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WEIGHTED HOMOGENEOUS POLYNOMIALS AND FUNDAMENTAL GROUPS

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§1. INTRODUCTION

A POLYNOMIAL $h(Z_o, \ldots, Z_n)$ is called *weighted homogeneous* if it is a sum of monomials $Z_o^{i_o} Z_1^{i_1} \cdots Z_n^{i_n}$with the property that for some fixed positive rational numbers $(w_o, \ldots w_n)$ called the weights of the variables,

$$\frac{i_o}{w_o} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1$$

holds for each monomial of h.

Suppose that $h(Z_o, Z_1, Z_2)$ is a weighted homogeneous polynomial in $\boldsymbol{C}^3$ and the variety

$$V = \{h(z_o, z_1, z_2) = 0\}$$

has an isolated singularity at the origin. For a sufficiently small sphere S_ε^5 we have that

$$K = V \cap S_\varepsilon^5$$

is a closed, orientable 3-manifold.

In [3, p. 80] Milnor conjectured that if

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} \leq 1,$$

then the 3-manifold K has infinite fundamental group and has an open 3-cell as universal covering space. Moreover, that this infinite group is nilpotent only if

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} = 1.$$

The purpose of this note is to prove these conjectures.

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§2. THE STRUCTURE OF K

Let $d = \langle w_o, w_1, w_2 \rangle$ denote the smallest positive integer so that

$$q_o = \frac{d}{w_o},\ q_1 = \frac{d}{w_1},\ q_2 = \frac{d}{w_2}$$

are integers.

If $V = \{h(z_o, z_1, z_2) = 0\}$ and h is weighted homogeneous with weights (w_o, w_1, w_2), then V is invariant under the C^* action on C^3 defined by $t \in C^*$ acting as

$$t(z_o, z_1, z_2) = (t^{q_o} z_o, t^{q_1} z_1, t^{q_2} z_2).$$

Now assume that V has an isolated singularity at the origin.

Restricting the action to $U(1) \subset C^*$ we see that K is invariant. Thus we have a closed, orientable 3-manifold with an action of $U(1) \cong SO(2)$. These were classified in [4] by their orbit invariants:

$$K = \{\beta; (\varepsilon, g, \bar{h}, t);\quad (\alpha_1\ \beta_1), \ldots, (\alpha_n, \beta_n)\}.$$

Since K is orientable and the action is fixed point free we have $\bar{h} = t = 0$ and $\varepsilon = o$. Thus K is a Seifert manifold [8]. For further details see [7].

§3. WEIGHTED HOMOGENEOUS POLYNOMIALS IN C^3

In [7] we proved that if $h(Z_o, Z_1, Z_2)$ is a weighted homogeneous polynomial in C^3 and $V = \{h = 0\}$ has an isolated singularity at the origin then up to equivariant diffeomorphism, K is determined by the weights and V is equivalent to a variety in one of the classes below:

Class I (Brieskorn varieties): $V(a_o, a_1, a_2; I) = \{z_o^{a_o} + z_1^{a_1} + z_2^{a_2} = 0\}$
Class II: $V(a_o, a_1, a_2; \mathrm{II}) = \{z_o^{a_o} + z_1^{a_1} + z_1 z_2^{a_2} = 0\}$, $a_1 > 1$
Class III. $V(a_o, a_1, a_2, \mathrm{III}) = \{z_o^{a_o} + z_1 z_2 (z_1^{a_1} + z_2^{a_2}) = 0\}$
Class IV. $V(a_o, a_1, a_2; \mathrm{IV}) = \{z_o^{a_o} + z_1^{a_1} z_2 + z_o z_2^{a_2} = 0\}$, $a_o > 1$
Class V: $V(a_o, a_1, a_2; V) = \{z_o^{a_o} z_1 + z_1^{a_1} z_2 + z_o z_2^{a_2} = 0\}$
Class VI: $V(a_o; \mathrm{VI}) = \{z_o^{a_o} + z_1 z_2 = 0\}$

The choice of class for h is not unique .

Define

$$K(a_o, a_1, a_2; \mathrm{I}) = V(a_o, a_1, a_2; \mathrm{I}) \cap S_\varepsilon^5$$

and use similar definitions for the other classes.

The weights (w_o, w_1, w_2) for these classes are computed as follows.

Class I: (a_o, a_1, a_2)

Class II: $\left(a_o, a_1, \dfrac{a_1 a_2}{a_1 - 1}\right)$

Class III: $\left(a_o, \dfrac{a}{a_2}, \dfrac{a}{a_1}\right)$ where $a = a_1 a_2 + a_1 + a_2$

Class IV: $\left(a_o, \dfrac{a_o a_1}{a_o - 1}, \dfrac{a_o a_1 a_2}{a_o a_1 - a_o + 1}\right)$

Class V: $\left(\dfrac{u}{u_o}, \dfrac{u}{u_1}, \dfrac{u}{u_2}\right)$ where

$$u = a_o a_1 a_2 + 1,\ u_o = a_1 a_2 - a_2 + 1,\ u_1 = a_o a_2 - a_o + 1,\ u_2 = a_o a_1 - a_1 + 1$$

Class VI: Here the weights are not unique. For any $w_1 > 1$ we may choose

$$\left(a_o, w_1, \frac{w_1}{w_1 - 1}\right).$$

In [4, Theorem 4] we proved that unless K is a lens space, K admits a *unique* $SO(2)$ action.

In [7, Chapter 3] we computed the orbit invariants of K for the classes above. They depend only on the weights [7; 3.1].

Finally in [6] and [4] we showed that unless K is a lens-space K is homeomorphic to K' if and only if $\pi_1(K)$ is isomorphic to $\pi_1(K')$,

From this we obtain that *unless K is a lens space the weights (w_o, w_1, w_2) and the fundamental group $\pi_1(K)$ determine each other.*

We shall see that if a lens space occurs then it is an $L(a_o, 1)$ and its possible distinct $SO(2)$ actions are given by the choices of w_1 in class VI.

§4. THE FUNDAMENTAL GROUP OF K

A presentation for $\pi_1(K)$ may be given as follows [8], [6], [4]. Let $a_1 b_1, \ldots, a_g, b_g$ be the standard generators of $\pi_1(K^*)$ and let $q_1, \ldots, q_n$ be the additional generators of $\pi_1(K_o{}^*)$. Let h be the homotopy element of a principal orbit. Then

$$\pi_1(K) = (a_i, b_i, q_j, h \mid \pi_* h^{-\beta}, [a_i, h], [b_i, h], [q_j, h], q_j^{\alpha_j} h^{\beta_j})$$

where $i = 1, \ldots, g; j = 1 \ldots, n$ and $\pi_* = q_1 \ldots q_n [a_1, b_1] \cdots [a_g, b_g]$.

In [8] it was proved that $\pi_1(K)$ is *finite* iff:

(1) $g = 0$, $n \leq 2$ (here K is a lens space); or

(2) $g = 0$, $n = 3$ and the three non-trivial stability groups have orders: $(2, 2, k)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$.

Moreover we proved in [6] that except for (1), (2) and

(3) $g = 1$, $n = 0$, $\beta = 0$

the center of $\pi_1(K)$ is the infinite cyclic group generated by (h). Assume this now.

If $\pi_1(K)$ is nilpotent then so is its quotient by the center

$$\pi_1(K)/(h) = (a_i, b_i, q_j \mid \pi_*, q_j^{\alpha_j}).$$

Now a nilpotent group is c-nil for some c (being the length of its upper and lower central series [2], which means that every commutator of $(c + 1)$ elements is trivial.

$$[\ldots[[x_1, x_2], x_3], \ldots, x_{c+1}] = 1$$

It is easily seen to happen for the above groups only if

(4) $g = 1, n = 0$.

Here is a different argument for the last statement. Thomas [10] showed that if $\pi_1(K)$ is infinite and nilpotent then it is an extension

$$1 \to Z(a) + Z(b) \to \pi_1(K) \to Z(c) \to 1$$

where the matrix of the automorphism defined by c has the form

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \; y \text{ an integer.}$$

Since all our manifolds are irreducible [12], [1] it follows from a result of Stallings [9] that K fibers over the circle with fiber the torus. For $y = 0$, $K = S^1 \times S^1 \times S^1 = \{0; (o, 1, 0, 0)\}$. For $y \neq 0$ the self-homeomorphism of the fiber given by the matrix above is of infinite order in its homeotopy group. Thus by [5; Corollary 1] we see that $K = \{\beta; (o, 1, 0, 0)\}$, that is $g = 1$, $n = 0$.

§5. PROOF OF THE CONJECTURES

Let $h(Z_0, Z_1, Z_2)$ be a weighted homogeneous polynomial in C^3. Let

$$V = \{h(z_0, z_1, z_2) = 0\}$$

have an isolated singularity at the origin and

$$K = V \cap S^5.$$

THEOREM 1. *The fundamental group $\pi_1(K)$ is infinite and the universal cover of K is an open 3-cell if and only if*

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} \leq 1.$$

Proof. First we show that if $\pi_1(K)$ is finite then

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} > 1.$$

(1) For $g = 0$, $n \leq 2$ K is a lens space. The only lens spaces that appear in our context are $L(k, 1)$, for $k \geq 1$.

They are given by $K(2, 2, k; \text{I})$ and by $K(a_o; \text{VI})$ for $k = a_o$ and different choices of w_1. For k odd we have

$K(2, 2, k; \text{I}) = \{-1; (o, 0, 0, 0); (k, k-2), (k, k-2)\}$ and for $k = 2r$

$$K(2, 2, 2r; \text{I}) = \{-2; (o, 0, 0, 0); (r, r-1), (r, r-1)\}.$$

The reader will easily compute the orbit invariants for K in class VI using [7].

These exhaust all possible $SO(2)$ actions on $L(k, 1)$.

Notice that always

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} = 1 + \frac{1}{k}.$$

Moreover

$$K = S^3/G$$

where G is the finite cyclic group Z_k.

(2) For $g = 0$, $n = 3$, $\alpha_1 = 2$, $a_2 = 2$, $\alpha_3 = k$ we have

$K(2, k+1, 2; \text{II}) = \{-2; (o, 0, 0, 0); (2, 1), (2, 1), (k, k-1)\}$ with weights

$$\left(2, k+1, \frac{2(k+1)}{k}\right).$$

Note that $K(2, k+1, 2; \text{II}) = S^3/G$ where G is the binary dihedral group.

For $g = 0$, $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 3$ we have

$$K(2, 3, 4; \text{I}) = \{-2; (o, 0, 0, 0); (2, 1), (3, 2), (3, 2)\}$$

with weights

$$(2, 3, 4).$$

Note that $K(2, 3, 4; \text{I}) = S^3/G$ where G is the binary tetrahedral group.

For $g = 0$, $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 4$ we have

$$K(2, 3, 3; \text{II}) = \{-2; (o, 0, 0, 0); (2, 1), (3, 2), (4, 3)\}$$

with weights

$$\left(2, 3, \frac{9}{2}\right).$$

Note that $K(2, 3, 3; \text{II}) = S^3/G$ where G is the binary octahedral group.

For $g = 0$, $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 3$, $a_3 = 5$ we have

$$K(2, 3, 5; \text{I}) = \{-2; (o, 0, 0, 0); (2, 1), (3, 2), (5, 4)\}$$

with weights

$$(2, 3, 5).$$

Note that $K(2, 3, 5; I) = S^3/G$ where G is the binary icosahedral group.

Since the weights and fundamental groups determine each other, the first assertion is proved. These special cases are well known, see [3].

The fact that the universal cover is an open 3-cell follows from results of Waldhausen [12] about Seifert-manifolds or results of Conner and Raymond [1] about $SO(2)$ actions on $K(\pi, 1)$'s.

Conversely, suppose that

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} > 1.$$

Using the weights determined by the exponents in Section 3 computation shows that only the above examples can occur.

This completes the proof.

Now let

$$M_b = \{-b; (o, 1, 0, 0)\}, \quad b = 1, 2, 3.$$

THEOREM 2. *Let V and K be as above with $\pi_1(K)$ infinite. Then $\pi_1(K)$ is nilpotent if and only if*

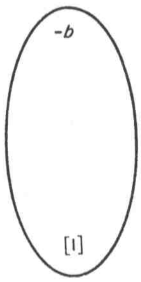
$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} = 1.$$

In that case K is equivariantly diffeomorphic to M_b for $b = 1, 2$ or 3.

Proof. As we have seen above, if $\pi_1(K)$ is nilpotent then $g = 1$ and $n = 0$. Let $\beta = -b$, then

$$K = \{-b; (o, 1, 0, 0)\}$$

for some integer b. According to [7] the resolution of the singularity of V at the origin is dual to the graph:



This is an elliptic curve with self-intersection $-b$. Since the quadratic form $(-b)$ is negative definite, $b \geq 1$.

Since V is embedded in $\mathbf{C}^3$, $b \leq 3$. This follows from results of Wagreich ([11] and unpublished) and I am indebted to him for communicating an explicit proof of this fact.

These manifolds are realized by

$$M_1 = K(2, 3, 6; \mathrm{I}),\ M_2 = K(2, 4, 4; \mathrm{I}),\ M_3 = K(3, 3, 3; \mathrm{I})$$

and by [4, Theorem 4] their $SO(2)$ action is unique, hence their weights and fundamental groups mutually determine each other.

Now suppose that

$$\frac{1}{w_o} + \frac{1}{w_1} + \frac{1}{w_2} = 1.$$

An easy computation using the weights determined by the exponents in Section 3 shows that there are only the following possibilities for exponents (a_o, a_1, a_2) in the six classes.

TABLE 1

Class	M_1	M_2	M_3
I	2, 3, 6	2, 4, 4	3, 3, 3
II	2, 3, 4	2, 4, 3	3, 3, 2
	3, 2, 3	4, 2, 2	
III	—	2, 2, 2	3, 1, 1
IV	—	2, 3, 2	3, 2, 2
V	—	—	2, 2, 2
VI	—	—	—

Computing the weights (w_o, w_1, w_2) associated to the exponents (a_o, a_1, a_2) shows that they are integers and equal to one of the unordered triples (2, 3, 6), (2, 4, 4), (3, 3, 3). Thus the corresponding K is equivariantly diffeomorphic to one of M_b, $b = 1, 2, 3$. The table above is arranged so that manifolds in the same column are equivariantly diffeomorphic.

This completes the proof.

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