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Additivity of free genus of knots

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Abstract

We show that free genus of knots is additive under connected sum.

Keywords: Free genus; Connected sum

1. Introduction

Let K be a knot in the 3-sphere S^3. A Seifert surface F for K in S^3 is said to be *free* if the fundamental group $\pi_1(S^3 - F)$ is a free group. We note that all knots bound free Seifert surfaces, e.g. canonical Seifert surfaces constructed by Seifert's algorithm. We define the *free genus* $g_f(K)$ of K as the minimal genus over all free Seifert surfaces for K [6].

Schubert [10, 2.10 Proposition] proved that the usual genus of knots is additive under connected sum. In general, the genus of a knot is not equal to its free genus. In fact, free genus may have arbitrarily high gaps with genus [8,7].

In this paper, we show the following theorem.

Theorem 1. *For two knots K_1, K_2 in S^3, $g_f(K_1) + g_f(K_2) = g_f(K_1 \# K_2)$.*

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2. Preliminaries

We can deform a Seifert surface F by an isotopy so that $F \cap N(K) = N(\partial F; F)$. We denote the exterior $cl(S^3 - N(K))$ by $E(K)$, and the exterior $cl(S^3 - N(F))$ or $cl(E(K) - N(F))$ by $E(F)$. We have the following proposition.

Proposition 2 (Hempel [4, 5.2]; Jaco [5, IV.15]; Ozawa [9, Lemma 2.2]). *A Seifert surface F is free if and only if E(F) is a handlebody.*

We have the following inequality.

Proposition 3. $g_f(K_1) + g_f(K_2) \geqslant g_f(K_1 \# K_2)$.

Proof. Let F_i $(i = 1,2)$ be a free Seifert surface of minimal genus for K_i. We construct a natural Seifert surface F for $K_1 \# K_2$ as the boundary connected sum of F_1 and F_2. Then $E(F)$ is obtained by a boundary connected sum of $E(F_1)$ and $E(F_2)$. Therefore the exterior of F is a handlebody, and F is free. Hence we have the desired inequality. $\square$

We can specify the $+$-*side* and $-$-*side* of a Seifert surface F for a knot K by an orientation of F. We say that a compressing disk D for F is a $+$-*compressing disk* (resp. $-$-*compressing disk*) if the collar of its boundary lies on the $+$-side (resp. $-$-side) of F, and F is called $+$-*compressible* (resp. $-$-*compressible*) if F has a $+$-*compressing disk* (resp. $-$-*compressing disk*). A Seifert surface is said to be *weakly reducible* if there exist a $+$-compressing disk D^+ and a $-$-compressing disk D^- for F such that $\partial D^+ \cap \partial D^- = \emptyset$. Otherwise F is *strongly irreducible*. The Seifert surface F is *reducible* if $\partial D^+ = \partial D^-$. Otherwise F is *irreducible*. If F is reducible, then by sliding ∂D^- on F slightly, we see that F is also weakly reducible.

Proposition 4. *A free Seifert surface of minimal genus is irreducible.*

Proof. Suppose that F is reducible. Then there exist a $+$-compressing disk D^+ and a $-$-compressing disk D^- for F such that $\partial D^+ = \partial D^-$. By a compression of F along D^+ (this is the same as a compression along D^-), we have a new Seifert surface F'. Since $E(F')$ is homeomorphic to a component of the manifold which is obtained by cutting $E(F)$ along $D^+ \cup D^-$, it is a handlebody. Hence F' is free, but it has a lower genus than F. This contradicts the minimality of F. $\square$

To prove Theorem 1, we require a version of Haken's lemma [2] by Casson and Gordon [1].

A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup 2$-handles $\cup$ 3-handles and $\partial_- W$ has no 2-sphere components. A *complete disk system* $\mathscr{D}$ for a connected compression body W is a disjoint union of disks $(\mathscr{D}, \partial \mathscr{D}) \subset (W, \partial_+ W)$ such that W cut along $\mathscr{D}$ is homeomorphic to $\partial_- W \times I$ if $\partial_- W \neq \emptyset$ or B^3 if $\partial_- W = \emptyset$. In general, a complete disk system for W is a union of complete disk systems for the components of W. A 3-*manifold triad* $(M; B, B')$ is a cobordism M rel ∂ between surfaces B and B'. A *Heegaard splitting* of $(M; B, B')$ is a pair (W, W') where W, W' are compression bodies such that $W \cup W' = M$,

$W \cap W' = \partial_+ W = \partial_+ W'$, and $\partial_- W = B$, $\partial_- W' = B'$. Let H be a surface and α a closed 1-manifold in H. We denote by $\rho(H; \alpha)$ the surface obtained from H by doing 1-surgeries along the components of α. Let H be a surface in a 3-manifold M, and let D be a disjoint union of disks in M such that $D \cap H = \partial D$. We may then do *ambient 1-surgery on H along D* to obtain a surface in M homeomorphism to $\rho(H; \partial D)$.

Proposition 5 (Casson and Gordon [1, Lemma 1.1]). *Let* (W, W') *be a Heegaard splitting of* $(M; B, B')$. *Let* $(S, \partial S) \subset (M, B \amalg B')$ *be a disjoint union of essential 2-spheres and disks. Then there exists a disjoint union of essential 2-spheres and disks* S^* *in* M *such that*

(i) S^* *is obtained from* S *by ambient 1-surgery and isotopy*;

(ii) *each component of* S^* *meets* F *in a single circle*;

(iii) *there exist complete disk systems* $\mathscr{D}$, $\mathscr{D}'$ *for* W, W', *respectively, such that* $\mathscr{D} \cap S^* = \mathscr{D}' \cap S^* = \emptyset$.

Note that if M is irreducible (in which case S must consist of disks) then it follows that S^* is isotopic to S.

For a free Seifert surface F of minimal genus for $K_1 \# K_2$ and a decomposing sphere S for the connected sum of K_1 and K_2, we will show ultimately that S can be deformed by an isotopy so that S intersects F in a single arc, and we have the equality in Theorem 1.

If a free Seifert surface F of minimal genus for $K_1 \# K_2$ is incompressible, then an innermost loop argument shows that a decomposing sphere S for $K_1 \# K_2$ can be deformed by an isotopy so that S intersects F in a single arc, and by Proposition 3, we have the equality in Theorem 1.

So, hereafter we suppose that F is compressible, and divide the proof of Theorem 1 into two cases; (1) F is strongly irreducible, (2) F is weakly reducible. Case (1) is treated in the next section and we consider case (2) in Section 4. Fig. 1

3. Proof of Theorem 1 (strongly irreducible case)

In this section, we suppose that F is strongly irreducible. Without loss of generality, we may assume that there is a $+$-compressing disk for F. Let $\mathscr{D}^+$ be a $+$-compressing disk system for F, and let F' be a surface obtained by compressing F along $\mathscr{D}^+$. Since $E(F)$ is a handlebody, we can choose $\mathscr{D}^+$ so that F' is connected. Take $\mathscr{D}^+$ to be maximal with respect to the above conditions. We deform F' by an isotopy so that $F' \cap F = K$. Put $A = \partial N(K_1 \# K_2) - \mathit{Int}\, N(F)$, and let H be a closed surface which is obtained by pushing $F \cup A \cup F'$ into the interior of $E(F')$. Let A_0 be a vertical annulus connecting a core of A and a core of the copy of A in H. Then H bounds a handlebody V in $E(F')$ since V is obtained from $E(F)$ by cutting along $\mathscr{D}^+$. The remainder $W = E(F') - \mathit{Int}\, V$ is a compression body since it is obtained from $N(\partial E(F'); E(F'))$ by adding 1-handles $N(\mathscr{D}^+)$.

Lemma 6. F' *is incompressible in* S^3.

Proof. We consider that F' inherits $\pm$-sides from F. Suppose that F' is $+$-compressible, and let E^+ be a $+$-compressible disk for F'. Then we can regard E^+ as a ∂-reducing disk for $E(F')$. By

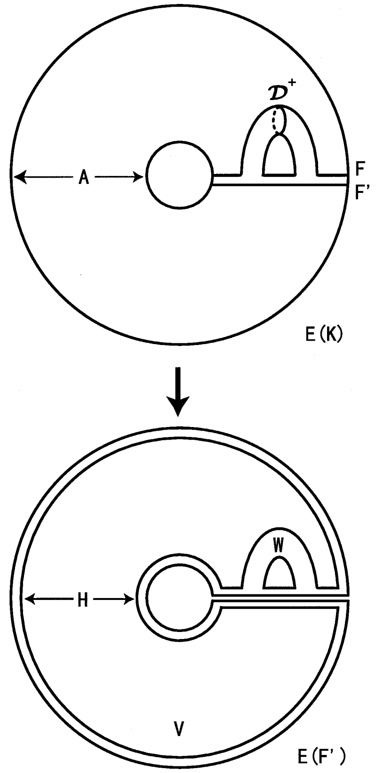

Fig. 1. Construction of a Heegaard splitting of $E(F')$.

applying our situation to Proposition 5, we may assume that $E^+ \cap \mathscr{D}^+ = \emptyset$. If ∂E^+ separates F', then E^+ cuts off a handlebody from $E(F')$, and there is a non-separating disk in it. So, we may assume that ∂E^+ is non-separating in F'. Then $\mathscr{D}^+ \cup E^+$ is a $+$-compressing disk system satisfying the previous conditions. This contradicts the maximality of $\mathscr{D}^+$.

Next, suppose that F' is $-$-compressible, and let E^- be a $-$-compressing disk for F'. Then we can regard E^- as a ∂-reducing disk for $E(F')$. By applying our situation to Proposition 5, we may assume that $E^- \cap H = E^- \cap F$ is a single loop, and by exchanging $\mathscr{D}^+$ if necessary, that E^- does not intersect $\mathscr{D}^+$. But this contradicts the strongly irreducibility of F. $\square$

By Lemma 6, we can deform the decomposing sphere S by an isotopy so that S intersects F' in a single arc. Put $E(S) = S \cap E(F')$. Then $E(S)$ is a ∂-reducing disk for $E(F')$. Otherwise, at least one of K_1 or K_2 is trivial, and Theorem 1 clearly holds. By applying our situation to Proposition 5, we may assume that $E(S)$ intersects H in a single loop, $E(S)$ intersects A_0 in two vertical arc, and (by exchanging $\mathscr{D}^+$ if necessary, preserving the previous conditions) $E(S)$ does not intersect $\mathscr{D}^+$. Then S intersects F in a single arc, hence we obtain the inequality $g_f(K_1) + g_f(K_2) \leqslant g_f(K_1 \# K_2)$. This and Proposition 3 complete the Proof of Theorem 1 in the strongly irreducible case. Fig. 2

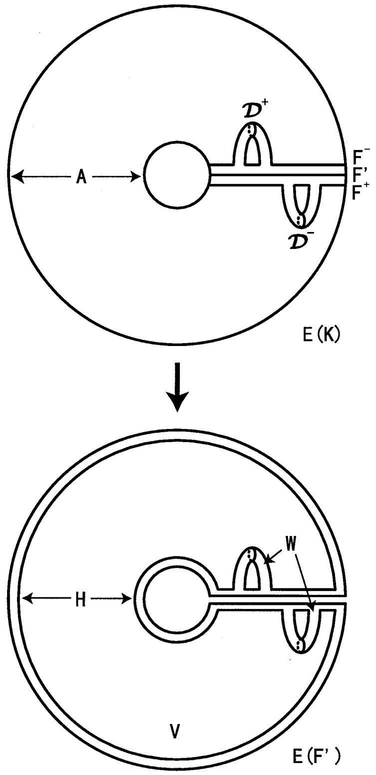

Fig. 2. Construction of a Heegaard splitting of $E(F')$.

4. Proof of Theorem 1 (weakly reducible case)

In this section, we consider the case that F is weakly reducible.

We use the *Hayashi–Shimokawa (HS-) complexity* [3]. Here we review it. Let H be a closed (possibly disconnected) 2-manifold. Put $w(H) = \{genus(T) | T \text{ is a component of } H\}$, where this "multi-set" may contain the same ordered pairs redundantly. We order finite multi-sets as follows: arrange the elements of each multi-set in monotonically non-increasing order, then compare the elements lexicographically. We define the HS-complexity $c(H)$ as a multi-set obtained from $w(H)$ by deleting all the 0 elements. We order $c(H)$ in the same way as w.

Since F is weakly reducible, there exist a $+$-compressing disk D^+ and a $-$-compressing disk D^- for F such that $\partial D^+ \cap \partial D^- = \emptyset$. If $c(\rho(F; \partial D^+ \cup \partial D^-)) = c(\rho(F; \partial D^+))$, say, then ∂D^- bounds a $+$-compressing disk for F. Hence F is reducible, and by Proposition 4, a contradiction.

Therefore, there exist a non-empty $+$-compressing disk system $\mathscr{D}^+$ and a non-empty $-$-compressing disk system $\mathscr{D}^-$ for F such that

1. $\partial \mathscr{D}^+ \cap \partial \mathscr{D}^- = \emptyset$,
2. $c(\rho(F; \partial \mathscr{D}^+ \cup \partial \mathscr{D}^-)) < c(\rho(F; \partial \mathscr{D}^+))$, $c(\rho(F; \partial \mathscr{D}^-))$,

and with $c(\rho(F; \partial\mathscr{D}^+ \cup \partial\mathscr{D}^-))$ minimal subject to these conditions. Moreover, we take $\mathscr{D}^\pm$ so that $|\mathscr{D}^\pm|$ is minimal.

Let $F^\pm$ be a 2-manifold obtained by compressing F along $\mathscr{D}^\pm$, and F' be a 2-manifold obtained by compressing F along $\mathscr{D}^+ \cup \mathscr{D}^-$. We deform F^+ and F^- by an isotopy so that $F^+ \cap F' \cap F^- = K$ and $F^\pm \cap N(K) = N(\partial F^\pm; F^\pm)$. Put $A = \partial N(K_1 \# K_2) - Int\, N(F)$, and let H be a closed 2-manifold which is obtained by pushing $F^+ \cup A \cup F^-$ into the interior of $E(F')$. Let A_0 be a vertical annulus connecting a core of A and a core of the copy of A in H. Then H bounds the union of handlebodies V in $E(F')$ since V is obtained from $E(F)$ by cutting along $\mathscr{D}^\pm$. The remainder $W = E(F') - Int\, V$ is a union of compression bodies since it is obtained from $N(\partial E(F'); E(F'))$ by adding 1-handles $N(\mathscr{D}^\pm)$.

Lemma 7. *There is no 2-sphere component of H.*

Proof. Suppose that there is a 2-sphere component H_i of H. We may assume that H does not contain A, and there is a copy of some component of $\mathscr{D}^+$ in H. Let $\mathscr{D}^+_s$ be a subsystem of $\mathscr{D}^+$ the union of whose boundaries separates F. If there is no copy of $\mathscr{D}^-$ in H_i, then we delete any one of $\mathscr{D}^+_s$. Then $\mathscr{D}^\pm$ holds the previous conditions, but this contradicts the minimality of $|\mathscr{D}^+|$. If there is a copy of $\mathscr{D}^-$ in H_i, then there is a simple closed curve in H_i which separates $N(\mathscr{D}^+) \cap H_i$ from $N(\mathscr{D}^-) \cap H_i$, and bounds a $+$-compressing disk and a $-$-compressing disk for F. Hence F is reducible, but this contradicts Proposition 4. □

Lemma 8. *Each component of F' is incompressible in S^3.*

Proof. We consider that $F^\pm$ and F' inherit $\pm$-sides from F. Suppose, without loss of generality, that F' is $+$-compressible, and let E^+ be a $+$-compressing disk for F'. Then we can regard E^+ as a ∂-reducing disk for $E(F')$. By applying our situation to Proposition 5, we may assume that E^+ intersects H in a single loop which does not intersect A_0. We deform E^+ by an isotopy so that $E^+ \cap \mathscr{D}^+ = \emptyset$ in S^3. We take a complete meridian disk system $\mathscr{C}$ of W which includes $\mathscr{D}^+$ and does not intersect E^+. Put $\mathscr{C}^- = \mathscr{C} - \mathscr{D}^+$. Then we have $c(\rho(F; \partial E^+ \cup \partial\mathscr{D}^+ \cup \partial\mathscr{C}^-)) < c(\rho(F; \partial\mathscr{D}^+ \cup \partial\mathscr{C}^-))$ since ∂E^+ is essential in F'. Suppose that $c(\rho(F; \partial E^+ \cup \partial\mathscr{D}^+ \cup \partial\mathscr{C}^-)) = c(\rho(F; \partial E^+ \cup \partial\mathscr{D}^+))$. Then each component of $\partial\mathscr{D}^-$ bounds both a $+$-compressing disk and a $-$-compressing disk for F. Hence F is reducible, but this contradicts Proposition 2.3. Similarly, if $c(\rho(F; \partial E^+ \cup \partial\mathscr{D}^+ \cup \partial\mathscr{C}^-)) = c(\rho(F; \partial\mathscr{C}^-))$, then we are done. Hence we obtain a $\pm$-compressing disk system $E^+ \cup \mathscr{D}^+$, $\mathscr{C}^-$ for F which satisfies conditions (1), (2) and has smaller complexity than $\mathscr{D}^+ \cup \mathscr{D}^-$. This contradicts the property of $\mathscr{D}^+ \cup \mathscr{D}^-$. □

By Lemma 8, we can deform the decomposing sphere S by an isotopy so that S intersects F' in a single arc. Put $E(S) = S \cap E(F')$. Then $E(S)$ is a ∂-reducing disk for $E(F')$. Otherwise, at least one of K_1 and K_2 is trivial, and Theorem 1 clearly holds. Let V_0 and W_0 be components of V and W, respectively, where V_0 contains A and W_0 is the next handlebody to V_0. Put $H_0 = V_0 \cap W_0$. Then H_0 gives a Heegaard splitting of $V_0 \cup W_0$. By Lemma 8, we can deform $E(S)$ by an isotopy so that

$E(S)$ is contained in $V_0 \cup W_0$. By applying this situation to Proposition 5, we may assume that $E(S)$ intersects H_0 in a single loop without moving $\partial E(S)$. Moreover, there exist a complete meridian disk system $\mathscr{E}_0$ of V_0 such that $\mathscr{E}_0 \cap E(S) = \emptyset$ and $\mathscr{E}_0 \cap A_0 = \emptyset$. Thus S intersects F in a single arc, hence we have the conclusion. $\square$

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