

## ACTIONS OF THE TORUS ON 4-MANIFOLDS—II

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IN THE first part [5] of this investigation of closed orientable 4-manifolds admitting an action of the 2-dimensional torus group,  $G = T^2$ , we obtained an equivariant classification of actions with no finite isotropy groups, provided the action was not free. The main tool there was a cross-sectioning theorem [5, 1.12 and 4.4]. Our first result in this sequel is to obtain an equivariant classification of all actions. This is accomplished by specifying a cross-section on the boundary of a tubular neighborhood of each orbit with finite isotropy group ( $E$ -orbit) and considering the problem of extending this partial cross-section to the rest of the orbit space. This approach is analogous to the equivariant classification of 3-manifolds with  $S^1$ -action [7] and in principle it goes back to Seifert [8]. We have to consider two cases. If there are points with infinite isotropy group,  $F \cup C \neq \emptyset$ , then there is no obstruction to extending this cross-section and the orbit data form a complete set of invariants. If the action has only finite isotropy groups,  $F \cup C = \emptyset$ , then  $M$  is a Seifert manifold and an additional invariant appears, representing the obstruction to extending the cross-section.

The remaining sections contain topological results. Since we have an equivariant classification, we may think of our manifolds given in terms of an action and ask when two such manifolds are homeomorphic (diffeomorphic). This problem was solved for simply connected manifolds in our first paper [5; §5]. In §2 we investigate the Seifert case,  $F \cup C = \emptyset$ , using the techniques of Conner and Raymond [1–4]. In “almost all” cases we see that two manifolds are homeomorphic if and only if they are equivariantly homeomorphic. This result was obtained independently by H. Zieschang [10] using different methods. We also observe that  $M$  fibers over  $S^1$  with fiber a 3-dimensional Seifert manifold. In §3 we show that in the presence of fixed points,  $F \neq \emptyset$ ,  $M$  can be represented as an equivariant connected sum of “elementary” 4-manifolds with  $G$ -action. Each elementary manifold has cyclic fundamental group. There are two unfortunate aspects of this result, however. First the mutual homeomorphism relationships of these elementary manifolds are not known. This difficulty resembles the classification problem of lens spaces. Next, the decomposition is not unique and in the light of [5; §5.8] this may turn out to be a rather serious obstacle. §4 is a partial answer to the first problem for manifolds with infinite cyclic fundamental group, i.e. for manifolds with no finite isotropy,  $E = \emptyset$ . We call  $M$

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and  $N$  stably homeomorphic (diffeomorphic) if there exist non-negative integers  $m, n$  so that the connected sums  $M \# mS^2 \times S^2$  and  $N \# nS^2 \times S^2$  are homeomorphic (diffeomorphic), and determine the stable homeomorphism (diffeomorphism) classification of elementary manifolds with  $E = \emptyset$ . At present we have only inconclusive results for the manifolds with finite cyclic fundamental groups and will not discuss them here.

### §1. EQUIVARIANT CLASSIFICATION

Let us recall the notation of [5]. The 2-dimensional torus group  $G = T^2$  acts on an oriented closed 4-manifold  $M^4$ . The set of orbits with non-trivial finite isotropy group is denoted  $E$ . The set of orbits with isotropy group isomorphic to  $S^1$  is denoted  $C$  and the set of fixed points  $F$ . Given a parametrization of  $G = \{\varphi, \theta \mid 0 \leq \varphi, \theta < 2\pi\}$  particular circle subgroups are denoted by  $G(m, n) = \{\varphi, \theta \mid m\varphi + n\theta = 0, (m, n) = 1\}$ .

The equivariant classification for  $E = \emptyset$ ,  $F \cup C \neq \emptyset$  was obtained in [5, §4]. The orbit space,  $M^*$  is a compact 2-manifold with non-empty boundary. Interior points correspond to principal orbits and  $\partial M^* = F^* \cup C^*$ . The components of  $F^*$  are isolated points, each in the closure of two components of  $C^*$  with isotropy groups  $G(m, n)$  and  $G(m', n')$  respectively, satisfying the condition  $mn' - m'n = \pm 1$ . We label each  $C^*$ -component by its "weight"  $(m, n)$ . With the orientation of  $G$  above and the given orientation on  $M$  a compatible orientation is chosen for  $M^*$  and this together with the weights give an equivariant classification of the action [5; 4.4].

To complete the equivariant classification for  $E \neq \emptyset$  we shall discuss two cases separately:

- (a)  $C \cup F \neq \emptyset$ , and
- (b)  $C \cup F = \emptyset$ .

First it is necessary to examine the  $E$ -orbits. Later, when we desire to give presentations of the fundamental group as well as a normal form for the action in case (b), we shall find it useful to give other equivalent presentations for the  $E$ -orbits.

1.1. *Oriented slice invariants for the  $E$ -orbits.* Let  $x_1, \dots, x_n$  be arbitrary points, one each on the distinct isolated  $E$ -orbits and let  $x_1^*, \dots, x_n^*$  denote their image in  $E^* \subset M^*$ . If  $x^* \in E^*$  is one of these points and  $D^*$  is a closed disk in the interior of  $M^*$  which contains only  $x^*$  from  $E^*$  in its interior, then  $\pi^{-1}(D^*)$  is an invariant tubular neighborhood of  $G(x)$  and the action is topologically equivalent to

$$(G, G \times_{\mathbb{Z}_\alpha} D^2)$$

where  $\mathbb{Z}_\alpha \cong G_x \subseteq G$  and the action of  $\mathbb{Z}_\alpha$  on  $G$  and  $D^2$  is given as follows:

$$\begin{aligned} \lambda \times z &\rightarrow \lambda^v z, & |z| &\leq 1, & z &\in D^2 \\ \lambda \times (z_1, z_2) &\rightarrow (z_1 \lambda^{\gamma_1}, z_2 \lambda^{\gamma_2}), & (z_1, z_2) &\in G, & \text{with} \\ 0 < v < \alpha, & (\alpha, v) = 1, & \lambda &= \exp(2\pi i/\alpha), & \text{and} \\ 0 \leq \gamma_1 < \alpha, & 0 \leq \gamma_2 < \alpha. \end{aligned}$$

We are indebted to the referee for pointing out that for  $\rho$  coprime to  $\alpha$  the map  $\lambda \rightarrow \lambda^\rho$  induces an equivariant map replacing  $(\gamma_1, \gamma_2, v)$  by  $(\gamma_1', \gamma_2', v')$ , with  $0 \leq \gamma_i' < \alpha$ ,  $\gamma_i' \equiv \rho\gamma_i \pmod{\alpha}$ ,  $0 \leq v' < \alpha$ ,  $v' \equiv \rho v \pmod{\alpha}$ . Thus we may normalize so that  $v = 1$ , and call the corresponding triple  $(\alpha; \gamma_1, \gamma_2)$  the *oriented slice invariants*. Note that it follows from the effectiveness of the action that  $(\alpha, (\gamma_1, \gamma_2)) = 1$ .

The action on  $D^2$  is the slice representation and the action of  $\mathbb{Z}_\alpha$  on  $G$  is really an embedding of  $\mathbb{Z}_\alpha$  in  $G$ ,  $\lambda \rightarrow (\lambda^{\gamma_1}, \lambda^{\gamma_2})$ . Note that if  $\alpha/(\alpha, \gamma_1) = r_1$  and  $\alpha/(\alpha, \gamma_2) = r_2$ , then  $[r_1, r_2] = \alpha$ , where “[ ]” means “least common multiple”. The projection of  $\mathbb{Z}_\alpha$  to the first factor is isomorphic to  $\mathbb{Z}_{r_1}$  while onto the second is isomorphic to  $\mathbb{Z}_{r_2}$ . The oriented slice invariants  $(\alpha; \gamma_1, \gamma_2)$  completely determine the action in a tubular neighborhood of  $G(x)$ . (This is really not difficult to check and of the various ways to see this perhaps the quickest is by the methods of [2; §8].) Now in addition to our weights of the form

$$\{\varepsilon; g; s; t; \langle p_1, q_1 \rangle, \dots, \langle p_s, q_s \rangle; \{m, n\}_1, \dots, \{m, n\}_t\}$$

of [5; §4] we add the collection of slice invariants

$$\{(\alpha_k; \gamma_{k,1}, \gamma_{k,2}), \quad k = 1, 2, \dots, n\}$$

and we call  $M^*$  and this total collection our *weighted orbit space*.

1.2. *Equivariant classification for  $C \cup F \neq \emptyset$ .* If  $M_1^*$  and  $M_2^*$  are two distinct oriented orbit spaces with weights, then an *isomorphism* between them

$$h: M_1^* \rightarrow M_2^*$$

will be an orientation preserving homeomorphism which preserves the weights. That is,  $E_1^* \rightarrow E_2^*$  and  $\partial M_1^* \rightarrow \partial M_2^*$  are also isomorphisms.

**THEOREM.** *Let  $(G, M_1)$  and  $(G, M_2)$  be actions with  $C \cup F \neq \emptyset$ . They are equivariantly homeomorphic through an orientation preserving homeomorphism if and only if there exists an isomorphism between their oriented weighted orbit spaces.*

*Proof.* The argument for this theorem is really the same as Corollary 2b of [7]. First if  $M_1$  and  $M_2$  are equivariantly homeomorphic then this homeomorphism induces an orientation and weight preserving map on the orbit spaces. On the other hand, given such an isomorphism

$$h: M_1^* \rightarrow M_2^*,$$

we shall show how one may construct an isomorphism from  $M_1$  to  $M_2$ . Obviously,  $h$  exists if the “numerical data” or weights are isomorphic. (Recall that the sets of numbers may be permuted.) Let us select closed 2-disks  $D_k^*$  about each  $x_k^*$ . They are mapped by  $h$  onto corresponding disks about  $h(x_k^*)$ . We can assume that  $x_k^*$  is in the interior of  $D_k^*$  which is in the interior of  $M_1^*$  and  $D_k^* \cap E^* = x_k^*$ . If we let

$$M_{1,1}^* = M_1^* - \bigcup_{k=1}^n \dot{D}_k^*$$

where  $\dot{D}_k^*$  is the interior of  $D_k^*$ , then

$$h_1: M_{1,1}^* \rightarrow M_{2,1}^*,$$

the restriction of  $h$ , is an isomorphism. Furthermore, as there are no  $E$  orbits in  $M_{1,1}$  any section of the orbit map over  $\bigcup_{k=1}^n \partial D_k^*$  can be extended to all of  $M_{1,1}^*$  by [5; §1.10]. Thus it is easy to find an isomorphism

$$H : M_{1,1} \rightarrow M_{2,1}.$$

To each  $M_{1,1}$  we must attach the  $(G, V_k) = (G, G \times_{\mathbb{Z}_{\alpha,k}} D_k)$  equivariantly. We have already a section over each  $\partial D_k^*$  and an action of  $G$  on  $G \times \partial D_k^*$ . We may attach  $(G, V_k)$  up to equivariant isomorphism in only one way and thereby extend  $H$  to all of  $M_1$ . This completes the proof.

1.3. *Invariants as obstructions to cross-sections*,  $C \cup F = \emptyset$ . In order to find the equivariant classification for case (b) we must find the analogue of the invariant “b” of Corollary 2b in [7]. Since  $\partial M^*$  is empty the weights, which are just the  $E$ -orbit data, are not enough. (For example, different principal actions can have the same orbit space.) The new invariants will be a pair of integers. When  $\partial M^*$  is empty the “weights” will include, in addition, this pair of integers and altogether will constitute a complete set of invariants for the oriented action.

Arbitrarily choose a principal orbit,  $x_0^*$ , and a closed 2-cell,  $D_0^*$  meeting only principal orbits in its interior. If we select sections about each  $\partial D_k^*$ ,

$$\chi_k : \partial D_k^* \rightarrow M, \quad k \geq 1,$$

these sections may be extended to

$$N^* = M^* - \bigcup_{k=0}^n \dot{D}_k^*.$$

We may without loss of generality assume that  $\pi^{-1}(D_k^*) = V_k$ , where  $(G, V_k)$  is  $(G, G \times_{\mathbb{Z}_{\alpha,k}} D^2)$ . That is by equivariant homeomorphism we can assume  $x_k^*$  is in the center of  $D_k^*$  and the action is equivalent to the standard linear action on the solid torus.

If we fix the sections  $\chi_k$  over each  $\partial D_k^*$ ,  $k > 0$ , then any two extensions over  $N^*$  are homotopic having the sections  $\chi_k$  fixed; [9, 34.8]. Thus we are interested in choosing *normalized* sections over  $\partial D_k^*$ ,  $k > 0$ , and measuring the obstruction to extending over all of  $M^* - \bigcup_{k=1}^n \dot{D}_k^*$ . This obstruction will be given by an element of

$$H^2\left(M^*, \bigcup_{k=0}^n D_k^*; \mathbb{Z} \oplus \mathbb{Z}\right) \approx \mathbb{Z} \oplus \mathbb{Z}$$

and can be thought of as a pair of integers  $b_1$  and  $b_2$ . We wish to interpret this pair of integers and show that it is an invariant of our action. To do this we first show how to normalize the section over  $\partial D_k^*$ .

Let  $h_1$  and  $h_2$  denote the oriented circle subgroups of  $G$  defined by  $h_1 = G(0, 1)$  and  $h_2 = G(1, 0)$ . On  $V_k$  which is equivariantly  $(G, G \times_{\mathbb{Z}_{\alpha,k}} D^2)$  the curve  $m = \partial D^2$ , that is the image of  $\partial D^2$  in  $G \times D^2 \rightarrow V_k$  is homotopically trivial. On  $\pi^{-1}(\partial D_k^*) = V_k = G \times \chi_k(\partial D_k^*)$ , the cross-sectional curve  $Q = \chi_k(\partial D_k^*)$  together with  $h_1$  and  $h_2$  form a mutually orthogonal

curve system for  $\partial(V_k)$ . Of course we may alter this section at will and obtain a new section  $Q'$ , which must satisfy the homological relation

$$Q \sim \varepsilon Q' + s_1 h_1 + s_2 h_2$$

for arbitrary integers  $s_1$  and  $s_2$  and  $\varepsilon = \pm 1$ . (We allow ourselves the convenience of using the symbols for the oriented curves as also the symbols for the generators of the first integral homology group and later for the fundamental group.)

The curve  $m$  satisfies the homology relation

$$m \sim \alpha Q + j_1 h_1 + j_2 h_2$$

for some integers  $\alpha, j_1$  and  $j_2$  and since it is connected,  $(\alpha, (j_1, j_2)) = 1$ . In fact, it is easy to see that  $|\alpha| = \alpha_k$ . By replacing  $Q$  by its equivalent  $\varepsilon Q' + s_1 h_1 + s_2 h_2$  we obtain

$$\begin{aligned} m &\sim \varepsilon \alpha Q' + \alpha s_1 h_1 + \alpha s_2 h_2 + j_1 h_1 + j_2 h_2 \\ &= \varepsilon \alpha Q' + (j_1 + \alpha s_1) h_1 + (j_2 + \alpha s_2) h_2. \end{aligned}$$

Since the choice of  $s_1$  and  $s_2$  is arbitrary we may choose them so that

$$\beta_{k,1} = \alpha s_1 + j_1, \quad \beta_{k,2} = \alpha s_2 + j_2$$

and

$$0 \leq \beta_{k,1} < |\varepsilon \alpha|, \quad 0 \leq \beta_{k,2} < |\varepsilon \alpha|.$$

In fact, this is our normalization. We have already implicitly oriented  $D_k^*$  and hence  $\partial D_k^*$  with the positive direction being measured by increasing angle. This orients both curves  $m$  and  $Q$  in the positive directions and so the choice of  $\varepsilon = 1$  and  $\alpha > 0$  can always be made. Our numbers  $\beta_{k,1}$  and  $\beta_{k,2}$  are then normalized invariants for the section  $Q'$  about  $\chi_k^*$ . It can be seen by a computation, similar to that employed in [7; §5] that we are always led to the same numbers regardless of the choices. (Of course, the numbers do depend upon the orientation and the choice of generators  $h_1$  and  $h_2$  of  $G$  since this is part of our data.) Thus the cross-section  $\chi$  on  $\partial D_k^*$ ,  $Q'$ , is normalized and well defined up to a homotopy. We call  $(\alpha_k; \beta_{k,1}, \beta_{k,2})$  the *oriented Seifert invariants* of the  $E$ -orbit. As above,  $(\alpha, (\beta_1, \beta_2)) = 1$ .

Now having been given normalized sections,  $\chi_k$ , over  $\partial D_k^*$ ,  $k > 0$ , we extend to  $N^*$ . This section intersects  $\partial V_0$  in a curve  $Q_0$ . Any other choice of extension,  $\chi'$  is homotopic by [9, 34]. Thus  $Q_0$  and  $Q_0'$  must be homologous on  $\partial V_0$  and so  $m_0$  is given by the homology relation

$$m_0 \sim Q_0 + b_1 h_1 + b_2 h_2.$$

The integers  $b_1$  and  $b_2$  are invariants of our action. In fact *given the normalized extensions on  $\{\partial D_k^*\}_{k=1,2,\dots,n}$ , the obstruction to extending this section over  $M^* - \bigcup_{k>0} \dot{D}_k^*$  is precisely the pair of integers*

$$(b_1, b_2) \in H^2 \left( M^* - \bigcup_{k>0} \dot{D}_k^*, \bigcup_{k>0} \partial D_k^*; \mathbb{Z} \oplus \mathbb{Z} \right) \approx \mathbb{Z} \oplus \mathbb{Z}.$$

This follows readily from the definitions of obstruction theory. In analogy with the notation above we define the “weights” of  $M^*$  as

$$\{(b_1, b_2); \varepsilon, g; (\alpha_1; \beta_{1,1}, \beta_{1,2}), \dots, (\alpha_r; \beta_{r,1}, \beta_{r,2})\}$$

where  $\varepsilon$  is a given orientation on the closed 2-manifold  $M^*$  of genus  $g$ ;  $(\alpha_i; \beta_{i,1}, \beta_{i,2})$  are the normalized oriented Seifert invariants of the  $E$ -orbits and  $(b_1, b_2) \in \mathbb{Z} \oplus \mathbb{Z}$  is the obstruction class defined above. Two sets of “weights” are called *isomorphic* if they agree up to a permutation of the indices  $1, \dots, r$ .

**THEOREM 1.4.** *If  $(G, M_1)$  and  $(G, M_2)$  are oriented actions, then they are equivariantly homeomorphic under an orientation preserving homeomorphism if and only if there exists an isomorphism between their (normalized) weighted orbit spaces.*

*Proof.* In (1.2) we settled the case when the orbit spaces have boundary. Since all the weights of an action are invariants, the necessity is clear. On the other hand, given an equivalence of the weights for the two actions, there is an orientation and weight preserving isomorphism  $h$  from  $M_1^*$  to  $M_2^*$ . By choosing disks  $D_k^*$  and  $h(D_k^*)$  we are led to choosing normalized sections  $\chi_{k,1} : \partial D_k^* \rightarrow M_1$  and  $\chi_{k,2} : h(\partial D_k^*) \rightarrow M_2$ . These sections are extendable to  $N_1^*$  and  $N_2^*$ , and are uniquely determined up to homotopy. With these two partial sections it is easy to define an orientation preserving equivariant homeomorphism  $H : M_1 \rightarrow M_2$  which covers  $h$ , via the orbit maps.

1.5. *A presentation of  $\pi_1(M, *)$  in terms of the weights.* In this section we give a presentation of the fundamental group  $\pi_1(M, *)$  in terms of our orbit invariants when  $F \cup C = \emptyset$ . For the  $E$ -orbits we use the oriented Seifert invariants.

In fact, it is just as easy to present the fundamental group of an action  $(T^m, M^{m+2})$ ,  $m > 0$  with only finite isotropy groups. We choose  $h_1, \dots, h_m$  as standard generators for  $\pi_1(T^m, *)$ ,  $a_1, b_1, \dots, a_g, b_g$  as standard generators for  $\pi_1(M^*, *)$ ,  $q_1, \dots, q_n$  generators for  $Q_1, \dots, Q_n$  or what is the same as  $\partial D_k^*$ . For each  $V_k$  we receive the relations:

$$q_k^{\alpha_k} h_1^{\beta_{k,1}} \dots h_m^{\beta_{k,m}}.$$

We may put all of them together via the cross-section and the Van Kampen theorem and obtain:

$$\begin{aligned} \pi_1(M^{m+2}, *) = & \{a_1, b_1, \dots, a_g, b_g; h_1, \dots, h_m; q_1, \dots, q_n | \\ & [a_1, b_1] \dots [a_g, b_g] \cdot q_1 \dots q_n \cdot h_1^{-b_1} \dots h_m^{-b_m}; \\ & q_k^{\alpha_k} h_1^{\beta_{k,1}} \dots h_m^{\beta_{k,m}}; \\ & [q_k, h_j]; [a_i, h_j]; [b_i, h_j]; [h_j, h_i]\}. \end{aligned}$$

Here  $l$  and  $j = 1, \dots, m$ ,  $k = 1, \dots, n$ ,  $i = 1, \dots, g$  and  $[x, y] = xyx^{-1}y^{-1}$ .

This is really the obvious analogue of Seifert's presentation of the fundamental group of  $(S^1, M^3)$ , see [8; 6]. Note that when  $m = 2$ , this yields the case that concerns us here, 4-manifolds. That this is the correct formula follows from the fairly obvious extension to  $G = T^m$  of our previous discussion on normalization when  $G = T^2$ .

1.6 *The tubular neighborhood of an  $E$ -orbit.* In this section we shall associate a third set of invariants with an  $E$ -orbit, describing the action in its tubular neighborhood. This neighborhood is homeomorphic to  $D^2 \times T^2$  with the  $E$ -orbit corresponding to  $O \times T^2$ , but the action is *not* translation in the second factor.

Our analysis here resembles [6; §2] where we considered  $S^1$  actions on  $D^2 \times S^1$  and it is for this analogy that we shall keep the order of the factors the reverse of the customary notation.

We let  $D^2 \times S^1 \times S^1$  be parameterized by  $(\rho, \omega_1, \omega_2, \omega_3)$ ,  $0 \leq \rho \leq 1$ ,  $0 \leq \omega_i < 2\pi$ . The  $G$ -action is given by

$$(\rho, \omega_1, \omega_2, \omega_3) \xrightarrow{(\varphi, \theta)} (\rho, \omega_1 + a_1\varphi + b_1\theta, \omega_2 + a_2\varphi + b_2\theta, \omega_3 + a_3\varphi + b_2\theta)$$

and we call the matrix

$$\Lambda = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

the *tube invariants* of the equivariant tubular neighborhood of the  $E$ -orbit. This action is effective only if the solution to the simultaneous equations

$$a_1\varphi + b_1\theta = 0, \quad a_2\varphi + b_2\theta = 0, \quad a_3\varphi + b_2\theta = 0$$

modulo the integers is  $\varphi = 0, \theta = 0$ . The isotropy group of  $(0, -, \omega_2, \omega_3)$  is finite if and only if  $\alpha = |a_2b_3 - a_3b_2| \neq 0$  and in this case the group is easily seen to be isomorphic to  $\mathbb{Z}_\alpha$ . A suitable sequence of equivariant diffeomorphisms normalizes the entries of  $\Lambda$  the following way:

- (i)  $a_3 = 0, a_2 > 0, b_3 > 0$ .
- (ii)  $0 \leq a_1 < a_2$ ,
- (iii)  $0 \leq b_1 < b_3$  and  $0 \leq b_2 < b_3$ .

The effectiveness of the action implies moreover

- (iv)  $(a_1, a_2) = 1$ ,
- (v)  $\left(b_3, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right) = 1$ .

This set of invariants will not be used in the sequel and we leave it to the reader to obtain conversion relations among the three sets of invariants associated with an  $E$ -orbit. Here is a sample. Given the normalized tube invariants  $\Lambda$  we obtain the oriented Seifert invariants by:

- (i)  $\alpha = a_2b_3$ ,
- (ii)  $\beta_2 = a_2\beta_2'$  where  $\beta_2'(a_1b_2 - a_2b_1) \equiv -1(b_3)$  and  $0 \leq \beta_2' < b_3$ ,
- (iii)  $\begin{vmatrix} \beta_1a_1 = ya_2 + 1 \\ \beta_1b_1 \equiv yb_2(b_3) \end{vmatrix}$  determine  $y$  and  $\beta_1$  so that  $0 \leq \beta_1 < \alpha$ .

## §2. TOPOLOGICAL CLASSIFICATION FOR $C \cup F = \emptyset$

An analysis of the topological and equivariant classification for  $C \cup F = \emptyset$  is given in §5 and §12 of Holomorphic Seifert fiberings [3] primarily as an illustration of the techniques of [3]. In fact considerably more general situations are investigated in the cited

sections and so it seems appropriate to extract and refine the relevant material here for the convenience of the reader. We shall briefly remind the reader of the terms injective, locally injective, homologically injective, holomorphically injective and Bieberbach classes and hint at their significance, but we must refer the reader to [3] for most of the details of the arguments.

This approach, based on the theory of locally injective actions developed by Conner and Raymond [1]–[4], yields, except for a few cases, that the manifolds in question are homeomorphic if and only if their fundamental groups are isomorphic (see also Zieschang [10]). It treats  $T^k$  actions on  $M^{k+2}$ , and we shall use this generality unless specific results for  $k = 2$  are called for. We shall also find conditions for  $M^4$  to fiber over  $T^2$  and show that  $M^4$  always fibers over  $S^1$ , although the fiber is not unique. The most striking aspect of this approach is that we do not need any of the local invariants defined in §1 to complete the equivariant and topological classifications. In fact §2 is completely independent of §1. We shall also observe that for almost any given manifold any two actions of  $T^2$  are *weakly equivalent*. (Two actions  $(G, X)$  and  $(G, Y)$  are *weakly equivalent* if there exists an automorphism  $\Phi: G \rightarrow G$  and a homeomorphism  $h: X \rightarrow Y$  so that  $h(g(x)) = \Phi(g)h(x)$ , for all  $x \in X, g \in G$ . That is, they are equivariantly homeomorphic up to an automorphism of  $G$ .)

**2.1. Locally injective and injective actions.** Let  $(T^k, X)$  be an action on a path-connected space. We need only now assume that the topology of  $X$  is sufficiently nice to admit covering space theory and use of the usual algebraic invariants of algebraic topology. In our applications,  $X$  will be locally Euclidean and these conditions will be obviously met.

Let  $f^x: (T^k, 1) \rightarrow (X, x)$  denote the *evaluation map*,  $f^x(t) = tx$ , and let  $\text{im}(f^x)$  denote the image

$$f_*^x: \pi_1(T^k, 1) \rightarrow \pi_1(X, x).$$

The image,  $\text{im}(f_*^x)$ , is a *central* subgroup of  $\pi_1(X, x)$ . If  $f_*^x$  is a *monomorphism* for some  $x$  (and hence all  $x \in X$ ), the action  $(T^k, X)$  is called *injective*, [1].

If  $g \in T_x^k$ , and  $g_t: (I, 0, 1) \rightarrow (T^k, 1, g)$  is a path in  $T^k$ , then  $f^x(g_t) = g_t x$  yields a loop in  $X$ , based at  $x$ , and induces a natural homomorphism

$$\eta_x: T_x^k \rightarrow \pi_1(X, x)/\text{im}(f_*^x).$$

If  $\eta_x$  is a *monomorphism* for each  $x \in X$ , the action  $(T^k, X)$  is called *locally injective* [3; §7]. *Injective actions are always locally injective*, [3; 7.4]. When a locally injective action is lifted to the covering space  $Y$  of  $X$  associated with the subgroup  $\text{im}(f_*^x)$ , the covering action  $(T^k, Y)$  is free, [3; §7]. (This property is actually a characterization of those actions which are locally injective.) The action commutes with the right action  $(Y, N)$  of covering transformations, where  $N = \pi_1(X, x)/\text{im}(f_*^x)$ . The orbit map  $(T^k, Y) \rightarrow (Y/T^k, N) = (W, N)$  induces a properly discontinuous action of  $N$  on the simply connected space  $W$  so that  $W/N = X/T^k$ . Of course we may divide first by  $N$  and obtain

$$(T^k, Y/N) = (T^k, X).$$

Associated therefore with the locally injective action  $(T^k, X)$  is the extension:  $0 \rightarrow \text{im}(f_*^x) \rightarrow \pi_1(X, x) \rightarrow N \rightarrow 1$ ; the left principal bundle  $(T^k, Y)$  over the simply connected space  $W$ ;



the group of covering transformations  $(Y, N)$ ; and the properly discontinuous action  $(W, N)$ . If  $(T^k, X)$  is injective, the covering action  $(T^k, Y)$  *splits*, [1]:

$$(T^k, T^k \times W, N).$$

In this splitting action,  $T^k$  acts by translation on the first factor. It is significant for our purposes that almost all of the actions to be considered in §2 are injective. The topological classification of these actions is accomplished by being able to interpret this problem entirely in terms of the ordinary second cohomology of planar groups.

**2.2. Bieberbach classes.** Locally injective actions may be completely described cohomologically by *Bieberbach classes*. These cohomology classes, in various situations, are readily computable and offer an algebraic method of classifying locally injective actions. In [3], a much more general concept of Bieberbach class is defined to describe "Seifert fiberings" of which the orbit map of a locally injective action is a special case. When we speak of Bieberbach classes here we shall be just referring to the special case arising from actions. The Bieberbach classes, in case of actions, are nothing but generalizations of the cohomological classification of principal toral bundles over  $W/N$ .

One begins with a properly discontinuous action  $(W, N)$  on a simply connected space. There is a natural one-one correspondence between equivalence classes of left principal  $T^k$  bundles over  $W$  with properly discontinuous right  $N$  operators,  $(T^k, Y, N)$  and the elements of a certain cohomology group

$$H^2(N; \mathcal{L}^k).$$

The coefficients belong to a sheaf with operators,  $\mathcal{L}^k$ , over  $W$ . For each commuting left  $T^k$ -right  $N$  action on  $Y$  there is induced a left  $T^k$  action,  $(T^k, Y/N) = (T^k, X)$ . Those classes  $a \in H^2(N; \mathcal{L}^k)$  which give rise to free actions  $(Y, N)$  and hence covering transformations, are called *Bieberbach classes*. They may be easily characterized both algebraically and geometrically, [3; §3 and §4].

If  $(Y, N)$  is a covering action, we obtain a locally injective action  $(T^k, Y/N) = (T^k, X)$ . Of course, when this action is lifted to the covering space associated with  $\text{im}(f_*^x)$  we obtain the left  $T^k$ -right  $N$  action prescribed by the Bieberbach class. In fact, all locally injective actions which induce on  $Y/T^k$  an action equivalent to  $(W, N)$  are necessarily constructed this way. We have thus described a converse to the procedure of 2.1.

For injective actions we again obtain a simplification. The cohomology group  $H^2(N; \mathcal{L}^k)$  becomes  $H^2(N; \mathbb{Z}^k)$ , the ordinary cohomology of the discrete group  $N$  with coefficients in  $\mathbb{Z}^k$  where  $N$  operates trivially on  $\mathbb{Z}^k$ . The Bieberbach class,  $a \in H^2(N; \mathbb{Z}^k)$ , which corresponds to the left-right splitting action

$$(T^k, T^k \times W, N),$$

also naturally corresponds to the central extension

$$0 \rightarrow \text{im}(f_*^x) \rightarrow \pi_1(X, x) \rightarrow N \rightarrow 1.$$

(We should also remark that, if in addition,  $H^2(W; \mathbb{Z}) = 0$ , then every locally injective action must be injective.)

For the rest of §2 assume that we have a codimension 2 toral action,  $(T^k, M^{k+2})$ , with only finite stability groups on a closed oriented manifold  $M^{k+2}$ . Let  $E$  be the set of exceptional orbits and  $x_1^*, \dots, x_n^*$  in  $M^*$  their images. For any  $x$  on an orbit in  $E$ , the stability group,  $T_x^k$  is a finite cyclic subgroup isomorphic to  $\mathbb{Z}_\alpha$  of order  $\alpha$ . Let  $g$  be the genus of the orbit space  $M^*$ .

**THEOREM.** *The action  $(T^k, M^{k+2})$  arises from a Bieberbach class if any one of the following holds:*

- (1)  $g > 0$ ,
- (2)  $g = 0$  and  $n > 3$ ,
- (3)  $g = 0, n = 3$  and  $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} \leq 1$ ,
- (4)  $g = 0, n = 3$  and  $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} > 1$ ,
- (5)  $g = 0, n = 2$  and  $\alpha_1 = \alpha_2$ ,
- (6)  $g = 0, n = 0$  (hence  $M^{k+2} = T^k \times S^2$ ).

*Conversely, those actions that do not arise from a Bieberbach class are just the remaining possibilities, i.e. where  $g = 0$  and  $n = 1$  or  $2$ . For these remaining possibilities  $M^{k+2}$  is homeomorphic (with two possible exceptions), to  $T^{k-1} \times L$ , where  $L$  is a 3-dimensional lens space.*

In 2.2.4 and in 2.2.5 (provided  $M^{k+2} \neq T^k \times S^2$ ) the action is locally injective but *not* injective. However, in all the other (Bieberbach) cases the action is injective. In 2.2.1, 2.2.2 and 2.2.3,  $W$  is homeomorphic to the Euclidean plane, the group  $N$  is a planar group (crystallographic or Fuchsian) and the action is topologically equivalent to a smooth action which, in turn, is smoothly equivalent to a holomorphic action. (If  $(T^k, M)$  is smooth or holomorphic then so is  $(W, N)$ .) In the remaining cases  $W$  is homeomorphic to the 2-sphere,  $N$  is finite, and  $(W, N)$  is topologically, smoothly, or holomorphically equivalent to a (finite) linear action.

The proof of the theorem follows, without much difficulty, from the results and methods of §12 of [3]. The reader who wishes to verify all the details, especially in case  $g = 0$ , will find it helpful to be familiar with the methods employed in §12 of [3] as well as the explicit results. (The two minor exceptions mentioned above will be discussed at the end of 2.5.)

2.3. We must dispense with a special case first for the topological classification. If  $g = 1, n = 0$  then  $M^{k+2}$  is a principal  $T^k$  bundle over the closed surface of genus 1. These principal bundles are classified by their Chern classes. All the possible actions can be described by fixing, say, the standard  $(\mathbb{R}^2, \mathbb{Z} \oplus \mathbb{Z})$  and taking the elements  $a \in \mathbb{Z}^k \cong H^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}^k)$ . The actions are determined by weak-bundle equivalences which, in this case, are almost the same as bundle equivalences. They are induced by automorphisms of  $\mathbb{Z} \oplus \mathbb{Z}$  which can at most change the sign of the projections,  $H^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}^k) \rightarrow H^2(\mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z})$ . There remains, for a topological classification, the automorphisms of  $T^k$  (which induce automorphisms of  $\mathbb{Z}^k$ ). Thus  $(T^k, M_1)$  and  $(T^k, N_2)$

are *weakly equivalent*, if and only if, there exists an automorphism  $\Phi: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$  so that  $\Phi^*(a_1) = \pm a_2$ . From an easy calculation of the first homology group we see that  $H_1(M_1; \mathbb{Z}) \cong H_1(M_2; \mathbb{Z})$  if and only if they are weakly equivalent.

**2.4. Topological classification.** We shall assume that  $(T^k, M^{k+2})$  satisfies 1, 2 or 3 of 2.2.

**THEOREM.**  $(T^k, M_1^{k+2})$  and  $(T^k, M_2^{k+2})$  are weakly equivalent actions if and only if they have isomorphic fundamental groups.

*Proof.* If  $g = 1$ , we assume that  $n > 0$  since we have already proved the theorem in case  $n = 0$ . The result is a consequence of [2, 8.6] and the generalized Nielsen theorem, [11]. The theorem is a special case of a much more general result due to Conner and Raymond mentioned at the end of §12 of [3]. However, not many details were given there. It has been obtained independently by H. Zieschang by a different approach in the announcement, [11]. The details of the proof of our stated theorem are given in [4; §9] for the case  $k = 1$ . We shall sketch the main points of the argument for the case  $k \geq 1$  and refer the reader to [4] for details.

Notice that  $M^{k+2}$  admits a (central) covering space of the form  $T^k \times R^2$ . Hence  $M$  is a closed aspherical manifold. (In all the other cases  $M$  is not aspherical.)

Since  $N \cong \pi_1(M, x)/(\text{im}(f_*^x))$  is an orientation preserving planar group with compact quotient and different from  $\mathbb{Z} \oplus \mathbb{Z}$ , it has trivial center. Consequently  $\text{im}(f_*^x) \cong \mathbb{Z}^k$  is precisely the center of  $\pi_1(M, x)$ . If

$$\Psi_*: \pi_1(M_1, x_1) \rightarrow \pi_1(M_2, x_2)$$

is an isomorphism, then for the torsion free central extensions given by the Bieberbach classes:

$$\begin{array}{ccccccc} a_1: 0 & \rightarrow & \mathbb{Z}^k & \rightarrow & \pi_1(M_1, x_1) & \rightarrow & N_1 \rightarrow 1 \\ & & \downarrow \Psi_* & & \downarrow \Psi_* & & \downarrow \Psi_* \\ a_2: 0 & \rightarrow & \mathbb{Z}^k & \rightarrow & \pi_1(M_2, x_2) & \rightarrow & N_2 \rightarrow 1 \end{array}$$

there are induced isomorphisms

$$\text{im}(f_*^{x_1}) = \text{center}(\pi_1(M_1, x_1)) \rightarrow \text{im}(f_*^{x_2}) = \text{center}(\pi_1(M_2, x_2))$$

and

$$N_1 = \pi_1(M_1, x_1)/\text{im}(f_*^{x_1}) \rightarrow N_2.$$

Since all actions of  $N$  on  $R^2$  are equivalent we can assume that  $N_1 = N_2$  and both actions  $(T^k, M_1)$   $(T^k, M_2)$  induce the same action of  $N$  on  $R^2$ . Because  $\Psi: N \rightarrow N$  is an automorphism we can, by the generalized Nielsen theorem, find an isomorphism  $\Phi: R^2 \rightarrow R^2$  so that

$$\Phi(w \cdot \alpha) = \Phi(w) \cdot \Psi(\alpha).$$

Let us suppose now that  $\Psi_*: \text{im}(f_*^{x_1}) \rightarrow \text{im}(f_*^{x_2})$  is the “identity”. Then  $a_1 = \Psi^*(a_2)$  and by [2; 8.6] we can find an equivariant homeomorphism  $(T^k, M_1^{k+2}) \rightarrow (T^k, M_2^{k+2})$ .

Of course, in general,  $\Psi_*$  restricted to the center,  $\text{im}(f(\ast^1))$ , is not the “identity”. This automorphism then corresponds to an automorphism of  $T^k$  onto itself. Denote this restriction by  $\theta$ . Then  $\Psi^*(\theta^{*-1}(a_2)) = a_1$ . By a slight modification of [2; 8.6] we are able to use covering space theory and the splitting actions to construct an equivariant homeomorphism between  $(\theta(T^k), M_1)$  and  $(T^k, M_2)$ . We refer the reader to [4; §9] for further details (for  $k = 1$ ).

*Remark 2.5.* We have alluded that the manifolds which satisfy 1, 2 or 3 of 2.2 can never admit any other type of action and so each has a unique action up to weak equivalence. Therefore we must distinguish these manifolds topologically from the remaining ones. Once again we refer to §12 of [3] to deduce the following. In 2.2.4 the fundamental group is non-abelian and has elements of finite order. All groups appearing in 1, 2 or 3 are torsionless and only 2.2.1 ( $M = T^{k+2}$ ) has abelian fundamental group. Furthermore, we know that the only time the fundamental group is abelian, in 2.2.1–2.2.6, other than  $T^{k+2}$ , is in 2.2.5 and 2.2.6. Here the manifold is  $T^k \times S^2$  or  $T^{k-1} \times L$ , where  $L$  is a lens space. Fortunately, we also know the manifolds where the action is not locally injective (they have  $g = 0$  and  $n = 1$  or 2 and do not satisfy 2.2.5 or 2.2.6) by virtue of [3; 12.17]. In fact it is shown that if  $(T^k, M^{k+k})$  is non-locally injective then  $\pi_1(M^{k+2})$  is abelian. Furthermore, except for two minor special cases, these manifolds may be identified with  $T^{k-1} \times L$ , where  $L$  is a 3-dimensional lens space. We described this action in terms of knowledge of the way the 2-dimensional torus acts on a lens space, which was discussed in the first paper of this series [5], (The two non-locally injective exceptions, alluded to in the concluding paragraph of Theorem 2.2, probably yield  $T^{k-1} \times L$  also. They do possess regular  $\mathbb{Z}_2$  or  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  covering spaces homeomorphic to  $T^{k-1} \times L$ , [3; 12.16].)

Notice that this section actually shows that the manifolds listed in 2.2 are all mutually non-homeomorphic with the exception of  $T^k \times S^2$  occurring both in 2.2.5 and 2.2.6.

*Remark 2.6.* We have mentioned nothing of differentiability since the techniques involved are valid in all of the categories: cohomological, topological, smooth and holomorphic. Of course, in the smooth category homeomorphism can always be replaced by diffeomorphism.

There are two more aspects of the approach of [1] and [3] that are of special interest here. We shall discuss these next.

*2.7. Reductions and fiberings over a torus.* The action  $(T^k, M^{k+2})$  for  $k > 1$  can be described in terms of those for  $k = 1$ . For  $(T^k, M^{k+2})$  one can always find [3; 12.3 and 12.15] a splitting  $T^k = T^1 \times T^{k-1}$  and a finite abelian subgroup  $\Delta \subset T^{k-1}$  so that  $(T^k, M^{k+2})$  has a  $\Delta$ -fold regular covering and a lifting of the action to

$$(T^1 \times T^{k-1}, T^{k-1} \times Y, \Delta).$$

The action of the  $T^{k-1}$  subgroup is by translation on the first factor. The action of  $\Delta$  is a diagonal action. Hence,

$$(T^{k-1}, M^{k+2}) = (T^{k-1}, T^{k-1} \times_{\Delta} Y)$$

and  $(T^{k-1}, M^{k+2})$  is fibered equivariantly over  $(T^{k-1}, T^{k-1}/\Delta)$  with fiber  $Y$  and structure group  $\Delta$ . The action of  $T^1$  on  $T^{k-1} \times Y$  induces an action of  $T^1$  on  $Y = T^{k-1} \times Y/T^{k-1}$  as well as an action  $(Y, \Delta)$ . Hence  $(T^1, Y)$  is an action of  $S^1$  on a closed oriented 3-manifold with only finite isotropy subgroups. The commuting actions  $(T^1, Y, \Delta)$  are called *associated actions* to  $(T^k, M^{k+2})$ . It is definitely true, as we shall see, that a given  $(T^k, M^{k+2})$  may have many different circle actions associated with it.

(Let us return, for a moment, to the general hypothesis of 2.1. If the evaluation map induces a *monomorphism on homology*,

$$f_{\#}^x : H_1(T^k; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})$$

the action  $(T^k, X)$  is called *homologically injective*. The paper [1] is devoted to the assertion that homological injectivity, Bieberbach classes of finite order, and equivariant fibering with finite abelian structure group, are all equivalent concepts.)

In summary we state:

**THEOREM [3; 12.15].** *Each associated action  $(T^1, Y, \Delta)$  to  $(T^k, M^{k+2})$  is an action of the circle on a connected 3-manifold without fixed points. Local injectivity, injectivity and homological injectivity of the associated induced actions are equivalent to the same for  $(T^k, M^{k+2})$ . The splitting  $T^1 \times T^{k-1}$  furthermore writes  $(T^{k-1}, M^{k+2})$  as an equivariant fiber bundle  $(T^{k-1}, T^{k-1} \times_{\Delta} Y)$  over  $(T^{k-1}, T^{k-1}/\Delta)$  with finite abelian structure group  $\Delta \subset T^{k-1}$ .*

Obviously, the last fact describes how one may reconstruct all the possible  $M^{k+2}$  topologically.

Each  $(T^k, M^{k+2})$  fibers over a  $T^{k-1}$  torus. However, it is possible to fiber  $M^{k+2}$  over  $T^k$  sometimes. In fact, this fibering question leads to very interesting connections with differential and algebraic geometry.

**THEOREM ([1] and [3]).** *For  $(T^k, M^{k+2})$  the following are equivalent:*

- (1)  $(T^k, M^{k+2})$  is homologically injective.
- (2)  $(T^k, M^{k+2})$  fibers equivariantly over  $(T^k, T^k/\Delta)$  with a closed oriented surface as fiber and a finite abelian structure group  $\Delta \subset T^k$ .
- (3) The action  $(T^k, M^{k+2})$  is represented by a Bieberbach class of finite order.
- (4) The circle action  $(T^1, Y)$  of our associated action  $(T; Y, \Delta)$  to  $(T^k, M^{k+2})$  is homologically injective.
- (5)  $\text{rank}(H_1(M^{k+2}; \mathbb{Z})) \equiv k(2)$ .

Where the group  $\Delta$  of (2) can be taken cyclic. We observe that  $\Delta$  can then be embedded in a circle subgroup, say  $T^1$  of  $T^k$ . Then  $(T^k, M^{k+2})$  can be written as  $(T^k, T^{k-1} \times (T^1 \times_{\Delta} S))$ , where  $S$  is an oriented closed surface. In particular,  $M^{k+2} = T^{k-1} \times X$  where  $X = T^1 \times_{\Delta} S$ . Now the action  $(T^1, X)$  can be represented by a Bieberbach class  $a$ , whose order is a divisor of  $|\Delta| = n$ . If  $(m, n) = 1$ , we may represent  $m_a$  by  $(T^1, Y) = (T^1, T^1 \times_{\Delta} S)$  with a different diagonal action of  $\Delta$  on  $T^1 \times S$ . In general  $m_a$  will represent a different action and hence

$\pi_1(X)$  is not isomorphic to  $\pi_1(Y)$ . Explicit calculations are given for every situation in [4; §10]. Now it is easy to see [4; §4], that  $T^1 \times X$  is diffeomorphic to  $T^1 \times Y$  and hence the closed  $M^{k+2}$  can be written as  $T^{k-1} \times X$  and simultaneously as  $T^{k-1} \times Y$  even though  $\pi_1(X) \not\cong \pi_1(Y)$ . To what extent this isomorphism is equivariant is also discussed in [4; §4].

It would be interesting, especially when  $k = 2$ , to determine in general when  $M^{k+2}$  may be written as a product of  $T^{k-1}$  with a 3-dimensional manifold.

**2.8. Holomorphic actions and elliptic fiber spaces.** We have already seen that those  $(T^k, M^{k+2})$  which were not locally injective were essentially “linear” actions on  $T^{k-1} \times L$ , where  $L$  is a lens space. Such actions, when  $k$  is even, are topologically equivalent to holomorphic actions. In all the remaining cases, where  $k$  is even, the topological actions are also topologically equivalent to holomorphic actions [3; §5 and §12].

Our observation is the following:

**THEOREM [3; 12.15].** *The topological action  $(T^k, M^{k+2})$  is topologically equivalent to a smooth action, which in turn when  $k$  is even is smoothly equivalent to a holomorphic action. In addition, for  $k$  even, the following are equivalent:*

- (1)  $(T^k, M^{k+2})$  is homologically injective.
- (2)  $M^{k+2}$  admits a structure of a Kähler manifold.
- (3)  $M^{k+2}$  admits a structure of a nonsingular projective algebraic manifold.
- (4)  $(T^k, M^{k+2})$  is topologically equivalent to a holomorphically injective action. (A holomorphic action is *holomorphically injective* if the evaluation map induces an *epimorphism* on the closed holomorphic 1-forms  $f_x^* : h^{1,0}(M) \rightarrow h^{1,0}(T)$ .)

When  $k = 2$ , holomorphic  $(T^2, M^4)$  are precisely the elliptic surfaces of Kodaira which admit a whole complex torus of automorphisms. In terms of Kodaira’s notation this means that all singular fibers are multiple fibers and the elliptic bundle away from the singular orbits is principal. Thus all projective algebraic surfaces with a complex torus of automorphisms are of the form  $T^2 \times_{\Delta} Y$  where  $Y$  is a nonsingular algebraic curve and  $\Delta$  is an abelian group of automorphisms. The analogous statement holds for a general  $(T^k, M^{k+2})$  with  $T^k$  algebraic.

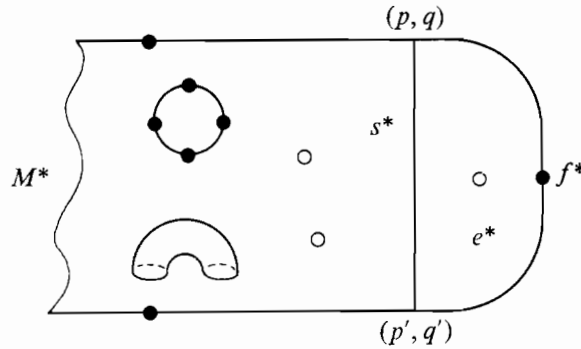
The important feature to remember about these complex manifolds for which the action is injective (and more generally whenever we have a Seifert fibering of real codimension 2 which arises from a Bieberbach class  $H_{\Phi}^2(N; \mathbb{Z}^k)$ , where  $N$  is planar and  $\Phi : N \rightarrow GL(k, \mathbb{Z})$ ) is that the homeomorphism type is completely determined by the fundamental group.

### §3. CONNECTED SUM DECOMPOSITION, $F \neq \emptyset$

Let  $M$  be a closed, connected, oriented 4-manifold with  $G$ -action so that  $F \neq \emptyset$ , i.e. the action has fixed points. It follows from Theorem 1.2 that the weighted orbit space  $M^*$  is a complete set of invariants up to equivariant homeomorphism. In this section we shall exhibit  $M$  as an equivariant connected sum of “elementary”  $G$ -manifolds.

We have seen that  $M^*$  is a compact, oriented 2-manifold. Its boundary is non-empty since  $F \neq \emptyset$ . Let  $f$  be a fixed point of the action and  $f^*$  its image in  $M^*$ . The connected sum decomposition is obtained by an inductive simplification of  $M^*$ .

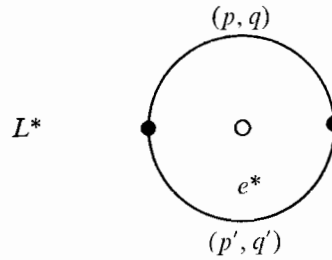
3.1. *Removing E-orbits.* Let  $E$  have  $r$  components. Move the image of an  $E$ -orbit,  $e^*$  into a small disk near  $f^*$  so that the orbit space is as below:



Then  $s^*$  is the image of  $S^3$  and  $M$  is an equivariant connected sum

$$M = N \# L$$

where the orbit space of  $N$  is that of  $M$  with  $e^*$  removed and the orbit space of  $L$  is:

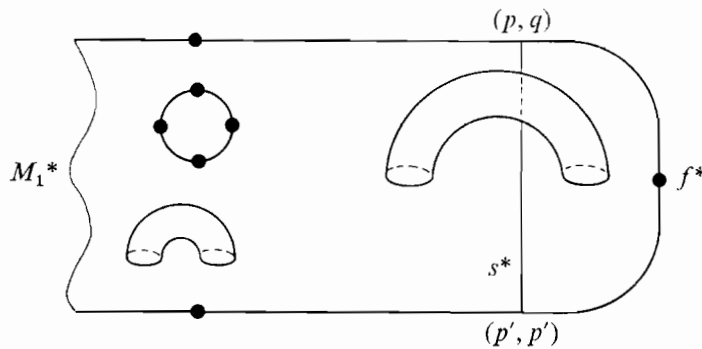


Repeated application of this step gives  $M$  as a connected sum

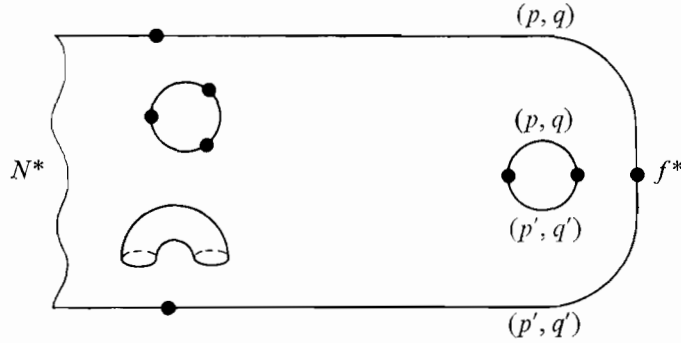
$$M = M_1 \# L_1 \# \cdots \# L_r$$

where the action on  $M_1$  has no  $E$ -orbits. We shall call the  $L_i$  *elementary manifolds of type L*.

3.2. *Removing handles.* Now consider  $M_1$  and assume that  $M_1^*$  has  $g$  handles. Move a handle into a small disk near  $f^*$  so that the orbit space is as below:



Consider the arc  $s^*$  whose inverse image is  $S^3$ . Cutting  $M_1$  along this sphere and adding two equivariant 4-cells results in a new manifold  $N$  whose orbit space has  $(g-1)$  handles and an additional boundary component:



Clearly

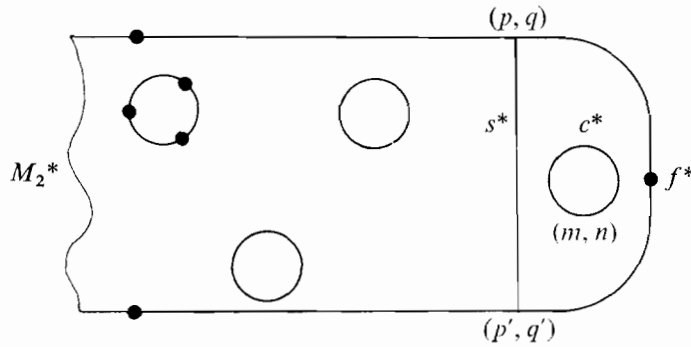
$$M_1 = N \# S^3 \times S^1$$

and repeated application decomposes  $M_1$  as a connected sum

$$M_1 = M_2 \# g(S^3 \times S^1)$$

so that the orbit space of  $M_2$  is a disk with holes.

**3.3. Removing C-orbits.** Now consider  $M_2$ . We shall remove the boundary components of  $M_2^*$  containing no fixed points. Let  $c^*$  be such a boundary component. Move it inside a small disk near  $f^*$  so that the orbit space is as below:

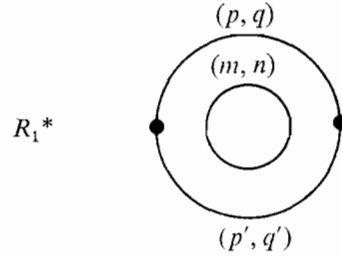


and consider the arc  $s^*$ . It corresponds to a sphere  $S^3$  in  $M_2$  and we get the connected sum decomposition of  $M_2$  as follows

$$M_2 = N \# R_1$$

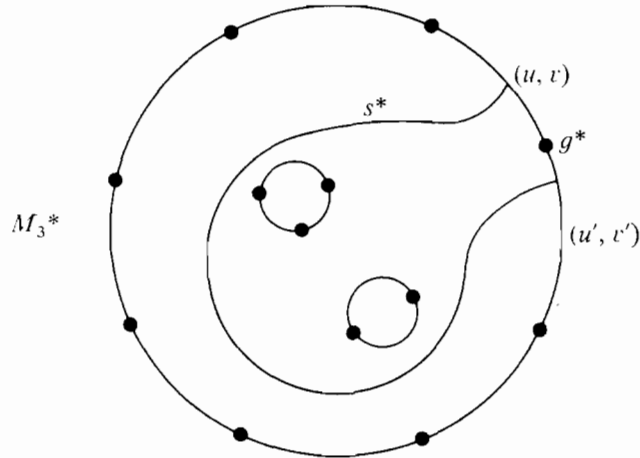
where  $N$  has the same orbit space as  $M_2$  except the boundary component  $c^*$  is deleted, and  $R_1$  is an *elementary manifold of type R* with orbit space





Repeated application yields  $M_2$  as a connected sum of  $M_3$  and *elementary manifolds of type R* so that the orbit space of  $M_3$  is a disk with holes and each boundary component contains images of fixed points.

3.4. *Removing fixed points.* Now consider  $M_3$ . Clearly each boundary component contains the images of at least two fixed points. Suppose we have a boundary component  $\partial$  with more than two fixed points. Let  $g^*$  be an arbitrary fixed point on  $\partial$  and run an arc  $s^*$

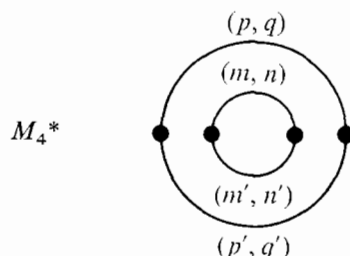


as indicated on  $M_3^*$ . Thus  $s^*$  meets  $\partial$  on either side of  $g^*$  and separates  $M_3^*$  into a simply connected component whose boundary contains all fixed points of  $\partial$  except  $g^*$  and another component containing all other boundary components of  $M_3^*$  and has  $g^*$  as only fixed point on the boundary containing  $s^*$ . Again,  $s^*$  is the image of  $S^3$  rendering  $M_3$  as a connected sum

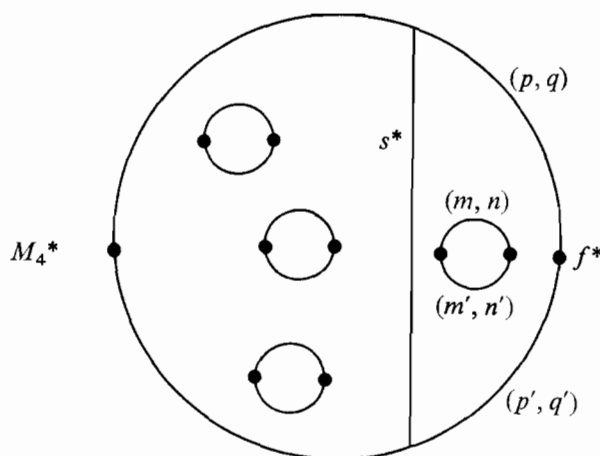
$$M_3 = U_1 \# N$$

where  $U_1$  is simply connected and  $N$  has the same orbit space as  $M_3$  except  $\partial$  now has only two fixed points. Repeated application yields  $M_3$  as a connected sum of simply connected 4-manifolds (classified in [5; §5]) and some  $M_4$  whose orbit space is a disk with holes and each boundary component contains the images of exactly two fixed points. By choosing  $g^* = f^*$  on the component of  $f^*$  we may assume that  $f \in M_4$ .

3.5. *Removing boundaries.* Consider  $M_4$  and suppose that  $M_4^*$  has  $k$  boundary components,  $k \geq 1$ . If  $k = 1$  then  $M_4 = S^4$  by [5; §5] and we have obtained  $M$  as a connected sum of simply connected manifolds, copies of  $S^3 \times S^1$  and manifolds of type  $L$  and type  $R$ . If  $k = 2$  then  $M_4$  has orbit space



and we call it an *elementary manifold of type T*. If  $k > 2$  we can decompose  $M_4$  as a connected sum of type  $T$  manifolds. Move the boundary component  $\partial$  into a small disk near  $f^*$  in  $M_4^*$  and consider the arc  $s^*$  as below:



Then  $s^*$  is the image of  $S^3$  and

$$M_4 = N \# T_1$$

where the orbit space of  $N$  is that of  $M_4$  with  $\partial$  deleted and  $T_1$  is a type of  $T$  manifold. Repeated application gives the required decomposition.

We may state the final result as follows:

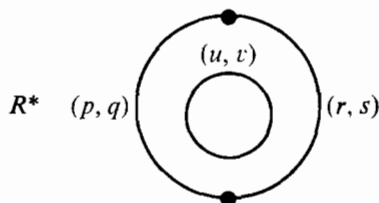
**THEOREM.** *Let  $M$  be a closed, connected, oriented 4-manifold with  $T^2$ -action. If the action has fixed points then  $M$  is an equivariant connected sum of simply connected manifolds, copies of  $S^3 \times S^1$  and manifolds of type  $L$ ,  $R$  and  $T$ .*

Note that if the orbit invariants of  $M$  are known, then a specific decomposition can be given, but the decomposition is not unique.

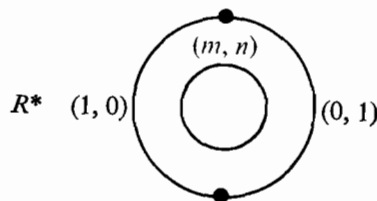
## §4. STABLE HOMEOMORPHISMS

In this section we shall investigate  $G$ -manifolds of type  $R$  and  $T$  and show that for every such manifold  $M$  there exists a nonnegative integer  $k$  so that the connected sum of  $M$  and  $k$  copies of  $S^2 \times S^2$ ,  $M \# kS^2 \times S^2$ , is homeomorphic to a well-known manifold. This is what we mean by *stable homeomorphism classification*.

4.1. Consider the closed, orientable 4-manifold  $R$  with  $T^2$ -action, whose orbit space is below. According to [5; 1.10] the manifold admits a cross-section and hence its equi-



variant classification is given there. We are interested in the homeomorphism classification. A suitable automorphism of  $T^2$  will change the orbit invariants to

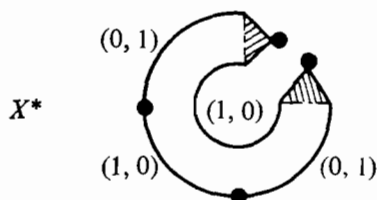


and we denote this manifold by  $R(m, n)$ . Note that if  $\bar{R}(m, n)$  is the reverse orientation, then

$$R(m, n) = \bar{R}(n, m) = \bar{R}(-m, n) = \bar{R}(m, -n) = R(-m, -n)$$

so we may assume  $m \geq n \geq 0$ .

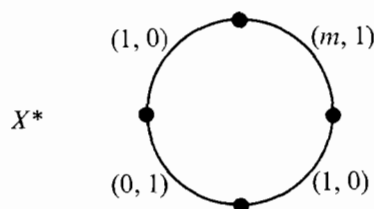
4.2. First consider two special cases. If  $n = 0$ , then the fact that  $(m, n) = 1$  requires that  $m = 1$ . Consider the invariant  $S^3$  whose orbit space is an arc  $(1, 0) = (m, n) \text{---} (0, 1)$  connecting the two boundary components of  $R^*(1, 0)$ . Cutting  $R(1, 0)$  open along this  $S^3$  and attaching two 4-cells, we obtain a simply connected 4-manifold  $X$  whose orbit space is below.



From [5; §5] we obtain that  $X \cong S^2 \times S^2 = Q_0$ . Removing two 4-cells from  $X$  and identifying the boundaries corresponds to taking the connected sum with  $S^3 \times S^1$ , hence we conclude that

$$R(1, 0) \cong Q_0 \# S^3 \times S^1.$$

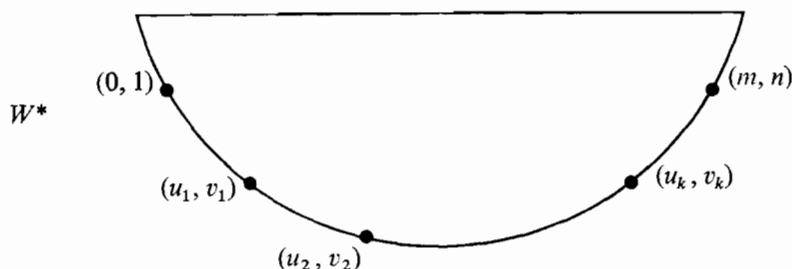
If  $n = 1$  we look at the invariant  $S^3$  whose orbit space is the arc  $(m, 1) \text{---}(1, 0)$  connecting the boundary components of  $R^*(m, 1)$ . Cutting open and attaching 4-cells we obtain  $X$  with orbit space:



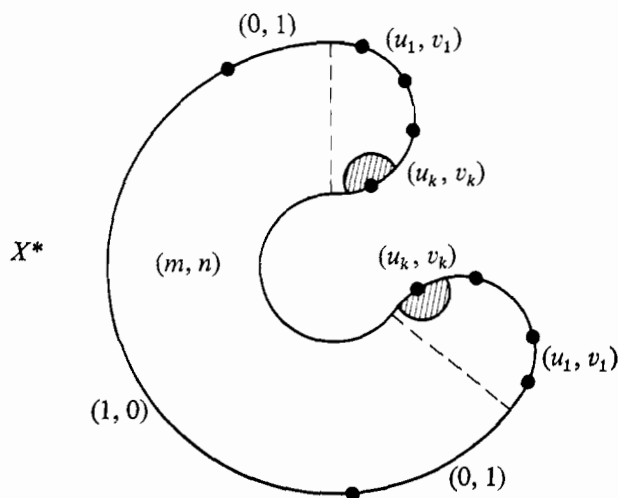
From [5; §5] we see that  $X = Q_0$  if  $m$  is even and  $X = P \# \bar{P} = Q_1$  if  $m$  is odd. Thus

$$R(m, 1) = \begin{cases} Q_0 \# S^3 \times S^1, & \text{for } m \text{ even.} \\ Q_1 \# S^3 \times S^1, & \text{for } m \text{ odd.} \end{cases}$$

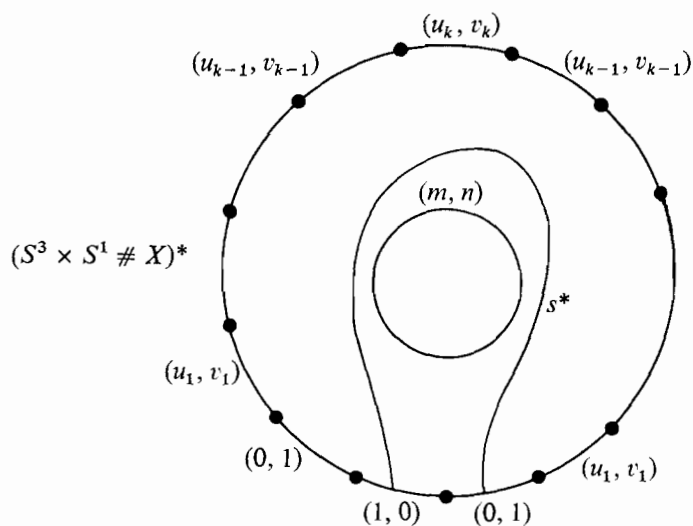
4.3. Assume now that  $m \geq n \geq 2$  and consider the invariant lens space  $L(m, n)$  whose orbit space is the arc  $(0, 1) \text{---}(m, n)$  connecting the two boundary components of  $R^*(m, n)$ . In order to do the analog of the construction above, we need a simply connected  $G$ -manifold  $W$  with  $\partial W = L(m, n)$  so that the action on the boundary is the one given above.



By [5; §5] any such 4-manifold has orbit space  $W^*$  and it is the result of a linear plumbing [5; §5]. Suppose we have such a manifold. We cut  $R(m, n)$  open along the lens space  $L(m, n)$  and sew equivariantly two copies of  $W$  onto the two boundary components. This is represented in the orbit space as follows.



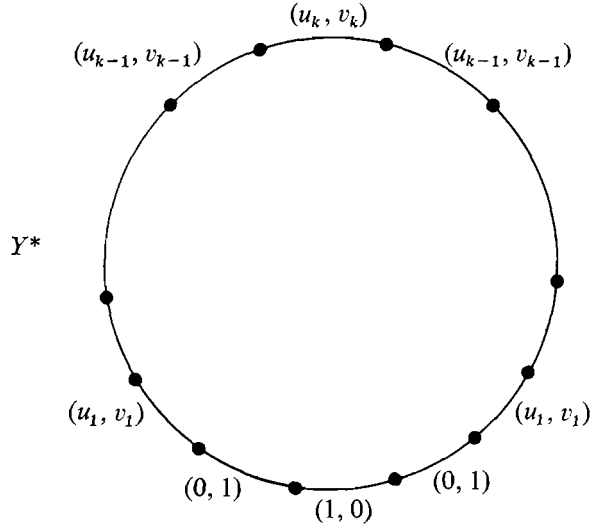
The manifold thus obtained,  $X$ , is simply connected and by [5; §5] we can find out what it is. Now identify the shaded 4-cells in  $X$ . The new manifold  $S^3 \times S^1 \# X$  has orbit space



This manifold is an equivariant connected sum along the 3-sphere whose orbit space is indicated by the arc  $s^*$ . In fact clearly

$$S^3 \times S^1 \# X \cong R(m, n) \# Y$$

where  $Y$  is a simply connected manifold with orbit space



According to [5; §5],  $Y$  is a connected sum of copies of  $Q_0$ ,  $P$  and  $\bar{P}$ . We wish to choose  $W$  so that  $Y$  is a connected sum of copies of  $Q_0$  only. This is clearly accomplished if we choose

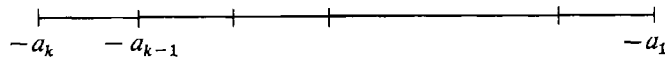
$$(*) \quad \begin{aligned} &u_1, u_3, u_5, \dots, v_2, v_4, v_6, \dots \text{ odd and} \\ &u_2, u_4, u_6, \dots, v_1, v_3, v_5, \dots \text{ even.} \end{aligned}$$

We have to consider two cases.

4.4. The case  $m \cdot n \equiv 0 \pmod{2}$ . Either  $m$  or  $n$  is even, the other is odd, since  $(m, n) = 1$ . If  $n$  is even and  $m$  is odd we expand  $m/n$  as a finite continued fraction with  $|a_i| \geq 2$

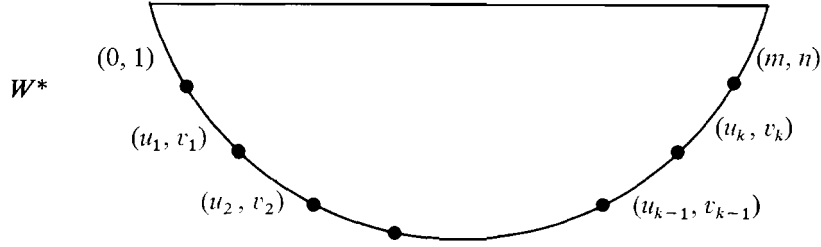
$$\frac{m}{n} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\ddots - \frac{1}{a_k}}}} = [a_1, \dots, a_k].$$

Such an expansion exists and according to [5; §5] there is a unique one with each  $a_i$  even and  $k$  even. We construct  $W$  as a linear plumbing [5, §5] according to the graph



whose action satisfies (\*).

That is,  $W^*$  is the figure below,



satisfying (\*) and

$$\begin{vmatrix} u_2 & 0 \\ v_2 & 1 \end{vmatrix} = -a_k, \quad \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix} = -a_{k-1}, \dots$$

$$\begin{vmatrix} u_k & u_{k-2} \\ v_k & v_{k-2} \end{vmatrix} = -a_2, \quad \begin{vmatrix} m & u_{k-1} \\ n & v_{k-1} \end{vmatrix} = -a_1.$$

It is a simple exercise in linear algebra to see that all these conditions may be satisfied. From this we have that

$$Y = \underbrace{Q_0 \# Q_0 \# \dots \# Q_0}_{k \text{ times}} = kQ_0$$

$$X = (k+1)Q_0$$

so that we have the stable result

$$R(m, n) \# kQ_0 \cong Q_0 \# S^3 \times S^1 \# kQ_0.$$

If  $m$  is even and  $n$  is odd, then we do the same with the arc  $(1, 0) \text{---} (m, n)$  and arrive at the same result.

4.5. The case  $m \cdot n \equiv 1 \pmod{2}$ . In this case we can find an odd integer  $a_1$  so that  $m/n = a_1 - m'/n$  where  $m'$  is even. Apply (4.4) to  $n/m'$  to obtain  $n/m' = [a_2, \dots, a_k]$  with unique even  $a_2, \dots, a_k$ . Then  $m/n = [a_1, a_2, \dots, a_k]$ . We construct  $W$  as above, except that  $\begin{vmatrix} m & u_{k-1} \\ n & v_{k-1} \end{vmatrix} = -a_1$  is odd. This is consistent with the condition of (\*), since both  $m$  and  $n$  are odd, while  $u_{k-1}$  and  $v_{k-1}$  have different parity. Thus  $Y \cong kQ_0$  and  $X \cong Q_1 \# kQ_0$  so

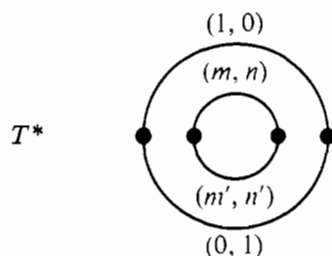
$$R(m, n) \# kQ_0 \cong Q_1 \# S^3 \times S^1 \# kQ_0.$$

We summarize these statements as follows:

**THEOREM.** *Given relatively prime integers  $(m, n)$ , there exists a non-negative integer  $k$  so that*

$$R(m, n) \# kQ_0 \cong \begin{cases} Q_0 \# S^3 \times S^1 \# kQ_0 & \text{if } m \cdot n \equiv 0 \pmod{2} \\ Q_1 \# S^3 \times S^1 \# kQ_0 & \text{if } m \cdot n \equiv 1 \pmod{2}. \end{cases}$$

4.6. A similar argument may be applied to the manifold  $T(m, n; m'n')$  whose orbit space is below:



Since  $\begin{vmatrix} m & m' \\ n & n' \end{vmatrix} = \pm 1$  at least two of these integers are odd and one is even. Let  $\kappa = 0$  if there are two even integers and  $\kappa = 1$  if three of the integers are odd. If one of the weights equals  $\pm 1$  then the cut and paste method of the earlier sections together with the fact that  $Q_0 \# Q_1 \cong Q_1 \# Q_1$  shows that

$$T(m, n; m'n') \cong \begin{cases} Q_0 \# Q_0 \# S^3 \times S^1 & \text{for } \kappa = 0 \\ Q_1 \# Q_1 \# S^3 \times S^1 & \text{for } \kappa = 1. \end{cases}$$

If none of the weights equals  $\pm 1$  we use the construction of (4.3) to obtain a stable conclusion:

**THEOREM.** *There, exists a non-negative integer  $k$  so that*

$$T(m, n; m'n') \# kQ_0 \cong \begin{cases} Q_0 \# Q_0 \# S^3 \times S^1 \# kQ_0 & \text{if } \kappa = 0 \\ Q_1 \# Q_1 \# S^3 \times S^1 \# kQ_0 & \text{if } \kappa = 1. \end{cases}$$

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