THE GROUPS $\pi_r(V_{n,m})$ (III)

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Introduction

THIS is the third of a sequence of five papers, the previous ones being (8) and (9), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those groups which are given in the following tables. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 1$. A full table of results can be found in (8) 249. Some of these, namely the values of $\pi^p_{k,m}$ for $2 \leqslant p \leqslant 5$ and $k \geqslant p+2$, have also been obtained, independently and by different methods, by Yoshihiro Saito (10).

For the notation used throughout the body of this paper please see (8), especially §§ 1, 2, and 3.1. Also please note that sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,4}$

k	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
1	Z_∞	Z_2	Z_2	0	Z_∞	0
3	Z_2	Z_2	Z_4+Z_∞	$Z_2+Z_2+Z_2$		
4	Z_2+Z_2	$Z_\infty+Z_{12}+$ $+Z_4+Z_\infty$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$	$Z_2+Z_2+Z_2+$ $+Z_2+Z_2$		
5	Z_8	Z_2+Z_2	Z_2+Z_2	Z_2		
6	0	$Z_{12}+Z_\infty$	Z_2	$Z_\infty+Z_4$		
$4s-1$	Z_2	Z_2+Z_2	Z_8	Z_2+Z_2		
$4s+1$	Z_8	Z_2+Z_2	Z_2	Z_2		
$4s$	Z_2+Z_2	$Z_{24}+Z_4+Z_\infty$	$Z_8+Z_2+Z_2$	Z_2+Z_2		
$4s+2$	0	$Z_{12}+Z_\infty$	Z_2	Z_4		

TABLE FOR $\pi^p_{2,3}$

$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
Z_∞	Z_2	Z_2	0	Z_∞	0

8. Calculation of $\pi^p_{k,4}$

We consider the fibring $V_{k+4,4}/V_{k+3,3} \to S^{k+3}$ and examine the sequence

$$\text{(C)} \qquad \to \pi_{k+p+1}(S^{k+3}) \xrightarrow{\Delta_*} \pi^p_{k,3} \xrightarrow{i_{k+p*}} \pi^p_{k,4} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+3}) \to.$$

8.1. $k \equiv 0 \pmod 4$.

In this case there is a three-field on S^{k+3} (5, 12), and so the fibring admits a cross-section p. Hence Theorem 1.1 gives

$$\pi^p_{k,4} = i_* \pi^p_{k,3} + \mathrm{p}_* \pi_{k+p}(S^{k+3}).$$

Using the values of $\pi^p_{k,3}$ as calculated in § 7.31, we obtain the values shown in the table for $\pi^p_{k,4}$ when $k \equiv 0 \pmod 4$.

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+4,4}\} = 0 \quad \text{for } k \equiv 0 \pmod 4.$$

8.2. $k \equiv 2 \pmod 4$ *and* $\geqslant 6$.

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+2*}} \pi^2_{k,4} \xrightarrow{p_{k+2*}} \pi_{k+2}(S^{k+3}) \to$$

i.e.
$$\to Z_\infty \to Z_2 \to \pi^2_{k,4} \to 0,$$

by § 7.32 (*b*). But $i^{-1}_{k+2*}(0) \neq 0$ since otherwise there would be a cross-section in the above fibring, by Theorem 1.2, and so a three-field on S^{k+3}, which is impossible (5, 12). Hence $i^{-1}_{k+2*}(0) = \pi^2_{k,3}$, whence we have that

$$\pi^2_{k,4} = 0.$$

Note that Δ_* is onto, whence the image of p_{k+3*} is the Z_∞ subgroup generated by $2\{h_{k+3,k+3}\}$. Further we have from Corollary 1.5 that $\{t_{k+4,4}\}$ generates $i^{-1}_{k+2*}(0)$. Hence, by § 7.32 (*b*),

$$\{t_{k+4,4}\} = \{i_{k+2,1}\, \mathrm{p} h_{k+1,k+2}\}.$$

(*b*) When $p = 3$, (C) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_*} \pi^3_{k,3} \xrightarrow{i_{k+3*}} \pi^3_{k,4} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \to,$$

i.e.
$$\to Z_2 \to Z_{12} + Z_2 \to \pi^3_{k,4} \to Z_\infty \to 0,$$

by § 7.32 (*c*) and since the image of p_{k+3*} is a Z_∞ subgroup by (*a*). Also

$$i^{-1}_{k+3*}(0) = t_{k+4,4*}\, \pi_{k+3}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+3}\{t_{k+4,4}\} = h^*_{k+2,k+3}\{i_{k+2,1}\, \mathrm{p} h_{k+1,k+2}\} = \{i_{k+2,1}\, \mathrm{p} h_{k+1,k+3}\}.$$

Hence $i^{-1}_{k+3*}(0)$ is the Z_2 summand of $\pi^3_{k,3}$, whence

$$\pi^3_{k,4} = Z_{12} + Z_\infty,$$

generated by $\{i_{k+1,3}\, h_{k,k+3}\}$ and a such that

$$p_{k+4,1*}\, a = 2\{h_{k+3,k+3}\}.$$

Note that $\Delta^{-1}_*(0) = 0$, and so that p_{k+4*} is trivial.

(*c*) When $p = 4$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \to,$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^4_{k,4} \to 0,$$

by § 7.32 (*d*) and since p_{k+4*} is trivial by (*b*). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\,\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{i_{k+2,1}\,\mathrm{p}h_{k+1,k+3}\} = i_{k+4*}\,\mathrm{p}_*12\{h_{k+1,k+4}\}$$
$$\in i_{k+4*}\,2\mathrm{p}_*\,\pi_{k+4}(S^{k+1}) = 0, \quad \text{by } 7.32\,(d).$$

Hence $i^{-1}_{k+4*}(0) = 0$, and so we have that

$$\pi^4_{k,4} = Z_2,$$

generated by $\{i_{k+2,2}\,\mathrm{p}h_{k+1,k+4}\}$. Note that thus Δ_* is trivial, whence p_{k+5*} is onto.

(*d*) When $p = 5$ and $k \geqslant 10$, (C) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+3}) \xrightarrow{\Delta_*} \pi^5_{k,3} \xrightarrow{i_{k+5*}} \pi^5_{k,4} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \to,$$

i.e.
$$\to Z_{24} \to Z_2 \to \pi^5_{k,4} \to Z_2 \to 0,$$

by § 7.32 (*e*) and since p_{k+5*} is onto by (*c*). Also

$$i^{-1}_{k+5*}(0) = t_{k+4,4*}\,\pi_{k+5}(S^{k+2}) = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+2}\}_*\,\pi_{k+5}(S^{k+2})$$
$$\subset i_{k+2,1*}\,\mathrm{p}_*\,\pi_{k+5}(S^{k+1}) = 0.$$

Hence $\pi^5_{k,4}$ has four elements. But, by Theorem 4.2 (*b*),

$$\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_4,$$

by § 7.2 (*e*). Thus $\qquad \pi^5_{k,4} = Z_4 \quad \text{when } k \geqslant 10$

and is generated by an element a such that $p_{k+4,1*}\,a = \{h_{k+3,k+5}\}$. Note that Δ_* is trivial, whence p_{k+6*} is onto.

(*e*) When $p = 5$ and $k = 6$, (C) gives

$$\xrightarrow{p_{12*}} \pi_{12}(S^9) \xrightarrow{\Delta_*} \pi^5_{6,3} \xrightarrow{i_{11*}} \pi^5_{6,4} \xrightarrow{p_{11*}} \pi_{11}(S^9) \to,$$

i.e.
$$\to Z_{24} \to Z_\infty + Z_2 \to \pi^5_{6,4} \to Z_2 \to 0,$$

by § 7.32 (*f*) and since p_{11*} is onto by (*c*). Also

$$i^{-1}_{11*}(0) = t_{10,4*}\,\pi_{11}(S^8) = \{i_{8,1}\mathrm{p}h_{7,8}\}_*\,\pi_{11}(S^8) \subset i_{8,1*}\,\mathrm{p}_*\,\pi_{11}(S^7) = 0.$$

Hence $\qquad \pi^5_{6,4}$ *is an extension of* $Z_\infty + Z_2$ *by* Z_2.

Note that thus Δ_* is trivial, whence p_{12*} is onto.

In determining the structure of $\pi^5_{6,4}$ we first consider the sequence associated with the fibring $V_{10,4}/S^6 \to V_{10,3}$, which is of the form

$$\to \pi_{11}(S^6) \to \pi^5_{6,4} \to \pi^4_{7,3} \to \pi_{10}(S^6) \to,$$

i.e.

$$\to Z_\infty \to \pi^5_{6,4} \to Z_4 \to 0,$$

by § 7.2 (e). Bearing in mind the result of the last paragraph, we see that $\pi^5_{6,4}$ is also an extension of Z_∞ by Z_4. Hence

$$\pi^5_{6,4} \textit{ is either } Z_\infty + Z_2 \textit{ or } Z_\infty + Z_4.$$

To determine which, we consider $\pi_{11}(P^9_5)$, $\approx \pi^5_{6,4}$ by Theorem 2.3 (g). When $k \geqslant 3$, let C^{k+2} be the space described in § 7.32 (f), and let D^{k+3}_λ be the space consisting of C^{k+2} to which a $(k+3)$-cell has been attached by a map ϕ_λ such that

$$\phi_\lambda | \dot{E}^{k+3} \to S^k \vee S^{k+1} \subset C^{k+2},$$

$$\{\delta\phi_\lambda\} = j_*\{h_{k+1,k+2}\} + i_*\lambda\{h_{k,k+2}\}.$$

It was shown in § (E) of the Appendix to (9) that, when $k \equiv 2 \pmod 4$ and $\geqslant 6$, $(P^{k+3}_{k-1}, P^{k+2}_{k-1})$ is of the same homotopy type as $(D^{k+3}_\lambda, C^{k+2})$ for some particular $\lambda = 0$ or 1. So consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{11}(C^8) & \xrightarrow{i_{11*}} & \pi_{11}(D^9_\lambda) & \xrightarrow{j_{11*}} & \pi_{11}(D^9_\lambda, C^8) & \xrightarrow{\delta_{11*}} & \pi_{10}(C^8) \to \\
\uparrow \mathfrak{E} & & \uparrow \mathfrak{E}_1 & & \uparrow \mathfrak{E}_2 & & \uparrow \mathfrak{E} \\
\to \pi_{10}(C^7) & \xrightarrow{i'_{10*}} & \pi_{10}(D^8_\lambda) & \xrightarrow{j'_{10*}} & \pi_{10}(D^8_\lambda, C^7) & \xrightarrow{\delta'_{10*}} & \pi_9(C^7) \to .
\end{array}$$

We know from § 7.32 (f) that $\pi_{11}(C^8)$ is $Z_\infty + Z_2$, and from above that $\pi_{11}(D^9_\lambda)$ is either $Z_\infty + Z_2$ or $Z_\infty + Z_4$. We first prove that $i^{-1}_{11*}(0) = 0$. Then we shall show that $\pi_{10}(D^8_\lambda)$ is finite, and that it contains an element a such that $j_{11*}\mathfrak{E}_1 a$ is non-zero. Thus $\mathfrak{E}_1 a$ is an element of $\pi_{11}(D^9_\lambda)$ which is non-zero, of finite order, and non-trivial under j_{11*}. Hence $\pi_{11}(D^9_\lambda)$ must be $Z_\infty + Z_4$.

The first assertion follows from the fact that

$$i^{-1}_{11*}(0) = \bar{i}_{11*}\{h_{7,8} + \lambda h_{6,8}\}_* \pi_{11}(S^8)$$

by Theorems 1 and 2 of (14) (where $\bar{i}$ denotes the identical injection of $S^6 \vee S^7$ into C^8), which is of finite order and yet a subgroup of $\bar{i}_{11*}\pi_{11}(S^7 \vee S^6)$ which equals Z_∞ by § 7.32 (f). Hence $i^{-1}_{11*}(0) = 0$.

Secondly, by § 7.32 (f), $\pi_{10}(C^7)$ has eight elements and, by Theorem 1 of (14),

$$\pi_{10}(D^8_\lambda, C^7) \approx \pi_9(S^7) = Z_2.$$

Thus $\pi_{10}(D^8_\lambda)$ is of finite order. Further, by Theorems 1 and 2 of (14),

$$\delta'_{10*}\pi_{10}(D^8_\lambda, C^7) = \bar{i}'_{9*}\{h_{6,7} + \lambda h_{5,7}\}_* \pi_9(S^7),$$

which is generated by

$$i'_{9*} h^*_{7,9}\{h_{6,7}+\lambda h_{5,7}\} = i'_{9*}(12\{h_{6,9}\}+12\lambda h_{5,8*}\{h_{8,9}\}) = i'_{9*}\, 12\{h_{6,9}\} = 0,$$

by the argument in § 7.32 (f). Thus $\delta'_{10*}(D^8_\lambda, C^7) = 0$, which implies that j'_{10*} is onto. But $\mathfrak{E}_2$ is an isomorphism by Theorem 3.2 (B). Thus $\tau_{10}(D^8_\lambda)$ contains an element a of finite order such that

$$j_{11*}\,\mathfrak{E}_1 a = \mathfrak{E}_2 j'_{10*}\, a \neq 0.$$

This completes the proof, and so we have that

$$\pi^5_{6,4} = Z_\infty + Z_4,$$

generated by $\{i_{7,3} h_{6,11}\}$ and a such that $p_{10,1*}\, a = \{h_{9,11}\}$ and $4a = 0$.

8.3. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

We first calculate

$$\{t_{k+4,4}\} \in \pi^2_{k,3} = Z_4 + Z_\infty$$

by §§ 7.1 and 5.2 (c). We have that

$$t_{k+4,4} | S^{k+1} = i_{k+2,1}\, t_{k+3,3}$$

by § 2.3 (b), and that $\{t_{k+3,3}\} = 0$ by § 7.1. Thus we can extend $i_{k+2,1}\, t_{k+3,3}$ over the hemisphere E^{k+2}_+ of S^{k+2}, and, since $t_{k+3,3}$ is a symmetric map (2.3 a), we can extend it symmetrically over E^{k+2}_-. Denote this extension by

$$g : S^{k+2} \to i_{k+2,1}\, V_{k+2,2} \subset V_{k+3,3}.$$

Now we use construction Q^r of § 6, with

$$r = k+2, \quad X = V_{k+3,3}, \quad f_1 = t_{k+4,4}, \quad f_2 = g.$$

Then we have $\quad 2\{h\} = \{f_1\}+\{f_2\} = \{t_{k+4,4}\}+\{g\}.$

Hence

$$p_{k+3,1*} 2\{h\} = p_{k+3,1*}\{t_{k+4,4}\}+p_{k+3,1*}\{g\}$$
$$= 2\{h_{k+2,k+2}\},$$

by § 2.3 (b) and since $\{g\} \in i_* \pi^2_{k,2}$. Thus

$$p_{k+3,1*}\{h\} = \{h_{k+2,k+2}\},$$

and

$$\{h\} = \{\mathrm{p} h_{k+2,k+2}\}+i_* w,$$

where $w \in \pi^2_{k,2}$.

Further $p_{k+2,1}\, g : S^{k+2} \to S^{k+1}$ is a symmetric map and such that

$$p_{k+2,1}\, g u^{-1}_{k+1} = p_{k+2,1}\, t_{k+3,3}\, u^{-1}_{k+1} : P^{k+1} \to S^{k+1},$$

which is essential by § 2.3 (c); whence $p_{k+2,1}\, g$ is essential by Theorem 6.1. Thus $\{g\}$ generates the Z_4 summand of $\pi^2_{k,3}$ (§§ 7.1 and 5.2 c).

Hence $\quad \{t_{k+4,4}\} = 2\{h\}-\{g\} = 2\{\mathrm{p} h_{k+2,k+2}\}+2i_* w-\{g\},$

i.e. $\quad \{t_{k+4,4}\} = 2\mathrm{p}_*\{h_{k+2,k+2}\}+i_{k+2,1*}\, a,$

where a is a generator of $\pi^2_{k,2}$, i.e.

$$p_{k+2,1*}a = \{h_{k+1,k+2}\}.$$

Thus we have the following:

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+2*}} \pi^2_{k,4} \to \pi_{k+2}(S^{k+3}) \to,$$

i.e.
$$\to Z_\infty \to Z_4 + Z_\infty \to \pi^2_{k,4} \to 0,$$

by §§ 7.1 and 5.2 (*c*), where $i^{-1}_{k+2*}(0)$ is generated by $\{t_{k+4,4}\}$. Using the above result we have that

$$\pi^2_{k,4} = Z_8,$$

generated by $\{i_{k+3,1}\, \mathrm{p}h_{k+2,k+2}\}$. Note that thus $\Delta^{-1}_*(0) = 0$, whence p_{k+3*} is trivial.

(*b*) When $p = 3$, (C) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_*} \pi^3_{k,3} \xrightarrow{i_{k+3*}} \pi^3_{k,4} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^3_{k,4} \to 0,$$

by §§ 7.1 and 5.2 (*d*), and since p_{k+3*} is trivial by (*a*). Also

$$i^{-1}_{k+3*}(0) = t_{k+4,4*}\,\pi_{k+3}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+3}\{t_{k+4,4}\} = h^*_{k+2,k+3}(i_{k+2,1*}a + 2\mathrm{p}_*\{h_{k+2,k+2}\})$$
$$= h^*_{k+2,k+3}\, i_{k+2,1*}a + 2\mathrm{p}_*\{h_{k+2,k+3}\} = i_{k+2,1*}\, h^*_{k+2,k+3}a.$$

But
$$p_{k+2,1*}\, h^*_{k+2,k+3}a = h^*_{k+2,k+3}\, p_{k+2,1*}a, \quad \text{by Lemma 3.1 } (b),$$
$$= h^*_{k+2,k+3}\{h_{k+1,k+2}\} = \{h_{k+1,k+3}\}.$$

Hence, by §§ 7.1 and 5.2 (*d*), $i^{-1}_{k+3*}(0)$ is a Z_2 summand of $\pi^3_{k,3}$. Thus

$$\pi^3_{k,4} = Z_2 + Z_2,$$

generated by $\{i_{k+1,3}\, h_{k,k+3}\}$ and $\{i_{k+3,1}\, \mathrm{p}h_{k+2,k+3}\}$. Note that thus

$$\Delta^{-1}_*(0) = 0,$$

whence p_{k+4*} is trivial.

(*c*) When $p = 4$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \to,$$

i.e.
$$\to Z_2 \to \pi^4_{k,3} \to \pi^4_{k,4} \to 0,$$

since p_{k+4*} is trivial by (*b*). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\,\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{t_{k+4,4}\} = h^*_{k+2,k+4}(i_*a+2p_*\{h_{k+2,k+2}\})$$
$$= h^*_{k+2,k+4}i_*a+2p_*\{h_{k+2,k+4}\} = i_*h^*_{k+2,k+4}a.$$

But
$$p_{k+2,1*}h^*_{k+2,k+4}a = h^*_{k+2,k+4}p_{k+2,1*}a \quad (3.1\,b)$$
$$= h^*_{k+2,k+4}\{h_{k+1,k+3}\} = 12\{h_{k+1,k+4}\}.$$

Hence, by §§ 7.1 and 5.2 (f) and (h), $i^{-1}_{k+4*}(0)$ is a Z_2 summand of $\pi^4_{k,3}$. Thus we have that

$$\pi^4_{k,4} = Z_2 \text{ for } k \geqslant 9; \quad \pi^4_{5,4} = Z_2+Z_2,$$

generated respectively by $\{i_{k+3,1}\,\mathrm{p}h_{k+2,k+4}\}$, and $\{i_{6,3}h_{5,9}\}$ and $\{i_{8,1}\,\mathrm{p}h_{7,9}\}$. Note that again p_{k+5*} is trivial.

(d) When $p = 5$ and $k \geqslant 9$, we use Theorem 4.2 (b) by which $\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_2$ by § 7.32 (d). Thus

$$\pi^5_{k,4} = Z_2 \quad \text{when } k \geqslant 9,$$

and is easily seen to be generated by $\{i_{k+3,1}\,\mathrm{p}h_{k+2,k+5}\}$.

(e) When $p = 5$ and $k = 5$, we consider the sequence associated with the fibring $V_{9,4}/S^5 \to V_{9,3}$, i.e.

$$\to \pi_{10}(S^5) \xrightarrow{i_{6,3*}} \pi^5_{5,4} \xrightarrow{p_{9,3*}} \pi^4_{6,3} \to \pi_9(S^5) \xrightarrow{i_{6,3*}},$$

i.e.
$$Z_2 \to \pi^5_{5,4} \to Z_2 \to Z_2 \to Z_2+Z_2 \to$$

by § 7.32 (d) and (c) above. But

(i) $i_{6,3*} = i_{7,2*}i_{6,1*}$ and $i_{6,1*}\pi_{10}(S^5) = 0$

by § 5.2 (k), whence $i_{6,3*}\pi_{10}(S^5) = 0$; and

(ii) $i_{6,3*}\pi_9(S^5)$ is non-trivial by (c) above.

Hence
$$\pi^5_{5,4} = Z_2,$$
generated by $\{i_{8,1}\,\mathrm{p}h_{7,10}\}$.

8.4. $k \equiv 3 \pmod 4$.

We have from § 2.3 (b) that

$$p_{k+3,1*}\{t_{k+4,4}\} = 2\{h_{k+2,k+2}\},$$

and so from § 7.2 (b) we have that $\{t_{k+4,4}\}$ generates an infinite summand of $\pi^2_{k,3} = Z_2+Z_\infty$. Thus we have the following:

(a) When $p = 2$, (C) gives

$$\xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+2*}} \pi^2_{k,4} \to \pi_{k+2}(S^{k+3}),$$

.e.
$$\to Z_\infty \to Z_2+Z_\infty \to \pi^2_{k,4} \to 0,$$

by § 7.2 (*b*), where $i_{k+2*}^{-1}(0)$ is generated by $\{t_{k+4,4}\}$. Using the above result we have that
$$\pi^2_{k,4} = Z_2,$$
generated by $i_{k+2,2*}a$, where $p_{k+2,1*}a = \{h_{k+1,k+2}\}$. Note that since $\Delta_*^{-1}(0) = 0$, p_{k+3*} is trivial.

(*b*) When $p = 3$ and $k \geqslant 7$, (C) gives
$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}) \xrightarrow{\Delta_*} \pi^2_{k,3} \xrightarrow{i_{k+3*}} \pi^2_{k,4} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+3}) \to,$$
i.e.
$$\to Z_2 \to Z_2 + Z_2 \to \pi^2_{k,4} \to 0,$$
by § 7.2 (*c*), and since p_{k+3*} is trivial by (*a*). Also
$$i_{k+3*}^{-1}(0) = t_{k+4,4*}\,\pi_{k+3}(S^{k+2}),$$
which is generated by $h^*_{k+2,k+2}\{t_{k+4,4}\}$. Any of the methods used previously in determining $i_{r*}^{-1}(0)$ will be seen to break down in this case. All we can say at the moment is that $\pi^2_{k,4}$ is either Z_2 or $Z_2 + Z_2$. I shall prove that it is the latter.

To determine $h^*_{k+2,k+3}\{t_{k+4,4}\}$, we consider first the sequence associated with the fibring $V_{k+3,3}/S^k \to V_{k+3,2}$, and operate with $h^*_{k+2,k+3}$ on the section
$$\to \pi^2_{k,3} \xrightarrow{p_{k+3,2*}} \pi^1_{k+1,2} \to,$$
giving the commutative diagram (3.1 *b*),
$$\begin{array}{ccc} \to \pi^3_{k,2} & \xrightarrow{p_{k+3,2*}} & \pi^2_{k+1,2} \to \\ \uparrow{\scriptstyle h^*} & & \uparrow{\scriptstyle h^*} \\ \to \pi^2_{k,3} & \xrightarrow{p_{k+3,2*}} & \pi^1_{k+1,2} \to. \end{array}$$
Then we have that
$$\begin{aligned} p_{k+3,2*}\,h^*\{t_{k+4,4}\} &= h^* p_{k+3,2*}\{t_{k+4,4}\} \\ &= h^*_{k+2,k+3}\{t_{k+4,3}\}, \quad \text{by § 2.3 } (b), \\ &= 0, \quad \text{by § 7.31 } (b). \end{aligned}$$
Hence
$$h^*\{t_{k+4,4}\} \in p_{k+3,2*}^{-1}(0).$$

Together with the above, consider now the commutative diagram (2.1),
$$\begin{array}{ccc} \to \pi^2_{k,3} & \xrightarrow{p_{k+3,2*}} & \pi^1_{k+1,2} \to \\ \uparrow{\scriptstyle i_{k+2,1*}} & & \uparrow{\scriptstyle i_{k+2,1*}} \\ \to \pi^2_{k,2} & \xrightarrow{p_{k+3,2*}} & \pi_{k+2}(S^{k+1}) \to. \end{array}$$

We have from § 7.2 (*b*) that, if a' is the generator of finite order in $\pi^2_{k,3} = Z_2 + Z_\infty$, then $a' = i_{k+2,1*}a$, where a generates $\pi^2_{k,2} = Z_4$. Further

$$p_{k+3,2*}a = \{h_{k+1,k+2}\}$$

by § 5.2 (*c*). Thus

$$\begin{aligned} p_{k+3,2*}h^*a' &= h^*p_{k+3,2*}a' = h^*p_{k+3,2*}i_{k+2,1*}a \\ &= h^*i_{k+2,1*}p_{k+3,2*}a \\ &= i_{k+2,1*}h^*_{k+2,k+3}\{h_{k+1,k+2}\} \\ &= i_{k+2,1*}\{h_{k+1,k+3}\} \\ &\neq 0, \end{aligned}$$

since $V_{k+3,2}/S^{k+1} \to S^{k+2}$ admits a cross-section (5.1). Hence

$$h^*a' \neq 0; \text{ and further, } h^*a' \neq h^*\{t_{k+4,4}\}.$$

We now turn our attention to P^{k+2}_{k-1}. By Theorem 2.3 (*g*),

$$\psi_{k+4,4*}: \pi_{k+2}(P^{k+2}_{k-1}) \to \pi^2_{k,3}$$

is an isomorphism for $k > 2$, and

$$\psi_{k+4,4*}: \pi_{k+3}(P^{k+2}_{k-1}) \to \pi^3_{k,3},$$

is an isomorphism for $k > 3$. Further, it was shown in (B) of the Appendix to (9) that, for $k \equiv 3 \pmod 4$, P^{k+2}_{k-1} is of the same homotopy type as B^{k+2}_2, the space defined in Corollary 3.2 (C). Hence we have a homotopy equivalence $i: P^{k+2}_{k-1} \to B^{k+2}_2$ when $k \equiv 3 \pmod 4$. Thus, dropping unambiguous subscripts, we have

$$\begin{aligned} h^*i_*\psi_*^{-1}\pi^2_{k,3} &= h^*i_*\pi_{k+2}(P^{k+2}_{k-1}) \\ &= h^*\pi_{k+2}(B^{k+2}_2) \\ &= h^*\mathfrak{E}^{k-3}\pi_5(B^5_2), \quad \text{by Corollary 3.2 (C)}, \\ &= \mathfrak{E}^{k-3}h^*_{5,6}\pi_5(B^5_2), \quad \text{by Lemma 3.1 } (b), \\ &\subset \mathfrak{E}^{k-3}\Omega, \end{aligned}$$

where Ω is the subgroup of $\pi_6(B^5_2)$ generated by the elements of order two. Now it will be shown in the Appendix that $\pi_6(B^5_2) = Z_4 + Z_2$, generated by $\bar{w}$ and $\bar{z}$ of orders 4 and 2 respectively, and further that

$\mathfrak{E}^2(2\bar{w}) = 0$. *Hence, since $k \geqslant 7$, there can be only one non-zero element in $h^*i_*\psi_*^{-1}\pi^2_{k,3}$, and that is $\mathfrak{E}^{k-3}(\bar{z})$.*

But
$$h^*i_*\psi_*^{-1}\pi^2_{k,3} = i_*\psi_*^{-1}h^*\pi^2_{k,3},$$

and i_* and ψ_*^{-1} are both isomorphisms. Hence there can be only one non-zero element in $h^*\pi^2_{k,3}$. But we have from above that $h^*a' \neq 0$, and that $h^*a' \neq h^*\{t_{k+4,4}\}$.

Thus $$h^*_{k+2,k+3}\{t_{k+4,4}\} = 0.$$

From this it follows that $\pi^2_{k,4} \approx \pi^2_{k,3}$, since we saw that

$$\pi^2_{k,4} \approx \pi^2_{k,3}/(\text{the subgroup generated by } h^*_{k+2,k+3}\{t_{k+4,4}\}).$$

Thus $$\pi^2_{k,4} = Z_2 + Z_2,$$

generated by $\{i_{k+1,3}h_{k,k+3}\}$, and $i_{k+2,2*}a$, where

$$p_{k+2,1*}a = \{h_{k+1,k+3}\}.$$

Note that, since Δ_* is trivial, p_{k+4*} is onto.

(*c*) When $p = 3$ and $k = 3$, (C) gives

$$\xrightarrow{p_{7*}} \pi_7(S^6) \xrightarrow{\Delta_*} \pi^3_{3,3} \xrightarrow{i_{6*}} \pi^3_{3,4} \xrightarrow{p_{6*}} \pi_6(S^6) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^3_{3,4} \to 0,$$

by § 7.2 (*d*) and since p_{6*} is trivial by (*a*). Hence $\pi^3_{3,4}$ is at most Z_2. I now show that it is at least Z_2. Consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which, starting with the term $\pi^3_{3,4}$, is of the form

$$\to \pi^3_{3,4} \to \pi^2_{4,3} \to \pi_5(S^3) \to \pi^2_{3,4} \to \pi^1_{4,3} \to \pi_4(S^3) \to,$$

i.e. $$\to \pi^3_{3,4} \to Z_2 + Z_2 \to Z_2 \to Z_2 \to Z_2 + Z_2 \to Z_2 \to$$

by the results of (*a*) above and of § 7.31 (*a*) and (*b*). From the exactness of this sequence we have that $\pi^3_{3,4}$ is at least Z_2. Hence

$$\pi^3_{3,4} = Z_2,$$

and is generated by $i_{5,3*}p^{-1}_{5,1*}\{h_{4,6}\}$. Note that, since Δ_* is trivial, p_{7*} is onto.

Further, since $i^{-1}_{6*}(0) = t_{7,4*}\pi_6(S^5)$ and since it also is zero from above, we have that $$h^*_{5,6}\{t_{7,4}\} = 0.$$

(*d*) When $p = 4$ and $k \geqslant 7$, (C) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \xrightarrow{\Delta_*} \pi^4_{k,3} \xrightarrow{i_{k+4*}} \pi^4_{k,4} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+3}),$$

i.e. $$\to Z_2 \to Z_4 \to \pi^4_{k,4} \to Z_2 \to 0,$$

by § 7.2 (*e*) and since p_{k+4*} is trivial by (*b*). Also

$$i^{-1}_{k+4*}(0) = t_{k+4,4*}\pi_{k+4}(S^{k+2}),$$

which is generated by

$$h^*_{k+2,k+4}\{t_{k+4,4}\} = h^*_{k+3,k+4}h^*_{k+2,k+3}\{t_{k+4,4}\} = 0$$

by (*b*).

Hence $\pi^4_{k,3}$ *is an extension of* Z_4 *by* Z_2.

Note that, since Δ_* is trivial, p_{k+5*} is onto.

To determine the extension consider the commutative diagram

$$\begin{array}{ccccccccc}
\rightarrow \pi_{k+4}(S^k) & \xrightarrow{i_{k+1,3*}} & \pi^4_{k,4} & \xrightarrow{p_{k+4,3*}} & \pi^3_{k+1,3} & \xrightarrow{\Delta_*} & \pi_{k+3}(S^k) \rightarrow \\
\uparrow i_{k+1,0*} & & \uparrow i_{k+3,1*} & & \uparrow i_{k+3,1*} & & \uparrow i_{k+1,0*} \\
\rightarrow \pi_{k+4}(S^k) & \xrightarrow{i_{k+1,2*}} & \pi^4_{k,3} & \xrightarrow{p_{k+3,2*}} & \pi^3_{k+1,2} & \xrightarrow{\Delta_*} & \pi_{k+3}(S^k) \rightarrow,
\end{array}$$

in which the horizontal sequences are associated with the fibrings

$$V_{k+4,4}/S^k \rightarrow V_{k+4,3}, \qquad V_{k+3,3}/S^k \rightarrow V_{k+3,2},$$

and the $i_{k+1,0*}$ are isomorphisms. Also, by § 7.31 (c) and by (d) above, the $i_{k+1,3*}$ are isomorphisms into. Then, when we use the results of §§ 7.2 (e), 5.1, and 7.31 (c), the diagram becomes

$$\begin{array}{ccccccccc}
\rightarrow 0 \rightarrow & \pi^4_{k,4} & \rightarrow & Z_{24}+Z_4 & \rightarrow & Z_{24} \rightarrow \\
\wr\wr & \uparrow & & \uparrow & & \wr\wr \\
\rightarrow 0 \rightarrow & Z_4 & \rightarrow & Z_{24}+Z_2 & \rightarrow & Z_{24} \rightarrow .
\end{array}$$

Let $a = p_*\{h_{k+2,k+4}\}$, the generator of a Z_2 summand of $\pi^3_{k+1,2} = Z_{24}+Z_2$, and b the generator of a Z_4 summand of $\pi^3_{k+1,3} = Z_{24}+Z_4$ as defined in § 7.31 (c). Then $i_{k+3,1*}\, a = 2b$. Now from the exactness of the lower sequence, or alternatively from § 7.2 (e), it follows that

$$\Delta_* a = 12\{h_{k,k+3}\}.$$

Hence

$$\Delta_* 2b = \Delta_* i_{k+3,1*}\, a = i_{k+1,0*} \Delta_* a = i_{k+1,0*}\, 12\{h_{k,k+3}\}.$$

Thus

$$\Delta_* 2b = 12\{h_{k,k+3}\} \neq 0.$$

But, if $\pi^4_{k,4}$ were Z_4+Z_2, $2b$ would be in $p_{k+4,3*}\pi^4_{k,4}$ since $2b$ is of order two and there are just three elements of order two in each of $Z_{24}+Z_4$ and the isomorphic image of Z_4+Z_2. Hence $\Delta_* 2b$ would have to be zero, and we have just proved the contrary. Thus

$$\pi^4_{k,4} = Z_8 \quad \text{for } k \geqslant 7,$$

generated by a such that $p_{k+4,1*}\, a = \{h_{k+3,k+4}\}$.

(e) When $p = 4$ and $k = 3$, (C) gives

$$\xrightarrow{p_{8*}} \pi_8(S^6) \xrightarrow{\Delta_*} \pi^4_{3,3} \xrightarrow{i_{7*}} \pi^4_{3,4} \xrightarrow{p_{7*}} \pi_7(S^6) \rightarrow,$$

i.e.

$$\rightarrow Z_2 \rightarrow Z_\infty + Z_4 \rightarrow \pi^4_{3,4} \rightarrow Z_2 \rightarrow 0,$$

by § 7.2 (f) and since p_{7*} is onto by (c). Further,

$$i_{7*}^{-1}(0) = t_{7,4*}\,\pi_7(S^5),$$

which is generated by

$$h^*_{5,7}\{t_{7,4}\} = h^*_{6,7}\, h^*_{5,6}\{t_{7,4}\} = 0,$$

by (*c*). Hence

$$\pi^4_{3,4} \textit{ is an extension of } Z_\infty + Z_4 \textit{ by } Z_2.$$

Note that Δ_* is trivial, whence p_{8*} is onto.

We determine the extension as in (*d*) above by considering the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_7(S^3) & \xrightarrow{i_{4,3*}} & \pi^4_{3,4} & \xrightarrow{p_{7,3*}} & \pi^3_{4,3} & \xrightarrow{\Delta_*} & \pi_6(S^3) \to \\
\uparrow{\scriptstyle i_{4,0*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{4,0*}} \\
\to \pi_7(S^3) & \xrightarrow{i_{4,2*}} & \pi^4_{3,3} & \xrightarrow{p_{6,2*}} & \pi^3_{4,2} & \xrightarrow{\Delta_*} & \pi_6(S^3) \to,
\end{array}$$

where the horizontal sequences are associated with the fibrings

$$V_{7,4}/S^3 \to V_{7,3}, \qquad V_{6,3}/S^3 \to V_{6,2},$$

and the $i_{4,0*}$ are isomorphisms. Also, by § 7.31 (*c*) and by (*e*) above, the $i_{6,1*}$ are isomorphisms into. We have further from § 7.2 (*f*) that $i_{4,2*}\pi_7(S^3) = 0$, whence it also follows that $i_{4,3*}\pi_7(S^3) = 0$. Then, by using the results of §§ 7.2 (*f*), 5.1, and 7.31 (*c*), the diagram becomes

$$\begin{array}{ccccccc}
0 \to & \pi^4_{3,4} & \to & Z_\infty + Z_{12} + Z_4 & \to & Z_{12} \to \\
 & \uparrow & & \uparrow & & \wr\wr \\
0 \to & Z_\infty + Z_4 & \to & Z_\infty + Z_{12} + Z_2 & \to & Z_{12} \to .
\end{array}$$

Let $a = p_*\{h_{5,7}\}$, the generator of a Z_2 summand in $\pi^3_{4,2}$, and b the generator of a Z_4 summand of $\pi^3_{4,3}$ as defined in § 7.31 (*c*). Then

$$i_{6,1*}a = 2b.$$

Further, we have from § 7.2 (*f*) that $\Delta_* a = 6\{h_{3,6}\}$. Hence

$$\Delta_* 2b = \Delta_* i_{6,1*} a = i_{4,0*}\Delta_* a = i_{4,0*} 6\{h_{3,6}\}.$$

Thus $$\Delta_* 2b = 6\{h_{3,6}\} \neq 0.$$

Now we had that $\pi^4_{3,4}$ is an extension of $Z_\infty + Z_4$ by Z_2; i.e. it is either $Z_\infty + Z_4 + Z_2$, $Z_\infty + Z_8$, or $Z_\infty + Z_4$. But, if $\pi^4_{3,4}$ were $Z_\infty + Z_4 + Z_2$, $2b$ would be in $p_{7,3*}\pi^4_{3,4}$ since $2b$ is of order two and there are just three elements of order two in each of $Z_\infty + Z_{12} + Z_4$ and the isomorphic image of $Z_\infty + Z_4 + Z_2$. Thus $\Delta_* 2b$ would have to be zero, and we have just proved the contrary. Thus $\pi^4_{3,4} \neq Z_\infty + Z_4 + Z_2$. Neither is it $Z_\infty + Z_8$ since it is impossible to map $Z_\infty + Z_8$ isomorphically into $Z_\infty + Z_{12} + Z_4$.

Hence $$\pi^4_{3,4} = Z_4 + Z_\infty,$$

generated by $i_{6,1*}a$, where a is of order four and $p_{6,1*}a = \{h_{5,7}\}$, and b, where $p_{7,1*}b = \{h_{6,7}\}$, of infinite order.

(*f*) When $p = 5$ and $k \geqslant 7$, (C) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+3}) \xrightarrow{\Delta_*} \pi^5_{k,3} \xrightarrow{i_{k+5*}} \pi^5_{k,4} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+3}) \to,$$

i.e. $$\to Z_{24} \to Z_{24} \to \pi^5_{k,4} \to Z_2 \to 0,$$

by § 7.2 (*g*) and since p_{k+5*} is onto by (*d*). Now we know from Theorem 4.2 (*b*) that $\pi^5_{k,4} \approx \pi^4_{k+1,3} = Z_2+Z_2$ by § 7.31 (*d*). Hence

$$\pi^5_{k,4} = Z_2+Z_2 \text{ when } k \geqslant 7,$$

and is generated by $i_{k+3,1*}\, p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$ and a such that

$$p_{k+4,1*}\, a = \{h_{k+3,k+5}\}.$$

Note that p_{k+6*} is thus onto the Z_2 subgroup of $\pi_{k+6}(S^{k+3})$.

(*g*) When $p = 5$ and $k = 3$, (C) gives

$$\xrightarrow{p_{9*}} \pi_9(S^6) \xrightarrow{\Delta_*} \pi^5_{3,3} \xrightarrow{i_{8*}} \pi^5_{3,4} \xrightarrow{p_{8*}} \pi_8(S^6) \to,$$

i.e. $$\to Z_{24} \to Z_2+Z_{24} \to \pi^5_{3,4} \to Z_2 \to 0,$$

by § 7.2 (*h*) and since p_{8*} is onto by (*e*). Further

$$i^{-1}_{8*}(0) = t_{7,4*}\, \pi_8(S^5),$$

which is generated by $h^*_{5,8}\{t_{7,4}\}$. But, by Lemma 3.1 (*b*), we have that

$$p_{6,1*}\, h^*_{5,8}\{t_{7,4}\} = h^*_{5,8}\, p_{6,1*}\{t_{7,4}\} = h^*_{5,8}\, 2\{h_{5,5}\}, \quad \text{by § 2.3 } (b),$$
$$= 2\{h_{5,8}\}.$$

Hence $$h^*_{5,8}\{t_{7,4}\} = 2a+\lambda b \quad (\lambda = 0 \text{ or } 1),$$

where a generates a Z_{24} summand of $\pi^5_{3,3}$, and b a Z_2 summand.

To determine λ we consider the sequence associated with the fibring $V_{6,3}/S^3 \to V_{6,2}$, which is of the form

$$\to \pi^5_{3,3} \xrightarrow{p_{6,2*}} \pi^4_{4,2} \xrightarrow{\Delta_*} \pi_7(S^3) \to,$$

i.e. $$\to Z_2+Z_{24} \to Z_2+Z_2+Z_{24} \to Z_2 \to,$$

by § 5.1, together with the sequence of § 5.1

$$0 \to \pi_8(S^4) \to \pi^4_{4,2} \underset{\mathrm{p}_*}{\overset{p_{6,1*}}{\rightleftarrows}} \pi_8(S^5) \to 0,$$

i.e. $$0 \to Z_2+Z_2 \to Z_2+Z_2+Z_{24} \longleftrightarrow Z_{24} \to 0.$$

Thus $$p_{6,1*}\, p_{6,2*}\, a = p_{6,1*}\, a = \{h_{5,8}\}.$$

Hence $$p_{6,2*}\, a = \mathrm{p}_*\{h_{5,8}\}+w,$$

where w is of order two. So

$$p_{6,2*}\, 2a = 2\mathrm{p}_*\{h_{5,8}\}.$$

Further, by § 2.4 (*b*), we have that

$$p_{6,2*}\{t_{7,4}\} = \{t_{7,3}\}.$$

Hence $$p_{6,2*}\, h^{*}_{5,8}\{t_{7,4}\} = h^{*}_{5,8}\{t_{7,3}\} = 2p_{*}\{h_{5,8}\},$$

by § 7.31 (*d*). Thus $$p_{6,2*}(2a+\lambda b) = p_{6,2*}\, 2a.$$

But $p_{6,2*}:\pi^{5}_{3,3}$ is an isomorphism into by exactness. Thus $\lambda = 0$, and

$$h^{*}_{5,8}\{t_{7,4}\} = 2a.$$

Hence $i^{-1}_{8*}(0)$ is the subgroup generated by $2a$, whence

$\pi^{5}_{3,4}$ is an extension of Z_2+Z_2 by Z_2.

Note that $\Delta_{*}\pi_9(S^6) = 2Z_{24}$, whence the image of p_{9*} is the Z_2 subgroup generated by $12\{h_{6,9}\}$.

To determine the extension, consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which is of the form:

$$\to \pi^{5}_{3,4} \xrightarrow{p_{7,3*}} \pi^{4}_{4,3} \xrightarrow{\Delta_*} \pi_7(S^3) \to,$$

i.e. $$\to \pi^{5}_{3,4} \to Z_2+Z_2+Z_2+Z_2 \to Z_2 \to$$

by § 7.31 (*d*). But $\pi^{5}_{3,4}$ has eight elements, whence by exactness $p_{7,3*}$ is an isomorphism into. Since $\pi^{4}_{4,3}$ has no elements of order greater than two, this implies that

$$\pi^{5}_{3,4} = Z_2+Z_2+Z_2,$$

and is generated by $i_{5,2*}\, p^{-1}_{5,1*}\, \mathfrak{E}\{h_{3,7}\}$, $i_{6,1*}\, a$, where $p_{6,1*}\, a = \{h_{5,8}\}$, and b, where $p_{7,1*}\, b = \{h_{6,8}\}$.

8.5. $k = 1$.

Again we first calculate

$$\{t_{5,4}\} \in \pi^{2}_{1,2} = Z_\infty + Z_\infty, \text{ by § 7.1.}$$

To do this we define a projection $p'\colon V_{4,3} \to V_{3,2}$ which maps $p^{-1}_{4,1}(s)$ homeomorphically onto $V_{3,2}$ for each $s \in S^3$ (which in R^4 has equation $x_0^2+x_1^2+x_2^2+x_3^2 = 1$). Let $v \in V_{4,3}$ be represented by

$$V = \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \end{bmatrix}.$$

Now form $$V^{*} = \begin{bmatrix} -v_{14} & v_{13} & -v_{12} & v_{11} \\ -v_{13} & -v_{14} & v_{11} & v_{12} \\ v_{12} & -v_{11} & -v_{14} & v_{13} \end{bmatrix}.$$

Then $V_1 = V \times V^{*\prime}$ is easily seen to represent a point $v_1 \in V_{3,2}$. So we define $p'\colon V_{4,3} \to V_{3,2}$ by $p'v = v_1$. Note that $p'i_{3,1} = 1\colon V_{3,2} \to V_{3,2}$, and further, that, if the cross-section $\mathrm{p}\colon S^3 \to V_{4,3}$ is chosen to be given by

$$\mathrm{p}(x_0, x_1, x_2, x_3) = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ x_2 & x_3 & -x_0 & -x_1 \end{bmatrix},$$

then $p'\mathrm{p}(S^3) = v_0$, the base point in $V_{3,2}$. Thus an element in $\pi_r(V_{4,3})$ is uniquely determined by its projection under p'_* and $p_{4,1*}$.

Now we have from § 2.2 that $t_{5,4}(x_0, x_1, x_2, x_3)$ is represented by

$$\begin{bmatrix} 2x_3x_0 & 2x_3x_1 & 2x_3x_2 & 2x_3^2-1 \\ 2x_2x_0 & 2x_2x_1 & 2x_2^2-1 & 2x_2x_3 \\ 2x_1x_0 & 2x_1^2-1 & 2x_1x_2 & 2x_1x_3 \end{bmatrix}.$$

Thus $p_{3,1}p't_{5,4}(x_0, x_1, x_2, x_3)$ is seen to be

$$\{2(x_2x_0+x_3x_1),\ 2(x_2x_1-x_3x_0),\ (x_3^2+x_2^2-x_1^2-x_0^2)\},$$

whence, by (7) 654, we see that $p_{3,1}p't_{5,4}\colon S^3 \to S^2$ is a map of Hopf invariant one. So

$$p_{3,1*}p'_*\{t_{5,4}\} = \{h_{2,3}\}.$$

But we have from § 2.3 (*b*) that $p_{4,1*}\{t_{5,4}\} = 2\{h_{3,3}\}$. Thus

$$\{t_{5,4}\} = i_{3,1*}p_{3,1*}^{-1}\{h_{2,3}\}+2\mathrm{p}_*\{h_{3,3}\}.$$

(*a*) When $p = 2$, (C) gives

$$\xrightarrow{p_{4*}} \pi_4(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^2 \xrightarrow{i_{3*}} \pi_{1,4}^2 \to \pi_3(S^4),$$

i.e.
$$\to Z_\infty \to Z_\infty + Z_\infty \to \pi_{1,4}^2 \to 0,$$

by § 7.1. Also $i_{3*}^{-1}(0)$ is generated by $\{t_{5,4}\}$. Using the above result, we have that

$$\pi_{1,4}^2 = Z_\infty,$$

generated by $\{i_{4,1}\mathrm{p}h_{3,3}\}$. Hence, by Theorem 4.2 (*a*), we also have that

$$\pi_{2,3}^1 = Z_\infty,$$

generated by $\{i_{4,1}\mathrm{p}h_{3,3}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{4*} is trivial.

(*b*) When $p = 3$, (C) gives

$$\xrightarrow{p_{5*}} \pi_5(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^3 \xrightarrow{i_{4*}} \pi_{1,4}^3 \xrightarrow{p_{4*}} \pi_4(S^4) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,4}^3 \to 0,$$

by § 7.1 and since p_{4*} is trivial by (*a*). Also

$$i_{4*}^{-1}(0) = t_{5,4*}\pi_4(S^3),$$

which is generated by

$$h_{3,4}^*\{t_{5,4}\} = h_{3,4}^*(i_{3,1*}p_{3,1*}^{-1}\{h_{2,3}\}+2\mathrm{p}_*\{h_{3,3}\}) = i_{3,1*}p_{3,1*}^{-1}\{h_{2,4}\}+2\mathrm{p}_*\{h_{3,4}\}$$
$$= i_{3,1*}p_{3,1*}^{-1}\{h_{2,4}\}.$$

Thus $i_{4*}^{-1}(0)$ is a Z_2 summand of $\pi_{1,3}^3$, whence

$$\pi_{1,4}^3 = Z_2,$$

generated by $\{i_{4,1}\,\mathrm{p}h_{3,4}\}$. Note that p_{5*} again is trivial. From Theorem 4.2 (a) we also have that

$$\pi_{2,3}^3 = Z_2,$$

generated by $\{i_{4,1}\,\mathrm{p}h_{3,4}\}$.

(c) When $p = 4$, (C) gives

$$\xrightarrow{p_{6*}} \pi_6(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^4 \xrightarrow{i_{5*}} \pi_{1,4}^4 \xrightarrow{p_{5*}} \pi_5(S^4) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,4}^4 \to 0,$$

by § 7.1 and since p_{5*} is trivial by (b). Also

$$i_{5*}^{-1}(0) = t_{5,4*}\,\pi_5(S^3),$$

which is generated by

$$\begin{aligned} h_{3,5}^*\{t_{5,4}\} &= h_{3,5}^*(i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathrm{p}_*\{h_{3,3}\}) \\ &= i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,5}\} + 2\mathrm{p}_*\{h_{3,5}\} = i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,5}\}. \end{aligned}$$

Thus $i_{5*}^{-1}(0)$ is a Z_2 summand of $\pi_{1,3}^4$, whence we have that

$$\pi_{1,4}^4 = Z_2,$$

generated by $\{i_{4,1}\,\mathrm{p}h_{3,5}\}$. Note that p_{6*} again is trivial. From Theorem 4.2 (a) we also have that

$$\pi_{2,3}^4 = Z_2,$$

generated by $\{i_{4,1}\,\mathrm{p}h_{3,5}\}$.

(d) When $p = 5$, (C) gives

$$\xrightarrow{p_{7*}} \pi_7(S^4) \xrightarrow{\Delta_*} \pi_{1,3}^5 \xrightarrow{i_{6*}} \pi_{1,4}^5 \xrightarrow{p_{6*}} \pi_6(S^4) \to,$$

i.e.

$$\to Z_\infty + Z_{12} \to Z_{12} + Z_{12} \to \pi_{1,4}^5 \to 0,$$

by § 7.1 and since p_{6*} is trivial by (c).

Unfortunately in this case we cannot make use of Theorem 1.3 since $\pi_7(S^4) \neq \mathfrak{E}\pi_6(S^3)$. On the other hand, Theorem 1.4 gives us that $\Delta_*\,\mathfrak{E}\pi_6(S^3) = t_{5,4*}\,\pi_6(S^3)$ which is generated by

$$\begin{aligned} h_{3,6}^*\{t_{5,4}\} &= h_{3,6}^*(i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,3}\} + 2\mathrm{p}_*\{h_{3,3}\}) \\ &= h_{3,6}^*\,i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,3}\} + 2h_{3,6}^*\,\mathrm{p}_*\{h_{3,3}\} \end{aligned}$$

[$h_{3,6}^*$ being a homomorphism since there is a multiplication on S^3: see (4) Theorem IV]

$$= i_{3,1*}\,p_{3,1*}^{-1}\{h_{2,6}\} + 2\mathrm{p}_*\{h_{3,6}\}.$$

Using the results of §§ 5.3 (b) and 7.1, we see that $\Delta_*\,\mathfrak{E}\pi_6(S^3)$ is the subgroup $\pi_{1,3}^5$ generated by $a+2b$, where a and b are generators of complementary Z_{12} summands. The order of $\pi_{1,4}^5$ thus depends on $\Delta_*\,\bar{p}_*\{h_{7,7}\}$. I have not been able to evaluate this directly. However,

A. Borel and J.-P. Serre showed in (2) Proposition 19.3 that $\pi^5_{1,4} = 0$. From this it follows that $\Delta_* \bar{p}_*\{h_{7,7}\}$ is an element which with $a+2b$ generates $\pi^5_{1,3}$, and so is of order twelve. Hence the image of p_{7*} is the Z_∞ subgroup of $\pi^5_{1,3}$ generated by $12\bar{p}_*\{h_{7,7}\}$.

The method used by A. Borel and J.-P. Serre is as follows. It is well known that $V_{n,n-1}$ is homeomorphic to R_n, the rotation group of S^{n-1}, and that the universal covering group of R_5 is the second symplectic group Sp_2. Thus $\pi^5_{1,4} \approx \pi_6(Sp_2)$. Now Sp_2 fibres over S^7 with fibre Sp_1 [see (11) 25.5], and the sequence associated with this fibring is of the form

$$\to \pi_7(S^7) \xrightarrow{\Delta_*} \pi_6(S^3) \xrightarrow{i_{6*}} \pi_6(Sp_2) \to \pi_6(S^7)$$

since $Sp_1 = S^3$. Now $i_{6*}^{-1}(0)$ is generated by the characteristic map of the above fibring, and A. Borel and J.-P. Serre show in Proposition 19.1 of (2) that this characteristic map is a generator of $\pi_6(S^3)$. Thus $\pi_6(Sp_2) = 0$, whence it follows that

$$\pi^5_{1,4} = 0 \quad \text{and} \quad \pi^4_{2,3} = 0,$$

the latter by virtue of Theorem 4.2 (a).

(e) When $p = 6$, (C) gives

$$\xrightarrow{p_{8*}} \pi_8(S^4) \xrightarrow{\Delta_*} \pi^6_{1,3} \xrightarrow{i_{7*}} \pi^6_{1,4} \xrightarrow{p_{7*}} \pi_7(S^4) \to,$$

i.e. $$\to Z_2+Z_2 \to Z_2+Z_2 \to \pi^6_{1,4} \to Z_\infty \to 0,$$

by § 7.1 and since the image of p_{7*} is a Z_∞ subgroup by (d). Again we cannot make use of Theorem 1.3. But, using the results of §§ 5.3 (b) and 7.1, we see that $\pi^6_{1,3}$ is generated by $i_{3,1*} p^{-1}_{3,1*}\{h_{2,7}\}$ and $\mathrm{p}_*\{h_{3,7}\}$. Now

$$\begin{aligned} i_{7*} i_{3,1*} p^{-1}_{3,1*}\{h_{2,7}\} &= i_{7*} i_{3,1*} p^{-1}_{3,1*} h^*_{6,7}\{h_{2,6}\} \\ &= h^*_{6,7} i_{6*} i_{3,1*} p^{-1}_{3,1*}\{h_{2,6}\}, \quad \text{by Lemma 3.1}\,(b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial by (d). Similarly

$$\begin{aligned} i_{7*}\, \mathrm{p}_*\{h_{3,7}\} &= i_{7*}\, \mathrm{p}_* h^*_{6,7}\{h_{3,6}\} \\ &= h^*_{6,7} i_{6*}\, \mathrm{p}_*\{h_{3,6}\}, \quad \text{by Lemma 3.1}\,(b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial. Thus $\qquad \pi^6_{1,4} = Z_\infty,$

generated by $p^{-1}_{5,1*} 12\bar{p}_*\{h_{7,7}\}$. Again, from Theorem 4.2 (a) we have

$$\pi^5_{2,3} = Z_\infty,$$

generated by $p^{-1}_{5,1*} 12\bar{p}_*\{h_{7,7}\}$. Note that, since i_{7*} is trivial, Δ_* is onto

$\pi^6_{1,3}$, whence it follows, since $\pi_8(S^4)$ is abstractly isomorphic to $\pi^6_{1,3}$, that $\Delta_*^{-1}(0) = 0$. Hence p_{8*} is trivial.

(*f*) When $p = 7$, (C) gives

$$\xrightarrow{p_{9*}} \pi_9(S^4) \xrightarrow{\Delta_*} \pi^7_{1,3} \xrightarrow{i_{8*}} \pi^7_{1,4} \xrightarrow{p_{8*}} \pi_8(S^4) \to,$$

i.e.
$$\to Z_2 + Z_2 \to Z_2 + Z_2 \to \pi^7_{1,4} \to 0,$$

by § 7.1 and since p_{8*} is trivial by (*e*). As above, instead of using Theorem 1.3, we note that, by §§ 5.3 (*b*) and 7.1, $\pi^7_{1,3}$ is generated by $i_{3,1*} p^{-1}_{3,1*}\{h_{2,8}\}$ and $p_*\{h_{3,8}\}$. Again

$$\begin{aligned} i_{8*} i_{3,1*} p^{-1}_{3,1*}\{h_{2,8}\} &= i_{8*} i_{3,1*} p^{-1}_{3,1*} h^*_{6,8}\{h_{2,6}\} \\ &= h^*_{6,8} i_{6*} i_{3,1*} p^{-1}_{3,1*}\{h_{2,6}\}, \quad \text{by Lemma 3.1}\,(b), \\ &= 0 \end{aligned}$$

since i_{6*} is trivial by (*d*). Similarly

$$\begin{aligned} i_{8*} p_*\{h_{3,8}\} &= i_{8*} p_* h^*_{6,8}\{h_{3,6}\} \\ &= h^*_{6,8} i_{6*} p_*\{h_{3,6}\} \\ &= 0. \end{aligned}$$

Thus i_{8*} is trivial, whence it follows that

$$\pi^7_{1,4} = 0 \quad \text{and} \quad \pi^6_{2,3} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that, by an argument similar to that at the end of (*e*), we have that p_{9*} is trivial.

Appendix

(A) *The group* $\pi_6(P^4_2)$

We have from § 2 that $V_{5,2}$ fibres over S^4 with fibre S^3. It is thus a seven-dimensional space, and the cellular decomposition of J. H. C. Whitehead's in **(13)** shows that in fact, besides the P^4_2 embedded in it, $V_{5,2}$ contains just one other cell, an E^7. Thus the pair $(V_{5,2}, P^4_2)$ is of the type considered in **(14)**, and its double sequence is of the form

$$\begin{tikzcd} \to \pi_7(E^7, S^6) \arrow[r] \arrow[d, "g_7"] & \pi_6(S^6) \arrow[r] \arrow[d, "h_6"] & \pi_6(E^7) \arrow[r] & \pi_6(E^7, S^6) \to \arrow[d, "g_6"] \\ \to \pi_7(V_{5,2}, P^4_2) \arrow[r] & \pi_6(P^4_2) \arrow[r] & \pi_6(V_{5,2}) \arrow[r] & \pi_6(V_{5,2}, P^4_2) \to. \end{tikzcd}$$

Now $\pi_7(E^7, S^6) = Z_\infty$ and $\pi_6(E^7, S^6) = 0$, and by Theorem I in **(14)** g_7 and g_6 are both isomorphisms. Also, by § 5.2 (*e*), $\pi_6(V_{5,2}) = Z_2$. Hence

$\pi_6(P^4_2)$ *is an extension of* Z_p *by* Z_2.

Consider next the commutative diagram

$$\begin{array}{ccccccccccc}
\to \pi_7(P_2^4, S^3) & \xrightarrow{\delta_*} & \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(P_2^4) & \xrightarrow{j_*} & \pi_6(P_2^4, S^3) & \xrightarrow{\delta_*} & \pi_5(S^3) \to \\
\downarrow p'_*\psi_{6,2*} & & \downarrow \psi_{5,2*} & & \downarrow \psi_{6,2*} & & \downarrow p'_*\psi_{6,2*} & & \downarrow \psi_{5,2*} \\
\to \pi_7(S^4) & \xrightarrow{\Delta_*} & \pi_6(S^3) & \xrightarrow{i_*} & \pi_6(V_{5,2}) & \xrightarrow{p_*} & \pi_6(S^4) & \xrightarrow{\Delta_*} & \pi_5(S^3) \to,
\end{array}$$

where the sequences are those of the pair (P_2^4, S^3) and the fibring $V_{5,2}/S^3 \to S^4$, and ψ and p'_* as defined in §§ 2.3 (d) and 1. From this follows first of all that $\delta_* \pi_6(P_2^4, S^3) = 0$, since $\psi_{5,2*}$ is an isomorphism and $\Delta_* \pi_6(S^4) = 0$ by § 5.2 (c).

Next we calculate $\pi_6(P_2^4, S^3)$. Let Y_2^{k+1} be the space consisting of an S^k to which one $(k+1)$-cell e^{k+1} has been attached by a map ϕ such that $\phi | \dot{E}^{k+1} \to S^k$ is of degree two. Then (Y_2^4, S^3) is of the same homotopy type as (P_2^4, S^3) [see (9) Appendix (A)]. Let S^{3*} be the closure of the space obtained from Y_2^4 by removing a 4-cell E^4 from the interior of the 4-cell of Y_2^4. Then (Y_2^4, S^{3*}) is of the same homotopy type as (Y_2^4, S^3). Consider thus the section of the upper sequence of the triad $(Y_2^4; E^4, S^{3*})$

$$\to \pi_6(E^4, \dot{E}^4) \xrightarrow{i_{6*}} \pi_6(Y_2^4, S^{3*}) \xrightarrow{j_{6*}} \pi_6(Y_2^4; E^4, S^{3*}) \xrightarrow{\delta_+} \pi_5(E^4, \dot{E}^4) \xrightarrow{i_{5*}}.$$

Since i_{r*} is homotopically equivalent to $g_r: \pi_r(E^4, \dot{E}^4) \to \pi_r(Y_2^4, S^3)$, the homomorphism induced by the attaching map ϕ, we have from Theorem 1.2 in (6) that both i_{5*} and i_{6*} are isomorphisms into, and moreover from Theorem 1.7 of the same paper that $i_{6*}\pi_6(E^4, \dot{E}^4)$ is a direct summand of $\pi_6(Y_2^4, S^{3*})$. Further, since $(S^{3*}, \dot{E}^4)$ is 2-connected and $(E^4, \dot{E}^4)$ is 3-connected, we have from Theorem 1 of (1) that

$$\pi_6(Y_2^4; E^4, S^{3*}) \approx \pi_3(S^{3*}, \dot{E}^4) \otimes \pi_4(E^4, \dot{E}^4).$$

But $\pi_4(E^4, \dot{E}^4) = Z_\infty$, and we have from the lower sequence of the above triad that $\pi_3(S^{3*}, \dot{E}^4) \approx \pi_3(Y_2^4, E^4)$, and from the homotopy sequence of the pair (Y_2^4, E^4) that $\pi_3(Y_2^4, E^4) \approx \pi_3(Y_2^4)$. Finally

$$\pi_3(Y_2^4) \approx \pi_3(P_2^4),$$

which by Theorem 2.3 (g) is isomorphic to $\pi_3(V_{5,2}) = Z_2$ by § 5.2 (a). Thus

$$\pi_6(Y_2^4; E^4, S^{3*}) \approx Z_2 \otimes Z_\infty = Z_2,$$

whence

$$\pi_6(P_2^4, S^3) \approx \pi_6(Y_2^4, S^3) = Z_2 + Z_2.$$

Finally consider the relative homotopy sequence of the triple $(V_{5,2}, P_2^4, S^3)$, which is of the form

$$\to \pi_7(P_2^4, S^3) \xrightarrow{p'_*\psi_{6,2*}} \pi_7(S^4) \xrightarrow{j_* p'^{-1}_*} \pi_7(V_{5,2}, P_2^4) \to \pi_6(P_2^4, S^3) \to,$$

i.e.

$$\to \pi_7(P_2^4, S^3) \to Z_\infty + Z_{12} \to Z_\infty \to Z_2 + Z_2 \to.$$

From this it follows by exactness that

$$p'_* \psi_{6,3*} \pi_7(P_2^4, S^3) = \mathfrak{E}\pi_6(S^3) \in \pi_7(S^4).$$

Hence, since $\psi_{5,2*}$ is an isomorphism, we have that

$$\delta_* \pi_7(P_2^4, S^3) = \psi_{5,2*}^{-1} \Delta_* \mathfrak{E}\pi_6(S^3) = 2\pi_6(S^3)$$

by § 5.2 (e). Thus $$i_* \pi_6(S^3) = Z_2.$$

From the above it follows that $\pi_6(P_2^4)$ is an extension of Z_2 by $Z_2 + Z_2$. But we saw that it was an extension of Z_p by Z_2. Hence

$$\pi_6(P_2^4) = Z_4 + Z_2.$$

We need to know something about the suspension properties of one of the elements in $\pi_6(P_2^4) \approx \pi_6(Y_2^4)$. So consider the diagram

$$\begin{array}{ccccc}
\to \pi_8(S^5) & \xrightarrow{i_{8*}} & \pi_8(Y_2^6) & \to \\
\uparrow \mathfrak{E}^2 & & \uparrow \mathfrak{E}^2 & \\
\to \pi_6(S^3) & \xrightarrow{i_{6*}} & \pi_6(Y_2^4) & \to
\end{array}$$

and let w and z generate respectively the Z_4 and Z_2 summands of $\pi_6(Y_2^4)$. Then $$2w = i_{6*}\{h_{3,6}\}.$$

Hence $$\mathfrak{E}^2\, 2w = \mathfrak{E}^2 i_{6*}\{h_{3,6}\} = i_{8*}\, \mathfrak{E}^2\{h_{3,6}\} = i_{8*}\, 2\{h_{5,8}\} = 0,$$

by (**14**) 4.5, since the attaching map is of degree two.

Thus $$\mathfrak{E}^2\, 2w = 0.$$

(B) *The group* $\pi_6(B_2^5)$

Let B_2^{k+2} be the space consisting of Y_2^{k+1} to which one $(k+2)$-cell has been attached by a map ϕ such that $\phi | \dot{E}^{k+2} : \dot{E}^{k+2} \to S^k \subset Y_2^{k+1}$ and is essential. Then it was shown in the appendix (B) of (9) that (B_2^{k+2}, Y_2^{k+1}) is of the same homotopy type as $(P_{k-1}^{k+2}, P_{k-1}^{k+1})$ when $k \equiv 3 \pmod 4$. Let

$$q : (B_2^5, Y_2^4) \to (P_2^5, P_2^4)$$

be a homotopy equivalence. Then we have the commutative diagram

$$\begin{array}{ccccccc}
\to \pi_6(Y_2^4) & \xrightarrow{i'_{6*}} & \pi_6(B_2^5) & \xrightarrow{j'_{6*}} & \pi_6(B_2^5, Y_2^4) & \to \\
\downarrow \psi_{6,3*}q_* & & \downarrow \psi_{7,4*}q_* & & \downarrow p'_*\psi_{7,4*}q_* & \\
\to \pi_6(V_{5,2}) & \xrightarrow{i_{6*}} & \pi_6(V_{6,3}) & \xrightarrow{p_{6*}} & \pi_6(S^5) & \to .
\end{array}$$

We have that

(a) $\psi_{7,4*} : \pi_6(P_2^5, P_2^4) \to \pi_6(V_{6,3}, V_{5,2})$ is an isomorphism by Lemma 2.3 (f), and so are p'_* and q_*. But p_{6*} is trivial by § 7.2 (b).

Thus $$j'_{6*} \textit{ is trivial.}$$

(*b*) By Theorem 1 in (3), $i_{6*}'^{-1}(0)$ is the union of $\phi_*\pi_6(S^4)$ and $[\phi_*\pi_4(S^4), \pi_3(Y_2^4)]$, where ϕ is the attaching map of the 5-cell in B_2^5. Thus

$$\phi_*\pi_6(S^4) = \bar{\imath}_* h_{3,4*}\pi_6(S^4) = \bar{\imath}_* 6\pi_6(S^3) = 0$$

by (A) above. Also

$$\pi_3(Y_2^4) = \bar{\imath}_*\pi_3(S^3)$$

since $\pi_3(Y_2^4, S^3) \approx \pi_2(S^3) = 0$. Thus

$$\begin{aligned}[\phi_*\pi_4(S^4), \pi_3(Y_2^4)] &= [\bar{\imath}_* h_{3,4*}\pi_4(S^4), \bar{\imath}_*\pi_3(S^3)] \\ &= \bar{\imath}_*[\pi_4(S^3), \pi_3(S^3)] \\ &= 0\end{aligned}$$

since all Whitehead products vanish on S^3.

Thus

$$i_{6*}'^{-1}(0) = 0.$$

Now let $\bar{w} = i_{6*}' w$, of order four, and $\bar{z} = i_{6*}' z$, of order two. Then

$$2\mathfrak{E}^2\bar{w} = \mathfrak{E}^2 i_{6*}' 2w = i_{8*}' \mathfrak{E}^2 2w = 0$$

by (A) above.

Hence $\pi_6(B_5^2) = Z_4 + Z_2$, *generated by* $\bar{w}$ *and* $\bar{z}$, *where* $\mathfrak{E}^2 2\bar{w} = 0$.

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