PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 919–922
S 0002-9939(99)05068-6
Article electronically published on May 6, 1999

SYNCHRONISM OF AN INCOMPRESSIBLE NON-FREE SEIFERT SURFACE FOR A KNOT AND AN ALGEBRAICALLY SPLIT CLOSED INCOMPRESSIBLE SURFACE IN THE KNOT COMPLEMENT

MAKOTO OZAWA

(Communicated by Ronald A. Fintushel)

ABSTRACT. We give a necessary and sufficient condition for knots to bound incompressible non-free Seifert surfaces.

1. INTRODUCTION

A Seifert surface for a knot in the 3-sphere is said to be *free* (*unknotted*) if the fundamental group of its complement is free, otherwise *non-free* (*knotted*). We note that any knot bounds both free and non-free Seifert surfaces if one admits Seifert surfaces to be compressible. For this reason, hereafter we consider only incompressible Seifert surfaces. Hatcher and Thurston classified incompressible surfaces in 2-bridge knot complements [3]. This result implies that any incompressible Seifert surface for 2-bridge knots is free. However, there exist knots which bound both free and non-free incompressible Seifert surfaces ([13], [10]). Moreover, there exist knots which bound only non-free incompressible Seifert surfaces ([9], [7]). In [6], C. H. Giffen and L. C. Siebenmann raised the following problem.

Problem 1.1 ([6, Problem 1.20 (B)])**.** Which knots bound an incompressible free Seifert surface?

In this paper, we give a necessary and sufficient condition for knots to bound incompressible non-free Seifert surfaces. As its corollary, we get a necessary condition for knots to bound incompressible free Seifert surfaces.

Let K be a knot in the 3-sphere S^3. For a closed surface S in $S^3 - K$, we define the *order* $o(S;K)$ of S for K as follows. Let $i : S \to S^3 - K$ be the inclusion map and let $i_* : H_1(S) \to H_1(S^3 - K)$ be the induced homomorphism. Since $Im(i_*)$ is a subgroup of $H_1(S^3 - K) = \mathbb{Z}\langle\text{meridian}\rangle$, there is an integer m such that $Im(i_*) = m\mathbb{Z}$. Then we define $o(S;K) = m$.

Theorem 1.2. *Let K be a knot in S^3. Then the following conditions are equivalent.*

(1) *There exists an incompressible non-free Seifert surface F for K.*
(2) *There exists a closed incompressible surface S in $S^3 - K$ with $o(S;K) = 0$.*

Received by the editors June 18, 1997 and, in revised form, May 4, 1998.
1991 *Mathematics Subject Classification.* Primary 57M25.
Key words and phrases. Seifert surface, closed incompressible surface.

Moreover, if these conditions hold, then we can take F and S so that they are disjoint.

For knots in S^3 whose complements contain no incompressible and meridionally incompressible closed surfaces, any closed incompressible surface in exteriors of those knots has a compressing disk in S^3 which intersects the knots in one point. Hence the closed incompressible surface has the order ± 1.

Corollary 1.3 ([1],[8])**.** *Let K be a toroidally alternating knot or closed* 3*-braid knot in S^3. Then any incompressible Seifert surface for K is free.*

Remark 1.4. Toroidally alternating knots ([1]) include alternating knots ([11]), almost alternating knots ([2]) and Montesinos knots ([12]).

2. Proof

All manifolds are assumed to be compact and orientable.

Lemma 2.1 ([5, IV.5, IV.10])**.** *Let M be a 3-manifold and F an incompressible surface properly embedded in M. Put $M' = cl(M - N(F))$. Then:*

1. *M is irreducible if and only if M' is irreducible.*
2. *Any closed incompressible surface embedded in $intM'$ is incompressible in M.*

Proof. (1) Suppose that M is irreducible and M' is reducible. Let S be a reducing 2-sphere for M'. Then S bounds a 3-ball B in M and F is contained in B. Hence F is a closed orientable surface, and hence π_1-injective in B. Therefore F is a 2-sphere and bounds a 3-ball in B, hence in M. This contradicts the incompressibility of F in M.

Suppose that M is reducible and M' is irreducible. Let S be a reducing 2-sphere for M. We may assume that $S \cap F$ consists of loops. Let D be an innermost disk in S. Since F is incompressible in M, ∂D bounds a disk D' in F. Since M' is irreducible, the 2-sphere $D \cup D'$ bounds a 3-ball B in M'. By using the 3-ball B, we can reduce $|S \cap F|$. Hence we may assume that $S \cap F = \emptyset$. Thus S is contained in M' and bounds a 3-ball in M', hence in M. This contradicts the supposition.

(2) Suppose that there exists a closed incompressible surface S properly embedded in M' which is compressible in M. Let D be a compressing disk for S in M. We may assume that $D \cap F$ consists of loops. Let d be an innermost disk in D. Since F is incompressible in M, ∂d bounds a disk d' in F. Let d'' be an innermost disk in d'. By cutting and pasting D along d'', we get a compressing disk for S in M with fewer intersections with F than D and a 2-sphere. By induction on $|D \cap F|$, we may assume that $D \cap F = \emptyset$. Thus D is contained in M'. This contradicts the supposition. □

Lemma 2.2 ([4, 5.2], [5, IV.15])**.** *Let M be an irreducible 3-manifold with connected boundary. Then the following conditions are mutually equivalent.*

1. *$\pi_1(M)$ is free.*
2. *M is a handlebody.*
3. *M does not contain any closed incompressible surface.*

Proof. (1)→(3). Suppose that M contains a closed incompressible surface S. Since S is 2-sided in M, S is π_1-injective in M. Therefore $\pi_1(S)$ is isomorphic to a subgroup of $\pi_1(M)$. Since any subgroup of a free group is free, $\pi_1(S)$ is free. Hence

S is a 2-sphere. By the irreducibility of M, S bounds a 3-ball in M. This contradicts the supposition.

(3)→(2). We will show this by induction on $g(\partial M)$. If $g(\partial M) = 0$, M is a 3-ball because M is irreducible. Suppose that (3)→(2) of Lemma 2.2 holds if $g(\partial M) < g$, and suppose that $g(\partial M) = g$. Since M does not contain any closed incompressible surface, ∂M is compressible in M. Let D be a compressing disk for ∂M in M. Put $M' = cl(M - N(D))$. Then by Lemma 2.1, M' is irreducible and does not contain any closed incompressible surface. By the supposition of induction, each component of M' is a handlebody. Hence M is a handlebody.

(2)→(1). By the definition of a handlebody. □

Proof of Theorem 1.2 (1) → (2). Suppose (1) of Theorem 1.2. Let F be an incompressible non-free Seifert surface for K. Put $M = cl(S^3 - N(K))$ and isotop F so that $F \cap N(K)$ is an annulus. We also denote $F \cap M$ by F. Since $\pi_1(S^3 - F) \cong \pi_1(cl(M - N(F)))$ is not free, by Lemma 2.2, it contains a closed incompressible surface S. By Lemma 2.1 (2), S is also incompressible in M, hence in $S^3 - K$. We take $2g(S)$ simple loops $l_1, \dots, l_{2g(S)}$ on S so that $H_1(S) = \mathbb{Z}\langle l_1\rangle \oplus \cdots \oplus \mathbb{Z}\langle l_{2g(S)}\rangle$. Since $S \cap F = \emptyset$, $lk(l_i, K) = 0$ $(i = 1, \dots, 2g(S))$, thus $i_*(\langle l_i\rangle) = 0$, where i_* denotes the induced homology homomorphism as in the definition of order. Hence $i_*(H_1(S)) = 0$ and we have condition (2).

Proof of Theorem 1.2 (2) → (1). Suppose (2) of Theorem 1.2.

Claim 2.3. There exists a Seifert surface F for K such that $F \cap S = \emptyset$.

Proof. Let F be any Seifert surface for K. Let G be a graph with one vertex v and $2g(S)$ edges $e_1, \dots, e_{2g(S)}$ embedded in S such that $S - G$ is an open disk. We may assume that F intersects S and G transversely and does not intersect v. Give an orientation to F and the edges of G arbitrarily. Suppose that $F \cap G \neq \emptyset$. Then there exists a pair of points p_1 and p_2 of $F \cap G$ such that p_1 and p_2 are adjacent in an edge e_i and have intersection numbers $+1$ and -1 respectively. Indeed, $lk(e_i, K) = 0$ since $o(S; K) = 0$. Let a be a subarc of e_i bounded by p_1 and p_2. By tubing F along a, we get a Seifert surface for K with fewer intersections with G than F. By induction on $|F \cap G|$, we may assume that $F \cap G = \emptyset$. Since $S - G$ is an open disk, by an innermost loop, and cut and paste argument, we can get a Seifert surface F for K with $F \cap S = \emptyset$. □

Claim 2.4. There exists an incompressible Seifert surface F for K such that $F \cap S = \emptyset$.

Proof. By Claim 2.3, there exists a Seifert surface F for K such that $F \cap S = \emptyset$. Suppose that F is compressible. Let D be a compressing disk for F in $S^3 - K$. By the argument which is similar to the proof of Lemma 2.1 (2), we may assume that $D \cap S = \emptyset$. Now we compress F along such D. Then by ignoring a closed component, we get a Seifert surface for K such that it does not intersect S and has fewer genus than F. By an induction on $g(F)$, we can get an incompressible Seifert surface F for K such that $F \cap S = \emptyset$. □

By Claim 2.4, there exists an incompressible Seifert surface F for K such that $F \cap S = \emptyset$. Then S is a closed incompressible surface in $cl(S^3 - N(F))$. Hence by Lemma 2.2, $\pi_1(cl(S^3 - N(F)))$ is not free. Thus F is an incompressible non-free Seifert surface for K. □

References

[1] C. Adams, *Toroidally alternating knots and links*, Topology **33**, (1994), 353-369. MR **95e:**57006

[2] C. Adams, J. Brock, J. Bugbee, T. Comar, K. Faigin, A. Huston, A. Joseph and D. Pesikoff, *Almost alternating links*, Topology and its Appl. **46**, (1992), 151-165. MR **93h:**57005

[3] A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79**, (1985), 225-246. MR **86g:**57003

[4] J. P. Hempel, *3-Manifolds*, Volume **86** of Ann. of Math. Stud., Princeton Univ. Press, 1976. MR **54:**3702

[5] W. H. Jaco, *Lectures on Three-manifold Topology*, Volume **43** of CBMS Regional Conf. Ser. in Math., American Math. Soc., 1980. MR **81k:**57009

[6] R. Kirby, *Problems in low-dimensional topology*, Part **2** of Geometric Topology (ed. W. H. Kazez), Studies in Adv. Math., Amer. Math. Soc. Inter. Press, 1997. MR **80g:**57002

[7] M. Kobayashi and T. Kobayashi, *On canonical genus and free genus of knot*, J. Knot Theory and its Ramifi. **5**, (1996), 77-85. MR **97d:**57008

[8] M. T. Lozano and J. H. Przytycki, *Incompressible surface in the exterior of a closed 3-braid*, Math. Proc. Camb. Phil. Soc. **98**, (1985), 275-299. MR **87a:**57013

[9] H. C. Lyon, *Knots without unknotted incompressible spanning surfaces*, Proc. Amer. Math. Soc. **35**, (1972), 617-620. MR **46:**2663

[10] H. C. Lyon, *Simple knots without unique minimal surfaces*, Proc. Amer. Math. Soc. **43**, (1974), 449-454. MR **51:**14019

[11] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology Vol. **23**, No. 1, (1984) 37-44. MR **86b:**57004

[12] U. Oertel, *Closed incompressible surfaces in complements of star links*, Pacific J. of Math., **111**, (1984), 209-230. MR **85j:**57008

[13] C. B. Schaufele, *The commutator group of a doubled knot*, Duke Math. J. **34**, (1967), 677-681. MR **35:**7328

Department of Science, School of Education, Waseda University, 1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan

E-mail address: ozawa@mn.waseda.ac.jp