THE GROUPS $\pi_r(V_{n,m})$ (II)

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[Received 20 November 1956]

Introduction

THIS is the second of a sequence of five papers, the first being (4), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those $\pi_r(V_{k+3,3})$ which are given in the following table. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 1$.† In the Appendix I examine the homotopy type of certain reduced projective spaces P^n_k which are required in this and the subsequent papers. Please note that paragraphs are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,3}$‡

k	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
1	0	$Z_\infty+Z_\infty$	Z_2+Z_2	Z_2+Z_2	$Z_{12}+Z_{12}$	Z_2+Z_2	Z_2+Z_2
3	0	Z_2+Z_∞	Z_2	$Z_\infty+Z_4$	Z_2+Z_{24}		
4	Z_2+Z_2	Z_2+Z_2	$Z_\infty+Z_{12}+Z_4$	$Z_2+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$		
5	Z_2	Z_4+Z_∞	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2$	Z_{24}		
6	Z_4	Z_2	$Z_{12}+Z_2$	Z_2	$Z_\infty+Z_2$		
$4s-1$	0	Z_2+Z_∞	Z_2+Z_2	Z_4	Z_{24}		
$4s+1$	Z_2	Z_4+Z_∞	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_{24}		
$4s$	Z_2+Z_2	Z_2+Z_2	$Z_{24}+Z_4$	Z_2+Z_2	Z_2		
$4s+2$	Z_4	Z_2	$Z_{12}+Z_2$	Z_2	Z_2		

6. The construction 'Q^r'

In § 7.1, I shall for the first time make use of the following construction, which is isolated here for easier subsequent reference.§

Let $r > 1$ and $r \equiv 1 \pmod 2$. Let S^r_1 and S^r_2 be given in R^{r+1} by the respective equations

$$x_0^2+x_1^2+\ldots+x_r^2 = 1, \qquad x_0^2+x_1^2+\ldots+2x_r^2 = 1.$$

Let
$$S^{r-1} = S^r_1 \cap S^r_2, \qquad Q^r = S^r_1 \cup S^r_2.$$

For $i = 1$ or 2 let E^r_{i+} and E^r_{i-} be the hemispheres of S^r_i for which respectively $x_r \geqslant 0$ and $x_r \leqslant 0$, oriented in accordance with S^r_i and in

† For a full table of results and for references see (4) 249.

‡ Since $\pi^p_{2,m} \approx \pi^{p+1}_{1,m+1}$ by Theorem 4.2 (a), the values of $\pi^p_{2,3}$ are obtained with those of $\pi^{p+1}_{1,4}$ in the third paper.

§ This is essentially the construction on p. 270 of (7).

Quart. J. Math. Oxford (2), 9 (1958), 8-27.

such a way that E^r_{1+} and E^r_{2+} induce opposite orientations of S^{r-1}. Thus the radial projection $S^r_2 \to S^r_1$ is of degree -1, and $(E^r_{1+} \cup E^r_{2+})$ and $(E^r_{1-} \cup E^r_{2-})$ are both oriented r-spheres.

Clearly $\pi_s(Q^r) = 0$ if $s < r$. Hence by a theorem due to Hurewicz [(3) Theorem 1] there is an isomorphism $\pi_r(Q^r) \approx H_r(Q^r)$. Thus, if

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \in \pi_r(Q^r),$$

are classes of the identical maps of

$$S^r_1, S^r_2, (E^r_{1+} \cup E^r_{2+}), (E^r_{1-} \cup E^r_{2-}),$$

then we have that $$\alpha_1+\alpha_2 = \beta_1+\beta_2.$$

Now let $f_i \colon S^r_i \to X$ ($i = 1, 2$; X r-simple) be symmetric maps which agree on S^{r-1}. Let

$$h_1 \colon (E^r_{1+} \cup E^r_{2+}) \to X \text{ be defined by } h_1 = f_i \text{ on } E^r_{i+},$$

$$h_2 \colon (E^r_{1-} \cup E^r_{2-}) \to X \text{ be defined by } h_2 = f_i \text{ on } E^r_{i-}.$$

Let $H \colon Q^r \to X$ be defined by h_1 on $(E^r_{1+} \cup E^r_{2+})$ and h_2 on $(E^r_{1-} \cup E^r_{2-})$. Then H induces a homomorphism $H_* \colon \pi_r(Q^r) \to \pi_r(X)$, and $H_* \alpha_i = \{f_i\}$, $H_* \beta_i = \{h_i\}$. But $\{h_1\} = \{h_2\} = \{h\}$, say, since the f_i are symmetric maps and $r \equiv 1 \pmod 2$. Hence we have

$$\{f_1\}+\{f_2\} = 2\{h\}.$$

Together with this construction we need the following theorem on symmetric maps of spheres. Let S^r, P^{r-1}, and u_{r-1} be defined as in § 2.3 (c), $u^{-1}_{r-1}(P^{r-1})$ being the equator of S^r. Then we have

THEOREM 6.1. *No symmetric map $f \colon S^r \to S^{r-1}$ is essential unless $r \equiv 3 \pmod 4$. If $r \equiv 3 \pmod 4$ and $fu^{-1}_{r-1} \colon P^{r-1} \to S^{r-1}$ is essential, then f is essential.*

For the proof see J. H. C. Whitehead (7) Theorem 7.

7. Calculation of $\pi^p_{k,3}$†

We consider the fibring $V_{k+3,3}/V_{k+2,2} \to S^{k+2}$, and examine the sequence

$$\text{(B)} \qquad \to \pi_{k+p+1}(S^{k+2}) \xrightarrow{\Delta_*} \pi^p_{k,2} \xrightarrow{i_{k+p*}} \pi^p_{k,3} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+2}) \to.$$

7.1. $k \equiv 1 \pmod 4$. In this case there is a two-field on S^{k+2} (2, 5) and so the fibring admits a cross-section p. Hence Theorem 1.1 gives

$$\pi^p_{k,3} = i_* \pi^p_{k,2} + p_* \pi_{k+p}(S^{k+2}).$$

Using the values of $\pi^p_{k,2}$ as calculated in § 5.2, we obtain the values shown in the table for $\pi^p_{k,3}$ when $k \equiv 1 \pmod 4$.

† For the notation used see (4), especially §§ 2 and 3.1.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+3,3}\} = 0 \quad \textit{for } k \equiv 1 \pmod 4.$$

7.2. $k \equiv 3 \pmod 4$

(a) When $p = 1$, (B) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$

i.e.

$$\to Z_\infty \to Z_2 \to \pi^1_{k,3} \to 0,$$

by § 5.2 (b). But $i^{-1}_{k+1*}(0) \neq 0$, for otherwise we should have a cross-section in the fibring by Theorem 1.2, and so a two-field on S^{k+2} which is impossible (2, 5). Hence $i^{-1}_{k+1*}(0) = \pi^1_{k,2}$, and

$$\pi^1_{k,3} = 0.$$

Note that Δ_* is onto, whence the image of p_{k+2*} is the Z_∞ subgroup generated by $2\{h_{k+2,k+2}\}$. We also have from Corollary 1.5 that $\{t_{k+3,3}\}$ generates $i^{-1}_{k+1*}(0)$.

Hence

$$\{t_{k+3,3}\} = \{i_{k+1,1}\, h_{k,k+1}\}.$$

(b) When $p = 2$, (B) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$

i.e.

$$\to Z_2 \to Z_4 \to \pi^2_{k,3} \to Z_\infty \to 0,$$

by § 5.2 (c) and since the image of p_{k+2*} is Z_∞ by (a). Also

$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\,\pi_{k+2}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+2}\{t_{k+3,3}\} = h^*_{k+1,k+2}\{i_{k+1,1}\, h_{k,k+1}\} = \{i_{k+1,1}\, h_{k,k+2}\},$$

which is non-zero and of order two by § 5.2 (c). Thus

$$\pi^2_{k,3} = Z_2 + Z_\infty,$$

and is generated by $i_{k+2,1*}\, a$ of order two, where

$$p_{k+2,1*}\, a = \{h_{k+1,k+2}\},$$

and b, of infinite order, which is such that $p_{k+3,1*}\, b = 2\{h_{k+2,k+2}\}$. Note also that the image of $\Delta_* = i^{-1}_{k+2*}(0)$ is Z_2, and thus that the image of p_{k+3*} is zero.

(c) When $p = 3$ and $k \geqslant 7$, (B) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi^3_{k,3} \to 0$$

by § 5.2 (d) and since p_{k+3*} is trivial by (b). Also

$$i_{k+3*}^{-1}(0) = t_{k+3,3*}\,\pi_{k+3}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+3}\{t_{k+3,3}\} = h^*_{k+1,k+3}\{i_{k+1,1}\,h_{k,k+1}\} = \{i_{k+1,1}(12h_{k,k+3})\} = 0, \text{by } 5.2\,(d).$$

Hence $$i_{k+3*}^{-1}(0) = 0,$$

and $$\pi^3_{k,3} = Z_2 + Z_2,$$

being generated by $\{i_{k+1,2}\,h_{k,k+3}\}$ and $i_{k+2,1*}\,a$, where

$$p_{k+2,1*}\,a = \{h_{k+1,k+3}\}.$$

Note also that since Δ_* is trivial, p_{k+4*} is onto.

(d) When $p = 3$ and $k = 3$, (B) gives

$$\cdot \xrightarrow{p_{7*}} \pi_7(S^5) \xrightarrow{\Delta_*} \pi^3_{3,2} \xrightarrow{i_{6*}} \pi^3_{3,3} \xrightarrow{p_{6*}} \pi_6(S^5) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^3_{3,3} \to 0,$$

by § 5.2 (e) and since p_{6*} is trivial by (b). Also $i_{6*}^{-1}(0) = t_{6,3*}\,\pi_6(S^4)$, which is generated by

$$h^*_{4,6}\{t_{6,3}\} = h^*_{4,6}\{i_{4,1}\,h_{3,4}\} = \{i_{4,1}(6h_{3,6})\} = 0,$$

by § 5.2 (e). Hence $$i_{6*}^{-1}(0) = 0,$$

and $$\pi^3_{3,3} = Z_2,$$

being generated by $i_{5,1*}\,a$, where $p_{5,1*}\,a = \{h_{4,6}\}$. Note also that since Δ_* is trivial, p_{7*} is onto $\pi_7(S^5)$.

(e) When $p = 4$ and $k \geqslant 7$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi^4_{k,2} \xrightarrow{i_{k+4*}} \pi^4_{k,3} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \to,$$

i.e. $$\to Z_{24} \to Z_2 \to \pi^4_{k,3} \to Z_2 \to 0,$$

by § 5.2 (f) and since p_{k+4*} is onto $\pi_{k+4}(S^{k+2})$ by (c). Also

$$i_{k+4*}^{-1}(0) = t_{k+3,3*}\,\pi_{k+4}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+4}\{t_{k+3,3}\} = h^*_{k+1,k+4}\{i_{k+1,1}\,h_{k,k+1}\} \in i_{k+1,1*}\,\pi_{k+4}(S^k) = 0.$$

Hence $$i_{k+4*}^{-1}(0) = 0,$$

and $\pi^4_{k,3}$ *has four elements.*

Note also that Δ_* is trivial and that thus p_{k+5*} is onto.

To determine the structure of $\pi^4_{k,3}$, consider the sequence associated with the fibring $$V_{4s+2,3}/S^{4s-1} \to V_{4s+2,2},$$

i.e. $$\to \pi_{4s+p}(S^{4s-1}) \to \pi^{p+1}_{4s-1,3} \to \pi^p_{4s,2} \xrightarrow{\Delta_*} \pi_{4s+p-1}(S^{4s-1}) \to,$$

which, starting at the term $\pi_{4s+3}(S^{4s-1})$, is, for $s \geqslant 2$, of the form

$$\to 0 \to \pi^4_{4s-1,3} \to Z_{24}+Z_2 \xrightarrow{\Delta_*} Z_{24} \to Z_2+Z_2 \to Z_2+Z_2 \xrightarrow{\Delta_*} Z_2 \to,$$

by virtue of the results of § 5.1 and (c). But, since $\pi^4_{4s-1,3}$ has four elements, we have, by working along the sequence from the left, that

$$\text{(i)} \quad \Delta_* \pi^3_{4s,2} = 2\pi_{4s+2}(S^{4s-1}),$$

$$\text{(ii)} \quad \Delta_* \pi^2_{4s,2} = \pi_{4s+1}(S^{4s-1}).$$

We now operate with $h^* = \{h_{4s+2,4s+3}, h_{4s+1,4s+2}\}^*$ on the portion

$$\to \pi^2_{4s,2} \xrightarrow{\Delta_*} \pi_{4s+1}(S^{4s-1}) \to$$

of the above sequence to obtain the commutative diagram (Lemma 3.1)

$$\begin{tikzcd} 0 \arrow[r] & \pi^4_{4s-1,3} \arrow[r] & \pi^3_{4s,2} \arrow[r, "\Delta_*"] & \pi_{4s+2}(S^{4s-1}) \arrow[r] & \pi^3_{4s-1,3} \arrow[r] & {} \\ & {} \arrow[r] & \pi^2_{4s,2} \arrow[u, "h^*"] \arrow[r, "\Delta_*"] & \pi_{4s+1}(S^{4s-1}) \arrow[u, "h^*"] \arrow[r] & {}. \end{tikzcd}$$

By virtue of the above relations this takes the form

$$\begin{tikzcd} 0 \arrow[r] & \pi^4_{4s-1,3} \arrow[r] & Z_{24}+Z_2 \arrow[r, "\Delta_*"] & Z_{24} \arrow[r] & Z_2 \subset Z_2+Z_2 \arrow[r] & 0 \\ & {} \arrow[r] & Z_2+Z_2 \arrow[u, "h^*"] \arrow[r, "\Delta_*"] & Z_2 \arrow[u, "h^*"] \arrow[r] & 0. \end{tikzcd}$$

Using the result of § 5.1, now choose generators of $\pi^2_{4s,2}$, $= Z_2+Z_2$, as

$$a = i_*\{h_{4s,4s+2}\} \quad \text{and} \quad b = p_*\{h_{4s+1,4s+2}\};$$

and of $\pi^3_{4s,2}$, $= Z_{24}+Z_2$, as

$$c = i_*\{h_{4s,4s+3}\} \quad \text{and} \quad d = p_*\{h_{4s+1,4s+3}\}.$$

Then we have, by Lemma 3.1, that

$$h^*a = h^*i_*\{h_{4s,4s+2}\} = i_* h^*_{4s+2,4s+3}\{h_{4s,4s+2}\} = i_*\{12h_{4s,4s+3}\} = 12c,$$

$$h^*b = h^*p_*\{h_{4s+1,4s+2}\} = p_* h^*_{4s+2,4s+3}\{h_{4s+1,4s+2}\} = p_*\{h_{4s+1,4s+3}\} = d.$$

But $\quad$ *either* $\Delta_* a$ *or* $\Delta_* b = \{h_{4s-1,4s+1}\}$, *or both.*

Hence

either $h^*\Delta_* a$ *or* $h^*\Delta_* b = 12\{h_{4s-1,4s+2}\}$, *or both,*

i.e. $\quad$ *either* $\Delta_* 12c$ *or* $\Delta_* d = 12\{h_{4s-1,4s+2}\} \neq 0$, *or both.*

But, if $\pi^4_{4s-1,3}$ were Z_2+Z_2, its (isomorphic) image in $Z_{24}+Z_2$ would be generated by $12c$ and d. Then both $\Delta_* 12c$ and $\Delta_* d$ would be zero by exactness, which we have just seen to be impossible.

Hence

$$\pi^4_{k,2} = Z_4 \quad \textit{when } k \equiv 3 \pmod 4) \textit{ and when } k \geqslant 7,$$

and is generated by an element a such that $p_{k+3,1*}a = \{h_{k+2,k+4}\}$. Note that it is $\Delta_* d$ which equals $12\{h_{4s-1,4s+2}\} \neq 0$.

(f) When $p = 4$ and $k = 3$, (B) gives

$$\xrightarrow{p_{8*}} \pi_8(S^5) \xrightarrow{\Delta_*} \pi^4_{3,2} \xrightarrow{i_{7*}} \pi^4_{3,3} \xrightarrow{p_{7*}} \pi_7(S^5) \rightarrow,$$

i.e. $$\rightarrow Z_{24} \rightarrow Z_\infty + Z_2 \rightarrow \pi^4_{3,3} \rightarrow Z_2 \rightarrow 0,$$

by § 5.2 (g) and since p_{7*} is onto by (d). Also

$$i_{7*}^{-1}(0) = t_{6,3*}\pi_7(S^4) = \{i_{4,1}h_{3,4}\}_*\pi_7(S^4) = i_{7*}\{h_{3,4}\}_*\pi_7(S^4)$$
$$\subset i_{7*}\pi_7(S^3) = 0, \quad \text{by 5.2 } (g).$$

Hence $\quad \pi^4_{3,3}$ *is an extension of* $Z_\infty + Z_2$ *by* Z_2.

Note that Δ_* is trivial, whence p_{8*} is onto $\pi_8(S^5)$.

There are now three possibilities for the structure of $\pi^4_{3,3}$: $Z_\infty + Z_2$, $Z_\infty + Z_2 + Z_2$, $Z_\infty + Z_4$. I shall prove that it is the last by eliminating the other two. As in (e) we consider the sequence associated with the fibring $V_{6,3}/S^3 \rightarrow V_{6,2}$, i.e.

$$\rightarrow \pi_{p+4}(S^3) \xrightarrow{i_{4,2*}} \pi^{p+1}_{3,3} \xrightarrow{p_{6,2*}} \pi^p_{4,2} \xrightarrow{\Delta_*} \pi_{p+3}(S^3) \rightarrow,$$

which, starting at the term $\pi_7(S^3)$, is of the form

$$\rightarrow Z_2 \rightarrow \pi^4_{3,3} \rightarrow Z_\infty + Z_{12} + Z_2 \xrightarrow{\Delta_*} Z_{12} \rightarrow Z_2 \rightarrow Z_2 + Z_2 \xrightarrow{\Delta_*} Z_2 \rightarrow,$$

by virtue of the results of § 5.1 and (d). Thus, working from the right, we have by exactness, that

(i) $\Delta_*\pi^2_{4,2} = \pi_5(S^3)$,

(ii) $i_{4,2*}\pi_6(S^3) = 0$,

(iii) $\Delta_*\pi^3_{4,2} = \pi_6(S^3)$.

We now operate with $h^*_{6,7}$ on the portion

$$\rightarrow \pi_6(S^3) \xrightarrow{i_{4,2*}} \pi^3_{3,3} \rightarrow$$

of the above sequence to obtain the commutative diagram (Lemma 3.1)

$$\begin{array}{ccccc} \rightarrow \pi_7(S^3) & \xrightarrow{i_{4,2*}} & \pi^4_{3,3} & \xrightarrow{p_{6,2*}} & \pi^3_{4,2} \rightarrow \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} & & \\ \rightarrow \pi_6(S^3) & \xrightarrow{i_{4,2*}} & \pi^3_{3,3} & \xrightarrow{p_{6,2*}} & . \end{array}$$

Thus $$i_{4,2*}\pi_7(S^3) = i_{4,2*}h^*\pi_6(S^3) = h^*i_{4,2*}\pi_6(S^3) = 0,$$

by (ii) above. Hence $p_{6,2*}\colon \pi^4_{3,3} \rightarrow \pi^3_{4,2}$ is a monomorphism.

Next we operate with $h^* = \{h_{6,7}, h_{5,6}\}^*$ on the portion

$$\to \pi^3_{4,2} \xrightarrow{\Delta_*} \pi_5(S^3) \to$$

of the sequence to obtain the commutative diagram

$$\begin{array}{ccccc} \to \pi^4_{3,3} & \xrightarrow{p_{4,3*}} & \pi^3_{4,2} & \xrightarrow{\Delta_*} & \pi_6(S^3) \to \\ & & \uparrow h^* & & \uparrow h^* \\ & \to & \pi^3_{4,2} & \xrightarrow{\Delta_*} & \pi_5(S^3) \to . \end{array}$$

By virtue of the above relations this takes the form

$$\begin{array}{ccccc} 0 \to \pi^4_{3,3} \to & & Z_\infty + Z_{12} + Z_2 & \to & Z_{12} \to 0 \\ & & \uparrow h^* & & \uparrow h^* \\ & \to & Z_2 + Z_2 & \longrightarrow & Z_2 \to 0. \end{array}$$

Using the result of § 5.1, now choose generators of $\pi^3_{4,2}, = Z_2 + Z_2$, as

$$a = i_*\{h_{4,6}\} \quad \text{and} \quad b = p_*\{h_{5,6}\};$$

and of the finite summand of $\pi^3_{4,2}, = Z_\infty + Z_{12} + Z_2$, as

$$c = i_* \mathfrak{E}\{h_{3,6}\} \quad \text{and} \quad d = p_*\{h_{5,7}\}.$$

Then, as in (*e*), we have by Lemma 3.1 that

$$h^*a = h^*i_*\{h_{4,6}\} = i_* h^*_{6,7}\{h_{4,6}\} = i_* 6\mathfrak{E}\{h_{3,6}\} = 6c,$$

and $$h^*b = h^*p_*\{h_{5,6}\} = p_* h^*_{6,7}\{h_{5,6}\} = p_*\{h_{5,7}\} = d.$$

But *either* $\Delta_* a$ *or* $\Delta_* b = \{h_{3,5}\}$, *or both.*

Hence *either* $h^*\Delta_* a$ *or* $h^*\Delta_* b = 6\{h_{3,6}\}$, *or both,*

i.e. *either* $\Delta_* 6c$ *or* $\Delta_* d = 6\{h_{3,6}\} \neq 0$, *or both.*

But, if $\pi^4_{3,3}$ were $Z_\infty + Z_2 + Z_2$, the (isomorphic) image of its finite subgroup in $Z_\infty + Z_{12} + Z_2$ would be generated by $6c$ and d. Then both $\Delta_* 6c$ and $\Delta_* d$ would be zero by exactness, which we have just seen to be impossible. So $\pi^4_{3,3}$ cannot be $Z_\infty + Z_2 + Z_2$.

Now consider the diagram, commutative by § 2.1,

$$\begin{array}{ccccccc} 0 \to & \pi^4_{3,3} & \xrightarrow{p_{4,3*}} & \pi^3_{4,2} & \xrightarrow{\Delta_*} & \pi_6(S^3) & \to 0 \\ & \uparrow i_{5,1*} & & \uparrow i_{5,1*} & & \uparrow i_{4,0*} & \\ \to & \pi^4_{3,2} & \xrightarrow{p_{5,1*}} & \pi_7(S^4) & \xrightarrow{\Delta_*} & \pi_6(S^3) & \to, \end{array}$$

where the upper sequence is that of the previous paragraph and the lower that associated with the fibring $V_{5,2}/S^3 \to S^4$. Then, in the notation of the previous paragraph, we have that

$$\Delta_* 6c = \Delta_* 6i_{5,1*}\mathfrak{E}\{h_{3,6}\} = i_{4,0*}\Delta_* 6\mathfrak{E}\{h_{3,6}\} = i_{4,0*}12\{h_{3,6}\},$$

by § 5.2 (*e*), $= 0$. Assume now that $\pi^4_{3,3} = Z_\infty + Z_2$, and let

$$G = \pi^2_{4,2}/p_{6,2*}\pi^4_{3,3}.$$

We know that G is of order 12. For this to be so the image under $p_{6,2*}$ of the infinite summand of $\pi^4_{3,3}$ must be an infinite summand of $\pi^2_{4,2}$. However, d is of order 2, and we have just seen that $\Delta_* 6c = 0$. Thus G can have no element of order 12. Since we know that in fact $G = Z_{12}$, we have arrived at a contradiction. Thus $\pi^4_{3,3}$ cannot be $Z_\infty + Z_2$.

Hence $$\pi^4_{3,3} = Z_\infty + Z_4,$$

and is generated by

$$i_{5,1*}p^{-1}_{5,1*}(2\bar{p}_*\{h_{7,7}\} - \lambda\mathfrak{E}\{h_{3,6}\}),$$

where λ is odd, and a, of order four, such that $p_{6,1*}a = \{h_{5,7}\}$. Note that $2a = i_{5,1*}p^{-1}_{5,1*}6\mathfrak{E}\{h_{3,6}\}$.

(*g*) When $p = 5$, and $k \geqslant 7$, (B) gives

$$\to \pi_{k+6}(S^{k+2}) \to \pi^5_{k,2} \to \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \to$$

i.e. $$\to 0 \to 0 \to \pi^5_{k,3} \to Z_{24} \to 0,$$

by § 5.2 (*i*) and since p_{k+5*} is onto by (*e*).

Thus $\pi^5_{k,3} = Z_{24}$ *for* $k \equiv 3 \pmod 4$ *and for* $k \geqslant 7$,

and is generated by $p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$.

(*h*) When $p = 5$ and $k = 3$, (B) gives

$$\xrightarrow{p_{9*}} \pi_9(S^5) \xrightarrow{\Delta_*} \pi^5_{3,2} \xrightarrow{i_{8*}} \pi^5_{3,3} \xrightarrow{p_{8*}} \pi_8(S^5) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^5_{3,3} \to Z_{24} \to 0,$$

by § 5.2 (*j*) and since p_{8*} is onto by (*f*). Also

$$i^{-1}_{8*}(0) = t_{6,3*}\pi_8(S^4) = \{i_{4,1}h_{3,4}\}_*\pi_8(S^4) = i_{8*}\{h_{3,4}\}_*\pi_8(S^4)$$
$$\subset i_{8*}\pi_8(S^3), \quad \text{which is zero by 5.2 } (j).$$

Hence $\pi^5_{3,3}$ *is an extension of* Z_2 *by* Z_{24}.

Note that Δ_* is trivial, whence p_{9*} is onto $\pi_9(S^5)$.

To calculate the extension we consider the sequence associated with the fibring $V_{6,3}/S^3 \to V_{6,2}$, which contains the portion

$$\to \pi^5_{3,3} \xrightarrow{p_*} \pi^4_{4,2} \xrightarrow{\Delta_*} \pi_7(S^3) \to .$$

By virtue of the results in § 5.1 this is of the form

$$\to \pi^5_{3,3} \xrightarrow{p_*} Z_2 + Z_2 + Z_{24} \xrightarrow{\Delta_*} Z_2 \to .$$

Since $\pi^5_{3,3}$ is of order 48, we see that p_* is a monomorphism, whilst Δ_* is onto. But it is impossible to map Z_{48} isomorphically into $Z_2+Z_2+Z_{24}$.

Hence $$\pi^5_{3,3} = Z_2+Z_{24},$$
and is generated by $i_{5,1*}p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$, and a such that $p_{6,1*}a = \{h_{5,8}\}$.

7.3. $k \equiv 0 \pmod 2$ *and* $\geqslant 4$. Our first task is to calculate
$$\{t_{k+3,3}\} \in \pi^1_{k,2} = Z_2+Z_\infty$$
by § 5.1. We have by (iii) of § 2.4 (*b*) that $t_{k+2,2} \mid S^k = i_{k+1,1}t_{k+2,2}$; and by § 5.1 that $\{t_{k+2,2}\} = 0$. Thus we can extend $i_{k+1,1}t_{k+2,2}$ over the hemisphere E^{k+1}_+ of S^{k+1}, and, since $t_{k+2,2}$ is a symmetric map [§ 2.4 (*a*)], we can extend it symmetrically over E^{k+1}_-. Denote this extension by
$$g\colon S^{k+1} \to i_{k+1,1}(S^k) \subset V_{k+2,2}.$$
We now use the construction 'Q^r' of § 6 with
$$r = k+1, \qquad X = V_{k+2,2}, \qquad f_1 = t_{k+3,3}, \qquad f_2 = g$$
as defined above. Then we have that
$$2\{h\} = \{f_1\}+\{f_2\} = \{t_{k+3,3}\}+\{g\}.$$
Hence
$$\begin{aligned} p_{k+2,1*}\,2\{h\} &= p_{k+2,1*}\{t_{k+3,3}\}+p_{k+2,1*}\{g\} \\ &= 2\{h_{k+1,k+1}\}, \text{ by 2.3 } (b) \text{ and since } \{g\} \in i_*\pi_{k+1}(S^k). \end{aligned}$$
Thus
$$p_{k+2,1*}\{h\} = \{h_{k+1,k+1}\}$$
and $$\{h\} = \{\mathrm{p}h_{k+1,k+1}\}+i_*w, \quad \text{where } w \in \pi_{k+1}(S^k).$$
Thus $$2\{h\} = 2\{\mathrm{p}h_{k+1,k+1}\},$$
whence $$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}-\{g\}.$$

7.31. $k \equiv 0 \pmod 4$. We have from § 7.3 that
$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}-\{g\}.$$
But $g\colon S^{k+1} \to S^k$ is a symmetric map by construction, and so is inessential by Theorem 6.1. Thus
$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}.$$
We are now ready to examine the sequence (B).

(*a*) When $p = 1$, (B) gives
$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$
i.e. $$\to Z_\infty \to Z_2+Z_\infty \to \pi^1_{k,3} \to 0,$$
by § 5.1. Also $i^{-1}_{k+1*}(0)$ is generated by $\{t_{k+3,3}\}$, i.e. by $2\{\mathrm{p}h_{k+1,k+1}\}$, which is twice the generator of the infinite summand.

Hence $$\pi^1_{k,3} = Z_2 + Z_2,$$
and is generated by $\{i_{k+1,2}h_{k,k+1}\}$ and $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+1}\}$. Note that thus $\Delta_*^{-1}(0) = 0$, whence p_{k+2*} is trivial.

(*b*) When $p = 2$, (B) gives
$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$
i.e. $$\to Z_2 \to Z_2 + Z_2 \to \pi^2_{k,3} \to 0,$$
by 5.1 and since p_{k+2*} is trivial by (*a*). Also
$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\,\pi_{k+2}(S^{k+1}),$$
which is generated by
$$\begin{aligned} h^*_{k+1,k+2}\{t_{k+3,3}\} = h^*_{k+1,k+2}\,2\{\mathrm{p}h_{k+1,k+1}\} &= 2h^*_{k+1,k+2}\{\mathrm{p}h_{k+1,k+1}\} \\ &= \{\mathrm{p}h_{k+1,k+1}\}_*\,2\{h_{k+1,k+2}\} = 0. \end{aligned}$$
Thus $$\pi^2_{k,3} = Z_2 + Z_2,$$
and it is generated by $\{i_{k+1,2}h_{k,k+2}\}$ and $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}$. Note that Δ_* is trivial, whence p_{k+3*} is onto.

(*c*) When $p = 3$, (B) gives
$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \to,$$
i.e. $$\to Z_2 \to \pi^3_{k,2} \to \pi^3_{k,3} \to Z_2 \to 0,$$
since p_{k+3*} is onto by (*b*). Also
$$i^{-1}_{k+3*}(0) = t_{k+3,3*}\,\pi_{k+3}(S^{k+1}),$$
which is generated by
$$\begin{aligned} h^*_{k+1,k+3}\{t_{k+3,3}\} = h^*_{k+1,k+3}\,2\{\mathrm{p}h_{k+1,k+1}\} &= 2h^*_{k+1,k+3}\{\mathrm{p}h_{k+1,k+1}\} \\ &= \{\mathrm{p}h_{k+1,k+1}\}_*\,2\{h_{k+1,k+3}\} = 0. \end{aligned}$$
Hence $i^{-1}_{k+3*}(0) = 0$, and so $\pi^3_{k,3}$ is an extension of $\pi^3_{k,2}$ by Z_2. Thus, using the results of § 5.1, we have that

$\pi^3_{k,3}$ *is an extension of* $Z_{24} + Z_2$ *by* Z_2 *for* $k \geqslant 8$;

and $\pi^3_{4,3}$ *is an extension of* $Z_\infty + Z_{12} + Z_2$ *by* Z_2.

Note that Δ_* is trivial, whence p_{k+4*} is onto.

This time we make use of Theorem 2.3 (*g*) to calculate the extension. This gives that $\pi^3_{k,3} \approx \pi_{k+3}(P^{k+2}_{k-1})$ since $k \geqslant 4$. It will be shown in the Appendix that P^{k+2}_{k-1} is of the same homotopy type as $S^k \vee P^{k+2}_k$ when $k \equiv 0 \pmod 4$, where $\vee$ denotes attachment at a point. Thus, by Theorem 3 of (9), we have that
$$\pi^3_{k,3} \approx \pi_{k+3}(S^k) + \pi_{k+3}(P^{k+2}_k) \approx \pi_{k+3}(S^k) + \pi^3_{k+1,2}$$

by Theorem 2.3 (g). But $\pi^2_{k+1,2} = Z_4$ by § 5.2 (c). Hence

$$\pi^2_{k,3} = Z_{24} + Z_4 \quad \text{for } k \equiv 0 \pmod 4 \text{ and } \geqslant 8,$$

and is generated by $\{i_{k+1,2} h_{k,k+3}\}$ and a such that

$$p_{k+3,1*} a = \{h_{k+2,k+3}\}, \qquad 2a = \{i_{k+2,1}\, \mathrm{p} h_{k+1,k+3}\}.$$

Also
$$\pi^3_{4,3} = Z_\infty + Z_{12} + Z_4,$$

generated by $\{i_{5,2} \bar{p} h_{7,7}\}$, $\{i_{5,2} \mathfrak{E} h_{3,6}\}$, and a such that $p_{7,1*} a = \{h_{6,7}\}$, $2a = \{i_{6,1}\, \mathrm{p} h_{5,7}\}$.

(d) When $p = 4$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi^4_{k,2} \xrightarrow{i_{k+4*}} \pi^4_{k,3} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \to,$$

i.e.
$$\to Z_{24} \to \pi^4_{k,2} \to \pi^4_{k,3} \to Z_2 \to 0$$

since p_{k+4*} is onto by (c). Also

$$i^{-1}_{k+4*}(0) = t_{k+3,3*}\, \pi_{k+4}(S^{k+1}),$$

which is generated by

$$\begin{aligned} h^*_{k+1,k+4}\{t_{k+3,3}\} &= h^*_{k+1,k+4}\, 2\{\mathrm{p} h_{k+1,k+1}\} = 2h^*_{k+1,k+4}\{\mathrm{p} h_{k+1,k+1}\} \\ &= 2\{\mathrm{p} h_{k+1,k+1}\}_*\{h_{k+1,k+4}\} = 2\mathrm{p}_*\{h_{k+1,k+4}\}. \end{aligned}$$

Hence
$$i^{-1}_{k+4*}(0) = 2\mathrm{p}_*\, \pi_{k+4}(S^{k+1}).$$

Using the results of § 5.1 we thus have that

$\pi^4_{k,3}$ is an extension of Z_2 by Z_2 when $k \geqslant 8$, and $\pi^4_{4,3}$ is an extension of $Z_2 + Z_2 + Z_2$ by Z_2.

To determine the extensions we operate with $h^*_{k+3,k+4}$ on the section of the sequence (B) given in (c)† to obtain the diagram, commutative by Lemma 3.1,

$$\begin{array}{ccccccc} \to \pi^4_{k,2} & \xrightarrow{i_{k+4*}} & \pi^4_{k,3} & \xrightarrow{p_{k+3,1*}} & \pi_{k+4}(S^{k+2}) \to \\ \uparrow h^* & & \uparrow h^* & & \uparrow h^* \\ \to \pi^3_{k,2} & \xrightarrow{i_{k+3*}} & \pi^3_{k,3} & \xrightarrow{p_{k+3,1*}} & \pi_{k+3}(S^{k+2}) \to . \end{array}$$

Let $a \in \pi^3_{k,3}$ and $a' \in \pi^4_{k,3}$ be such that

$$p_{k+3,1*} a = \{h_{k+2,k+3}\}, \qquad p_{k+3,1*} a' = \{h_{k+2,k+4}\}.$$

Then

$$p_{k+3,1*} h^* a = h^* p_{k+3,1*} a = h^*_{k+3,k+4}\{h_{k+2,k+3}\} = \{h_{k+2,k+4}\} = p_{k+3,1*} a'.$$

Hence
$$h^* a = a' + i_{k+4*} b, \quad \text{where } b \in \pi^4_{k,2}.$$

Accordingly
$$2a' = 2h^* a - 2i_{k+4*} b = 0$$

since $2h^* a = a_* 2\{h_{k+3,k+4}\} = 0$, and since we have just seen that

† I do not use Theorem 2.3 (g) again since it no longer holds for $p = k = 4$.

$i_{k+4*}\pi^4_{k,2}$ has only elements of order two. Thus the extension is trivial, and

$$\pi^4_{k,3} = Z_2 + Z_2 \quad \textit{when } k \geqslant 8,$$

generated by $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+4}\}$ and a' such that $p_{k+3,1*}a' = \{h_{k+2,k+4}\}$ and

$$\pi^4_{4,3} = Z_2 + Z_2 + Z_2 + Z_2,$$

generated by $\{i_{5,2}\,\bar{p}h_{7,8}\}$, $\{i_{5,2}\,\mathfrak{E}h_{3,7}\}$, $\{i_{6,1}\,\mathrm{p}h_{5,8}\}$ and a' such that

$$p_{7,1*}a' = \{h_{6,8}\}.$$

Note that in either case

$$\Delta_* \pi_{k+5}(S^{k+2}) = 2\mathrm{p}_* \pi_{k+4}(S^{k+1}),$$

whence we have by exactness that the image of p_{k+5*} is the Z_2 subgroup generated by $12\{h_{k+2,k+5}\}$.

(*e*) When $p = 5$, (B) gives

$$\to \pi_{k+6}(S^{k+2}) \xrightarrow{\Delta_*} \pi^5_{k,2} \xrightarrow{i_{k+5*}} \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \to,$$

i.e.

$$\to 0 \to \pi^5_{k,2} \to \pi^5_{k,3} \to Z_2 \to 0$$

since p_{k+5*} is onto the Z_2 subgroup of $\pi_{k+5}(S^{k+2})$ by (*d*). Further, by § 5.1, $\pi^5_{k,2} = 0$ when $k \geqslant 8$. Hence

$$\pi^5_{k,3} = Z_2 \quad \textit{when } k \geqslant 8,$$

and is generated by $p^{-1}_{k+3,1*}\,12\{h_{k+2,k+5}\}$.

When $k = 4$, we have from § 5.1 that $\pi^5_{4,2} = Z_2 + Z_2 + Z_2$, whence $\pi^5_{4,3}$ is an extension of $Z_2 + Z_2 + Z_2$ by Z_2. We determine this extension as in (*d*) by operating, this time with $h^*_{7,9}$, on the section of the sequence (B) for $k = 4$ given in (*c*). This gives the commutative diagram

$$\begin{array}{ccccc}
\to \pi^5_{4,2} & \xrightarrow{i_{9*}} & \pi^5_{4,3} & \xrightarrow{p_{7,1*}} & \pi_9(S^6) \to \\
\uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} \\
\to \pi^3_{4,2} & \xrightarrow{i_{7*}} & \pi^3_{4,3} & \xrightarrow{p_{7,1*}} & \pi_7(S^6) \to .
\end{array}$$

Let $a \in \pi^3_{4,3}$ and $\bar{a} \in \pi^5_{4,3}$ be such that $p_{7,1*}a = \{h_{6,7}\}$ and $p_{7,1*}\bar{a} = 12\{h_{6,9}\}$. Then

$$p_{7,1*}h^*a = h^*p_{7,1*}a = h^*_{7,9}\{h_{6,7}\} = 12\{h_{6,9}\} = p_{7,1*}\bar{a}.$$

Hence

$$h^*a = \bar{a} + i_{9*}b, \quad \text{where } b \in \pi^5_{4,2},$$

and

$$2\bar{a} = 2h^*a - 2i_{9*}b = 0$$

since $2h^*a = a_*2\{h_{7,9}\} = 0$ and since $\pi^5_{4,2}$ has only elements of order two. Thus the extension is trivial, and

$$\pi^5_{4,3} = Z_2 + Z_2 + Z_2 + Z_2,$$

generated by $\{i_{5,2}\,\bar{p}h_{7,9}\}$, $\{i_{5,2}\,\mathfrak{E}h_{3,8}\}$, $\{i_{6,1}\,\mathrm{p}h_{5,9}\}$, and $\bar{a}$ such that

$$p_{7,1*}\,\bar{a} = 12\{h_{6,9}\}.$$

7.32. $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$. We first obtain the value of $\{t_{k+3,3}\}$. From § 7.3 we have that

$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} - \{g\},$$

where $g\colon S^{k+1} \to i_{k+1,1}(S^k)$ is a symmetric map, $i_{k+1,1}$ being a homeomorphism into. Further, using the notation of §§ 2.3 (*c*) and (*d*), we have from the definition of g that

$$gu_k^{-1} = \phi_{k+2,2}\colon P^k \to i(S^k),$$

which is of degree one (mod 2) and therefore essential. Thus g is essential by Theorem 6.1, i.e. $\{g\} = \{i_{k+1,1}\,h_{k,k+1}\}$.

Hence $$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}.$$

We are now ready to examine the sequence (B).

(*a*) When $p = 1$, (B) gives

$$\xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \xrightarrow{\Delta_*} \pi^1_{k,2} \xrightarrow{i_{k+1*}} \pi^1_{k,3} \to \pi_{k+1}(S^{k+2}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_\infty \to \pi^1_{k,3} \to 0,$$

by § 5.1. Also $i^{-1}_{k+1*}(0)$ is generated by $\{t_{k+3,3}\}$: that is, by

$$2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}.$$

Hence $$\pi^1_{k,3} = Z_4,$$

and is generated by $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+1}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{k+2*} is trivial.

(*b*) When $p = 2$, (B) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \xrightarrow{\Delta_*} \pi^2_{k,2} \xrightarrow{i_{k+2*}} \pi^2_{k,3} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+2}) \to,$$

i.e. $$\to Z_2 \to Z_2 + Z_2 \to \pi^2_{k,3} \to 0,$$

by § 5.1, and since p_{k+2*} is trivial by (*a*). Also

$$i^{-1}_{k+2*}(0) = t_{k+3,3*}\,\pi_{k+2}(S^{k+1}),$$

which is generated by

$$\begin{aligned} &h^*_{k+1,k+2}(2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1}\,h_{k,k+1}\}) \\ &\quad = \{\mathrm{p}h_{k+1,k+1}\}_*\,2\{h_{k+1,k+2}\} + \{i_{k+1,1}\,h_{k,k+1}\,h_{k+1,k+2}\} = \{i_{k+1,1}\,h_{k,k+2}\}. \end{aligned}$$

Thus $$\pi^2_{k,3} = Z_2,$$

generated by $\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}$. Note that p_{k+3*} is trivial.

When $p = 3$, (B) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \xrightarrow{\Delta_*} \pi^3_{k,2} \xrightarrow{i_{k+3*}} \pi^3_{k,3} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+2}) \rightarrow,$$

i.e.
$$\rightarrow Z_2 \rightarrow Z_{24}+Z_2 \rightarrow \pi^3_{k,3} \rightarrow 0$$

by § 5.1, and since p_{k+3*} is trivial by (*b*). Also

$$i^{-1}_{k+3*}(0) = \{t_{k+3,3}\}_* \pi_{k+3}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+3}(2\{\mathrm{p}h_{k+1,k+1}\}+\{i_{k+1,1}h_{k,k+1}\}) = 12\{i_{k+1,1}h_{k,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0)$ is the Z_2 subgroup generated by $12\{i_{k+1,1}h_{k,k+3}\}$.

Thus
$$\pi^3_{k,3} = Z_{12}+Z_2,$$

generated by $\{i_{k+1,2}h_{k,k+3}\}$ and $\{i_{k+2,1}\mathrm{p}h_{k+1,k+3}\}$. Note that again p_{k+4*} is trivial.

(*d*) When $p = 4$, (B) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \xrightarrow{\Delta_*} \pi^4_{k,2} \xrightarrow{i_{k+4*}} \pi^4_{k,3} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+2}) \rightarrow,$$

i.e.
$$\rightarrow Z_{24} \rightarrow Z_{24} \rightarrow \pi^4_{k,3} \rightarrow 0$$

by § 5.1, and since p_{k+4*} is trivial by (*c*). Also

$$i^{-1}_{k+4*}(0) = \{t_{k+3,3}\}_* \pi_{k+4}(S^{k+1}),$$

which is generated by

$$h^*_{k+1,k+4}(2\{\mathrm{p}h_{k+1,k+1}\}+\{i_{k+1,1}h_{k,k+1}\}$$
$$= 2\{\mathrm{p}h_{k+1,k+4}\}+\{i_{k+1,1}h_{k,k+1}h_{k+1,k+4}\} = 2\{\mathrm{p}h_{k+1,k+4}\}.$$

Thus
$$i^{-1}_{k+4*}(0) = 2\mathrm{p}_* \pi_{k+4}(S^{k+1}),$$

and
$$\pi^4_{k,3} = Z_2,$$

generated by $\{i_{k+2,1}\mathrm{p}h_{k+1,k+4}\}$. Note that, since

$$\Delta_* \pi_{k+5}(S^{k+2}) = 2\mathrm{p}_* \pi_{k+4}(S^{k+1}),$$

the image of p_{k+5*} is the Z_2 subgroup of $\pi_{k+5}(S^{k+2})$.

(*e*) When $p = 5$ and $k \geqslant 10$, (B) gives

$$\rightarrow \pi_{k+6}(S^{k+2}) \xrightarrow{\Delta_*} \pi^5_{k,2} \xrightarrow{i_{k+5*}} \pi^5_{k,3} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+2}) \rightarrow,$$

i.e.
$$\rightarrow 0 \rightarrow 0 \rightarrow \pi^5_{k,3} \rightarrow Z_2 \rightarrow 0$$

by § 5.1, and since the image of p_{k+5*} is Z_2 by (*d*).

Hence
$$\pi^5_{k,3} = Z_2 \quad \textit{when } k \geqslant 10,$$

and is generated by $p^{-1}_{k+3,1*}12\{h_{k+2,k+5}\}$.

(*f*) When $p = 5$ and $k = 6$, (B) gives

$$\to \pi_{12}(S^8) \xrightarrow{\Delta_*} \pi^5_{6,2} \xrightarrow{i_{11*}} \pi^5_{6,3} \xrightarrow{p_{11*}} \pi_{11}(S^8) \to,$$

i.e.
$$\to 0 \to Z_\infty \to \pi^5_{6,3} \to Z_2 \to 0$$

by § 5.1, and since the image of p_{11*} is Z_2 by (*d*).

Hence $\pi^5_{6,3}$ *is an extension of* Z_∞ *by* Z_2.

To calculate the extension we consider $\pi_{11}(P^8_5)$, $\approx \pi^5_{6,3}$ by Theorem 2.3 (*g*). Let C^{k+2} ($k \geqslant 3$) be the space consisting of an S^k and an S^{k+1} having just one point in common, to which a $k+2$ cell e^{k+2} has been attached by a map ϕ such that

$$\phi \mid \dot{E}^{k+2} \to S^k \vee S^{k+1}$$

is essential over S^k and of degree two over S^{k+1}. It will be shown in the Appendix that, when $k \equiv 2 \pmod 4$ and $k \geqslant 6$, $(P^{k+2}_{k-1}, P^{k+1}_{k-1})$ is of the same homotopy type as $(C^{k+2}, S^{k+1} \vee S^k)$. So consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{11}(S^7 \vee S^6) & \xrightarrow{i_{11*}} & \pi_{11}(C^8) & \xrightarrow{j_{11*}} & \pi_{11}(C^8, S^7 \vee S^6) & \xrightarrow{\delta_{11*}} & \pi_{10}(S^7 \vee S^6) \to \\
\uparrow \mathfrak{E} & & \uparrow \mathfrak{E}_1 & & \uparrow \mathfrak{E}_2 & & \uparrow \mathfrak{E} \\
\to \pi_{10}(S^6 \vee S^5) & \xrightarrow{i'_{10*}} & \pi_{10}(C^7) & \xrightarrow{j'_{10*}} & \pi_{10}(C^7, S^6 \vee S^5) & \xrightarrow{\delta'_{10*}} & \pi_9(S^6 \vee S^5) \to.
\end{array}$$

We know that $\pi_{11}(C^8)$ is an extension of Z_∞ by Z_2. I shall show that $\pi_{10}(C^7)$ is of finite order and that it contains an element a such that $j_{11*}\mathfrak{E}_1 a$ is non-zero. Thus $\mathfrak{E}_1 a$ is a non-zero element of finite order in $\pi_{11}(C^8)$, which must then be $Z_\infty + Z_2$.

We have by Theorem 3 of (9) that

$$\pi_9(S^6 \vee S^5) \approx \pi_9(S^6) + \pi_9(S^5) = Z_{24} + Z_2,$$

by Theorem 1 of (9) that

$$\pi_{10}(C^7, S^6 \vee S^5) \approx \pi_9(S^6) = Z_{24},$$

and by Theorem 2 of (6) that

$$\pi_{10}(S^6 \vee S^5) \approx \pi_{10}(S^6) + \pi_{10}(S^5) + [\pi_6(S^6), \pi_5(S^5)] = Z_2 + Z_\infty.$$

Thus the lower sequence of the diagram is of the form

$$\to Z_2 + Z_\infty \xrightarrow{i'_{10*}} \pi_{10}(C^7) \xrightarrow{j'_{10*}} Z_{24} \xrightarrow{\delta'_{10*}} Z_{24} + Z_2 \to.$$

Now we have by Theorem 1 of (1) that $i'^{-1}_{10*}(0)$ is the union of

$$\{2h_{6,6} + h_{5,6}\}_* \pi_{10}(S^6) \quad \text{and} \quad [\{2h_{6,6} + h_{5,6}\}_* \pi_6(S^6), \pi_5(S^5)],$$

which, since $\pi_{10}(S^6) = 0$, is generated by

$$[\{2h_{6,6}+h_{5,6}\}_*\{h_{6,6}\}, \{h_{5,5}\}] = 2[\{h_{6,6}\}, \{h_{5,5}\}]+[\{h_{5,6}\}, \{h_{5,5}\}]$$
$$= 2[\{h_{6,6}\}, \{h_{5,5}\}]+\{h_{5,10}\}.$$

Thus $i'_{10*}\pi_{10}(S^6 \vee S^5) = Z_4$, whence $\pi_{10}(C^7)$ is of finite order.

Further, we have by Theorems 1 and 2 of (9) that

$$\delta'_{10*} Z_{24} = \{2h_{6,6}+h_{5,6}\}_* \pi_9(S^6),$$

which is generated by

$$h^*_{6,9}\{2h_{6,6}+h_{5,6}\} = 2\{h_{6,9}\}+h^*_{6,9}\{h_{5,6}\} = 2\{h_{6,9}\}$$

since

$$h^*_{6,9}\{h_{5,6}\} = \{\mathfrak{E}^2 h_{3,4}\,\mathfrak{E}^2\bar{p}h_{7,7}\} = \mathfrak{E}^2\{h_{3,7}\} = \{\mathfrak{E}^2 h_{3,6}\,\mathfrak{E}^2 h_{6,7}\}$$
$$= 2h_{5,8*}\{h_{8,9}\} = h_{5,8*}\,2\{h_{8,9}\} = 0.$$

Thus the image of $\delta'_{10*} = 2Z_{24}$, whence $j'_{10*}\pi_{10}(C^7)$ is the Z_2 subgroup of

$$\pi_{10}(C^7, S^6 \vee S^5).$$

This implies that $\pi_{10}(C^7)$ has eight elements and contains an element a, of finite order, such that $j'_{10*}a \neq 0$. But $\mathfrak{E}_2$ is an isomorphism by Theorem 3.2 (B). Hence

$$j_{11*}\mathfrak{E}_1 a = \mathfrak{E}_2 j'_{10*} a \neq 0, \quad \text{whence } \mathfrak{E}_1 a \neq 0.$$

This is what we set out to prove.

Thus $$\pi^5_{6,3} = Z_\infty + Z_2,$$

and is generated by $\{i_{7,2}h_{6,11}\}$ and a such that $p_{9,1*}a = 12\{h_{8,11}\}$ and $2a = 0$.

Appendix. The homotopy type of certain P^{k+s}_{k-1}.†

(A) *The homotopy type of* P^{k+1}_{k-1}. Let $\psi_{n+1,m+1}$: $P^{n-1}_{k-1} \to V_{n,m}$ be the map defined in § 2.3 (d). Then $\psi_{k+2,2}$: $P^k_{k-1} \to S^k$ is a homeomorphism. We also have from § 2.3 (d) that

$$g = \psi^{-1}_{k+3,3}t'_{k+2,2}\colon (E^{k+1}, \dot{E}^{k+1}) \to (P^{k+1}_{k-1}, P^k_{k-1})$$

is a characteristic map for the $(k+1)$-cell in P^{k+1}_{k-1}, and, further, that

$$\delta g = \psi^{-1}_{k+2,2}t_{k+2,2}\colon \dot{E}^{k+1} \to S^k.‡$$

We thus have, by § 2.3 (b) (ii), that

$$\{\delta g\} = \begin{cases} \psi^{-1}_{k+2,2*}\,2\{h_{k,k}\} & (k \text{ odd}), \\ 0 & (k \text{ even}). \end{cases}$$

Now let Y^{k+1}_2 be the space consisting of an S^k to which one $(k+1)$-cell e^{k+1} has been attached by a map ϕ such that

$$\phi \mid \dot{E}^{k+1} \to S^k$$

† P^{k+s}_{k-1} is the projective $k+s$ space P^{k+s} in which a P^{k-1} has been shrunk to a point.

‡ If $f\colon (E^n, \dot{E}^n) \to (K, L)$, then $\delta f\colon \dot{E}^n \to L$ is defined by $f \mid \dot{E}^n$.

is of degree two, i.e.

$$\{\delta\phi\} = 2\{h_{k,k}\}, = \psi_{k+2,2*}\{\delta g\}$$

when k is odd. Thus, by Lemma 3 in (8), $\psi_{k+2,2}$ can be extended to a homotopy equivalence

$$f_{k+2,2}\colon (P^{k+1}_{k-1}, P^k_{k-1}) \to (Y^{k+1}_2, S^k) \textit{ for all odd } k.$$

Note also that (Y^{k+2}_2, S^{k+1}) is of the same homotopy type as $\mathfrak{E}(Y^{k+1}_2, S^k)$.

On the other hand, if $X \vee Y$ denotes the union of two spaces X and Y having just one point in common, then, by the same lemma, we can extend $\psi_{k+2,2}$ to a homotopy equivalence

$$f_{k+2,2}\colon (P^{k+1}_{k-1}, P^k_{k-1}) \to (S^{k+1} \vee S^k, S^k) \textit{ for all even } k.$$

Note again that $(S^{k+2} \vee S^{k+1}, S^{k+1})$ is of the same homotopy type as $\mathfrak{E}(S^{k+1} \vee S^k, S^k)$. Also, by Theorem 3 of (9), we have that

$$\pi_{k+1}(S^{k+1} \vee S^k) = i_* \pi_{k+1}(S^k) + j_* \pi_{k+1}(S^{k+1}),$$

where i_* and j_* are the monomorphisms induced by the identical injections of S^k and S^{k+1}. Thus, when k is even, $\pi^1_{k,2}$ and $\pi_{k+1}(S^{k+1} \vee S^k)$ are abstractly isomorphic (see § 5.1), this isomorphism being realized by

$$f_{k+2,2*}\psi^{-1}_{k+3,3*}\colon \pi^1_{k,2} \to \pi_{k+1}(S^{k+1} \vee S^k).$$

Therefore $\qquad f_{k+2,2*}\psi^{-1}_{k+3,3*}\{i_{k+1,1} h_{k,k+1}\} = i_*\{h_{k,k+1}\},$

$$f_{k+2,2*}\psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+1}\} = j_*\{h_{k+1,k+1}\} + \lambda i_*\{h_{k,k+1}\} \quad (\lambda = 0 \text{ or } 1)$$

provided that the S^{k+1} are suitably oriented.

(B) *The homotopy type of P^{k+2}_{k-1} when $k \equiv 3 \pmod 4$.* We have from § 2.3 (*d*) that

$$g = \psi^{-1}_{k+4,4} t'_{k+3,3}\colon (E^{k+2}, \dot{E}^{k+2}) \to (P^{k+2}_{k-1}, P^{k+1}_{k-1})$$

is a characteristic map for the $k+2$ cell in P^{k+2}_{k-1}, and that

$$\delta g = \psi^{-1}_{k+3,3} t_{k+3,3}\colon \dot{E}^{k+2} \to P^{k+1}_{k-1}.$$

But, when $k \equiv 3 \pmod 4$, $\{t_{k+3,3}\} = \{i_{k+1,1} h_{k,k+1}\}$ by § 7.2 (*a*), and $\psi_{k+3,3*}$ is an isomorphism in this dimension by Theorem 2.3 (*g*). So

$$\{\delta g\} = \psi^{-1}_{k+3,3*}\{i_{k+1,1} h_{k,k+1}\} = i'_*\psi^{-1}_{k+2,2*}\{h_{k,k+1}\},$$

where i'_* is induced by the identical injection of $P^k_{k-1} \to P^{k+1}_{k-1}$.

Now let B^{k+2}_2 be the space consisting of Y^{k+2}_2 to which one $k+2$ cell has been attached by a map ϕ such that

$$\phi \mid \dot{E}^{k+2}\colon \dot{E}^{k+2} \to S^k \subset Y^{k+1}_2$$

and is essential: that is

$$\{\delta\phi\} = i_*\{h_{k,k+1}\} = f_{k+2,2*} i'_* \psi^{-1}_{k+2,2*}\{h_{k,k+1}\} = f_{k+2,2*}\{\delta g\},$$

where i is the identical injection and $f_{k+2,2}$ the homotopy equivalence defined in (A). Hence, by Lemma 3 of (8), we can extend $f_{k+2,2}$ to a homotopy equivalence

$$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (B^{k+2}_2, Y^{k+1}_2), \; \textit{for } k \equiv 3 \pmod 4.$$

Note also that (B^{k+3}_2, Y^{k+2}_2) is of the same homotopy type as $\mathfrak{E}(B^{k+2}, Y^{k+1})$ when $k \geqslant 2$. (When $k = 2$, the attaching map of the 4-cell is to be of Hopf invariant unity.)

(C) *The homotopy type of* P^{k+2}_{k-1} *when* $k \equiv 0 \pmod 4$. We have again from § 2.3 (d) that

$$g = \psi^{-1}_{k+4,4} t'_{k+3,3}\colon (E^{k+2}, \dot{E}^{k+2}) \to (P^{k+2}_{k-1}, P^{k+1}_{k-1})$$

is a characteristic map for the $k+2$ cell in P^{k+2}_{k-1}, and that

$$\delta g = \psi^{-1}_{k+3,3} t_{k+3,3}\colon \dot{E}^{k+2} \to P^{k+1}_{k-1}.$$

But, when $k \equiv 0 \pmod 4$, $\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\}$ by § 7.31. Thus, if $f_{k+2,2*}$ and j_* are as defined in (A), we have that

$$f_{k+2,2*}\{\delta g\} = f_{k+2,2*}\psi^{-1}_{k+3,3*} 2\{\mathrm{p}h_{k+1,k+1}\} = 2j_*\{h_{k+1,k+1}\}.$$

Hence $f_{k+2,2}$ can be extended to a homotopy equivalence

$$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (Y^{k+2}_2 \vee S^k, S^{k+1} \vee S^k) \; \textit{for } k \equiv 0 \pmod 4.$$

But we have from (A) that Y^{k+2}_2 is of the same homotopy type as P^{k+2}_k. Thus P^{k+2}_{k-1} is of the same homotopy type as $P^{k+2}_k \vee S^k$ when

$$k \equiv 0 \pmod 4.$$

(D) *The homotopy type of* P^{k+2}_{k-1} *when* $k \equiv 2 \pmod 4$, *and* $k \geqslant 6$. When $k \geqslant 3$, let C^{k+2} be the space consisting of $S^{k+1} \vee S^k$ to which one $k+2$ cell has been attached by a map ϕ such that

$$\phi \mid \dot{E}^{k+2}\colon \dot{E}^{k+2} \to S^{k+1} \vee S^k$$

is essential over S^k and of degree two over S^{k+1}: that is,

$$\{\delta\phi\} = i_*\{h_{k,k+1}\} + 2j_*\{h_{k+1,k+1}\},$$

where i_* and j_* are as defined in (A). If g is the characteristic map for the $k+2$ cell in P^{k+2}_{k-1}, we have, by § 2.3 (d) as above, that

$$\{\delta g\} = \{\psi^{-1}_{k+3,3} t_{k+3,3}\} \in \pi_{k+1}(P^{k+1}_{k-1}).$$

But, when $k \equiv 2 \pmod 4$,

$$\{t_{k+3,3}\} = 2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\}$$

by § 7.32. Thus, if $f_{k+2,2}$ is the homotopy equivalence defined in (A), we have that

$$\begin{aligned} f_{k+2,2*}\{\delta g\} &= f_{k+2,2*}\psi^{-1}_{k+3,3*}(2\{\mathrm{p}h_{k+1,k+1}\} + \{i_{k+1,1} h_{k,k+1}\}) \\ &= 2j_*\{h_{k+1,k+1}\} + (2\lambda+1)i_*\{h_{k,k+1}\}, \quad \text{by (A)}, \\ &= \{\delta\phi\}. \end{aligned}$$

Hence, by Lemma 3 in (8), we can extend $f_{k+2,2}$ to a homotopy equivalence

$f'_{k+2,2}\colon (P^{k+2}_{k-1}, P^{k+1}_{k-1}) \to (C^{k+2}, S^{k+1} \vee S^k)$ *when* $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$.

Note also that $(C^{k+3}, S^{k+2} \vee S^{k+1})$ is of the same homotopy type as $\mathfrak{E}(C^{k+2}, S^{k+1} \vee S^k)$ when $k \geqslant 3$.

(E) *The homotopy type of* P^{k+3}_{k-1} *when* $k \equiv 2 \pmod 4$ *and* $k \geqslant 6$. When $k \geqslant 3$, let D^{k+3}_λ be the space consisting of C^{k+2} to which one $k+3$ cell has been attached by a map ϕ_λ such that

$$\phi_\lambda \mid \dot{E}^{k+3}\colon \dot{E}^{k+3} \to S^{k+1} \vee S^k \subset C^{k+2},$$

and such that
$$\{\delta\phi_\lambda\} = j_*\{h_{k+1,k+2}\} + \lambda i_*\{h_{k,k+2}\}.$$

Again we have from § 2.3 (*d*) that, if g is a characteristic map for the $k+3$ cell in P^{k+3}_{k-1}, then

$$\{\delta g\} = \{\psi^{-1}_{k+4,4}\, t_{k+4,4}\} \in \pi_{k+2}(P^{k+2}_{k-1}).$$

But I shall show in § 8.2 of the third paper that, when $k \equiv 2 \pmod 4$ and $k \geqslant 6$,
$$\{t_{k+4,4}\} = \{i_{k+2,1}\, \mathrm{p}h_{k+1,k+2}\}.$$

Also, by Theorem 2.3 (*g*), $\psi_{k+4,4*}$ is an isomorphism in this dimension if $k > 2$. Thus we see that

$$\{\delta g\} = \psi^{-1}_{k+4,4*}\, i_{k+2,1*}\{\mathrm{p}h_{k+1,k+2}\} = i'_*\, \psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+2}\} \in i'_*\pi_{k+2}(P^{k+1}_{k-1}),$$

where i'_* is induced by the identical injection of P^{k+1}_{k-1} into P^{k+2}_{k-1}. If we now consider $\{\delta g\}$ in $\pi_{k+2}(P^{k+1}_{k-1})$ and if $f'_{k+2,2}$ and $f_{k+2,2}$ are the homotopy equivalences defined in (D) and (A), we have that

$$\begin{aligned} f'_{k+2,2*}\{\delta g\} &= f_{k+2,2*}\, \psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+2}\} \\ &= h^*_{k+1,k+2} f_{k+2,2*}\, \psi_{k+3,3*}\{\mathrm{p}h_{k+1,k+1}\} \\ &= h^*_{k+1,k+2}(j_*\{h_{k+1,k+1}\} + \lambda i_*\{h_{k,k+1}\}) \\ &= j_*\{h_{k+1,k+2}\} + \lambda i_*\{h_{k,k+2}\} \\ &= \{\delta\phi_\lambda\}. \end{aligned}$$

Thus, by Lemma 3 of (8), we can extend $f'_{k+2,2}$ to a homotopy equivalence

$$\bar{f}_{k+2,2}\colon (P^{k+3}_{k-1}, P^{k+2}_{k-1}) \to (D^{k+3}_\lambda, C^{k+2}) \quad \textit{for } k \equiv 2 \pmod 4 \textit{ and } k \geqslant 6,$$

where the λ is that determined by $f_{k+2,2*}\,\psi^{-1}_{k+3,3*}\{\mathrm{p}h_{k+1,k+1}\}$. Note also that $(D^{k+4}_\lambda, C^{k+3})$ is of the same homotopy type as $\mathfrak{E}(D^{k+3}_\lambda, C^{k+2})$ when $k > 2$.

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