THE GROUPS $\pi_r(V_{n,m})$ (I)

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Introduction

THE points of a Stiefel manifold $V_{n,m}$ are ordered sets of m mutually orthogonal unit vectors in euclidean n-space R^n. $V_{n,m}$ is topologized as a subset of R^{nm}.

In this sequence of papers I calculate the homotopy groups $\pi_{k+p}(V_{k+m,m})$ for all k when $0 \leqslant p \leqslant 5$ (except for an ambiguity in the cases $p = 5$, $k = 6$, $m \geqslant 5$), for $k = 1$ and 2 when $p = 6$, and for $k = 1$ when $p = 7$. These results are collected in the following tables, wherein $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$.† As each group is calculated in the text, I have specified generators in terms of elements of the homotopy groups of spheres, whose structure is assumed to be well known in the relevant cases.

TABLES FOR $\pi^p_{k,m}$

$(a): p = 0$

	$k = 1$	2	$4s-1$	$4s+1$	$4s$	$4s+2$
$m = 1$	Z_∞	Z_∞	Z_∞	Z_∞	Z_∞	Z_∞
$m \geqslant 2$	Z_2	Z_∞	Z_2	Z_2	Z_∞	Z_∞

$(b): p = 1$

$m = 1$	0	Z_∞	Z_2	Z_2	Z_2	Z_2
$m = 2$	0	$Z_\infty + Z_\infty$	Z_2	Z_2	$Z_2 + Z_\infty$	$Z_2 + Z_\infty$
$m \geqslant 3$	0	Z_∞	0	Z_2	$Z_2 + Z_2$	Z_4

$(c): p = 2$

$m = 1$	0	Z_2	Z_2	Z_2	Z_2	Z_2
$m = 2$	Z_∞	$Z_2 + Z_2$	Z_4	Z_4	$Z_2 + Z_2$	$Z_2 + Z_2$
$m = 3$	$Z_\infty + Z_\infty$	Z_2	$Z_2 + Z_\infty$	$Z_4 + Z_\infty$	$Z_2 + Z_2$	Z_2
$m \geqslant 4$	Z_∞	0	Z_2	Z_2	$Z_2 + Z_2$	0

† Not all these results are new. For $p = 0$ see Stiefel (**18**); for $p = 1$ see J. H. C. Whitehead (**22, 23**). Using the erroneous announcement by Pontrjagin that $\pi_4(S^3)$ was trivial (**14**), the following cases had been calculated: $p = 2$ by J. H. C. Whitehead, $m = 2$ and $0 \leqslant p \leqslant 2$ by Eckmann (**6**), and $k = 1$ and $0 \leqslant p \leqslant 4$ by Eckmann (**7**), and G. W. Whitehead (**19**), independently. The groups for $k = 1$ are well known for $p \leqslant 4$, e.g. (**17**), and for $5 \leqslant p \leqslant 7$ have been obtained independently by Borel and Serre (**5, 16**).

$(d): p = 3$

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m \geqslant 5$
1	0	Z_2	Z_2+Z_2	Z_2	0
2	Z_2	Z_2+Z_2	Z_2	Z_∞	0
3	Z_{12}	Z_2	Z_2	Z_2	Z_2
4	$Z_\infty+Z_{12}$	$Z_\infty+Z_{12}+Z_2$	$Z_\infty+Z_{12}+Z_4$	$Z_\infty+Z_{12}+Z_4+Z_\infty$	$Z_{12}+Z_4+Z_\infty$
$8s-1$	Z_{24}	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	Z_2
$8s+3$	Z_{24}	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2	Z_2+Z_2
$4s+1$	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s$	Z_{24}	$Z_{24}+Z_2$	$Z_{24}+Z_4$	$Z_{24}+Z_4+Z_\infty$	$Z_{24}+Z_2$
$8s+4$	Z_{24}	$Z_{24}+Z_2$	$Z_{24}+Z_4$	$Z_{24}+Z_4+Z_\infty$	Z_4+Z_{48}
$4s+2$	Z_{24}	$Z_{24}+Z_2$	$Z_{12}+Z_2$	$Z_{12}+Z_\infty$	Z_{12}

$(e): p = 4$

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m \geqslant 6$
1	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
2	Z_{12}	$Z_{12}+Z_{12}$	0	0	0	0
3	Z_2	$Z_\infty+Z_2$	$Z_\infty+Z_4$	Z_4+Z_∞	$Z_4+Z_\infty+Z_\infty$	Z_4+Z_∞
4	Z_2+Z_2	$Z_2+Z_2+Z_{24}$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2$
5	Z_2	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2+Z_∞	Z_2
$8s-1$	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_8
$8s+3$	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_{16}
$4s+5$	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$4(s+1)$	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s-2$	0	Z_{24}	Z_2	Z_2	Z_2	0
$8s+2$	0	Z_{24}	Z_2	Z_2	Z_2	Z_2

$(f): p = 5$

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m \geqslant 7$
1	0	Z_{12}	$Z_{12}+Z_{12}$	0	0	0	0
2	Z_2	Z_2+Z_2	Z_∞	Z_∞	Z_∞	$Z_\infty+Z_\infty$	Z_∞
3	Z_2	Z_2	Z_2+Z_{24}	$Z_2+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2$	Z_2+Z_2
4	Z_2+Z_2	Z_2+Z_2+ $+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_2$	Z_2+Z_2+ $+Z_2+Z_\infty$	Z_2+Z_2+ $+Z_2$
5	Z_2	0	Z_{24}	Z_2	Z_2	Z_2	0
6	Z_∞	Z_∞	$Z_\infty+Z_2$	$Z_\infty+Z_4$	$Z_\infty+Z_8$ or Z_4+Z_∞	$Z_\infty+Z_8+Z_\infty$ or $Z_4+Z_\infty+Z_\infty$	$Z_\infty+Z_8$ or Z_4+Z_∞
$4s+3$	0	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2
$8s+1$	0	0	Z_{24}	Z_2	Z_2	Z_2	Z_2
$8s+5$	0	0	Z_{24}	Z_2	Z_2	Z_2	0
$4(s+1)$	0	0	Z_2	Z_2+Z_2	Z_2	Z_∞	0
$8s+6$	0	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_8
$8s+2$	0	0	Z_2	Z_4	Z_8	Z_8+Z_∞	Z_{16}

$(g): p = 6$

k	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m \geqslant 8$
1	0	Z_2	Z_2+Z_2	Z_∞	Z_∞	Z_∞	$Z_\infty+Z_\infty$	Z_∞
2	Z_2	Z_2+Z_2	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

(h): $p = 7$ *and* $k = 1$

$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m \geq 9$
0	Z_2	Z_2+Z_2	0	Z_{24}	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

Whenever possible in the calculations I have used the fact that $V_{n+1,m+1}$ fibres over the n-sphere S^n with fibre $V_{n,m}$ and that this fibring admits an element of section which can be exhibited in a particularly simple form (a generalization of those of Eckmann in (6) 15d and (7) 15a). Thus the calculations of $\pi_r(V_{n,m})$ proceed by induction on m for fixed $k = n-m$ and r, using the exact homotopy sequence for a fibre space. These calculations take place in §§ 4, 5 of this paper and in the subsequent papers—the earlier sections, and the appendixes to all the papers, being devoted to the assembly of the necessary machinery.

In conclusion I would like to take this opportunity of expressing my gratitude to Prof. J. H. C. Whitehead for his guidance and encouragement. I am also deeply indebted to Dr. M. G. Barratt, for his many suggestions in general, and in particular for his collaboration in obtaining the important result in 5.2 c.

1. Various theorems in fibre-space theory

The triple (X, p, B) is to be a fibre-space X, in the sense of Serre (15), over the base-space B, with fibre-mapping $p\colon X \to$ onto B. For $b \in B$, $p^{-1}(b)$ is called the *fibre standing* over b. If b_0 is the base point in B, and $A = p^{-1}(b_0)$, then we write the fibring as $X/A \xrightarrow{p} B$. We thus have the well-known exact homotopy sequences (base points omitted)

$$\begin{array}{c} \longrightarrow \pi_r(A) \xrightarrow{i_{r*}} \pi_r(X) \xrightarrow{j_{r*}} \pi_r(X, A) \xrightarrow{\delta_{r*}} \pi_{r-1}(A) \longrightarrow \\ \downarrow p'_{r*} \\ \pi_r(B) \end{array}$$

and

$$\longrightarrow \pi_r(A) \xrightarrow{i_{r*}} \pi_r(X) \xrightarrow{p_{r*}} \pi_r(B) \xrightarrow{\Delta_{r*}} \pi_{r-1}(A) \longrightarrow,$$

where i_{r*} and j_{r*} are the natural injection homomorphisms, δ_{r*} the boundary homomorphism, p'_{r*} the isomorphism induced by p, $p_{r*} = p'_{r*} j_{r*}$, and $\Delta_{r*} = \delta_{r*} p'^{-1}_{r*}$.

We say that the fibring $X/A \xrightarrow{p} B$ admits a *cross-section* $\mathfrak{p}$ if there is a map $\mathfrak{p}\colon B \to X$ such that $p\mathfrak{p}\colon B \to B$ is the identity map. Then $\mathfrak{p}$ induces natural homomorphisms $\mathfrak{p}_{r*}\colon \pi_r(B) \to \pi_r(X)$ with $p_{r*}\mathfrak{p}_{r*}$ the identity isomorphism. Since the homotopy groups are abelian for $r > 1$, we have, by the exactness of the second sequence, THEOREM 1.1. *A necessary condition for $X/A \to B$ to admit a cross-section is that* (*a*) $i_{r*}^{-1}(0) = 0$ *for* $r > 0$, (*b*) $\pi_r(X) = i_{r*}\pi_r(A) + p_{r*}\pi_r(B)$ *for* $r > 1$.

Now let B be a sphere S^n. Then we have

THEOREM 1.2. *A necessary and sufficient condition that $X/A \to S^n$ admit a cross-section is that* $i_{n-1*}^{-1}(0) = 0$.

For a proof see (6), Theorem 11.

An *element of section* (Schnittelement) is a map $t'\colon (E^n, \dot{E}^n) \to (X, A)$ such that $T = pt'\colon (E^n, \dot{E}^n) \to (S^n, s)$, where $s = p(A)$, is topological on $E^n - \dot{E}^n$. Let the orientation be such that T is of degree 1. Let $t\colon S^{n-1} \to A$ be the map defined by $t' \mid \dot{E}^n$. Then, for each r, t induces a homomorphism $t_*\colon \pi_{r-1}(S^{n-1}) \to \pi_{r-1}(A)$. Then we have

THEOREM 1.3. $t_*\pi_{r-1}(S^{n-1}) = i_{r-1*}^{-1}(0)$ *when* $r < 2n-1$ ($r = 2n-1$ *if n is odd*).

The proof of this theorem follows from the fact that

$$\mathfrak{E}\pi_{r-1}(S^{n-1}) = \pi_r(S^n) \quad \text{when } r < 2n-1$$

($r = 2n-1$ if n is odd), where $\mathfrak{E}$ denotes the Freudenthal suspension homomorphism (9), and the following theorem:

THEOREM 1.4. $t_* = \Delta_{r*}\mathfrak{E}\colon \pi_{r-1}(S^{n-1}) \to \pi_{r-1}(A)$.

For the proof of this theorem see (6) 172 et seq. In consequence of 1.3 we have

COROLLARY 1.5. *If $r = n$, then $\{t\}$ generates* $i_{n-1*}^{-1}(0)$.†

2. Application to the Stiefel manifolds $V_{n,m}$

2.1. *General properties of $V_{n,m}$.* The points z of $V_{n,m}$ are ordered sets of m mutually orthogonal unit vectors $(z_1, z_2, \dots, z_m)$ in R^n (Euclidean n-space). Thus $V_{n,1} = S^{n-1}$, the unit sphere in R^n. We topologize $V_{n,m}$ as a subset of R^{nm}. In (6) Eckmann showed that $V_{n,m}$ is a fibre-space over $V_{n,k}$, with fibre homeomorphic to $V_{n-k,m-k}$, and fibre mapping $p_{n,k}\colon V_{n,m} \to V_{n,k}$ given by $p_{n,k}(z_1, z_2, \dots, z_m) = (z_1, z_2, \dots, z_k)$.

Let a point z of $V_{n,m}$ be represented by the matrix $\|v_{i,j}\|$, having as its rows the ordered unit vectors of z. As base point in $V_{r,s}$ we take the point $v_0 = \|v_{i,j}\|$, where $v_{i,j} = -\delta_{i,s-j+1}$. We define the identical map $i_{n-k,k}\colon V_{n-k,m-k} \to V_{n,m}$ as

$$i_{n-k,k}(\|w_{i,j}\|) = (\|v_{i,j}\|) \quad (\|w_{i,j}\| \in V_{n-k,m-k};\ \|v_{i,j}\| \in V_{n,m}),$$

$$v_{i,j} = \begin{cases} -\delta_{i,n-j+1} & (i \leqslant k), \\ w_{i\cdot} & (i > k;\ j \leqslant m-k), \\ 0 & (i > k;\ j > m-k). \end{cases}$$

† $\{k\}$ denotes the homotopy class of a map k.

Then we see that $i_{n-k,k}(V_{n-k,m-k})$ is the fibre standing over the base point v_0 of $V_{n,k}$. Further, for $s \leqslant k$, $i_{n-s,s}$ induces the fibre-preserving map

$$\bar{i}_{n-s,s}\colon (V_{n-s,m-s}, p_{n-s,k-s}, V_{n-s,k-s}) \to (V_{n,m}, p_{n,k}, V_{n,k}),$$

the induced maps being $i_{n-s,s}$ on the base space $V_{n-s,k-s}$, and the identity $i_{n-k,0}$ on the fibre $V_{n-k,m-k}$. Similarly, for $k \leqslant s$, $p_{n,s}$ induces the fibre-preserving map

$$\bar{p}_{n,s}\colon (V_{n,m}, p_{n,k}, V_{n,k}) \to (V_{n,s}, p_{n,k}, V_{n,k}),$$

the induced maps being $p_{n,k}$ on the base space $V_{n,k}$, and $p_{n-k,s-k}$ on the fibre $V_{n-k,m-k}$. It follows that the homomorphisms of homotopy groups induced by the injections $i_{r,s}$ and the projections $p_{n,k}$ commute with each other and with the homotopy boundary homomorphism whenever the composition makes sense.

2.2. *Construction of an element of section for $V_{n+1,m+1}/V_{n,m} \to S^n$.* Let E^n be given by the equations

$$\sum_{i=0}^{n} x_i^2 = 1, \qquad x_n \geqslant 0,$$

in R^{n+1}, $S^{n-1} = \dot{E}^n$ being its intersection with $x_n = 0$. Then we define

$$t'_{n+1,m+1}\colon (E^n, \dot{E}^n) \to (V_{n+1,m+1}, V_{n,m})$$

by
$$t'_{n+1,m+1}(x_0, x_1,..., x_n) = v_{i,j}$$

in $V_{n+1,m+1}$, where

$$v_{i,j} = 2x_{n+1-i}\,x_{j-1} - \delta_{n+1-i,j-1} \quad (i = 1,..., m+1;\, j = 1,..., n+1).$$

It is easily seen that $t'_{n+1,m+1}$ has all the properties required of an element of section, and we observe that $t_{n+1,m+1}\colon S^{n-1} \to V_{n,m}$ is given by

$$t_{n+1,m+1}(x_0, x_1,..., x_{n-1}) = u_{i,j}$$

in $V_{n,m}$, where

$$u_{i,j} = 2x_{n-i}\,x_{j-1} - \delta_{n-i,j-1} \quad (i = 1,..., m;\, j = 1,..., n).$$

2.3. *Properties of $t_{n+1,m+1}$.*

(*a*) $t_{n+1,m+1}$ is a symmetric map; i.e. $tx = tx^*$, where x and x^* are diametrically opposite points of S^{n-1};

(*b*) (i) $p_{n,k}\,t_{n+1,m+1} = t_{n+1,k+1}\colon S^{n-1} \to V_{n,k}$.

Thus (ii) $p_{n,1}\,t_{n+1,k+1} = t_{n+1,2}\colon S^{n-1} \to S^{n-1}$,

which is of degree 2 if n is even, and of zero degree if n is odd. Geometrically this is the map which assigns to every point s of S^{n-1} the reflection of $(0, 0,..., -1)$ in the $n-1$ flat through the centre of S^{n-1} which is orthogonal to the line joining s to the centre. It maps the S^{n-2} given by $x_{n-1} = 0$ onto the point $(0,..., 0, -1)$.

Note also that

(iii) $t_{n+1,m+1} \mid S^{n-i}$

$$= i_{n-i+1,i-1}\, t_{n-i+2,m-i+2}\colon S^{n-i} \to i_{n-i+1,i-1}(V_{n-i+1,m-i+1}),$$

where S^{n-i} is the intersection of S^{n-1} with

$$x_{n-1} = x_{n-2} = \ldots = x_{n-i+1} = 0.$$

(c) If $u_r\colon S^r \to P^r$ is the identification map which identifies x and x^* in S^r, P^r being the real r-dimensional projective space, then

$$p_{n,1}\, t_{n+1,m+1}\, u_{n-1}^{-1}\colon P^{n-1} \to S^{n-1}$$

is single-valued and so continuous (24), and of degree 1 (mod 2) and therefore essential. Further, it maps the P^{n-2} given by $x_{n-1} = 0$ onto the point $s = (0,\ldots,0,-1)$, and is topological on $(P^{n-1}-P^{n-2})$. We choose the orientation of the latter such that this map, when restricted to $(P^{n-1}-P^{n-2})$, is of degree 1.

LEMMA 2.3 (*d*). *The image of S^{n-1} under $t_{n+1,m+1}$ in $V_{n,m}$ is the homeomorphic image of a P_{k-1}^{n-1}, where $k = n-m$,† and P_{k-1}^{n-1} is the projective space P^{n-1} with a subspace P^{k-1} shrunk to a point.*

Proof. Let

$$\phi_{n+1,m+1} = t_{n+1,m+1}\, u_{n-1}^{-1}\colon P^{n-1} \to V_{n,m}.$$

Then we see that $\phi_{n+1,m+1}$ maps the P^{k-1} which is the intersection of P^{n-1} with $x_{n-1} = x_{n-2} = \ldots = x_k = 0$ onto a point, while it is one-to-one on $P^{n-1}-P^{k-1}$. If $w_{r,s}\colon P^r \to P_s^r$ is the identification map which shrinks the subspace P^s to a point p (P_s^r being given the identification topology), then

$$\psi_{n+1,m+1} = \phi_{n+1,m+1}\, w_{n-1,m-1}^{-1}\colon P_{k-1}^{n-1} \to V_{n,m}$$

is single-valued and so continuous, and one-to-one. But it is a map of a compact space into a Hausdorff space: that is, it is a homeomorphism 'into'. Note that

$$t_{n+1,m+1} = \psi_{n+1,m+1}\, w_{n-1,k-1}\, u_{n-1}\colon S^{n-1} \to V_{n,m},$$

so that t and ψ do map onto the same space. Hereafter I shall refer to this image as 'the P_{k-1}^{n-1} imbedded in $V_{n,m}$'. We also see that § 2.3 (*b*) (iii) implies a similar relation for $\psi_{n+1,m+1} \mid P_{k-i}^{n-i}$.

Further note that, if E^n is the hemisphere of S^n given by $x_n \geqslant 0$, then

$$t_{n+2,m+2} \mid E^n = t'_{n+1,m+1}\colon (E^n, \dot{E}^n) \to (V_{n+1,m+1}, V_{n,m}).$$

Since restricting $t_{n+2,m+2}$ to E^n has no effect on $\phi_{n+2,m+2}$, we see that the image of $t'_{n+1,m+1}$ is the P_{k-1}^n imbedded in $V_{n+1,m+1}$. Let us consider P^n

† This was proved originally explicitly in (22) 250.

and P^n_{k-1} as CW complexes. (I shall drop subscripts of maps whenever ambiguity cannot arise.) Then we have from 2.3 (c) that

$$p_{n+1,1}\phi_{n+2,m+2} = p_{n+1,1}t_{n+2,m+2}u_n^{-1}: (P^n, P^{n-1}) \to (S^n, s)$$

is topological and of degree one when restricted to $(P^n - P^{n-1})$. Since the same is true of

$$w_{n,k-1}: (P^n, P^{k-1}) \to (P^n_{k-1}, p),$$

we have that $\quad p\psi = p\phi w^{-1}: (P^n_{k-1}, P^{n-1}_{k-1}) \to (S^n, s)$

is topological and of degree one when restricted to $P^n_{k-1} - P^{n-1}_{k-1}$.

But $\quad p\psi\psi^{-1}t' = p_{n+1,1}t'_{n+1,m+1}: (E^n, \dot{E}^n) \to (S^n, s),$

which is topological and of degree one when restricted to $(E^n - \dot{E}^n)$. Thus

$$g = \psi^{-1}t': (E^n, \dot{E}^n) \to (P^n_{k-1}, P^{n-1}_{k-1})$$

is a characteristic map for the n-cell of P^n_{k-1}. Let

$$\delta'_{r*}: \pi_r(E^n, \dot{E}^n) \to \pi_{r-1}(S^{n-1})$$

be the boundary homomorphism. Then P. J. Hilton showed [(**10**), proof of Theorem 1.1] that

$$(p\psi)_{r*}g_{r*}\delta'^{-1}_{r*} = \mathfrak{E}: \pi_{r-1}(S^{n-1}) \to \pi_r(S^n),$$

i.e. $$p'_{r*}\psi_{r*}g_{r*}\delta'^{-1}_{r*} = \mathfrak{E}.$$

But (i) p'_{r*} and δ'^{-1}_{r*} are always isomorphisms 'onto',

(ii) g_{r*} is 'onto' if $1 < r < n+k-1$, by (**25**) Theorem 1, since $\pi_r(P^{n-1}_{k-1}) = 0$ for $r < k$.

Hence we have the following lemmas:

LEMMA 2.3 (*e*). *If* $\pi_r(S^n) = \mathfrak{E}\pi_{r-1}(S^{n-1})$, *then*

$$\psi_{n+2,m+2*}: \pi_r(P^n_{k-1}, P^{n-1}_{k-1}) \to \pi_r(V_{n+1,m+1}, V_{n,m})$$

is 'onto'.

LEMMA 2.3 (*f*). *If* $1 < r < n+k-1$, *and* $\mathfrak{E}: \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n)$, *then*

$$\psi_{n+2,m+2*}: \pi_r(P^n_{k-1}, P^{n-1}_{k-1}) \approx \pi_r(V_{n+1,m+1}, V_{n,m}).$$

By use of induction on m and the 'five' lemma these lemmas lead to a proof of

THEOREM 2.3 (*g*). *If* $r < 2k$ $(k = n-m > 0)$,

then $\quad \psi_{n+1,m+1*}: \pi_r(P^{n-1}_{k-1}) \approx \pi_r(V_{n,m}),$

and $\quad \psi_{n+1,m+1*}: \pi_{2k}(P^{n-1}_{k-1}) \to \pi_{2k}(V_{n,m})$

is 'onto'.

I omit this proof. A different proof can be found in (**22**), Theorem 3.

3.1. *Notation and properties of homotopy groups.* The following conventions will be used throughout.

$\pi^p_{k,m}$ denotes the group $\pi_{p+k}(V_{k+m,m})$, $\mathfrak{E}$ the suspension functor. If $f\colon (E^r, \dot{E}^r, e_0) \to (X, A, a_0)$, where A may equal a_0, then $\{f\}$ denotes the homotopy class of f in $\pi_r(X, A, a_0)$. As a rule base points will be omitted. $[a, b]$ in $\pi_{p+q-1}(X)$ denotes the Whitehead product of a in $\pi_p(X)$ and b in $\pi_q(X)$ (**21**). It will sometimes be used also with a and b as maps instead of classes. A similar notation will be used for generalized Whitehead products.

If $\pi_r(S^n)$ is a cyclic group, then $h_{n,r}\colon S^r \to S^n$ will be such that $\{h_{n,r}\}$ generates $\pi_r(S^n)$; $h_{r,r}$ will be of degree one, $h_{2,3}$ of Hopf invariant one, and $\{h_{3,6}\}$ will be the Blakers–Massey element. Also $\bar{p}$ will denote the Hopf fibre map : $S^7 \to S^4$, of invariant one. Thus, for instance, we write generators of $\pi_7(S^4)$, $= Z_\infty + Z_{12}$, as $\{\bar{p}h_{7,7}\}$ and $\mathfrak{E}\{h_{3,6}\}$. It will be assumed that the reader is familiar with the properties of $\pi_{n+k}(S^n)$ for $k \leqslant 5$: that is, with their structure, their behaviour under $\mathfrak{E}$, and the values of compositions $\{h_{n,s} h_{s,r}\}$ and Whitehead products.

Let $\{h\} \in \mathfrak{E}\pi_{r-1}(S^{n-1})$; $\{h'\} = \delta_*^{-1}\{h\}$, where

$$\delta_*\colon \pi_{r+1}(E^{n+1}, \dot{E}^{n+1}) \approx \pi_r(S^n).$$

Then, if A is arcwise connected, we have

LEMMA 3.1 (*a*). *$\{h\}$ and $\{h'\}$ induce homomorphisms $h^*\colon \pi_n(A) \to \pi_r(A)$ and $h'^*\colon \pi_{n+1}(X, A) \to \pi_{r+1}(X, A)$;*

(*b*) *these homomorphisms commute with $\mathfrak{E}$, a fibre mapping, a cross-section, and the homomorphisms of the exact homotopy sequences of* § 1.

The proof is straightforward and therefore omitted. When an h^* appears in the sequel, it will always be a homomorphism, and this in virtue of the above lemma unless otherwise stated. Note also that

$$h^*\{k\} = \{kh\} = k_*\{h\}.$$

3.2. *A theorem on suspension.* Let X be a CW complex, X^p its p-section. Let X be $(k-1)$-connected, where $k > 1$. Then we have the following theorems:

THEOREM 3.2 (A).

$$(a)\quad \mathfrak{E}\colon \pi_{r-1}(X) \to \pi_r(\mathfrak{E}X)$$

is an isomorphism for $r < 2k$.

(*b*) *When $r = 2k$, $\mathfrak{E}$ is 'onto' and $\mathfrak{E}^{-1}(0)$ is generated by the products $[\alpha, \beta]$, where α and β are any elements in $\pi_k(X)$.*

THEOREM 3.2 (B).

$$\mathfrak{E}\colon \pi_{r-1}(X^{p+1}, X^p) \to \pi_r(\mathfrak{E}X^{p+1}, \mathfrak{E}X^p)$$

is an isomorphism when $2 < r < p+k+1$, and 'onto' when $r = p+k+1$.

Proof of Theorem 3.2 (A). Let X_1 and X_2 be two cones on X intersecting in X, so that $X_1 \cup X_2 = \mathfrak{E}X$, and $X_1 \cap X_2 = X$. Then consider the diagram

$$\begin{tikzcd}
\to \pi_{r+1}(\mathfrak{E}X; X_1, X_2) \arrow[r, "\delta_*"] & \pi_r(X_1, X) \arrow[r, "i_*"] \arrow[d, "\delta_{1*}"] & \pi_r(\mathfrak{E}X, X_2) \arrow[r, "j_*"] & \pi_r(\mathfrak{E}X; X_1, X_2) \to \\
& \pi_{r-1}(X) \arrow[r, "\mathfrak{E}"] & \pi_r(\mathfrak{E}X). \arrow[u, "i'_*"] &
\end{tikzcd}$$

By Theorem 1 of (13) the sequence is exact. Since X_1 and X_2 are contractible, δ_{1*} and i'_* are isomorphisms. Also, by an argument similar to that on p. 375 of (13), the diagram is commutative. Hence i_* and $\mathfrak{E}$ are equivalent. Since (X_1, X) and (X_2, X) are both k-connected,

$$\pi_r(\mathfrak{E}X; X_1, X_2) = 0 \quad \text{for } r \leqslant 2k$$

by Theorem 1 of (3) (II). This proves (a) and the first part of (b).

Now by Theorem 1 of (3) (III), $\pi_{2k+1}(\mathfrak{E}X; X_1, X_2)$ is generated by the products $[\alpha', \beta']$ for any elements $\alpha' \in \pi_{k+1}(X_2, X)$ and $\beta' \in \pi_{k+1}(X_1, X)$. Further, if $\delta_{2*}\colon \pi_r(X_2, X) \approx \pi_{r-1}(X)$, then

$$\delta_*[\alpha', \beta'] = -[\delta_{2*}\alpha', \beta'], \quad \text{by 4.3 of (4),}$$

and $$\delta_{1*}[\delta_{2*}\alpha', \beta'] = (-1)^k[\delta_{2*}\alpha', \delta_{1*}\beta'], \quad \text{by 3.5 of (4).}$$

Putting $\alpha = \delta_{2*}\alpha'$ and $\beta = \delta_{1*}\beta'$, we obtain the last part of (b).

Proof of Theorem 3.2 (B). This is omitted. It follows directly from the special case of (A) when X is a bunch of p-spheres having a single common point, and Theorem 1 of (25).

Now let Y_2^{k+1} be the space consisting of an S^k to which one $k+1$ cell has been attached by a map ϕ such that $\phi \mid \dot{E}^{k+1} \to S^k$ is of degree 2. Let B_2^{k+2} be the space consisting of Y_2^{k+1} to which one $k+2$ cell has been attached by a map ϕ such that $\phi \mid \dot{E}^{k+2} \to S^k \subset Y_2^{k+1}$ and is essential on S^k. Then Theorem 3.2 (A) yields

COROLLARY 3.2 (C). (a) $\mathfrak{E}\colon \pi_{k+2}(Y_2^{k+1}) \approx \pi_{k+3}(Y_2^{k+2}) \quad (k \geqslant 3);$

(b) $\mathfrak{E}\colon \pi_{k+2}(B_2^{k+2}) \approx \pi_{k+3}(B_2^{k+3}) \quad (k \geqslant 3).$

When $k = 3$, all products in $\pi_6(Y_2^4)$ and $\pi_6(B_2^5)$ must be injections of products in $\pi_6(S^3)$, which are all zero.)

3.3. *An algebraic theorem*

Definition. A subgroup $U \subset G$ is *partially division-closed* if and only if for every $g \in G$ such that $mg \in U$, there is a $u \in U$ such that $mu = mg$.

THEOREM 3.3 (A). *Let G be a finitely generated abelian group, U a subgroup. Then a necessary and sufficient condition for U to be a direct summand of G is that U be partially division-closed.*

Proof. (*a*) Let $G = U+V$ and let $mg \in U$, where

$$g = u+v \ (u \in U,\ v \in V).$$

Then $mv = mg-mu \in U$, whence $mv = 0$. Thus $mg = mu$, and U is partially division-closed.

(*b*) Let U be partially division-closed, $V = G-U$, and $f\colon G \to V$ the natural homomorphism. Since G is a finitely generated abelian group, so is V. Therefore $V = V_1+...+V_n$, where each V_i is cyclic. Let v_i, of order m_i, be a generator of V_i and let $g_i \in f^{-1}v_i$. Then $m_i g_i \in U$ since $m_i v_i = 0$. Therefore $m_i g_i = m_i u_i$ for some $u_i \in U$. Let $g'_i = g_i-u_i$. Then $fg'_i = v_i$ and $m_i g'_i = 0$. Therefore a homomorphism $h\colon V \to G$, such that $fh = 1$, is defined by $hv_i = g'_i$ for each $i = 1$ to n. Hence, since G is abelian, $G = U+hV$, and the theorem is proved.

Let G be a finitely generated abelian group, $U \subset G$ finite, and, for every prime p, let $N(G,p)$ be the maximal order of elements in the p-component of G.

LEMMA 3.3 (B). *If $U = U_1+...+U_r$, where U_i is cyclic of order*

$$N(G,p_i) > 1,$$

for some prime p_i, then U is partially division-closed in G.

Proof.† If

$$G = G'+G'', \qquad U = U'+U'', \qquad U' \subset G', \qquad U'' \subset G'',$$

and if U' and U'' are partially division-closed in G' and G'', then so obviously is U in G. Therefore we may consider the p-components of G separately. Let X be the p-component of G for some prime p, V that of U, and suppose that $N(G,p) = p^t$. Suppose that $V = V_1+...+V_r$, where V_i is cyclic of order p^t, and let v_i be a generator of V_i. Suppose that $g \in X$ is such that

$$mg = k_1 v_1+...+k_r v_r,$$

where $m = hp^s$, h being prime to p and $s \leqslant t$. Then there exists an h' such that $hh' \equiv 1 \pmod{p^t}$. Also $p^{t-s}mg = 0$, whence every $k_i = l_i p^s$. Thus

$$mg = p^s u = p^s hh'u = m(h'u),$$

† I am indebted to the referee for this simplified version of the original proof.

where $u = l_1 v_1 + ... + l_r v_r$. By the above observation, this proves the lemma.

COROLLARY 3.3 (C). *Under the conditions of Lemma* 3.3 (B), U *is a direct summand of* G.

4. Calculation of $\pi^p_{k,m}$: preliminaries

The calculations are based on the examination of the exact sequences associated with the fibrings

$$V_{k+m+1,m+1}/V_{k+m,m} \to S^{k+m}.$$

When these are insufficient, we turn in the first place to those associated with the fibrings

$$V_{k+m+1,m+1}/S^k \to V_{k+m+1,m}.$$

Constant use will be made of Theorem 1.3; because of this frequency it will not be referred to every time it is applied.

I first prove two theorems:

THEOREM 4.1. (*a*) $\pi^p_{k,m} \approx \pi^p_{k,p+2}$ *for* $m \geqslant p+2$.

(*b*) $\pi^p_{k,m} = 0$ *for* $p < 0$ *and all* k *and* m.

Proof. (*a*) Consider the sequence associated with the fibring

$$V_{k+m+1,m+1}/V_{k+m,m} \to S^{k+m} \quad \text{when } r = k+p;$$

i.e. $$\to \pi_{k+p+1}(S^{k+m}) \xrightarrow{\Delta_*} \pi^p_{k,m} \xrightarrow{i_*} \pi^p_{k,m+1} \xrightarrow{p_*} \pi_{k+p}(S^{k+m}) \to.$$

Hence, since $\pi_r(S^{k+m}) = 0$ when $r < k+m$, $\pi^p_{k,m+1} \approx \pi^p_{k,m}$ when $p \leqslant m-2$; and (*a*) follows.

(*b*) By (*a*), $\pi^p_{k,m} \approx \pi^p_{k,1}$ if $p < 0$, i.e. $\approx \pi_{k+p}(S^k) = 0$ since $p < 0$.

THEOREM 4.2. (*a*) $\pi^p_{1,m} \approx \pi^{p+1}_{1,m+1}$ $(p \geqslant 1)$.

(*b*) $\pi^5_{k,m} \approx \pi^4_{k+1,m-1}$ $(k \geqslant 7)$.

Proof. (*a*) Consider the sequence associated with the fibring

$$V_{m+2,m+1}/S^1 \to V_{m+2,m},$$

i.e. $$\to \pi_{p+2}(S^1) \to \pi^{p+1}_{1,m+1} \xrightarrow{p_{2+m,m*}} \pi^p_{2,m} \to \pi_{p+1}(S^1) \to.$$

Hence, since $\pi_r(S^1) = 0$ for $r > 1$, (*a*) follows.

(*b*) Consider the sequence associated with the fibring

$$V_{k+m,m}/S^k \to V_{k+m,m-1}, \quad \text{when } r = k+5,$$

i.e. $$\to \pi_{k+5}(S^k) \to \pi^5_{k,m} \xrightarrow{p_{k+m,m-1*}} \pi^4_{k+1,m-1} \to \pi_{k+4}(S^k) \to.$$

But $\pi_{k+5}(S^k) = 0 = \pi_{k+4}(S^k)$ when $k \geqslant 7$; whence (*b*) follows.

5. Calculation of $\pi^p_{k,2}$

We consider the fibring $V_{k+2,2}/S^k \to S^{k+1}$, and examine the sequence

$$\text{(A)} \qquad \to \pi_{k+p+1}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+p}(S^k) \xrightarrow{i_{k+p*}} \pi^p_{k,2} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+1}) \to.$$

5.1. $k \equiv 0 \pmod 2$. In this case there is a one-field on S^{k+1} [see (8), (19)] and so the fibring admits a cross-section $\mathfrak{p}$. Hence Theorem 1.1 gives

$$\pi^p_{k,2} = i_* \pi_{k+p}(S^k) + \mathfrak{p}_* \pi_{k+p}(S^{k+1}).$$

The values of $\pi^p_{k,2}$ for even k are then as shown in the tables.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+2,2}\} = 0 \quad \textit{for } k \equiv 0 \pmod 2.$$

5.2. $k \equiv 1 \pmod 2)$ *and* $k \geqslant 3$. We have from 2.3 (*b*) that

$$t_{k+2,2}\colon S^k \to S^k$$

is of degree 2. Hence $\{t_{k+2,2}\} = 2\{h_{k,k}\}$. Then

(*a*) When $p = 0$, (A) gives

$$\xrightarrow{p_{k+1*}} \pi_{k+1}(S^{k+1}) \xrightarrow{\Delta_*} \pi_k(S^k) \xrightarrow{i_{k*}} \pi^0_{k,2} \to \pi_k(S^{k+1}) \to$$

i.e.
$$\to Z_\infty \to Z_\infty \to \pi^0_{k,2} \to 0,$$

where $i^{-1}_{k*}(0)$ is generated by $\{t_{k+2,2}\}$: that is, by $\{2h_{k,k}\}$.

Hence $\qquad \pi^0_{k,2} = Z_2$, *generated by* $\{i_{k+1,1} h_{k,k}\}$.

Note that $\Delta^{-1}_*(0) = 0$, and hence that p_{k+1*} is trivial.

(*b*) When $p = 1$, (A) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+1}(S^k) \xrightarrow{i_{k+1*}} \pi^1_{k,2} \xrightarrow{p_{k+1*}} \pi_{k+1}(S^{k+1}).$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^1_{k,2} \to 0$$

since p_{k+1*} is trivial by (*a*). Also

$$i^{-1}_{k+1*}(0) = t_{k+2,2*}\, \pi_{k+1}(S^k),$$

which is generated by

$$h^*_{k,k+1}\{t_{k+2,2}\} = h^*_{k,k+1}\, 2\{h_{k,k}\} = 2h^*_{k,k+1}\{h_{k,k}\} = 2\{h_{k+1,k}\} = 0.$$

Hence $i^{-1}_{k+1*}(0) = 0$, and $\pi^1_{k,2} = Z_2$, *generated by* $\{i_{k+1,1} h_{k,k+1}\}$.

Note that thus Δ_* is trivial, and hence p_{k+2*} is 'onto'.

(*c*) When $p = 2$, (A) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+2}(S^k) \xrightarrow{i_{k+2*}} \pi^2_{k,2} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+1}),$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^2_{k,2} \to Z_2 \to 0,$$

since p_{k+2*} is onto by (*b*). Also

$$i^{-1}_{k+2*}(0) = t_{k+2,2*}\, \pi_{k+2}(S^k),$$

which is generated by

$$h^*_{k,k+2}\{t_{k+2,2}\} = h^*_{k,k+2}\,2\{h_{k,k}\} = 2h^*_{k,k+2}\{h_{k,k}\} = 2\{h_{k+2,k}\} = 0.$$

Hence $i^{-1}_{k+2*}(0) = 0$; whence $\pi^2_{k,2}$ *has four elements.*

Note that thus Δ_* is trivial, and hence p_{k+3*} is 'onto'.

We now need to determine the structure of $\pi^2_{k,2}$. For this we look at the space P^{k+1}_{k-1} [see 2.3 (*d*)], which is of the same homotopy type as the Y^{k+1}_2 defined in 3.2 [see Appendix to part (II)]. M. G. Barratt showed in (1) that the homotopy classes of maps of P^{k+1}_{k-1} into itself can be turned into a group $(P^{k+1}_{k-1})^0(P^{k+1}_{k-1})$ when $k \geqslant 3$, and that this group is Z_4 [(1) 10.61]; and more particularly that, if α is the identical map of P^{k+1}_{k-1} onto itself, i' the identical map

$$\psi^{-1}_{k+3,3}\, i_{k+1,1} \colon S^k \to P^{k+1}_{k-1},$$

and p' the shrinking map

$$p_{k+2,1}\,\psi_{k+3,3} \colon P^{k+1}_{k-1} \to S^{k+1},$$

then α generates $(P^{k+1}_{k-1})^0(P^{k+1}_{k-1})$ and $2\alpha = \{i' h_{k,k+1} p'\}$.

Next consider the sequence

$$\to \pi_{k+2}(S^k) \xrightarrow{i'_*} \pi_{k+2}(P^{k+1}_{k-1}) \xrightarrow{p'_*} \pi_{k+2}(S^{k+1}) \to .$$

By virtue of the isomorphisms of Theorems 2.3 (*f*) and (*g*), this is exact and reduces to

$$(0) \to Z_2 \to \pi_{k+2}(P^{k+1}_{k-1}) \to Z_2 \to (0).$$

Let $\beta\colon S^{k+2} \to P^{k+1}_{k-1}$ be such that $p'^*\{\beta\} = \{h_{k+1,k+2}\}$. Then since

$$\{\beta\} \in \mathfrak{E}\pi_{k+2}(P^{k+1}_{k-1}) \quad \text{for } k \geqslant 5$$

by Corollary 3.2 (C) (k is odd), it induces a homomorphism

$$\beta^* \colon (P^{k+1}_{k-1})^0(P^{k+1}_{k-1}) \to \pi_{k+2}(P^{k+1}_{k-1}).$$

Thus, for $k \geqslant 5$,

$$\begin{aligned} 2\beta^*\alpha = \beta^*\,2\alpha = \beta^*\{i' h_{k,k+1} p'\} &= \{i' h_{k,k+1} p' \beta\} \\ &= \{i' h_{k,k+1} h_{k+1,k+2}\} = \{i' h_{k,k+2}\} \neq 0 \end{aligned}$$

from (*c*) above. Thus, when $k \geqslant 5$, we have found an element, $\beta^*\alpha$, in $\pi_{k+2}(P^{k+1}_{k-1})$ twice which is non-zero. Hence $\pi_{k+2}(P^{k+1}_{k-1}) = Z_4$ for $k \geqslant 5$. For $k = 3$, the result follows from this and Corollary 3.2 (C). By Theorem 2.3 (*g*) we therefore have that

$$\pi^2_{k,2} = Z_4, \quad \textit{for odd } k \geqslant 3,$$

and is generated by any a such that $p_{k+2,1*}\,a = \{h_{k+1,k+2}\}$.

(*d*) When $p = 3$, (A) gives, when $k \geqslant 5$,

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+1}) \xrightarrow{\Delta_*} \pi_{k+3}(S^k) \xrightarrow{i_{k+3*}} \pi^3_{k,2} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+1}) \to$$

i.e. $$\to Z_{24} \to Z_{24} \to \pi^3_{k,2} \to Z_2 \to 0$$

since p_{k+3*} is 'onto' by (*c*). Also

$$i^{-1}_{k+3*}(0) = t_{k+2,2*}\,\pi_{k+3}(S^k),$$

which is generated by

$$h^*_{k,k+3}\{t_{k+2,2}\} = h^*_{k,k+3}\,2\{h_{k,k}\} = 2\{h_{k,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0) = 2Z_{24}$; *i.e.* $\pi^3_{k,2}$ *has four elements.*

Note that exactness implies that $\Delta_*\pi_{k+4}(S^{k+1}) = 2\pi_{k+3}(S^k)$, whence $\Delta_*^{-1}(0)$, and so the image of p_{k+4*}, is Z_2 generated by $12\{h_{k+1,k+4}\}$.

To determine the structure of $\pi^3_{k,2}$, we operate with $h^*_{k+2,k+3}$ on the section of the sequence given in 5.2 (*c*) and obtain the commutative diagram [cf. (3.2)]

$$\begin{array}{ccccccc}
\to \pi_{k+3}(S^k) & \xrightarrow{i_{k+3*}} & \pi^3_{k,2} & \xrightarrow{p_{k+2,1*}} & \pi_{k+3}(S^{k+1}) & \to & 0 \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* & & \\
\to \pi_{k+2}(S^k) & \xrightarrow{i_{k+2*}} & \pi^2_{k,2} & \xrightarrow{p_{k+2,1*}} & \pi_{k+2}(S^{k+1}) & \to & 0.
\end{array}$$

Now let a be a generator of $\pi^3_{k,2}$ such that

$$p_{k+2,1*}\,a = \{h_{k+1,k+3}\},$$

and a' a generator of $\pi^2_{k,2}$. Then

$$h^*p_{k+2,2*}\,a' = h^*\{h_{k+1,k+2}\} = \{h_{k+1,k+2}\,h_{k+2,k+3}\} = \{h_{k+1,k+3}\}$$

Hence $$p_{k+2,1*}\,h^*a' = \{h_{k+1,k+3}\}.$$

Thus $$h^*a' = a + i_{k+3*}\,b,$$

where $b \in \pi_{k+3}(S^k)$; and $2(h^*a') = 2a + 2i_{k+3}$.

But $$2i_{k+3*}\,b = i_{k+3*}\,2b = 0,$$

from above; and further,

$$2(h^*a') = 2a'_*\{h_{k+2,k+3}\} = a'_*\,2\{h_{k+2,k+3}\} = 0.$$

Hence $$2a = 0,$$

i.e. $$\pi^3_{k,2} = Z_2 + Z_2 \textit{ for odd } k \geqslant 5,$$

and is generated by $\{i_{k+1,1}\,h_{k,k+3}\}$ and any a such that $p_{k+2,1*}\,a = \{h_{k+1,k+3}\}$.

(*e*) When $p = 3$, and $k = 3$, (A) gives

$$\xrightarrow{p_{7*}} \pi_7(S^4) \xrightarrow{\Delta_{7*}} \pi_6(S^3) \xrightarrow{i_{6*}} \pi^3_{3,2} \xrightarrow{p_{6*}} \pi_6(S^4) \to$$

i.e. $$\to Z_\infty + Z_{12} \to Z_{12} \to \pi^3_{3,2} \to Z_2 \to 0$$

since p_{6*} is 'onto' by (c). Unfortunately in this case we cannot make use of Theorem 1.3 since $\pi_7(S^4) \neq \mathfrak{E}\pi_6(S^3)$, but at any rate we have from Theorem 1.4 that

$$\Delta_{7*}\,\mathfrak{E}\pi_6(S^3) = t_{5,2*}\,\pi_6(S^3),$$

which is generated by

$$h^*_{3,6}\{t_{5,2}\} = h^*_{3,6}\,2\{h_{3,3}\} = 2\{h_{3,6}\}$$

since $h^*_{3,6}$ is a homomorphism by Theorem IV of (7). Hence

$$\Delta_{7*}\,\mathfrak{E}\pi_6(S^3) = 2\pi_6(S^3).$$

I have not been able to find a direct method of calculating $\Delta_{7*}\,\tilde{p}_*\{h_{7,7}\}$. There is however another way of evaluating $\pi^3_{3,2}$. Consider the sequence associated with the fibring $V_{5,3}/S^2 \to V_{5,2}$:

i.e. $$\to \pi^4_{2,3} \to \pi^3_{3,2} \to \pi_5(S^2) \to \pi^3_{2,3} \to \pi^2_{3,2} \to \pi_4(S^2) \to.$$

I shall show in § 8.5 (c) and (d) respectively that $\pi^3_{2,3} = Z_2$ and $\pi^4_{2,3} = 0$. Hence the sequence is of the form

$$\to 0 \to \pi^3_{3,2} \to Z_2 \to Z_2 \to Z_4 \to Z_2 \to$$

since $\pi^2_{3,2} = Z_4$ by (c). Thus, by exactness, we have

$$\pi^3_{3,2} = Z_2,$$

generated by any a such that $p_{5,1*}\,a = \{h_{4,6}\}$. Note that i_{6*} is trivial and that this implies that $\Delta_{7*}\,\bar{p}_*\{h_{7,7}\} = \lambda\{h_{3,6}\}$, where λ is odd. Exactness then gives that the image of p_{7*} is the subgroup $Z_\infty + Z_2$ which is generated by $(2\bar{p}_*\{h_{7,7}\} - \lambda\mathfrak{E}\{h_{3,6}\})$ and $6\mathfrak{E}\{h_{3,6}\}$.

(f) When $p = 4$, and $k \geqslant 7$, (A) gives

$$\to \pi_{k+5}(S^{k+1}) \to \pi_{k+4}(S^k) \to \pi^4_{k,2} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+1}) \to.$$

But $\pi_{k+4}(S^k) = 0$, and by (d) the image of p_{k+4*} is Z_2. Hence

$$\pi^4_{k,2} = Z_2 \quad \textit{for odd } k \geqslant 7,$$

and is generated by $p^{-1}_{k+2,1*}\,12\{h_{k+1,k+4}\}$.

(g) When $p = 4$ and $k = 3$, (A) gives

$$\xrightarrow{p_{8*}} \pi_8(S^4) \xrightarrow{\Delta_{8*}} \pi_7(S^3) \xrightarrow{i_{7*}} \pi^4_{3,2} \xrightarrow{p_{7*}} \pi_7(S^4) \to$$

i.e. $$\to Z_2 + Z_2 \to Z_2 \to \pi^4_{3,2} \to Z_\infty + Z_2 \to 0$$

since the image of p_{7*} is $Z_\infty + Z_2$ by (e). Again we cannot make use of Theorem 1.3. But consider

$$i_{7*}\{h_{3,7}\} = i_{7*}\{h_{3,6}\,h_{6,7}\} = i_{7*}\,h^*_{6,7}\{h_{3,6}\} = h^*_{6,7}\,i_{6*}\{h_{3,6}\} = 0$$

since i_{6*} is trivial by (e). Hence $i_{7*}\,\pi_7(S^3) = 0$; and we have the result

$$\pi^4_{3,2} = Z_\infty + Z_2,$$

and is generated by

$$p_{5,1*}^{-1}(2\bar{p}_*\{h_{7,7}\}-\lambda\mathfrak{E}\{h_{3,6}\})\ (\lambda \text{ odd}) \quad \text{and} \quad p_{5,1*}^{-1}6\mathfrak{E}\{h_{3,6}\}.$$

Note that, by Theorem 1.4,

$$\Delta_{8*}\mathfrak{E}\pi_7(S^3) = i_{5,2*}\pi_7(S^3),$$

which is generated by

$$h_{3,7}^*2\{h_{3,3}\} = 2h_{3,7}^*\{h_{3,3}\},$$

$h_{3,7}^*$ being a homomorphism since [(7) Theorem IV] there is a multiplication on $S^3 = 2\{h_{3,7}\} = 0$. Thus $\Delta_{8*}\mathfrak{E}\pi_7(S^3) = 0$. But $i_{7*}\pi_7(S^3) = 0$, whence, by exactness,

$$\Delta_{8*}\bar{p}_*\{h_{7,8}\} = \{h_{3,7}\}.$$

So we have that the image of p_{8*} is the Z_2 subgroup $\mathfrak{E}\pi_7(S^3)$.

(*h*) When $p = 4$ and $k = 5$, (A) gives

$$\to \pi_{10}(S^6) \to \pi_9(S^5) \to \pi_{5,2}^4 \xrightarrow{p_{9*}} \pi_9(S^6) \to,$$

i.e. $$\to 0 \to Z_2 \to \pi_{5,2}^4 \to Z_2 \subset Z_{24} \to 0$$

since by (*d*) the image of p_{9*} is Z_2. Thus $\pi_{5,2}^4$ *has four elements.* To determine its structure we operate with $h_{7,9}^*$ on that section of the sequence given in (*c*) to obtain the diagram

$$\begin{tikzcd}
\to \pi_9(S^5) \arrow[r, "i_{9*}"] & \pi_{5,2}^4 \arrow[r, "p_{7,1*}"] & \pi_9(S^6) \to \\
\to \pi_7(S^5) \arrow[r, "i_{7*}"] \arrow[u, "h^*"] & \pi_{5,2}^2 \arrow[r, "p_{7,1*}"] \arrow[u, "h^*"] & \pi_7(S^6) \to . \arrow[u, "h^*"]
\end{tikzcd}$$

Let a be a generator of $\pi_{5,2}^4$ such that $p_{7,1*}a = 12\{h_{6,9}\}$, and let a' be a generator of $\pi_{5,2}^2$ $(= Z_4)$. Then

$$h^*p_{7,1*}a' = h^*\{h_{6,7}\} = \{h_{6,7}h_{7,9}\} = 12\{h_{6,9}\}.$$

Hence $$p_{7,1*}h^*a' = 12\{h_{6,9}\}.$$

Thus $h^*a' = a+i_{9*}b$, where $b \in \pi_9(S^5)$; and $2h^*a' = 2a+2i_{9*}b$.

But $$2i_{9*}b = i_{9*}2b = 0;$$

and further $$2h^*a' = 2a'_*\{h\} = a'_*2\{h_{7,9}\} = 0.$$

Hence $$2a = 0,$$

i.e. $$\pi_{5,2}^4 = Z_2+Z_2,$$

and is generated by $\{i_{6,1}h_{5,9}\}$ and any a such that $p_{7,1*}a = 12\{h_{6,9}\}$.

(*i*) When $p = 5$ and $k \geqslant 7$, (A) gives

$$\to \pi_{k+6}(S^{k+1}) \to \pi_{k+5}(S^k) \to \pi_{k,2}^5 \to \pi_{k+5}(S^{k+1}) \to.$$

But $\pi_{k+5}(S^{k+1}) = 0 = \pi_{k+5}(S^k)$ when $k \geqslant 7$. Hence

$$\pi^5_{k,2} = 0 \quad \textit{for odd } k \geqslant 7.$$

(*j*) When $p = 5$ and $k = 3$, (A) gives

$$\xrightarrow{p_{9*}} \pi_9(S^4) \xrightarrow{\Delta_{9*}} \pi_8(S^3) \xrightarrow{i_{8*}} \pi^5_{3,2} \xrightarrow{p_{8*}} \pi_8(S^4) \to,$$

i.e.
$$\to Z_2 + Z_2 \to Z_2 \to \pi^5_{3,2} \to Z_2 \subset Z_2 + Z_2 \to 0$$

since by (*g*) the image of p_{8*} is Z_2. Again we cannot make use of Theorem 1.3; but as in (*g*) we consider

$$i_{8*}\{h_{3,8}\} = i_{8*}\{h_{3,6}\, h_{6,8}\} = i_{8*}\, h^*_{6,8}\{h_{3,6}\} = h^*_{6,8}\, i_{6*}\{h_{3,6}\} = 0$$

since i_{6*} is trivial by (*e*). Hence $i_{8*}\,\pi_8(S^3) = 0$, and we have the result

$$\pi^5_{3,2} = Z_2,$$

and is generated by $p^{-1}_{5,1*}\,\mathfrak{E}\{h_{3,7}\}$. Note that, by Theorem 1.4,

$$\Delta_{9*}\,\mathfrak{E}\pi_8(S^3) = t_{5,2*}\,\pi_8(S^3),$$

which is generated by

$$h^*_{3,8}\, 2\{h_{3,3}\} = 2h^*_{3,8}\{h_{3,3}\} = 2\{h_{3,8}\} = 0,$$

$h^*_{3,8}$ being a homomorphism since there is a multiplication on S^3. Thus $\Delta_{9*}\,\mathfrak{E}\pi_8(S^3) = 0$. But $i_{8*}\,\pi_8(S^3) = 0$, whence, by exactness,

$$\Delta_{9*}\,\bar{p}_*\{h_{7,9}\} = \{h_{3,8}\}.$$

So the image of p_{9*} is the Z_2 subgroup generated by $\mathfrak{E}\{h_{3,8}\}$.

(*k*) When $p = 5$ and $k = 5$, (A) gives

$$\to \pi_{11}(S^6) \to \pi_{10}(S^5) \xrightarrow{i_{10*}} \pi^5_{5,2} \to \pi_{10}(S^6) \to,$$

i.e.
$$\to Z_\infty \to Z_2 \to \pi^5_{5,2} \to 0.$$

Since $\pi_{11}(S^6) \neq \mathfrak{E}\pi_{10}(S^5)$, we cannot use Theorem 1.3 to determine the kernel of i_{10*}, but have to use a special method. Note, however, that $\pi^5_{5,2}$ is at most Z_2.

Consider first the P^6_4 imbedded in $V_{7,2}$ [2.3 (*d*)], which is of the same homotopy type as Y^6_2 which consists of an S^5 to which a 6-cell has been attached by a map of degree two on its boundary. Then (Y^6_2, S^5) is a pair of the type considered in § 2 of (12), and so we have the exact sequence

$$\to \pi_{11}(S^6) \xrightarrow{H_\alpha} \pi_5(S^5) \xrightarrow{Q} \pi_{10}(Y^6_2, S^5) \to \pi_{10}(S^6) \to,$$

where H_α is defined as $\alpha_*\,\mathfrak{E}^{-6}H$ by

$$\pi_{11}(S^6) \xrightarrow{H} \pi_{11}(S^{11}) \xleftarrow{\mathfrak{E}^6} \pi_5(S^5) \xrightarrow{\alpha_*} \pi_5(S^5),$$

H being the Hopf invariant and α the attaching map, of degree 2. Since there exist maps : $S^{11} \to S^6$ of Hopf invariant 2 (**11**), but not of Hopf invariant 1 (**20**), and, since α is of degree 2, the image of Q is Z_4. But $\pi_{10}(S^6) = 0$. Hence

$$\pi_{10}(P^6_4, S^5) \approx \pi_{10}(Y^6_4, S^5) = Z_4.$$

Now $V_{7,2}$, being a fibre space over S^6 with fibre S^5, is an eleven-dimensional space, and the cellular decomposition of (**22**) shows that, besides P^6_4, $V_{7,2}$ contains just one other cell, an E^{11}. Thus

$$\pi_{11}(V_{7,2}, P^6_4) = Z_\infty, \qquad \pi_{10}(V_{7,2}, P^6_4) = 0,$$

and the homotopy sequence of the triple $(V_{7,2}, P^6_4, S^5)$:

$$\to \pi_{11}(P^6_4, S^5) \xrightarrow{\psi_{8,3*}} \pi_{11}(V_{7,2}, S^5) \to \pi_{11}(V_{7,2}, P^6_4) \xrightarrow{\delta''_*} \pi_{10}(P^6_4, S^5)$$
$$\to \pi_{10}(V_{7,2}, S^5) \to$$

becomes $$\to \pi_{11}(P^6_4, S^5) \to Z_\infty \to Z_\infty \to Z_4 \to 0 \to.$$

Hence $$\psi_{8,3*}\,\pi_{11}(P^6_4, S^5) = 0,$$

and δ''_*: $\pi_{11}(V_{7,2}, P^6_4)$ is 'onto'.

Next consider the commutative diagram

$$\begin{tikzcd}
 & & & 0 & \\
{} \arrow[r] & \pi_{11}(S^6) \arrow[r, "\Delta_*"] & \pi_{10}(S^5) \arrow[r, "i_*"] & \pi_{10}(V_{7,2}) \arrow[r] \arrow[u] & \pi_{10}(S^6) \arrow[r] & {} \\
{} \arrow[r] & \pi_{11}(P^6_4, S^5) \arrow[r, "\delta'_*"] \arrow[u, "p'_*\psi_{8,3*}"'] & \pi_{10}(S^5) \arrow[r, "i'_*"] \arrow[u, "\psi_{7,2*}"'] & \pi_{10}(P^6_4) \arrow[r, "j'_*"] \arrow[u, "\psi_{8,3*}"'] & \pi_{10}(P^6_4, S^5) \arrow[r] \arrow[u, "p'_*\psi_{8,3*}"'] & {} \\
 & & & \pi_{11}(V_{7,2}, P^6_4), \arrow[u, "\delta_*"'] \arrow[ur, "\delta_*"'] & \\
 & & & {} \arrow[u] &
\end{tikzcd}$$

in which both the horizontal and vertical sequences are exact. Since δ''_* is 'onto', so is j'_*. Since $\psi_{8,3*}$ is trivial and $\psi_{7,2*}$ is an isomorphism,

$$\delta'_*\,\pi_{11}(P^6_4, S^5) = 0.$$

Hence $i'^{-1}_*(0) = 0$, and thus $\pi_{10}(P^6_4)$ is an extension of Z_2 by Z_4. I show below that $i'_*\{h_{5,10}\}$ can be halved in $\pi_{10}(P^6_4)$, whence $\pi_{10}(P^6_4) = Z_8$. But $j'_*\,\delta_* = \delta''_*$, which is 'onto'. Hence δ_* is 'onto', and

$$\pi_{10}(V_{7,2}) = 0.$$

To prove that $i'_*\{h_{5,10}\}$ can be halved in $\pi_{10}(P^6_4)$, consider Y^6_2 and let $\langle Y^6_2$ and $Y^6_2\rangle$ be two cones with different vertices based on Y^6_2, so that

$\langle Y_2^6 \cap Y_2^6 \rangle = Y_2^6$ and $\langle Y_2^6 \cup Y_2^6 \rangle = \mathfrak{E} Y_2^6 \simeq Y_2^7$.† Then consider the diagram

$$\begin{array}{ccc} \to \pi_{12}(Y_2^7; \langle Y_2^6, Y_2^6 \rangle) & \xrightarrow{\delta_+} & \pi_{11}(\langle Y_2^6, Y_2^6 \rangle \to \\ & & \downarrow \delta_{1*} \\ \pi_{10}(S^5) & \xrightarrow{i'_*} & \pi_{10}(Y_2^6), \end{array}$$

where δ_+, δ_{1*}, and i'_* are the triad-boundary, homotopy-boundary, and injection homomorphisms. Since $\langle Y_2^6$ is contractible, δ_{1*} is an isomorphism. Let δ_{2*} be the corresponding isomorphism on $\pi_r(Y_2^6\rangle, Y_2^6)$, and let $h'_{5,5}$ and $h'_{5,6}$ be generators of $\pi_6(Y_2^6\rangle, Y_2^6)$ and $\pi_7(\langle Y_2^6, Y_2^6)$. Then

$$\begin{aligned} \delta_{1*}\delta_+[h'_{5,5}, h'_{5,6}] &= -\delta_{1*}[\delta_{2*}h'_{5,5}, h'_{5,6}], \quad \text{by 4.3 of (4)}, \\ &= -[\delta_{2*}h'_{5,5}, \delta_{1*}h'_{5,6}], \quad \text{by 3.5 of (4)}. \end{aligned}$$

But $$\delta_{2*}h'_{5,5} = i'_*\{h_{5,5}\}, \qquad \delta_{1*}h'_{5,6} = i'_*\{h_{5,6}\}.$$

Hence $$\delta_{1*}\delta_+[h'_{5,5}, h'_{5,6}] = i'_*\{h_{5,10}\}.$$

If we now shrink Y_2^6 to a point in $(Y_2^7; \langle Y_2^6, Y_2^6 \rangle)$, we obtain a triad $(X; A, B)$, where A and B are two different copies of Y_2^7 having only a single point in common. By §§ 5, 6, and especially p. 403 of (2), we have that

$$\pi_{12}(Y_2^7; \langle Y_2^6, Y_2^6 \rangle) \approx \pi_{12}(X; A, B) = Z_4;$$

and further that, if α is a generator, $[h'_{5,5}, h'_{5,6}] = 2\alpha$. Hence

$$i'_*\{h_{5,10}\} = 2\delta_{1*}\delta_+\alpha,$$

and so can be halved in $\pi_{10}(Y_2^6)$.

5.3. $k = 1$. As in § 5.2, we have from § 2.3 (*b*) that $\{t_{3,2}\} = 2\{h_{1,1}\}$. Then

(*a*) when $p = 0$, (A) gives

$$\xrightarrow{p_{2*}} \pi_2(S^2) \xrightarrow{\Delta_*} \pi_1(S^1) \xrightarrow{i_{1*}} \pi_{1,2}^0 \to \pi_1(S^2),$$

i.e. $$\to Z_\infty \to Z_\infty \to \pi_{1,2}^0 \to 0,$$

where $i_{1*}^{-1}(0)$ is generated by $\{t_{3,2}\}$, i.e. by $2\{h_{1,1}\}$.

Thus $\pi_{1,2}^0 = Z_2$, *generated by* $\{i_{2,1}h_{1,1}\}$.

Note that $\Delta_*^{-1}(0) = 0$, and thus p_{2*} is trivial.

(*b*) When $p \geqslant 1$, (A) gives

$$\to \pi_{p+1}(S^1) \xrightarrow{i_{p+1*}} \pi_{1,2}^p \xrightarrow{p_{p+1*}} \pi_{p+1}(S^2) \xrightarrow{\Delta_*} \pi_p(S^1) \to.$$

But $\pi_{p+1}(S^1) = 0$ when $p \geqslant 1$; and we also have from (*a*) that p_{2*} is trivial.

Thus $\pi_{1,2}^1 = 0$; and $\pi_{1,2}^p \approx \pi_{p+1}(S^2)$ *for* $p \geqslant 2$. The values of $\pi_{1,2}^p$ for $p \geqslant 2$ are then as shown in the tables.

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