THE GROUPS $\pi_r(V_{n,m})$ (IV)

By G. F. PAECHTER (*Oxford*)

[Received 12 December 1957]

Introduction

THIS is the fourth of a sequence of five papers, the previous ones being (2), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. The present paper contains the calculations of those groups which are given in the following tables. There $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$. A full table of results can be found in (2) (I) 249. For the notation used throughout the body of this paper please see (2), especially §§ 1, 2, and 3.1. Also please note that sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

TABLE FOR $\pi^p_{k,5}$.

k	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
1	0	Z_∞	0	Z_∞	Z_{24}
3	Z_2	$Z_4+Z_\infty+Z_\infty$	$Z_2+Z_2+Z_2+Z_2$		
4	$Z_{12}+Z_4+Z_\infty$	$Z_2+Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_2$		
5	Z_2	Z_2+Z_∞	Z_2		
6	Z_{12}	Z_2	$Z_\infty+Z_2$ or Z_4+Z_∞		
$8s-1$	Z_8	Z_8+Z_∞	$Z_2+Z_2+Z_2$		
$8s+3$	Z_2+Z_2	Z_8+Z_∞	$Z_2+Z_2+Z_2$		
$4s+5$	Z_2	Z_∞	Z_2		
$8s$	$Z_{24}+Z_2$	Z_2+Z_2	Z_2		
$8s+4$	Z_4+Z_{48}	Z_2+Z_2	Z_2		
$4s+6$	Z_{12}	Z_2	Z_2		

TABLE FOR $\pi^p_{2,4}$.

$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
0	Z_∞	0	Z_∞	Z_{24}

9. Calculation of $\pi^p_{k,5}$

We consider the fibring $V_{k+5,5}/V_{k+4,4} \to S^{k+4}$ and examine the sequence

$$\text{(D)} \qquad \to \pi_{k+p+1}(S^{k+4}) \xrightarrow{\Delta_*} \pi^p_{k,4} \xrightarrow{i_{k+p*}} \pi^p_{k,5} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+4}) \to.$$

9.1. $k \equiv 3 \pmod 8$.

In this case there is a four-field on S^{k+4} (1, 4), and so the fibring admits a cross-section $\mathfrak{p}$. Hence Theorem 1.1 gives that

$$\pi^p_{k,5} = i_*\pi^p_{k,4} + \mathfrak{p}_*\pi_{k+p}(S^{k+4}).$$

Using the values of $\pi^p_{k,4}$ as calculated in § 8.4, we obtain the values shown in the table for $\pi^p_{k,5}$ when $k \equiv 3 \pmod 8$.

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+5,5}\} = 0 \quad \text{for } k \equiv 3 \pmod 8.$$

9.2. $k \equiv 7 \pmod 8$.

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_2 \to \pi^3_{k,5} \to 0,$$

by § 8.4 (*b*). But $i^{-1}_{k+3*}(0) \neq 0$ since otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a four-field on S^{k+4}, which is impossible by Theorem 1.1 of (3). Also $i^{-1}_{k+3*}(0) = \Delta_* Z_\infty$, and so must be cyclic. Hence

$$\pi^3_{k,5} = Z_2.$$

Note that $\Delta_* \pi_{k+4}(S^{k+4})$ is of order two, whence the image of p_{k+4*} is the Z_∞ subgroup generated by $2\{h_{k+4,k+4}\}$.

To determine the generator of $\pi^3_{k,5}$ we must evaluate $\{t_{k+5,5}\}$ which generates $i^{-1}_{k+3*}(0)$. Consider the sequence associated with the fibring $V_{k+4,4}/S^k \to V_{k+4,3}$, which is of the form

$$\to \pi_{k+3}(S^k) \xrightarrow{i_{k+1,3*}} \pi^3_{k,4} \xrightarrow{p_{k+4,3*}} \pi^3_{k+1,3} \to.$$

Then, by § 2.3 (*b*), $p_{k+4,3*}\{t_{k+5,5}\} = \{t_{k+5,4}\}$, which is zero by § 8.1. Hence, by exactness,

$$\{t_{k+5,5}\} \in i_{k+1,3*}\, \pi_{k+3}(S^k),$$

and is non-zero by the previous paragraph. Hence, using the result of § 8.4 (*b*), we have that

$$\{t_{k+5,5}\} = \{i_{k+1,3}\, h_{k,k+3}\},$$

and the generator of $\pi^3_{k,5}$ is $i_{k+2,3*}\, a$, where $p_{k+2,1*}\, a = \{h_{k+1,k+3}\}$.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_8 \to \pi^4_{k,5} \to Z_\infty \to 0,$$

by § 8.4 (*d*), and since the image of p_{k+4*} is a Z_∞ subgroup by (*a*). Also $i^{-1}_{k+4*}(0) = t_{k+5,5*}\, \pi_{k+4}(S^{k+3})$, which is generated by

$$h^*_{k+3,k+4}\{t_{k+5,5}\} = h^*_{k+3,k+4}\, i_{k+1,3*}\{h_{k,k+3}\} = i_{k+1,3*}\, h^*_{k+3,k+4}\{h_{k,k+3}\}$$
$$\in i_{k+1,3*}\, \pi_{k+4}(S^k) = 0 \quad \text{since } k \geqslant 7.$$

Hence $i_{k+4*}^{-1}(0) = 0$, whence

$$\pi_{k,5}^4 = Z_8 + Z_\infty,$$

generated by $i_{k+4,1*}a$, where $p_{k+4,1*}a = \{h_{k+3,k+4}\}$, and b such that $p_{k+5,1*}b = 2\{h_{k+4,k+4}\}$. Note that Δ_* is trivial, whence p_{k+5*} is onto.

(c) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^5 \xrightarrow{i_{k+5*}} \pi_{k,5}^5 \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e. $$\to Z_2 \to Z_2 + Z_2 \to \pi_{k,5}^5 \to Z_2 \to 0,$$

by § 8.4 (f), and since p_{k+5*} is onto by (b). But, by Theorem 4.2 (b),

$$\begin{aligned}\pi_{k,5}^5 &\approx \pi_{k+1,4}^4 \\ &= Z_2 + Z_2 + Z_2\end{aligned}$$

by § 8.1. Hence $i_{k+5*}^{-1}(0) = 0$, whence Δ_* is trivial and p_{k+6*} onto, and

$$\pi_{k,5}^5 = Z_2 + Z_2 + Z_2,$$

generated by $\quad i_{k+3,2*}\, p_{k+3,1*}^{-1}\{h_{k+2,k+5}\}, \quad i_{k+4,1*}a, \quad$ and b,
where $p_{k+4,1*}a = \{h_{k+3,k+5}\}$, and $p_{k+5,1*}b = \{h_{k+4,k+5}\}$.

9.3. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^3 \xrightarrow{i_{k+3*}} \pi_{k,5}^3 \to \pi_{k+3}(S^{k+4}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_2 \to \pi_{k,5}^3 \to 0,$$

by § 8.3 (b). As in § 9.2 (a) above, $i_{k+3*}^{-1}(0) \neq 0$ since otherwise the fibring $V_{k+5,5}/V_{k+4,4} \to S^{k+4}$ would admit a cross-section by Theorem 1.2, which would imply a four-field on S^{k+4}, which is impossible by Theorem 1.1 in (3). Also $i_{k+3*}^{-1}(0) = \Delta_* Z_\infty$, and so must be cyclic. Hence

$$\pi_{k,5}^3 = Z_2.$$

Note also that $\Delta_* \pi_{k+4}(S^{k+4})$ is of order two, whence the image of p_{k+4*} is the Z_∞ subgroup generated by $2\{h_{k+4,k+4}\}$.

To determine the generator of $\pi_{k,5}^3$ we must evaluate $\{t_{k+5,5}\}$ which generates $i_{k+3*}^{-1}(0)$. Consider the sequence associated with the fibring $V_{k+4,4}/S^k \to V_{k+4,3}$, which is of the form

$$\to \pi_{k+3}(S^k) \xrightarrow{i_{k+1,3*}} \pi_{k,4}^3 \xrightarrow{p_{k+4,3*}} \pi_{k+1,3}^2 \to.$$

Then, by § 2.3 (b),

$$\begin{aligned} p_{k+4,3*}\{t_{k+5,5}\} &= \{t_{k+5,4}\} \\ &= \{i_{k+3,1}\, p h_{k+2,k+3}\}, \quad \text{by 8.2 (a)}, \\ &\neq 0. \end{aligned}$$

Hence $$\{t_{k+5,5}\} \notin i_{k+1,3*}\, \pi_{k+3}(S^k),$$

whence, using the result of § 8.3 (*b*) we have that

$$\{t_{k+5,5}\} = \{i_{k+3,1}\,\mathrm{p}h_{k+2,k+3}\}+\lambda\{i_{k+1,3}\,h_{k,k+3}\}, \quad \textit{where } \lambda = 0 \textit{ or } 1.$$

Thus the generator of $\pi^3_{k,5}$ is $\{i_{k+1,4}\,h_{k,k+3}\}$.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^4_{k,5} \to Z_\infty \to 0 \quad (k > 5)$$

and $$\to Z_2 \to Z_2+Z_2 \to \pi^4_{5,5} \to Z_\infty \to 0 \quad (k = 5)$$

by § 8.3 (*c*), and since the image of p_{k+4*} is a Z_∞ subgroup by (*a*). Further, $i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3})$, which is generated by

$$\begin{aligned} h^*_{k+3,k+4}\{t_{k+5,5}\} &= h^*_{k+3,k+4}(i_{k+3,1*}\,\mathrm{p}_*\{h_{k+2,k+3}\}+\lambda i_{k+1,3*}\{h_{k,k+3}\}) \\ &= i_{k+3,1*}\,\mathrm{p}_*\,h^*_{k+3,k+4}\{h_{k+2,k+3}\}+\lambda i_{k+1,3*}\,h^*_{k+3,k+4}\{h_{k,k+3}\} \\ &= \begin{cases} i_{k+3,1*}\,\mathrm{p}_*\{h_{k+2,k+4}\} & (k \geqslant 9) \\ i_{8,1*}\,\mathrm{p}_*\{h_{7,9}\}+\lambda i_{6,3*}\{h_{5,9}\} & (k = 5). \end{cases} \end{aligned}$$

Thus $$\pi^4_{k,5} = Z_\infty \quad \textit{when } k \geqslant 9,$$

generated by a such that $p_{k+5,1*}\,a = 2\{h_{k+4,k+4}\}$;

and $$\pi^4_{5,5} = Z_2+Z_\infty,$$

generated by $\{i_{6,4}\,h_{5,9}\}$, and a such that $p_{10,1*}\,a = 2\{h_{9,9}\}$. Note that in either case $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e. $$\to Z_2 \to Z_2 \to \pi^5_{k,5} \to 0,$$

by §§ 8.3 (*d*) and (*e*), and since p_{k+5*} is trivial by (*b*). Also

$$i^{-1}_{k+5*}(0) = t_{k+5,5*}\,\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+5}\{t_{k+5,5}\} &= h^*_{k+3,k+5}(i_{k+3,1*}\,\mathrm{p}_*\{h_{k+2,k+3}\}+\lambda i_{k+1,3*}\{h_{k,k+3}\}) \\ &= i_{k+3,1*}\,\mathrm{p}_*\,h^*_{k+3,k+5}\{h_{k+2,k+3}\}+\lambda i_{k+1,3*}\,h^*_{k+3,k+5}\{h_{k,k+3}\} \\ &= i_{k+3,1*}\,\mathrm{p}_*\,12\{h_{k+2,k+5}\}, \end{aligned}$$

since $\pi_{k+5}(S^k) = 0$ $(k > 6)$, and $i_{6,3*}\,\pi_{10}(S^5) = 0$ $(k = 5)$ by § 8.3 (*e*),

$$= 0, \quad \text{since } \pi^5_{k,4} \text{ is of order two.}$$

Thus $i^{-1}_{k+5*}(0) = 0$ and $$\pi^5_{k,5} = Z_2,$$

generated by $\{i_{k+3,2}\,\mathrm{p}h_{k+2,k+5}\}$. Note that Δ_* is trivial, whence p_{k+6*} is onto.

9.4. $k = 1$.

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{5*}} \pi_5(S^5) \xrightarrow{\Delta_*} \pi^3_{1,4} \xrightarrow{i_{4*}} \pi^3_{1,5} \to \pi_4(S^5),$$

i.e.

$$\to Z_\infty \to Z_2 \to \pi^3_{1,5} \to 0,$$

by § 8.5 (*b*). Again $i_{4*}^{-1}(0) \neq 0$ since otherwise the fibring $V_{6,5}/V_{5,4} \to S^5$ would admit a cross-section by Theorem 1.2, which would imply a four-field on S^5, which is impossible by Theorem 1.1 in (3). Hence

$$\pi^3_{1,5} = 0 \quad \textit{and} \quad \pi^2_{2,4} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). But, by Corollary 1.5, $\{t_{6,5}\}$ generates $i_{4*}^{-1}(0)$, $= \pi^3_{1,4}$. Thus from § 8.5 (*b*) we have that

$$\{t_{6,5}\} = \{i_{4,1}\, \mathrm{p} h_{3,4}\}.$$

Note that the image of Δ_* is of order two, whence the image of p_{5*} is the Z_∞ subgroup generated by $2\{h_{5,5}\}$.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{6*}} \pi_6(S^5) \xrightarrow{\Delta_*} \pi^4_{1,4} \xrightarrow{i_{5*}} \pi^4_{1,5} \xrightarrow{p_{5*}} \pi_5(S^5) \to,$$

i.e.

$$\to Z_2 \to Z_2 \to \pi^4_{1,5} \to Z_\infty \to 0,$$

by § 8.5 (*c*), and since the image of p_{5*} is a Z_∞ subgroup by (*a*). Also

$$i_{5*}^{-1}(0) = t_{6,5*}\, \pi_5(S^4),$$

which is generated by

$$h^*_{5,4}\{t_{6,5}\} = h^*_{4,5}\, i_{4,1*}\, \mathrm{p}_*\{h_{3,4}\} = i_{4,1*}\, \mathrm{p}_*\{h_{3,5}\}, \text{ the generator of } \pi^4_{1,4}.$$

Thus

$$i_{5*}^{-1}(0) = \pi^4_{1,4},$$

whence

$$\pi^4_{1,5} = Z_\infty,$$

generated by $p_{6,1*}^{-1}\, 2\{h_{5,5}\}$. Theorem 4.2 (*a*) then gives that

$$\pi^3_{2,4} = Z_\infty,$$

generated by $p_{6,1*}^{-1}\, 2\{h_{5,5}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{6*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{7*}} \pi_7(S^5) \xrightarrow{\Delta_*} \pi^5_{1,4} \xrightarrow{i_{5*}} \pi^5_{1,5} \xrightarrow{p_{6*}} \pi_6(S^5) \to,$$

i.e.

$$\to Z_2 \to 0 \to \pi^5_{1,5} \to 0,$$

by § 8.5 (*d*), and since p_{6*} is trivial by (*b*). Thus

$$\pi^5_{1,5} = 0 \quad \textit{and} \quad \pi^4_{2,4} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that, since Δ_* is trivial, p_{7*} is onto.

(*d*) When $p = 6$ (D) gives

$$\xrightarrow{p_{8*}} \pi_8(S^5) \xrightarrow{\Delta_*} \pi^6_{1,4} \xrightarrow{i_{7*}} \pi^6_{1,5} \xrightarrow{p_{7*}} \pi_7(S^5) \rightarrow,$$

i.e.
$$\rightarrow Z_{24} \rightarrow Z_\infty \rightarrow \pi^6_{1,5} \rightarrow Z_2 \rightarrow 0,$$

by § 8.5 (*e*), and since p_{7*} is onto $\pi_7(S^5)$ by (*c*). Further, $i_{7*}^{-1}(0) = 0$, since it is impossible to map a finite group essentially into an infinite cyclic one. Thus $\pi^6_{1,5}$ *is an extension of* Z_∞ *by* Z_2, *as, by Theorem* 4.2 (*a*), *is* $\pi^5_{2,4}$. Note that Δ_* is trivial, whence p_{8*} is onto.

To calculate the extension we operate with $h^*_{r,r+1}$ on the section of the sequence associated with the fibring $V_{6,4}/S^2 \rightarrow V_{6,3}$ for which $r = 5$ and 6, to obtain the diagram

$$\begin{array}{ccccccccc}
\rightarrow \pi_7(S^2) & \xrightarrow{i_{7*}} & \pi^5_{2,4} & \xrightarrow{p_{7*}} & \pi^4_{3,3} & \xrightarrow{\Delta_{7*}} & \pi_6(S^2) \rightarrow \\
\uparrow h^* & & \uparrow h^* & & \uparrow h^* & & \uparrow h^* \\
\rightarrow \pi_6(S^2) & \xrightarrow{i_{6*}} & \pi^4_{2,4} & \xrightarrow{p_{6*}} & \pi^3_{3,3} & \xrightarrow{\Delta_{6*}} & \pi_5(S^2) \rightarrow,
\end{array}$$

which is commutative by Lemma 3.1 (*b*). Using the results of §§ 7:2 (*d*), (*f*), and (*c*) above, the diagram becomes

$$\begin{array}{ccccccc}
\rightarrow Z_2 & \rightarrow & \pi^5_{2,4} & \rightarrow & Z_\infty + Z_4 & \rightarrow & Z_{12} \rightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\rightarrow Z_{12} & \rightarrow & 0 & \rightarrow & Z_2 & \rightarrow & Z_2 \rightarrow.
\end{array}$$

Now let a be the generator of $\pi^3_{3,3}$ and a' a generator of order four in $\pi^4_{3,3}$. Then from the exactness of the lower line it follows that

$$\Delta_* a = \{h_{2,5}\}.$$

Hence
$$\Delta_* h^* a = h^* \Delta_* a = h^*_{5,6}\{h_{2,5}\} = 6\{h_{2,6}\}.$$

Thus $h^* a \neq 0$, whence it follows that

$$h^* a = 2a',$$

the only element of order two in $\pi^4_{3,3}$, and that

$$\Delta_* 2a' = 6\{h_{2,6}\} \neq 0.$$

Again,
$$i_{7*}\pi_7(S^2) = i_{7*} h^* \pi_6(S^2) = h^* i_{6*} \pi_6(S^2) = 0,$$

whence p_{7*} is an isomorphism into. Hence, if $\pi^5_{2,4}$ were $Z_\infty + Z_2$, $2a'$ would be in $p_{7*}\pi^5_{2,4}$ since $2a'$ is the only element of order two in $\pi^4_{3,3}$ and so must be the image of the element of order two in $\pi^5_{2,4}$. Thus $\Delta_* 2a'$ would have to be zero, and we have just proved the contrary. Hence

$$\pi^5_{2,4} = Z_\infty,$$

generated by a such that $p_{6,1*} a = \{h_{5,7}\}$, and

$$\pi^6_{1,5} = Z_\infty,$$

generated by a such that $p_{6,1*} a = \{h_{5,7}\}$.

(e) When $p = 7$, (D) gives

$$\xrightarrow{p_{9*}} \pi_9(S^5) \xrightarrow{\Delta_*} \pi^7_{1,4} \xrightarrow{i_{8*}} \pi^7_{1,5} \xrightarrow{p_{8*}} \pi_8(S^5) \to,$$

i.e.
$$\to Z_2 \to 0 \to \pi^7_{1,5} \to Z_{24} \to 0,$$

by § 8.5 (f), and since p_{8*} is onto $\pi_8(S^5)$ by (d). Thus

$$\pi^7_{1,5} = Z_{24},$$

generated by $p^{-1}_{6,1*}\{h_{5,8}\}$. Hence, by Theorem 4.2 (a), we have that

$$\pi^6_{2,4} = Z_{24},$$

generated by $p^{-1}_{6,1*}\{h_{5,8}\}$. Note that since Δ_* is trivial, p_{9*} is onto.

9.5. $k \equiv 2 \pmod 4$ *and* $\geqslant 6$.

We first calculate $\{t_{k+5,5}\} \in \pi^3_{k,4}$. We have from § 2.3 (b) that

$$p_{k+4,1*}\{t_{k+5,5}\} = 2\{h_{k+3,k+3}\}.$$

Using the result of § 8.2 (b) we thus have that $\{t_{k+5,5}\}$ generates a cyclic infinite summand of $\pi^3_{k,4} = Z_{12}+Z_\infty$.

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \xrightarrow{p_{k+3*}} \pi_{k+3}(S^{k+4}),$$

i.e.
$$\to Z_\infty \to Z_{12}+Z_\infty \to \pi^3_{k,5} \to 0,$$

by § 8.2 (b). Also $i^{-1}_{k+3*}(0)$ is generated by $\{t_{k+5,5}\}$, i.e. $i^{-1}_{k+3*}(0)$ is a cyclic infinite summand of $\pi^3_{k,4}$. Hence

$$\pi^3_{k,5} = Z_{12},$$

generated by $\{i_{k+1,4}\, h_{k,k+3}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{k+4*} is trivial.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^4_{k,5} \to 0,$$

by § 8.2 (c), and since p_{k+4*} is trivial by (a). Further,

$$i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3}),$$

which is generated by $h^*_{k+3,k+4}\{t_{k+5,5}\}$. To determine this we consider the section of the sequence associated with the fibring $V_{k+4,4}/S^k \to V_{k+4,3}$ which is of the form

$$\to \pi_{k+4}(S^k) \to \pi^4_{k,4} \xrightarrow{p_{k+4,3*}} \pi^3_{k+1,3} \to.$$

We have from § 2.3 (b) that

$$p_{k+4,3*}\{t_{k+5,5}\} = \{t_{k+5,4}\}.$$

Thus
$$p_{k+4,3*}h^*_{k+3,k+4}\{t_{k+5,5}\} = h^*_{k+3,k+4}p_{k+4,3*}\{t_{k+5,5}\}$$
$$= h^*_{k+3,k+4}\{t_{k+5,4}\}$$
$$= 0 \quad \text{by } 8.4\,(b).$$
Thus, since $\pi_{k+4}(S^k) = 0$ $(k \geqslant 6)$,
$$h^*_{k+3,k+4}\{t_{k+5,5}\} = 0.$$
Hence $i^{-1}_{k+4*}(0) = 0$, and so
$$\pi^4_{k,5} = Z_2,$$
generated by $\{i_{k+2,3}\,ph_{k+1,k+4}\}$. Note that Δ_* is trivial, whence p_{k+5*} is onto.

(*c*) When $p = 5$, and $k \geqslant 10$, (D) gives
$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \rightarrow,$$
i.e.
$$\rightarrow Z_2 \rightarrow Z_4 \rightarrow \pi^5_{k,5} \rightarrow Z_2 \rightarrow 0,$$
by § 8.2 (*d*), and since p_{k+5*} is onto $\pi_{k+5}(S^{k+4})$ by (*b*). But we have from Theorem 4.2 (*b*) that
$$\pi^5_{k,5} \approx \pi^4_{k+1,4}$$
$$= Z_8$$
by § 8.4 (*d*). Hence $i^{-1}_{k+5*}(0) = 0$, whence Δ_* is trivial and so p_{k+6*} is onto, and
$$\pi^5_{k,5} = Z_8,$$
generated by a such that $p_{k+5,1*}\,a = \{h_{k+4,k+5}\}$.

(*d*) When $p = 5$, and $k = 6$, (D) gives
$$\xrightarrow{p_{12*}} \pi_{12}(S^{10}) \xrightarrow{\Delta_*} \pi^5_{6,4} \xrightarrow{i_{11*}} \pi^5_{6,5} \xrightarrow{p_{11*}} \pi_{11}(S^{10}) \rightarrow,$$
i.e.
$$\rightarrow Z_2 \rightarrow Z_\infty + Z_4 \rightarrow \pi^5_{6,5} \rightarrow Z_2 \rightarrow 0,$$
by § 8.2 (*e*), and since p_{11*} is onto $\pi_{11}(S^{10})$ by (*b*). Further,
$$i^{-1}_{11*}(0) = t_{11,5*}\,\pi_{11}(S^9),$$
which is generated by
$$h^*_{9,11}\{t_{11,5}\} = h^*_{10,11}\,h^*_{9,10}\{t_{11,5}\} = 0$$
by (*b*) above. Thus $i^{-1}_{11*}(0) = 0$, whence
$$\pi^5_{6,5} \textit{ is an extension of } Z_\infty + Z_4 \textit{ by } Z_2.$$
Note that Δ_* is trivial, whence p_{12*} is onto.

To determine the extension we consider first the section of the sequence associated with the fibring $V_{11,5}/S^6 \rightarrow V_{11,4}$, which is of the form
$$\rightarrow \pi_{11}(S^6) \rightarrow \pi^5_{6,5} \rightarrow \pi^4_{7,4} \rightarrow \pi_{10}(S^6) \rightarrow,$$

which becomes, with the result of § 8.4 (d),

$$\rightarrow Z_\infty \rightarrow \pi^5_{6,5} \rightarrow Z_8 \rightarrow 0.$$

From this we see, bearing in mind the result of the last paragraph, that $\pi^5_{6,5}$ is also an extension of Z_∞ by Z_8. Thus we have two possibilities for $\pi^5_{6,5}$: $Z_\infty + Z_8$ or $Z_4 + Z_\infty$. But the method used in §§ 7.32 (f) and 8.2 (e) does not yield a result in this case, and so all we can say is that

$$\pi^5_{6,5} \textit{ is either } Z_\infty + Z_8 \textit{ or } Z_4 + Z_\infty,$$

generated respectively by $\{i_{7,4} h_{6,11}\}$, and a of order eight such that $p_{11,1*} a = \{h_{10,11}\}$; or $i_{10,1*} a$, where a is of order four and $p_{10,1*} a = \{h_{9,11}\}$, and b such that $p_{11,1*} b = \{h_{10,11}\}$.

9.6. $k \equiv 0 \pmod 4$.

Our first task is to calculate $\{t_{k+5,5}\}$ in $\pi^3_{k,4}$. We have that

$$t_{k+5,5} | S^{k+2} = i_{k+3,1} t_{k+4,4}$$

by § 2.3 (b), and that $$\{t_{k+4,4}\} = 0$$

by § 8.1. Thus we can extend $i_{k+3,1} t_{k+4,4}$ over the hemisphere E^{k+3}_+ of S^{k+3}, and, since $t_{k+4,4}$ is a symmetric map (2.3 a), we can extend it symmetrically over E^{k+3}_-. Denote this extension by

$$g : S^{k+3} \rightarrow i_{k+3,1}(V_{k+3,3}) \subset V_{k+4,4}.$$

Now we use construction 'Q' of § 6, with $r = k+3$, $X = V_{k+4,4}$, $f_1 = t_{k+5,5}$, and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\} + \{f_2\} = \{t_{k+5,5}\} + \{g\}.$$

Hence

$$p_{k+4,1*} 2\{h\} = p_{k+4,1*}\{t_{k+5,5}\} + p_{k+4,1*}\{g\} = 2\{h_{k+3,k+3}\},$$

by § 2.3 (b) and since $\{g\} \in i_{k+3,1*} \pi^3_{k,3}$. Thus

$$p_{k+4,1*}\{h\} = \{h_{k+3,k+3}\},$$

and $$\{h\} = \{\mathrm{p} h_{k+3,k+3}\} + i_{k+3,1*} w, \quad \textit{where } w \in \pi^3_{k,3}.$$

Further, if we consider $\{g\}$ for the moment as in $\pi^3_{k,3}$, we see that $p_{k+3,1} g : S^{k+3} \rightarrow S^{k+2}$ is a symmetric map such that, in the notation of § 2.3 (c),

$$p_{k+3,1} g u^{-1}_{k+2} = p_{k+3,1} t_{k+4,4} u^{-1}_{k+2} : P^{k+2} \rightarrow S^{k+2},$$

which is essential by § 2.3 (c); whence $p_{k+3,1} g$ is essential by Theorem 6.1. We thus have, using the results of § 7.31 (c), that, in $\pi^3_{k,3}$,

$$\{g\} = (i_{k+1,2*} \bar{x} + \bar{z}),$$

where $\bar{x} \in \pi_{k+3}(S^k)$, and $\bar{z}$ generates a Z_4 summand with

$$2\bar{z} \in i_{k+2,1*} \mathrm{p}_* \pi_{k+3}(S^{k+1}).$$

Thus

$$\{t_{k+5,5}\} = 2\{h\}-\{g\} = 2\{\mathrm{p}h_{k+3,k+3}\}+2i_{k+3,1*}w-i_{k+1,3*}\bar{x}-i_{k+3,1*}\bar{z}.$$

Hence $$\{t_{k+5,5}\} = i_{k+1,3*}x+i_{k+3,1*}z+2\mathrm{p}_*\{h_{k+3,k+3}\},$$

where $x \in \pi_{k+3}(S^k)$ and z generates a Z_4 summand of $\pi^3_{k,3}$ with

$$2z = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+3}\}.$$

What remains to be done is to determine x (mod 2) in $\pi_{k+3}(S^k)$. We shall see that there is a distinction between the cases $k \equiv 0$ and 4 (mod 8), quite apart from the special case $k = 4$.

9.61. $k \equiv 0$ (mod 8).

First let us pick generators in $\pi^3_{k,3}$, $\pi^3_{k,4}$, $\pi^4_{k-1,4}$, $\pi^4_{k-1,5}$. By § 7.31 (c), $\pi^3_{k,3} = Z_{24}+Z_4$. Pick generators

$\bar{a}$, of order twenty-four, as $i_{k+1,2*}\{h_{k,k+3}\}$;

$\bar{b}$, of order four, as the z defined in § 9.6 above.

By §§ 8.1 and 7.31 (c), $\pi^3_{k,4} = Z_{24}+Z_4+Z_\infty$. Pick generators

a, of order twenty-four, as $i_{k+3,1*}\bar{a}$;

b, of order four, as $i_{k+3,1*}\bar{b}$;

c, of infinite order, as $\mathrm{p}_*\{h_{k+3,k+3}\}$.

By § 8.4 (d), $\pi^4_{k-1,4} = Z_8$. Let

$\bar{v}$ be any particular generator.

By § 9.2 (b), $\pi^4_{k-1,5} = Z_8+Z_\infty$. Pick generators

v, of order eight, as $i_{k+3,1*}\bar{v}$;

w, of infinite order, as $\{t_{k+5,6}\}$.

This latter choice is possible since

$$p_{k+4,1*}\{t_{k+5,6}\} = 2\{h_{k+3,k+3}\}$$

by § 2.3 (b).

Now consider the diagram:

$$\begin{tikzcd} \rightarrow \pi_{k+3}(S^{k-1}) \arrow[r, "i_{k,3*}"] & \pi^4_{k-1,4} \arrow[r, "p_{k+3,3*}"] \arrow[dr, "p_{k+3,1*}"'] & \pi^3_{k,3} \rightarrow, \arrow[d, "p_{k+3,1*}"] \\ & & \pi_{k+3}(S^{k+2}) \end{tikzcd}$$

where the horizontal sequence is associated with the fibring

$$V_{k+3,4}/S^{k-1} \rightarrow V_{k+3,3},$$

and the triangle is commutative since $p_{k+3,1}p_{k+3,3} = p_{k+3,1}$. Further, since $\pi_{k+3}(S^{k-1}) = 0$, $p_{k+3,3*}$ is a monomorphism. Thus

$$p_{k+3,3*}\bar{v} = 3\epsilon\bar{a}+\mu\bar{b},$$

where $\epsilon \equiv 1 \pmod 2$, and μ unknown. But

$$p_{k+3,1*}\bar{v} = \{h_{k+2,k+3}\},$$

by § 8.4 (*d*) and

$$p_{k+3,1*}\bar{a} = 0, \quad p_{k+3,1*}\bar{b} = \{h_{k+2,k+3}\}, \quad p_{k+3,1*}2\bar{b} = 0,$$

by § 7.31 (*c*). Hence we have by commutability that μ is odd, i.e.

$$p_{k+3,3*}\bar{v} = 3\epsilon\bar{a}\mp\bar{b}, \quad \textit{where } \epsilon \equiv 1 \pmod 2).$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{k+3}(S^{k-1}) & \xrightarrow{i_{k,3*}} & \pi^4_{k-1,4} & \xrightarrow{p_{k+3,3*}} & \pi^3_{k,3} & \xrightarrow{\Delta_*} & \pi_{k+2}(S^{k-1}) \to \\
\downarrow{\scriptstyle i_{k,0*}} & & \downarrow{\scriptstyle i_{k+3,1*}} & & \downarrow{\scriptstyle i_{k+3,1*}} & & \downarrow{\scriptstyle i_{k,0*}} \\
\to \pi_{k+3}(S^{k-1}) & \xrightarrow{i_{k,4*}} & \pi^4_{k-1,5} & \xrightarrow{p_{k+4,4*}} & \pi^3_{k,4} & \xrightarrow{\Delta_*} & \pi_{k+2}(S^{k-1}) \to,
\end{array}$$

in which the horizontal sequences are associated with the fibrings $V_{k+3,4}/S^{k-1} \to V_{k+3,3}$ and $V_{k+4,5}/S^{k-1} \to V_{k+4,4}$. Substitution for the groups changes the diagram into

$$\begin{array}{ccccccc}
\to 0 \to & Z_8 & \to & Z_{24}+Z_4 & \to Z_{24} \to \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
\to 0 \to & Z_8+Z_\infty & \to & Z_{24}+Z_4+Z_\infty & \to Z_{24} \to.
\end{array}$$

Thus

$$\begin{aligned}
p_{k+4,4*}v &= p_{k+4,4*}i_{k+3,1*}\bar{v} = i_{k+3,1*}p_{k+3,3*}\bar{v} \\
&= i_{k+3,1*}(3\bar{a}\mp\bar{b}) \\
&= 3a\mp b.
\end{aligned}$$

Further, since by § 2.3 (*b*) $p_{k+4,4*}\{t_{k+5,6}\} = \{t_{k+5,5}\}$, we have from § 9.6 that

$$p_{k+4,4*}w = \lambda a+b+2c,$$

where λ is to be determined.

Now the next stage of the lower sequence is

$$\xrightarrow{\Delta_*} \pi_{k+2}(S^{k-1}) \to \pi^3_{k-1,5} \to \pi^2_{k,4} \xrightarrow{\Delta_*} \pi_{k+1}(S^{k-1}) \to,$$

which becomes, with the results of §§ 9.2 (*a*) and 8.1 and 7.31 (*b*),

$$\to Z_{24} \to Z_2 \to Z_2+Z_2 \to Z_2 \to,$$

whence we have by exactness that $\Delta_*\pi^3_{k,4} = \pi_{k+2}(S^{k-1})$.

To recapitulate, the position now is this:

$$\begin{array}{lccccc}
 & 0 \to Z_8+Z_\infty & \xrightarrow{p_{k+4,4*}} & Z_{24}+Z_4+Z_\infty \to & Z_{24} & \to 0 \\
\text{generated by} & v, \quad w, & & a, \quad b, \quad c, & \{h_{k-1,k+2}\}, &
\end{array}$$

where

$$\begin{aligned}
p_{k+4,4*}v &= 3\epsilon a\mp b \qquad (\epsilon \equiv 1 \pmod 2)), \\
p_{k+4,4*}w &= \lambda a+b+2c \quad (\lambda \textit{ to be determined}).
\end{aligned}$$

Hence the factor group $\pi^3_{k,4}/p_{k+4,4*}\pi^4_{k-1,5}$ is generated by a, b, c with the relations:

$$24a = 0, \quad 4b = 0, \quad 3\epsilon a \mp b \equiv 0, \quad \lambda a+b+2c \equiv 0.$$

Hence
$$4(3\epsilon a \mp b) \equiv 0;$$
whence
$$12a \equiv 0.$$
Further
$$2(3\epsilon a \mp b) \equiv 0;$$
whence
$$6a+2b \equiv 0.$$
But
$$6(\lambda a+b+2c) \equiv 0;$$
whence
$$6\lambda' a+2b+12c \equiv 0,$$
where

$$\lambda' = 1 \text{ if } \lambda \equiv 1 \pmod 2 \quad \text{and} \quad \lambda' = 0 \text{ if } \lambda \equiv 0 \pmod 2.$$

But, if $\lambda' = 1$, we have, since $6a+2b \equiv 0$, that $12c \equiv 0$, i.e.

$$12a \equiv 12b \equiv 12c \equiv 0,$$

and the factor group cannot possibly be cyclic of order 24. Hence

$$\lambda' = 0, \quad \text{and thus} \quad \lambda \equiv 0 \pmod 2),$$

i.e.
$$\lambda = 2\sigma.$$
Thus
$$\{\iota_{k+5,5}\} = b+2(\sigma a+c).$$

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e.
$$\to Z_\infty \to Z_{24}+Z_4+Z_\infty \to \pi^3_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (c). But i^{-1}_{k+3*} is generated by $\{\iota_{k+5,5}\}$, that is by $b+2(\sigma a+c)$ in the notation of the last paragraph. Hence

$$\pi^3_{k,5} = Z_{24}+Z_8,$$

generated by $\{i_{k+1,4}\, h_{k,k+3}\}$ and $\{i_{k+4,1}\, \mathrm{p} h_{k+3\,k+3}\}$. Note that, since $\Delta_*^{-1}(0) = 0$, p_{k+4*} is trivial.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2 \to \pi^4_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (*d*), and since p_{k+4*} is trivial by (*a*). Also

$$i_{k+4*}^{-1}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h_{k+3,k+4}^*[b+2(\sigma a+c)] &= h_{k+3,k+4}^*\,b+2(\sigma a+c)_*\{h_{k+3,k+4}\} \\ &= h_{k+3,k+4}^*\,b \\ &= h_{k+3,k+4}^*\,i_{k+3,1*}\,\bar{b} = i_{k+3,1*}\,h_{k+3,k+4}^*\,\bar{b}, \end{aligned}$$

by Lemma 3.1 (*b*). But

$$\begin{aligned} p_{k+3,1*}\,h_{k+3,k+4}^*\,\bar{b} = h_{k+3,k+4}^*\,p_{k+3,1*}\,\bar{b} &= h_{k+3,k+4}^*\{h_{k+2,k+3}\} \\ &= \{h_{k+2,k+4}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (*d*), that $i_{k+4*}^{-1}(0)$ is a Z_2 subgroup of $\pi_{k,4}^4$, and that

$$\pi_{k,5}^4 = Z_2+Z_2,$$

generated by $\{i_{k+2,3}\,\mathrm{p}h_{k+1,k+4}\}$ and $\{i_{k+4,1}\,\mathrm{p}h_{k+3,k+4}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi_{k,4}^5 \xrightarrow{i_{k+5*}} \pi_{k,5}^5 \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e.

$$\to Z_2 \to Z_2+Z_2 \to \pi_{k,5}^5 \to 0$$

by § 8.1 and § 7.31 (*e*), and since p_{k+5*} is trivial by (*b*). Further

$$i_{k+5*}^{-1}(0) = t_{k+5,5*}\,\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h_{k+3,k+5}^*[b+2(\sigma a+c)] &= h_{k+3,k+5}^*\,b+2(\sigma a+c)_*\{h_{k+3,k+5}\} \\ &= h_{k+3,k+5}^*\,b \\ &= h_{k+3,k+5}^*\,i_{k+3,1*}\,\bar{b} = i_{k+3,1*}\,h_{k+3,k+5}^*\,\bar{b}. \end{aligned}$$

But

$$\begin{aligned} p_{k+3,1*}\,h_{k+3,k+5}^*\,\bar{b} = h_{k+3,k+5}^*\,p_{k+3,1*}\,\bar{b} &= h_{k+3,k+5}^*\{h_{k+2,k+3}\} \\ &= 12\{h_{k+2,k+5}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (*e*), that $i_{k+5*}^{-1}(0)$ is a Z_2 subgroup of $\pi_{k,4}^5$, and that

$$\pi_{k,5}^5 = Z_2,$$

generated by $\{i_{k+4,1}\,\mathrm{p}h_{k+3,k+5}\}$. Note that again p_{k+6*} is trivial.

9.62. $k \equiv 4 \pmod 8$ *and* $\geqslant 12$.

We have from § 9.1 that $\pi_{k-1,5}^4 \approx \pi_{k-1,4}^4+Z_\infty$, where the infinite summand is generated by $\mathrm{p}_*\{h_{k+3,k+3}\}$, and from § 8.4 (*d*) that $\pi_{k-1,4}^4 = Z_8$.

I shall show in § 10.7 that

$$\{t_{k+5,6}\} = v + 2\mathrm{p}_*\{h_{k+3,k+3}\} \in \pi^4_{k-1,5},$$

where v generates the finite summand. Thus we can pick generators in $\pi^4_{k-1,5}$ as

v, of order eight, as defined above by $\{t_{k+5,6}\}$;

w, of infinite order, as $\mathrm{p}_*\{h_{k+3,k+3}\}$.

By §§ 8.1 and 7.31 (c), $\pi^3_{k,4} = Z_{24}+Z_4+Z_\infty$. Pick generators

a, of order twenty-four, as $i_{k+1,3*}\{h_{k,k+3}\}$;

b, of order four, as $i_{k+3,1*}z$, where z is as defined in 9.6;

c, of infinite order, as $\mathrm{p}_*\{h_{k+3,k+3}\}$.

Consider now the section of the sequence associated with the fibring $V_{k+4,5}/S^{k-1} \to V_{k+4,4}$, which is of the form

$$\to \pi_{k+3}(S^{k-1}) \to \pi^4_{k-1,5} \xrightarrow{p_{k+4,4*}} \pi^3_{k,4} \to,$$

i.e. $$\to 0 \to Z_8+Z_\infty \to Z_{24}+Z_4+Z_\infty \to$$

generated by $\qquad v, \quad w, \quad a, \quad b, \quad c.$

Thus we have, since v is of order eight,

$$p_{k+4,4*}\, v = 3\epsilon a+\mu b, \quad \text{where } \epsilon \equiv 1 \pmod 2,$$

and, since w is of infinite order,

$$p_{k+4,4*}\, w = \alpha a+\beta b+\gamma c, \quad \text{where } \gamma \neq 0.$$

But, by § 2.3 (b),

$$\begin{aligned}\{t_{k+5,5}\} &= p_{k+4,4*}\{t_{k+5,6}\}\\ &= p_{k+4,4*}(v+2w)\\ &= (3\epsilon+2\alpha)a+(\mu+2\beta)b+2\gamma c.\end{aligned}$$

But we have from § 9.6 that

$$\{t_{k+5,5}\} = \lambda a+b+2c.$$

Hence

$$\lambda \equiv 1 \pmod 2, \quad \text{i.e.} \quad \lambda = 2\sigma+1, \quad \mu+2\beta \equiv 1 \pmod 4, \quad \gamma = 1.$$

Thus $$\{t_{k+5,4}\} = (a+b)+2(\sigma a+c).$$

(a) When $p = 3$, (D) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \xrightarrow{\Delta_*} \pi^3_{k,4} \xrightarrow{i_{k+3*}} \pi^3_{k,5} \to \pi_{k+3}(S^{k+4}),$$

i.e. $$\to Z_\infty \to Z_{24}+Z_4+Z_\infty \to \pi^3_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (c). Also $i^{-1}_{k+3*}(0)$ is generated by $\{t_{k+5,5}\}$: that is, in the notation of the last paragraph, by $(a+b)+2(\sigma a+c)$. Hence

$$\pi^3_{k,5} = Z_4+Z_{48},$$

generated by $i_{k+3,2*}a$, where

$$p_{k+3,1*}a = \{h_{k+2,k+3}\} \quad\text{and}\quad 2a = \{i_{k+2,1}\,\mathrm{p}h_{k+1,k+3}\},$$

and
$$\{i_{k+4,1}\,\mathrm{p}h_{k+3,k+3}\}+\sigma\{i_{k+1,4}\,h_{k,k+3}\}.$$

Note that, since $\Delta_*^{-1}(0) = 0$, p_{k+4*} is trivial.

(b) When $p = 4$, (D) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \xrightarrow{\Delta_*} \pi^4_{k,4} \xrightarrow{i_{k+4*}} \pi^4_{k,5} \xrightarrow{p_{k+4*}} \pi_{k+4}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2 \to \pi^4_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (d), and since p_{k+4*} is trivial by (a). Also

$$i^{-1}_{k+4*}(0) = t_{k+5,5*}\,\pi_{k+4}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+4}[(a+b)+2(\sigma a+c)] &= h^*_{k+3,k+4}(a+b)+2(\sigma a+c)_*\{h_{k+3,k+4}\} \\ &= h^*_{k+3,k+4}\,a+h^*_{k+3,k+4}\,b. \end{aligned}$$

But

$$h^*_{k+3,k+4}\,a = h^*_{k+3,k+4}\,i_{k+1,3*}\{h_{k,k+3}\} = i_{k+1,3*}\,h^*_{k+3,k+4}\{h_{k,k+3}\} = 0.$$

Also, if $b = i_{k+3,1*}\bar{b}$, we have that

$$h^*_{k+3,k+4}\,b = h^*_{k+3,k+4}\,i_{k+3,1*}\bar{b} = i_{k+3,1*}\,h^*_{k+3,k+4}\,\bar{b},$$

by Lemma 3.1 b,

$$\begin{aligned} p_{k+3,1*}\,h^*_{k+3,k+4}\,\bar{b} = h^*_{k+3,k+4}\,p_{k+3,1*}\,\bar{b} &= h^*_{k+3,k+4}\{h_{k+2,k+3}\} \\ &= \{h_{k+2,k+4}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (d), that $i^{-1}_{k+4*}(0)$ is a Z_2 subgroup of $\pi^4_{k,4}$, and that

$$\pi^4_{k,5} = Z_2+Z_2,$$

generated by $\{i_{k+2,3}\,\mathrm{p}h_{k+1,k+4}\}$ and $\{i_{k+4,1}\,\mathrm{p}h_{k+3,k+4}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(c) When $p = 5$, (D) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+4}) \xrightarrow{\Delta_*} \pi^5_{k,4} \xrightarrow{i_{k+5*}} \pi^5_{k,5} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+4}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2 \to \pi^5_{k,5} \to 0,$$

by §§ 8.1 and 7.31 (e), and since p_{k+5*} is trivial by (b). Also

$$i^{-1}_{k+5*}(0) = t_{k+5,5*}\pi_{k+5}(S^{k+3}),$$

which is generated by

$$\begin{aligned} h^*_{k+3,k+5}[(a+b)+2(\sigma a+c)] &= h^*_{k+3,k+5}(a+b)+2(\sigma a+c)_*\{h_{k+3,k+5}\} \\ &= h^*_{k+3,k+5}\,a+h^*_{k+3,k+5}\,b \\ &= h^*_{k+3,k+5}\,b \end{aligned}$$

since $$h^*_{k+3,k+5}\,a = h^*_{k+4,k+5}\,h^*_{k+3,k+4}\,a = 0,$$

by (b). Further, if $b = i_{k+3,1*}\bar{b}$, we have

$$h^*_{k+3,k+5}\,b = h^*_{k+3,k+5}\,i_{k+3,1*}\bar{b} = i_{k+3,1*}\,h^*_{k+3,k+5}\bar{b},$$

$$\begin{aligned} p_{k+3,1*}\,h^*_{k+3,k+5}\bar{b} = h^*_{k+3,k+5}\,p_{k+3,1*}\bar{b} &\doteq h^*_{k+3,k+5}\{h_{k+2,k+3}\} \\ &= 12\{h_{k+2,k+5}\}. \end{aligned}$$

Thus we see, by looking at the results of §§ 8.1 and 7.31 (e), that $i^{-1}_{k+5*}(0)$ is a Z_2 summand of $\pi^5_{k,4}$, and that

$$\pi^5_{k,5} = Z_2,$$

generated by $\{i_{k+4,1}\,\mathrm{p}h_{k+3,k+5}\}$. Note that again p_{k+6*} is trivial.

9.63. $k = 4$.

We first consider the diagram

$$\begin{tikzcd} \arrow[r] & \pi^4_{3,4} \arrow[r, "i_{7,1*}"] \arrow[d, "p_{7,3*}"] & \pi^4_{3,5} \arrow[r, "p_{8,1*}"] \arrow[d, "p_{8,4*}"] & \pi_7(S^7) \arrow[r] \arrow[d, "\bar{p}_{8,1*}"] & {} \\ \arrow[r] & \pi^3_{4,3} \arrow[r, "i_{7,1*}"] & \pi^3_{4,4} \arrow[r, "p_{8,1*}"] & \pi_7(S^7) \arrow[r] & {}, \end{tikzcd}$$

where the horizontal sequences are those associated with the fibrings $V_{8,5}/V_{7,4} \to S^7$ and $V_{8,4}/V_{7,3} \to S^7$. By § 2.1 the diagram is commutative and $\bar{p}_{8,1*}$ an isomorphism. By §§ 8.1, 9.1 the $i_{7,1*}$ are monomorphisms and the $p_{8,1*}$ are onto. Hence, with the results of §§ 8.1 and 9.1, together with those of §§ 7.31 (c) and 8.4 (e), the diagram becomes

$$\begin{array}{ccccccc} 0 \to & \underset{\bar{u}}{Z_4} + \underset{\bar{v}}{Z_\infty} & \to & \underset{u}{Z_4} + \underset{v}{Z_\infty} + \underset{w}{Z_\infty} & \to & \underset{\{h_{7,7}\}}{Z_\infty} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \underset{\bar{a}}{Z_\infty} + \underset{\bar{b}}{Z_{12}} + \underset{\bar{c}}{Z_4} & \to & \underset{a}{Z_\infty} + \underset{b}{Z_{12}} + \underset{c}{Z_4} + \underset{d}{Z_\infty} & \to & \underset{\{h_{7,7}\}}{Z_\infty} & \to 0, \end{array}$$

the summands being generated by the elements displayed below them. These generators are chosen as follows.

In § 10.7 it will be shown that $\{t_{9,6}\} \in \pi^4_{3,5}$ is of the form

$$\{t_{9,6}\} = (2\rho+1)i_{7,1*}\bar{v} + 2\mathrm{p}_*\{h_{7,7}\},$$

where $\bar{v}$ generates an infinite summand of $\pi^4_{3,4}$. Thus we can choose generators as follows. In $\pi^4_{3,4} = Z_4 + Z_\infty$, choose

$\bar{u}$, of order four (any such);

$\bar{v}$, of infinite order, as defined above by $\{t_{9,6}\}$.

In $\pi^4_{3,5} = Z_4+Z_\infty+Z_\infty$, choose

u, of order four, as $i_{7,1*}\bar{u}$;
v, of infinite order, as $i_{7,1*}\bar{v}$;
w, of infinite order, as $p_*\{h_{7,7}\}+\rho v$.

In $\pi^3_{4,3} = Z_\infty+Z_{12}+Z_4$, choose

$\bar{a}$, of infinite order, as $i_{5,2*}\bar{p}_*\{h_{7,7}\}$ ($\bar{p}$ is Hopf map);
$\bar{b}$, of order twelve, as $i_{5,2*}\mathfrak{E}\{h_{3,6}\}$;
$\bar{c}$, of order four, as the z defined in § 9.6.

In $\pi^3_{4,4} = Z_\infty+Z_{12}+Z_4+Z_\infty$, choose

a, of infinite order, as $i_{7,1*}\bar{a}$;
b, of order twelve, as $i_{7,1*}\bar{b}$;
c, of order four, as $i_{7,1*}\bar{c}$;
d, of infinite order, as $p_*\{h_{7,7}\}$.

Now consider the sequence associated with the fibring $V_{7,4}/S^3 \to V_{7,3}$, which is of the form

$$\to \pi^4_{3,4} \xrightarrow{p_{7,3*}} \pi^3_{4,3} \xrightarrow{\Delta_*} \pi_6(S^3) \to \pi^3_{3,4} \to \pi^2_{4,3} \to \pi_5(S^3) \to.$$

From the second paragraph of § 8.4 (*e*) we have that $p_{7,3*}$ is an isomorphism into. Further, since $\pi^3_{3,4} = Z_2$ by § 8.4 (c), $\pi^2_{4,3} = Z_2+Z_2$ by § 7.31 (*b*), and $\pi_5(S^3) = Z_2$, it follows by exactness that Δ_* is onto $\pi_6(S^3)$. Now both $\pi^4_{3,4}$ and $\pi^3_{4,3}$ project onto $\pi_7(S^6)$, so that we have the diagram

$$\begin{tikzcd} \to \pi^4_{3,4} \arrow[r, "p_{7,3*}"] \arrow[dr, "p_{7,1*}"'] & \pi^3_{4,3} \arrow[r, "\Delta_*"] \arrow[d, "p_{7,1*}"] & \pi_6(S^3) \to 0, \\ & \pi_7(S^6) & \end{tikzcd}$$

which is commutative. Now from § 7.31 (*c*) we have that

$$p_{7,1*}\bar{b} = 0, \quad p_{7,1*}\bar{c} \neq 0, \quad p_{7,1*}2\bar{c} = 0.$$

Thus, since, by § 8.4 (*e*), $p_{7,1*}\bar{u} = 0$ and since $\bar{u}$ is of order four,

$$p_{7,3*}\bar{u} = 3\epsilon\bar{b}+2\mu\bar{c}, \quad \textit{where} \quad \epsilon = \mp 1 \quad \textit{and} \quad \mu = 0 \textit{ or } 1.$$

Again, by § 7.31 (*c*), $p_{7,1*}\bar{a} = 0$. Thus, since $p_{7,1*}\bar{v} \neq 0$ and $\bar{v}$ is of infinite order,

$$p_{7,3*}\bar{v} = \alpha\bar{a}+\beta\bar{b}+(2\gamma+1)\bar{c}, \quad \textit{where } \alpha \neq 0 \textit{ and } \gamma = 0 \textit{ or } 1.$$

Now change the basis of $\pi^3_{4,3}$ to $\{\bar{a}, \bar{b}_1, \bar{c}_1\}$, where

$$\bar{b}_1 = \bar{b}+2\mu\bar{c}, \qquad \bar{c}_1 = [2(\gamma-\beta\mu)+1]\bar{c}.$$

Then the position becomes

$$0 \to Z_4 + Z_\infty \xrightarrow{p_{7,3*}} Z_\infty + Z_{12} + Z_4 \to Z_{12} \to 0$$

generated by $\bar{u}$ $\bar{v}$ $\bar{a}$ $\bar{b}_1$ $\bar{c}_1$

with
$$p_{7,3*}\bar{u} = 3\epsilon\bar{b}_1, \quad \textit{where } \epsilon = \mp 1,$$
$$p_{7,3*}\bar{v} = \alpha\bar{a} + \beta\bar{b}_1 + \bar{c}_1, \quad \textit{where} \quad \alpha \neq 0.$$

Thus the factor group, $\pi^3_{4,3}/p_{7,3*}\pi^4_{3,4}$, is generated by $\bar{a}$, $\bar{b}_1$, $\bar{c}_1$ with the relations

$$12\bar{b}_1 = 0, \quad 4\bar{c}_1 = 0, \quad 3\bar{b}_1 \equiv 0, \quad \alpha\bar{a} + \beta\bar{b}_1 + \bar{c}_1 \equiv 0.$$

There are now two cases

(*a*) $\beta \equiv 0 \,(\mathrm{mod}\, 3)$, (*b*) $\beta \equiv \epsilon_1 \,(\mathrm{mod}\, 3)$, *where* $\epsilon_1 = \mp 1$.

In the case (*a*) we see that the factor group becomes $Z_3 + Z_{4\alpha}$. But the factor group is in fact $Z_{12} = Z_3 + Z_4$. Hence

$$\alpha = 1.$$

In case (*b*), we first change the basis to $\{\bar{a}, \bar{b}_2, \bar{c}_1\}$, where

$$\bar{b}_2 = \epsilon_1 \bar{b}_1 + \bar{c}_1,$$

and then to $\{\bar{a}, \bar{b}_2, \bar{c}_2\}$, where

$$\bar{c}_2 = \bar{c}_1 + 3\bar{b}_2.$$

The factor group will then be generated by $\bar{a}$, $\bar{b}_2$, $\bar{c}_2$, with the relations

$$12\bar{b}_2 = 0, \quad 4\bar{c}_2 = 0, \quad \bar{c}_2 \equiv 0, \quad \alpha\bar{a} + \bar{b}_2 \equiv 0.$$

Thus in this case the factor group becomes $Z_{12\alpha}$. But it is Z_{12}. So again

$$\alpha = 1.$$

Thus we have that

$$p_{7,3*}\bar{v} = \bar{a} + \beta\bar{b} + (2\gamma+1)\bar{c}.$$

Hence
$$p_{8,4*}v = p_{8,4*}i_{7,1*}\bar{v} = i_{7,1*}p_{7,3*}\bar{v}$$
$$= a + \beta b + (2\gamma+1)c.$$

Also, since $p_{8,1*}w = \{h_{7,7}\}$, $p_{8,1*}p_{8,4*}w = \{h_{7,7}\}$. Hence

$$p_{8,4*}w = \alpha' a + \beta' b + \gamma' c + d.$$

But, by § 2.3 (*b*),

$$\{t_{9,5}\} = p_{8,4*}\{t_{9,6}\}$$
$$= p_{8,4*}[(2\rho+1)v + 2p_*\{h_{7,7}\}]$$
$$= p_{8,4*}(v + 2w).$$

Thus
$$\{t_{9,5}\} = (2\alpha'+1)a + (2\beta'+\beta)b + [2(\gamma'+\gamma)+1]c + 2d.$$

But we have from § 9.6 that

$$\{t_{9,5}\} = \lambda a + \mu b + c + 2d,$$

whence, by comparing coefficients, we see that

$$\lambda = 2\alpha'+1, \qquad 2(\gamma'+\gamma) \equiv 0 \pmod 4,$$
$$\{t_{9,5}\} = a+\mu b+c+2(\sigma a+d) \quad (\sigma = \alpha').$$

(*a*) When $p = 3$, (D) gives

$$\xrightarrow{p_{8*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^3_{4,4} \xrightarrow{i_{7*}} \pi^3_{4,5} \to \pi_7(S^8),$$

i.e.
$$\to Z_\infty \to Z_\infty + Z_{12} + Z_4 + Z_\infty \to \pi^3_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*c*). But $i_{7*}^{-1}(0)$ is generated by $\{t_{9,5}\}$: that is, in the notation of the previous paragraph, by $[(a+\mu b+c)+2(\sigma a+d)]$. Thus

$$\pi^3_{4,5} = Z_{12}+Z_4+Z_\infty,$$

generated by $\{i_{5,4}\, \mathfrak{E}h_{3,6}\}$, $i_{7,3*}\, a$, where $p_{7,1*}\, a = \{h_{6,7}\}$ and $2a = \{i_{6,1}\, \mathrm{p}h_{5,7}\}$, and $\{i_{8,1}\, \mathrm{p}h_{7,7}\}+\sigma\{i_{5,4}\, \bar{p}h_{7,7}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{8*} is trivial.

(*b*) When $p = 4$, (D) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^4_{4,4} \xrightarrow{i_{8*}} \pi^4_{4,5} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2+Z_2+Z_2 \to \pi^4_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*d*), and since p_{8*} is trivial by (*a*). Further

$$i_{8*}^{-1}(0) = t_{9,5*}\, \pi_8(S^7),$$

which is generated by

$$h^*_{7,8}[(a+\mu b+c)+2(\sigma a+d)] = h^*_{7,8}\, a+\mu h^*_{7,8}\, b+h^*_{7,8}\, c+2(\sigma a+d)_*\{h_{7,8}\}$$
$$= h^*_{7,8}\, a+h^*_{7,8}\, b+h^*_{7,8}\, c.$$

But

$$h^*_{7,8}\, a = h^*_{7,8}\, i_{5,3*}\, \bar{p}_*\{h_{7,7}\} = i_{5,3*}\, \bar{p}_*\, h^*_{7,8}\{h_{7,7}\} = i_{5,3*}\, \bar{p}_*\{h_{7,8}\},$$
$$h^*_{7,8}\, b = h^*_{7,8}\, i_{5,3*}\, \mathfrak{E}\{h_{3,6}\} = i_{5,3*}\, \mathfrak{E}h^*_{6,7}\{h_{3,6}\} = i_{5,3*}\, \mathfrak{E}\{h_{3,7}\}.$$

Further
$$h^*_{7,8}\, c = h^*_{7,8}\, i_{7,1*}\, \bar{c} = i_{7,1*}\, h^*_{7,8}\, \bar{c},$$
$$p_{7,1*}\, h^*_{7,8}\, \bar{c} = h^*_{7,8}\, p_{7,1*}\, \bar{c} = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Thus, using the results of §§ 8.1 and 7.31 (*d*), we see that $i_{8*}^{-1}(0) \neq 0$, and, since it must be cyclic, that it is a Z_2 summand, and that

$$\pi^4_{4,5} = Z_2+Z_2+Z_2+Z_2,$$

generated by $\{i_{8,1}\, \mathrm{p}h_{7,8}\}$, $\{i_{6,3}\, \mathrm{p}h_{5,8}\}$, $\{i_{5,4}\, \mathfrak{E}h_{3,7}\}$, and $\{i_{5,4}\, \bar{p}h_{7,8}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{9*} is trivial.

(*c*) When $p = 5$, (D) gives

$$\xrightarrow{p_{10*}} \pi_{10}(S^8) \xrightarrow{\Delta_*} \pi^5_{4,4} \xrightarrow{i_{9*}} \pi^5_{4,5} \xrightarrow{p_{9*}} \pi_9(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2+Z_2+Z_2 \to \pi^5_{4,5} \to 0,$$

by §§ 8.1 and 7.31 (*e*), and since p_{9*} is trivial by (*b*). Further,

$$i_{9*}^{-1}(0) = t_{9,5*}\,\pi_9(S^7),$$

which is generated by

$$h^*_{7,9}[(a+\mu b+c)+2(\sigma a+d)] = h^*_{7,9}\,a+\mu h^*_{7,9}\,b+h^*_{7,9}\,c+2(\sigma a+d)_*\{h_{7,9}\}$$
$$= h^*_{7,9}\,a+h^*_{7,9}\,b+h^*_{7,9}\,c.$$

But

$$h^*_{7,9}\,a = h^*_{7,9}\,i_{5,3*}\,\bar{p}_*\{h_{7,7}\} = i_{5,3*}\,\bar{p}_*\,h^*_{7,9}\{h_{7,7}\} = i_{5,3*}\,\bar{p}_*\{h_{7,9}\},$$
$$h^*_{7,9}\,b = h^*_{7,9}\,i_{5,3*}\,\mathfrak{E}\{h_{3,6}\} = i_{5,3*}\,\mathfrak{E}h^*_{6,8}\{h_{3,6}\} = i_{5,3*}\,\mathfrak{E}\{h_{3,8}\}.$$

Further
$$h^*_{7,9}\,c = h^*_{7,9}\,i_{7,1*}\,\bar{c} = i_{7,1*}\,h^*_{7,9}\,\bar{c},$$
$$p_{7,1*}\,h^*_{7,9}\,\bar{c} = h^*_{7,9}\,p_{7,1*}\,\bar{c} = h^*_{7,9}\{h_{6,7}\} = 12\{h_{6,9}\}.$$

Thus, by looking at the results of §§ 8.1 and 7.31 (*e*), we see that $i_{9*}^{-1}(0) \neq 0$, whence it must be a Z_2 summand since it is cyclic, and further that
$$\pi^5_{4,5} = Z_2+Z_2+Z_2+Z_2,$$
generated by $\{i_{6,3}\,\mathrm{p}h_{5,9}\}$, $\{i_{8,1}\,\mathrm{p}h_{7,9}\}$, $\{i_{5,4}\,\bar{p}h_{7,9}\}$, and $\{i_{5,4}\,\mathfrak{E}_{3,8}\}$. Note that again $\Delta_*^{-1}(0) = 0$, whence p_{10*} is trivial.

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