigma_1$ *if and only if* $q(R_1, R_2) = 0$.

PROOF. Apply 1.6.3 and 1.6.5. Q.E.D.

1.7. Some Theorems about SL_n

Our purpose here is to quote some theorems about the abstract case. See Chapters 2 and 3 of the *Linear Lectures* for proofs. Corresponding theorems for the regular alternating situation will be developed later.

1.7.1. THEOREM. $\mathrm{SL}_n(F)$ *is generated by elementary matrices when* $n \geqslant 2$.
$\mathrm{SL}_n(V)$ *is generated by transvections, indeed by elementary transvections with respect to a given base, when* $n \geqslant 2$.

1.7.2. *If* $\sigma \in \mathrm{GL}_n(V)$ *is expressed in the form* $\sigma = \sigma_1 \cdots \sigma_t$ *with* $\sigma_i \in \mathrm{GL}_n(V)$ *and* $\operatorname{res} \sigma_i = 1$ *for* $1 \leqslant i \leqslant t$, *then* $t \geqslant \operatorname{res} \sigma$.

We say that an element σ in $\mathrm{GL}_n(V)$ is a big dilation if there is a splitting $V = U \oplus W$ with $W \neq 0$ such that $\sigma = (1_U) \oplus (\alpha 1_W)$ for some $\alpha \neq 1$. If σ is a big dilation as above, then

$$R = W, \qquad P = U.$$

1.7.3. THEOREM. *Each nontrivial* σ *in* $\mathrm{SL}_n(V)$, *other than a big dilation, is a product of* $\operatorname{res} \sigma$ *transvections in* $\mathrm{SL}_n(V)$. *A big dilation* σ *in* $\mathrm{SL}_n(V)$ *is a product of* $(\operatorname{res} \sigma) + 1$, *but not of* $\operatorname{res} \sigma$, *transvections. No* σ *in* $\mathrm{SL}_n(V)$ *is a product of fewer than* $\operatorname{res} \sigma$ *transvections.*

1.7.4. THEOREM. *The order of* $\mathrm{GL}_n(\mathbf{F}_q)$ *is*

$$q^{n(n-1)/2} \prod_1^n (q^i - 1),$$

of $\mathrm{SL}_n(\mathbf{F}_q)$ *and* $\mathrm{PGL}_n(\mathbf{F}_q)$ *it is*

$$\frac{q^{n(n-1)/2} \prod_1^n (q^i - 1)}{(q-1)}$$

and of $\mathrm{PSL}_n(\mathbf{F}_q)$ *it is*

$$\frac{q^{n(n-1)/2} \prod_1^n (q^i - 1)}{(q-1) \cdot \gcd(q-1, n)}.$$

1.7.5. THEOREM. *The group* $\mathrm{PSL}_n(F)$ *is simple for any natural number* $n \geqslant 2$ *and any field* F *but for two exceptions, namely the groups* $\mathrm{PSL}_2(\mathbf{F}_2)$ *and* $\mathrm{PSL}_2(\mathbf{F}_3)$ *which are not simple.*

1.7.6. *If* X *is any subgroup of* GL_n *that is invariant under conjugation by elements of* SL_n, *then* $X \subseteq \mathrm{RL}_n$ *or* $X \supseteq \mathrm{SL}_n$, *but for the two exceptional situations* $n = 2$ *with* $F = \mathbf{F}_2, \mathbf{F}_3$ *which clearly do not possess this property.*

1.8. Comments

The symplectic groups are the second of the four large families which make up the classical groups, the other families being the linear groups, the unitary groups, and the orthogonal groups. Closely related to these families are the Chevalley groups and algebraic groups. All these groups have been studied extensively over fields, and with varying degrees of success over rings, often over rings that come from algebraic number theory. In these notes we concentrate on the symplectic family which is almost as well-behaved as the linear family which we studied in the *Linear Lectures*. Our philosophy is the same as in the *Linear Lectures*. We ask, what are the generators of the symplectic groups, what is their structure, and what are their isomorphisms? We prove what we can over arbitrary fields and integral domains but we stay clear of theories which depend on special properties of the underlying rings since that would take us too far afield. As supplementary reading matter we suggest:

E. ARTIN, *Geometric algebra*, Interscience, New York, 1957,
J. DIEUDONNÉ, *La géométrie des groupes classiques*, 3ième éd., Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Springer-Verlag, Berlin and New York, 1971,

for the classical groups over fields;

R. W. CARTER, *Simple groups of Lie type*, Wiley, New York, 1972,
A. BOREL, *Linear algebraic groups*, Benjamin, New York, 1969,

for Chevalley and algebraic groups over fields; and

L. E. DICKSON, *Linear groups*, Teubner, Leipzig, 1901,
B. L. VAN DER WAERDEN, *Gruppen von linearen Transformationen*, Julius Springer, Berlin, 1935,
J. DIEUDONNÉ, *Sur les groupes classiques*, Actualités Scientifiques et Industrielles, no. 1040, Hermann, Paris, 1948,

for historical perspective.

2. GENERATION THEOREMS

Recall our general assumption that, starting with Chapter 2, V is a regular alternating space over F with associated alternating form q: $V \times V \to F$ and $n = \dim V \geqslant 2$. Similarly with V_1, F_1, n_1, q_1.

2.1. Generation by Transvections in Sp_n

2.1.1. *Let a be an element of V, and let τ be a transvection in $\mathrm{Sp}_n(V)$. Then $\tau a = a$ if and only if $q(a, \tau a) = 0$.*

PROOF. If $\tau a = a$, then of course $q(a, \tau a) = 0$. Conversely, suppose $q(a, \tau a) = 0$. Express τ in the form $\tau = \tau_{b,\lambda}$ using 1.4.12. Then

$$q(a, a + \lambda q(a, b)b) = q(a, \tau a) = 0$$

implies $\lambda = 0$ or $q(a, b) = 0$, whence $\tau a = a$. Q.E.D.

2.1.2. *If τ_1 and τ_2 are transvections in $\mathrm{Sp}_n(V)$ with $\tau_1 a = \tau_2 a \neq a$ for some a in V, then $\tau_1 = \tau_2$.*

PROOF. Let the transvections in question be expressed in the usual form τ_{a_1,λ_1} and τ_{a_2,λ_2}. Then $\tau_1 a = \tau_2 a \neq a$ becomes

$$\lambda_1 q(a, a_1)a_1 = \lambda_2 q(a, a_2)a_2 \neq 0,$$

whence $Fa_1 = Fa_2$. We can therefore assume that the transvections in question are actually τ_{a_1,λ_1} and τ_{a_1,λ_2}. Then

$$\tau_{a_1,\lambda_1}^{-1}\tau_{a_1,\lambda_2} = \tau_{a_1,\lambda_2-\lambda_1}$$

is identity on the hyperplane $(Fa_1)^*$, and also on the vector a which falls outside this hyperplane since $\tau_1 a \neq a$, so $\tau_1^{-1}\tau_2 = 1_V$, so $\tau_1 = \tau_2$. Q.E.D.

2.1.3. *Suppose a and b are distinct vectors in V. Then there is a transvection τ in $\mathrm{Sp}_n(V)$ such that $\tau a = b$ if and only if $q(a, b) \neq 0$. If this condition is satisfied, then*

$$\tau = \tau_{b-a,(q(a,b))^{-1}}$$

is the one and only τ that will do the job.

PROOF. If $q(a, b) = 0$, then $q(a, \tau a) = 0$, so $\tau a = a$ by 2.1.1, i.e., $b = a$, so the existence of τ implies $q(a, b) \neq 0$. Conversely, suppose $q(a, b) \neq 0$. Then direct substitution shows that the τ specified in the statement of the proposition does the job. Uniqueness follows from 2.1.2. Q.E.D.

2.1.4. *Let* σ *be an element of* $\mathrm{Sp}_n(V)$. *Then there is a transvection* τ *in* $\mathrm{Sp}_n(V)$ *such that* $\operatorname{res}\tau\sigma < \operatorname{res}\sigma$ *if and only if* $q(a, \sigma a) \neq 0$ *for some* a *in* V. *If this condition is satisfied, then*

$$\tau = \tau^{-1}_{\sigma a - a,(q(a,\sigma a))^{-1}}$$

will do the job. The fixed space of this $\tau\sigma$ *is equal to* $P + Fa$ *(which* $\supset P$*).*

PROOF. First let us suppose that we have a τ with $\operatorname{res}\tau\sigma < \operatorname{res}\sigma$. Let $L \subseteq H$ be the residual and fixed spaces of τ. Of course $H = L^*$. Put $\sigma_1 = \tau\sigma$. Then the equation $\sigma = \tau^{-1}\sigma_1$ implies that $L \not\subseteq R_1$, hence $H \not\supseteq P_1$, hence there is an a in V with $\sigma_1 a = a$ and $\tau a \neq a$. Then

$$q(a, \sigma a) = q\big(a, \tau^{-1}\sigma_1 a\big) = q(a, \tau a) \neq 0$$

by 2.1.1.

Conversely, let there be an a in V with $q(a, \sigma a) \neq 0$. Put

$$\tau_1 = \tau_{\sigma a - a,(q(a,\sigma a))^{-1}}, \qquad \tau = \tau_1^{-1}.$$

Then $\tau_1 a = \sigma a$ by 2.1.3. Hence $\tau\sigma a = a$. Now the residual space of $\tau\sigma$ is contained in $F(\sigma a - a) + R$ which is R since $\sigma a - a \in R$; hence the fixed space of $\tau\sigma$ contains P; but a is in the fixed space of $\tau\sigma$ though not in the fixed space P of σ; hence the fixed space of $\tau\sigma$ contains $P + Fa$ which strictly contains P. So $\operatorname{res}\tau\sigma < \operatorname{res}\sigma$. If the fixed space of $\tau\sigma$ strictly contained $P + Fa$, then

$$\operatorname{res}\tau\sigma + 2 \leqslant \operatorname{res}\sigma = \operatorname{res}(\tau^{-1}\tau\sigma) \leqslant 1 + \operatorname{res}\tau\sigma,$$

which is absurd. Q.E.D.

2.1.5. DEFINITION. A geometric transformation k of V, i.e., an element k of $\Xi\mathrm{L}_n(V)$, is called hyperbolic if $q(x, kx) = 0$ for all x in V.

2.1.6. *Every hyperbolic transformation in* $\mathrm{Sp}_n(V)$ *is an involution.*

PROOF. Consider a hyperbolic σ in $\mathrm{Sp}_n(V)$. Then for all x, y in V we have

$$\begin{aligned} q(\sigma^2 x - x, \sigma y) &= q(\sigma^2 x, \sigma y) - q(x, \sigma y) \\ &= q(\sigma x, y) + q(\sigma y, x) \\ &= q(\sigma x + \sigma y, x + y) = 0; \end{aligned}$$

hence $q(\sigma^2 x - x, V) = 0$; hence $\sigma^2 x - x = 0$ for all x in V by regularity. Q.E.D.

2.1.7. *If* σ *is hyperbolic and a transvection in* $\mathrm{Sp}_n(V)$, *then* $\sigma = 1_V$.

PROOF. Apply 2.1.1. Q.E.D.

2.1.8. *Let* σ *be a nontrivial hyperbolic transformation in* $\mathrm{Sp}_n(V)$ *and let* τ *be any transvection in* $\mathrm{Sp}_n(V)$ *whose residual space is a line in* R. *Then* $\tau\sigma$ *still has residual space* R, *but it is not hyperbolic.*

PROOF. Express τ in usual form $\tau = \tau_{a,\lambda}$. Here $a \in \dot{R}$. Since V is never the union of two of its hyperplanes, there is a b in V with $b \notin (Fa)^*$ and $b \notin (F(\sigma^{-1}a))^*$, i.e., with

$$q(a, b) \neq 0, \qquad q(a, \sigma b) \neq 0.$$

Then

$$\begin{aligned} q(b, \tau\sigma b) &= q(\tau^{-1}b, \sigma b) \\ &= q(b - \lambda q(b, a)a, \sigma b) \\ &= -\lambda q(b, a)q(a, \sigma b) \neq 0. \end{aligned}$$

So $\tau\sigma$ is not hyperbolic. Its residual space is clearly contained in R, and it is actually equal to R by 2.1.4. Q.E.D.

2.1.9. THEOREM. *The transvections in* $\mathrm{Sp}_n(V)$ *generate* $\mathrm{Sp}_n(V)$.

PROOF. Successive application of 2.1.4 and 2.1.8. Q.E.D.

2.1.10. $\mathrm{Sp}_n(V) \subseteq \mathrm{SL}_n(V)$ *and* $\mathrm{Sp}_n(F) \subseteq \mathrm{SL}_n(F)$.

PROOF. Apply Theorem 2.1.9. Q.E.D.

2.1.11. THEOREM. *Suppose* $F \neq \mathbf{F}_2$ *and let* σ *be an element of* $\mathrm{Sp}_n(V)$ *with* $\sigma \neq 1_V$. *If* σ *is not hyperbolic, it is a product of* res σ *transvections in* $\mathrm{Sp}_n(V)$. *If* σ *is hyperbolic, it is a product of* (res σ) + 1, *but not of* res σ, *transvections in* $\mathrm{Sp}_n(V)$. *Of course,* σ *is not the product of fewer than* res σ *transvections.*

This result will be established in several steps in the course of §2.1.

2.1.12. *If* char $F \neq 2$, *and if* σ *is any element of* $\mathrm{Sp}_n(V)$, *then the following assertions are equivalent*:

(1) σ *is hyperbolic.*
(2) σ *is an involution.*
(3) $V = R \perp P$ *with* $\sigma = (-1_R)\perp(1_P)$.
(4) σ *is a big dilation or* 1_V.

PROOF. To prove that (1) implies (2), apply 2.1.6. To prove (2) implies (3), apply 1.3.7. That (3) implies (4) is a consequence of the definition of a big dilation. Finally let us show that (4) implies (1). By definition, we have a direct sum $V = U \oplus W$ with $\sigma = (1_U) \oplus (\alpha 1_W)$ for some α in F with $\alpha \neq 0, 1$. We can clearly assume that $W \neq 0$. Then for any $u \in U$, $w \in W$,

$$q(u, w) = q(\sigma u, \sigma w) = q(u, \alpha w) = \alpha q(u, w),$$

whence $q(U, W) = 0$, so $V = U \perp W$. We then have, for any $u \in U$, $w \in W$,

$$q(u + w, \sigma(u + w)) = q(u + w, u + \alpha w) = 0. \quad \text{Q.E.D.}$$

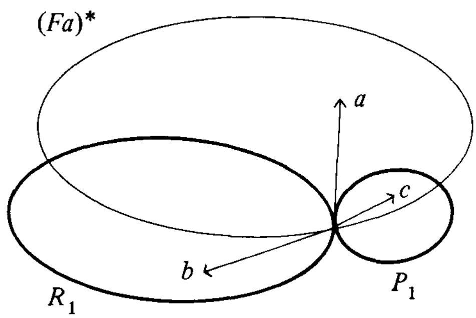
2.1.13. *If* Char $F \neq 2$, *then Theorem* 2.1.11 *is true.*

PROOF. (1) First suppose the given σ is not hyperbolic. If we can find a transvection τ in $\mathrm{Sp}_n(V)$ such that res $\tau\sigma <$ res σ with $\tau\sigma$ either nonhyperbolic or 1_V, then by a successive application of this fact, we will be through. So our purpose in step (1) is to establish this fact.

(1a) We can suppose that $P \subseteq R$. For suppose this case is already known and consider a σ with $P \nsubseteq R$. Then P is not totally degenerate. Take a radical splitting $P = P_0 \perp \operatorname{rad} P$ and write $V = P_0 \perp P_0^*$. Here P_0 and P_0^* will be nonzero, regular, alternating spaces over F. Then $(\sigma | P_0^*)$ is a nonhyperbolic transformation in $\mathrm{Sp}(P_0^*)$ whose fixed space is equal to $\operatorname{rad} P$ and therefore totally degenerate. We therefore have a τ_0 for $\sigma | P_0^*$. Then $1_{P_0} \perp \tau_0$ is a τ for σ.

(1b) We can assume, in addition, that P is a line. For if $\dim P = 0$ or $\dim P \geqslant 2$, we can apply 2.1.4 and 2.1.12–the $\tau\sigma$ resulting from 2.1.4 will then have a degenerate fixed space and will therefore be nonhyperbolic.

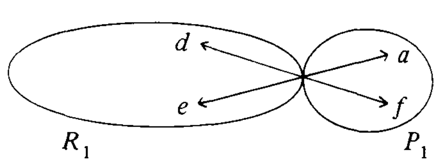
(1c) We therefore have to prove (1) under the assumption that P is a line with $P \subseteq R = P^*$. So $\operatorname{res} \sigma = n - 1$. Apply 2.1.4 to σ. If the resulting $\tau\sigma$ is nonhyperbolic or 1_V we are through. If not, some adjustments become necessary. So let us assume that our given σ has the form $\sigma = \tau_1\sigma_1$ with τ_1 a nontrivial transvection in $\mathrm{Sp}_n(V)$ and with σ_1 a hyperbolic transformation in $\mathrm{Sp}_n(V)$ having $\operatorname{res} \sigma_1 = n - 2 > 0$. This makes $n \geqslant 4$. And $V = R_1 \perp P_1$ with $\dim R_1 = n - 2$, $\dim P_1 = 2$, and $\sigma_1 = (-1_{R_1}) \perp (1_{P_1})$, by 2.1.12. Express $\tau_1 = \tau_{a,\lambda}$ in the usual way. We must have



$a \in V - R_1$ since otherwise $\operatorname{res} \sigma \leqslant n - 2$. If $a \notin P_1$, the adjustment is as follows. Since $a \notin P_1$ we must have $R_1 \nsubseteq (Fa)^*$, so we can pick a vector $b \in R_1 - (Fa)^*$. Since $(Fa)^*$ is a hyperplane and P_1 is a plane we can pick $c \in P_1 \cap (Fa)^*$ with $c \neq 0$. Clearly $P = Fc$ by 1.3.11. Then

$$q(b, \sigma b) = q(b, \tau_1\sigma_1 b) = -q(b, \tau_1 b) \neq 0$$

by 2.1.1. By 2.1.4 we have a transvection τ in $\mathrm{Sp}_n(V)$ with $\tau\sigma$ having fixed space $P + Fb = Fc + Fb$ and $\operatorname{res} \tau\sigma < \operatorname{res} \sigma$. Then $\tau\sigma$ cannot be hyperbolic since its fixed space is totally degenerate, by 2.1.12. So we must adjust the case where $a \in P_1$. This time pick d and e in R_1 with $q(d, e) = 1$, and f in P_1 with $q(a, f) = 1$. This time note that $\sigma(d + f) - (d + f) = -2d - \lambda a$ and



then $q(d + f, \sigma(d + f)) = \lambda$. Accordingly, by 2.1.4, if we let τ be the transvection

$$\tau = \tau_{-2d-\lambda a, -\lambda^{-1}},$$

we find that res $\tau\sigma <$ res σ, and that the fixed space of $\tau\sigma$ is equal to $Fa + F(d + f)$. If $\tau\sigma$ were hyperbolic its action on its residual space would have to be -1 by 2.1.12. But it is easily verified that $e + a$ is orthogonal to $Fa + F(d + f)$, i.e., that $e + a$ is in the residual space of $\tau\sigma$; and also that $\tau\sigma(e + a) \neq -(e + a)$. So $\tau\sigma$ is not hyperbolic, as required.

(2) Now suppose that the given σ is hyperbolic. Then 2.1.8 and the first part of the present proposition imply that σ is a product of (res σ) + 1 transvection in $\mathrm{Sp}_n(V)$. Since σ is hyperbolic it is a big dilation by 2.1.12; hence it is not a product of res σ transvections by Theorem 1.7.3. Q.E.D.

2.1.14. *Let* char $F = 2$ *and let A be any symmetric matrix over F with $A \neq 0$. Then there is an invertible matrix T over F such that*

$${}^tTAT = \left(\begin{array}{c|c} A' & \\ \hline & 0 \end{array}\right)$$

with A' invertible and diagonal if A is not alternating, and A' of the form

$$\left(\begin{array}{ccc} \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & & \\ & \ddots & \\ & & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \end{array}\right)$$

if A is alternating.

PROOF. If we add a multiple of one column of A to another and then add the same multiple of the first corresponding row to the second corresponding row, or if we interchange two columns of A and then interchange the corresponding rows, or if we multiply a column of A by a nonzero scalar and then multiply the corresponding row by the same scalar, then, in each of these cases, it is easily seen that the matrix obtained is a nonzero symmetric matrix of the form ${}^tTAT$ for some invertible matrix T over F. We leave it as an exercise in elementary matrix theory to verify that A can be put in the desired form

$$\left(\begin{array}{c|c} A' & \\ \hline & 0 \end{array}\right)$$

using a sequence of operations of the type just described, and hence that there is a T of the desired type such that ${}^tTAT$ has the desired form. Q.E.D.

2.1.15. *Let* char $F = 2$, *let* $\mathfrak{X}$ *be a symplectic base for* V, *and let* σ *be the involution in* $\mathrm{GL}_n(V)$ *that is defined matrically by*

$$\sigma \sim \left(\begin{array}{c|c} I & A \\ \hline & I \end{array}\right) \quad \textit{in } \mathfrak{X}$$

where A *is any given* $\frac{1}{2}n \times \frac{1}{2}n$ *matrix over* F. *Then*

(1) σ *is in* $\mathrm{Sp}_n(V)$ *if and only if* A *is symmetric.*

(2) *If* σ *is in* $\mathrm{Sp}_n(V)$, *it is hyperbolic if and only if* A *is alternating.*

PROOF. The first part is a special case of 1.1.17. Now suppose that σ is in $\mathrm{Sp}_n(V)$. So $A = (a_{ij})$ is symmetric. If σ is hyperbolic, then

$$\begin{aligned} a_{jj} &= q\left(y_j, \sum_{i=1}^{n/2} a_{ij}x_i\right) \\ &= q(y_j, \sigma y_j) \\ &= 0, \end{aligned}$$

so A is alternating. Conversely, if A is alternating, express a typical z in V in the form

$$z = \sum_i \alpha_i x_i + \sum_j \beta_j y_j$$

and show, by direct computation, that $q(z, \sigma z) = 0$, i.e., that σ is hyperbolic. Q.E.D.

2.1.16. *Let* char $F = 2$. *If* $\sigma \neq 1_V$ *is an involution in* $\mathrm{Sp}_n(V)$, *there is a symplectic base* $\mathfrak{X}$ *for* V *in which*

$$\sigma \sim \left(\begin{array}{c|c} I & A \\ \hline & I \end{array}\right) \quad \textit{with } A = \left(\begin{array}{c|c} A' & \\ \hline & 0 \end{array}\right)$$

where A' *is invertible and diagonal if* σ *not hyperbolic, and* A' *has the form*

$$\begin{pmatrix} \begin{array}{cc|} 0 & 1 \\ 1 & 0 \\ \hline \end{array} & & \\ & \ddots & \\ & & \begin{array}{|cc} \hline 0 & 1 \\ 1 & 0 \end{array} \end{pmatrix}$$

if σ *is hyperbolic.*

PROOF. Since σ is an involution we have $R \subseteq P$ by 1.3.13. Now $R = P^*$ so $R = \operatorname{rad} P$, so it follows easily that P contains a totally degenerate subspace of dimension $\frac{1}{2}n$ that contains R. Accordingly, by 1.1.14, there is a symplectic base

$$\mathfrak{Y} = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

for V in which

$$\begin{aligned} R &= Fx_1 + \cdots + Fx_r \\ &\subseteq Fx_1 + \cdots + Fx_r + \cdots + Fx_{n/2} \\ &\subseteq P. \end{aligned}$$

By 1.1.17 we know that

$$\sigma \sim \left(\begin{array}{c|c} I & D \\ \hline & I \end{array}\right) \quad \text{in } \mathfrak{Y}$$

for some $\frac{1}{2}n \times \frac{1}{2}n$ symmetric matrix D. By 2.1.14 there is an invertible $\frac{1}{2}n \times \frac{1}{2}n$ matrix T over F such that ${}^tTDT = A$ with

$$A = \left(\begin{array}{c|c} A' & \\ \hline & 0 \end{array}\right)$$

for some A' of the form given in 2.1.14. Put

$$S = \left(\begin{array}{c|c} {}^tT^{-1} & \\ \hline & T \end{array}\right)$$

and let S carry $\mathfrak{Y}$ to some base $\mathfrak{X}$ for V. We have

$${}^t\left(\begin{array}{c|c} {}^tT^{-1} & \\ \hline & T \end{array}\right)\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)\left(\begin{array}{c|c} {}^tT^{-1} & \\ \hline & T \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

so that $\mathfrak{X}$ is also a symplectic base for V. And the matrix of σ in the base $\mathfrak{X}$ is equal to

$$\left(\begin{array}{c|c} {}^tT^{-1} & \\ \hline & T \end{array}\right)^{-1}\left(\begin{array}{c|c} I & D \\ \hline & I \end{array}\right)\left(\begin{array}{c|c} {}^tT^{-1} & \\ \hline & T \end{array}\right) = \left(\begin{array}{c|c} I & {}^tTDT \\ \hline & I \end{array}\right) = \left(\begin{array}{c|c} I & A \\ \hline & I \end{array}\right).$$

By 2.1.15, if σ is not hyperbolic, then A is not alternating, so A' is invertible and diagonal by 2.1.14; and similarly if σ is hyperbolic. Q.E.D.

2.1.17. *If* char $F = 2$ *and if* σ *is a hyperbolic transformation in* $\mathrm{Sp}_n(V)$, *then* res σ *is even.*

Proof. The results follow by suitably interpreting 2.1.16. Q.E.D.

2.1.18. *If* char $F = 2$ *and* σ *is an involution, then Theorem* 2.1.11 *is true* (*even if* $F = \mathbf{F}_2$).

Proof. First let σ be nonhyperbolic. By 2.1.16 there is a symplectic base $\mathfrak{X}$ for V in which

$$\sigma \sim \left(\begin{array}{c|c} I & D \\ \hline & I \end{array}\right) \quad \text{with } D = \mathrm{diag}(d_1, \ldots, d_{n/2})$$

where $d_1, \ldots, d_r \in \dot{F}$ and $d_{r+1}, \ldots, d_{n/2}$ are 0. Note that we must have $r = \mathrm{res}\,\sigma$. For $1 \leqslant i \leqslant r$ let D_i be the $\frac{1}{2}n \times \frac{1}{2}n$ diagonal matrix with d_i in the ith position and 0 everywhere else. Then

$$\tau_i \sim \left(\begin{array}{c|c} I & D_i \\ \hline & I \end{array}\right) \quad \text{in } \mathfrak{X}$$

defines a transvection in $\mathrm{Sp}_n(V)$, and $\sigma = \tau_1 \cdots \tau_r$, so σ is a product of res σ transvections in $\mathrm{Sp}_n(V)$, as required. Now assume that σ is hyperbolic. If σ were a product of res σ transvections in $\mathrm{Sp}_n(V)$, say $\sigma = \tau_1 \cdots \tau_r$, then res $\tau_1^{-1}\sigma <$ res σ, and this would contradict 2.1.4. On the other hand, it follows from 2.1.8 and the first part of Theorem 2.1.11 for involutions which we have just proved, that σ is a product of (res σ) + 1 transvections in $\mathrm{Sp}_n(V)$. Q.E.D.

2.1.19. *If* char $F = 2$ (*with* $F \neq \mathbf{F}_2$, *of course*) *then Theorem* 2.1.11 *is true.*

Proof. (1) First suppose the given σ is not hyperbolic. If we can find a transvection τ in $\mathrm{Sp}_n(V)$ such that res $\tau\sigma <$ res σ with $\tau\sigma$ either nonhyperbolic or 1_V, then by a successive application of this fact, we will be through. So our purpose in step (1) is to establish this fact.

(1a) Proceeding as in step (1a) of the proof of 2.1.13 allows us to assume that $P \subseteq R$ for the given σ.

(1b) We can assume, in addition, that dim P is $\frac{1}{2}n - 1$ or $\frac{1}{2}n$. For P, being totally degenerate, must have dim $P \leqslant \frac{1}{2}n$. If dim $P \leqslant \frac{1}{2}n - 2$ we can apply 2.1.4–the $\tau\sigma$ resulting from 2.1.4 will have a fixed space of dimension $\leqslant \frac{1}{2}n - 1$ and so res $\tau\sigma \geqslant \frac{1}{2}n + 1$; such a $\tau\sigma$ cannot be an involution, let alone a hyperbolic transformation, since involutions in characteristic 2, being totally degenerate by 1.3.13, must have residue $\leqslant \frac{1}{2}n$.

(1c) Let us now prove the case where dim $P = \frac{1}{2}n - 1$. If $n = 2$, then an application of 2.1.4 gives us a τ whose res $\tau\sigma = 1$ is odd and therefore whose $\tau\sigma$ cannot be hyperbolic by 2.1.17. So let $n \geqslant 4$. Then $P \neq 0$. So there are vectors in $V - R$. In fact we can find $a \in V - R$ with $q(a, \sigma a) \neq 0$; to see this, pick $b \in V - R$; we can assume that $q(b, \sigma b) = 0$, else b would be a suitable a; since σ is not hyperbolic, there is a vector r with $q(r, \sigma r) \neq 0$, and we can assume that r is in R, else r would be a suitable a; then

$$q(b + \lambda r, \sigma(b + \lambda r)) = \lambda(q(b, \sigma r) + q(r, \sigma b) + \lambda q(r, \sigma r))$$

cannot be 0 for more than a single λ in $\dot{F}$; since $F \neq \mathbf{F}_2$ we have a λ in $\dot{F}$ with

$$q(b + \lambda r, \sigma(b + \lambda r)) \neq 0;$$

$b + \lambda r$ with this value of λ is our a. Let τ be obtained for the given σ using this a in the manner of 2.1.4. Then the fixed space of $\tau\sigma$, being $P + Fa$, is not totally degenerate since $a \notin R$. Now $\dim(P + Fa) = \frac{1}{2}n$, so res $\tau\sigma = \frac{1}{2}n$, so if the residual space of $\tau\sigma$ were totally degenerate it would be equal to $P + Fa$ which is not totally degenerate. So $\tau\sigma$ is not totally degenerate. So $\tau\sigma$ is not an involution. So $\tau\sigma$ is not hyperbolic.

(1d) Finally the case dim $P = \frac{1}{2}n$. Here we actually have $P = R$. So σ is totally degenerate. So σ is an involution. So $\sigma = \tau_1 \cdots \tau_{n/2}$ by 2.1.18. Put $\tau = \tau_1^{-1}$. Then res $\tau\sigma = \frac{1}{2}n - 1 <$ res σ. If $\tau\sigma$ were hyperbolic it would be an involution, hence 2.1.18 would apply, so $\tau\sigma$ would not be a product of $\frac{1}{2}n - 1$

transvections in $\mathrm{Sp}_n(V)$, and this is absurd. So $\tau\sigma$ is nonhyperbolic, as required.

(2) If σ is hyperbolic it is an involution; hence 2.1.18 applies, so σ is a product of $(\text{res}\,\sigma) + 1$, but not of $\text{res}\,\sigma$, transvections in $\mathrm{Sp}_n(V)$. Q.E.D.

2.2. Elementary Generation of Sp_n

Is $\mathrm{Sp}_n(V)$ generated by elementary transvections in $\mathrm{Sp}_n(V)$ with respect to a fixed symplectic base

$$\mathfrak{X} = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

for V? Using the fact that

$$\tau_{a,\lambda} = \tau_{a,\lambda q(\,\bullet\,,a)}$$

where $\tau_{a,\lambda}$ denotes a transvection in the symplectic situation, while $\tau_{a,\rho}$ with ρ a linear functional denotes a transvection in the linear theory, we see that the elementary transvections with respect to $\mathfrak{X}$ that fall in $\mathrm{Sp}_n(V)$ are precisely the transvections

$$\tau_{x_1,\lambda}, \ldots, \tau_{y_{n/2},\lambda}$$

as λ runs through F. Do these elements generate $\mathrm{Sp}_n(V)$? If we let e_{ij} denote the $\frac{1}{2}n \times \frac{1}{2}n$ matrix over F with 1 in the (i,j) position and 0 elsewhere, then it is clear that the elementary transvections in $\mathrm{Sp}_n(V)$ correspond to all the matrices

$$\left(\begin{array}{c|c} I & \lambda e_{ii} \\ \hline & I \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} I & \\ \hline \lambda e_{ii} & I \end{array}\right)$$

as λ runs through F and $1 \leqslant i \leqslant \frac{1}{2}n$. These matrices generate $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$, so our question is answered in the affirmative when $n = 2$. So consider $n \geqslant 4$ for the remainder of §2.2. Then any product of the above matrices will be of the form

$$\left(\begin{array}{c|c} \text{diag} & \text{diag} \\ \hline \text{diag} & \text{diag} \end{array}\right),$$

in particular the matrix

$$\left(\begin{array}{c|c} I & X \\ \hline & I \end{array}\right)$$

with X a nondiagonal symmetric $\frac{1}{2}n \times \frac{1}{2}n$ matrix over F cannot be obtained as a product of such matrices. So the above matrices do not generate $\mathrm{Sp}_n(F)$, i.e., the elementary transvections in $\mathrm{Sp}_n(V)$ do not generate $\mathrm{Sp}_n(V)$. In fact let us show that $\mathrm{Sp}_n(F)$ is generated by the matrices

$$\left(\begin{array}{c|c} I & \lambda e_{ii} \\ \hline & I \end{array}\right), \quad \left(\begin{array}{c|c} I & \\ \hline \lambda e_{ii} & I \end{array}\right),$$

$$\left(\begin{array}{c|c} I & \lambda(e_{ij} + e_{ji}) \\ \hline & I \end{array}\right), \quad \left(\begin{array}{c|c} I & \\ \hline \lambda(e_{ij} + e_{ji}) & I \end{array}\right),$$

$$\left[\begin{array}{c|c} I + \lambda e_{ij} & \\ \hline & I - \lambda e_{ji} \end{array}\right],$$

as λ runs through F and i and j run through $1 \leqslant i \leqslant \frac{1}{2}n$, $1 \leqslant j \leqslant \frac{1}{2}n$ with $i \neq j$. All these matrices are, of course, in $\mathrm{Sp}_n(F)$; let G denote the subgroup of $\mathrm{Sp}_n(F)$ which they generate. Clearly G is the group generated by

$$\left(\begin{array}{c|c} A & \\ \hline & {}^tA^{-1} \end{array}\right), \quad \left(\begin{array}{c|c} I & B \\ \hline & I \end{array}\right), \quad \left(\begin{array}{c|c} I & \\ \hline B & I \end{array}\right),$$

as A runs through $\mathrm{SL}_{n/2}(F)$ and B runs through all $\frac{1}{2}n \times \frac{1}{2}n$ symmetric matrices over F. G can also be described as the group generated by the matrices

$$\left(\begin{array}{c|c} A & \\ \hline & {}^tA^{-1} \end{array}\right), \quad \left(\begin{array}{c|c} I & B \\ \hline & I \end{array}\right), \quad \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

with A and B as above, in view of the equations

$$\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right) = \left(\begin{array}{c|c} I & I \\ \hline & I \end{array}\right)\left(\begin{array}{c|c} I & \\ \hline -I & I \end{array}\right)\left(\begin{array}{c|c} I & I \\ \hline & I \end{array}\right),$$

$$\left(\begin{array}{c|c} I & \\ \hline B & I \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)^{-1}\left(\begin{array}{c|c} I & -B \\ \hline & I \end{array}\right)\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right).$$

We must show that $G = \mathrm{Sp}_n(F)$, or that $G_0 = \mathrm{Sp}_n(V)$ where G_0 consists of the transformations corresponding to G. Now G_0 contains all transvections in $\mathrm{Sp}_n(V)$ with residual line Fx_1 so, since $\mathrm{Sp}_n(V)$ is generated by transvections, it is enough to prove that there is a σ in G_0 such that

$$\sigma x_1 = \alpha_1 x_1 + \cdots + \alpha_{n/2}x_{n/2} + \beta_1 y_1 + \cdots + \beta_{n/2}y_{n/2}$$

whenever the α's and β's are given in F with not all 0. This will certainly be accomplished if we can find a matrix in G whose first column is equal to

$$(\alpha_1, \ldots, \alpha_{n/2}, \beta_1, \ldots, \beta_{n/2}).$$

If all α's are 0 pick A in $\mathrm{SL}_{n/2}(F)$ with first column equal to

$$(-\beta_1, \ldots, -\beta_{n/2})$$

and observe that

$$\left(\begin{array}{c|c} 0 & {}^tA^{-1} \\ \hline -A & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)\left(\begin{array}{c|c} A & \\ \hline & {}^tA^{-1} \end{array}\right) \in G.$$

If some $\alpha_i \neq 0$, let A be a matrix in $\mathrm{SL}_{n/2}(F)$ whose first column is equal to

$$(\alpha_1, \ldots, \alpha_{n/2})$$

and let B be the $\frac{1}{2}n \times \frac{1}{2}n$ symmetric matrix

$$\begin{pmatrix} & & \beta_1\alpha_i^{-1} & & \\ & & \vdots & & \\ \beta_1\alpha_i^{-1} & \cdots & x & \cdots & \beta_{n/2}\alpha_i^{-1} \\ & & \vdots & & \\ & & \beta_{n/2}\alpha_i^{-1} & & \end{pmatrix}$$

where x is given by

$$(\beta_1\alpha_i^{-1})\alpha_1 + \cdots + (x)\alpha_i + \cdots + (\beta_{n/2}\alpha_i^{-1})\alpha_{n/2} = \beta_i.$$

Then the first column of

$$\left(\begin{array}{c|c} I & \\ \hline B & I \end{array}\right)\left(\begin{array}{c|c} A & \\ \hline & {}^tA^{-1} \end{array}\right) \in G$$

is equal to

$$(\alpha_1, \ldots, \alpha_{n/2}, \beta_1, \ldots, \beta_{n/2}),$$

as required.

2.3. Comments

The main theorem of this chapter is due to Dieudonné. If the underlying field is $\mathbf{F}_2$, then the theorem fails, i.e., it is no longer possible to express every σ in $\mathrm{Sp}_n(V)$ as a product of res σ or of (res σ) + 1 transvections in $\mathrm{Sp}_n(V)$. There is a theorem for $\mathbf{F}_2$, but it is considerably more complicated. See

J. Dieudonné, *Sur les générateurs des groupes classiques*, Summa Basil. Math. **3** (1955), 149–179

for Dieudonné's original proof when $F \neq \mathbf{F}_2$,

D. Callan, *The generation of* $\mathrm{Sp}(\mathbf{F}_2)$ *by transvections*, J. Algebra **42** (1976), 378–390

for the case of $F = \mathbf{F}_2$ (Callan points out that Dieudonné's treatment of the case $\mathbf{F}_2$ is incomplete). For generation theorems for an alternating V that is not necessarily regular, see

U. Spengler, *Relationen zwischen symplektischen Transvektionen*, J. reine angew. Math. **274/275** (1975), 141–149.
U. Spengler and H. Wolff, *Die Länge einer symplektischen Abbildung*, J. reine angew. Math. **274/275** (1975), 150–157.

Generation questions over rings are of the following form. What is a good set of generators? For example, do transvections generate symplectic groups over rings? Which groups are finitely generated? Results are highly dependent on the underlying ring. See the survey article:

Yu. I. Merzlyakov, *Linear groups*, J. Soviet Math. **1** (1973), 571–593

for more information on the generation theory over special rings.

3. STRUCTURE THEOREMS

3.1. Orders of Symplectic Groups

3.1.1. *If F has an infinite number of elements, then so do the groups* Sp_n *and* PSp_n *over F.*

PROOF. The number of transvections $\tau_{a,\lambda}$ in $\mathrm{Sp}_n(V)$ is infinite. Q.E.D.

3.1.2. THEOREM. *The order of* $\mathrm{Sp}_n(\mathbf{F}_q)$ *is*

$$q^{(n/2)^2}\prod_{1}^{n/2}(q^{2i}-1),$$

and the order of $\mathrm{PSp}_n(\mathbf{F}_q)$ *is*

$$\frac{q^{(n/2)^2}\prod_1^{n/2}(q^{2i}-1)}{\gcd(2,\, q-1)}.$$

PROOF. The second part follows from the first since, by §1.5, $\mathrm{PSp}_n(F)$ is isomorphic to $\mathrm{Sp}_n(F)/(\pm I)$. We prove the first part by induction on n. If $n = 2$, then $\mathrm{SL}_2 = \mathrm{Sp}_2$ by 1.2.6. Apply Theorem 1.7.4. So let $n \geqslant 4$.

By a pair we will mean an ordered pair of vectors x, y with $q(x, y) = 1$. If x is fixed in $\dot{V}$, then there is exactly one pair (x, y) with y belonging to a given line that is not orthogonal to x. So the number of pairs with x in first position is equal to the number of lines that do not fall in $(Fx)^*$, and this number is equal to

$$\frac{q^n-1}{q-1}-\frac{q^{n-1}-1}{q-1}=q^{n-1}.$$

Therefore there are q^{n-1} pairs with x in first position. Therefore there are $q^{n-1}(q^n-1)$ pairs in all.

Fix a pair (i, j). For each pair (x, y) we have at least one element of $\mathrm{Sp}_n(V)$ carrying (i, j) to (x, y) by Witt's Theorem. Hence there are exactly

$$\operatorname{card}\mathrm{Sp}_{n-2}((Fx+Fy)^*)$$

elements of $\mathrm{Sp}_n(V)$ which will carry the pair (i, j) to the pair (x, y). By induction, this number is equal to

$$q^{(n/2-1)^2}\prod_{1}^{n/2-1}(q^{2i}-1).$$

Now every element of $\mathrm{Sp}_n(V)$ carries (i, j) to exactly one pair. Hence $\mathrm{Sp}_n(V)$ contains

$$q^{(n/2-1)^2}\prod_{1}^{n/2-1}(q^{2i}-1)\cdot q^{n-1}(q^n-1) = q^{(n/2)^2}\prod_{1}^{n/2}(q^{2i}-1)$$

elements. Q.E.D.

3.1.3. *The number of maximal totally degenerate subspaces of V is equal to*

$$\prod_{1}^{n/2}(q^i+1)$$

when $F = \mathbf{F}_q$.

PROOF. (1) First let us show that the subgroup G_M of $\mathrm{Sp}_n(V)$ which stabilizes a typical maximal totally degenerate subspace M of V has order

$$q^{(n/2)^2}\prod_{1}^{n/2}(q^i-1).$$

To see this, fix a symplectic base

$$\mathfrak{X} = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

for V in which the x's span M. This can be done because of 1.1.14. Then it follows from 1.1.17 that the matrix of a typical $\sigma \in G_M$ can be expressed in the form

$$\left(\begin{array}{c|c} C & \\ \hline & {}^tC^{-1} \end{array}\right)\left(\begin{array}{c|c} I & B \\ \hline & I \end{array}\right)$$

with $C \in \mathrm{GL}_{n/2}(F)$ and with B a symmetric $\frac{1}{2}n \times \frac{1}{2}n$ matrix over F; and these C and B are uniquely determined by σ; and every such C and B come from some σ in G_M. The result then follows by multiplying the order of $\mathrm{GL}_{n/2}(\mathbf{F}_q)$ (which is given in Theorem 1.7.4) by the number of $\frac{1}{2}n \times \frac{1}{2}n$ symmetric matrices over $\mathbf{F}_q$.

(2) Fix a maximal totally degenerate subspace M of V. By Witt's Theorem, every maximal totally degenerate subspace of V can be obtained by forming σM as σ runs through $\mathrm{Sp}_n(V)$. It follows easily from step (1) that each maximal totally degenerate subspace is duplicated exactly

$$q^{(n/2)^2}\prod_{1}^{n/2}(q^i-1)$$

times by this process. So the number of such subspaces is equal to the order of $\mathrm{Sp}_n(V)$ divided by the above quantity. This is clearly the desired number. Q.E.D.

3.1.4. *The number of regular planes in V is equal to*

$$q^{n-2}\left(\frac{q^n-1}{q^2-1}\right)$$

when $F = \mathbf{F}_q$.

PROOF. By arguing as in the proof of 3.1.3 we find that V must contain

$$\frac{\text{card Sp}_n}{\text{card Sp}_2 \cdot \text{card Sp}_{n-2}}$$

regular planes. This number turns out to be the number given above, by Theorem 3.1.2. Q.E.D.

3.1.5. $\text{Sp}_4(\mathbf{F}_2)$ *is isomorphic to the symmetric group* $\mathfrak{S}_6$.

PROOF. By a configuration let us mean any subset C of 5 elements in the 4-dimensional regular alternating space V over $\mathbf{F}_2$ with the property that no two distinct elements of C are orthogonal. Every nonzero vector x in V belongs to exactly 2 configurations C and C', and these two configurations must then intersect in the set $\{x\}$ itself. To see this take a symplectic base $\{x_1, x_2 | y_1, y_2\}$ for V in which $x = x_1$. Then it is obvious that

$$\{x_1, y_1, x_1 + y_1 + x_2, x_1 + y_1 + y_2, x_1 + y_1 + x_2 + y_2\}$$

and

$$\{x_1, x_1 + y_1, y_1 + x_2, y_1 + y_2, y_1 + x_2 + y_2\}$$

are two distinct configurations intersecting in $\{x\}$. And a quick check by the process of elimination will show that there are no other configurations containing x. If we now list all distinct configurations $C_1, \ldots, C_j$ in V, then every x in $\dot{V}$ appears in exactly 2 of the C's; hence $5j = 2.15$; hence $j = 6$. We let $\Gamma = \{C_1, \ldots, C_6\}$ denote the set of all configurations of V.

If σ is any element of $\text{Sp}_4(V)$, then σC is a configuration if and only if C is, so σ induces a mapping $\tilde{\sigma}: \Gamma \rightarrowtail \Gamma$. It is clear that $\tilde{\sigma}$ is surjective, hence a permutation of Γ. So $\sigma \rightarrowtail \tilde{\sigma}$ defines a mapping, indeed a homomorphism, $\text{Sp}_4(V) \rightarrowtail \mathfrak{S}_6$. To find the kernel take σ in $\text{Sp}_4(V)$ with $\sigma \neq 1$. Then there is an x in $\dot{V}$ with $\sigma x \neq x$. Let C and C' be the two configurations containing x. Then σx does not belong to one of them, say $\sigma x \notin C$. So $\sigma C \neq C$. So $\tilde{\sigma} \neq 1$. In other words, the kernel is trivial and we have an injective homomorphism $\text{Sp}_4(V) \rightarrowtail \mathfrak{S}_6$. But $\text{Sp}_4(V)$ has 6! elements by Theorem 3.1.2. So $\text{Sp}_4(V)$ is isomorphic to $\mathfrak{S}_6$, as required. Q.E.D.

3.2. Centers

Note that $\text{PSp}_n(V)$ is not commutative. To see this take nontrivial projective transvections in $\text{PSp}_n(V)$ with nonorthogonal residual lines and apply 1.6.8. So $\text{Sp}_n(V)$ is not commutative too.

3.2.1. $\text{PSp}_n(V)$ *is centerless and* $\text{cen Sp}_n(V) = (\pm 1_V)$.

PROOF. Consider a typical σ in the center of $\mathrm{PSp}_n(V)$. Let L be a typical line in V. Let τ be a projective transvection in $\mathrm{PSp}_n(V)$ with residual line L. Then the residual line of $\sigma\tau\sigma^{-1}$ is equal to σL. But $\sigma\tau\sigma^{-1} = \tau$ since σ is in the center. So $\sigma L = L$ for all L. So $\sigma = 1$. So $\mathrm{PSp}_n(V)$ is indeed centerless. The second part follows by applying P and using 1.2.5. Q.E.D.

3.3. Commutator Subgroups

3.3.1. *If L and L' are any two lines in V, then the set of transvections in* $\mathrm{Sp}_n(V)$ *with residual line L is conjugate under* $\mathrm{Sp}_n(V)$ *to the set with residual line L'.*

PROOF. By Witt's Theorem there is a Σ in $\mathrm{Sp}_n(V)$ such that $\Sigma L = L$. Then conjugation by Σ carries the transvections in $\mathrm{Sp}_n(V)$ with residual line L into, indeed onto, those with residual line L'. Q.E.D.

3.3.2. EXAMPLE. Two transvections in $\mathrm{Sp}_n(V)$ need not be conjugate in $\mathrm{Sp}_n(V)$. For example, the conjugates of $\tau_{a,\lambda}$ with residual line Fa are the transvections $\tau_{a,\alpha^2\lambda}$ as α runs through $\dot{F}$.

3.3.3. REMARK. Let $\mathfrak{X}$ be a symplectic base for V. If S is any $\frac{1}{2}n \times \frac{1}{2}n$ symmetric matrix over F, and if σ is the linear transformation defined matrically by

$$\sigma \sim \left(\begin{array}{c|c} I & S \\ \hline & I \end{array}\right) \quad \text{in } \mathfrak{X},$$

then we know from 1.1.17 that σ is an element of $\mathrm{Sp}_n(V)$. If we now derive S' from S by (1) adding a multiple of one column to another and then doing the same thing with the corresponding rows, or (2) interchanging two columns and then interchanging the corresponding rows, then the linear transformation σ' with

$$\sigma' \sim \left(\begin{array}{c|c} I & S' \\ \hline & I \end{array}\right) \quad \text{in } \mathfrak{X}$$

is still in $\mathrm{Sp}_n(V)$ since S' is still symmetric. But in fact σ and σ' are conjugate in $\mathrm{Sp}_n(V)$. To see this observe that in either case S' has the form $S' = {}^tTST$ for some T in $\mathrm{GL}_{n/2}(F)$. Then Σ defined by

$$\Sigma \sim \left(\begin{array}{c|c} {}^tT & \\ \hline & T^{-1} \end{array}\right) \quad \text{in } \mathfrak{X}$$

is in $\mathrm{Sp}_n(V)$ by 1.1.17, and $\sigma' = \Sigma\sigma\Sigma^{-1}$ since

$$\left(\begin{array}{c|c} I & S' \\ \hline & I \end{array}\right) = \left(\begin{array}{c|c} {}^tT & \\ \hline & T^{-1} \end{array}\right)\left(\begin{array}{c|c} I & S \\ \hline & I \end{array}\right)\left(\begin{array}{c|c} {}^tT & \\ \hline & T^{-1} \end{array}\right)^{-1}.$$

3.3.4. *Suppose $n \geqslant 4$ with $F \neq \mathbf{F}_2, \mathbf{F}_3$, and let G be a normal subgroup of* $\mathrm{Sp}_n(V)$ *which contains a regular element σ of residue 2 which is a product of two transvections in* $\mathrm{Sp}_n(V)$. *Then $G =$* $\mathrm{Sp}_n(V)$.

PROOF. Here $V = R \perp P$ with R a regular plane. Let G_2 be the group

$$G_2 = \{ g \in \mathrm{Sp}_2(R) | g \perp 1_P \in G \}.$$

Then $(\sigma|R) \in G_2 \lhd \mathrm{Sp}_2(R)$. And $(\sigma|R) \neq \pm 1_R$: This is obvious when char $F = 2$; if char $F \neq 2$ apply 2.1.12 and Theorem 2.1.11. So G_2 is a normal subgroup of $\mathrm{SL}_2(R)$ which is not contained in $\mathrm{RL}_2(R)$. This implies that $G_2 = \mathrm{SL}_2(R)$ by 1.7.6. In particular, if we fix a line L in R, then G_2 contains all transvections on R with residual line L. Hence G contains all transvections in $\mathrm{Sp}_n(V)$ with residual line L. Hence G contains all transvections in $\mathrm{Sp}_n(V)$, by 3.3.1. Hence $G = \mathrm{Sp}_n(V)$, by Theorem 2.1.9. Q.E.D.

3.3.5. *Suppose $n \geqslant 4$ with $F = \mathbf{F}_3$, or $n \geqslant 6$ with $F = \mathbf{F}_2$, and let G be a normal subgroup of* $\mathrm{Sp}_n(V)$ *which contains a degenerate element σ of residue 2 which is a product of two transvections in* $\mathrm{Sp}_n(V)$. *Then $G = \mathrm{Sp}_n(V)$.*

PROOF. (1) A variation of the argument used in the proof of 3.3.4 allows us to assume that $n = 4$ if F is $\mathbf{F}_3$, and $n = 6$ if F is $\mathbf{F}_2$.

(2) First the case $n = 4$ with $F = \mathbf{F}_3$. Here σ has the form $\sigma = \tau_{a,*}\tau_{b,*}$ with $R = Fa \perp Fb$ and with the stars equal to ± 1. Now the τ's permute since $q(a, b) = 0$, so we can replace σ by its square if necessary and thereby assume that in fact $\sigma = \tau_{a,1}, \tau_{b,*}$. We can then assume that this new $*$ is -1. For if $\sigma = \tau_{a,1}\tau_{b,1}$ use Witt's Theorem and pick $\Sigma \in \mathrm{Sp}_4(V)$ with $\Sigma b = b$, $\Sigma a = a + b$. Then

$$\Sigma\sigma^{-1}\Sigma^{-1} = \tau_{b,-1}\tau_{\Sigma a,-1}.$$

Replace σ by

$$\sigma\Sigma\sigma^{-1}\Sigma^{-1} = \tau_{a,1}\tau_{\Sigma a,-1}.$$

So indeed let us assume that $\sigma = \tau_{a,1}\tau_{b,-1}$. Extend $\{a, b\}$ to a symplectic base

$$\mathfrak{X} = \{a, b | c, d\}$$

for V and note that

$$\sigma \sim \left(\begin{array}{cc|cc} 1 & & -1 & \\ & 1 & & 1 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \quad \text{in } \mathfrak{X}.$$

By suitably conjugating and applying Remark 3.3.3 we can find linear transformations in G whose matrices with respect to $\mathfrak{X}$ are equal to

$$\left(\begin{array}{cc|cc} 1 & & -1 & 1 \\ & 1 & 1 & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|cc} 1 & & -1 & -1 \\ & 1 & -1 & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right).$$

Multiplying these transformations gives us an element of G with matrix

$$\left(\begin{array}{cc|cc} 1 & & 1 & \\ & 1 & & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right).$$

So G contains $\tau_{a,-1}$. So G contains all (= both) transvections in $\mathrm{Sp}_4(V)$ with residual line Fa. So G contains all transvections in $\mathrm{Sp}_4(V)$ by 3.3.1. So $G = \mathrm{Sp}_4(V)$ by Theorem 2.1.9.

(3) Now the case $n = 6$ with $F = \mathbf{F}_2$. Here $\sigma = \tau_{a,1}\tau_{b,1}$ with $R = Fa \perp Fb$. Extend $\{a, b\}$ to a symplectic base

$$\mathfrak{X} = \{a, b, c | d, e, f\}.$$

Then

$$\sigma \sim \left(\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & & 1 & \\ & & 1 & & & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \quad \text{in } \mathfrak{X}.$$

Conjugating and applying Remark 3.3.3 gives us linear transformations in G with matrices

$$\left(\begin{array}{ccc|ccc} 1 & & & 1 & 1 & 0 \\ & 1 & & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|ccc} 1 & & & 1 & 1 & 1 \\ & 1 & & 1 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right),$$

hence with matrices

$$\left(\begin{array}{ccc|ccc} 1 & & & 0 & 0 & 1 \\ & 1 & & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|ccc} 1 & & & 0 & 1 & 0 \\ & 1 & & 1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right),$$

hence with matrix

$$\left(\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right).$$

In other words, G contains $\tau_{a,1}$. So G contains all transvections in $\mathrm{Sp}_n(V)$. So $G = \mathrm{Sp}_n(V)$. Q.E.D.

3.3.6. *If* $n \geqslant 4$, *then* $D\mathrm{Sp}_n = \mathrm{Sp}_n$ *but for the following exception*:

$$(\mathrm{Sp}_4(\mathbf{F}_2)\colon D\mathrm{Sp}_4(\mathbf{F}_2)) = 2.$$

PROOF. Put $\tau = \tau_{a,1}$ for some a in $\dot{V}$. By Witt's Theorem there is a Σ in $\mathrm{Sp}_n(V)$ such that $Fa + F\Sigma a$ is a plane with

$$q(a, \Sigma a) = \begin{cases} 1 & \text{if } F \neq \mathbf{F}_2, \mathbf{F}_3, \\ 0 & \text{if } F = \mathbf{F}_2, \mathbf{F}_3. \end{cases}$$

Define

$$\sigma = \tau\Sigma\tau^{-1}\Sigma^{-1} \in D\mathrm{Sp}_n(V).$$

Apply 3.3.4 and 3.3.5. In the exceptional case apply 3.1.5 plus the well-known behavior of $\mathfrak{S}_6$. Q.E.D.

3.3.7. *If* $n \geqslant 4$, *then* $D\mathrm{PSp}_n = \mathrm{PSp}_n$ *but for the following exception*:

$$(\mathrm{PSp}_4(\mathbf{F}_2)\colon D\mathrm{PSp}_4(\mathbf{F}_2)) = 2.$$

3.4. Simplicity Theorems

3.4.1. THEOREM. *The group* $\mathrm{PSp}_n(F)$ *is simple for any natural number* $n \geqslant 4$ *and any field* F *but for the group* $\mathrm{PSp}_4(\mathbf{F}_2)$ *which is not simple.*

PROOF. (1) The exceptional behavior of $\mathrm{PSp}_4(\mathbf{F}_2)$ follows from 3.3.7. So let us assume that $n \geqslant 4$ in general, and $n \geqslant 6$ when F is $\mathbf{F}_2$. Instead of working in the projective group we will work in Sp_n. It is enough to consider a normal subgroup G of Sp_n with $G \not\subseteq (\pm 1_V)$ and deduce that $G = \mathrm{Sp}_n$.

(2) First let us show that we have $a \in \dot{V}$, $\Sigma \in G$ such that $Fa + F\Sigma a$ is a regular plane. To this end pick $\Phi \in G$ with $\Phi \neq \pm 1_V$. Then Φ moves at least one line in V by 1.2.3 and 1.2.5, so there is a line L in V such that $\Phi L \neq L$. Let T be a nontrivial transvection in $\mathrm{Sp}_n(V)$ with residual line L. Then

$$\Sigma = (\mathrm{T}\Phi\mathrm{T}^{-1})\Phi^{-1} = \mathrm{T}(\Phi\mathrm{T}^{-1}\Phi^{-1})$$

is an element of G; it is also a product of two transvections in $\mathrm{Sp}_n(V)$ whose residual spaces are the distinct lines L and ΦL. So the residual space of Σ is the plane $L + \Phi L$, in particular res $\Sigma = 2$. If Σ were a hyperbolic transformation, then Σ would be an involution by 2.1.6; so 2.1.18 would apply in characteristic 2; and 2.1.13 would apply in characteristic not 2; in particular Σ would not be a product of res $\Sigma = 2$ transvections in $\mathrm{Sp}_n(V)$; and this is absurd. So Σ is not hyperbolic. There therefore exists an a in $\dot{V}$ such that $q(a, \Sigma a) \neq 0$, i.e., such that $Fa + F\Sigma a$ is a regular plane.

(3) We can also show that we have $a \in \dot{V}$, $\Sigma \in G$ such that $Fa + F\Sigma a$ is a degenerate plane. Pick $\Phi \in G$ with $\Phi \neq \pm 1_V$. Then there is an a in V such that $Fa \neq F\Phi a$. If $q(a, \Phi a) = 0$ we are done, so assume $q(a, \Phi a) \neq 0$. Pick $b \in V - (Fa + F\Phi a)$ with

$$q(b, a) = 0 \quad \text{and} \quad q(\Phi a, b) = q(a, \Phi a) \neq 0.$$

By Witt's Theorem there is a T in $\mathrm{Sp}_n(V)$ with $\mathrm{T}a = \Phi a$ and $\mathrm{T}\Phi a = b$. Then $\Sigma = \mathrm{T}\Phi\mathrm{T}^{-1}\Phi$ is an element of G which carries a to b, so $Fa + F\Sigma a$ is a degenerate plane.

(4) Pick $a \in \dot{V}$, $\Sigma \in G$ such that $Fa + F\Sigma a$ is a regular plane when $F \neq \mathbf{F}_2, \mathbf{F}_3$, a degenerate plane when $F = \mathbf{F}_2, \mathbf{F}_3$. Then

$$\begin{aligned}\sigma &= \left(\tau_{a,1}\Sigma\tau_{a,1}^{-1}\right)\Sigma^{-1}\\ &= \tau_{a,1}\left(\Sigma\tau_{a,1}^{-1}\Sigma^{-1}\right)\\ &= \tau_{a,1}\tau_{\Sigma a,-1}\end{aligned}$$

is an element of G, it is a product of two transvections in $\mathrm{Sp}_n(V)$, and its residual space is the plane $Fa + F\Sigma a$. So $G = \mathrm{Sp}_n(V)$ by 3.3.4 and 3.3.5. Q.E.D.

3.4.2. *If X is a normal subgroup of* Sp_n *with $n \geqslant 4$, then $X \subseteq (\pm I)$ or $X = \mathrm{Sp}_n$, but for the exceptional situation* $\mathrm{Sp}_4(\mathbf{F}_2)$ *which clearly does not possess this property.*

PROOF. For the exceptional situation see 3.3.6. By applying Theorem 3.4.1 to PX we see that $X \subseteq (\pm I)$ or $X \cdot (\pm I) = \mathrm{Sp}_n$. Assume the latter. Then

$$X \supseteq DX = D(X \cdot (\pm I)) = D\mathrm{Sp}_n = \mathrm{Sp}_n. \quad \text{Q.E.D.}$$

The Simplicity Theorem 3.4.1 can also be proved using the permutation group approach given in §3.5 of the *Linear Lectures*. Recall that a permutation group G on a nonempty set A is a subgroup of the group of all permutations of A. Recall that G is called transitive if, given $a \in A$ and $b \in A$, there is a σ in G such that $\sigma a = b$. Recall that a partition $\mathcal{P}$ of A is a set of nonoverlapping subsets of A whose union is A. The trivial partitions of A are the ones consisting of A itself on the one hand, and of every point of A on the other. A transitive group of permutations on the set A is called imprimitive if there is a nontrivial partition $\mathcal{P}$ of A such that $\sigma X \in \mathcal{P}$ for all σ in G and all X in $\mathcal{P}$. Otherwise it is called primitive. The key result (see 3.5.4 of the *Linear Lectures* for the proof) is the following.

3.4.3. *A primitive permutation group G on a set A is simple if it satisfies the following two conditions*:

(1) *$DG = G$.*

(2) *For some $a \in A$ there is a normal abelian subgroup H_a of the stabilizer S_a of a such that G is the group generated by gH_ag^{-1} as g runs through G.*

In order to prove Theorem 3.4.1 using this result one considers $\mathrm{PSp}_n(V)$ as a permutation group on the set of lines $\mathfrak{L}$ of V. This is possible since $\mathrm{PSp}_n(V)$, being a subgroup of the group of projectivities of V, acts faithfully on $\mathfrak{L}$; hence $\mathrm{PSp}_n(V)$ is naturally isomorphic to a permutation group on the set $\mathfrak{L}$. We know that $\mathrm{PSp}_n(V)$ is transitive by Witt's Theorem, that $D\mathrm{PSp}_n(V) = \mathrm{PSp}_n(V)$ by 3.3.7, and that the set of projective transvections in $\mathrm{PSp}_n(V)$ with residual line L, plus 1, is a normal abelian subgroup of the stabilizer of L in $\mathrm{PSp}_n(V)$ which, along with its conjugates in $\mathrm{PSp}_n(V)$, generate $\mathrm{PSp}_n(V)$, by §1.6, 3.3.1 and

Theorem 2.1.9. So all that remains before we can apply 3.4.3 is for us to verify that $\mathrm{PSp}_n(V)$ is primitive.

3.4.4. *The permutation group* $\mathrm{PSp}_n(V)$ *acting on the set of lines* $\mathfrak{L}$ *of* V *is primitive when* $n \geqslant 4$.

PROOF. (1) We must consider a partition $\mathcal{P}$ of $\mathfrak{L}$ which contains at least two cosets such that at least one coset, say C_0, has at least two lines. And we must find an element of $\mathrm{PSp}_n(V)$ that will disrupt this partition. Suppose, if possible, there is no such element.

(2) First let C_0 contain two distinct lines L_1, L_2 which are nonorthogonal. Then every pair of distinct lines K_1, K_2 in C_0 must be nonorthogonal. Otherwise we would have distinct K_1, K_2 in C_0 with $q(K_1, K_2) = 0$. Pick a line J of V that is not in the coset C_0. If $q(L_1, J) = 0$, then it follows from Witt's Theorem that there is a Σ in PSp_n such that $\Sigma K_1 = L_1$, $\Sigma K_2 = J$, and this disrupts the partition. If $q(L_1, J) \neq 0$, it again follows from Witt's Theorem that we have Σ in PSp_n with $\Sigma L_1 = L_1$, $\Sigma L_2 = J$, and this is again absurd. So, indeed, every pair of distinct lines in C_0 is nonorthogonal. The argument just used then shows that if L is any line in C_0, then C_0 contains all lines of V that are nonorthogonal to L. Now we can clearly find a line M in V that is nonorthogonal to L_1 and orthogonal to L_2; the first condition puts M in C_0, the second puts it outside C_0, and this is absurd.

(3) We may therefore assume that all lines in C_0 are mutually orthogonal. The arguments used in step (2) then show that if L is any line in C_0, then C_0 contains all lines orthogonal to L. This is again impossible. Q.E.D.

3.5. Comments

The simplicity of PSp_n is due to Dickson and Dieudonné. See

J. DIEUDONNÉ, *Sur les groupes classiques*, Actualités Scientifiques et Industrielles, no. 1040, Hermann, Paris, 1948.

for details. For the Chevalley groups see

C. CHEVALLEY, *Sur certain groupes simples*, Tôhoku Math. J. (2) **7** (1955), 14–66.

For simplicity theorems in a unified algebraic framework see

J. TITS, *Algebraic and abstract simple groups*, Ann. of Math. (2) **80** (1964), 313–329.

If we pass from fields to rings there is no longer any hope for simplicity. Structure theory then becomes not a question of simplicity but a question of describing all normal subgroups. Results are highly dependent on the underlying ring. See

YU. I. MERZLYAKOV, *Linear groups*, J. Soviet Math. **1** (1973), 571–593

for the extensive literature on structure theory over special rings.

4. SYMPLECTIC COLLINEAR TRANSFORMATIONS

4.1. Collinear Transformations

In §4.1 we assume that V and V_1 are just abstract vector spaces, i.e., we ignore the alternating forms which our general assumptions say they possess. For proofs see the *Linear Lectures*.

Consider a field isomorphism $\mu: F \rightarrowtail F_1$. We will write α^μ for the action $\mu\alpha$ of μ on any α in F. Thus

$$(\alpha + \beta)^\mu = \alpha^\mu + \beta^\mu, \qquad (\alpha\beta)^\mu = \alpha^\mu\beta^\mu.$$

And if $\mu_1: F_1 \rightarrowtail F_2$ is a second such situation, then

$$(\alpha^\mu)^{\mu_1} = \alpha^{\mu_1\mu}.$$

A map $k: V \to V_1$ is called semilinear with respect to $\mu: F \rightarrowtail F_1$ if

$$k(x + y) = kx + ky, \qquad k(\alpha x) = \alpha^\mu(kx)$$

for all x, y in V and all α in F. A map $k: V \to V_1$ is called semilinear if it is semilinear with respect to some μ. If $k \neq 0$, then the associated μ is unique. If $k: V \to V_1$ is semilinear with respect to $\mu: F \rightarrowtail F_1$ and $k_1: V_1 \to V_2$ is semilinear with respect to $\mu_1: F_1 \rightarrowtail F_2$, then $k_1k: V \to V_2$ is semilinear with respect to $\mu_1\mu$. If the bijection $k: V \rightarrowtail V_1$ is semilinear with respect to $\mu: F \rightarrowtail F_1$, then $k^{-1}: V_1 \rightarrowtail V$ is semilinear with respect to $\mu^{-1}: F_1 \rightarrowtail F$.

4.1.1. *Let $k: V \to V_1$ be semilinear and let W and W_1 be subspaces of V and V_1 respectively. Then*

(1) *kW is a subspace of V_1.*

(2) *$k^{-1}W_1$ is a subspace of V.*

(3) $\dim_{F_1} kV + \dim_F k^{-1}0 = \dim_F V$.

4.1.2. *If $x_1, \ldots, x_n$ is a base for V, and $v_1, \ldots, v_n$ are any n vectors in V_1, and if a field isomorphism $\mu: F \rightarrowtail F_1$ is given, then there is a unique semilinear map $k: V \to V_1$ with associated field isomorphism μ which carries x_i to v_i for $1 \leqslant i \leqslant n$. The defining equation of this k is*

$$k\left(\sum_1^n \alpha_i x_i\right) = \sum_1^n \alpha_i^\mu v_i.$$

The field isomorphism $\mu: F \rightarrowtail F_1$ is extended to matrices in obvious way, entry by entry. With the usual restrictions on matching rows and columns we have

$$(A+B)^\mu = A^\mu + B^\mu, \qquad (AB)^\mu = A^\mu B^\mu,$$
$$\det A^\mu = (\det A)^\mu, \qquad (A^{-1})^\mu = (A^\mu)^{-1}$$

where the last equation means that A is invertible over F if and only if A^μ is invertible over F_1, and then $(A^{-1})^\mu = (A^\mu)^{-1}$. Under composition we have $(A^\mu)^{\mu_1} = A^{\mu_1\mu}$.

If $k: V \rightarrowtail V_1$ is semilinear with respect to $\mu: F \rightarrowtail F_1$, and if bases $\mathfrak{X} = \{x_1, \ldots, x_n\}$ and $\mathfrak{X}_1 = \{y_1, \ldots, y_n\}$ are fixed in V and V_1 respectively, then each kx_j can be expressed in the form

$$kx_j = \sum_{i=1}^{n_1} a_{ij} y_i \qquad (a_{ij} \in F_1)$$

and the resulting $n_1 \times n$ matrix $A = (a_{ij})$ over F_1 is called the matrix of k with respect to the pair of bases $\mathfrak{X}$, $\mathfrak{X}_1$. If we consider a second such situation $k_1: V_1 \rightarrowtail V_2$ with respect to $\mu_1: F_1 \rightarrowtail F_2$ with bases $\mathfrak{X}_1$, $\mathfrak{X}_2$, then the matrix of k_1k with respect to $\mathfrak{X}$, $\mathfrak{X}_2$ is easily seen to be $A_1 A^{\mu_1}$. We have

$$k \text{ bijective} \Leftrightarrow A^{\mu^{-1}} \text{ invertible} \Leftrightarrow A \text{ invertible}.$$

If k is invertible, then the matrix of k^{-1} with respect to $\mathfrak{X}_1$, $\mathfrak{X}$ is $(A^{-1})^{\mu^{-1}}$.

All this holds if we take $F = F_1$, $V = V_1$, $\mathfrak{X} = \mathfrak{X}_1$, and $\mu: F \rightarrowtail F$ an automorphism of F. We then call the $n \times n$ matrix $A = (a_{ij})$ over F the matrix of k with respect to the base $\mathfrak{X}$. If $\mathfrak{Z} = \{z_1, \ldots, z_n\}$ is a second base for V that is related to the first by the $n \times n$ matrix $T = (t_{ij})$ over F that is given by

$$z_j = \sum_{i=1}^{n} t_{ij} x_i,$$

and if B is the matrix of k with respect to $\mathfrak{Z}$, then

$$B = T^{-1} A T^\mu.$$

A collinear transformation k of V onto V_1 is, by definition, a semilinear bijection $k: V \rightarrowtail V_1$. Composites and inverses of collinear transformations are themselves collinear. Clearly

$$k \text{ collinear} \Rightarrow k \text{ geometric}.$$

In particular we can form $\bar{k}$ for any collinear k and thereby obtain a projectivity, indeed a projective geometric transformation, $\bar{k}: \mathrm{P}(V) \rightarrowtail \mathrm{P}(V_1)$ of V onto V_1. A projectivity $\pi: \mathrm{P}(V) \rightarrowtail \mathrm{P}(V_1)$ which has the form $\pi = \bar{k}$ for some collinear $k: V \rightarrowtail V_1$ is called a projective collinear transformation of V onto V_1. Clearly composites and inverses of projective collinear transformations are themselves projective collinear transformations, and

$$\pi \text{ projective collinear} \Rightarrow \pi \text{ projective geometric}.$$

4.1.3. *Let π be a bijection of the lines of V onto the lines of V_1, and let*

$\dim_F V = \dim_{F_1} V_1 \geqslant 3$. *Suppose that for each hyperplane H of V there is a hyperplane H_1 of V_1 such that πL is a line in H_1 whenever L is a line in H. Then π can be extended uniquely to a projectivity* Π: $\mathrm{P}(V) \rightarrowtail \mathrm{P}(V_1)$.

4.1.4. FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY. *Suppose* $\dim V \geqslant 3$. *Then every projectivity of V onto V_1 is a projective collinear transformation.*

A collinear transformation of V is, by definition, a collinear transformation of V onto V. The field isomorphism μ associated with a collinear transformation of V is therefore an automorphism of the field F, i.e., if k: $V \rightarrowtail V$ is a collinear transformation of V with associated field isomorphism μ: $F \rightarrowtail F$, then

$$\mu \in \operatorname{aut} F$$

where aut F denotes the group of automorphisms of F. The set of collinear transformations of V is a subgroup of the general geometric group $\Xi\mathrm{L}_n(V)$. It will be written $\Gamma\mathrm{L}_n(V)$ and will be called the collinear group of V. By a group of collinear transformations of V we mean any subgroup of $\Gamma\mathrm{L}_n(V)$. Clearly $\mathrm{GL}_n(V) \subseteq \Gamma\mathrm{L}_n(V)$; so any group of linear transformations of V is a group of collinear transformations of V.

A projective collinear transformation of V is, by definition, a projective collinear transformation of V onto V. The set of projective collinear transformations of V is exactly the subgroup of projectivities of V consisting of the images of $\Gamma\mathrm{L}_n(V)$ under the bar mapping, i.e., it is the group $\mathrm{P}\Gamma\mathrm{L}_n(V)$. This group is called the projective collinear group of V. By a projective group of collinear transformations of V we mean any subgroup of $\mathrm{P}\Gamma\mathrm{L}_n(V)$. Clearly $\mathrm{PGL}_n(V) \subseteq \mathrm{P}\Gamma\mathrm{L}_n(V)$; so any projective group of linear transformations is a projective group of collinear transformations of V.

4.1.5. RL_n, SL_n and GL_n *are normal subgroups of* $\Gamma\mathrm{L}_n$; *and* PSL_n *and* PGL_n *are normal subgroups of* $\mathrm{P}\Gamma\mathrm{L}_n$.

4.1.6. *Suppose $n \geqslant 2$ and let $k \in \Gamma\mathrm{L}_n$ be such that $kL = L$ for all lines L in V. Then* $k \in \mathrm{RL}_n$.

4.1.7. $\mathrm{P}\Xi\mathrm{L}_n(V) = \mathrm{P}\Gamma\mathrm{L}_n(V)$ *for* $n \geqslant 3$, *and* $\ker(\mathrm{P}|\Gamma\mathrm{L}_n) = \mathrm{RL}_n$ *for* $n \geqslant 2$.

By a representative of an element Σ in $\mathrm{P}\Gamma\mathrm{L}_n(V)$ we mean an element k of $\Gamma\mathrm{L}_n(V)$ such that $\bar{k} = \Sigma$. If k_1 and k_2 are elements of $\Gamma\mathrm{L}_n(V)$, then the following statements are equivalent when $n \geqslant 2$:

(1) k_1 and k_2 represent the same Σ in $\mathrm{P}\Gamma\mathrm{L}_n$.
(2) $k_1 = k_2 r$ for some r in RL_n.
(3) $k_1 = r k_2$ for some r in RL_n.

If $n \geqslant 2$, then all representatives of an element Σ of PGL_n fall in GL_n.

We now introduce a group isomorphism Φ_g where g is first a collinear transformation g: $V \rightarrowtail V_1$ of V onto V_1, and secondly a projective collinear transformation g: $\mathrm{P}(V) \rightarrowtail \mathrm{P}(V_1)$ of V onto V_1.

First consider a collinear transformation g: $V \rightarrowtail V_1$. Let μ: $F \rightarrowtail F_1$ be the associated field isomorphism. Here $n = n_1$ follows. Then Φ_g defined by

$$\Phi_g k = gkg^{-1} \qquad \forall\, k \in \Gamma L_n(V)$$

is a group isomorphism

$$\Phi_g\colon \Gamma L_n(V) \rightarrowtail \Gamma L_{n_1}(V_1).$$

Under composition and inversion,

$$\Phi_{g_1 g} = \Phi_{g_1}\Phi_g, \qquad \Phi_g^{-1} = \Phi_{g^{-1}}.$$

And Φ_g induces

$$\Phi_g\colon GL_n(V) \rightarrowtail GL_{n_1}(V_1),$$

$$\Phi_g\colon SL_n(V) \rightarrowtail SL_{n_1}(V_1),$$

$$\Phi_g\colon RL_n(V) \rightarrowtail RL_{n_1}(V_1).$$

If σ is in $GL_n(V)$, then the residual and fixed spaces of $\Phi_g\sigma$ are gR and gP respectively; in particular

$$\text{res}\, \Phi_g\sigma = \text{res}\, \sigma.$$

If H is a hyperplane and L is a line with $L \subseteq H$, then gL is a line contained in the hyperplane gH of V_1, and Φ_g carries the set of transvections with spaces $L \subseteq H$ onto the set of transvections with spaces $gL \subseteq gH$. If σ is the transvection $\sigma = \tau_{a,\rho}$ in usual form, then

$$\Phi_g \tau_{a,\rho} = \tau_{ga,\mu\rho g^{-1}}.$$

Now consider a projective collinear transformation $g\colon P(V) \rightarrowtail P(V_1)$ of V onto V_1. We again have $n = n_1$. This time define

$$\Phi_g k = gkg^{-1} \qquad \forall\, k \in P\Gamma L_n(V)$$

and obtain a group isomorphism

$$\Phi_g\colon P\Gamma L_n(V) \rightarrowtail P\Gamma L_{n_1}(V_1).$$

Under composition and inversion,

$$\Phi_{g_1 g} = \Phi_{g_1}\Phi_g, \qquad \Phi_g^{-1} = \Phi_{g^{-1}}.$$

Since g is projective collinear it is of the form $g = \bar{h}$ for some collinear $h\colon V \rightarrowtail V_1$. We have

$$\Phi_g \bar{j} = \Phi_{\bar{h}} \bar{j} = \overline{\Phi_h j} \qquad \forall\, j \in \Gamma L_n(V).$$

And Φ_g induces

$$\Phi_g\colon PGL_n(V) \rightarrowtail PGL_{n_1}(V_1),$$

$$\Phi_g\colon PSL_n(V) \rightarrowtail PSL_{n_1}(V_1).$$

And Φ_g carries the set of projective transvections with spaces $L \subseteq H$ onto the set with spaces $gL \subseteq gH$.

4.1.8. *Suppose $n = n_1 \geqslant 2$. If g_1 and g_2 are collinear transformations of V onto V_1, then the following statements are equivalent*:

(1) $\Phi_{\bar{g}_1} = \Phi_{\bar{g}_2}$.

(2) $\bar{g}_1 = \bar{g}_2$.

(3) $g_1 = g_2 r$ *for some* r *in* $\mathrm{RL}_n(V)$.
(4) $g_1 = r_1 g_2$ *for some* r_1 *in* $\mathrm{RL}_{n_1}(V_1)$.

4.1.9. *Let* $\sigma \in \mathrm{GL}_n(V)$ *and let* g *be a collinear transformation of* V *onto* V_1 *with associated isomorphism* $\mu\colon F \rightarrowtail F_1$. *Let* $\mathfrak{X}$ *and* $\mathfrak{X}_1$ *be bases for* V *and* V_1 *respectively. Let* A *be the matrix of* σ *in* $\mathfrak{X}$, *and let* X *be the matrix of* g *in* $\mathfrak{X}$, $\mathfrak{X}_1$. *Then the matrix of* $\Phi_g\sigma$ *in* $\mathfrak{X}_1$ *is*

$$XA^{\mu}X^{-1}.$$

In particular, $\det \Phi_g\sigma = (\det \sigma)^{\mu}$.

We say that two elements k_1 and k_2 of $\Gamma\mathrm{L}_n(V)$ permute projectively if $\bar{k}_1$ and $\bar{k}_2$ permute. Obviously

$$\text{permutability} \Rightarrow \text{projective permutability}.$$

4.1.10. *Let* σ *be any element of* $\mathrm{GL}_n(V)$ *which satisfies any one of the following conditions*:

(1) $\operatorname{res} \sigma < \frac{1}{2}n$;

(2) $\operatorname{res} \sigma = \frac{1}{2}n$ *with* σ *not a big dilation*;

(3) σ *is a transvection.*

If σ *permutes projectively with some* k *in* $\Gamma\mathrm{L}_n(V)$, *then* σ *permutes with* k.

4.1.11. *Suppose that* $\sigma \in \mathrm{SL}_n$ *and* $k \in \Gamma\mathrm{L}_n$, *or that* σ *and* k *are in* GL_n. *If* σ *permutes projectively with* k, *then* σ^n *permutes with* k.

4.2. Symplectic Collinear Transformations

We now return to our general assumptions that V and V_1 are regular alternating spaces.

A collinear transformation k of V onto V_1 (with associated field isomorphism $\mu\colon F \rightarrowtail F_1$) is said to be a symplectic collinear transformation if there is a constant m_k in $\dot{F}_1$, dependent on k, such that

$$q_1(kx, ky) = m_k(q(x, y))^{\mu}$$

holds for all x and y in V. This constant is clearly uniquely determined by k (since V is regular and nonzero) and it is called the multiplier of the symplectic collinear k. It is obvious that composites and inverses of symplectic collinear transformations are themselves symplectic collinear transformations with

$$m_{k_1k} = m_{k_1}m_k^{\mu_1}, \qquad m_{k^{-1}} = \left(m_k^{-1}\right)^{\mu^{-1}}.$$

By a projective symplectic collinear transformation we mean a projective collinear transformation which can be expressed in the form $\bar{g}$ for some symplectic collinear transformation g of V onto V_1. Composites and inverses of projective symplectic collinear transformations are, of course, projective symplectic collinear transformations.

4.2.1. *The following statements are equivalent for a collinear transformation* k *of* V *onto* V_1:

(1) *k is symplectic collinear.*
(2) *$\bar{k}$ is projective symplectic collinear.*
(3) $q(x,y) = 0 \Leftrightarrow q_1(kx, ky) = 0 \; \forall \; x, y \in V$.
(4) $kU^* = (kU)^* \; \forall \; U \in P(V)$.

PROOF. (1) implies (2) by the definition of projective symplectic collinear. And (2) implies (1) follows easily from 4.1.8 and the definition of symplectic collinear. So (1) is equivalent to (2). And the equivalence of (3) and (4) follows easily from the definitions. Clearly (1) implies (3). To prove that (3) implies (1) we observe (after some checking) that we can define a regular alternating form $q_2\colon V_1 \times V_1 \twoheadrightarrow F_1$ by the equation

$$q_2(kx, ky) = (q(x,y))^{\mu} \qquad \forall \; x, y \in V.$$

Then

$$q_1(kx, ky) = 0 \Leftrightarrow q(x,y) = 0 \Leftrightarrow q_2(kx, ky) = 0,$$

so q_1 and q_2 determine the same polarity on V_1. Apply 1.2.7. Q.E.D.

A symplectic collinear transformation of V is, by definition, a symplectic collinear transformation of V onto V. The set of symplectic collinear transformations of V forms a subgroup of $\Gamma L_n(V)$, denoted $\Gamma Sp_n(V)$, and called the symplectic collinear group of V. By a group of symplectic collinear transformations of V we mean any subgroup of $\Gamma Sp_n(V)$.

A symplectic similitude of V is, by definition, a symplectic collinear transformation of V that is actually linear. Thus

$$q(\sigma x, \sigma y) = m_\sigma q(x,y) \qquad \forall \; x, y \in V.$$

The set of similitudes of V forms a subgroup of $\Gamma Sp_n(V)$, denoted $GSp_n(V)$, and called the group of symplectic similitudes of V. By a group of symplectic similitudes of V we mean any subgroup of $GSp_n(V)$.

4.2.2. *Let $V \cong A$ in some base $\mathfrak{X}$. Let k be an element of $\Gamma L_n(V)$, let μ be its associated field automorphism, let K be its matrix with respect to $\mathfrak{X}$. Then $k \in \Gamma Sp_n(V)$ if and only if ${}^tKAK = \beta A^{\mu}$ holds for some β in $\dot{F}$. If this condition is satisfied, then $m_k = \beta$.*

Accordingly $GSp_n(V)$ is isomorphic to the group of $n \times n$ matrices K over F which satisfy ${}^tKAK = \alpha_K A$. And, of course, $Sp_n(V)$ is isomorphic to the subgroup of these matrices with $\alpha_K = 1$.

Of course $GSp_n(V)$ is the normal subgroup $GSp_n(V) = \Gamma Sp_n(V) \cap GL_n(V)$ of $\Gamma Sp_n(V)$, and it follows from Theorem 1.1.13, 4.1.2 and 4.2.2 that

$$\Gamma Sp_n(V)/GSp_n(V) \cong \text{aut } F.$$

For similitudes we have

$$m_{\sigma_1 \sigma} = m_{\sigma_1} m_\sigma, \qquad m_{\sigma^{-1}} = m_\sigma^{-1},$$

so the mapping $\sigma \twoheadrightarrow m_\sigma$ is a homomorphism of $GSp_n(V)$ into $\dot{F}$; and it follows easily from Theorem 1.1.13 and 4.2.2 that this mapping is surjective. Its kernel is clearly $Sp_n(V)$. Therefore $Sp_n(V) \lhd GSp_n(V)$ with

$$\mathrm{GSp}_n(V)/\mathrm{Sp}_n(V) \cong \dot{F}.$$

4.2.3([2]). (1) $\Gamma\mathrm{Sp}_2(V) = \Gamma\mathrm{L}_2(V)$.
(2) $\mathrm{GSp}_2(V) = \mathrm{GL}_2(V)$.
(3) $\mathrm{Sp}_2(V) = \mathrm{SL}_2(V)$.
(4) $m_\sigma = \det \sigma$ *for all* σ *in* $\mathrm{GSp}_2(V)$.

PROOF. Take a symplectic base for V and apply 4.2.2. Q.E.D.

To define the projective symplectic collinear group $\mathrm{P}\Gamma\mathrm{Sp}_n(V)$, apply P to $\Gamma\mathrm{Sp}_n(V)$. By a projective group of symplectic collinear transformations we mean any subgroup of $\mathrm{P}\Gamma\mathrm{Sp}_n(V)$. To define the projective group of symplectic similitudes $\mathrm{PGSp}_n(V)$ apply P to $\mathrm{GSp}_n(V)$. By a projective group of symplectic similitudes we mean any subgroup of $\mathrm{PGSp}_n(V)$.

Clearly every element r_α of $\mathrm{RL}_n(V)$ is a similitude with multiplier α^2, in particular $\mathrm{RL}_n(V)$ is a normal subgroup of $\Gamma\mathrm{Sp}_n(V)$, and so of $\mathrm{GSp}_n(V)$. So

$$\mathrm{P}\Gamma\mathrm{Sp}_n \cong \Gamma\mathrm{Sp}_n/\mathrm{RL}_n, \qquad \mathrm{PGSp}_n \cong \mathrm{GSp}_n/\mathrm{RL}_n.$$

And

$$\mathrm{P}\Gamma\mathrm{Sp}_n/\mathrm{PGSp}_n \cong \Gamma\mathrm{Sp}_n/\mathrm{GSp}_n \cong \text{aut } F.$$

The kernel of the composite homomorphism

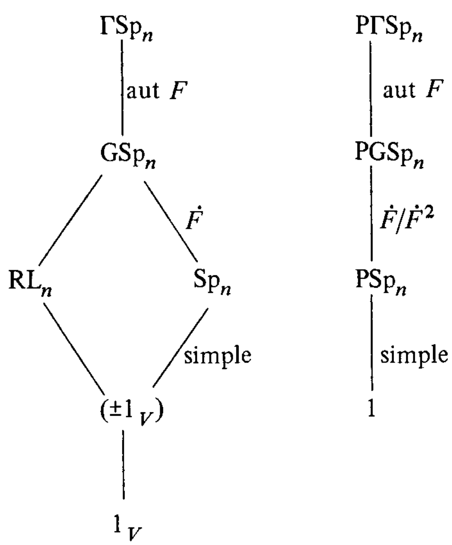
$$\mathrm{GSp}_n \to \mathrm{PGSp}_n \to \mathrm{PGSp}_n/\mathrm{PSp}_n$$

is easily seen to be $\mathrm{RL}_n \cdot \mathrm{Sp}_n$, whence

$$\mathrm{PGSp}_n/\mathrm{PSp}_n \cong \dot{F}/\dot{F}^2.$$

4.2.4. *If* $\sigma \in \mathrm{GSp}_n(V)$, *then* $\bar{\sigma} \in \mathrm{PSp}_n(V)$ *if and only if* $m_\sigma \in \dot{F}^2$.

We can summarize the above information in the following diagram (where we assume $n \geqslant 4$, and also $F \neq \mathbf{F}_2$ when $n = 4$).



([2])We can therefore adopt the philosophy that symplectic groups really begin in 4-dimensions.

4.2.5. (1) *The centralizer of* PSp_n *in* $\mathrm{P\Gamma L}_n$ *is trivial.*
(2) $\mathrm{P\Gamma Sp}_n$, PGSp_n, PSp_n *are centerless.*
(3) *The centralizer of* RL_n *in* $\mathrm{\Gamma Sp}_n$ *is* GSp_n.
(4) *The centralizer of* Sp_n *in* $\mathrm{\Gamma L}_n$ *is* RL_n.
(5) $\mathrm{cen}\ \mathrm{GSp}_n = \mathrm{RL}_n$.

PROOF. Left as an exercise. Q.E.D.

Note that all representatives in $\Gamma\mathrm{L}_n$ of an element of $\mathrm{P\Gamma Sp}_n$ actually fall in $\Gamma\mathrm{Sp}_n$; and all representatives of an element of PGSp_n fall in GSp_n; of course, the representatives of an element of PSp_n will fall in GSp_n, but not necessarily in Sp_n.

If we consider the isomorphism Φ_g of §4.1 with $g\colon V \rightarrowtail V_1$ symplectic collinear (not just collinear) then, in addition to the properties stated in §4.1, we find that Φ_g induces

$$\Phi_g\colon \Gamma\mathrm{Sp}_n(V) \rightarrowtail \Gamma\mathrm{Sp}_{n_1}(V_1),$$
$$\Phi_g\colon \mathrm{GSp}_n(V) \rightarrowtail \mathrm{GSp}_{n_1}(V_1),$$
$$\Phi_g\colon \mathrm{Sp}_n(V) \rightarrowtail \mathrm{Sp}_{n_1}(V_1);$$

in particular if we take a transvection in $\mathrm{Sp}_n(V)$ and express it in the usual form $\tau_{a,\lambda}$, and if μ denotes the field isomorphism associated with g, then we find that

$$\Phi_g \tau_{a,\lambda} = \tau_{ga,\lambda^\mu m_g^{-1}}.$$

Similarly we find that if g is a projective symplectic collinear transformation $g\colon \mathrm{P}(V) \rightarrowtail \mathrm{P}(V_1)$ of V onto V_1 (not just projective collinear) then Φ_g induces

$$\Phi_g\colon \mathrm{P\Gamma Sp}_n(V) \rightarrowtail \mathrm{P\Gamma Sp}_{n_1}(V_1),$$
$$\Phi_g\colon \mathrm{PGSp}_n(V) \rightarrowtail \mathrm{PGSp}_{n_1}(V_1),$$
$$\Phi_g\colon \mathrm{PSp}_n(V) \rightarrowtail \mathrm{PSp}_{n_1}(V_1).$$

If σ is an element of $\mathrm{GSp}_n(V)$, then σ is in $\mathrm{GL}_n(V)$ so that R, P, $\mathrm{res}\,\sigma$ are already defined.

4.2.6. *Let* σ *be any element of* $\mathrm{GSp}_n(V)$. *Then*
(1) $R = P^* \Leftrightarrow \sigma \in \mathrm{Sp}_n(V)$ *or* $P = 0$.
(2) $q(P, P) \neq 0 \Rightarrow \sigma \in \mathrm{Sp}_n(V)$.

PROOF. (1) If $\sigma \in \mathrm{Sp}_n(V)$ then $R = P^*$ by 1.3.10. If $P = 0$ then $P^* = V$, but $\dim R + \dim P = n$, so $R = V$, so $R = P^*$. Finally suppose $R = P^*$ and let us show that $\sigma \in \mathrm{Sp}_n(V)$ if $P \neq 0$. Pick $x \in V$ and $p \in P$ with $q(x, p) = 1$. Then $\sigma x - x \in R$, so $q(\sigma x - x, p) = 0$, so $q(\sigma x, p) = q(x, p) = 1$. So

$$1 = q(x, p) = m_\sigma^{-1} q(\sigma x, \sigma p) = m_\sigma^{-1} q(\sigma x, p) = m_\sigma^{-1}.$$

So σ is in $\mathrm{Sp}_n(V)$.

(2) Take p_1, p_2 in P with $q(p_1, p_2) = 1$. Then

$$1 = q(p_1, p_2) = q(\sigma p_1, \sigma p_2) = m_\sigma q(p_1, p_2) = m_\sigma. \quad \text{Q.E.D.}$$

4.2.7. *Every transvection in* $\Gamma\mathrm{Sp}_n(V)$ *is already in* $\mathrm{Sp}_n(V)$. *Every projective* *transvection in* $\mathrm{P\Gamma Sp}_n(V)$ *is already in* $\mathrm{PSp}_n(V)$, *and its representative transvection is in* $\mathrm{Sp}_n(V)$.

PROOF. If $n = 2$, apply 4.2.3. So assume $n \geqslant 4$. If σ is a transvection in $\Gamma\mathrm{Sp}_n(V)$, then $\sigma \in \mathrm{GSp}_n(V)$, and $q(P, P) \neq 0$ since $\dim P \geqslant n - 1 > \frac{1}{2} n$, so σ is in $\mathrm{Sp}_n(V)$ by 4.2.6. A projective transvection in $\mathrm{P\Gamma Sp}_n(V)$ has the form $\bar{\tau} = \bar{k}$ with τ a transvection in $\mathrm{SL}_n(V)$ and k an element of $\Gamma\mathrm{Sp}_n(V)$. Then $\tau = r_\alpha k$ by 4.1.7, so τ is in $\Gamma\mathrm{Sp}_n(V)$, so τ is in $\mathrm{Sp}_n(V)$, so the representative transvection τ of a given projective transvection in $\mathrm{P\Gamma Sp}_n(V)$ is in $\mathrm{Sp}_n(V)$, and the given projective transvection $\bar{\tau}$ is in $\mathrm{PSp}_n(V)$. Q.E.D.

4.2.8. EXAMPLE. If $\mathfrak{X}$ is a symplectic base for V and if

$$\sigma \sim \left(\begin{array}{c|c} \alpha I_{n/2} & \\ \hline & I_{n/2} \end{array}\right) \quad \text{in } \mathfrak{X},$$

with $\alpha \neq 0, 1$, then $\sigma \in \mathrm{GSp}_n(V)$, $m_\sigma = \alpha$,

$$R = Fx_1 + \cdots + Fx_{n/2}, \qquad P = Fy_1 + \cdots + Fy_{n/2},$$

and $q(R, P) = F \neq 0$.

4.2.9. EXAMPLE. If $k \in \Gamma\mathrm{Sp}_n(V)$ and $r_\alpha \in \mathrm{RL}_n(V)$, then k and r_α obviously permute projectively, but $kr_\alpha = (r_{\alpha^\mu \alpha^{-1}})r_\alpha k$ so that k need not permute with r_α. On the other hand, if σ_1 and σ_2 are elements of $\mathrm{GSp}_n(V)$ that permute projectively, then applying multipliers shows that $\sigma_1\sigma_2 = \pm\,\sigma_2\sigma_1$.

4.3. Hyperbolic Transformations

4.3.1. *Let k be a hyperbolic transformation in* $\Gamma\mathrm{Sp}_n(V)$. *Then*

(1) *rk and kr are hyperbolic for all r in* $\mathrm{RL}_n(V)$.

(2) $k \in \mathrm{GSp}_n(V)$.

(3) $k^2 = m_k 1_V$.

(4) *$\bar{k}$ is an involution in* $\mathrm{PGSp}_n(V)$.

PROOF. (1) follows immediately from the definition of hyperbolic. In order to prove (2) we note that

$$q(x, ky) + q(y, kx) = 0$$

is a consequence of $q(x + y, k(x + y)) = 0$. Replacing y by αy ($\alpha \in F$) gives

$$\alpha^\mu q(x, ky) + \alpha q(y, kx) = 0.$$

Therefore $(\alpha^\mu - \alpha)q(x, ky) = 0$. Therefore $\alpha^\mu = \alpha$. So k is in $\mathrm{GSp}_n(V)$, i.e., we have (2). We have

$$\begin{aligned} q(k^2x - m_kx, ky) &= q(k^2x, ky) - m_k q(x, ky) \\ &= m_k q(kx, y) + m_k q(ky, x) \\ &= m_k q(x + y, k(x + y)) \\ &= 0; \end{aligned}$$

hence $q(k^2x - m_kx, V) = 0$; hence $k^2x = m_kx$, and we have (3). So $\bar{k}^2 = 1$ and we have (4). Q.E.D.

4.3.2. *If k is a hyperbolic transformation in $\Gamma\mathrm{Sp}_n(V)$, then k is in $\mathrm{Sp}_n(V)$ if and only if k is an involution.*

PROOF. Apply 4.3.1. Q.E.D.

4.3.3. *If a hyperbolic transformation k in $\Gamma\mathrm{Sp}_n(V)$ stabilizes a line in V, then $\bar{k}$ is in $\mathrm{PSp}_n(V)$.*

PROOF. Apply 4.3.1 and 4.2.4. Q.E.D.

4.3.4. *Let k be a hyperbolic transformation in $\Gamma\mathrm{Sp}_n(V)$, and hence in $\mathrm{GSp}_n(V)$, such that $m_k \in \dot{F} - \dot{F}^2$. Then $n \equiv 0 \bmod 4$, and there is a symplectic base $\mathfrak{X}$ for V in which*

$$k \sim \left(\begin{array}{c|c} A & \\ \hline & {}^tA \end{array}\right)$$

where A is the $\frac{1}{2}n \times \frac{1}{2}n$ matrix

$$\begin{pmatrix} \begin{array}{cc} 0 & m_k \\ 1 & 0 \end{array} & & \\ & \ddots & \\ & & \begin{array}{cc} 0 & m_k \\ 1 & 0 \end{array} \end{pmatrix}$$

PROOF. By 4.2.4 we know that $\bar{k} \notin \mathrm{PSp}_n(V)$, hence by 4.3.3 we know that k can stabilize no line in V. So by 4.3.1 we know that $L + kL$ is a totally degenerate plane for which

$$k(L + kL) = L + kL,$$

for any line L in V. We therefore have totally degenerate planes Π_1 and Π_1' in V with

$$k\Pi_1 = \Pi_1, \qquad k\Pi_1' = \Pi_1',$$
$$q(\Pi_1, \Pi_1') \neq 0, \qquad \Pi_1 \cap \Pi_1' = 0.$$

Then $Q = \Pi_1 + \Pi_1'$ is a 4-dimensional subspace of V with $kQ = Q$. The space Q is clearly not totally degenerate, but can it be degenerate? If this were so, then rad Q would have to be a plane (take a radical splitting and recall that regular subspaces are even-dimensional). Now $\Pi_1 \cap \operatorname{rad} Q = 0$ would make Q totally degenerate; so would $\Pi_1 = \operatorname{rad} Q$ since then $\Pi_1' \cap \operatorname{rad} Q = 0$; so $\Pi_1 \cap \operatorname{rad} Q$ is a line, and this is impossible since k would then have to stabilize this line. Therefore Q is indeed regular. Repeat the entire argument on Q^* which is clearly stabilized by k. Continue. Ultimately we obtain totally degenerate planes

$$\Pi_1, \Pi'_1, \ldots, \Pi_t, \Pi'_t$$

with $\Pi_i \cap \Pi'_i = 0$, and $\Pi_i + \Pi'_i$ regular and 4-dimensional, such that

$$V = (\Pi_1 + \Pi'_1) \perp \cdots \perp (\Pi_t + \Pi'_t),$$

and such that each plane is stabilized by k. At this point we know that $n \equiv 0$ mod 4. Fix a nonzero vector x_1 in Π_1 and define x_2 in Π_1 by $kx_1 = x_2$. Thus

$$kx_1 = x_2,$$
$$kx_2 = m_k x_1.$$

Similarly pick x_3 and define x_4 in Π_2. And so on. Then $x_1, \ldots, x_{n/2}$ is a base for the maximal totally degenerate subspace

$$W = \Pi_1 + \cdots + \Pi_t$$

of V and the matrix of k in this base is equal to A. Put

$$W' = \Pi'_1 + \cdots + \Pi'_t.$$

By 1.1.16 we can find a base $y_1, \ldots, y_{n/2}$ for W' such that

$$\mathfrak{X} = \{x_1, \ldots, x_{n/2} | y_1, \ldots, y_{n/2}\}$$

is a symplectic base for V. By 4.2.2,

$$k \sim \left(\begin{array}{c|c} A & \\ \hline & {}^tA \end{array}\right) \quad \text{in } \mathfrak{X}. \quad \text{Q.E.D.}$$

4.3.5. *Suppose* $m \in \dot{F} - \dot{F}^2$ *and* $n \equiv 0$ mod 4. *Let* $\mathfrak{X}$ *be a symplectic base for* V, *let* A *be the* $\frac{1}{2}n \times \frac{1}{2}n$ *matrix*

$$A = \left(\begin{array}{cc|c|c} 0 & m & & \\ 1 & 0 & & \\ \hline & & \ddots & \\ \hline & & & \begin{matrix} 0 & m \\ 1 & 0 \end{matrix} \end{array}\right)$$

and let k *be the linear transformation defined by*

$$k \sim \left(\begin{array}{c|c} A & \\ \hline & {}^tA \end{array}\right) \quad \text{in } \mathfrak{X}.$$

Then k *is a hyperbolic transformation in* $\mathrm{GSp}_n(V)$ *with* $m_k = m$.

PROOF. k is in $\mathrm{GSp}_n(V)$ with $m_k = m$, by 4.2.2. And $q(x, kx) = 0$ for all x in V by direct calculation. Q.E.D.

A projective hyperbolic transformation in $\mathrm{P\Gamma Sp}_n(V)$ is, by definition, an element of $\mathrm{P\Gamma Sp}_n(V)$ of the form $\bar{k}$ for some hyperbolic transformation in $\Gamma\mathrm{Sp}_n(V)$. It follows from 4.3.1 that every projective hyperbolic transformation is in fact an involution in $\mathrm{PGSp}_n(V)$, and all representatives in $\Gamma\mathrm{Sp}_n(V)$ of a projective hyperbolic transformation are hyperbolic.

4.3.6. *There are projective hyperbolic transformations in* $\mathrm{PGSp}_n(V) - \mathrm{PSp}_n(V)$ *if and only if* $n \equiv 0 \bmod 4$ *with* $\dot{F}^2 \subset \dot{F}$.

5. THE ISOMORPHISMS OF SYMPLECTIC GROUPS

5.1. Groups with Enough Projective Transvections

We say that a subgroup Δ of $\mathrm{P\Gamma Sp}_n(V)$ has enough projective transvections if for each line L in V there is at least one projective transvection σ in Δ with $R = L$, at least two when $n = 4$ with char $F = 2$ and $F \neq \mathbf{F}_2$.

5.1.1. EXAMPLE. $\mathrm{PSp}_n(V)$ has enough projective transvections.

From now on Δ *will denote a subgroup of* $\mathrm{P\Gamma Sp}_n(V)$ *which has enough projective transvections. And* Δ_1 *will denote a subgroup of* $\mathrm{P\Gamma Sp}_{n_1}(V_1)$ *with enough projective transvections. And* Λ *will denote a group isomorphism* $\Lambda\colon \Delta \rightarrowtail\!\!\!\rightarrow \Delta_1$ *of* Δ *onto* Δ_1. *Our purpose is to describe* Λ.

We call F the underlying field, char F the underlying characteristic, V the underlying alternating space, and dim V the underlying dimension, of such a Δ.

5.1.2. EXAMPLE. If $F = \mathbf{F}_2$, then Δ has exactly one projective transvection with given residual line. On the other hand, if char $F \neq 2$, then Δ will have at least two projective transvections with given residual line.

We say that Λ preserves the projective transvection σ in Δ if $\Lambda\sigma$ is a projective transvection in Δ_1, that it preserves the projective transvection σ_1 in Δ_1 if $\Lambda^{-1}\sigma_1$ is a projective transvection in Δ, and that it preserves projective transvections if it preserves all projective transvections in Δ and Δ_1.

5.1.3. *If* R_0 *is any subspace of* V, *then there is a* σ *in* $\mathrm{Sp}_n(V)$ *with* $R = R_0$ *such that* $\bar{\sigma}$ *is in* Δ.

PROOF. Use 1.3.11 and the definitions. Q.E.D.

5.1.4. *We have* card $\Delta > \frac{1}{2}(8!)$ *when* $n \geqslant 4$ *but for the following exception*: card $\Delta = 6!$ *when* $n = 4$ *with* $F = \mathbf{F}_2$.

PROOF. (1) If $n = 4$ with $F = \mathbf{F}_2$, then $\Delta = \mathrm{PSp}_4(V)$ by §4.2 and Theorem 2.1.9, so card $\Delta = 6!$ by 3.1.5. If F is a prime field $\mathbf{F}_p$, then taking powers of a nontrivial projective transvection will produce all projective transvections with the same residual line, so here $\Delta \supseteq \mathrm{PSp}_n(V)$, so card $\Delta > \frac{1}{2}(8!)$ (if we exclude $n = 4$ with $p = 2$) by Theorem 3.1.2. If F is infinite, then card Δ is clearly infinite. So we may assume for the rest of the proof that $n \geqslant 4$ with $F = \mathbf{F}_q$ where q is not a prime.

(2) It is enough if we can find more than $\frac{1}{2}(8!)$ elements in $\mathrm{Sp}_n(V)$ which are not big dilations, whose residual spaces are regular planes in V, and which are projectively in Δ, since any two distinct elements of this type remain distinct when read projectively in Δ. Consider a regular plane Π in V, fix a line L_0 in Π, and let τ_{L_0} be a transvection in $\mathrm{Sp}_n(V)$ which is projectively in Δ and whose residual line is L_0. For each line X that falls in Π but is distinct from L_0 let τ_X be defined in the same way. Then the elements $\tau_{L_0}\tau_X$ with variable X are distinct elements of $\mathrm{Sp}_n(V)$ with residual space Π by 1.3.11, they are not big dilations by Theorem 1.7.3, and they are projectively in Δ. Now the number of such elements associated with a given Π is equal to the number of lines in Π distinct from L_0, i.e., it is equal to q. But there are

$$q^{n-2}\left(\frac{q^n-1}{q^2-1}\right)$$

such planes in V by 3.1.4. Therefore

$$\operatorname{card}\Delta \geqslant q^{n-1}\left(\frac{q^n-1}{q^2-1}\right).$$

If $n \geqslant 6$ this gives $\operatorname{card}\Delta > \frac{1}{2}(8!)$, also if $n=4$ with $F \neq \mathbf{F}_4$. If $n=4$ with $F=\mathbf{F}_4$ we see that Δ contains all projective transvections, so $\Delta \supseteq \mathrm{PSp}_4(V)$, apply Theorem 3.1.2. Q.E.D.

5.2. Preservation of Projective Transvections(³)

5.2.1. *Suppose $n \geqslant 4$, $n_1 \geqslant 4$. Let τ be a transvection in $\mathrm{Sp}_n(V)$ that is projectively in Δ, and let k_1 be an element of $\Gamma\mathrm{Sp}_{n_1}(V_1)$ with $\Lambda\bar{\tau} = \bar{k}_1$. Suppose there is a transvection τ_1 in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 such that $k_1\tau_1k_1^{-1}\tau_1^{-1}$, which is always an element of $\mathrm{Sp}_{n_1}(V_1)$ of residue $\leqslant 2$, is actually regular of residue 2. Then $\bar{k}_1$ is a projective transvection, i.e., Λ preserves the projective transvection $\bar{\tau}$.*

PROOF. (1) Pick k in $\Gamma\mathrm{Sp}_n(V)$ with $\Lambda\bar{k} = \bar{\tau}_1$ and put

$$\sigma = \tau k\tau^{-1}k^{-1}, \qquad \sigma_1 = k_1\tau_1k_1^{-1}\tau_1^{-1},$$

so

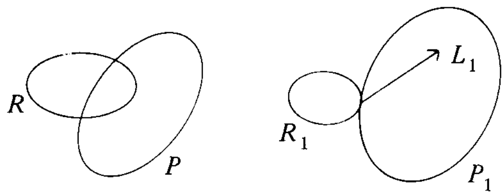
$$\Lambda\bar{\sigma} = \bar{\sigma}_1.$$

By hypothesis R_1 is a regular plane. And the equation $\sigma = \tau k\tau^{-1}k^{-1} \in \mathrm{Sp}_n(V)$ shows that R is either a line or plane containing the residual line of τ. By Theorem 1.7.3 we know that σ_1 is not a big dilation; so any element of $\Gamma\mathrm{L}_{n_1}(V_1)$ which permutes projectively with σ_1 will also permute with σ_1, by 4.1.10. And any element of $\Gamma\mathrm{L}_n(V)$ that permutes projectively with σ will also permute with σ.

(2) First let us show that if we consider a line L_1 in P_1, and if we pick a

(³)This paragraph extends to the symplectic group a new and more general approach to the isomorphisms of the classical groups. See §5.7 for references.

transvection τ_{L_1} in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 and has residual line L_1, and if j is an element of $\Gamma\mathrm{Sp}_n(V)$ for which $\Lambda\bar{j} = \bar{\tau}_{L_1}$, then $jP = P$ and j moves at least one line in P.



The equation $jP = P$ is easy: $L_1 \subseteq P_1$, so $q(R_1, L_1) = 0$ by 1.3.10, so τ_{L_1} permutes with σ_1 by 1.3.11, so $\bar{\tau}_{L_1}$ permutes with $\bar{\sigma}_1$, so $\bar{j}$ permutes with $\bar{\sigma}$, so j permutes with σ by step (1), so $jP = P$ by §4.1. Suppose, if possible, that j fixes all lines in P. Then j is a radiation on P by 4.1.6 and, by changing the representative j of $\bar{j}$ if necessary, we can assume that $(j|P) = 1_P$. Let e_1 be a transvection in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 such that e_1 permutes with σ_1 but $e_1\tau_{L_1}e_1^{-1}$ does not permute with τ_{L_1} (any transvection in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 and whose residual line is nonorthogonal to L_1 and contained in P_1 would be a suitable e_1, by 1.4.15). Let e be an element of $\Gamma\mathrm{Sp}_n(V)$ with $\Lambda\bar{e} = \bar{e}_1$. We have seen that j permutes with σ. By a similar argument e permutes with σ; and clearly eje^{-1} does not permute with j. Since e permutes with σ we have $eP = P$, so eje^{-1} is identity on P. We therefore have found two elements j and j' $(= eje^{-1})$ with the following properties:

$$(j|P) = (j'|P) = 1_P,$$
$$j, j' \in G\mathrm{Sp}_n(V), \bar{j}, \bar{j}' \in \Delta,$$
$$j\sigma = \sigma j, j'\sigma = \sigma j',$$
$$jj' \neq j'j.$$

If R is regular, we note that $jR = R$ since $j\sigma j^{-1} = \sigma$, and similarly $j'R = R$; so $(j|R)$ and $(j'|R)$ are elements of $\mathrm{GL}_2(R)$ which permute with $(\sigma|R)$; but $(\sigma|R)$ is a nonradiation since σ, being a product of two transvections, cannot be a big dilation; so $(j|R)$ and $(j'|R)$ permute by 1.2.4; but $(j|P) = (j'|P) = 1_P$; so j and j' permute; and this is absurd. So R cannot be regular. So R is either a line or a totally degenerate plane. If R is a line, then P is a hyperplane, and j and j' have fixed space P, but $q(P, P) \neq 0$ so that j is in $\mathrm{Sp}_n(V)$ by 4.2.6, so j and j' are transvections in $\mathrm{Sp}_n(V)$ with the same residual line, so j and j' permute by 1.4.15, and this is absurd. Therefore R has to be a totally degenerate plane. If $n \geqslant 6$ we again find that j and j' are elements of $\mathrm{Sp}_n(V)$ whose fixed spaces contain P, hence with residual spaces in R, hence with orthogonal residual spaces; but then j and j' permute; and this is absurd. So the only thing that can happen is that R is a totally degenerate plane and $n = 4$; let $\mathfrak{X} = \{x_1, x_2 | y_1, y_2\}$ be a symplectic base for V with $Fx_1 \perp Fx_2 = R = P$. By 4.2.2, σ, j, j' will have matrices

$$\left(\begin{array}{c|c} I & A \\ \hline & I \end{array}\right), \quad \left(\begin{array}{c|c} I & B \\ \hline & \beta I \end{array}\right), \quad \left(\begin{array}{c|c} I & B' \\ \hline & \beta' I \end{array}\right),$$

in $\mathfrak{X}$ with A, B, B' symmetric and β, β' in F. Of course $A \neq 0$. The equation $j\sigma = \sigma j$ then makes $\beta = 1$; similarly $\beta' = 1$; but then $jj' = j'j$; even this is absurd. Therefore, in all cases, the assumption that j fixes all lines in P leads to a contradiction. So step (2) is established.

(3) Given any line L_1 in P_1, there are elements σ_2 and σ_3 with $\Lambda\bar{\sigma}_2 = \bar{\sigma}_3$ such that

$$\begin{array}{ll} \sigma_2 \in \mathrm{Sp}_n(V) & \sigma_3 \in \mathrm{Sp}_{n_1}(V_1) \\ \bar{\sigma}_2 \in \Delta & \bar{\sigma}_3 \in \Delta_1 \\ \sigma_2 \text{ not big dilation} & \sigma_3 \text{ not big dilation} \\ \mathrm{res}\,\sigma_2 = 2 & 1 \leqslant \mathrm{res}\,\sigma_3 \leqslant 2 \\ R_2 \subseteq P & L_1 \subseteq R_3 \subseteq P_1. \end{array}$$

To see this, pick τ_{L_1} and j as in step (2). Then $jP = P$ and $jL \neq L$ for some line L in P, by step (2). Let T_L be a transvection in $\mathrm{Sp}_n(V)$ that is projectively in Δ and has residual line L, let J_1 be a representative in $\Gamma\mathrm{Sp}_{n_1}(V_1)$ of $\Lambda\bar{\mathrm{T}}_L$. Then

$$\sigma_2 = j\mathrm{T}_L j^{-1}\mathrm{T}_L^{-1}, \qquad \sigma_3 = \tau_{L_1} J_1 \tau_{L_1}^{-1} J_1^{-1},$$

have the desired properties.

(4) Let us show that k_1 stabilizes R_1 and P_1. Pick σ_2 and σ_3 as in step (3), for any choice of L_1 in P_1. Then the residual line of τ is contained in R by step (1); hence it is orthogonal to P, hence to R_2; hence τ permutes with σ_2; hence k_1 permutes with σ_3; hence $k_1 R_3 = R_3$; hence $k_1 L_1 \subseteq k_1 R_3 \subseteq P_1$; hence $k_1 P_1 \subseteq P_1$ since L_1 is arbitrary in P_1. Therefore $k_1 P_1 = P_1$ and $k_1 R_1 = R_1$, as required.

(5) We shall now show that k_1 stabilizes all lines in P_1.

(5a) The case $n_1 \geqslant 6$. Let L_1 be a typical line in P_1. Choose σ_2 and σ_3 as in step (3). Then, as in step (4), we have $k_1 R_3 = R_3$. If $\mathrm{res}\,\sigma_3 = 1$, then $L_1 = R_3$ and k_1 stabilizes L_1. So let $\mathrm{res}\,\sigma_3 = 2$, i.e., suppose that R_3 is a plane. It is easily seen that there is a transvection in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1, which permutes with σ_1, and which carries R_3 to a plane in P_1 which intersects R_3 in L_1. Conjugating σ_3 by this transvection and carrying things back to $\Gamma\mathrm{Sp}_n(V)$ in the usual way, we obtain a new situation σ_2', σ_3', etc., with $R_3' \cap R_3 = L_1$. Then k_1 stabilizes R_3 and R_3', hence L_1, as required.

(5b) The case $n_1 = 4$ with $F_1 \neq \mathbf{F}_2$. Suppose, if possible, that k_1 moves a line in P_1. We then have a base $\mathfrak{X} = \{x_1, y_1\}$ for P_1 such that $k_1 x_1 = y_1$. Let T_1 and T_1' be distinct transvections in $\mathrm{Sp}_{n_1}(V_1)$ which are projectively in Δ_1 and which have residual space equal to the line $F_1 x_1$ (possible because Δ_1 has enough projective transvections). Then $k_1\mathrm{T}_1^{-1}k_1^{-1}$ and $k_1(\mathrm{T}_1')^{-1}k_1^{-1}$ are transvections in $\mathrm{Sp}_{n_1}(V_1)$ which are projectively in Δ_1 and have the same residual line $F_1 y_1$. Of course T_1, T_1', $k_1\mathrm{T}_1^{-1}k_1^{-1}$, $k_1(\mathrm{T}_1')^{-1}k_1^{-1}$, all act on R_1 and P_1, and are identity on R_1. And

$$(\mathrm{T}_1|P_1) \sim \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}, \quad (\mathrm{T}_1'|P_1) \sim \begin{pmatrix} 1 & \alpha' \\ & 1 \end{pmatrix},$$

$$\left(k_1\mathrm{T}_1^{-1}k_1^{-1}|P_1\right) \sim \begin{pmatrix} 1 & \\ \beta & 1 \end{pmatrix}, \quad \left(k_1(\mathrm{T}_1')^{-1}k_1^{-1}|P_1\right) \sim \begin{pmatrix} 1 & \\ \beta' & 1 \end{pmatrix}$$

in the base $\mathfrak{X}$, with $\alpha \neq \alpha'$ and $\beta \neq \beta'$ in $\dot{F}_1$. By direct matrix computation we find that $\mathrm{T}_1 k_1 \mathrm{T}_1^{-1} k_1^{-1}$ and $\mathrm{T}_1' k_1 (\mathrm{T}_1')^{-1} k_1^{-1}$ do not permute on P_1, and hence they do not permute at all. They do, of course, permute with σ_1. Let g and g' be representatives in $\Gamma\mathrm{Sp}_n(V)$ of $\Lambda^{-1}\overline{\mathrm{T}}_1$ and $\Lambda^{-1}\overline{\mathrm{T}}_1'$ respectively. So

$$\Lambda\left(\overline{g\tau g^{-1}\tau^{-1}}\right) = \overline{\mathrm{T}_1 k_1 \mathrm{T}_1^{-1} k_1^{-1}}, \qquad \Lambda\left(\overline{g'\tau(g')^{-1}\tau^{-1}}\right) = \overline{\mathrm{T}_1' k_1 (\mathrm{T}_1')^{-1} k_1^{-1}}.$$

By standard arguments, $g\tau g^{-1}\tau^{-1}$ and $g'\tau(g')^{-1}\tau^{-1}$ permute with σ but not with each other. We shall now obtain a contradiction by proving that they permute with each other too. Since T_1 permutes with σ_1 we find, by standard arguments, that g permutes with σ, so $gR = R$ and $gP = P$. Of course $\tau R = R$ and $\tau P = P$ with $\tau|P = 1_P$ since the residual line of τ is contained in R. So $g\tau g^{-1}\tau^{-1}$ is an element of $\mathrm{Sp}_n(V)$, it acts on R and P, it is identity on P, and it permutes with σ. Similarly with $g'\tau(g')^{-1}\tau^{-1}$. If R is regular, then $(g\tau g^{-1}\tau^{-1}|R)$ and $(g'\tau(g')^{-1}\tau^{-1}|R)$ are in the centralizer of $(\sigma|R)$, and $(\sigma|R)$ is a nonradiation since σ is not a big dilation, so $g\tau g^{-1}\tau^{-1}$ and $g'\tau(g')^{-1}\tau^{-1}$ permute on R by 1.2.4, so they permute on V, and we have our contradiction. If R is degenerate it is totally degenerate, and in this case the residual spaces of $g\tau g^{-1}\tau^{-1}$ and $g'\tau(g')^{-1}\tau^{-1}$ are contained in R and therefore orthogonal, $g\tau g^{-1}\tau^{-1}$ and $g'\tau(g')^{-1}\tau^{-1}$ permute, and we again have our contradiction.

(5c) The case $n_1 = 4$ with $F_1 = \mathbf{F}_2$. In this case $\mathrm{P}\Gamma\mathrm{Sp}_4(V_1) = \mathrm{PSp}_4(V_1) = \Delta_1$, so card $\Delta_1 = 6!$. Hence $n = 4$, $F = \mathbf{F}_2$ and $\Delta = \mathrm{PSp}_4(V)$, by 5.1.4. If R were degenerate, then σ would be an involution by 1.3.13, so σ_1 would be an involution, so R_1 would be degenerate, which it is not. Therefore R is a regular plane. Take an arbitrary line L_1 in P_1 and let τ_{L_1} be a transvection in $\mathrm{Sp}_4(V_1)$ with $\bar{\tau}_{L_1}$ in Δ_1 and with residual line L_1. Pick j in $\mathrm{Sp}_4(V)$ with $\Lambda\bar{j} = \bar{\tau}_{L_1}$. Then by standard methods j is an involution permuting with σ, so $jR = R$ and $jP = P$. Then $(j|R)$ is an involution permuting with $(\sigma|R)$ and a study of the possibilities involved shows that $(j|R) = 1_R$. But then τ permutes with j since the residual line of τ is contained in R. So k_1 permutes with τ_{L_1}. So k_1 stabilizes a typical line L_1 in P_1.

(6) So k_1 stabilizes all lines in P_1. So k_1 is a radiation on P_1. Replacing k_1 by another representative allows us to assume that the k_1 in the statement of the proposition is such that $(k_1|P_1) = 1_{P_1}$. In particular, k_1 is an element of $\mathrm{Sp}_{n_1}(V_1)$ with $1 \leqslant \operatorname{res} k_1 \leqslant 2$. If $\operatorname{res} k_1 = 1$, then k_1 is a transvection and we are through. So let $\operatorname{res} k_1 = 2$. The residual space of k_1 is then the regular plane R_1. If k_1 were a big dilation, then $k_1\tau_1 k_1^{-1}\tau_1^{-1}$ could not have residual space R_1. So k_1 is not a big dilation. So k_1 moves a line in R_1–in fact it moves it to a nonorthogonal line in R_1. So there is a line L_1' in V_1 with

$$L_1' \not\subseteq R_1 \cup P_1, \qquad q(L_1', kL_1') \neq 0.$$

Let τ_1' be a transvection in $\mathrm{Sp}_{n_1}(V_1)$ which is projectively in Δ_1 and has residual line L_1'. The statement of the proposition then applies to the τ, k_1, τ_1' situation. Therefore, by step (5), k_1 is a radiation on the $(n_1 - 2)$-dimensional space $(L_1' + k_1 L_1')^*$. This space is distinct from the $(n_1 - 2)$-dimensional space P_1.

This clearly implies that res $k_1 = 1$ when $n_1 \geqslant 6$; and that k_1 is a transvection or a big dilation, hence a transvection, when $n_1 = 4$. Q.E.D.

5.2.2. *Suppose $n \geqslant 4$, $n_1 \geqslant 4$. Let $\bar{\tau}$ be a projective transvection in Δ. Then $\Lambda\bar{\tau}$ is either a projective transvection or a projective hyperbolic transformation in Δ_1.*

PROOF. Suppose $\Lambda\bar{\tau}$ is not a projective hyperbolic transformation in Δ_1. Let k_1 be a representative of $\Lambda\bar{\tau}$ in $\Gamma\mathrm{Sp}_{n_1}(V_1)$. Then k_1 is not a hyperbolic transformation, so we have a line L_1 in V_1 with $q(L_1, kL_1) \neq 0$. Let τ_1 be a transvection in $\mathrm{Sp}_{n_1}(V_1)$ which is projectively in Δ_1 and which has residual line L_1. Then the residual space of $k_1\tau_1k_1^{-1}\tau_1^{-1}$ is the regular plane $L_1 + kL_1$. So 5.2.1 applies. So $\bar{k}_1$ is a projective transvection. Q.E.D.

5.2.3. *If $n \geqslant 4$, $n_1 \geqslant 4$, then* char F = char F_1, *i.e., the two underlying fields have a common characteristic.*

PROOF. If exactly one of the fields has characteristic 2, let it be the second. We may therefore assume that char $F \neq 2$. Let $\bar{\tau}$ be a nontrivial projective transvection in Δ. Since char $F \neq 2$, $\bar{\tau}$ is not an involution. So $\Lambda\bar{\tau}$ is a projective transvection by 5.2.2 and 4.3.1. If char $F = p \neq 0$, then $\bar{\tau}^p = 1$, so $(\Lambda\bar{\tau})^p = 1$, so char $F_1 = p$. Similarly if char $F_1 = p_1 \neq 0$. So char F = char F_1. Q.E.D.

5.2.4. *If $n \geqslant 4$, $n_1 \geqslant 4$, and if the common characteristic is not 2, then Λ preserves projective transvections.*

PROOF. See the proof of 5.2.3. Q.E.D.

5.2.5. *Suppose $n \geqslant 4$, $n_1 \geqslant 6$, and that the common characteristic is 2. Let $\bar{\tau}$ be a nontrivial projective transvection in Δ. Then $\Lambda\bar{\tau}$ has a representative k_1 in* $\mathrm{Sp}_{n_1}(V_1)$ *with* $1 \leqslant$ res $k_1 \leqslant 2$.

PROOF. (1) Let τ be the representative transvection of $\bar{\tau}$. So τ is in $\mathrm{Sp}_n(V)$. Let L be the residual line of τ. Let k_1 be a representative (arbitrarily chosen to begin with) of $\Lambda\bar{\tau}$ in $\Gamma\mathrm{Sp}_{n_1}(V_1)$. By 5.2.2 we may assume that k_1 is a hyperbolic transformation.

(2) Since $\bar{k}_1 \neq 1$, k_1 will move at least one line in V_1. Let L_1 be any line in V_1 that is moved by k_1 (the line L_1 will be varied later). Let τ_1 be a transvection in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 and with residual line L_1. Let k be a representative in $\Gamma\mathrm{Sp}_n(V)$ of $\Lambda^{-1}\bar{\tau}_1$. Put

$$\sigma = \tau k \tau^{-1} k^{-1}, \qquad \sigma_1 = k_1\tau_1k_1^{-1}\tau_1^{-1}.$$

So $\Lambda\bar{\sigma} = \bar{\sigma}_1$; and R_1 is a totally degenerate plane containing L_1; so σ_1 is an involution; so $\bar{\sigma}$ is an involution; so σ is an involution with res $\sigma \leqslant 2$; so R is a totally degenerate line or plane containing L. By standard methods we can find a conjugate σ' to σ in $\Gamma\mathrm{Sp}_n(V)$ and a conjugate σ_1' to σ_1 in $\Gamma\mathrm{Sp}_{n_1}(V_1)$, such that

$$\Lambda\bar{\sigma}' = \bar{\sigma}_1, \qquad R \cap R' = L$$

(start by suitably conjugating σ by a transvection in $\mathrm{Sp}_n(V)$ that is projectively in Δ). So $\dim(R_1 + R_1') \leqslant 4$. Consider a typical line K_1 in the space $(R_1 + R_1')^*$

of dimension $\geqslant n_1 - 4$. Take a transvection τ_3 in $\mathrm{Sp}_{n_1}(V_1)$ which is projectively in Δ_1 with residual line K_1. Then τ_3 permutes with σ_1 and σ_1' so, by standard methods, a representative j of $\Lambda^{-1}\bar{\tau}_3$ in $\Gamma\mathrm{Sp}_n(V)$ will permute with σ and σ', so j will stabilize the line $L = R \cap R'$. Now if $\bar{j}$ is not a hyperbolic transformation, applying 5.2.2 to Λ^{-1} shows that $\bar{j}$ is a projective transvection; so in this case j can be chosen as a transvection, in particular as an involution, in $\mathrm{Sp}_n(V)$ that stabilizes the line L; if, on the other hand, $\bar{j}$ is a hyperbolic transformation, then $\bar{j}$ is in $\mathrm{PSp}_n(V)$ by 4.3.3, i.e., j can be chosen as a hyperbolic transformation in $\mathrm{Sp}_n(V)$, i.e., j is an involution in $\mathrm{Sp}_n(V)$ by 4.3.2. In either event, our representative j has now been chosen to be an involution in $\mathrm{Sp}_n(V)$ that stabilizes the line L, and hence that acts like 1_L on L. But then j permutes with τ. Hence, by standard methods, k_1 permutes with τ_3. Hence k_1 stabilizes K_1 which is a typical line in $(R_1 + R_1')^*$. We may therefore assume that our representative k_1 of $\Lambda\bar{\tau}$ has been so chosen that it acts like identity on $(R_1 + R_1')^*$ which has dimension $\geqslant n_1 - 4$. We have $m_{k_1} = 1$ by 4.3.1, i.e., k_1 is in $\mathrm{Sp}_{n_1}(V_1)$. And $\operatorname{res} k_1 \leqslant 4$. If $\operatorname{res} k_1 = 4$, then the fixed and residual spaces of k_1 will be $(R_1 + R_1')^*$ and $(R_1 + R_1')$; pick a new line L_{11} (instead of the original L_1) that is contained in V_1 but not in

$$(R_1 + R_1') \cup (R_1 + R_1')^*;$$

then k_1 moves L_{11} since k_1, being an involution by 4.3.2, has all its characteristic values equal to 1; so we can repeat the entire operation using L_{11} instead of L_1: this shows that k_1 stabilizes all lines in $(R_{11} + R_{11}')^*$; since $L_{11} \subseteq (R_{11} + R_{11}')$ we have $(R_{11} + R_{11}') \neq (R_1 + R_1')$; so

$$(R_{11} + R_{11}') \not\supseteq (R_1 + R_1')$$

since $\dim(R_{11} + R_{11}') \leqslant 4 = \dim(R_1 + R_1')$; so

$$(R_{11} + R_{11}')^* \not\subseteq (R_1 + R_1')^*;$$

in other words, k_1 stabilizes a line outside $(R_1 + R_1')^*$; since k_1 is an involution it must therefore fix a vector outside $(R_1 + R_1')^*$; but $(R_1 + R_1')^*$ is the fixed space of k_1; this is absurd. Therefore $\operatorname{res} k_1 \leqslant 3$. But k_1 is a hyperbolic transformation in $\mathrm{Sp}_{n_1}(V_1)$ so that $\operatorname{res} k_1$ is even by 2.1.17. Therefore $\operatorname{res} k_1 = 2$. Q.E.D.

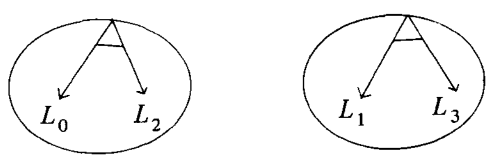
5.2.6. *It is impossible to have* $n = 4$, $n_1 \geqslant 6$, *in characteristic* 2.

PROOF. Suppose to the contrary that we had an isomorphism $\Lambda: \Delta \rightarrowtail \Delta_1$ in characteristic 2 with $n = 4$ and $n_1 \geqslant 6$.

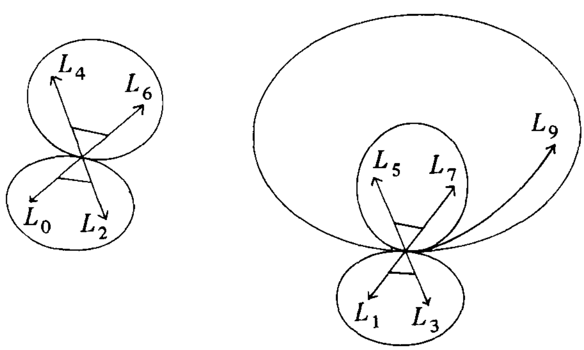
(1) First we observe that if $\bar{k}$ is a projective hyperbolic transformation in Δ which permutes with a pair of nonpermuting projective transvections in Δ, then $\bar{k} = 1$. To see this pick a representative k of $\bar{k}$ in $\Gamma\mathrm{Sp}_n(V)$. Then k stabilizes the residual lines of the projective transvections in question, so k can be chosen in $\mathrm{Sp}_n(V)$ by 4.3.3. So the hyperbolic transformation k is an involution by 4.3.2. But the residual lines in question are nonorthogonal by 1.6.8. So k is identity on a regular plane Π in V. Then the residual space of $(k|\Pi^*)$ has dimension 0 or 2 by 2.1.17, and it is totally degenerate by 1.3.13, so it is 0. So $k = 1_V$. So $\bar{k} = 1$.

(2) Next let us show that Λ preserves at least one nontrivial projective transvection. Start with a nontrivial transvection τ in $\mathrm{Sp}_n(V)$ which is projectively in Δ. By 5.2.5 we can pick σ_1 in $\mathrm{Sp}_{n_1}(V_1)$ with $1 \leqslant \operatorname{res} \sigma_1 \leqslant 2$ such that $\Lambda\bar{\tau} = \bar{\sigma}_1$. By standard methods we can find a conjugate τ' to τ in $\Gamma\mathrm{Sp}_n(V)$ that does not permute with τ, and a conjugate σ'_1 to σ_1 in $\Gamma\mathrm{Sp}_{n_1}(V_1)$, such that $\Lambda\bar{\tau} = \overline{\sigma'_1}$. Then $\dim(R_1 + R'_1) \leqslant 4$, so there is a transvection τ_1 in $\Gamma\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 whose residual line is orthogonal to R_1 and R'_1. Then τ_1 permutes with σ_1 and σ'_1. So $\Lambda^{-1}\bar{\tau}_1$ permutes with $\bar{\tau}$ and $\overline{\tau'}$. So $\Lambda^{-1}\bar{\tau}_1$ is not a projective hyperbolic transformation by step (1). So $\Lambda^{-1}\bar{\tau}_1$ is a projective transvection by 5.2.2. In other words, Λ preserves the nontrivial projective transformation $\Lambda^{-1}\bar{\tau}_1$, as required.

(3) In the argument that follows the τ_i will denote nontrivial transvections; for even i they will be in $\mathrm{Sp}_n(V)$ and projectively in Δ, for odd i they will be in $\mathrm{Sp}_{n_1}(V_1)$ and projectively in Δ_1; L_i will denote the residual line of τ_i. By step (2) we can find τ_0 and τ_1 with $\Lambda\bar{\tau}_0 = \bar{\tau}_1$. By suitably conjugating τ_0 we can find τ_2 and τ_3 with $\Lambda\bar{\tau}_2 = \bar{\tau}_3$ such that $\bar{\tau}_0$ does not permute with $\bar{\tau}_2$ and $\bar{\tau}_1$ does not permute with $\bar{\tau}_3$. Then L_0 and L_2 are nonorthogonal, so are L_1 and L_3, so $L_0 + L_2$ and $L_1 + L_3$ are



regular planes. Choose $\tau_5$with L_5 orthogonal to $(L_1 + L_3)$, and then define τ_4 by $\Lambda\bar{\tau}_4 = \bar{\tau}_5$ (the fact that τ_4 can be chosen as a transvection follows from step (1) and 5.2.2). Clearly L_4 is orthogonal to $(L_0 + L_2)$ by the usual argument of permutability. Now



pick L_7 orthogonal to $L_1 + L_3$ but not to L_5 and thereby introduce τ_7 and τ_6. Note that

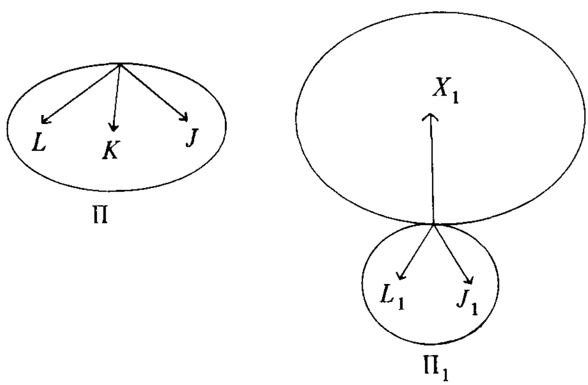
$$V = L_0 + L_2 + L_4 + L_6, \qquad V_1 \supset L_1 + L_3 + L_5 + L_7.$$

Repeat this procedure one last time with L_9 orthogonal to $L_1 + L_3 + L_5 + L_7$ to obtain $\Lambda\bar{\tau}_8 = \bar{\tau}_9$. Then τ_8 is a transvection permuting with $\tau_0, \tau_2, \tau_4, \tau_6$, as such it must stabilize L_0, L_2, L_4, L_6, so $\tau_8 = 1_V$, and this is absurd. Q.E.D.

5.2.7. *Suppose* $n \geqslant 4$, $n_1 \geqslant 4$. *If* Λ *preserves one projective transvection it preserves all.*

PROOF. By 5.2.4 we can assume that the common characteristic is 2. We must consider two transvections τ_L and τ_K in $\mathrm{Sp}_n(V)$ which are projectively in Δ and whose residual lines are L and K respectively, and show that if $\Lambda\bar{\tau}_L$ is a projective transvection in Δ_1 then so is $\Lambda\bar{\tau}_K$. If L and K are orthogonal, conjugate τ_L in an appropriate way to obtain a new L which is nonorthogonal to K. This allows us to assume that the given L and K satisfy $q(L, K) \neq 0$.

Let Π denote the regular plane $\Pi = L + K$. Then $J = \tau_K L$ is a line in Π that is distinct from L and K. Since $\tau_K \tau_L \tau_K^{-1}$ is a transvection with residual line J we can denote it $\tau_J = \tau_K \tau_L \tau_K^{-1}$. Of course τ_J is in $\mathrm{Sp}_n(V)$ and projectively in Δ. And $\Lambda\bar{\tau}_J$ is a projective transvection in Δ_1 by conjugation. Let L_1 and J_1 be the residual lines of the projective transvections $\Lambda\bar{\tau}_L$ and $\Lambda\bar{\tau}_J$ respectively, and write $\Lambda\bar{\tau}_L = \bar{\tau}_{L_1}$ and $\Lambda\bar{\tau}_J = \bar{\tau}_{J_1}$ with τ_{L_1} and τ_{J_1} transvections in $\mathrm{Sp}_{n_1}(V_1)$. The nonorthogonality of L and J leads to the nonorthogonality of L_1 and J_1 in the usual way. Let Π_1 denote the regular plane $\Pi_1 = L_1 + J_1$. Consider an arbitrary line X_1 in Π_1^*. Let τ_{X_1} be a transvection which is in $\mathrm{Sp}_{n_1}(V_1)$,

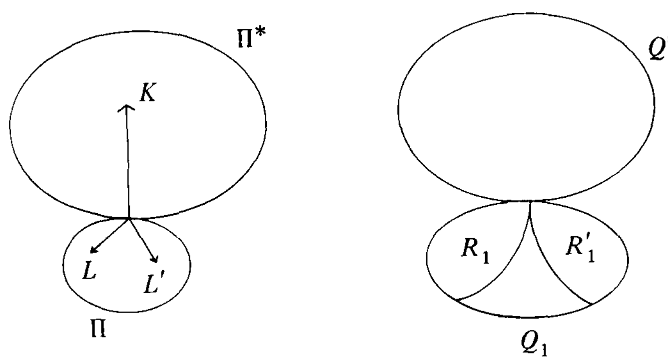


which is projectively in Δ_1, and which has residual line X_1, and let k be a representative in $\Gamma\mathrm{Sp}_n(V)$ of $\Lambda^{-1}\bar{\tau}_{X_1}$. If $\Lambda^{-1}\bar{\tau}_{X_1}$ is a projective transvection, then k can be chosen as a transvection, in particular as an involution in $\mathrm{Sp}_n(V)$; otherwise $\Lambda^{-1}\bar{\tau}_{X_1}$ is a projective hyperbolic transformation by 5.2.2, and k stabilizes L by standard arguments, so k can be chosen in $\mathrm{Sp}_n(V)$ by 4.3.3, and then k is an involution by 4.3.2. In either event k can be chosen as an involution in $\mathrm{Sp}_n(V)$ that stabilizes L and J. Hence $(k|\Pi) = 1_\Pi$. Hence k permutes with τ_K. Hence $\bar{\tau}_{X_1}$ permutes with $\Lambda\bar{\tau}_K$. So any representative of $\Lambda\bar{\tau}_K$ in $\Gamma\mathrm{Sp}_{n_1}(V_1)$ will stabilize X_1. So $\Lambda\bar{\tau}_K$ has a representative k_1 in $\Gamma\mathrm{Sp}_{n_1}(V_1)$ with $(k_1|\Pi_1^*) = 1_{\Pi_1^*}$. Then k_1 is an involution; hence $(k_1|\Pi_1)$ is an involution; hence $(k_1|\Pi_1)$ is a transvection; hence k_1 is a transvection, i.e., Λ preserves the projective transvection $\bar{\tau}_K$. Q.E.D.

5.2.8. *Suppose* $n \geqslant 4$, $n_1 \geqslant 4$, *but rule out the case where* $n = n_1 = 4$ *with* char F = char $F_1 = 2$. *Then* Λ *preserves projective transvections.*

PROOF. If the common characteristic is not 2, apply 5.2.4. So let char $F =$ char $F_1 = 2$. This makes $n \geqslant 6$, $n_1 \geqslant 6$ by 5.2.6. By 5.2.7 it suffices to prove that Λ preserves at least one nontrivial projective transvection. Suppose, to the contrary, that Λ preserves none. Then, by 5.2.5, Λ sends each projective transvection in Δ to an element of Δ_1 having a representative in $\mathrm{Sp}_{n_1}(V_1)$ of residue 2. And similarly with Λ^{-1}.

Fix a nontrivial transvection τ in $\mathrm{Sp}_n(V)$ that is projectively in Δ and let σ_1 be an element of residue 2 in $\mathrm{Sp}_{n_1}(V_1)$ such that $\Lambda\bar{\tau} = \bar{\sigma}_1$. Here R_1, the residual space of σ_1, is a totally degenerate plane by 1.3.13. Let L denote the residual line of τ. Let Q_1 denote a regular quaternary subspace of V_1 that contains R_1. It is easy to find a Σ that is in $\mathrm{Sp}_{n_1}(V_1)$ and projectively in Δ_1 such that $Q_1 = R_1 \oplus \Sigma R_1$ (in fact a product of two suitable transvections will do the job). Using this Σ and standard methods we have a conjugate σ_1' to σ_1 in $\Gamma\mathrm{Sp}_{n_1}(V_1)$ and a conjugate τ' to τ in $\Gamma\mathrm{Sp}_n(V)$ such that $\Lambda\bar{\tau}' = \overline{\sigma_1'}$ and such that the residual space R_1' of σ_1' is a totally degenerate plane with $Q_1 = R_1 \oplus R_1'$. Let L' denote the residual line of τ'. Then σ_1 and σ_1' do not permute since $R_1 \cap R_1' = 0$ with $q_1(R_1, R_1') \neq 0$, by 1.3.6. Hence τ and τ' do not permute. Hence $\Pi = L + L'$ is a regular plane.



Consider a typical line K in Π^*. Let τ_K be a transvection in $\mathrm{Sp}_n(V)$ with residual line K that is projectively in Δ. Write $\Lambda\bar{\tau}_K = \bar{\sigma}_3$ with σ_3 an element of residue 2 in $\mathrm{Sp}_{n_1}(V_1)$. By standard methods, σ_3 will stabilize R_1 and R_1'; hence it will stabilize Q_1 and Q_1^*. It is easily seen that σ_3 is identity on either Q_1 or Q_1^*: Otherwise $(\sigma_3|Q_1)$ and $(\sigma_3|Q_1^*)$ will have residue 1 and so be transvections, so σ_3 will be a product of two transvections in $\mathrm{Sp}_n(V)$, so σ_3 will not be a hyperbolic transformation by 2.1.18, so $\bar{\sigma}_3$ will not be a projective hyperbolic transformation, so $\bar{\sigma}_3$ will be a projective transvection by 5.2.2, and this is absurd. Therefore, as K is allowed to vary through Π^*, all the corresponding R_3's will vary through Q_1 and Q_1^*. A simple logical argument then shows that you cannot have some R_3's in Q_1, and others in Q_1^*: If K gives $R_3 \subseteq Q_1$ and K' gives R_3' in Q_1^* (say), take K'' orthogonal to neither K nor K'; then where is R_3'' to fall? Let us therefore assume that all the R_3's fall in Q_1^* (the argument is similar if they all fall in Q_1). Let T_1 be a transvection in $\mathrm{Sp}_{n_1}(V_1)$ that is projectively in Δ_1 and whose residual line falls in Q_1. Let Σ be an element of residue 2 in $\mathrm{Sp}_n(V)$ with

$\Lambda\bar{\Sigma} = \bar{T}_1$. Of course Σ is a hyperbolic transformation and an involution. Then T_1 permutes with all σ_3's, so Σ permutes with all τ_K's, so Σ stabilizes all lines in Π^*, so $(\Sigma|\Pi^*) = 1_{\Pi^*}$, and $\Sigma\Pi = \Pi$ so that $(\Sigma|\Pi)$ is a hyperbolic transformation and an involution in $Sp_2(\Pi)$, but this makes $(\Sigma|\Pi) = 1_\Pi$, so $\Sigma = 1_V$, and this is absurd. Q.E.D.

5.3. The Isomorphism Theorems in General

We let $\mathfrak{L}$ denote the set of lines of V, i.e., $\mathfrak{L}$ will stand for the subset $P^1(V)$ of the projective space $P(V)$. For each L in $\mathfrak{L}$ define $\Delta(L)$ to be the group consisting of all projective transvections in Δ with residual line L, plus 1. The quantities $\mathfrak{L}_1$, $\Delta_1(L_1)$ are defined in the same way for the V_1 situation.

5.3.1. *If L and L' are lines in $\mathfrak{L}$, then*:
(1) $\Delta(L) = \Delta(L') \Leftrightarrow L = L'$.
(2) $\Delta(L) \cap \Delta(L') \supset 1 \Leftrightarrow L = L'$.
(3) $\Delta(L)$ *is a maximal group of projective transvections in* Δ.
(4) *Every maximal group of projective transvections in Δ is a $\Delta(L)$.*

PROOF. (1) and (2) are obvious. Then (3) follows from 1.6.7. So does (4). Q.E.D.

We can now derive a mapping $\pi: \mathfrak{L} \rightarrowtail \mathfrak{L}_1$ from the group isomorphism $\Lambda: \Delta \rightarrowtail \Delta_1$ in the following way, provided Λ is known to preserve projective transvections. For each L in $\mathfrak{L}$, $\Delta(L)$ is a maximal group of projective transvections in Δ; hence $\Lambda\Delta(L)$ is a maximal group of projective transvections in Δ_1; hence there is a unique line L_1 in $\mathfrak{L}_1$ such that $\Lambda\Delta(L) = \Delta_1(L_1)$. Define $\pi L = L_1$.

5.3.2. *The above mapping $\pi: \mathfrak{L} \rightarrowtail \mathfrak{L}_1$ associated with a Λ which preserves projective transvections satisfies the following properties*:
(1) $\pi: \mathfrak{L} \rightarrowtail \mathfrak{L}_1$ *is a bijection.*
(2) *Its defining equation is $\Lambda\Delta(L) = \Delta_1(\pi L)$ for all L in $\mathfrak{L}$.*
(3) *L is orthogonal to L' if and only if πL is orthogonal to $\pi L'$.*

PROOF. (1) and (2) are immediate. To prove (3) we take nontrivial projective transvections τ_L and $\tau_{L'}$ in Δ with residual lines L and L' respectively. Then

$$\begin{aligned} q(L, L') = 0 &\Leftrightarrow \tau_L \text{ permutes with } \tau_{L'} \\ &\Leftrightarrow \Lambda\tau_L \text{ permutes with } \Lambda\tau_{L'} \\ &\Leftrightarrow \tau_{\pi L} \text{ permutes with } \tau_{\pi L'} \\ &\Leftrightarrow q_1(\pi L, \pi L') = 0. \quad \text{Q.E.D.} \end{aligned}$$

5.3.3. *If Λ preserves projective transvections so that the above bijection π of $\mathfrak{L}$ onto $\mathfrak{L}_1$ is defined, and if Φ is any isomorphism of Δ into $P\Gamma Sp_{n_1}(V_1)$ such that every element of $\Phi\Delta(L)$ is a projective transvection with residual line πL for each line L in $\mathfrak{L}$, then $\Phi = \Lambda$.*

PROOF. Let k be a typical element of Δ. We must show that $\Phi k = \Lambda k$. Consider a typical line L in V. Then πL is a typical line in V_1. Let τ_L denote a projective transvection in Δ with residual line L. Then $k\tau_L k^{-1}$ is a projective transvection in Δ with residual line kL and we write $k\tau_L k^{-1} = \tau_{kL}$. Now $\Phi\tau_L$ is a projective transvection in $\mathrm{P\Gamma Sp}_{n_1}(V_1)$ with residual line πL; accordingly we can write $\Phi\tau_L = \tau_{\pi L}$. Similarly $\Phi\tau_{kL}$ can be written $\Phi\tau_{kL} = \tau_{\pi(kL)}$. We have

$$\tau_{\pi(kL)} = \Phi\tau_{kL} = \Phi(k\tau_L k^{-1})$$
$$= (\Phi k)(\Phi\tau_L)(\Phi k)^{-1} = (\Phi k)(\tau_{\pi L})(\Phi k)^{-1}$$

and so $(\Phi k)(\pi L) = \pi(kL)$. Now Λ is a Φ so that $(\Lambda k)(\pi L) = \pi(kL)$. Hence

$$(\Phi k)(\pi L) = (\Lambda k)(\pi L).$$

In other words, Φk and Λk have the same action on the lines of V_1. So $\Phi k = \Lambda k$. So $\Phi = \Lambda$. Q.E.D.

5.3.4. *If Λ preserves projective transvections, then $n = n_1$.*

PROOF. By interchanging the two groups if necessary, we can assume that $n \geqslant n_1$. Since Λ preserves projective transvections, the bijection π: $\mathfrak{L} \rightarrowtail\!\!\!\!\rightarrow \mathfrak{L}_1$ is now available. For any subspace U of V define ΠU as the subspace of V_1 spanned by all πL as L runs through all lines in U. Then Π agrees with π on $\mathfrak{L}$. And

$$U \subseteq W \Rightarrow \Pi U \subseteq \Pi W.$$

By considering a strictly ascending chain of $n + 1$ subspaces of V we see that we will be through if we can verify that

$$U \subset W \Rightarrow \Pi U \subset \Pi W.$$

To prove this pick a line L in V that is orthogonal to U but not to W–this is possible since $W^* \subset U^*$. Then L is orthogonal to all lines in U but not to all lines in W, so πL is orthogonal to ΠU but not to πW by 5.3.2, so $\Pi U \neq \Pi W$, so $\Pi U \subset \Pi W$ as asserted. Q.E.D.

5.3.5. *Suppose $n \geqslant 4$, $n_1 \geqslant 4$ and that Λ preserves projective transvections. Then there is a unique projective symplectic collinear transformation g of V onto V_1 such that $\Lambda k = gkg^{-1}$ for all k in Δ.*

PROOF. By 5.3.4 we have $n = n_1$. Once again the bijection π: $\mathfrak{L} \rightarrowtail\!\!\!\!\rightarrow \mathfrak{L}_1$ is available. For any hyperplane H of V define a hyperplane H_1 of V_1 by the equation $H_1 = (\pi H^*)^*$. Then for any line L in $\mathfrak{L}$ which falls in H we have L orthogonal to H^*, so πL is orthogonal to πH^*, so $\pi L \subseteq H_1$. Therefore π can be extended uniquely to a projectivity Π: $\mathrm{P}(V) \rightarrowtail\!\!\!\!\rightarrow \mathrm{P}(V_1)$ by 4.1.3. So by the Fundamental Theorem of Projective Geometry there is a projective collinear transformation g: $\mathrm{P}(V) \rightarrowtail\!\!\!\!\rightarrow \mathrm{P}(V_1)$ such that $gL = \pi L$ for all L in $\mathfrak{L}$. Of course L is orthogonal to L' if and only if gL is orthogonal to gL', so g is a projective symplectic collinear transformation by 4.2.1. The group isomorphism

$$\Phi_g\colon \mathrm{P\Gamma Sp}_n(V) \rightarrowtail\!\!\!\!\rightarrow \mathrm{P\Gamma Sp}_{n_1}(V_1)$$

of §4.2 now becomes available. It sends $\Delta(L)$ into a group of projective transvections in $\mathrm{P\Gamma Sp}_{n_1}(V_1)$ with residual line $gL = \pi L$. So $(\Phi_g|\Delta) = \Lambda$ by 5.3.3. In other words, $\Lambda k = gkg^{-1}$ for all k in Δ.

Now the question of uniqueness. If we have two projective symplectic collinear transformations g and j of V onto V_1 such that

$$gkg^{-1} = \Lambda k = jkj^{-1} \qquad \forall\, k \in \Delta,$$

then for any line L in $\mathfrak{L}$ we have

$$g\tau_L g^{-1} = j\tau_L j^{-1}$$

for a nontrivial projective transvection τ_L in $\Delta(L)$, so $gL = jL$, so $g = j$. Q.E.D.

5.3.6. THEOREM. *Let Δ and Δ_1 be subgroups of* $\mathrm{P\Gamma Sp}_n(V)$ *and* $\mathrm{P\Gamma Sp}_{n_1}(V_1)$ *respectively. Suppose that Δ and Δ_1 have enough projective transvections, and that the underlying dimensions are $\geqslant 4$. Let Λ be a group isomorphism of Δ onto Δ_1. If one of the Δ's has underlying characteristic 2 and underlying dimension 4, then so does the other. Exclude this situation. Then there is a unique projective symplectic collinear transformation g of V onto V_1 such that*

$$\Lambda k = gkg^{-1} \qquad \forall\, k \in \Delta.$$

PROOF. If one underlying characteristic is 2, then so is the other by 5.2.3; if, in addition, one underlying dimension is 4, then so is the other by 5.2.6. Now exclude this situation. Then Λ preserves projective transvections by 5.2.8. Apply 5.3.5. Q.E.D.

5.3.6A. THEOREM. *Isomorphic projective groups of symplectic collinear transformations with enough projective transvections and with underlying dimensions $\geqslant 4$ have equal underlying characteristics and equal underlying dimensions. If we exclude characteristic 2 with dimension 4, then the underlying fields are isomorphic.*

Now let us extend the isomorphism theorems to the nonprojective case. We say that a subgroup Γ of $\Gamma\mathrm{Sp}_n(V)$ has enough transvections if for each line L in V there is at least one transvection σ in Γ with $R = L$, at least two when $n = 4$ with char $F = 2$ and $F \neq \mathbf{F}_2$. We let Γ denote a subgroup of $\Gamma\mathrm{Sp}_n(V)$ which has enough transvections. And Γ_1 will be a subgroup of $\Gamma\mathrm{Sp}_{n_1}(V_1)$ with enough transvections. And Φ will denote a group isomorphism $\Phi\colon \Gamma \rightarrowtail \Gamma_1$. Note that $\overline{\Gamma} = \mathrm{P}\Gamma$ and $\overline{\Gamma}_1 = \mathrm{P}\Gamma_1$ are subgroups of $\mathrm{P\Gamma Sp}_n(V)$ and $\mathrm{P\Gamma Sp}_{n_1}(V_1)$ with enough projective transvections so that the preceding theorem for Δ and Δ_1 applies to $\overline{\Gamma}$ and $\overline{\Gamma}_1$.

5.3.7. *Φ naturally induces an isomorphism $\overline{\Phi}\colon \overline{\Gamma} \rightarrowtail \overline{\Gamma}_1$ by the equation*

$$\overline{\Phi}\bar{k} = \overline{\Phi k} \qquad \forall\, k \in \Gamma$$

when $n \geqslant 4$, $n_1 \geqslant 4$.

PROOF. It is enough to verify that $\Phi(\Gamma \cap \mathrm{RL}_n) = \Gamma_1 \cap \mathrm{RL}_1$, in fact that $\Phi(\Gamma \cap \mathrm{RL}_n)$ is contained in $\Gamma_1 \cap \mathrm{RL}_{n_1}$.

Suppose, if possible, that a radiation r in Γ is such that Φr is not a radiation in Γ_1. Put $\Phi r = k_1$. Then k_1 moves a line L_1 in V_1. So if we form the commutator $k_1\tau_1k_1^{-1}\tau_1^{-1}$ with τ_1 a transvection in Γ_1 with residual line L_1, and then pull things back to Γ, we obtain transformations σ and σ_1 with $\Phi\sigma = \sigma_1$ such that R_1 is a plane containing L_1 and

$$\sigma \in \Gamma \cap \mathrm{GSp}_n(V), \qquad \sigma_1 \in \Gamma_1 \cap \mathrm{Sp}_{n_1}(V_1).$$

By suitably conjugating on the right and pulling things back to the left we find another pair σ', σ_1' with the same properties as the pair σ, σ_1, and such that $R_1 \cap R_1' = L_1$. Then r permutes with σ and σ'; hence k_1 permutes with σ_1 and σ_1'; hence k_1 stabilizes R_1' and R_1'; hence k_1 stabilizes L_1, and this is absurd. Q.E.D.

5.3.8. THEOREM. *Let Γ and Γ_1 be subgroups of $\Gamma\mathrm{Sp}_n(V)$ and $\Gamma\mathrm{Sp}_{n_1}(V_1)$ respectively. Suppose that Γ and Γ_1 have enough transvections, and that the underlying dimensions are $\geqslant 4$. Let Φ be a group isomorphism of Γ onto Γ_1. If one of the Γ's has underlying characteristic 2 and underlying dimension 4, then so does the other. Exclude this situation. Then there is a symplectic collinear transformation g of V onto V_1 and there is a mapping χ of Γ into $\mathrm{RL}_{n_1}(V_1)$ such that*

$$\Phi k = \chi(k)\, gkg^{-1} \qquad \forall\, k \in \Gamma.$$

5.3.8A. THEOREM. *Isomorphic groups of symplectic collinear transformations with enough transvections and with underlying dimensions $\geqslant 4$ have equal underlying characteristics and equal underlying dimensions. If we exclude characteristic 2 with dimension 4, then the underlying fields are isomorphic.*

5.4. 4-Dimensional Groups in Characteristic 2([4])

Throughout §5.4 we assume that we are in the exceptional situation where

$$n = n_1 = 4 \quad \text{and} \quad \mathrm{char}\, F = \mathrm{char}\ F_1 = 2.$$

By our general assumptions, Δ is a subgroup of $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ with the property that for each line L in V there is at least one projective transvection σ in Δ with $R = L$, at least two if $F \neq \mathbf{F}_2$. Similarly with Δ_1. Let us fix a symplectic base

$$\mathfrak{X} = \{x_1, x_2 | y_1, y_2\}$$

for V. We let mon V stand for the multiplicative monoid of semilinear maps of V into V. So $\Gamma\mathrm{Sp}_4$ is a submonoid of mon V.

For each $k \in \Gamma\mathrm{Sp}_4$ with associated field automorphism μ and with matrix $A = (a_{ij})$ in $\mathfrak{X}$, define $\mathrm{E}(k) \in$ mon V to be the semilinear map of V into V whose associated field automorphism is μ and whose matrix in the base $\mathfrak{X}$ is equal to

$$\left(\begin{array}{cc|cc} a_{12}a_{21} - a_{11}a_{22} & a_{14}a_{21} - a_{11}a_{24} & a_{13}a_{24} - a_{14}a_{23} & a_{13}a_{22} - a_{12}a_{23} \\ a_{12}a_{41} - a_{11}a_{42} & a_{14}a_{41} - a_{11}a_{44} & a_{13}a_{44} - a_{14}a_{43} & a_{13}a_{42} - a_{12}a_{43} \\ \hline a_{42}a_{31} - a_{41}a_{32} & a_{44}a_{31} - a_{41}a_{34} & a_{43}a_{34} - a_{44}a_{33} & a_{43}a_{32} - a_{42}a_{33} \\ a_{22}a_{31} - a_{21}a_{32} & a_{24}a_{31} - a_{21}a_{34} & a_{23}a_{34} - a_{24}a_{33} & a_{23}a_{32} - a_{22}a_{33} \end{array}\right).$$

If we say that the action of E on a certain element of $\Gamma\mathrm{Sp}_4$ can be described

([4])The isomorphism theory for subgroups of $\mathrm{P}\Gamma\mathrm{Sp}_4$ which is developed in this paragraph is new. See §5.7 for references.

by

$$A \text{ with } \mu \rightarrow B \text{ with } \mu$$

we mean that E carries the element of ΓSp_4 whose matrix (in $\mathfrak{X}$) is the matrix A and whose field automorphism is the automorphism μ, to an element k_1 of mon V with matrix B and with field automorphism μ. If we omit the μ's we mean that the k and k_1 in question are actually linear. It follows immediately from the definition of E that E has the following action on the following elements of ΓSp_4:

(5.4.1a) $$\left(\begin{array}{cc|cc} 1 & & \alpha & 0 \\ & 1 & 0 & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & & 0 & \alpha \\ & 1 & \alpha & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

(5.4.1b) $$\left(\begin{array}{cc|cc} 1 & & 0 & 0 \\ & 1 & 0 & \alpha \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & \alpha & & \\ & 1 & & \\ \hline & & 1 & \\ & & \alpha & 1 \end{array}\right),$$

(5.4.1c) $$\left(\begin{array}{cc|cc} 1 & & 0 & \alpha \\ & 1 & \alpha & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & & \alpha^2 & 0 \\ & 1 & 0 & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

(5.4.1d) $$\left(\begin{array}{cc|cc} 1 & \alpha & & \\ & 1 & & \\ \hline & & 1 & \\ & & \alpha & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & & 0 & 0 \\ & 1 & 0 & \alpha^2 \\ \hline & & 1 & \\ & & & 1 \end{array}\right),$$

(5.4.1e) $$\left(\begin{array}{cc|cc} 1 & & & \\ \alpha & 1 & & \\ \hline & & 1 & \alpha \\ & & & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline 0 & 0 & 1 & \\ 0 & \alpha^2 & & 1 \end{array}\right),$$

(5.4.1f) $$\left(\begin{array}{cc|cc} & 1 & & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right),$$

(5.4.1g) $$\left(\begin{array}{cc|cc} \alpha & & & \\ & \alpha & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} \alpha^2 & & & \\ & \alpha & & \\ \hline & & 1 & \\ & & & \alpha \end{array}\right),$$

(5.4.1h) $$I \text{ with } \mu' \rightarrow I \text{ with } \mu',$$

(5.4.1i) $$\alpha I \rightarrow \alpha^2 I.$$

Let J denote the set of elements of the type listed on the left-hand side of (a)–(h) above. Note that it follows from §2.2 that

$$\text{Types (a)–(f) generate } \mathrm{Sp}_4;$$

hence, by Example 4.2.8,

$$\text{Types (a)–(g) generate } \mathrm{GSp}_4;$$

hence

$$\text{Types (a)–(h) generate } \Gamma\mathrm{Sp}_4,$$

in other words, J generates $\Gamma\mathrm{Sp}_4$. Also note that (g) gives an example of a similitude of multiplier α on the left that becomes a similitude of multiplier α^2 on the right. It is easily verified, by direct computation, that

$$\mathrm{E}(kj) = \mathrm{E}(k)\mathrm{E}(j) \qquad \forall\, k \in \Gamma\mathrm{Sp}_4, j \in J;$$

note that in verifying this for j of the types (c), (d), (e) one needs the identities

$$\begin{aligned} a_{14}a_{22} + a_{13}a_{21} + a_{12}a_{24} + a_{11}a_{23} &= 0,\\ a_{14}a_{42} + a_{13}a_{41} + a_{11}a_{43} + a_{12}a_{44} &= 0,\\ a_{32}a_{44} + a_{31}a_{43} + a_{33}a_{41} + a_{34}a_{42} &= 0,\\ a_{32}a_{24} + a_{31}a_{23} + a_{33}a_{21} + a_{34}a_{22} &= 0 \end{aligned}$$

for the matrix $A = (a_{ij})$ of a typical k in $\Gamma\mathrm{Sp}_4$; these identities follow from the equation

$$A\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right){}^tA = * \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

which is obtained from the usual

$${}^tA\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)A = * \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

by taking inverses.

If we now take typical k, k' in $\Gamma\mathrm{Sp}_4$ and express $k' = j_1 \cdots j_t$ with the j's in J we find that

$$\begin{aligned} \mathrm{E}(kk') &= \mathrm{E}(kj_1 \cdots j_{t-1})\mathrm{E}(j_t)\\ &\ \ \vdots\\ &= \mathrm{E}(k)\mathrm{E}(j_1)\cdots \mathrm{E}(j_t)\\ &\ \ \vdots\\ &= \mathrm{E}(k)\mathrm{E}(k'). \end{aligned}$$

In particular $\mathrm{E}(k')$ is in $\Gamma\mathrm{Sp}_4$ since all $\mathrm{E}(j)$ are. The mapping E is therefore a group homomorphism

$$\mathrm{E}\colon \Gamma\mathrm{Sp}_4 \rightarrowtail \Gamma\mathrm{Sp}_4.$$

Now it follows from the definition of E that the field automorphism of $\mathrm{E}(k)$ is equal to the field automorphism of k; in particular, $\mathrm{EGSp}_4 \subseteq \mathrm{GSp}_4$; and it follows from (a)–(f) that $\mathrm{ESp}_4 \subseteq \mathrm{Sp}_4$. We also have

$$m_{\mathrm{E}k} = (m_k)^2 \qquad \forall\, k \in \Gamma\mathrm{Sp}_4.$$

And

$$\mathrm{E}(\Gamma\mathrm{Sp}_4 - \mathrm{GSp}_4) \subseteq \Gamma\mathrm{Sp}_4 - \mathrm{GSp}_4, \qquad \mathrm{E}(\mathrm{GSp}_4 - \mathrm{Sp}_4) \subseteq \mathrm{GSp}_4 - \mathrm{Sp}_4.$$

In particular $\ker \mathrm{E} \subseteq \mathrm{Sp}_4$ so that, by 3.4.2, $\ker \mathrm{E} = 1$ (note that 3.4.2 does not apply when $F = \mathbf{F}_2$ but in this case the result follows by showing that the right-hand side contains a set of generators for Sp_4, so $\ker \mathrm{E} = 1$ by counting). We therefore have group monomorphisms

$$\mathrm{E}\colon \Gamma\mathrm{Sp}_4 \rightarrowtail \Gamma\mathrm{Sp}_4,$$
$$\mathrm{E}\colon \mathrm{GSp}_4 \rightarrowtail \mathrm{GSp}_4,$$
$$\mathrm{E}\colon \mathrm{Sp}_4 \rightarrowtail \mathrm{Sp}_4.$$

By studying equations (a)–(f) and using the fact that

$$(a_{ij})(b_{ij}) = (c_{ij}) \Rightarrow (a_{ij}^2)(b_{ij}^2) = (c_{ij}^2)$$

holds for square matrices in characteristic 2, we see that E^2 has the following action on a typical element of Sp_4:

$$(a_{ij}) \rightarrowtail (a_{ij}^2).$$

It follows from this fact, in conjunction with (g) and (h), that the matrix of E^2k for any k in $\Gamma\mathrm{Sp}_4$ has the form

$$(\alpha c_{ij}^2) \quad \text{in } \mathfrak{X}$$

with α in $\dot{F}$ and the c_{ij} in F.

We can associate with the above isomorphism E a mapping $\bar{\mathrm{E}}$ of $\mathrm{P}\Gamma\mathrm{Sp}_4$ into $\mathrm{P}\Gamma\mathrm{Sp}_4$ by defining

$$\bar{\mathrm{E}}\bar{k} = \overline{\mathrm{E}k} \qquad \forall\, k \in \Gamma\mathrm{Sp}_4.$$

Note that $\bar{\mathrm{E}}$ is well defined since, by (i), we have $\mathrm{E}(\mathrm{RL}_4) \subseteq \mathrm{RL}_4$. So $\bar{\mathrm{E}}$ is a group homomorphism. Now if $\mathrm{E}k \in \mathrm{RL}_4$, then $k \in \mathrm{GSp}_4$, so we can write $k = \sigma_1\sigma_2$ with σ_1 of type (g) and σ_2 in Sp_4; let the matrices of σ_1 and σ_2 in the base $\mathfrak{X}$ be

$$\left(\begin{array}{c|c} \alpha I & \\ \hline & I \end{array}\right) \quad \text{and} \quad (a_{ij})$$

respectively; then $\mathrm{E}^2k = \mathrm{E}(\mathrm{E}k)$ is in RL_4, so (a_{ij}^2) is diagonal, so (a_{ij}) is diagonal; so

$$k \sim \mathrm{diag}[\alpha_1, \ldots, \alpha_4] \quad \text{in } \mathfrak{X};$$

so

$$\mathrm{E}k \sim \mathrm{diag}[\alpha_1\alpha_2, \alpha_1\alpha_4, \alpha_4\alpha_3, \alpha_2\alpha_3] \quad \text{in } \mathfrak{X};$$

but $\mathrm{E}k$ is in RL_4; from this it follows that $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$; then $\alpha_1 = \alpha_2$ by 4.2.2; so k is in RL_4. We have therefore proved that $\bar{\mathrm{E}}$ is injective. Note however that $\bar{\mathrm{E}}\mathrm{PGSp}_4 \subseteq \mathrm{PSp}_4$ since the multipliers of all elements of EGSp_4 are squares. So we have group monomorphisms

$$\mathrm{E}\colon \mathrm{P}\Gamma\mathrm{Sp}_4 \rightarrowtail \mathrm{P}\Gamma\mathrm{Sp}_4, \qquad \bar{\mathrm{E}}\colon \mathrm{PGSp}_4 \rightarrowtail \mathrm{PSp}_4.$$

Of course we still have

$$\bar{\mathrm{E}}(\mathrm{P}\Gamma\mathrm{Sp}_4 - \mathrm{PGSp}_4) \subseteq \mathrm{P}\Gamma\mathrm{Sp}_4 - \mathrm{PGSp}_4.$$

5.4.2. *The following statements are equivalent for the monomorphism* E *under discussion*:

(1) $\mathrm{E}\Gamma\mathrm{Sp}_4 = \Gamma\mathrm{Sp}_4$.
(2) $\mathrm{EGSp}_4 = \mathrm{GSp}_4$.
(3) $\mathrm{ESp}_4 = \mathrm{Sp}_4$.
(4) *F is perfect.*

PROOF. (1) implies (2) is a consequence of the inclusions $\mathrm{EGSp}_4 \subseteq \mathrm{GSp}_4$ and

$$\mathrm{E}(\Gamma\mathrm{Sp}_4 - \mathrm{GSp}_4) \subseteq \Gamma\mathrm{Sp}_4 - \mathrm{GSp}_4.$$

And (2) implies (3) is done in a similar way. Next let us prove that (3) implies (4). Let α be a typical element of F. Then there is an element σ of Sp_4 with matrix

$$\left(\begin{array}{cc|cc} 1 & & \alpha & 0 \\ & 1 & 0 & 0 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \quad \text{in } \mathfrak{X},$$

but σ is in $\mathrm{E}^2\mathrm{Sp}_4$ by hypothesis, and E^2 has action $(a_{ij}) \rightarrowtail (a_{ij}^2)$ on elements of Sp_4, so α is a square, so F is perfect. Finally we have to prove that (4) implies (1). Here F is perfect. If we can prove that $\mathrm{ESp}_4 = \mathrm{Sp}_4$, then $\mathrm{EGSp}_4 = \mathrm{GSp}_4$ by (5.4.1g), and then $\mathrm{E}\Gamma\mathrm{Sp}_4 = \Gamma\mathrm{Sp}_4$ by (5.4.1h), so we will be through. Let us prove $\mathrm{ESp}_4 = \mathrm{Sp}_4$. Consider a typical σ in Sp_4 and let σ have matrix (a_{ij}) in $\mathfrak{X}$. Then (a_{ij}) satisfies the matrix equation

$$^{t}X\left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)X = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right),$$

hence $(\sqrt{a_{ij}})$ satisfies the same equation; hence the linear transformation defined by $\Sigma \sim (\sqrt{a_{ij}})$ in $\mathfrak{X}$ is in Sp_4. But $\mathrm{E}^2\Sigma = \sigma$. Q.E.D.

5.4.3. *The following statements are equivalent for the monomorphism* $\bar{\mathrm{E}}$ *under discussion*:

(1) $\bar{\mathrm{E}}\mathrm{P}\Gamma\mathrm{Sp}_4 = \mathrm{P}\Gamma\mathrm{Sp}_4$.
(2) $\bar{\mathrm{E}}\mathrm{PGSp}_4 = \mathrm{PGSp}_4$.
(3) $\bar{\mathrm{E}}\mathrm{PSp}_4 = \mathrm{PSp}_4$.
(4) *F is perfect.*

PROOF. It is an easy consequence of 5.4.2 that (4) implies each of the others. If we have (1), then $\bar{\mathrm{E}}\mathrm{P}\Gamma\mathrm{Sp}_4 = \mathrm{P}\Gamma\mathrm{Sp}_4$, but

$$\bar{\mathrm{E}}(\mathrm{P}\Gamma\mathrm{Sp}_4 - \mathrm{PGSp}_4) \subseteq \mathrm{P}\Gamma\mathrm{Sp}_4 - \mathrm{PGSp}_4,$$

and $\bar{\mathrm{E}}\mathrm{PGSp}_4 \subseteq \mathrm{PGSp}_4$, so $\bar{\mathrm{E}}\mathrm{PGSp}_4 = \mathrm{PGSp}_4$, i.e., (1) implies (2). If we have (2), then

$$\mathrm{PGSp}_4 = \bar{\mathrm{E}}\mathrm{PGSp}_4 \subseteq \mathrm{PSp}_4 \subseteq \mathrm{PGSp}_4,$$

so $\mathrm{PGSp}_4 = \mathrm{PSp}_4$, so

$$\bar{\mathrm{E}}\mathrm{PSp}_4 = \bar{\mathrm{E}}\mathrm{PGSp}_4 = \mathrm{PGSp}_4 = \mathrm{PSp}_4,$$

so (2) implies (3). If we have (3), then $\mathrm{ESp}_4 = \mathrm{Sp}_4$, so F is perfect by 5.4.2. Q.E.D.

5.4.4. *If τ is the transvection $\tau_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda}$ in* Sp_4, *then*

$$\tau \sim \left(\begin{array}{cc|cc} 1+\lambda\alpha\gamma & \lambda\alpha\delta & \lambda\alpha^2 & \lambda\alpha\beta \\ \lambda\beta\gamma & 1+\lambda\beta\delta & \lambda\alpha\beta & \lambda\beta^2 \\ \hline \lambda\gamma^2 & \lambda\gamma\delta & 1+\lambda\alpha\gamma & \lambda\beta\gamma \\ \lambda\gamma\delta & \lambda\delta^2 & \lambda\alpha\delta & 1+\lambda\beta\delta \end{array}\right),$$

$$\mathrm{E}\tau \sim \left(\begin{array}{cc|cc} 1+\lambda(\alpha\gamma+\beta\delta) & \lambda\beta^2 & 0 & \lambda\alpha^2 \\ \lambda\delta^2 & 1+\lambda(\alpha\gamma+\beta\delta) & \lambda\alpha^2 & 0 \\ \hline 0 & \lambda\gamma^2 & 1+\lambda(\alpha\gamma+\beta\delta) & \lambda\delta^2 \\ \lambda\gamma^2 & 0 & \lambda\beta^2 & 1+\lambda(\alpha\gamma+\beta\delta) \end{array}\right)$$

in the base $\mathfrak{X}$.

Proof. From the definitions. Q.E.D.

Given any α, β, γ, δ in F and any λ which satisfies

$$\lambda \in \dot{F}, \qquad \lambda + (\alpha\gamma + \beta\delta) \in F^2,$$

define $\Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$ to be the linear transformation with matrix

$$\left(\begin{array}{cc|cc} \sqrt{\lambda+(\alpha\gamma+\beta\delta)} & \beta & 0 & \alpha \\ \delta & \sqrt{\lambda+(\alpha\gamma+\beta\delta)} & \alpha & 0 \\ \hline 0 & \gamma & \sqrt{\lambda+(\alpha\gamma+\beta\delta)} & \delta \\ \gamma & 0 & \beta & \sqrt{\lambda+(\alpha\gamma+\beta\delta)} \end{array}\right)$$

in the base $\mathfrak{X}$. Using 4.2.2 one verifies that this transformation is an element of GSp_4 with multiplier λ. With a little calculation one sees that in fact it is hyperbolic. We have

$$\Sigma_{0,0,0,0;\lambda} = r_{\sqrt{\lambda}} \quad \text{if } \lambda \in \dot{F}^2.$$

Note that if

$$\Sigma_{\alpha',\beta',\gamma',\delta';\lambda'} \quad \text{and} \quad \Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$$

are defined, then

$$\Sigma_{\alpha',\beta',\gamma',\delta';\lambda'} = \Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$$

if and only if

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma, \quad \delta' = \delta; \quad \lambda' = \lambda.$$

Projectively we find that

$$\bar{\Sigma}_{\alpha',\beta',\gamma',\delta';\lambda'} = \bar{\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda}$$

if and only if we have a proportion

$$\alpha' = \alpha\xi, \quad \beta' = \beta\xi, \quad \gamma' = \gamma\xi, \quad \delta' = \delta\xi; \quad \lambda' = \lambda\xi^2$$

for some ξ in $\dot{F}$. For transvections the situation, as we know, is slightly different. If

$$\tau_{\alpha' x_1 + \beta' x_2 + \gamma' y_1 + \delta' y_2, \lambda'} \quad \text{and} \quad \tau_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2, \lambda}$$

are not equal to 1_V, then

$$\tau_{\alpha' x_1 + \beta' x_2 + \gamma' y_1 + \delta' y_2, \lambda'} = \tau_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2, \lambda}$$

if and only if

$$\bar{\tau}_{\alpha' x_1 + \beta' y_2 + \gamma' y_1 + \delta' y_2, \lambda'} = \bar{\tau}_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2, \lambda}$$

if and only if we have a proportion

$$\alpha' = \alpha\xi, \quad \beta' = \beta\xi, \quad \gamma' = \gamma\xi, \quad \delta' = \delta\xi; \quad \lambda' = \lambda\xi^{-2}$$

for some ξ in $\dot{F}$.

5.4.5. *If k is any hyperbolic transformation in $\Gamma\mathrm{Sp}_4$, then there are α, β, γ, δ in F and λ in $\dot{F}$ with $\lambda + (\alpha\gamma + \beta\delta)$ in F^2 such that $k = \Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$.*

PROOF. We know from 4.3.1 that k is in GSp_4. Since

$$q(x_1, kx_1) = \cdots = q(y_2, ky_2) = 0,$$

the matrix of k in the base $\mathfrak{X}$ will have the form

$$\left(\begin{array}{cc|cc} G_1 & B_1 & 0 & A_1 \\ D_1 & G_2 & A_2 & 0 \\ \hline 0 & C_1 & G_3 & D_2 \\ C_2 & 0 & B_2 & G_4 \end{array}\right).$$

The six equations

$$q(x_1 + x_2, k(x_1 + x_2)) = 0, \quad q(x_1 + y_1, k(x_1 + y_1)) = 0,$$
$$q(x_1 + y_2, k(x_1 + y_2)) = 0, \quad q(x_2 + y_1, k(x_2 + y_1)) = 0,$$
$$q(x_2 + y_2, k(x_2 + y_2)) = 0, \quad q(y_1 + y_2, k(y_1 + y_2)) = 0,$$

then give in turn

$$\begin{aligned} C_1 = C_2 &= \gamma, & G_3 &= G_1, \\ D_2 = D_1 &= \delta, & B_2 = B_1 &= \beta, \\ G_4 &= G_2, & A_1 = A_2 &= \alpha, \end{aligned}$$

where α, β, γ, δ are defined to be the common values indicated in the equations. Now

$$q(kx_1, ky_1) = q(kx_2, ky_2) = \lambda$$

where λ is the multiplier of k, and hence an element of $\dot{F}$, so

$$G_1G_3 + (\alpha\gamma + \beta\delta) = G_2G_4 + (\alpha\gamma + \beta\delta) = \lambda.$$

Therefore $G_1^2 = G_2^2$. Therefore $G_1 = G_2$. Therefore $G_1 = G_2 = G_3 = G_4$, and $\lambda + (\alpha\gamma + \beta\delta)$ is in F^2, and all diagonal entries are equal to

$$\sqrt{\lambda + (\alpha\gamma + \beta\delta)}\,. \quad \text{Q.E.D.}$$

5.4.6. REMARK. The set of hyperbolic transformations in $\Gamma\mathrm{Sp}_4$ is the set of $\Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$. And the set of projective hyperbolic transformations in $\mathrm{P}\Gamma\mathrm{Sp}_4$ is the set of $\overline{\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda}$.

5.4.7. *For any* α, β, γ, δ *in* F, *and any* λ *in* $\dot{F}$ *satisfying* $\lambda + (\alpha\gamma + \beta\delta) \in F^2$,

$$\mathrm{E}\Sigma_{\alpha,\beta,\gamma,\delta;\lambda} \sim \left(\begin{array}{cc|cc} \lambda + \alpha\gamma & \alpha\delta & \alpha^2 & \alpha\beta \\ \beta\gamma & \lambda + \beta\delta & \alpha\beta & \beta^2 \\ \hline \gamma^2 & \gamma\delta & \lambda + \alpha\gamma & \beta\gamma \\ \gamma\delta & \delta^2 & \alpha\delta & \lambda + \beta\delta \end{array}\right)$$

in the base $\mathfrak{X}$.

PROOF. From the definitions. Q.E.D.

5.4.8. *For any* α, β, γ, δ *in* F *and any* λ *in* $\dot{F}$ *we have*

$$\mathrm{E}\tau_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2,\lambda} = \Sigma_{\lambda\alpha^2,\lambda\beta^2,\lambda\gamma^2,\lambda\delta^2;1} = r_\lambda \Sigma_{\alpha^2,\beta^2,\gamma^2,\delta^2;\lambda^{-2}}$$

and so

$$\overline{\mathrm{E}}\bar{\tau}_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2,\lambda} = \overline{\Sigma}_{\alpha^2,\beta^2,\gamma^2,\delta^2;\lambda^{-2}}.$$

5.4.9. *For any* α, β, γ, δ *in* F *and any* λ *in* $\dot{F}$ *satisfying* $\lambda + (\alpha\gamma + \beta\delta) \in F^2$, *we have*

$$\mathrm{E}\Sigma_{\alpha,\beta,\gamma,\delta;\lambda} = r_\lambda \tau_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2,\lambda^{-1}}$$

and so

$$\overline{\mathrm{E}}\overline{\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda} = \bar{\tau}_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2,\lambda^{-1}}.$$

5.4.10. $\overline{\mathrm{E}}\mathrm{PGSp}_4$ *has enough projective transvections.*

5.4.11. REMARK. We know that PGSp_4 has enough projective transvections, and we have just seen that $\overline{\mathrm{E}}\mathrm{PGSp}_4$ has enough projective transvections, so $\overline{\mathrm{E}}\colon \mathrm{PGSp}_4 \rightarrowtail \overline{\mathrm{E}}\mathrm{PGSp}_4$ is an isomorphism between two groups with enough projective transvections. But $\overline{\mathrm{E}}$ clearly does not preserve projective transvections!

5.4.12. *Given* α, β, γ, δ *in* F, *not all* 0, *and given* λ *in* $\dot{F}$, *then*

$$\bar{\tau}_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2,\lambda} \in \overline{\mathrm{E}}\mathrm{PGSp}_4$$

if and only if

$$\lambda^{-1} + (\alpha\gamma + \beta\delta) \in F^2.$$

If this condition is satisfied, then

$$\bar{\tau}_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda} = \bar{\mathrm{E}}\bar{\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda^{-1}}.$$

PROOF. If the condition is satisfied we just apply 5.4.9. So assume that the given transvection is in $\bar{\mathrm{E}}\mathrm{PGSp}_4$. Since $\mathrm{P\Gamma Sp}_4$ and $\bar{\mathrm{E}}\mathrm{P\Gamma Sp}_4$ have enough projective transvections by 5.4.10, and since we have an isomorphism $\bar{\mathrm{E}}$ of $\mathrm{P\Gamma Sp}_4$ onto $\bar{\mathrm{E}}\mathrm{P\Gamma Sp}_4$, it follows from 5.2.7 and 5.2.2 that the given projective transvection comes from a projective hyperbolic transformation, so by 5.4.5 we have α_1, β_1, γ_1, δ_1 in F and λ_1 in $\dot{F}$ with $\lambda_1^{-1} + (\alpha_1\gamma_1 + \beta_1\delta_1)$ in F^2 such that

$$\bar{\mathrm{E}}\bar{\Sigma}_{\alpha_1,\beta_1,\gamma_1,\delta_1;\lambda_1^{-1}} = \bar{\tau}_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda}.$$

So by 5.4.9

$$\bar{\tau}_{\alpha_1 x_1+\beta_1 x_2+\gamma_1 y_1+\delta_1 y_2,\lambda_1} = \bar{\tau}_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda}.$$

There is then a proportion

$$\alpha_1 = \alpha\xi,\quad \beta_1 = \beta\xi,\quad \gamma_1 = \gamma\xi,\quad \delta_1 = \delta\xi;\quad \lambda_1 = \lambda\xi^{-2}$$

for some ξ in $\dot{F}$, whence the result. Q.E.D.

5.4.13. *Given α, β, γ, δ in F, not all 0, and given λ in $\dot{F}$, then*

$$\bar{\tau}_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda} \in \bar{\mathrm{E}}\mathrm{PSp}_4$$

if and only if

$$\lambda^{-1} \in F^2 \quad \textit{and} \quad \alpha\gamma + \beta\delta \in F^2.$$

PROOF. If the condition is satisfied apply 5.4.12 observing that the $\bar{\Sigma}$ in question is in PSp_4 since the multiplier of the Σ in question is a square (see 4.2.4). Conversely, suppose the given $\bar{\tau}$ is in $\bar{\mathrm{E}}\mathrm{PSp}_4$. Then $\lambda^{-1} + (\alpha\gamma + \beta\delta)$ is in F^2 by 5.4.12, and $\bar{\Sigma} = \bar{\mathrm{E}}^{-1}\bar{\tau}$ is in PSp_4, so λ^{-1} is in F^2 by 4.2.4 again. Q.E.D.

5.4.14. $\bar{\mathrm{E}}\mathrm{PSp}_4$ *has enough projective transvections if and only if F is perfect.*

PROOF. If F is perfect, then $\mathrm{PSp}_4 = \mathrm{PGSp}_4$ by 4.2.4, but $\bar{\mathrm{E}}\mathrm{PGSp}_4$ has enough projective transvections by 5.4.10, hence $\bar{\mathrm{E}}\mathrm{PSp}_4$ does. Conversely, suppose $\bar{\mathrm{E}}\mathrm{PSp}_4$ has enough projective transvections. Let α be any element of F. Then $\bar{\tau}_{\alpha x_1+y_1,\lambda}$ must be in $\bar{\mathrm{E}}\mathrm{PSp}_4$ for some λ in $\dot{F}$. So $\alpha \in F^2$ by 5.4.13. So F is perfect. Q.E.D.

5.4.15. *The residual space of a nontrivial hyperbolic transformation in* $\mathrm{Sp}_4(V)$ *is a totally degenerate plane.*

PROOF. The given transformation is an involution by 2.1.6, and involutions in characteristic 2 are totally degenerate by 1.3.13, and hyperbolic transformations in $\mathrm{Sp}_4(V)$ have even residue by 2.1.17. Q.E.D.

We define $\Pi(\alpha, \beta, \gamma, \delta)$ for any α, β, γ, δ in F that are not all 0 and which satisfy $\alpha\gamma + \beta\delta \in F^2$ to be the subspace of V that is spanned by the vectors

$$\left\{\begin{array}{ccccccc} \sqrt{\alpha\gamma + \beta\delta}\, x_1 & + & \delta x_2 & & & + & \gamma y_2 \\ \beta x_1 & + & \sqrt{\alpha\gamma + \beta\delta}\, x_2 & + & \gamma y_1 & & \\ & & \alpha x_2 & + & \sqrt{\alpha\gamma + \beta\delta}\, y_1 & + & \beta y_2 \\ \alpha x_1 & + & & + & \delta y_1 & + & \sqrt{\alpha\gamma + \beta\delta}\, y_2 \end{array}\right.$$

where $\mathfrak{X}$ is the fixed symplectic base $\{x_1, x_2 | y_1, y_2\}$ of §5.4.

5.4.16. $\Pi(\alpha, \beta, \gamma, \delta)$ *is a totally degenerate plane; indeed it is the residual space of the nontrivial hyperbolic transformation* $\Sigma_{\alpha,\beta,\gamma,\delta;1}$ *in* $\mathrm{Sp}_4(V)$.

PROOF. Directly from the definitions. Q.E.D.

5.4.17. *Every totally degenerate plane is a* $\Pi(\alpha, \beta, \gamma, \delta)$.

PROOF. Let Π be a typical totally degenerate plane. Choose a symplectic base $\{x_1', x_2' | y_1', y_2'\}$ for V with $\Pi = Fx_1' + Fx_2'$. By 2.1.15 there is a hyperbolic transformation σ in Sp_4 with residual space Π. So by 5.4.5 we have $\alpha, \beta, \gamma, \delta$ in F and λ in $\dot{F}$ with $\lambda + (\alpha\gamma + \beta\delta)$ in F^2 such that $\sigma = \Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$. Then $\lambda = 1$ since σ is in Sp_4, and not all $\alpha, \beta, \gamma, \delta$ are 0 since $\sigma \neq 1$. Therefore $\alpha\gamma + \beta\delta \in F^2$ and $\Pi(\alpha, \beta, \gamma, \delta)$ is defined. It is equal to Π by 5.4.16. Q.E.D.

5.4.18. *Let* Π_0 *denote* Π *defined with respect to a second symplectic base* $\mathfrak{X}_0$ *for* V. *Then* $\Pi(\alpha, \beta, \gamma, \delta)$ *is equal to*

(1) $\Pi_0(\alpha, \delta, \gamma, \beta)$ *if* $\mathfrak{X}_0 = \{x_2, x_1 | y_2, y_1\}$,
(2) $\Pi_0(\gamma, \beta, \alpha, \delta)$ *if* $\mathfrak{X}_0 = \{y_2, y_1 | x_2, x_1\}$,
(3) $\Pi_0(\beta, \alpha, \delta, \gamma)$ *if* $\mathfrak{X}_0 = \{x_1, y_2 | y_1, x_2\}$,
(4) $\Pi_0(\delta, \gamma, \beta, \alpha)$ *if* $\mathfrak{X}_0 = \{y_1, x_2 | x_1, y_2\}$.

PROOF. Directly from the definitions. Q.E.D.

5.4.19. $\Pi(\alpha, \beta, \gamma, \delta) = \Pi(\alpha', \beta', \gamma', \delta')$ *if and only if there is a proportion*

$$\alpha' = \alpha\xi, \quad \beta' = \beta\xi, \quad \gamma' = \gamma\xi, \quad \delta' = \delta\xi,$$

for some ξ *in* $\dot{F}$.

PROOF. If ξ exists then the Π's are obviously equal. So assume conversely that the Π's are equal. By 5.4.18 we can assume that $\alpha \neq 0$, and hence that $\alpha = 1$. So by definition the two vectors

$$\left\{\begin{array}{ccccc} x_2 & + & \sqrt{\gamma + \beta\delta}\, y_1 & + & \beta y_2 \\ x_1 & + & \delta y_1 & + & \sqrt{\gamma + \beta\delta}\, y_2 \end{array}\right.$$

form a base for $\Pi(1, \beta, \gamma, \delta)$.

Let us show that $\alpha' \neq 0$. Suppose, to the contrary, that $\alpha' = 0$. If $\beta' \neq 0$, then the two vectors

$$\left\{\begin{array}{ccc} \beta' x_1 + \sqrt{\beta'\delta'}\, x_2 + & \gamma' y_1 & \\ & \sqrt{\beta'\delta'}\, y_1 & + \beta' y_2 \end{array}\right.$$

form a base for $\Pi(0, \beta', \gamma', \delta')$, hence for $\Pi(1, \beta, \gamma, \delta)$, and this is clearly false. Similarly $\alpha' = \beta' = 0$, $\gamma' \neq 0$ is impossible. Similarly $\alpha' = \beta' = \gamma' = 0$, $\delta' \neq 0$, is impossible. So indeed $\alpha' \neq 0$.

We may therefore assume that $\alpha' = 1$. So

$$\begin{cases} x_2 + \sqrt{\gamma' + \beta'\delta'}\, y_1 + \beta' y_2 \\ x_1 + \delta' y_1 + \sqrt{\gamma' + \beta'\delta'}\, y_2 \end{cases}$$

is a base for $\Pi(1, \beta', \gamma', \delta')$, and hence for $\Pi(1, \beta, \gamma, \delta)$. This implies $\beta = \beta'$, $\delta = \delta'$, $\sqrt{\gamma + \beta\delta} = \sqrt{\gamma' + \beta'\delta'}$, $\gamma = \gamma'$, and hence the result. Q.E.D.

5.4.20. $\Pi(\alpha, \beta, \gamma, \delta) \cap \Pi(\alpha', \beta', \gamma', \delta') \neq 0$ *if and only if*

$$\alpha\gamma' + \beta\delta' + \gamma\alpha' + \delta\beta' = 0.$$

PROOF. (1) First let us assume that the given Π's have at least one of $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$ nonzero. By 5.4.18 we can assume that in fact $\alpha\alpha' \neq 0$, hence that $\alpha = \alpha' = 1$. So $\Pi(1, \beta, \gamma, \delta)$ has a base

$$\begin{cases} x_2 + \sqrt{\gamma + \beta\delta}\, y_1 + \beta y_2 \\ x_1 + \delta y_1 + \sqrt{\gamma + \beta\delta}\, y_2 \end{cases}$$

and $\Pi(1, \beta', \gamma', \delta')$ has a base

$$\begin{cases} x_2 + \sqrt{\gamma' + \beta'\delta'}\, y_1 + \beta' y_2 \\ x_1 + \delta' y_1 + \sqrt{\gamma' + \beta'\delta'}\, y_2 \end{cases}.$$

So the two Π's intersect if and only if there are scalars λ, ν, not both 0, such that

$$\begin{cases} \lambda\sqrt{\gamma + \beta\delta} + \nu\delta = \lambda\sqrt{\gamma' + \beta'\delta'} + \nu\delta', \\ \lambda\beta + \nu\sqrt{\gamma + \beta\delta} = \lambda\beta' + \nu\sqrt{\gamma' + \beta'\delta'}, \end{cases}$$

i.e., such that

$$\begin{cases} \lambda\left(\sqrt{\gamma + \beta\delta} + \sqrt{\gamma' + \beta'\delta'}\right) = \nu(\delta + \delta'), \\ \nu\left(\sqrt{\gamma + \beta\delta} + \sqrt{\gamma' + \beta'\delta'}\right) = \lambda(\beta + \beta'). \end{cases}$$

If $\beta = \beta'$, the existence of the above λ, ν is easily seen to be equivalent to the required condition. Similarly if $\delta = \delta'$. We may therefore assume that $\beta \neq \beta'$ and $\delta \neq \delta'$. In this case the nontrivial intersection of the planes in question is equivalent to the existence of λ, ν, both nonzero, satisfying the last pair of equations, hence to the existence of λ, ν in $\dot{F}$ such that

$$\begin{cases} \dfrac{\nu}{\lambda} = \dfrac{\sqrt{\gamma + \beta\delta} + \sqrt{\gamma' + \beta'\delta'}}{\delta + \delta'}, \\[2ex] \dfrac{\lambda}{\nu} = \dfrac{\sqrt{\gamma + \beta\delta} + \sqrt{\gamma' + \beta'\delta'}}{\beta + \beta'}. \end{cases}$$

This is equivalent to the condition

$$\frac{\sqrt{\gamma+\beta\delta}+\sqrt{\gamma'+\beta'\delta'}}{\delta+\delta'}\cdot\frac{\sqrt{\gamma+\beta\delta}+\sqrt{\gamma'+\beta'\delta'}}{\beta+\beta'}=1,$$

i.e., to the condition

$$\gamma'+\beta\delta'+\gamma+\delta\beta'=0,$$

as required.

(2) By step (1) we may now assume that the given Π's satisfy

$$\alpha\alpha'=\beta\beta'=\gamma\gamma'=\delta\delta'=0.$$

By 5.4.18 we may assume that $\alpha\neq 0$, hence that $\alpha=1$ and $\alpha'=0$. By 5.4.18 again we may assume that $\delta'=0$ if $\beta'=0$. Therefore we may assume that we are in one of the following situations:

(i) $\Pi(1,0,\gamma,\delta)$ and $\Pi(0,1,\gamma',\delta')$,

(ii) $\Pi(1,\beta,0,\delta)$ and $\Pi(0,0,1,0)$.

In case (i) the two planes have bases

$$\begin{cases} x_2 + \sqrt{\gamma}\,y_1 \\ x_1 + \delta y_1 + \sqrt{\gamma}\,y_2 \end{cases}$$

and

$$\begin{cases} x_1+\sqrt{\delta'}\,x_2+\gamma' y_1 \\ \sqrt{\delta'}\,y_1+y_2 \end{cases}$$

respectively, and these two planes intersect if and only if there are ξ, η in F such that

$$x_1+\xi x_2+(\xi\sqrt{\gamma}+\delta)y_1+\sqrt{\gamma}\,y_2=x_1+\sqrt{\delta'}\,x_2+(\gamma'+\eta\sqrt{\delta'})y_1+\eta y_2$$

i.e., if and only if there are ξ, η in F such that

$$\xi=\sqrt{\delta'},\quad \xi\sqrt{\gamma}+\delta=\gamma'+\eta\sqrt{\delta'},\quad \sqrt{\gamma}=\eta,$$

i.e., if and only if

$$\sqrt{\delta'}\sqrt{\gamma}+\delta=\gamma'+\sqrt{\gamma}\sqrt{\delta'},$$

i.e., if and only if $\gamma'+\delta=0$, i.e., if and only if

$$\alpha\gamma'+\beta\delta'+\gamma\alpha'+\delta\beta'=0.$$

In case (ii) the same sort of argument shows that the given planes never intersect while the required equation is never satisfied. Q.E.D.

We define $L(\alpha,\beta,\gamma,\delta)$ for any $\alpha,\beta,\gamma,\delta$ in F that are not all 0 to be the subspace of V that is spanned by the vector

$$\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,$$

where $\mathfrak{X}$ is the fixed symplectic base $\{x_1, x_2|y_1, y_2\}$ of §5.4. The following facts are self-evident–they are listed in order to emphasize the analogy between $L(\alpha,\beta,\gamma,\delta)$ and $\Pi(\alpha,\beta,\gamma,\delta)$. We know that $L(\alpha,\beta,\gamma,\delta)$ is a (totally degenerate) line, indeed it is the residual space of the nontrivial transvection

$\tau_{\alpha x_1+\beta x_2+\gamma y_1+\delta y_2,\lambda}$ for any λ in $\dot{F}$; every (totally degenerate) line is an $L(\alpha, \beta, \gamma, \delta)$;

$$L(\alpha, \beta, \gamma, \delta) = L(\alpha', \beta', \gamma', \delta')$$

if and only if there is a proportion

$$\alpha' = \alpha\xi, \quad \beta' = \beta\xi, \quad \gamma' = \gamma\xi, \quad \delta' = \delta\xi,$$

for some ξ in $\dot{F}$; and

$$q(L(\alpha, \beta, \gamma, \delta), L(\alpha', \beta', \gamma', \delta')) = 0$$

if and only if

$$\alpha\gamma' + \beta\delta' + \gamma\alpha' + \delta\beta' = 0.$$

As usual, $\mathfrak{L}$ stands for the set of all lines of V, regarded as a subset of the projective space $P(V)$; we let $\mathfrak{P}$ denote the set of totally degenerate planes of V, also regarded as a subset of projective space $P(V)$. Of course $\mathfrak{P}$ consists of all $\Pi(\alpha, \beta, \gamma, \delta)$ by 5.4.16 and 5.4.17. And $\mathfrak{L}$ consists of all $L(\alpha, \beta, \gamma, \delta)$.

5.4.21. *The mapping* l: $\mathfrak{P} \rightarrowtail \mathfrak{L}$ *defined by* $\Pi(\alpha, \beta, \gamma, \delta) \rightarrowtail L(\alpha, \beta, \gamma, \delta)$ *for any given scalars* $\alpha, \beta, \gamma, \delta$ *which are not all* 0 *and satisfy* $\alpha\gamma + \beta\delta \in F^2$, *is an injection* l: $\mathfrak{P} \rightarrowtail \mathfrak{L}$ *with the property that*

$$\Pi \cap \Pi' \neq 0 \Leftrightarrow q(l\Pi, l\Pi') = 0.$$

It is bijective if and only if F is perfect.

PROOF. At this point, the proof of this result is obvious. Q.E.D.

5.4.22. LEMMA. *Let* $\mathfrak{L}_0$ *be a family of lines in V with these properties*:

(1) $L(\alpha, \beta, \gamma, \delta) \in \mathfrak{L}_0$ *whenever* $\alpha\gamma + \beta\delta \in F^2$;

(2) *whenever three independent lines in* $\mathfrak{L}_0$ *are orthogonal to a given line L in* $\mathfrak{L}$, *L is also in* $\mathfrak{L}_0$.

Then $\mathfrak{L}_0 = \mathfrak{L}$.

PROOF. (1) $\mathfrak{L}_0$ clearly contains all lines $L(\alpha, \beta, \gamma, \delta)$ with exactly one of the scalars nonzero, by the first property.

(2) Let us show that $\mathfrak{L}_0$ contains all $L(\alpha, \beta, \gamma, \delta)$ with exactly two scalars nonzero. We need only consider situations in which $\alpha\gamma + \beta\delta \neq 0$, and these are the situations $L(\alpha, 0, \gamma, 0)$ with $\alpha\gamma \neq 0$ and $L(0, \beta, 0, \delta)$ with $\beta\delta \neq 0$.The three lines

$$L(0, 1, 0, 0), \qquad L(0, 0, 0, 1), \qquad L(\alpha, \alpha, \gamma, \gamma)$$

are all in $\mathfrak{L}_0$, they are independent, and they are orthogonal to $L(\alpha, 0, \gamma, 0)$. So $L(\alpha, 0, \gamma, 0)$ is in $\mathfrak{L}_0$ by the second property. Similarly with $L(0, \beta, 0, \delta)$.

(3) Next we verify that $L(\alpha, \beta, \gamma, \delta)$ is in $\mathfrak{L}_0$ whenever exactly three of the scalars are nonzero. These are the situations

$$L(\alpha, \beta, \gamma, 0); \quad L(\alpha, \beta, 0, \delta); \quad L(\alpha, 0, \gamma, \delta); \quad L(0, \beta, \gamma, \delta).$$

In the first situation we note that

$$L(0, 0, \alpha^{-1}, \beta^{-1}), \quad L(\gamma^{-1}, 0, \alpha^{-1}, 0), \quad L(0, 1, 0, 0)$$

are in $\mathfrak{L}_0$ by steps (1) and (2); these three lines are clearly independent; and they are orthogonal to $L(\alpha, \beta, \gamma, 0)$. So $L(\alpha, \beta, \gamma, 0)$ is in $\mathfrak{L}_0$. Similarly with the other situations.

(4) Finally, for $L(\alpha, \beta, \gamma, \delta)$ use

$$L(\alpha, 0, \gamma, 0), \qquad L(0, \beta, 0, \delta), \qquad L(\beta, 0, 0, \gamma). \quad \text{Q.E.D.}$$

5.4.23. *There exists a bijection* $i\colon \mathfrak{P} \rightarrowtail \mathfrak{L}$ *with the property that*

$$\Pi \cap \Pi' \neq 0 \Leftrightarrow q(i\Pi, i\Pi') = 0,$$

if and only if F is perfect.

PROOF. If F is perfect apply 5.4.21. So assume, conversely, that we have a bijection i with the stated properties. Let φ be the injection

$$\varphi\colon \mathfrak{L} \overset{i^{-1}}{\rightarrowtail} \mathfrak{P} \overset{l}{\rightarrowtail} \mathfrak{L}$$

where l is the injection of 5.4.21, and put $\mathfrak{L}_0 = \varphi\mathfrak{L}$. If we can prove that $\mathfrak{L}_0 = \mathfrak{L}$, then l will be surjective, hence bijective, and so F will be perfect by 5.4.21. By definition of l we know that

$$\mathfrak{L}_0 = \left\{L(\alpha, \beta, \gamma, \delta) | \alpha\gamma + \beta\delta \in F^2\right\}.$$

So, by Lemma 5.4.22, we will be through if we can prove that L is in $\mathfrak{L}_0$ whenever there are three independent lines K_1, K_2, K_3 in $\mathfrak{L}_0$ which are orthogonal to L. These three K's are, of course, in $\varphi\mathfrak{L}$, so there are lines J_1, J_2, J_3 in $\mathfrak{L}$ with

$$\varphi J_1 = K_1, \qquad \varphi J_2 = K_2, \qquad \varphi J_3 = K_3.$$

Pick J in $\mathfrak{L}$ orthogonal to J_1, J_2, J_3. It follows from the nature of i and l that $\varphi = li^{-1}$ sends orthogonal lines to orthogonal lines; hence φJ is a line in $\mathfrak{L}_0$ that is orthogonal to K_1, K_2, K_3. So

$$L = (K_1 + K_2 + K_3)^* = \varphi J \in \mathfrak{L}_0. \quad \text{Q.E.D.}$$

5.4.24. *The following statements are equivalent*:

(1) $\Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$ *permutes with* $\Sigma_{\alpha',\beta',\gamma',\delta';\lambda'}$.

(2) $\overline{\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda}$ *permutes with* $\overline{\Sigma}_{\alpha',\beta',\gamma',\delta';\lambda'}$.

(3) $\alpha\gamma' + \beta\delta' + \gamma\alpha' + \delta\beta' = 0$.

PROOF. (1) obviously implies (2). And (2) implies (1) by Example 4.2.9. The proof that (2) is equivalent to (3) is as follows: by 5.4.9 we have

$$\overline{\mathrm{E}\Sigma}_{\alpha,\beta,\gamma,\delta;\lambda} = \bar{\tau}_{\alpha x_1 + \beta x_2 + \gamma y_1 + \delta y_2, \lambda^{-1}}$$

and similarly with the second $\overline{\Sigma}$. Then

$$\begin{aligned} \overline{\Sigma}\overline{\Sigma}' = \overline{\Sigma}'\overline{\Sigma} &\Leftrightarrow (\overline{\mathrm{E}\Sigma})(\overline{\mathrm{E}\Sigma}') = (\overline{\mathrm{E}\Sigma}')(\overline{\mathrm{E}\Sigma}) \\ &\Leftrightarrow \bar{\tau}\bar{\tau}' = \bar{\tau}'\bar{\tau} \\ &\Leftrightarrow \alpha\gamma' + \beta\delta' + \gamma\alpha' + \delta\beta' = 0, \end{aligned}$$

using 1.6.8. Q.E.D.

5.4.25. *If σ and σ' are nontrivial hyperbolic transformations in* Sp_4, *then* $\sigma\sigma' = \sigma'\sigma$ *if and only if* $R \cap R' \neq 0$.

PROOF. By 5.4.5 we can write

$$\sigma = \Sigma_{\alpha,\beta,\gamma,\delta;1}, \qquad \sigma' = \Sigma_{\alpha',\beta',\gamma',\delta';1};$$

by 5.4.16 we have

$$R = \Pi(\alpha, \beta, \gamma, \delta), \qquad R' = \Pi(\alpha', \beta', \gamma', \delta');$$

by 5.4.24 we have $\sigma\sigma' = \sigma'\sigma$ if and only if

$$\alpha\gamma' + \beta\delta' + \gamma\alpha' + \delta\beta' = 0,$$

and by 5.4.20 this is equivalent to $R \cap R' \neq 0$. Q.E.D.

5.4.26. *Let* $\Sigma = \Sigma_{\alpha,\beta,\gamma,\delta;\lambda}$ *and* $\Sigma' = \Sigma_{\alpha',\beta',\gamma',\delta';\lambda'}$ *be typical hyperbolic transformations in* $\Gamma\mathrm{Sp}_4$ *which are not radiations. So* $\bar{\Sigma}$ *and* $\bar{\Sigma}'$ *are typical nontrivial projective hyperbolic transformations in* $\mathrm{P}\Gamma\mathrm{Sp}_4$. *Then the following statements are equivalent*:

(1) $\Sigma\Sigma'$ *is a hyperbolic transformation.*

(2) $\bar{\Sigma}\bar{\Sigma}'$ *is a projective hyperbolic transformation.*

(3) $\alpha' = \alpha\xi$, $\beta' = \beta\xi$, $\gamma' = \gamma\xi$, $\delta' = \delta\xi$ *for some* $\xi \in \dot{F}$.

If these equivalent conditions are satisfied, then

$$\Sigma_{\sqrt{\lambda\alpha'^2+\lambda'\alpha^2}, \ldots, \sqrt{\lambda\delta'^2+\lambda'\delta^2};\, \lambda\lambda'}$$

is defined (over F) and equal to $\Sigma\Sigma'$.

PROOF. That (1) implies (2) is obvious. To see that (2) implies (3) express $\bar{\Sigma}\bar{\Sigma}' = \bar{\Sigma}''$ using 5.4.5; then $(\overline{E\Sigma})(\overline{E\Sigma'})$ is a product of two nontrivial projective transvections which is equal to a third projective transvection, namely $(\overline{E\Sigma''})$, by 5.4.9; so the residual lines of $\overline{E\Sigma}$ and $\overline{E\Sigma'}$ are equal by 1.6.6; so the proportion given in (3) holds.

We will now prove that if (3) holds, then so does the last part of the proposition. This will of course show that (3) implies (1). So assume (3). Since Σ and Σ' are defined we must have

$$\lambda + (\alpha\gamma + \beta\delta) \in F^2, \qquad \lambda' + (\alpha'\gamma' + \beta'\delta') \in F^2.$$

Substituting from the proportion gives $\lambda\xi^2 + \lambda'$ in F^2. Multiplying by $\alpha^2, \ldots, \delta^2$ gives

$$\lambda\alpha'^2 + \lambda'\alpha^2, \ldots, \lambda\delta'^2 + \lambda'\delta^2 \in F^2.$$

So

$$\sqrt{\lambda\alpha'^2 + \lambda'\alpha^2}, \ldots, \sqrt{\lambda\delta'^2 + \lambda'\delta^2} \in F.$$

On the other hand,

$$\begin{aligned}
\lambda\lambda' &+ \sqrt{\lambda\alpha'^2 + \lambda'\alpha^2}\sqrt{\lambda\gamma'^2 + \lambda'\gamma^2} + \sqrt{\lambda\beta'^2 + \lambda'\beta^2}\sqrt{\lambda\delta'^2 + \lambda'\delta^2} \\
&= \lambda\lambda' + (\lambda\alpha'\gamma' + \lambda'\alpha\gamma) + (\lambda\beta'\delta' + \lambda'\beta\delta) \\
&= \lambda\lambda' + \lambda(\alpha'\gamma' + \beta'\delta') + \lambda'(\alpha\gamma + \beta\delta) \\
&= (\lambda + (\alpha\gamma + \beta\delta))(\lambda' + (\alpha'\gamma' + \beta'\delta')) + \text{square} \\
&= \text{square}.
\end{aligned}$$

So the big new Σ is indeed defined. By applying E and the formula for the product of two transvections given in §1.4 we find it is equal to $\Sigma\Sigma'$. Q.E.D.

5.4.27. *Let σ and σ' be typical hyperbolic transformations in* Sp_4 *which are not* 1_V. *So $\bar{\sigma}$ and $\overline{\sigma'}$ are typical nontrivial projective hyperbolic transformations in* PSp_4. *And R and R' are totally degenerate planes. Then the following statements are equivalent*:

(1) *$\sigma\sigma'$ is a hyperbolic transformation.*

(2) *$\overline{\sigma\sigma'}$ is a projective hyperbolic transformation.*

(3) *$R = R'$.*

PROOF. By 5.4.5 and 5.4.16 we can express

$$\sigma = \Sigma_{\alpha,\beta,\gamma,\delta;1}, \qquad \sigma' = \Sigma_{\alpha',\beta',\gamma',\delta';1}$$

with

$$\alpha\gamma + \beta\delta \in F^2, \qquad \alpha'\gamma' + \beta'\delta' \in F^2,$$

and

$$R = \Pi(\alpha, \beta, \gamma, \delta), \qquad R' = \Pi(\alpha', \beta', \gamma', \delta').$$

Then (1) is obviously the same as (2). And $\sigma\sigma'$ is a hyperbolic transformation if and only if there is a proportion

$$\alpha' = \alpha\xi, \quad \beta' = \beta\xi, \quad \gamma' = \gamma\xi, \quad \delta' = \delta\xi$$

by 5.4.26, and this is true if and only if $R = R'$ by 5.4.19. So (1) is equivalent to (3). Q.E.D.

Recall from 5.4.15 that the residual space of a nontrivial hyperbolic transformation in Sp_4 is a totally degenerate plane. Accordingly, in the 4-dimensional situation in characteristic 2 that is now under discussion, we say that a subgroup X of $\mathrm{P\Gamma Sp}_4(V)$ has enough projective hyperbolic transformations if, for each totally degenerate plane Π in V, there is at least one hyperbolic transformation σ in $\mathrm{Sp}_4(V)$ with $R = \Pi$ and $\bar{\sigma} \in X$. If we say that our subgroup Δ that is currently under discussion has enough projective hyperbolic transformations, then we mean that Δ has this property in addition to the general assumption that it has enough projective transvections. Similarly with Δ_1.

5.4.28. EXAMPLE. $\mathrm{PSp}_4(V)$ has enough projective hyperbolic transformations.

5.4.29. *Let Π be a totally degenerate plane in V, and let L be a line in Π. Suppose Δ has enough projective hyperbolic transformations. Then there is a hyperbolic transformation σ in* Sp_4 *with $\bar{\sigma} \in \Delta$ such that $\sigma L \neq L$ and $\sigma\Pi = \Pi$.*

PROOF. Take a symplectic base $\mathfrak{X} = \{x_1, x_2 | y_1, y_2\}$ for V with $L = Fx_1$ and $\Pi = Fx_1 + Fx_2$. Here $\Pi = \Pi(1, 0, 0, 0)$. Now $\Pi(0, 1, 0, 1)$ is a totally degenerate plane in V. So there is a hyperbolic transformation σ in Sp_4 with $R = \Pi(0, 1, 0, 1)$ and $\bar{\sigma} \in \Delta$. Then it follows from 5.4.5, 5.4.16 and 5.4.19 that $\sigma = \Sigma_{0,\beta,0,\beta;1}$ for some β in $\dot{F}$. This σ does the job. Q.E.D.

5.4.30. *If Δ and Δ_1 have enough projective hyperbolic transformations, then Λ sends all projective transvections in Δ into* $\mathrm{PSp}_4(V_1)$.

PROOF. If Λ preserves at least one nontrivial projective transvection, then it preserves all projective transvections by 5.2.7; hence it sends all projective transvections in Δ into $\mathrm{PSp}_4(V_1)$. We may therefore assume that Λ does not preserve a single nontrivial projective transvection in Δ.

(1) First let us show that there is at least one nontrivial projective transvection in Δ which is sent into $\mathrm{PSp}_4(V_1)$ by Λ.

In order to verify this we first prove that it is possible to find nontrivial transvections τ_2 and τ_4 in $\mathrm{Sp}_4(V)$ having distinct and orthogonal residual lines L_2 and L_4 such that $\bar{\tau}_2$ and $\bar{\tau}_4$ are in Δ, and such that $\Lambda\bar{\tau}_2$ and $\Lambda\bar{\tau}_4$ can be conjugated into each other using a projective transvection in Δ_1. To see this start with an arbitrary nontrivial projective transvection in Δ_1, pull it back to a hyperbolic transformation in $\mathrm{GSp}_4(V)$ using 5.2.2, take a line that is moved by this hyperbolic transformation (into a distinct orthogonal line, of course), let τ_2 be a transvection in $\mathrm{Sp}_4(V)$ with residual line the first of these lines and with $\bar{\tau}_2$ in Δ, conjugate τ_2 in the usual obvious way to define τ_4, then push things back to V_1 in the obvious way.

So we have the desired τ_2 and τ_4. By 5.2.2, $\Lambda\bar{\tau}_2$ and $\Lambda\bar{\tau}_4$ are projective hyperbolic transformations in Δ_1. Since they can be conjugated into each other by a projective transvection in Δ_1, we can find hyperbolic transformations Σ_1 and Σ_3 having the same multiplier in $\mathrm{GSp}_4(V_1)$ such that

$$\Lambda\bar{\tau}_2 = \bar{\Sigma}_1, \qquad \Lambda\bar{\tau}_4 = \bar{\Sigma}_3.$$

We can assume that $m_{\Sigma_1} = m_{\Sigma_3} = \beta$ with β in $F_1 - F_1^2$ since otherwise $\bar{\Sigma}_1$ would be in $\mathrm{PSp}_4(V_1)$ by 4.2.4 and we would be through. Then by 4.3.4 and the definitions involved there is a symplectic base $\mathfrak{X}_1 = \{x_1, x_2 | y_1, y_2\}$ for V_1 such that

$$\Sigma_1 = \Sigma_{0,\beta,0,1;\beta}.$$

(Note that in this proof $\mathfrak{X}_1$ and the Σ's, etc., appear in the context of V_1 rather than in the context of V.) By 5.4.5 and 5.4.24 we have α, γ, ξ in F_1 with

$$\beta + (\alpha\gamma + \beta\xi^2) \in F_1^2$$

such that

$$\Sigma_3 = \Sigma_{\alpha,\beta\xi,\gamma,\xi;\beta}.$$

Now α, $\beta(\xi + 1)$, γ, $(\xi + 1)$ are elements of F_1, they are not all 0, and they satisfy

$$\alpha\gamma + \beta(\xi + 1)^2 \in F_1^2$$

(if all were 0 we would have $\Sigma_3 = \Sigma_1$). Therefore $\Pi(\alpha, \beta(\xi + 1), \gamma, (\xi + 1))$ defines a totally degenerate plane in V_1. But Δ_1 has enough projective hyperbolic transformations. Hence there is a hyperbolic transformation σ_5 in $\mathrm{Sp}_4(V_1)$ with residual space $\Pi(\alpha, \beta(\xi + 1), \gamma, (\xi + 1))$ such that $\bar{\sigma}_5 \in \Delta_1$. Using 5.4.5, 5.4.16 and 5.4.19 we see that σ_5 must have the form

$$\sigma_5 = \Sigma_{\alpha\eta,\beta(\xi+1)\eta,\gamma\eta,(\xi+1)\eta;1}$$

for some η in $\dot{F}_1$. Let k be an element of $\Gamma\mathrm{Sp}_4(V)$ with $\bar{k}$ in Δ such that $\Lambda\bar{k} = \bar{\sigma}_5$. Let us show that k stabilizes all lines in the totally degenerate plane $L_2 + L_4$. It is enough to show that $\bar{k}$ permutes with $\bar{\tau}$ whenever $\bar{\tau}$ is a projective transvection in Δ whose residual line is contained in $L_2 + L_4$. By 5.2.2, $\Lambda\bar{\tau}$ is a projective hyperbolic transformation, so by 5.4.5 we can express

$$\Lambda\bar{\tau} = \bar{\Sigma}_{p,q,r,s;\lambda}$$

with p, q, r, s in F_1, not all 0, and with λ in $\dot{F}_1$, and with

$$\lambda + pr + qs \in F_1^2.$$

Now $\bar{\tau}$ permutes with $\bar{\tau}_2$ and $\bar{\tau}_4$; hence $\Lambda\bar{\tau}$ permutes with $\bar{\Sigma}_1$ and $\bar{\Sigma}_3$, so by 5.4.24

$$\begin{cases} \beta s + q = 0, \\ \alpha r + \beta\xi s + \gamma p + \xi q = 0, \end{cases}$$

so

$$\alpha r + \beta(\xi + 1)s + \gamma p + (\xi + 1)q = 0,$$

so, again by 5.4.24, $\bar{\sigma}_5$ permutes with $\Lambda\bar{\tau}$. Hence $\bar{k}$ permutes with $\bar{\tau}$. So k does indeed stabilize all lines in the totally degenerate plane $L_2 + L_4$. We can, in fact, assume that the action of k on this plane is 1, by 4.1.6. This puts k in $\mathrm{GSp}_4(V)$. Now $\bar{k}$ is an involution since $\bar{\sigma}_5$ is, so k^2 is a radiation, so $k^2 = 1_V$, so $(m_k)^2 = 1$. So k is in fact in $\mathrm{Sp}_4(V)$ with $\operatorname{res} k \leqslant 2$. If $\operatorname{res} k = 1$, we have proved our assertion. So let $\operatorname{res} k = 2$.

Taking a fresh notation we see that we now have elements σ and σ_1 with

$$\begin{array}{ll} \sigma \in \mathrm{Sp}_4(V) & \sigma_1 \in \mathrm{Sp}_4(V_1) \\ \bar{\sigma} \in \Delta & \bar{\sigma}_1 \in \Delta_1 \\ R \in \mathcal{P} & R_1 \in \mathcal{P}_1 \\ & \sigma_1 \text{ hyperbolic} \end{array}$$

and $\Lambda\bar{\sigma} = \bar{\sigma}_1$. By standard arguments, starting with an appropriate conjugation on the right by a suitable transvection in $\mathrm{Sp}_4(V_1)$ that is projectively in Δ_1, we can find another pair σ', σ_1' having the same properties as σ, σ_1, and such that

$$R_1 \cap R_1' = \text{a line}.$$

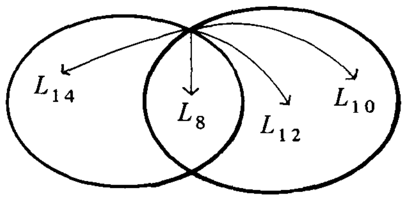
Then σ_1 and σ_1' permute by 5.4.25; hence σ and σ' permute; hence $R \cap R' \neq 0$ by 1.3.6. Let T be a nontrivial transvection in $\mathrm{Sp}_4(V)$ that is projectively in Δ and has residual line contained in $R \cap R'$. Then T permutes with σ and σ' by 1.3.11; hence $\Lambda\bar{\mathrm{T}}$ permutes with $\bar{\sigma}_1$ and $\overline{\sigma_1'}$; hence $\Lambda\bar{\mathrm{T}}$ has a representative in $\Gamma\mathrm{Sp}_4(V_1)$ which is both a hyperbolic transformation and permutes with σ and σ_1'; this representative must therefore stabilize the line $R_1 \cap R_1'$; this puts $\Lambda\bar{\mathrm{T}}$ in $\mathrm{PSp}_4(V_1)$ by 5.4.3. We have proved (1).

(2) Next let us show that if we have nontrivial projective transvections $\bar{\tau}_8$ and $\bar{\tau}_{10}$ in Δ with orthogonal and distinct residual lines L_8 and L_{10}, and if $\Lambda\bar{\tau}_8$ is in $\mathrm{PSp}_4(V_1)$, then $\Lambda\bar{\tau}_{10}$ is also in $\mathrm{PSp}_4(V_1)$. Let $\Pi = L_8 + L_{10}$. By 5.4.29 there is a nontrivial projective transvection $\bar{\tau}_{12}$ in Δ which is conjugate to $\bar{\tau}_8$ and such that $\Pi = L_8 + L_{12}$. Clearly $\Lambda\bar{\tau}_{12} \in \mathrm{PSp}_4(V_1)$. Since $\Lambda\bar{\tau}_8$ and $\Lambda\bar{\tau}_{12}$ are projective

hyperbolic transformations in $\mathrm{PSp}_4(V_1)$, there are hyperbolic transformations σ_7 and σ_{11} in $\mathrm{Sp}_4(V_1)$ with

$$\Lambda\bar{\tau}_8 = \bar{\sigma}_7 \quad \text{and} \quad \Lambda\bar{\tau}_{12} = \bar{\sigma}_{11}.$$

Of course R_7 and R_{11} are totally degenerate planes by 5.4.15. Now by conjugating $\bar{\tau}_8$ by a suitable projective



hyperbolic transformation in Δ (use 5.4.29) we can find $\bar{\tau}_{14}$ permuting with $\bar{\tau}_8$ but not with $\bar{\tau}_{12}$. Hence we can find a hyperbolic transformation which is in $\mathrm{Sp}_4(V_1)$ and projectively in Δ_1, and which permutes with σ_7 but not with σ_{11}. Therefore $R_7 \neq R_{11}$ by 5.4.25. So $R_7 \cap R_{11}$ is a line, again by 5.4.25. Then $\bar{\tau}_{10}$ permutes with $\bar{\tau}_8$ and $\bar{\tau}_{12}$; hence $\Lambda\bar{\tau}_{10}$ is a projective hyperbolic transformation permuting with $\bar{\sigma}_7$ and $\bar{\sigma}_{11}$; hence $\Lambda\bar{\tau}_{10}$ has a representative which is a hyperbolic transformation permuting with σ_7 and σ_{11}, so this representative must stabilize the line $R_7 \cap R_{11}$, so $\Lambda\bar{\tau}_{10}$ is in $\mathrm{PSp}_4(V_1)$ by 4.3.3, as required.

(3) To complete the proof we must show that a typical nontrivial projective transvection $\bar{\tau}$ in Δ is carried into $\mathrm{PSp}_4(V_1)$ by Λ. By step (1) we know that this happens at least once, say to $\bar{\tau}_0$. Let L and L_0 be the residual lines of $\bar{\tau}$ and $\bar{\tau}_0$. If L and L_0 are distinct and orthogonal, apply step (2). If L and L_0 are distinct and nonorthogonal, take $\bar{\tau}_*$ with L_* orthogonal to L and L_0, and apply step (2) twice. Similarly if $L = L_0$. Q.E.D.

5.4.31. Remark. Note that 5.4.30 need not hold if we drop the assumption about enough projective hyperbolic transformations. To illustrate this consider $V = V_1$ with $F = F_1$ imperfect and pick $\lambda \in F - F^2$. We saw in Remark 5.4.11 that $\bar{\mathrm{E}}$ provides an isomorphism $\bar{\mathrm{E}}$: $\mathrm{PGSp}_4 \rightarrowtail \bar{\mathrm{E}}\mathrm{PGSp}_4$ between the two groups PGSp_4 and $\bar{\mathrm{E}}\mathrm{PGSp}_4$ which have enough projective transvections. Then

$$\bar{\mathrm{E}}\bar{\Sigma}_{\lambda,0,1,0;\lambda} = \bar{\tau}_{\lambda x_1 + y_1, \lambda^{-1}}$$

by 5.4.9, so $\bar{\mathrm{E}}^{-1}$ carries the projective transvection on the right to the $\bar{\Sigma}$ on the left which, by 4.2.4, is not in PSp_4.

5.4.32. Remark. It follows from 5.4.30 and Remark 5.4.31 that there exist groups which have enough projective transvections but do not have enough projective hyperbolic transformations.

5.4.33. *If Δ and Δ_1 have enough projective hyperbolic transformations and if Λ does not preserve projective transvections, then F and F_1 are perfect.*

Proof. It is enough to prove that the field F_1 is perfect.

(1) First let us establish a bijection j: $\mathfrak{L} \rightarrowtail \mathfrak{P}_1$ of the lines $\mathfrak{L}$ of V onto the totally degenerate planes $\mathfrak{P}_1$ of V_1, with the property that

$$q(L, L') = 0 \Leftrightarrow jL \cap jL' \neq 0.$$

To define j, take a typical line L in V, let $\bar{\tau}$ be a projective transvection in Δ with residual line L; then $\Lambda\bar{\tau}$ is a projective hyperbolic transformation in $\mathrm{PSp}_4(V_1)$ by 5.2.7, 5.2.2 and 5.4.30, so there is a hyperbolic transformation σ_1 in $\mathrm{Sp}_4(V_1)$ with $\Lambda\bar{\tau} = \bar{\sigma}_1 \in \Delta_1$; the residual space of σ_1 is a totally degenerate plane Π_1 by 5.4.15; we define $jL = \Pi_1$. That jL is well defined is a consequence of 1.6.6 and 5.4.27. To see that j is injective consider $L \neq L'$; pick L'' orthogonal to L but not to L'; construct appropriate $\bar{\tau}, \overline{\tau'}, \overline{\tau''}$; these lead to $\sigma, \sigma', \sigma''$ with their totally degenerate planes Π_1, Π_1', Π_1''; then $\overline{\tau''}$ permutes with $\bar{\tau}$ but not with $\overline{\tau'}$; hence $\Pi_1'' \cap \Pi_1 \neq 0$ but $\Pi_1'' \cap \Pi_1' = 0$, by 5.4.25; hence $\Pi_1 \neq \Pi_1'$; but $jL = \Pi_1$ and $jL' = \Pi_1'$; so j is indeed injective. To prove that j is surjective consider a totally degenerate plane Π_1 in V_1; let $\bar{\tau}_3$ and $\bar{\tau}_5$ be nontrivial projective transvections in Δ_1 whose residual lines span Π_1; move them to the left and up to a pair of hyperbolic transformations σ_2 and σ_4 in $\mathrm{Sp}_4(V)$; on grounds of permutability we must have $R_2 \cap R_4 \neq 0$; let $\bar{\tau}$ be a projective transvection in Δ whose residual line L is contained in $R_2 \cap R_4$; move $\bar{\tau}$ to the right and up to a hyperbolic transformation σ_1 in $\mathrm{Sp}_4(V_1)$; then σ_1 permutes with τ_3 and τ_5; so σ_1 stabilizes L_3 and L_5; but σ_1 is an involution being totally degenerate; so $R_1 = L_3 + L_5 = \Pi_1$; so j is surjective, as asserted. The condition

$$q(L, L') = 0 \Leftrightarrow jL \cap jL' \neq 0$$

follows easily by interpreting each side in terms of the permutability of corresponding transformations.

(2) Now fix a symplectic base $\mathfrak{X}_1 = \{x_1, x_2 | y_1, y_2\}$ for V_1 and let $l\colon \mathcal{P}_1 \rightarrowtail \mathfrak{L}_1$ be the injection defined in 5.4.21 (for V_1, not for V, of course). Then $l \circ j$ is an injection

$$l \circ j\colon \mathfrak{L} \overset{j}{\rightarrowtail} \mathcal{P}_1 \overset{l}{\rightarrowtail} \mathfrak{L}_1$$

which preserves orthogonality. Let $\mathfrak{L}_0$ be the set of lines of $\mathfrak{L}_1$ which are images under $l \circ j$, i.e., but $\mathfrak{L}_0 = l\mathcal{P}_1$. If we can prove that $\mathfrak{L}_0 = \mathfrak{L}_1$ we will be through by 5.4.21. Now if $\alpha_1, \beta_1, \gamma_1, \delta_1$ are elements of F_1, not all 0, with $\alpha_1\gamma_1 + \beta_1\delta_1 \in F_1^2$, then $\Pi_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ is defined and in $\mathcal{P}_1$; hence $L_1(\alpha_1, \beta_1, \gamma_1, \delta_1)$ is in $\mathfrak{L}_0$ by definition of l; so the first condition of Lemma 5.4.22 holds for $\mathfrak{L}_0$: we must verify the second, and this is done as in the proof of 5.4.23. Q.E.D.

5.4.34. *If Δ and Δ_1 are actually subgroups of* $\mathrm{PSp}_4(V)$ *and* $\mathrm{PSp}_4(V_1)$ *respectively, and if Λ does not preserve projective transvections, then Δ and Δ_1 have enough projective hyperbolic transformations, and F and F_1 are perfect.*

Proof. By 5.4.33 it is enough to show that Δ and Δ_1 have enough projective hyperbolic transformations. On grounds of symmetry it is enough to prove that Δ_1 has enough. Accordingly consider a totally degenerate plane Π_1 in V_1. Take two lines L_3 and L_5 with $\Pi_1 = L_3 + L_5$. Let $\bar{\tau}_3$ and $\bar{\tau}_5$ be projective transvections in Δ_1 with residual lines L_3 and L_5, respectively. Pull $\bar{\tau}_3$ and $\bar{\tau}_5$ to the left and up to hyperbolic transformations σ_2 and σ_4 in $\mathrm{Sp}_4(V)$ (note that we have just used the fact that $\Delta \subseteq \mathrm{PSp}_4(V)$). On grounds of permutability we must have $R_2 \cap R_4 \neq 0$. Let $\bar{\tau}$ be a nontrivial projective transvection in Δ whose residual line is contained in $R_2 \cap R_4$. Then $\bar{\tau}$ permutes with $\bar{\sigma}_2$ and $\bar{\sigma}_4$; hence $\Lambda\bar{\tau}$ lifts to a hyperbolic transformation σ_1 in $\mathrm{Sp}_4(V_1)$ that permutes with τ_3 and τ_5. From this it follows that $R_1 = L_3 + L_5 = \Pi_1$. So we have a hyperbolic transformation σ_1 in $\mathrm{Sp}_4(V_1)$ with $\bar{\sigma}_1 = \Lambda\bar{\tau}$ in Δ_1 and $R_1 = \Pi_1$. So Δ_1 has enough projective hyperbolic transformations. Q.E.D.

5.4.35. *If Δ and Δ_1 have enough projective hyperbolic transformations, and if Λ does not preserve projective transvections, then Λ carries the set of projective transvections in Δ onto the set of projective hyperbolic transformations in Δ_1, and the set of projective hyperbolic transformations in Δ onto the set of projective transvections in Δ_1.*

PROOF. By considering Λ^{-1} instead of Λ we see that it suffices to prove the first part of the proposition. By 5.2.7 and 5.2.2 we see that it suffices to prove that every projective hyperbolic transformation $\bar{\sigma}_1$ in Δ_1 is the image under Λ of a projective transvection in Δ. Since F_1 is perfect by 5.4.33, we have $\mathrm{PGSp}_4(V_1) = \mathrm{PSp}_4(V_1)$ by 4.2.4, so we may assume that our σ_1 actually is a hyperbolic transformation in $\mathrm{Sp}_4(V_1)$. By the argument used in proving surjectivity in the proof of 5.4.33, there is a nontrivial projective transvection $\bar{\tau}$ in Δ such that $\Lambda\bar{\tau} = \bar{\sigma}_3 \in \Delta_1$ for some hyperbolic transformation σ_3 in $\mathrm{Sp}_4(V_1)$ having $R_3 = R_1$. Let L be the residual line of $\bar{\tau}$. If L_* is any line in V that is orthogonal to L, take a nontrivial projective transvection $\bar{\tau}_*$ in Δ with residual line L_*; then $\bar{\tau}_*$ permutes with $\bar{\tau}$, so $\Lambda\bar{\tau}_*$ permutes with $\Lambda\bar{\tau}$, but $\Lambda\bar{\tau}_*$ has the form $\Lambda\bar{\tau}_* = \bar{\sigma}_* \in \Delta_1$ for some hyperbolic transformation σ_* in $\mathrm{Sp}_4(V_1)$ by arguments already used; then σ_* permutes with σ_3, so $R_* \cap R_3 \neq 0$, so $R_* \cap R_1 \neq 0$, so σ_* permutes with σ_1, so $\bar{\tau}_*$ permutes with $\Lambda^{-1}\bar{\sigma}_1$; hence $\Lambda^{-1}\bar{\sigma}_1$ has a representative in $\Gamma\mathrm{Sp}_4(V)$ which stabilizes all lines in the hyperplane L^* of V; hence $\Lambda^{-1}\bar{\sigma}_1$ has a representative in $\Gamma\mathrm{Sp}_4(V)$ which is the identity map on L^*; this representative is in $\mathrm{Sp}_4(V)$ by 4.2.6, and therefore a transvection in $\mathrm{Sp}_4(V)$; so $\Lambda^{-1}\bar{\sigma}_1$ is a projective transvection in Δ, i.e., $\bar{\sigma}_1$ is the image under Λ of a projective transvection in Δ. Q.E.D.

5.4.36. DEFINITION. An automorphism A of $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ is called exceptional if it does not preserve projective transvections.

5.4.37. *The group* $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ *has an exceptional automorphism if and only if the field F is perfect.*

PROOF. If F is perfect, then the isomorphism $\bar{\mathrm{E}}$ of §5.4 provides an automorphism of $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ by 5.4.3, and this automorphism does not preserve projective transvections by 5.4.8, so $\bar{\mathrm{E}}$ is exceptional. Conversely, if $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ has an exceptional automorphism, then F is perfect by 5.4.33. Q.E.D.

5.4.38. REMARK. There is no point in defining exceptional automorphisms of $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$ unless $n = 4$ with F perfect of characteristic 2, by 5.2.8 and 5.4.37.

5.4.39. *If A is an exceptional automorphism of* $\mathrm{P}\Gamma\mathrm{Sp}_4(V)$, *and if Δ has enough projective hyperbolic transformations, then AΔ has a projective transvection with residual line L for any line L in V.*

PROOF. Here F is perfect since Λ exists. By 5.4.35 and 4.2.4 there is a hyperbolic transformation σ in $\mathrm{Sp}_4(V)$ such that $\Lambda\bar{\sigma}$ is a nontrivial projective transvection with residual line L. Since Δ has enough projective hyperbolic transformations there is a hyperbolic transformation σ_2 in $\mathrm{Sp}_4(V)$ with $\bar{\sigma}_2$ in Δ and $R_2 = R$. Then $\bar{\sigma}$, $\bar{\sigma}_2$, $\bar{\sigma}\bar{\sigma}_2$ are projective hyperbolic transformations in $\mathrm{P\Gamma Sp}_4(V)$, by 5.4.27. Hence $\Lambda\bar{\sigma}$, $\Lambda\bar{\sigma}_2$, $\Lambda\bar{\sigma}\bar{\sigma}_2$ are projective transvections by 5.4.35; hence the residual lines of $\Lambda\bar{\sigma}$ and $\Lambda\bar{\sigma}_2$ are equal by 1.6.6. In other words, $\Lambda\bar{\sigma}_2$ is a projective transvection in $\Lambda\Delta$ with residual line L. Q.E.D.

5.4.40. THEOREM. *Let Δ and Δ_1 be subgroups of $\mathrm{P\Gamma Sp}_4(V)$ and $\mathrm{P\Gamma Sp}_4(V_1)$ over fields F and F_1 of characteristic 2. Suppose Δ and Δ_1 have enough projective transvections and enough projective hyperbolic transformations. If F is perfect, let E_0 denote a fixed exceptional automorphism of $\mathrm{P\Gamma Sp}_4(V)$. Then each isomorphism $\Lambda\colon \Delta \rightarrowtail \Delta_1$ has exactly one of the forms*

$$\Lambda k = gkg^{-1} \qquad \forall\, k \in \Delta$$

or

$$\Lambda k = g(\mathrm{E}_0 k)g^{-1} \qquad \forall\, k \in \Delta$$

for a unique projective symplectic collinear transformation g of V onto V_1, the second possibility appearing only when F is perfect.

PROOF. If Λ preserves projective transvections, apply 5.3.5. If Λ does not preserve projective transvections, then F and F_1 are perfect by 5.4.33, in particular E_0 is defined. By 5.4.39 we know that $\mathrm{E}_0\Delta$ has at least one projective transvection with residual line L for each line L in V (in particular $\mathrm{E}_0\Delta$ has enough projective transvections if $F = \mathbf{F}_2$). If we take a nontrivial projective transvection $\bar{\mathrm{T}}$ in $\mathrm{E}_0\Delta$, then $\mathrm{E}_0^{-1}\bar{\mathrm{T}}$ is a projective hyperbolic transformation in Δ by 5.2.7 and 5.2.2; hence $\Lambda\mathrm{E}_0^{-1}\bar{\mathrm{T}}$ is a projective transvection in Δ_1 by 5.4.35; hence $\Lambda\mathrm{E}_0^{-1}\colon \mathrm{E}_0\Delta \rightarrowtail \Delta_1$ sends projective transvections in $\mathrm{E}_0\Delta$ to projective transvections in Δ_1. If $F \neq \mathbf{F}_2$, then it follows from 5.1.4 that $F_1 \neq \mathbf{F}_2$, so Δ_1 has at least two projective transvections with residual line L_1 for each line L_1 in V_1, and it then follows by a standard argument that $\mathrm{E}_0\Delta$ has the same property, i.e., that $\mathrm{E}_0\Delta$ has enough projective transvections. So, whether F is $\mathbf{F}_2$ or not, $\mathrm{E}_0\Delta$ has enough projective transvections. Hence $\Lambda\mathrm{E}_0^{-1}\colon \mathrm{E}_0\Delta \rightarrowtail \Delta_1$ preserves projective transvections by 5.2.7. So, by 5.3.5, there is a unique projective collinear transformation g of V onto V_1 such that

$$\Lambda k = \Lambda\mathrm{E}_0^{-1}(\mathrm{E}_0 k) = g(\mathrm{E}_0 k)g^{-1} \qquad \forall\, k \in \Delta.$$

Finally we must show that Λ cannot have both the $\Lambda k = g_* k g_*^{-1}$ and $\Lambda k = g(\mathrm{E}_0 k)g^{-1}$ forms. For the first equation would imply that $\Lambda\mathrm{T}$ is a projective transvection whenever T is, while the second implies that it is not. Q.E.D.

5.4.40A. THEOREM. *Isomorphic projective groups of symplectic collinear transformations with underlying characteristics 2 and underlying dimensions 4, and with enough projective transvections and enough projective hyperbolic transformations, have isomorphic underlying fields.*

Recall that in §5.3 we defined what we meant by saying that a subgroup of $\Gamma Sp_4(V)$ had enough transvections. Also recall that the residual space of a nontrivial hyperbolic transformation in $Sp_4(V)$ is a totally degenerate plane. Accordingly, in the 4-dimensional situation in characteristic 2 that is now under discussion, we say that a subgroup X of $\Gamma Sp_4(V)$ has enough hyperbolic transformations if, for each totally degenerate plane Π in V, there is at least one hyperbolic transformation σ in $Sp_4(V)$ with $R = \Pi$ and $\sigma \in X$. We let Γ denote a subgroup of $\Gamma Sp_4(V)$ that has enough transvections and enough hyperbolic transformations. Similarly with Γ_1 in $\Gamma Sp_4(V_1)$. And we let $\Phi\colon \Gamma \rightarrowtail \Gamma_1$ denote a group isomorphism between them. Note that $\bar{\Gamma} = P\Gamma$ is a subgroup of $P\Gamma Sp_4(V)$ with enough projective transvections and enough projective hyperbolic transformations. Similarly with $\bar{\Gamma}_1 = P\Gamma_1$ in $P\Gamma Sp_4(V_1)$.

5.4.41. DEFINITION. Let B be any automorphism of $\Gamma Sp_4(V)$. Then B naturally induces an automorphism $\bar{B}$ of $P\Gamma Sp_4(V)$ by defining $\bar{B}\bar{k} = \overline{Bk}$ for all $\bar{k}$ in $P\Gamma Sp_4(V)$, by 5.3.7. We call B an exceptional automorphism of $\Gamma Sp_4(V)$ if $\bar{B}$ is an exceptional automorphism of $P\Gamma Sp_4(V)$.

5.4.42. *The group* $\Gamma Sp_4(V)$ *has an exceptional automorphism if and only if* F *is perfect.*

PROOF. See 5.4.37 and its proof. Q.E.D.

5.4.43. THEOREM. *Let* Γ *and* Γ_1 *be subgroups of* $\Gamma Sp_4(V)$ *and* $\Gamma Sp_4(V_1)$ *over fields* F *and* F_1 *of characteristic* 2. *Suppose* Γ *and* Γ_1 *have enough transvections and enough hyperbolic transformations. If* F *is perfect, let* E_2 *denote a fixed exceptional automorphism of* $\Gamma Sp_4(V)$. *Then each isomorphism* $\Phi\colon \Gamma \rightarrowtail \Gamma_1$ *has exactly one of the forms*

$$\Phi k = \chi(k)gkg^{-1} \qquad \forall\, k \in \Gamma$$

or

$$\Phi k = \chi(k)g(E_2k)g^{-1} \qquad \forall\, k \in \Gamma$$

for a mapping χ *of* Γ *into* $RL_4(V_1)$ *and a symplectic collinear transformation* g *of* V *onto* V_1, *the second possibility appearing only when* F *is perfect.*

PROOF. Apply 5.3.7, Theorem 5.4.40 and 4.2.1. Q.E.D.

5.4.43A. THEOREM. *Isomorphic groups of symplectic collinear transformations with underlying characteristics* 2 *and underlying dimensions* 4, *and with enough transvections and enough hyperbolic transformations, have isomorphic underlying fields.*

5.5. Bounded Modules over Integral Domains

We now consider an arbitrary (commutative) integral domain $\mathfrak{o}$. We let F be a field of quotients of $\mathfrak{o}$. So $1 \in \mathfrak{o} \subseteq F$. Later $\mathfrak{o}_1$ will be a second integral domain with field of quotients F_1. In §5.5 we assume that V is just an abstract vector space over F with $1 \leqslant \dim < \infty$, i.e., we ignore the alternating form which our general assumptions say it must possess.

By a (fractional) ideal $\mathfrak{a}$ with respect to $\mathfrak{o}$ we mean a nonzero subset $\mathfrak{a}$ of F which is an $\mathfrak{o}$-module in the natural way, and which satisfies $\lambda\mathfrak{a} \subseteq \mathfrak{o}$ for some nonzero λ in $\mathfrak{o}$. Here $\lambda\mathfrak{a}$ stands for the $\mathfrak{o}$-module

$$\lambda\mathfrak{a} = \{\lambda x | x \in \mathfrak{a}\}.$$

It is clear that $\alpha\mathfrak{o}$ is a fractional ideal for any α in $\dot{F}$. Any fractional ideal which can be expressed in the form $\alpha\mathfrak{o}$ for some α in $\dot{F}$ will be called a principal ideal. If $\mathfrak{a}$ is a fractional ideal, then so is $\alpha\mathfrak{a}$ for all α in $\dot{F}$. Every finitely generated nonzero $\mathfrak{o}$-module that is contained in F in the natural way is a fractional ideal. For any two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ it is easily seen that there is a nonzero λ in $\mathfrak{o}$ such that $\lambda\mathfrak{a} \subseteq \mathfrak{b}$. If F is $\mathfrak{o}$, then F is the only fractional ideal with respect to $\mathfrak{o}$.

An integral ideal is a fractional ideal that is contained in $\mathfrak{o}$. Thus the integral ideals are the ideals of $\mathfrak{o}$ in the usual sense of the word, with the exception of 0. Every integral ideal satisfies $0 \subset \mathfrak{a} \subseteq \mathfrak{o}$.

For any two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ define:

g.c.d.: $\mathfrak{a} + \mathfrak{b} = \{\alpha + \beta | \alpha \in \mathfrak{a}, \beta \in \mathfrak{b}\}$,

l.c.m.: $\mathfrak{a} \cap \mathfrak{b}$,

product: $\mathfrak{a}\mathfrak{b} = \left\{ \sum_{\text{finite}} \alpha\beta | \alpha \in \mathfrak{a}, \beta \in \mathfrak{b}\right\}$.

It is easily verified that $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$ are again fractional ideals. The following laws are evident:

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}, \qquad (\alpha\mathfrak{a})(\beta\mathfrak{b}) = (\alpha\beta)(\mathfrak{a}\mathfrak{b}),$$
$$\mathfrak{a}(\mathfrak{b}\mathfrak{c}) = (\mathfrak{a}\mathfrak{b})\mathfrak{c}, \qquad \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}, \qquad \mathfrak{a}\mathfrak{o} = \mathfrak{a}.$$

We say that the fractional ideal $\mathfrak{a}$ is invertible if there is a fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathfrak{o}$; this $\mathfrak{b}$, if it exists, must be unique and we then write $\mathfrak{a}^{-1} = \mathfrak{b}$ for the inverse of $\mathfrak{a}$. Every principal ideal is invertible with $(\alpha\mathfrak{o})^{-1} = \alpha^{-1}\mathfrak{o}$. So the set of invertible ideals forms a commutative group under product formation, and the set of principal ideals is a subgroup.

By an $\mathfrak{o}$-module M in the vector space V we mean a subset M of V that is an $\mathfrak{o}$-module in the natural way. We say that an $\mathfrak{o}$-module M in V is on V if it spans V over F. It is easily seen that if M is in V, then it is on V if and only if it contains a base for V.

Consider an $\mathfrak{o}$-module M in V. Define

$$FM = \{\alpha x | \alpha \in F, x \in M\}.$$

We have

$$FM = \{\alpha^{-1}x | \alpha \in \mathfrak{o}, \alpha \neq 0, x \in M\}$$

since F is a quotient field of $\mathfrak{o}$. Hence FM is a subspace of V, in fact the subspace of V that is spanned by M. Thus M is on V if and only if $FM = V$.

For any α in F, and any fractional ideal $\mathfrak{a}$, and any M in V, define

$$\alpha M = \{\alpha x | x \in M\}, \quad \mathfrak{a}M = \left\{ \sum_{\text{finite}} \beta x | \beta \in \mathfrak{a}, x \in M\right\}.$$

Both αM and $\mathfrak{a}M$ are again modules in V, in fact in FM, and the following laws are easily seen to hold:

$$\alpha(M \cap N) = (\alpha M) \cap (\alpha N), \qquad \alpha(M + N) = \alpha M + \alpha N,$$

$$\alpha(\mathfrak{a}M) = (\alpha\mathfrak{a})M = \mathfrak{a}(\alpha M),$$

$$(\mathfrak{a} + \mathfrak{b})M = \mathfrak{a}M + \mathfrak{b}M, \qquad (\mathfrak{a}\mathfrak{b})M = \mathfrak{a}(\mathfrak{b}M),$$

$$\mathfrak{a}(M + N) = \mathfrak{a}M + \mathfrak{a}N,$$

$$F(M + N) = FM + FN.$$

A subset of vectors of V is independent over $\mathfrak{o}$ if and only if it is independent over F. In particular, an $\mathfrak{o}$-module M on V is free if and only if there is a base $x_1, \ldots, x_n$ for V such that $M = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$.

We say that an $\mathfrak{o}$-module M on V is bounded if it is contained in a free $\mathfrak{o}$-module on V. Thus free $\mathfrak{o}$-modules on V are bounded. And a bounded module on V contains a free module on V, and is contained in one too. A submodule of a bounded module is bounded if it is on V.

5.5.1. *Let M and N be $\mathfrak{o}$-modules on V with N free. Then M is bounded if and only if $\alpha M \subseteq N$ holds for some nonzero α in $\mathfrak{o}$.*

PROOF. If $\alpha M \subseteq N$, then M is contained in the free module $\alpha^{-1}N$, so M is bounded. Conversely, suppose M is bounded. So $M \subseteq P$ with P free on V. Take bases in which

$$N = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n, \qquad P = \mathfrak{o}y_1 + \cdots + \mathfrak{o}y_n$$

and write

$$y_j = \sum_{i=1}^{n} a_{ij}x_i \qquad (a_{ij} \in F).$$

Let α be the product of the denominators in the a_{ij}. So α is a nonzero element of $\mathfrak{o}$ with $\alpha a_{ij} \in \mathfrak{o}$ for all i, j. We have $\alpha M \subseteq \alpha P \subseteq N$. Q.E.D.

5.5.2. *Let M and N be $\mathfrak{o}$-modules on V with N bounded. Then M is bounded if and only if $\alpha M \subseteq N$ holds for some nonzero α in $\mathfrak{o}$.*

It follows from the above results that if M and N are bounded $\mathfrak{o}$-modules on V, then so are αM, $\mathfrak{a}M$, $M \cap N$, and $M + N$. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are fractional ideals, and $z_1, \ldots, z_r$ are vectors spanning V, then

$$\mathfrak{a}_1 z_1 + \cdots + \mathfrak{a}_r z_r$$

is a bounded $\mathfrak{o}$-module on V. In particular, any finitely generated $\mathfrak{o}$-module on V is a bounded $\mathfrak{o}$-module on V. Note that if z_1 and z_2 are independent in V, then

$$\mathfrak{a}_1 z_1 + \mathfrak{a}_2 z_2 = \mathfrak{a}_1 z_1 + \mathfrak{a}_2(z_2 + \lambda z_1)$$

if and only if $\lambda\mathfrak{a}_2 \subseteq \mathfrak{a}_1$. Also note that $\mathfrak{a}(\mathfrak{b}\mathfrak{c}) = (\mathfrak{a}\mathfrak{b})\mathfrak{c}$. In particular,

$$\mathfrak{a}(\mathfrak{a}_1 z_1 + \cdots + \mathfrak{a}_r z_r) = (\mathfrak{a}\mathfrak{a}_1)z_1 + \cdots + (\mathfrak{a}\mathfrak{a}_r)z_r.$$

For any bounded $\mathfrak{o}$-module M on V, and for any nonzero x in V, the coefficient $\mathfrak{c}_x$ of x with respect to M is defined as the set

$$\mathfrak{c}_x = \{\alpha \in F | \alpha x \in M\}.$$

Clearly $\mathfrak{c}_x x = M \cap Fx$ and, using 5.5.2, we find that $\mathfrak{c}_x$ is actually a fractional ideal. Note that

$$\alpha \mathfrak{c}_{\alpha x} = \mathfrak{c}_x \qquad \forall\, \alpha \in \dot{F}.$$

If $x_1, \ldots, x_n$ is a base for V and

$$M = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n$$

with the $\mathfrak{a}_i$ fractional ideals, then we know that M is a bounded $\mathfrak{o}$-module on V, so the coefficient $\mathfrak{c}_x$ is defined for any nonzero x in V. If we express

$$x = \beta_1 x_1 + \cdots + \beta_p x_p$$

with $\beta_1, \ldots, \beta_p$ nonzero, then it is easily seen that

$$\mathfrak{c}_x = (\beta_1^{-1}\mathfrak{a}_1) \cap \cdots \cap (\beta_p^{-1}\mathfrak{a}_p).$$

In particular $\mathfrak{c}_{x_1} = \mathfrak{a}_1$.

5.5.3. Definition. By an $\mathfrak{o}$-lattice M on V we mean a subset M of V which can be expressed in the form

$$M = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n$$

for some base $x_1, \ldots, x_n$ for V, and some invertible fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$.

So for any $\mathfrak{o}$-module M on V we have

$$M \text{ is free} \Rightarrow M \text{ is a lattice}$$
$$\Rightarrow M \text{ is bounded.}$$

If M is a lattice on V, then so are αM and $\mathfrak{a} M$ lattices on V, for any α in $\dot{F}$ and any invertible ideal $\mathfrak{a}$.

5.5.4. *Let M and N be $\mathfrak{o}$-modules on V of the form*

$$M = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n, \qquad N = \mathfrak{b}_1 y_1 + \cdots + \mathfrak{b}_n y_n,$$

with the x's and y's bases for V, and the $\mathfrak{a}$'s and $\mathfrak{b}$'s fractional ideals. Let (a_{ij}) be the matrix defined by $y_j = \sum_i a_{ij} x_i$. Then

(1) *$N \subseteq M$ if and only if $a_{ij}\mathfrak{b}_j \subseteq \mathfrak{a}_i$ for all i, j.*

(2) *If $N = M$, then $\mathfrak{a}_1 \cdots \mathfrak{a}_n = \mathfrak{b}_1 \cdots \mathfrak{b}_n \det(a_{ij})$.*

(3) *M is a lattice if and only if each $\mathfrak{a}_i$ is invertible.*

(4) *If N and M are lattices with $N \subseteq M$, then $N = M$ if and only if*

$$\mathfrak{a}_1 \cdots \mathfrak{a}_n = \mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot \det(a_{ij}).$$

Proof. Let (b_{ij}) denote the inverse of the matrix (a_{ij}). So $x_j = \sum_i b_{ij} y_i$. The proof of (1) is easy and consists of a comparison of coefficients in the x-base. Let us prove (2). Here $N = M$. So by part (1),

$$\begin{aligned} \mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot \det(a_{ij}) &= \mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot (\Sigma \pm a_{1\alpha} \cdots a_{n\omega}) \\ &\subseteq \Sigma(a_{1\alpha}\mathfrak{b}_\alpha) \cdots (a_{n\omega}\mathfrak{b}_\omega) \\ &\subseteq \Sigma \mathfrak{a}_1 \cdots \mathfrak{a}_n, \end{aligned}$$

i.e.,

$$\mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot \det(a_{ij}) \subseteq \mathfrak{a}_1 \cdots \mathfrak{a}_n.$$

On grounds of symmetry,

$$\mathfrak{a}_1 \cdots \mathfrak{a}_n \det(b_{ij}) \subseteq \mathfrak{b}_1 \cdots \mathfrak{b}_n.$$

But (a_{ij}) is the inverse of (b_{ij}). So (2) follows. Now (3). If each $\mathfrak{a}_i$ is invertible, then M is a lattice by definition. Conversely let M be a lattice. Then M has the form of N with all $\mathfrak{b}_i$ invertible, by definition of a lattice. So

$$\mathfrak{a}_1 \cdots \mathfrak{a}_n = \mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot \det(a_{ij})$$

by part (2). So $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ is an invertible fractional ideal. So each $\mathfrak{a}_i$ is invertible. Finally let us prove (4). By part (2) it is enough to consider $N \subseteq M$ with

$$\mathfrak{a}_1 \cdots \mathfrak{a} = \mathfrak{b}_1 \cdots \mathfrak{b}_n \cdot \det(a_{ij})$$

and deduce that $N = M$. Here the $\mathfrak{a}$'s and $\mathfrak{b}$'s are invertible by part (3). We have $a_{ij} \in \mathfrak{a}_i\mathfrak{b}_j^{-1}$ for all relevant i, j. The cofactor A_{ij} of a_{ij} is equal to

$$A_{ij} = \Sigma \pm a_{1\alpha} \cdots a_{n\omega}$$

in which the first index avoids i and the second j. Hence

$$\begin{aligned} A_{ij}\mathfrak{a}_i\mathfrak{b}_j^{-1} &\subseteq (\mathfrak{a}_1 \cdots \mathfrak{a}_n)(\mathfrak{b}_1 \cdots \mathfrak{b}_n)^{-1} \\ &= \mathfrak{o} \cdot \det a_{ij}. \end{aligned}$$

Therefore

$$b_{ji}\mathfrak{a}_i = \frac{A_{ij}}{\det(a_{ij})} \cdot \mathfrak{a}_i \subseteq \mathfrak{b}_j.$$

This is true for all i, j. Hence $M \subseteq N$ by part (1). Hence $M = N$. Q.E.D.

Consider a bounded $\mathfrak{o}$-module M on V. Define the integral linear groups

$$\mathrm{GL}_n(M) = \{\sigma \in \mathrm{GL}_n(V) | \sigma M = M\},$$

$$\mathrm{SL}_n(M) = \mathrm{GL}_n(M) \cap \mathrm{SL}_n(V).$$

We say that a linear transformation is on M if it is in $\mathrm{GL}_n(M)$, i.e., if $\sigma M = M$. For any nonzero integral ideal $\mathfrak{a}$, define the linear congruence groups

$$\mathrm{GL}_n(M; \mathfrak{a}) = \{\sigma \in \mathrm{GL}_n(M) | (\sigma - 1_V)M \subseteq \mathfrak{a}M\},$$

$$\mathrm{SL}_n(M; \mathfrak{a}) = \mathrm{GL}_n(M; \mathfrak{a}) \cap \mathrm{SL}_n(V).$$

It is clear that $\mathrm{SL}_n(M; \mathfrak{a})$ and $\mathrm{GL}_n(M; \mathfrak{a})$ are normal subgroups of $\mathrm{GL}_n(M)$. We have

$$\mathrm{GL}_n(M; \mathfrak{o}) = \mathrm{GL}_n(M), \qquad \mathrm{SL}_n(M; \mathfrak{o}) = \mathrm{SL}_n(M).$$

The projective integral linear groups $\mathrm{PGL}_n(M)$, $\mathrm{PSL}_n(M)$, and the projective linear congruence groups $\mathrm{PGL}_n(M; \mathfrak{a})$, $\mathrm{PSL}_n(M; \mathfrak{a})$ are, of course, obtained by applying P.

If we consider any nontrivial transvection τ and express it in the usual form $\tau_{a,\rho}$ then we find that

$$\tau M = M \Leftrightarrow \tau M \subseteq M \Leftrightarrow (\rho M)a \subseteq M$$

and

$$\tau \in \mathrm{SL}_n(M; \mathfrak{a}) \Leftrightarrow (\rho M)a \subseteq \mathfrak{a}M.$$

An $n \times n$ matrix over a field F is called integral (with respect to the underlying integral domain $\mathfrak{o}$) if its entries are in $\mathfrak{o}$; it is called unimodular if it is integral with determinant a unit of $\mathfrak{o}$. By looking at cofactors we see that an integral matrix A is unimodular if and only if it is invertible with A^{-1} integral. We let $\mathrm{GL}_n(\mathfrak{o})$ denote the subgroup of unimodular matrices in $\mathrm{GL}_n(F)$, and put

$$\mathrm{SL}_n(\mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o}) \cap \mathrm{SL}_n(F).$$

The groups $\mathrm{GL}_n(\mathfrak{o})$ and $\mathrm{SL}_n(\mathfrak{o})$ are called integral matrix groups. For any nonzero integral ideal $\mathfrak{a}$, define the matrix congruence groups

$$\mathrm{GL}_n(\mathfrak{o}; \mathfrak{a}) = \{X \in \mathrm{GL}_n(\mathfrak{o}) | X \equiv I \bmod \mathfrak{a}\},$$
$$\mathrm{SL}_n(\mathfrak{o}; \mathfrak{a}) = \mathrm{GL}_n(\mathfrak{o}; \mathfrak{a}) \cap \mathrm{SL}_n(F),$$

where congruence of two $n \times n$ matrices mod $\mathfrak{a}$ means congruence mod $\mathfrak{a}$ entrywise. It is clear that $\mathrm{SL}_n(\mathfrak{o}; \mathfrak{a})$ and $\mathrm{GL}_n(\mathfrak{o}; \mathfrak{a})$ are normal subgroups of $\mathrm{GL}_n(\mathfrak{o})$. We have

$$\mathrm{GL}_n(\mathfrak{o}; \mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o}), \qquad \mathrm{SL}_n(\mathfrak{o}; \mathfrak{o}) = \mathrm{SL}_n(\mathfrak{o}).$$

The projective integral matrix groups $\mathrm{PGL}_n(\mathfrak{o})$, $\mathrm{PSL}_n(\mathfrak{o})$ and the projective matrix congruence groups $\mathrm{PGL}_n(\mathfrak{o}; \mathfrak{a})$, $\mathrm{PSL}_n(\mathfrak{o}; \mathfrak{a})$ are, of course, defined by applying P.

Now assume that M is actually free on V, say

$$M = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$$

with $x_1, \ldots, x_n$ a base for V. Let $\rho_1, \ldots, \rho_n$ denote the dual base. If a linear transformation σ in $\mathrm{GL}_n(V)$ has matrix S in the above base, then it follows easily that

$$\sigma M \subseteq M \Leftrightarrow S \text{ is integral}$$

and

$$\sigma M = M \Leftrightarrow S \text{ is unimodular}.$$

In particular, the elementary transvection $\tau_{\lambda x_i, \rho_j}$ is on M if and only if λ is in $\mathfrak{o}$, i.e.,

$$\tau_{\lambda x_i, \rho_j} \in \mathrm{SL}_n(M) \Leftrightarrow \lambda \in \mathfrak{o}.$$

Since a unimodular matrix has determinant a unit, we easily see that in the free case $\mathrm{GL}_n(M)/\mathrm{SL}_n(M)$ is isomorphic to the group of units of $\mathfrak{o}$. We find that the matrix isomorphism associated with the above base (for M and V) induces

$$\mathrm{GL}_n(M) \rightarrowtail \mathrm{GL}_n(\mathfrak{o}), \quad \mathrm{GL}_n(M; \mathfrak{a}) \rightarrowtail \mathrm{GL}_n(\mathfrak{o}; \mathfrak{a}),$$
$$\mathrm{SL}_n(M) \rightarrowtail \mathrm{SL}_n(\mathfrak{o}), \quad \mathrm{SL}_n(M; \mathfrak{a}) \rightarrowtail \mathrm{SL}_n(\mathfrak{o}; \mathfrak{a}),$$

so the associated projective isomorphism $\mathrm{PGL}_n(V) \rightarrowtail \mathrm{PGL}_n(F)$ induces

$$\mathrm{PGL}_n(M) \rightarrowtail \mathrm{PGL}_n(\mathfrak{o}), \quad \mathrm{PGL}_n(M; \mathfrak{a}) \rightarrowtail \mathrm{PGL}_n(\mathfrak{o}; \mathfrak{a}),$$
$$\mathrm{PSL}_n(M) \rightarrowtail \mathrm{PSL}_n(\mathfrak{o}), \quad \mathrm{PSL}_n(M; \mathfrak{a}) \rightarrowtail \mathrm{PSL}_n(\mathfrak{o}; \mathfrak{a}).$$

5.6. The Isomorphism Theorems over Integral Domains

We now return to our general assumptions that V and V_1 are regular alternating spaces over fields F and F_1 with dimensions n and n_1. We also consider arbitrary integral domains $\mathfrak{o}$ and $\mathfrak{o}_1$ with quotient fields F and F_1. And we consider bounded $\mathfrak{o}$- and $\mathfrak{o}_1$-modules M and M_1 on V and V_1.

We define the integral symplectic group $\mathrm{Sp}_n(M)$ by the equation

$$\mathrm{Sp}_n(M) = \mathrm{GL}_n(M) \cap \mathrm{Sp}_n(V).$$

For any nonzero integral ideal $\mathfrak{a}$, we define the symplectic congruence group

$$\mathrm{Sp}_n(M;\, \mathfrak{a}) = \mathrm{GL}_n(M;\, \mathfrak{a}) \cap \mathrm{Sp}_n(V).$$

It is clear that $\mathrm{Sp}_n(M;\, \mathfrak{a})$ is a normal subgroup of $\mathrm{Sp}_n(M)$. We have

$$\mathrm{Sp}_n(M;\, \mathfrak{o}) = \mathrm{Sp}_n(M).$$

The projective integral symplectic group $\mathrm{PSp}_n(M)$, and the projective symplectic congruence groups $\mathrm{PSp}_n(M;\, \mathfrak{a})$ are, of course, obtained by applying P.

If we consider any nontrivial transvection τ in $\mathrm{Sp}_n(V)$ and express it in the usual form $\tau_{a,\lambda}$ then we find that

$$\tau M = M \Leftrightarrow \tau M \subseteq M \Leftrightarrow \lambda q(M, a)a \subseteq M$$

and

$$\tau \in \mathrm{Sp}_n(M;\, \mathfrak{a}) \Leftrightarrow \lambda q(M, a)a \subseteq \mathfrak{a}M.$$

5.6.1. *Let M and N be bounded $\mathfrak{o}$-modules on V and let $\mathfrak{a}$ be a nonzero integral ideal. Then there is a nonzero integral ideal $\mathfrak{b}$ such that* $\mathrm{Sp}_n(N;\, \mathfrak{b}) \subseteq \mathrm{Sp}_n(M;\, \mathfrak{a})$.

Proof. By 5.5.2 there is a nonzero α in F such that $\alpha M \subseteq N$. But it is easily seen that $\mathrm{Sp}_n(M;\, \mathfrak{a}) = \mathrm{Sp}_n(\alpha M;\, \mathfrak{a})$. In effect this allows us to assume that $M \subseteq N$. Let us do so. Let $\mathfrak{b}$ be any nonzero integral ideal for which

$$\mathfrak{b}N \subseteq \mathfrak{a}M \subseteq M \subseteq N.$$

For any σ in $\mathrm{Sp}_n(N;\, \mathfrak{b})$ we have

$$(\sigma - 1_V)N \subseteq \mathfrak{b}N \subseteq \mathfrak{a}M;$$

hence $(\sigma - 1_V)M \subseteq \mathfrak{a}M$. In particular, $\sigma M \subseteq M$. But $\mathrm{Sp}_n(N;\, \mathfrak{b})$ is a group. So $\sigma^{-1}M \subseteq M$. So $\sigma M = M$. So $\sigma \in \mathrm{Sp}_n(M;\, \mathfrak{a})$. Q.E.D.

5.6.2. *If M is a bounded $\mathfrak{o}$-module on V and $\mathfrak{a}$ is a nonzero integral ideal, then* $\mathrm{Sp}_n(M;\, \mathfrak{a})$ *has enough transvections.*

Proof. If $\mathfrak{o}$ is a field, then $\mathrm{Sp}_n(M;\, \mathfrak{a}) = \mathrm{Sp}_n(V)$ and the result is obvious. So assume $0 \subset \mathfrak{o} \subset F$; in particular that $\mathfrak{o}$ is infinite. Given a nonzero a in V it is enough to find two distinct λ's in $\dot{F}$ such that each $\tau_{a,\lambda}$ is in $\mathrm{Sp}_n(M;\, \mathfrak{a})$. It is easily verified that $q(M, a)$ is a fractional ideal. There is therefore a nonzero λ in $\mathfrak{o}$ such that $\lambda q(M, a) \subseteq \mathfrak{a}\mathfrak{c}_a$, where $\mathfrak{c}_a$ denotes the coefficient of a with respect to M. Since $\mathfrak{o}$ is infinite it contains an element $\neq 0, 1$, and multiplying λ by this element gives us a second λ with the same properties as the first. Then for either λ we have

$$\lambda q(M, a)a \subseteq (\mathfrak{a}\mathfrak{c}_a)a \subseteq \mathfrak{a}(\mathfrak{c}_a a) \subseteq \mathfrak{a}M,$$

so $\tau_{a,\lambda} \in \mathrm{Sp}_n(M; \mathfrak{a})$, so $\mathrm{Sp}_n(M; \mathfrak{a})$ has enough transvections. Q.E.D.

5.6.3. *If M is a bounded $\mathfrak{o}$-module on V, and $\mathfrak{a}$ is a nonzero integral ideal, and if we are in the 4-dimensional situation in characteristic 2, then $\mathrm{Sp}_4(M; \mathfrak{a})$ has enough hyperbolic transformations.*

PROOF. Let $\mathfrak{X} = \{x_1, x_2 | y_1, y_2\}$ be a symplectic base for V and put

$$N = \mathfrak{o}x_1 + \mathfrak{o}x_2 + \mathfrak{o}y_1 + \mathfrak{o}y_2.$$

By 5.6.1 it is enough to prove that $\mathrm{Sp}_4(N; \mathfrak{b})$ has enough hyperbolic transformations. So we must consider a totally degenerate plane Π in V and find a hyperbolic transformation σ in $\mathrm{Sp}_4(N; \mathfrak{b})$ with $R = \Pi$. By 5.4.17 and 5.4.19 we can assume that

$$\Pi = \Pi(\alpha, \beta, \gamma, \delta)$$

with $\alpha, \beta, \gamma, \delta$ elements of $\mathfrak{b}$, not all 0, which satisfy

$$\alpha\gamma + \beta\delta \in F^2, \qquad \sqrt{\alpha\gamma + \beta\delta} \in \mathfrak{b}.$$

Then

$$\sigma = \Sigma_{\alpha,\beta,\gamma,\delta;1}$$

is a hyperbolic transformation in $\mathrm{Sp}_4(V)$ with $R = \Pi$, by 5.4.16. If σ has matrix X in the base $\mathfrak{X}$, then the defining matrix of $\sigma = \Sigma$ shows that X has integral entries, and $\det X = 1$ since σ is in $\mathrm{Sp}_4(V)$, so X is a unimodular matrix with respect to $\mathfrak{o}$. Again referring to the defining matrix of $\sigma = \Sigma$ we see that $X \equiv I \bmod \mathfrak{b}$. So σ is in $\mathrm{GL}_4(N; \mathfrak{b})$ by §5.5. So σ is an element of $\mathrm{Sp}_4(N; \mathfrak{b})$ with $R = \Pi$. Q.E.D.

5.6.4. THEOREM. *Let $\mathfrak{o}$ be an integral domain with quotient field F, let M be a bounded $\mathfrak{o}$-module on the regular alternating space V over F, let $\mathfrak{a}$ be a nonzero integral ideal, let Δ be a group with*

$$\mathrm{PSp}_n(M; \mathfrak{a}) \subseteq \Delta \subseteq \mathrm{P\Gamma Sp}_n(V),$$

and let $n \geqslant 4$. Let $\mathfrak{o}_1, F_1, M_1, V_1, \mathfrak{a}_1, \Delta_1, n_1$ be a second such situation. In the exceptional situation where F is a perfect field of characteristic 2 and $n = 4$, let E_0 denote a fixed exceptional automorphism of $\mathrm{P\Gamma Sp}_4(V)$. Then each isomorphism $\Lambda: \Delta \rightarrowtail \Delta_1$ has exactly one of the forms

$$\Lambda k = gkg^{-1} \qquad \forall\, k \in \Delta$$

or

$$\Lambda k = g(\mathrm{E}_0 k)g^{-1} \qquad \forall\, k \in \Delta$$

for a unique projective symplectic collinear transformation g of V onto V_1, the second possibility appearing only in the exceptional situation.

PROOF. It follows from 5.6.2 that Δ and Δ_1 have enough projective transvections. If $n \geqslant 6$ or char $F \neq 2$, apply Theorem 5.3.6. So let $n = 4$ with char $F = 2$. Then $n_1 = 4$ with char $F_1 = 2$, by Theorem 5.3.6. And Δ and Δ_1

have enough projective hyperbolic transformations, by 5.6.3. Apply Theorem 5.4.40. Q.E.D.

5.6.5. THEOREM. *Let* $\mathfrak{o}$ *be an integral domain with quotient field* F, *let* M *be a bounded* $\mathfrak{o}$*-module on the regular alternating space* V *over* F, *let* $\mathfrak{a}$ *be a nonzero integral ideal, let* Γ *be a group with*

$$\mathrm{Sp}_n(M;\,\mathfrak{a}) \subseteq \Gamma \subseteq \Gamma\mathrm{Sp}_n(V),$$

and let $n \geqslant 4$. *Let* $\mathfrak{o}_1, F_1, M_1, V_1, \mathfrak{a}_1, \Gamma_1, n_1$ *be a second such situation. In the exceptional situation where* F *is a perfect field of characteristic* 2 *and* $n = 4$, *let* E_2 *denote a fixed exceptional automorphism of* $\Gamma\mathrm{Sp}_4(V)$. *Then each isomorphism* $\Phi\colon \Gamma \rightarrowtail \Gamma_1$ *has exactly one of the forms*

$$\Phi k = \chi(k)\, gkg^{-1} \qquad \forall\, k \in \Gamma$$

or

$$\Phi k = \chi(k)\, g(\mathrm{E}_2 k)\, g^{-1} \qquad \forall\, k \in \Gamma$$

for a mapping χ *of* Γ *into* $\mathrm{RL}_{n_1}(V_1)$ *and a symplectic collinear transformation* g *of* V *onto* V_1, *the second possibility appearing only in the exceptional situation.*

5.6.6. DEFINITION. We say that a free $\mathfrak{o}$-module M on V has a symplectic base if M has a base $\mathfrak{X}$ which, when viewed as a base for V, is symplectic.

5.6.7. *If* $\mathfrak{a}$ *is an invertible ideal, then* $\mathrm{GL}_n(\mathfrak{a}M) = \mathrm{GL}_n(M)$ *and* $\mathrm{Sp}_n(\mathfrak{a}M) = \mathrm{Sp}_n(M)$.

PROOF. For each σ in $\mathrm{GL}_n(M)$ we have $\sigma M = M$; hence $\sigma(\mathfrak{a}M) \subseteq \mathfrak{a}M$, the same for σ^{-1}, so $\mathrm{GL}_n(M) \subseteq \mathrm{GL}_n(\mathfrak{a}M)$. Equality follows from the fact that, since $\mathfrak{a}$ is injective, $\mathrm{GL}_n(\mathfrak{a}M) \subseteq \mathrm{GL}_n(\mathfrak{a}^{-1}\mathfrak{a}M)$. Finally

$$\begin{aligned}\mathrm{Sp}_n(\mathfrak{a}M) &= \mathrm{GL}_n(\mathfrak{a}M) \cap \mathrm{Sp}_n(V)\\ &= \mathrm{GL}_n(M) \cap \mathrm{Sp}_n(V) = \mathrm{Sp}_n(M). \quad \text{Q.E.D.}\end{aligned}$$

5.6.8. *Let* M *be a free* $\mathfrak{o}$*-module with a symplectic base on* V, *let* M' *be any* $\mathfrak{o}$*-lattice on* V, *and suppose* $\mathrm{Sp}_n(M) \subseteq \mathrm{Sp}_n(M')$. *Then there is an invertible ideal* $\mathfrak{a}$ *such that* $M' = \mathfrak{a}M$. *And in fact* $\mathrm{Sp}_n(M) = \mathrm{Sp}_n(M')$.

PROOF. Let $\mathfrak{X} = \{x_1, \ldots, x_n\}$ be a symplectic base for M. So

$$M = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n.$$

Let $\mathfrak{c}_i$ $(1 \leqslant i \leqslant n)$ denote the coefficient of x_i with respect to M'. So each $\mathfrak{c}_i$ is a fractional ideal and

$$\mathfrak{c}_1x_1 + \cdots + \mathfrak{c}_nx_n \subseteq M'.$$

We shall prove that $\mathfrak{c}_1 = \cdots = \mathfrak{c}_n$, and then that the above inclusion is an equality. Once this is done we will be through since $\mathfrak{a} = \mathfrak{c}_1 = \cdots = \mathfrak{c}_n$ will be invertible by 5.5.4, and

$$\mathfrak{a}M = \mathfrak{a}(\mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n) = \mathfrak{c}_1x_1 + \cdots + \mathfrak{c}_nx_n = M';$$

and $\mathrm{Sp}_n(M) = \mathrm{Sp}_n(M')$ by 5.6.7.

So consider $\mathfrak{c}_i$ for some i $(1 \leqslant i \leqslant n)$. Take that j $(1 \leqslant j \leqslant n)$ for which $q(x_i, x_j) = \pm 1$. Then

$$\tau_{x_i+x_j,1} \in \mathrm{Sp}_n(M) \subseteq \mathrm{Sp}_n(M'),$$

so

$$\tau_{x_i+x_j,1}(\gamma x_i) \in M'$$

for any γ in $\mathfrak{c}_i$, and from this we find that $\gamma \in \mathfrak{c}_j$, so $\mathfrak{c}_i \subseteq \mathfrak{c}_j$. A similar argument using $\tau_{x_k+x_i,1}$ instead of $\tau_{x_i+x_j,1}$ shows that $\mathfrak{c}_i \subseteq \mathfrak{c}_k$ for $k \neq i, j$. Hence $\mathfrak{c}_i \subseteq \mathfrak{c}_k$ for all k. Hence $\mathfrak{c}_i \subseteq \mathfrak{c}_k$ for all i and k. Hence $\mathfrak{c}_i = \mathfrak{c}_k$ for all i and k. Let $\mathfrak{a}$ stand for the common value of all the $\mathfrak{c}_i$.

We still have to show that each $m' \in M'$ is in $\mathfrak{c}_1x_1 + \cdots + \mathfrak{c}_nx_n$. Write

$$m' = p_1x_1 + \cdots + p_nx_n \qquad (p_i \in F).$$

We must show that each p_i is in $\mathfrak{c}_i = \mathfrak{a}$. Again take j with $q(x_i, x_j) = \pm 1$. Then

$$\tau_{x_j,1} \in \mathrm{Sp}_n(M) \subseteq \mathrm{Sp}_n(M');$$

hence

$$\tau_{x_j,1}(m') \in M';$$

hence

$$m' \pm p_ix_j \in M';$$

hence $p_ix_j \in M'$; hence $p_i \in \mathfrak{c}_j = \mathfrak{a}$, as required. Q.E.D.

5.6.9. *If M is a bounded $\mathfrak{o}$-module on V, and if g is a symplectic collinear transformation $g\colon V \rightarrowtail V_1$ with associated field isomorphism $\mu\colon F \rightarrowtail F_1$, then*

(1) *$\mathfrak{o}^\mu$ is an integral domain with quotient field F_1;*

(2) *gM is a bounded $\mathfrak{o}^\mu$-module on V_1;*

(3) $\Phi_g\mathrm{Sp}_n(M) = \mathrm{Sp}_{n_1}(gM)$.

Proof. Only (3) really needs proof. Since we already know from §4.2 that $\Phi_g\mathrm{Sp}_n(V) = \mathrm{Sp}_{n_1}(V_1)$, we need only check that $\Phi_g\mathrm{GL}_n(M) = \mathrm{GL}_{n_1}(gM)$. But it is clear that Φ_g sends $\mathrm{GL}_n(M)$ into $\mathrm{GL}_{n_1}(gM)$. Equality then follows by considering g^{-1} instead of g. Q.E.D.

5.6.10. *Let M be a free $\mathfrak{o}$-module with a symplectic base on V, let M_1 be a free $\mathfrak{o}_1$-module with a symplectic base on V_1, and let g be a symplectic collinear transformation of V onto V_1 with associated field isomorphism μ. Then the following assertions are equivalent:*

(1) $\Phi_g\mathrm{Sp}_n(M) = \mathrm{Sp}_{n_1}(M_1)$.

(2) $\Phi_{\bar g}\mathrm{PSp}_n(M) = \mathrm{PSp}_{n_1}(M_1)$.

(3) *$\mathfrak{o}^\mu = \mathfrak{o}_1$ and $gM = \mathfrak{a}_1M_1$ for some invertible ideal $\mathfrak{a}_1$ with respect to $\mathfrak{o}_1$.*

Proof. Express $M = \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n$ with $\mathfrak{X} = \{x_1, \ldots, x_n\}$ a symplectic base for V. The proof that (1) is equivalent to (2) is straightforward. And

$$\Phi_g\mathrm{Sp}_n(M) = \mathrm{Sp}_{n_1}(gM) = \mathrm{Sp}_{n_1}(\mathfrak{a}_1M_1) = \mathrm{Sp}_{n_1}(M_1)$$

so that (3) implies (1). We must prove that (1) implies (3). We have $\tau_{x_1,1} \in$

$\mathrm{Sp}_n(M)$; therefore by §4.2

$$\tau_{gx_1,\varepsilon_1'} = g\tau_{x_1,1}g^{-1} \in \mathrm{Sp}_{n_1}(M_1),$$

where ε_1 denotes the multiplier of g. So

$$\varepsilon_1^{-1}q_1(M_1, gx_1)gx_1 \subseteq M_1,$$

so

$$\xi_1\varepsilon_1^{-1}q_1(M_1, gx_1)gx_1 \subseteq M_1$$

for any ξ_1 in $\mathfrak{o}_1$, so

$$\tau_{gx_1,\xi_1\varepsilon_1^{-1}} \in \mathrm{Sp}_{n_1}(M_1),$$

so

$$\tau_{x_1,\xi_1^{\mu^{-1}}} = g^{-1}\tau_{gx_1,\xi_1\varepsilon_1^{-1}}g \in \mathrm{Sp}_n(M),$$

so

$$\xi_1^{\mu^{-1}}q(M, x_1)x_1 \subseteq M,$$

so

$$\xi_1^{\mu^{-1}}x_1 \in \mathfrak{o}x_1 + \cdots + \mathfrak{o}x_n,$$

so $\xi_1^{\mu^{-1}} \in \mathfrak{o}$, so $\xi_1 \in \mathfrak{o}^\mu$, so $\mathfrak{o}_1 \subseteq \mathfrak{o}^\mu$. Considering g^{-1} instead of g gives us equality, i.e., $\mathfrak{o}_1 = \mathfrak{o}^\mu$.

To show that gM has the desired form $gM = \mathfrak{a}_1M_1$ with $\mathfrak{a}_1$ invertible, observe that

$$\mathrm{Sp}_{n_1}(M_1) = \Phi_g\mathrm{Sp}_n(M) = \mathrm{Sp}_{n_1}(gM) \subseteq \mathrm{Sp}_{n_1}(gM)$$

with gM an $\mathfrak{o}^\mu$-lattice, i.e., with gM an $\mathfrak{o}_1$-lattice, and that M_1 is a free $\mathfrak{o}_1$-module with a symplectic base on V_1. Apply 5.6.8. Q.E.D.

5.6.11. *Let F and F_1 be perfect of characteristic 2 and let $n = n_1 = 4$. Let M be a free $\mathfrak{o}$-module with a symplectic base $\mathfrak{X}$ on V, let M_1 be a free $\mathfrak{o}_1$-module with a symplectic base $\mathfrak{X}_1$ on V_1. Let g be a symplectic collinear transformation of V onto V_1 with associated field isomorphism μ. Let* E *be the exceptional automorphism of* $\Gamma\mathrm{Sp}_4(V)$ *of* 5.4.2 (*defined with respect to $\mathfrak{X}$*). *Finally suppose that*

$$(\Phi_g \circ \mathrm{E})\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M_1).$$

Then

(1) $\mathfrak{o}^\mu = \mathfrak{o}_1$,
(2) $\mathrm{E}\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M)$,
(3) $\Phi_g\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M_1)$.

PROOF. Write $\mathfrak{X} = \{x_1, x_2 | y_1, y_2\}$ and $\mathfrak{X}_1 = \{x_1', x_2' | y_1', y_2'\}$. Let ε_1 denote the multiplier of g.

Consider the hyperbolic transformation $\Sigma = \Sigma_{1,0,1,0;1}$ defined with respect to $\mathfrak{X}$. The matrix of Σ with respect to $\mathfrak{X}$ is clearly integral, and $\det \Sigma = 1$ since Σ is in $\mathrm{Sp}_4(V)$, so Σ is unimodular, so Σ is in $\mathrm{Sp}_4(M)$. Therefore

$$\tau_{g(x_1+y_1),\varepsilon_1^{-1}} = (\Phi_g \circ \mathrm{E})\Sigma \in \mathrm{Sp}_4(M_1).$$

So

$$\tau_{\xi_1 g(x_1+y_1),\varepsilon_1^{-1}} \in \mathrm{Sp}_4(M_1)$$

for any ξ_1 in $\mathfrak{o}_1$. But

$$(\Phi_g \circ \mathrm{E})\Sigma_{\xi_1^{\mu^{-1}},0,\xi_1^{\mu^{-1}},0;1} = \tau_{\xi_1 g(x_1+y_1),\varepsilon_1^{-1}}.$$

So

$$\Sigma_{\xi_1^{\mu^{-1}},0,\xi_1^{\mu^{-1}},0;1} \in \mathrm{Sp}_4(M).$$

So $\xi_1^{\mu^{-1}} \in \mathfrak{o}$. So $\mathfrak{o}_1 \subseteq \mathfrak{o}^{\mu}$.

Now consider the hyperbolic transformation

$$\Sigma' = \Sigma'_{1,0,1,0;1} \in \mathrm{Sp}_4(M_1)$$

defined with respect to $\mathfrak{X}_1$ (primes on the Σ's indicate that they occur in the V_1 situation). Then $\Phi_g^{-1}\Sigma'$ is a hyperbolic transformation in $\mathrm{Sp}_4(V)$ since g is symplectic collinear, so there are α, β, γ, δ in F with

$$\Phi_g \Sigma_{\alpha,\beta,\gamma,\delta;1} = \Sigma'_{1,0,1,0;1}$$

by 5.4.5. By 5.4.8 we have

$$\mathrm{E}\tau_{x,1} = \Sigma_{\alpha,\beta,\gamma,\delta;1}$$

where $x = \sqrt{\alpha}\, x_1 + \sqrt{\beta}\, x_2 + \sqrt{\gamma}\, y_1 + \sqrt{\delta}\, y_2$. Then

$$(\Phi_g \circ \mathrm{E})\tau_{x,1} = \Sigma'_{1,0,1,0;1} \in \mathrm{Sp}_4(M_1),$$

so $\tau_{x,1}$ is in $\mathrm{Sp}_4(M)$, so for any ξ in $\mathfrak{o}$ we have $\tau_{x,\xi}$ in $\mathrm{Sp}_4(M)$. Hence

$$\Phi_g \Sigma_{\xi\alpha,\xi\beta,\xi\gamma,\xi\delta;1} = (\Phi_g \circ \mathrm{E})\tau_{x,\xi} \in \mathrm{Sp}_4(M_1).$$

It then follows from a matrix computation using 4.1.9 and the action of Φ_g on $\Sigma_{\alpha,\beta,\gamma,\delta;1}$ that

$$\Phi_g \Sigma_{\xi\alpha,\xi\beta,\xi\gamma,\xi\delta;1} = \Sigma'_{\xi^{\mu},0,\xi^{\mu},0;1}.$$

Since this element is in $\mathrm{Sp}_4(M_1)$ we must have $\xi^{\mu} \in \mathfrak{o}_1$. Hence $\mathfrak{o}^{\mu} \subseteq \mathfrak{o}_1$.

We now have $\mathfrak{o}_1 \subseteq \mathfrak{o}^{\mu} \subseteq \mathfrak{o}_1$, so $\mathfrak{o}^{\mu} = \mathfrak{o}_1$, so (1) is true.

If σ is any element of $\mathrm{Sp}_4(M)$ its matrix with respect to $\mathfrak{X}$ is unimodular; hence the matrix of $\mathrm{E}\sigma$ is integral, but $\det \mathrm{E}\sigma = 1$ since $\mathrm{E}\sigma$ is in $\mathrm{Sp}_4(V)$; hence $\mathrm{E}\sigma$ also has a unimodular matrix with respect to $\mathfrak{X}$; hence $\mathrm{E}\sigma$ is in $\mathrm{Sp}_4(M)$; hence $\mathrm{ESp}_4(M) \subseteq \mathrm{Sp}_4(M)$; hence

$$\mathrm{Sp}_4(M_1) = (\Phi_g \circ \mathrm{E})\mathrm{Sp}_4(M) \subseteq \Phi_g \mathrm{Sp}_4(M) = \mathrm{Sp}_4(gM).$$

Here gM is a free $\mathfrak{o}_1$-module on V_1 since we already know that $\mathfrak{o}_1 = \mathfrak{o}^{\mu}$. But then $\mathrm{Sp}_4(M_1) = \mathrm{Sp}_4(gM)$ by 5.6.8. Hence $\Phi_g\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M_1)$. Hence $\mathrm{ESp}_4(M) = \mathrm{Sp}_4(M)$. Q.E.D.

For any even integer $n \geqslant 2$ and any nonzero integral ideal $\mathfrak{a}$ we define

$$\mathrm{Sp}_n(\mathfrak{o}) = \mathrm{GL}_n(\mathfrak{o}) \cap \mathrm{Sp}_n(F),$$

$$\mathrm{Sp}_n(\mathfrak{o};\, \mathfrak{a}) = \mathrm{GL}_n(\mathfrak{o};\, \mathfrak{a}) \cap \mathrm{Sp}_n(F).$$

The projective groups $\mathrm{PSp}_n(\mathfrak{o})$ and $\mathrm{PSp}_n(\mathfrak{o};\, \mathfrak{a})$ are, of course, obtained by applying P. Note that $\mathrm{Sp}_n(\mathfrak{o})$ consists of all unimodular $n \times n$ matrices X over $\mathfrak{o}$

which satisfy the equation

$$ {}^tX\left[\begin{array}{c|c} 0 & I_{n/2} \\ \hline -I_{n/2} & 0 \end{array}\right]X = \left[\begin{array}{c|c} 0 & I_{n/2} \\ \hline -I_{n/2} & 0 \end{array}\right] $$

while $\mathrm{PSp}_n(\mathfrak{o})$ is isomorphic to this group reduced modulo $(\pm I)$. If M is a free $\mathfrak{o}$-module with a symplectic base on V, then the associated matrix isomorphism induces

$$ \mathrm{Sp}_n(M) \rightarrowtail \mathrm{Sp}_n(\mathfrak{o}), \qquad \mathrm{Sp}_n(M;\mathfrak{a}) \to \mathrm{Sp}_n(\mathfrak{o};\mathfrak{a}), $$

so projectively we obtain

$$ \mathrm{PSp}_n(M) \rightarrowtail \mathrm{PSp}_n(\mathfrak{o}), \qquad \mathrm{PSp}_n(M;\mathfrak{a}) \rightarrowtail \mathrm{PSp}_n(\mathfrak{o};\mathfrak{a}). $$

5.6.12. THEOREM([5]). *Let n and n_1 be even integers $\geqslant 4$, and let $\mathfrak{o}$ and $\mathfrak{o}_1$ be any two integral domains. Then the following statements are equivalent*:

(0) $n = n_1$ *and* $\mathfrak{o} \cong \mathfrak{o}_1$.

(1) $\mathrm{Sp}_n(\mathfrak{o}) \cong \mathrm{Sp}_{n_1}(\mathfrak{o}_1)$.

(2) $\mathrm{PSp}_n(\mathfrak{o}) \cong \mathrm{PSp}_{n_1}(\mathfrak{o}_1)$.

PROOF. (0) implies (1) is trivial. To see that (1) implies (2), convert to modules, apply 5.6.2 and 5.3.7, convert back to matrices. We must prove that (2) implies (0). Converting to modules gives us a situation

$$ \mathrm{PSp}_n(M) \rightarrowtail \mathrm{PSp}_{n_1}(M_1) $$

with M, M_1 free with symplectic bases. If there is a projective symplectic collinear transformation g of V onto V_1 such that

$$ \Phi_g \mathrm{PSp}_n(M) = \mathrm{PSp}_{n_1}(M_1), $$

then obviously $n = n_1$, and $\mathfrak{o} \cong \mathfrak{o}_1$ by 5.6.10. So assume there is no such g. Then by Theorem 5.6.4, F must be a perfect field of characteristic 2 and n must be equal to 4, and

$$ \left(\Phi_{\bar g} \circ \bar{\mathrm{E}}\right)\mathrm{PSp}_n(M) = \mathrm{PSp}_{n_1}(M_1), $$

for some symplectic collinear transformation g of V onto V_1, where $\bar{\mathrm{E}}$ is the exceptional automorphism of $\mathrm{P\Gamma Sp}_4(V)$ associated with a fixed symplectic base for M in the manner of 5.4.3. This of course makes $n = n_1 = 4$. It follows easily that

$$ (\Phi_g \circ \mathrm{E})\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M_1). $$

Apply 5.6.11. Q.E.D.

5.6.13. REMARK. Let us consider the group of automorphisms of the group $\mathrm{PSp}_n(\mathfrak{o})$ over an arbitrary integral domain $\mathfrak{o}$ or, more exactly, of $\mathrm{PSp}_n(M)$ where M is a free $\mathfrak{o}$-module with a symplectic base $\mathfrak{X}$ and $n \geqslant 4$. It follows from Theorem 5.6.4 that every automorphism of $\mathrm{PSp}_n(M)$ can be lifted to an automorphism of $\mathrm{P\Gamma Sp}_n(V)$ (also to one of $\mathrm{PSp}_n(V)$). Furthermore, if the characteristic is not 2, or if the characteristic is 2 and $n \geqslant 6$, or if the

([5])This result is known for fields, new for integral domains.

characteristic is 2 with F imperfect and $n = 4$, then every automorphism of $\mathrm{PSp}_n(M)$ is induced by a Φ_g for some projective symplectic collinear transformation g of V. Consider the exceptional 4-dimensional situation over a perfect field of characteristic 2. Here the automorphisms of $\mathrm{PSp}_4(M)$ which are induced by Φ_g's form a subgroup of the entire group of automorphisms of $\mathrm{PSp}_4(M)$; they are precisely the automorphisms which preserve projective transvections; hence by 5.4.35 they form a subgroup of index 1 or 2 in the entire group of automorphisms of $\mathrm{PSp}_4(M)$; if $\mathfrak{o}$ is "perfect", i.e., if $\mathfrak{o} = \mathfrak{o}^2$ where $\mathfrak{o}^2$ denotes the integral domain consisting of the set of squares of $\mathfrak{o}$, then the mapping E of §5.4 is easily seen to induce an automorphism of $\mathrm{Sp}_4(M)$, so $\bar{\mathrm{E}}$ induces an automorphism of $\mathrm{PSp}_4(M)$ which does not preserve projective transvections, so the Φ_g's induce a subgroup of index 2 in this case; conversely, if the automorphisms of $\mathrm{PSp}_4(M)$ that are induced by Φ_g's form a subgroup of index 2 in the entire group of automorphisms of $\mathrm{PSp}_4(M)$, then it follows from Theorem 5.6.4 that there is a symplectic collinear transformation g of V onto V such that

$$\left(\Phi_{\bar{g}} \circ \bar{\mathrm{E}}\right)\mathrm{PSp}_4(M) = \mathrm{PSp}_4(M).$$

This implies that

$$(\Phi_g \circ \mathrm{E})\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M),$$

and so $\mathrm{E}\,\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M)$ by 5.6.11, in particular $\mathrm{E}^2\mathrm{Sp}_4(M) = \mathrm{Sp}_4(M)$. Consider a typical α in $\mathfrak{o}$. Then $\tau_{x_1,\alpha}$ is in $\mathrm{Sp}_4(M)$. And

$$\mathrm{E}^2\tau_{x_1,\sqrt{\alpha}} = \tau_{x_1,\alpha}$$

by 5.4.8 and 5.4.9. So $\tau_{x_1,\sqrt{\alpha}}$ is in $\mathrm{Sp}_4(M)$. So $\sqrt{\alpha}$ is in $\mathfrak{o}$. So $\mathfrak{o}^2 = \mathfrak{o}$. In other words, *in the exceptional case of a perfect field of characteristic* 2 *with* $n = 4$, *the automorphisms of* $\mathrm{PSp}_4(M)$ *which are induced by* Φ_g*'s form a subgroup of index* 2 *in the entire group of automorphisms of* $\mathrm{PSp}_4(M)$ *if and only if* $\mathfrak{o}$ *is equal to its own set of squares*([6]). We will now see that both $\mathfrak{o}^2 = \mathfrak{o}$ and $\mathfrak{o}^2 \subset \mathfrak{o}$ are possible.

5.6.14. Remark. Let us produce a situation $0 \subset \mathfrak{o} \subset F$ with F perfect of characteristic 2 and $\mathfrak{o}^2 = \mathfrak{o}$. Start with a fixed perfect field k of characteristic 2, let x be transcendental over k, let C be an algebraic closure of $k(x)$. Then

$$k \cdot x^{p/2^q} = \left\{\alpha \cdot x^{p/2^q} \mid \alpha \in k\right\}$$

is an additive subgroup of C for any given integers $p \geqslant 0$, $q \geqslant 0$. (Here $x^{1/2^q}$, and so $x^{p/2^q}$, are well defined since the characteristic is 2.) Let $\mathfrak{o}$ be the additive subgroup

$$\mathfrak{o} = \sum_{p \geqslant 0, q \geqslant 0} k \cdot x^{p/2^q}$$

generated by all the $k \cdot x^{p/2^q}$. It is easily seen that $\mathfrak{o}$ is closed under multiplication in C, so $\mathfrak{o}$ is in fact an integral domain in C. It is also easily seen that $\mathfrak{o}^2 = \mathfrak{o}$. The quotient field of $\mathfrak{o}$ in C will then produce the desired situation provided we can prove that $\mathfrak{o}$ is not a field. But x is clearly in $\mathfrak{o}$; and if x^{-1} were

([6])This answers a question raised in a paper of mine in 1968.

in $\mathfrak{o}$ we would immediately get an algebraic equation over k satisfied by x, which is impossible. So $\mathfrak{o}$ is not a field, as desired.

5.6.15. REMARK. Now let us produce $0 \subset \mathfrak{o} \subset F$ with F perfect of characteristic 2 and $\mathfrak{o}^2 \subset \mathfrak{o}$. By Remark 5.6.14 we have an integral domain I with perfect quotient field k of characteristic 2 such that $0 \subset I \subset k$ and $I^2 = I$. Fix a nonzero nonunit α in I. So

$$0 \subset \alpha^2 I \subset \alpha I \subset I.$$

Let x be transcendental over k and let C be an algebraic closure of $k(x)$. Define

$$\mathfrak{o} = I + \sum_{p > 0, q > 0} I \cdot \alpha \cdot x^{p/2^q}.$$

This is an integral domain that contains I and is contained in C. Let F be the quotient field of $\mathfrak{o}$ in C. Then F contains k; hence F contains the larger integral domain

$$k + \sum_{p > 0, q > 0} k \cdot x^{p/2^q}.$$

This new integral domain is equal to its set of squares; hence F is perfect. We will be through if we can prove that $\mathfrak{o}^2 \subset \mathfrak{o}$. Now αx is clearly in $\mathfrak{o}$. If $\mathfrak{o}^2 = \mathfrak{o}$, then αx is in $\mathfrak{o}^2$, so there is an expression

$$\alpha x = \left(A_0 + \alpha\varphi(x^{1/2^q})\right)^2$$

with A_0 in I and $\varphi(x^{1/2^q})$ a polynomial in $x^{1/2^q}$ for some $q \geqslant 0$ with coefficients in I. Putting $t = x^{1/2^q}$,

$$\alpha t^{2q} = \left(A_0 + \alpha\varphi(t)\right)^2,$$

so $\alpha \in \alpha^2 I$ since t is transcendental over k, so $\alpha I \subseteq \alpha^2 I \subset \alpha I$, and this is absurd.

5.7. Comments

The isomorphism theory of the classical groups over fields was initiated by

O. SCHREIER AND B. L. VAN DER WAERDEN, *Die Automorphismen der projektiven Gruppen*, Abh. Math. Sem. Univ. Hamburg **6** (1928), 303–322,

in which the automorphisms of PSL_n were determined over arbitrary commutative fields. Several years later

L.-K. HUA, *On the automorphisms of the symplectic group over any field*, Ann. of Math. (2) **49** (1948), 739–759

extended the theory to symplectic groups in characteristic not 2. Then

J. DIEUDONNÉ, *On the automorphisms of the classical groups*, Mem. Amer. Math. Soc., No. 2, Amer. Math. Soc., Providence, R.I., 1951,

C. E. RICKART, *Isomorphisms of infinite-dimensional analogues of the classical groups*, Bull. Amer. Math. Soc. **57** (1951), 435–448

introduced their method of involutions which was subsequently used by them and a large number of other authors to determine isomorphisms and nonisomorphisms among big classical groups. (A big classical group is one like $PSL_n(V)$, $PSp_n(V)$, etc.) For finite fields

E. ARTIN, *The orders of the linear groups*, Comm. Pure Appl. Math. **8** (1955), 355–365.
E. ARTIN, *The orders of the classical simple groups*, Comm. Pure Appl. Math. **8** (1955), 455–472

gave an entirely different argument, based on comparing group orders, to establish the expected nonisomorphisms. Automorphisms and isomorphisms of Chevalley groups have been found over various fields in

R. STEINBERG, *Automorphisms of finite linear groups*, Canad. J. Math. **12** (1960), 606–615.
R. STEINBERG, *Lectures on Chevalley groups*, Yale Lecture Notes, 1967.
J. E. HUMPHREYS, *On the automorphisms of infinite Chevalley groups*, Canad. J. Math. **21** (1969), 908–911.

An isomorphism theory for a wide class of groups that includes big linear and symplectic groups over infinite fields, as well as big classical groups in the isotropic case over infinite fields, is provided by

A. BOREL AND J. TITS, *Homomorphismes "abstraits" de groupes algébriques simples*, Ann. of Math. (2) **97** (1973), 499–571.

The first move towards an automorphism theory over rings, in fact for linear groups over **Z**, was made by Hua and Reiner in 1951. This was extended to the symplectic groups over **Z** in

I. REINER, *Automorphisms of the symplectic modular group*, Trans. Amer. Math. Soc. **80** (1955), 35–50.

The automorphisms of standard symplectic groups over arbitrary integral domains for $n \geqslant 4$ were determined in

O. T. O'MEARA, *The automorphisms of the standard symplectic group over any integral domain*, J. reine angew. Math. **230** (1968), 104–138.

The automorphisms of certain groups of integral points of certain split groups over algebraic number fields were determined by

A. BOREL, *On the automorphisms of certain subgroups of semi-simple Lie groups*, Proc. Conf. on Algebraic Geometry, Bombay, 1968, 43–73, thereby giving the automorphisms of symplectic groups over arithmetic domains of number fields as a special case. In

O. T. O'MEARA, *The automorphisms of the orthogonal groups* $\Omega_n(V)$ *over fields*, Amer. J. Math. **90** (1968), 1260–1306,

a method called CDC was introduced which was subsequently used by a number of authors to find the isomorphisms of several of the classical groups over integral domains.The automorphisms of subgroups of $\mathrm{PSp}_n(V)$ with enough projective transvections were determined in

R. E. SOLAZZI, *The automorphisms of the symplectic congruence groups*, J. Algebra **21** (1972), 91–102,

for char $F \neq 2$ with. $n \geqslant 6$, while

A. J. HAHN, *The isomorphisms of certain subgroups of the isometry groups of reflexive spaces*, J. Algebra **27** (1973), 205–242

determined the isomorphisms and nonisomorphisms between such groups, indeed gave a unified treatment including linear, symplectic and unitary groups, when $n \geqslant 5$. Their methods depend on CDC. So do the methods in my *Linear Lectures*. In the summer of 1974, after unsuccessfully trying to adapt CDC to noncommutative fields, I developed a new approach which is less group-theoretic, more geometric, than CDC and which works for subgroups of linear groups over division rings. These results have appeared in

O. T. O'MEARA, *A general isomorphism theory for linear groups*, J. Algebra **44** (1977), 93–142.

I extended this approach to the symplectic group in lectures at Notre Dame during the academic year in 1974–1975 (these are the notes of the lectures), thereby extending the isomorphism theory from subgroups of PSp_n ($n \geqslant 6$) to subgroups of $\mathrm{P\Gamma Sp}_n$ ($n \geqslant 4$). The exceptional automorphisms of the big groups $\mathrm{PSp}_4(V)$ over perfect fields of characteristic 2 date back to

J. TITS, *Les groupes simples de Suzuki et de Ree*, Sem. Bourbaki, 13ᵉ annee, no. 210, 1960/61,
Z.-X. WAN AND Y.-X. WANG, *On the automorphisms of symplectic groups over a field of characteristic* 2, Sci. Sinica **12** (1963), 289–315.

I wish to thank Warren Wong for bringing these exceptional automorphisms to my attention several years ago, and also for giving me the explicit description of the monomorphism E of §5.4. The isomorphism theory for subgroups of $\mathrm{P\Gamma Sp}_4$ which is developed in §5.4 is new.

L. MCQUEEN AND B. R. MCDONALD, *Automorphisms of the symplectic group over a local ring*, J. Algebra **30** (1974), 485–495,

have moved the automorphism theory of the symplectic group to rings with zero divisors. For further references to the extensive literature on the isomorphism theory see the *Linear Lectures* and

> J. DIEUDONNÉ, *La géométrie des groupes classiques*, 3ième ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 5, Springer-Verlag, Berlin and New York, 1971,

and the survey articles

> J. TITS, *Homomorphismes et automorphismes "abstraits" de groupes algébriques et arithmétiques*. Actes Congrès Int. Math. Nice **2** (1970), 349–355,
>
> O. T. O'MEARA, *The integral classical groups and their automorphisms*, Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., Providence, R.I., 1971, pp. 76–85,
>
> YU. I. MERZLYAKOV, *Linear groups*, J. Soviet Math. **1** (1973), 571–593.

6. THE NONISOMORPHISMS BETWEEN LINEAR AND SYMPLECTIC GROUPS

6.1. The Nonisomorphisms

We continue to consider the symplectic situation of a subgroup Δ of $\mathrm{P\Gamma Sp}_n(V)$ which has enough projective transvections, where V is a nonzero n-dimensional regular alternating space over a field F with underlying form q.

And we introduce the following abstract situation: V_3 will be an n_3-dimensional vector space over an arbitrary field F_3, and Δ_3 will be a subgroup of $\mathrm{P\Gamma L}_{n_3}(V_3)$ that is full of projective transvections. Recall from the *Linear Lectures* that a subgroup Δ_3 of $\mathrm{P\Gamma L}_{n_3}(V_3)$ is said to be full of projective transvections if $n_3 \geqslant 2$ and, for each hyperplane H of V_3 and each line $L \subseteq H$, there is at least one projective transvection σ in Δ_3 with $R = L$ and $P = H$.

We let Λ denote an isomorphism $\Lambda\colon \Delta \rightarrowtail\!\!\!\!\rightarrow \Delta_3$. Our goal is to show that, generally speaking, Λ does not exist.

6.1.1. *Let $n_3 \geqslant 3$ and $F_3 \neq \mathbf{F}_2$. Then for each hyperplane H in V_3 and each line $L \subseteq H$ there are at least two distinct projective transvections in Δ_3 with residual line L and fixed hyperplane H.*

PROOF. See 5.2.8 of the *Linear Lectures*. Q.E.D.

6.1.2. *Suppose $n \geqslant 4$, $n_3 \geqslant 4$. Let τ be a transvection in $\mathrm{Sp}_n(V)$ that is projectively in Δ, and let k_1 be an element of $\Gamma\mathrm{L}_{n_3}(V_3)$ with $\Lambda_3\bar{\tau} = \bar{k}_1$. Suppose there is a transvection τ_1 in $\mathrm{GL}_{n_3}(V_3)$ that is projectively in Δ_3 such that $\sigma_1 = k_1\tau_1k_1^{-1}\tau_1^{-1}$ has residue 2 with $R_1 \cap P_1 = 0$. Then $\bar{k}_1$ has a representative in $\mathrm{GL}_{n_3}(V_3)$ with residue 1.*

PROOF. (1) Pick k in $\Gamma\mathrm{Sp}_n(V)$ with $\Lambda\bar{k} = \bar{\tau}_1$ and put

$$\sigma = \tau k \tau^{-1} k^{-1}$$

so that

$$\Lambda\bar{\sigma} = \bar{\sigma}_1.$$

By hypothesis R_1 is a plane, and P_1 is an $(n_3 - 2)$-space, and $R_1 \cap P_1 = 0$. And the equation $\sigma = \tau k\tau^{-1}k^{-1}$ shows that R is either a line or plane containing the residual line of τ. By Theorem 1.7.3 we know that σ_1 is not a big dilation; so any element of $\Gamma L_{n_3}(V_3)$ which permutes projectively with σ_1 will also permute with σ_1, by 4.1.10. And any element of $\Gamma L_n(V)$ that permutes projectively with σ will also permute with σ.

(2) If L_1 is an arbitrary line in P_1, and if H_1 is an arbitrary hyperplane of V_3 which contains L_1 and R_1, and if we pick a transvection τ_{L_1,H_1} in $SL_{n_3}(V_3)$ that is projectively in Δ_3 and has spaces $L_1 \subseteq H_1$, and if j is an element of $\Gamma Sp_n(V)$ for which $\Lambda\bar{j} = \bar{\tau}_{L_1,H_1}$, then $jP = P$ and j moves at least one line in P. The proof of all this is almost identical to the proof of step (2) of 5.2.1.

(3) Given any line L_1 in P_1 and any hyperplane H_1 of V_3 which contains L_1 and R_1, there are elements σ_2 and σ_3 with $\Lambda\bar{\sigma}_2 = \bar{\sigma}_3$ such that

$$\begin{array}{ll} \sigma_2 \in Sp_n(V), & \sigma_3 \in SL_{n_3}(V_3), \\ \bar{\sigma}_2 \in \Delta, & \bar{\sigma}_3 \in \Delta_3, \\ \sigma_2 \text{ not big dilation}, & \sigma_3 \text{ not big dilation}, \\ \operatorname{res} \sigma_2 = 2, & 1 \leqslant \operatorname{res} \sigma_3 \leqslant 2, \\ R_2 \subseteq P, & R_3 \subseteq P_1 \text{ and } P_3 \supseteq R_1, \\ & L_1 \subseteq R_3 \text{ or } H_1 = P_3, \\ & H_1 \supseteq P_3 \text{ or } L_1 = R_3. \end{array}$$

To see this, pick τ_{L_1,H_1} and j as in step (2). Then $jP = P$ and $jL \neq L$ for some line L in P, by step (2). Let T_L be a transvection in $Sp_n(V)$ that is projectively in Δ and has residual line L, let J_1 be a representative in $\Gamma L_{n_3}(V_3)$ of $\Lambda\bar{T}_L$. Then

$$\sigma_2 = jT_Lj^{-1}T_L^{-1}, \qquad \sigma_3 = \tau_{L_1,H_1}J_1\tau_{L_1,H_1}^{-1}J_1^{-1},$$

have the desired properties.

(4) Let us show that k_1 stabilizes P_1 and R_1. Pick σ_2 and σ_3 as in step (3) (for any choice of L_1 and H_1). Then $R_2 \subset P$ implies that τ permutes with σ_2; hence k_1 permutes with σ_3, so k_1 stabilizes R_3. In particular k_1 stabilizes a nonzero subspace of P_1. Let W_1 be any nonzero subspace of P_1 that is stabilized by k_1. If $W_1 = P_1$ we have $k_1P_1 = P_1$. Otherwise $0 \subset W_1 \subset P_1$. It is easily seen that there is a transvection in $SL_{n_3}(V_3)$ that is projectively in Δ_3, that permutes with σ_1, and that carries R_3 outside W_1 (but still in P_1). Conjugating σ_3 by this transvection and carrying things back to $Sp_n(V)$ in the usual way, we obtain a new situation L_1', H_1', σ_2', σ_3', etc., as in step (3), with $R_3' \nsubseteq W_1$. Then k_1 stabilizes W_1 and R_3', i.e., stabilizes a subspace of P_1 that is larger than W_1. So k_1 stabilizes P_1. To prove that k_1 stabilizes R_1, proceed in an analogous way, working with spaces that contain R_1 instead of with spaces that are contained in P_1.

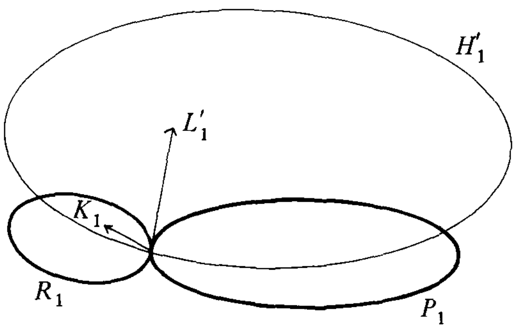
(5) We shall now show that k_1 stabilizes all lines in P_1.

(5a) First let $n_3 \geqslant 5$. Let L_1 be a typical line in P_1. If k_1 stabilizes all hyperplanes H_1 of V_3 containing L_1 and R_1, then k_1 stabilizes all resulting $H_1 \cap P_1$, so k_1 stabilizes all hyperplanes of P_1 containing L_1, so k_1 stabilizes L_1. We may therefore assume, for the sake of argument, that there is a hyperplane H_1 of V_3 which contains $L_1 + R_1$, but which is not stabilized by k_1. Let σ_2 and σ_3 be associated with L_1 and this H_1 in the manner of step (3). Then τ permutes with σ_2, and this leads to $k_1R_3 = R_3$ and $k_1P_3 = P_3$. Since H_1 is not stabilized by k_1 we must have $H_1 \neq P_3$, and hence $L_1 \subseteq R_3$ by step (3). It is easily seen that there is a transvection in $\mathrm{SL}_{n_3}(V_3)$ that is projectively in Δ_3, that permutes with σ_1, and that carries R_3 to a subspace of P_1 which intersects R_3 in L_1. Conjugating σ_3 by this transvection and carrying things back to $\mathrm{Sp}_n(V)$ in the usual way, we obtain a new situation L_1', H_1', σ_2', σ_3', etc., as in step (3), with $R_3' \cap R_3 = L_1$. Then k_1 stabilizes R_3' as well as R_1'; hence k_1 stabilizes L_1, as required.

(5b) Next consider $n_3 = 4$ with $F_3 \neq \mathbf{F}_2$. Proceed as in step (5b) of the proof of 5.2.1. Use 6.1.1.

(5c) Now $n_3 = 4$ with $F_3 = \mathbf{F}_2$. In this situation $\mathrm{P\Gamma L}_4(V_3) = \mathrm{PSL}_4(V_3) = \Delta_3$, so card $\Delta_3 = \frac{1}{2}\cdot(8!)$. Therefore card $\Delta = \frac{1}{2}\cdot(8!)$. This is impossible by 5.1.4.

(6) So k_1 stabilizes all lines in P_1. So k_1 is a radiation on P_1. Replacing k_1 by another representative of $\bar{k}_1$ therefore allows us to assume that the k_1 in the statement of the proposition is such that $(k_1|P_1) = 1_{P_1}$. In particular, k_1 is an element of $\mathrm{GL}_{n_3}(V_3)$ with $1 \leqslant \operatorname{res} k_1 \leqslant 2$. If $\operatorname{res} k_1 = 1$, we are through. Therefore assume that $\operatorname{res} k_1 = 2$. The residual space of k_1 is therefore the plane R_1. If k_1 were a big dilation, then $k_1\tau_1k_1^{-1}\tau_1^{-1}$ could not have residual space R_1. So k_1 is not a big dilation. So k_1 moves a line K_1 in R_1. Let H_1' be a hyperplane of V_3 which intersects R_1 in K_1 and P_1 in a hyperplane of P_1. Let L_1' be a line



in $K_1 + (H_1' \cap P_1)$ that falls neither in K_1 nor in $(H_1' \cap P_1)$. We find

$$L_1' \subseteq H_1',$$
$$k_1L_1' \neq L_1', \quad k_1H_1' \neq H_1',$$
$$k_1L_1' \not\subseteq H_1', \quad L_1' \not\subseteq k_1H_1',$$

whence

$$\dim(L_1' + k_1L_1') = 2, \qquad \dim(H_1' \cap k_1H_1') = n_3 - 2,$$

and

$$V_3 = (L_1' + k_1L_1') \oplus (H_1' \cap k_1H_1').$$

Let τ_1' be a transvection in $\mathrm{GL}_{n_3}(V_3)$ which is projectively in Δ_3 and which has spaces $L_1' \subseteq H_1'$. Then it is easily verified that the residual and fixed spaces of $\sigma_1' = k_1(\tau_1')k_1^{-1}(\tau_1')^{-1}$ are, respectively,

$$R_1' = (L_1' + k_1L_1'), \qquad P_1' = (H_1' \cap k_1H_1').$$

The statement of the proposition therefore applies to the τ, k_1, τ_1', σ_1' situation. Therefore, by step (5), k_1 is a radiation on the $(n_3 - 2)$-space $(H_1' \cap k_1'H_1')$. This space is distinct from the $(n_3 - 2)$-space P_1. This clearly implies that res $k_1 = 1$ if $n_3 \geqslant 5$; and that k_1 is a big dilation or has residue 1, hence has residue 1, when $n_3 = 4$. Q.E.D.

6.1.3. *Suppose $n_3 \geqslant 3$. Let k_1 be any element of $\Gamma L_{n_3}(V_3)$ that moves a line L_1 of V_3. Then there is a hyperplane H_1 of V_3 containing L_1 such that for every transvection τ_1 in $GL_{n_3}(V_3)$ with residual line L_1 and fixed hyperplane H_1, $\sigma_1 = k_1\tau_1k_1^{-1}\tau_1^{-1}$ is an element of $GL_{n_3}(V_3)$ of residue 2 with $R_1 = (L_1 + k_1L_1)$ and $R_1 \cap P_1 = 0$.*

PROOF. Pick a hyperplane H_1 of V_1 such that $L_1 \subseteq H_1$, $k_1L_1 \not\subseteq H_1$, $k_1^{-1}L_1 \not\subseteq H_1$ (why possible?). We have

$$L_1 \not\subseteq k_1H_1, \qquad H_1 \neq k_1H_1,$$

and

$$\dim(L_1 + k_1L_1) = 2, \qquad \dim(H_1 \cap k_1H_1) = n_3 - 2,$$

and

$$V_3 = (L_1 + k_1L_1) \oplus (H_1 \cap k_1H_1).$$

Any τ_1 with spaces $L_1 \subseteq H_1$ will satisfy the required conditions. Q.E.D.

6.1.4. *Suppose $n \geqslant 4$, $n_3 \geqslant 4$. Let τ be a transvection in $Sp_n(V)$ that is projectively in Δ. Then $\Lambda\bar{\tau}$ is a projective transvection in Δ_3.*

PROOF. Suppose if possible that we have a τ for which $\Lambda\bar{\tau}$ is not a projective transvection. We will show that this will lead to a contradiction. Of course τ is nontrivial. Let L be its residual line. By 6.1.3 and 6.1.2 there is a k_1 in $GL_{n_3}(V_3)$ with res $k_1 = 1$ such that $\bar{k}_1 = \Lambda\bar{\tau} \in \Delta_3$. By our hypothesis k_1 is not a transvection, so, by 1.3.8, $V_3 = L_1 \oplus H_1$ where L_1 denotes the residual line of k_1 and H_1 denotes its fixed hyperplane. Let I denote a set of indices labelling the lines L_i $(i \in I)$ of H_1. It is easily seen that if we commutate k_1 with a transvection in $GL_{n_3}(V_3)$ with spaces $L_i \subseteq H_1$ that is projectively in Δ_3 we obtain a transvection, call it τ_i, that is projectively in Δ_3 and whose spaces are $L_i \subseteq H_1$. Carrying this commutator back to $\Gamma Sp_n(V)$ in the usual way we obtain an element σ_i with the following properties:

$$\sigma_i \in Sp_n(V), \qquad \bar{\sigma}_i \in \Delta,$$

$$\sigma_i \text{ is not a big dilation,}$$

$$R_i \text{ is a regular plane,}$$

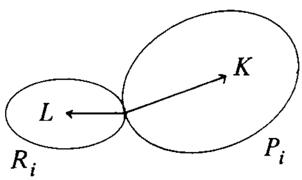
$$V = R_i \perp P_i, \qquad L \subseteq R_i,$$

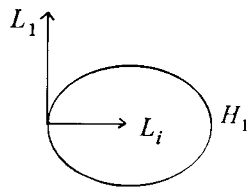
$$\Lambda\bar{\sigma}_i = \bar{\tau}_i.$$

Let us show that all R_i $(i \in I)$ must be equal. Consider α, β in I. Then τ_β permutes with τ_α; hence σ_β permutes with σ_α; hence σ_β stabilizes R_α and P_α. If $(\sigma_\beta|P_\alpha) = 1_{P_\alpha}$ we are through. If not, then res$(\sigma_\beta|R_\alpha)$ is 0 or 1 and so, since

$L \subseteq R_\alpha \cap R_\beta$, $\operatorname{res}(\sigma_\beta | R_\alpha) = 1$. This makes $\operatorname{res}(\sigma_\beta | P_\alpha) = 1$. But then R_β is degenerate, which it is not. So $(\sigma_\beta | P_\alpha) = 1_{P_\alpha}$, i.e., $R_\beta = R_\alpha$, as required. Of course this makes all P_i $(i \in I)$ equal too.

Now let τ_K be a transvection in $\mathrm{Sp}_n(V)$ whose residual line K is a typical line in P_i and which is projectively in Δ. Let k_K be a representative in $\Gamma\mathrm{L}_{n_3}(V_3)$ of $\Lambda\bar{\tau}_K$. Then by standard methods





k_K stabilizes all L_i $(i \in I)$. So in fact we can assume that k_K stabilizes H_1 with $(k_K | H_1) = 1_{H_1}$. Now τ_K permutes with τ since K and L are orthogonal, so all k_K permute with k_1, so all k_K have fixed space H_1 and residual space L_1. This implies that all k_K permute as K runs through P_i. But all τ_K clearly do not. So we have our desired contradiction. So $\Lambda\bar{\tau}$ is indeed a projective transvection. Q.E.D.

6.1.5. *Λ does not exist when $n \geqslant 4$, $n_3 \geqslant 4$.*

PROOF. We suppose that Λ does exist and we show that this leads to a contradiction. In what follows we will construct nontrivial transvections τ, τ_2, τ_4, τ_6 that belong to $\mathrm{Sp}_n(V)$, that belong projectively to Δ, and whose residual lines will be written L, L_2, L_4, L_6. And we will construct transvections τ_1, τ_3, τ_5, τ_7 that belong to $\mathrm{SL}_{n_3}(V_3)$, that belong projectively to Δ_3, with spaces

$$L_1 \subseteq H_1, \quad L_3 \subseteq H_3, \quad L_5 \subseteq H_5, \quad L_7 \subseteq H_7,$$

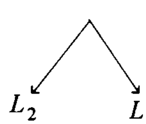
and such that

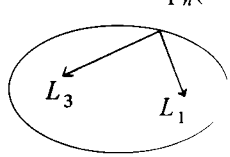
$$\Lambda\bar{\tau} = \bar{\tau}_1, \quad \Lambda\bar{\tau}_2 = \bar{\tau}_3, \quad \Lambda\bar{\tau}_4 = \bar{\tau}_5, \quad \Lambda\bar{\tau}_6 = \bar{\tau}_7.$$

Pick τ in any way at all (but subject to the above specifications, of course). By 6.1.4 we have a τ_1 (again as above) such that $\Lambda\bar{\tau} = \bar{\tau}_1$. Define τ_3 by conjugating τ_1 by a suitable transvection in such a way that

$$L_1 \neq L_3 \subseteq H_3 = H_1,$$

and then define τ_2 by pulling this conjugation back to $\mathrm{Sp}_n(V)$. It



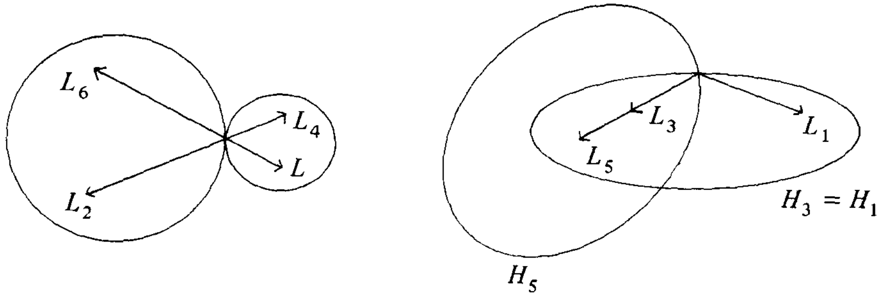


$H_3 = H_1$

follows from the fact that τ_3 permutes with τ_1 that $q(L_2, L) = 0$. Now construct τ_5 from τ_3, and then τ_4 from τ_5, by a similar construction, in such a way that

$$L_5 = L_3, \qquad L_1 \nsubseteq H_5.$$

Then τ_5 permutes with τ_3 but not with τ_1; from this it follows that



$q(L_4, L_2) = 0$ and $q(L_4, L) \neq 0$. Now let τ_6 be any transvection (of the above form) for which L_6 is orthogonal to the regular plane $(L_4 + L)$, but not to the line L_2, and then define τ_7 by $\Lambda\bar{\tau}_6 = \bar{\tau}_7$ (use 6.1.4 again). Since τ_6 permutes with τ, we have τ_7 permuting with τ_1, so

$$L_7 \subseteq H_1 = H_3;$$

similarly

$$H_7 \supseteq L_5 = L_3;$$

but then τ_7 permutes with τ_3, so τ_6 permutes with τ_2, so $q(L_6, L_2) = 0$, i.e., L_6 is orthogonal to L_2, and this is absurd. Q.E.D.

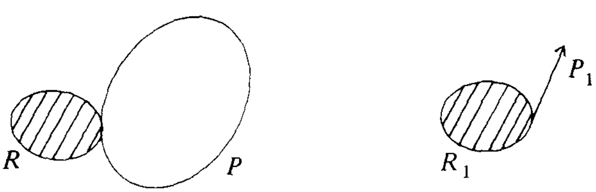
6.1.6. Λ *does not exist when* $n \geqslant 4$, $n_3 = 3$, *and* $\Delta_3 \subseteq \mathrm{PGL}_3(V_3)$.

Proof. We assume that Λ does exist and we show that this leads to a contradiction. If $F_3 = \mathbf{F}_2$, then card $\Delta_3 = 168$, so card $\Delta = 168$, and this is impossible by 5.1.4. Therefore we can assume from now on that $F_3 \neq \mathbf{F}_2$.

(1) Let us show that we have elements σ and σ_1 with $\Lambda\bar{\sigma} = \bar{\sigma}_1$ such that

$\sigma \in \mathrm{Sp}_n(V)$,	$\sigma_1 \in \mathrm{SL}_3(V_3)$,
$\bar{\sigma} \in \Delta$,	$\bar{\sigma}_1 \in \Delta_3$,
R regular plane,	R_1 plane, P_1 line,
	$R_1 \cap P_1 = 0$,
	σ_1 not big dilation.

Pictorially, then, we are looking for



(1a) Pick a transvection τ in $\mathrm{Sp}_n(V)$ that is projectively in Δ, and fix it. Then pick k_1 in $\mathrm{GL}_3(V_3)$ with $\Lambda\bar{\tau} = \bar{k}_1$. Since $\bar{k}_1$ is nontrivial there will be a line $L_1 = F_3 a$ in V_3 with $k_1 L_1 \neq L_1$. Pick a hyperplane H_1 of V_3 such that

$$L_1 \subseteq H_1, \qquad k_1 L_1 \nsubseteq H_1, \qquad k_1^{-1} L_1 \nsubseteq H_1$$

(why possible?). We have

$$L_1 \nsubseteq k_1 H_1, \qquad H_1 \neq k_1 H_1,$$

and

$$\dim(L_1 + k_1L_1) = 2, \qquad \dim(H_1 \cap k_1H_1) = 1,$$

and

$$V_3 = (L_1 + k_1L_1) \oplus (H_1 \cap k_1H_1).$$

(1b) Let ρ be a linear functional describing the hyperplane H_1 of V_3. Of course there are several nonzero a in L_1 for which $\bar{\tau}_{a,\rho} \in \Delta_3$ since Δ_3 is full of projective transvections. We claim that there is at least one such a for which k_1 does not permute projectively with $\tau_{a,\rho}k_1\tau_{a,\rho}^{-1}$. Suppose this does not hold for a first choice of a. Then there is a scalar α in F_3 such that

$$k_1\tau_{a,\rho}k_1^{-1}\tau_{a,\rho}^{-1} = r_\alpha\tau_{a,\rho}k_1^{-1}\tau_{a,\rho}^{-1}k_1,$$

i.e.,

$$\tau_{k_1a,\rho k_1^{-1}}\tau_{-a,\rho} = r_\alpha\tau_{a,\rho}\tau_{-k_1^{-1}a,\rho k_1}.$$

Hence

$$(\alpha - 1)x + \big((\alpha + 1)(\rho x) - \alpha(\rho k_1x)(\rho k_1^{-1}a)\big)a$$
$$+ \big((\rho x)(\rho k_1^{-1}a) - (\rho k_1^{-1}x)\big)k_1a = \alpha(\rho k_1x)k_1^{-1}a,$$

for all x in V_3. Putting $x = a$ shows that $a, k_1a, k_1^{-1}a$ are dependent, i.e., they all fall in a plane; taking x outside this plane shows that $\alpha = 1$; so

$$\big(2(\rho x) - (\rho k_1x)(\rho k_1^{-1}a)\big)a + \big((\rho x)(\rho k_1^{-1}a) - (\rho k_1^{-1}x)\big)k_1a = (\rho k_1x)k_1^{-1}a.$$

Since $F_3 \neq \mathbf{F}_2$ we can replace a by λa for some $\lambda \neq 0, 1$, by 6.1.1. The last equation, along with its counterpart for λa, then yields

$$(\rho k_1x)(\rho k_1^{-1}a) = \lambda(\rho k_1x)(\rho k_1^{-1}a),$$

and this is absurd since $\lambda \neq 1$ and $\rho k_1^{-1}a \neq 0$. Therefore, if a does not work, λa will. Our claim is established.

(1c) We now have a linear functional ρ describing H_1 and a nonzero vector a in L_1 such that $\bar{\tau}_{a,\rho} \in \Delta_3$ with k_1 not permuting projectively with $\tau_{a,\rho}k_1^{-1}\tau_{a,\rho}^{-1}$. Choose k in $\Gamma\mathrm{Sp}_n(V)$ with $\bar{k}$ in Δ and $\Lambda\bar{k} = \bar{\tau}_{a,\rho}$. Define

$$\sigma = \tau k\tau^{-1}k^{-1}, \qquad \sigma_1 = k_1\tau_{a,\rho}k_1^{-1}\tau_{a,\rho}^{-1}.$$

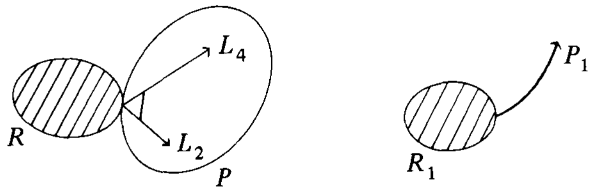
It is easily verified, using standard arguments, that σ and σ_1 have the properties desired in step (1).

(2) Note that

$$\operatorname{char} F = 3 \Rightarrow \sigma_1^3 \neq 1;$$

for the equation $\sigma_1^3 = 1$ implies $\sigma^3 = 1$; hence $(\sigma|R)^3 = 1$; hence $(\sigma|R)$ has 1 as a characteristic root; hence res $\sigma \leqslant 1$, and this is absurd.

(3) Let τ_2 and τ_4 be transvections which belong to $\mathrm{Sp}_n(V)$, which belong projectively to Δ, and whose residual spaces are nonorthogonal lines in P. Pick k_3 and k_5 in $\mathrm{GL}_3(V_3)$ with $\Lambda\bar{\tau}_2 = \bar{k}_3$



and $\Lambda\bar{\tau}_4 = \bar{k}_5$. If char $F \neq 3$, then τ_2^3 and τ_4^3 are transvections with residual spaces L_2 and L_4; so τ_2 and τ_4 permute with σ, but τ_2^3 and τ_4^3 do not permute with each other; so $\bar{k}_3$ and $\bar{k}_5$ permute with $\bar{\sigma}_1$, but $\bar{k}_3^3$ and $\bar{k}_5^3$ do not permute with each other; so k_3^3 and k_5^3 permute with σ_1 (use 4.1.11) but not with each other. This is impossible by 1.2.4. If char $F = 3$ proceed in the same way. This time we find that $\bar{k}_3$ and $\bar{k}_5$ permute with $\bar{\sigma}_1$ but not with each other; so k_3 and k_5 do not permute with each other. If k_3 did not permute with σ_1, then $k_3\sigma_1 k_3^{-1} = \zeta\sigma_1$ for some ζ in F_3 with $\zeta \neq 0, 1$ and $\zeta^3 = 1$. This implies that σ_1 has characteristic roots $1, \zeta, \zeta^2$ in F_3. From this it follows that $\sigma_1^3 = 1$, and this is impossible by step (2); so k_3 does permute with σ_1; similarly for k_5; so k_3 and k_5 are elements of $\mathrm{GL}_3(V_3)$ which permute with σ_1 but not with each other. This is impossible by 1.2.4.

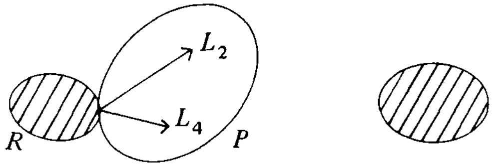
We have our desired contradiction. Q.E.D.

6.1.7. Λ *does not exist when* $n \geqslant 4$, $n_3 = 2$, *and* $\Delta_3 \subseteq \mathrm{PGL}_2(V_3)$.

Proof. Pick elements σ and σ_1 with $\Lambda\bar{\sigma} = \bar{\sigma}_1$ such that

$$\sigma \in \mathrm{Sp}_n(V), \qquad \sigma_1 \in \mathrm{GL}_2(V_3),$$
$$\bar{\sigma} \in \Delta, \qquad \bar{\sigma}_1 \in \Delta_3,$$
$$R \text{ regular plane.}$$

Pictorially, we are looking at



Let τ_2 and τ_4 be transvections which belong to $\mathrm{Sp}_n(V)$, which belong projectively to Δ, and whose residual spaces are nonorthogonal lines L_2 and L_4 in P. Pick k_3 and k_5 in $\mathrm{GL}_2(V_3)$ with $\Lambda\bar{\tau}_3 = \bar{k}_3$ and $\Lambda\bar{\tau}_4 = \bar{k}_5$. If char $F \neq 2$, then τ_2^2 and τ_4^2 are transvections with residual spaces L_2 and L_4; so τ_2 and τ_4 permute with σ, but τ_2^2 and τ_4^2 do not permute with each other; so $\bar{k}_3$ and $\bar{k}_5$ permute with $\bar{\sigma}_1$, but $\bar{k}_3^2$ and $\bar{k}_5^2$ do not permute with each other; so k_3^2 and k_5^2 permute with σ_1 (use 4.1.11) but not with each other. This is impossible by 1.2.4. If char $F = 2$ we find that $\bar{k}_3$ and $\bar{k}_5$ permute with $\bar{\sigma}_1$ but not with each other; so k_3 and k_5 permute with σ_1^2 but not with each other; but $\bar{\sigma}^2 \neq 1$ since char $F = 2$, so $\sigma_1^2 \notin RL_2(V_3)$; again impossible by 1.2.4. Q.E.D.

6.1.8. Theorem. *A projective group of symplectic collinear transformations which has enough projective transvections and whose underlying dimension is* $\geqslant 4$ *cannot* *be isomorphic to a projective group of collinear transformations (resp. linear transformations) that is full of projective transvections and whose underlying dimension is* $\geqslant 4$ *(resp.* $\geqslant 2$).

We now extend the nonisomorphism theorems to the nonprojective case. Accordingly consider a subgroup Γ of $\Gamma\mathrm{Sp}_n(V)$ with enough transvections, and a subgroup Γ_3 of $\Gamma\mathrm{L}_{n_3}(V_3)$ that is full of transvections. Recall from the *Linear Lectures* that a subgroup Γ_3 of $\Gamma\mathrm{L}_{n_3}(V_3)$ is said to be full of transvections if $n_3 \geqslant 2$ and, for each hyperplane H of V_3 and each line $L \subseteq H$, there is at least one transvection σ in Γ_3 with $R = L$ and $P = H$.

We let Φ denote an isomorphism $\Phi: \Gamma \rightarrowtail \Gamma_3$. Our goal is to show that, generally speaking, Φ does not exist.

Note that $\bar{\Gamma} = P\Gamma$ is a subgroup of $P\Gamma\mathrm{Sp}_n(V)$ that has enough projective transvections, and $\bar{\Gamma}_3 = P\Gamma_3$ is a subgroup of $P\Gamma\mathrm{L}_{n_3}(V_3)$ that is full of projective transvections. The preceding nonisomorphism theory therefore applies to $\bar{\Gamma}$ and $\bar{\Gamma}_3$.

6.1.9. Φ *naturally induces an isomorphism of* $\bar{\Gamma}$ *onto* $\bar{\Gamma}_3$ *when* $n \geqslant 4$, $n_3 \geqslant 3$.

PROOF. It is enough to verify that $\Phi(\Gamma \cap \mathrm{RL}_n) = \Gamma_3 \cap \mathrm{RL}_3$.

(1) First let us verify that $\Phi(\Gamma \cap \mathrm{RL}_n) \subseteq \Gamma_3 \cap \mathrm{RL}_{n_3}$. Suppose to the contrary that there is a radiation r in Γ such that Φr is a nonradiation in Γ_3. Put $\Phi r = k_1$. Then k_1 moves a line L_1 in V_3. So using 6.1.3 we can find elements σ and σ_1 with $\Phi\sigma = \sigma_1$ such that

$$\begin{aligned} \sigma \in \Gamma \cap \mathrm{GSp}_n(V), \quad & \sigma_1 \in \Gamma_3 \cap \mathrm{SL}_{n_3}(V_3), \\ & R_1 \text{ plane}, \\ & R_1 \cap P_1 = 0, \\ & L_1 \subseteq R_1. \end{aligned}$$

By suitably conjugating on the right and pulling things back to the left, we can find another pair σ', σ_1' with the same properties as the pair σ, σ_1, and such that

$$R_1 \cap R_1' = L_1.$$

Then r permutes with σ and σ'; hence k_1 permutes with σ_1 and σ_1'; hence k_1 stabilizes R_1 and R_1'; hence k_1 stabilizes L_1, and this is absurd. So indeed $\Phi(\Gamma \cap \mathrm{RL}_n) \subseteq \Gamma_3 \cap \mathrm{RL}_{n_3}$.

(2) Conversely, consider r_1 in $\Gamma_3 \cap \mathrm{RL}_{n_3}$ and suppose, if possible, that $\Phi^{-1}r_1 = k$ is not in RL_n. Then k moves a line L in V. Let τ be a transvection in Γ with residual line L. Putting $\sigma = \tau k \tau^{-1} k^{-1}$ shows that we have elements σ and σ_1 with $\Phi\sigma = \sigma_1$ such that

$$\begin{aligned} & \sigma \in \Gamma \cap \mathrm{Sp}_n(V), \quad \sigma_1 \in \Gamma_3 \cap \mathrm{GL}_{n_3}(V_3), \\ & R \text{ plane}, \\ & L \subseteq R. \end{aligned}$$

By suitably conjugating on the left and pushing things to the right we find another pair σ', σ_1' with the same properties as σ, σ_1, and such that $R \cap R' = L$. Then r_1 permutes with σ_1 and σ_1'; hence k permutes with σ and σ'; hence k

stabilizes R and R'; hence k stabilizes L, and this is also absurd. So $\Phi^{-1}(\Gamma_3 \cap \mathrm{RL}_{n_3}) \subseteq \Gamma \cap \mathrm{RL}_n$. Q.E.D.

6.1.10. Φ *naturally induces an isomorphism of* $\bar{\Gamma}$ *onto* $\bar{\Gamma}_3$ *when* $n \geqslant 4$, $n_1 = 2$, *and* $\Gamma_3 \subseteq \mathrm{GL}_3(V_3)$.

Proof. Again it is enough to verify that $\Phi(\Gamma \cap \mathrm{RL}_n) = \Gamma_3 \cap \mathrm{RL}_2$.

(1) In order to see that $\Phi(\Gamma \cap \mathrm{RL}_n) \subseteq \Gamma_3 \cap \mathrm{RL}_2$, consider $r \in \Gamma \cap \mathrm{RL}_n$. Let τ_1 and τ_2 be transvections in Γ with nonorthogonal lines. Then τ_1 and τ_2 permute with r but not with each other. So the elements $\Phi\tau_1$ and $\Phi\tau_2$ of GL_2 permute with $\Phi r \in \mathrm{GL}_2$ but not with each other. So $\Phi r \in \mathrm{RL}_2$ by 1.2.4.

(2) Conversely consider r_1 in $\Gamma_3 \cap \mathrm{RL}_2$. Then r_1 is in the center of Γ_1, so $\Phi^{-1}r_1$ is in the center of Γ, so $\Phi^{-1}r_1$ permutes with all transvections in Γ, so $\Phi^{-1}r_1$ stabilizes all lines in V, so $\Phi^{-1}r_1$ is in RL_n, i.e., $\Phi^{-1}(\Gamma_3 \cap \mathrm{RL}_2) \subseteq \Gamma \cap \mathrm{RL}_n$. Q.E.D.

6.1.11. Theorem. *A group of symplectic collinear transformations which has enough transvections and whose underlying dimension is* $\geqslant 4$, *cannot be isomorphic to a group of collinear transformations* (*resp. linear transformations*) *that is full of transvections and whose underlying dimension is* $\geqslant 4$ (*resp.* $\geqslant 2$).

Recall from the *Linear Lectures* that if M_3 is a bounded $\mathfrak{o}_3$-module on V_3 where $\mathfrak{o}_3$ is an integral domain with quotient field F_3, and if $\mathfrak{a}_3$ is a nonzero integral ideal, then $\mathrm{SL}_{n_3}(M_3;\mathfrak{a}_3)$ is full of transvections if $n_3 \geqslant 2$. We also know that in the analogous alternating situation, $\mathrm{Sp}_n(M;\,\mathfrak{a})$ has enough transvections. We therefore have the following special case of Theorems 6.1.8 and 6.1.11.

6.1.12. Theorem. *Let* $n \geqslant 4$ *and* $n_1 \geqslant 2$ *be natural numbers, let* $\mathfrak{o}$ *and* $\mathfrak{o}_1$ *be any two integral domains with quotient fields* F *and* F_1, *let* $\mathfrak{a}$ *and* $\mathfrak{a}_1$ *be nonzero integral ideals with respect to* $\mathfrak{o}$ *and* $\mathfrak{o}_1$. *If* Δ *and* Δ_1 *are groups such that*

$$\mathrm{PSp}_n(\mathfrak{o};\,\mathfrak{a}) \subseteq \Delta \subseteq \mathrm{PSp}_n(F),$$

$$\mathrm{PSL}_{n_1}(\mathfrak{o}_1;\,\mathfrak{a}_1) \subseteq \Delta_1 \subseteq \mathrm{PGL}_{n_1}(F_1),$$

then Δ *is not isomorphic to* Δ_1. *If* Γ *and* Γ_1 *are groups such that*

$$\mathrm{Sp}_n(\mathfrak{o};\,\mathfrak{a}) \subseteq \Gamma \subseteq \mathrm{Sp}_n(F),$$

$$\mathrm{SL}_{n_1}(\mathfrak{o}_1;\,\mathfrak{a}_1) \subseteq \Gamma_1 \subseteq \mathrm{GL}_{n_1}(F_1),$$

then Γ *is not isomorphic to* Γ_1.

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