THE GROUPS $\pi_r(V_{n,m})$ (V)

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Introduction

THIS is the last of a sequence of five papers, the previous ones being (**2**), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. A full table of these results can be found in (**2**) (I) 249. The present paper contains the calculations of those groups which are given in the tables (*a*)–(*c*) below. In these tables $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$. For the notation used throughout the body of this paper please see (**2**), especially §§ 1, 2, and 3.1. Also please note that the sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

(*a*) TABLE FOR $\pi^4_{k,6}$ $(k \geqslant 3)$

$k = 3$	4	5	$8s-1$	$8s+3$	$4s+5$	$4(s+1)$	$8s-2$	$8s+2$
Z_4+Z_∞	$Z_2+Z_2+Z_2$	Z_2	Z_8	Z_{16}	0	Z_2	0	Z_2

(*b*) TABLE FOR $\pi^5_{k,m}$ $(k \geqslant 3)$

	$k = 3$	4	6
$m = 6$	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_\infty$	$Z_\infty+Z_8+Z_\infty$ or $Z_4+Z_\infty+Z_\infty$
$m = 7$	Z_2+Z_2	$Z_2+Z_2+Z_2$	$Z_\infty+Z_8$ or Z_4+Z_∞

	$k = 4s+3$	$8s+1$	$8s-3$	$4(s+1)$	$8s+6$	$8s+2$
$m = 6$	Z_2+Z_2	Z_2	Z_2	Z_∞	Z_2+Z_∞	Z_2+Z_∞
$m = 7$	Z_2	Z_2	0	0	Z_8	Z_{16}

(*c*) TABLE FOR $\pi^p_{1,m} \approx \pi^p_{2,m-1}$

	$m = 6$	$m = 7$	$m = 8$	$m \geqslant 9$
$p = 4, 5$	0	0	0	0
$p = 6$	Z_∞	$Z_\infty+Z_\infty$	Z_∞	Z_∞
$p = 7$	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

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10. Calculation of $\pi^p_{k,6}$

We consider the fibring $V_{k+6,6}/V_{k+5,5} \to S^{k+5}$, and examine the sequence

$$(\mathrm{E}) \qquad \to \pi_{k+p+1}(S^{k+5}) \xrightarrow{\Delta_*} \pi^p_{k,5} \xrightarrow{i_{k+p*}} \pi^p_{k,6} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+5}) \to .$$

10.1. $k \equiv 2 \pmod 8$.

In this case there is a five-field on S^{k+5} (1), and so the fibring admits a cross-section $\mathfrak{p}$. Hence Theorem 1.1 gives that

$$\pi^p_{k,6} = i_* \pi^p_{k,5} + \mathfrak{p}_* \pi_{k+p}(S^{k+5}).$$

Using the values of $\pi^p_{k,5}$ as calculated in § 9.5, we obtain the values shown in the tables for $\pi^p_{k,5}$ when $k\ (>2) \equiv 2 \pmod 8$.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+6,6}\} = 0 \textit{ for } k \equiv 2 \pmod 8.$$

10.2. $k \equiv 6 \pmod 8$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e. $$\to Z_\infty \to Z_2 \to \pi^4_{k,6} \to 0,$$

by § 9.5 (*b*). But $i^{-1}_{k+4*}(0) \neq 0$, for otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a five-field on S^{k+5}, which is impossible by Theorem 1.1. of (3). Thus $i^{-1}_{k+4*}(0) = \pi^4_{k,5}$, whence

$$\pi^4_{k,6} = 0.$$

Further, we have from Corollary 1.5 that $\{t_{k+6,6}\}$ generates $i^{-1}_{k+4*}(0)$. Hence

$$\{t_{k+6,6}\} = \{i_{k+2,3}\, \mathfrak{p} h_{k+1,k+4}\}.$$

Note that $\Delta_* \pi_{k+5}(S^{k+5})$ is of order two, whence it follows that the image of p_{k+5*} is the Z_∞ subgroup generated by $2\{h_{k+5,k+5}\}$.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e. $$\to Z_2 \to \pi^5_{k,5} \to \pi^5_{k,6} \to Z_\infty \to 0,$$

since p_{k+5*} is onto a Z_∞ subgroup by (*a*). Also

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\, \pi_{k+5}(S^{k+4}) = i_{k+2,3*}\, \mathfrak{p}_* h_{k+1,k+4*}\, \pi_{k+5}(S^{k+4})$$
$$\in i_{k+2,3*}\, \mathfrak{p}_* \pi_{k+5}(S^{k+1}).$$

Thus $$i^{-1}_{k+5*}(0) = 0 \quad (\text{since } k \geqslant 6).$$

Hence $$\pi^5_{k,6} \approx \pi^5_{k,5} + Z_\infty.$$

But we have from § 9.5 (c) that, when $k > 6$, $\pi^5_{k,5} = Z_8$, whence

$$\pi^5_{k,6} = Z_8 + Z_\infty \quad (k > 6),$$

generated by $i_{k+5,1*}a$, where $p_{k+5,1*}a = \{h_{k+4,k+5}\}$, and b such that $p_{k+6,1*}b = 2\{h_{k+5,k+5}\}$. When $k = 6$, we have from § 9.5 (d) that $\pi^5_{6,5}$ is either $Z_\infty + Z_8$ or $Z_4 + Z_\infty$, whence

$$\pi^5_{6,6} \textit{ is either } Z_\infty + Z_8 + Z_\infty \textit{ or } Z_4 + Z_\infty + Z_\infty,$$

generated respectively by $\{i_{7,5}h_{6,11}\}$, $i_{11,1*}a$ where $8a = 0$ and

$$p_{11,1*}a = \{h_{10,11}\},$$

and b such that $p_{12,1*}b = 2\{h_{11,11}\}$; or $i_{10,2*}a$, where $4a = 0$ and $p_{10,1*}a = \{h_{9,11}\}$, $i_{11,1*}b$, where $p_{11,1*}b = \{h_{10,11}\}$, and c such that

$$p_{12,1*}c = 2\{h_{11,11}\}.$$

Note that in all cases Δ_* is trivial, whence p_{k+6*} is onto.

10.3. $k \equiv 0 \pmod 4$.

(a) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_2 \to \pi^4_{k,6} \to 0 \quad \text{when } k > 4,$$

and $$\to Z_\infty \to Z_2 + Z_2 + Z_2 + Z_2 \to \pi^4_{4,6} \to 0 \quad \text{when } k = 4,$$

by §§ 9.61 (b), 9.62 (b), and 9.63 (b). Also again $i^{-1}_{k+4*}(0) \neq 0$, since otherwise there would be a cross-section in the fibring $V_{k+6,6}/V_{k+5,5} \to S^{k+5}$ by Theorem 1.2, and so a five-field on S^{k+5}, which is impossible by Theorem 1.1 of (3). But, by exactness, $i^{-1}_{k+4*}(0)$ must be a cyclic subgroup. Hence we have that

$$\pi^4_{k,6} = Z_2 \quad (k > 4), \qquad \pi^4_{4,6} = Z_2 + Z_2 + Z_2.$$

Note that the image of Δ_* is of order two, whence it follows that the image of p_{k+5*} is the Z_∞ subgroup generated by $2\{h_{k+5,k+5}\}$.

To determine the generator(s) of $\pi^4_{k,6}$ we have to evaluate $\{t_{k+6,6}\}$ which generates $i^{-1}_{k+4*}(0)$. Consider the section of the sequence associated with the fibring $V_{k+5,5}/V_{k+2,2} \to V_{k+5,3}$ which is of the form

$$\to \pi^4_{k,2} \xrightarrow{i_{k+2,2*}} \pi^4_{k,5} \xrightarrow{p_{k+5,3*}} \pi^2_{k+2,3} \to.$$

Then we have by § 2.3 (b) that

$$p_{k+5,3*}\{t_{k+6,6}\} = \{t_{k+6,4}\} = \{i_{k+4,1}\, p h_{k+3,k+4}\}, \quad \text{by § 8.2 } (a),$$
$$\neq 0.$$

Thus $$\{t_{k+6,6}\} \notin i_{k+2,3*}\pi^4_{k,2}.$$

If we now look at the results and relations in §§ 9.61 (*b*), 9.62 (*b*), and 9.63 (*b*), we see that

$$\{t_{k+6,6}\} = \{i_{k+4,1}\,\mathrm{p}h_{k+3,k+4}\}+w, \quad \text{where } w \in i_{k+2,3*}\pi^4_{k,2}.$$

Thus, when $k > 4$, $\pi^4_{k,6}$ is generated by $\{i_{k+2,4}\,\mathrm{p}h_{k+1,k+4}\}$, and $\pi^4_{4,6}$ is generated by $\{i_{5,5}\,\mathfrak{E}h_{3,7}\}$, $\{i_{5,5}\,\bar{p}h_{7,8}\}$, and $\{i_{6,4}\,\mathrm{p}h_{5,8}\}$.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \rightarrow,$$

i.e. $$\rightarrow Z_2 \rightarrow Z_2 \rightarrow \pi^5_{k,6} \rightarrow Z_\infty \rightarrow 0 \quad \text{when } k > 4$$

and

$$\rightarrow Z_2 \rightarrow Z_2+Z_2+Z_2+Z_2 \rightarrow \pi^5_{4,6} \rightarrow Z_\infty \rightarrow 0 \quad \text{when } k = 4,$$

by §§ 9.61 (*c*), 9.62 (*c*), and 9.63 (*c*), and since p_{k+5*} is onto a Z_∞ subgroup by (*a*). Further,

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}),$$

which is generated by

$$h^*_{k+4,k+5}(i_{k+4,1*}\,\mathrm{p}_*\{h_{k+3,k+4}\}+w) = i_{k+4,1*}\,\mathrm{p}_*\{h_{k+3,k+5}\}+h^*_{k+4,k+5}\,w.$$

Now, when $k > 4$, we have that $i_{k+4,1*}\,\mathrm{p}_*\{h_{k+3,k+5}\}$ generates $\pi^5_{k,5}$, whilst $h^*_{k+4,k+5}\,w \in i_{k+2,3*}\pi^5_{k,2} = 0$ by § 5.1. Thus

$$i^{-1}_{k+5*}(0) = \pi^5_{k,5},$$

and $$\pi^5_{k,6} = Z_\infty \quad \textit{when } k > 4,$$

generated by a such that $p_{k+6,1*}\,a = 2\{h_{k+5,k+5}\}$.

When $k = 4$, we have that $i_{8,1*}\,\mathrm{p}_*\{h_{7,9}\}$ generates one summand Z_2 which is not in $i_{6,3*}\pi^5_{4,2}$. However, $h^*_{8,9}\,w \in i_{6,3*}\pi^5_{4,2}$. Thus $i^{-1}_{9*}(0) \neq 0$, and by exactness it must also be cyclic. Hence it is Z_2 and thus

$$\pi^5_{4,6} = Z_2+Z_2+Z_2+Z_\infty,$$

generated by $\{i_{5,5}\,\mathfrak{E}h_{3,8}\}$, $\{i_{5,5}\,\bar{p}h_{7,9}\}$, $\{i_{6,4}\,\mathrm{p}h_{5,9}\}$ and a such that

$$p_{10,1*}\,a = 2\{h_{9,9}\}.$$

Note that in both cases $\Delta^{-1}_*(0) = 0$, whence p_{k+6*} is trivial.

10.4. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \rightarrow \pi_{k+4}(S^{k+5}),$$

i.e. $$\rightarrow Z_\infty \rightarrow Z_\infty \rightarrow \pi^4_{k,6} \rightarrow 0 \quad \text{when } k > 5,$$

and $$\rightarrow Z_\infty \rightarrow Z_2+Z_\infty \rightarrow \pi^4_{5,6} \rightarrow 0 \quad \text{when } k = 5,$$

by § 9.3 (*b*). Also $i^{-1}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$, and we have from § 2.3 (*b*) that

$$p_{k+5,1*}\{t_{k+6,6}\} = 2\{h_{k+4,k+4}\}.$$

Hence $\{t_{k+6,6}\}$ generates an infinite summand of $\pi^4_{k,6}$, whence

$$\pi^4_{k,6} = 0 \quad (k > 5), \qquad \pi^4_{5,6} = Z_2,$$

generated by $\{i_{6,5} h_{5,9}\}$. Note that thus $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \rightarrow,$$

i.e.
$$\rightarrow Z_2 \rightarrow Z_2 \rightarrow \pi^5_{k,6} \rightarrow 0,$$

by § 9.3 (*c*), and since p_{k+5*} is trivial by (*a*). Also

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}),$$

which is generated by $h^*_{k+4,k+5}\{t_{k+6,6}\}$. To evaluate this consider the portion of the sequence associated with the fibring $V_{k+5,5}/V_{k+2,2} \rightarrow V_{k+5,3}$ which is of the form

$$\rightarrow \pi^5_{k,2} \rightarrow \pi^5_{k,5} \xrightarrow{p_{k+5,3*}} \pi^3_{k+2,3} \rightarrow.$$

From §§ 5.2 (*i*) and (*k*) we have that $\pi^5_{k,2} = 0$, and, from § 2.3 (*b*), that $p_{k+5,3*}\{t_{k+6,6}\} = \{t_{k+6,4}\}$. Hence

$$\begin{aligned} p_{k+5,3*}\, h^*_{k+4,k+5}\{t_{k+6,6}\} &= h^*_{k+4,k+5}\, p_{k+5,3*}\{t_{k+6,6}\} \\ &= h^*_{k+4,k+5}\{t_{k+6,4}\} \\ &= 0, \quad \text{by § 8.4 } (b). \end{aligned}$$

Thus
$$h^*_{k+4,k+5}\{t_{k+6,6}\} = 0,$$
and
$$\pi^5_{k,6} = Z_2,$$

generated by $\{i_{k+3,3}\, p h_{k+2,k+5}\}$. Note that, since Δ_* is trivial, p_{k+6*} is onto.

10.5. $k = 1$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{6*}} \pi_6(S^6) \xrightarrow{\Delta_*} \pi^4_{1,5} \xrightarrow{i_{5*}} \pi^4_{1,6} \rightarrow \pi_5(S^6),$$

i.e.
$$\rightarrow Z_\infty \rightarrow Z_\infty \rightarrow \pi^4_{1,6} \rightarrow 0,$$

by § 9.4 (*b*). Also $i^{-1}_{5*}(0)$ is generated by $\{t_{7,6}\}$. But

$$p_{6,1*}\{t_{7,6}\} = 2\{h_{5,5}\}$$

by § 2.3 (*b*), whence we see that $\{t_{7,6}\}$ generates $\pi^4_{1,5}$. Hence

$$\pi^4_{1,6} = 0, \qquad \pi^3_{2,5} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that p_{6*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{7*}} \pi_7(S^6) \xrightarrow{\Delta_*} \pi^5_{1,5} \xrightarrow{i_{6*}} \pi^5_{1,6} \xrightarrow{p_{6*}} \pi_6(S^6) \to,$$

i.e.

$$\to Z_2 \to 0 \to \pi^5_{1,6} \to 0,$$

by § 9.4 (*c*), and since p_{6*} is trivial by (*a*). Hence

$$\pi^5_{1,6} = 0, \qquad \pi^4_{2,5} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that, since Δ_* is trivial, p_{7*} is onto.

(*c*) When $p = 6$, (E) gives

$$\xrightarrow{p_{8*}} \pi_8(S^6) \xrightarrow{\Delta_*} \pi^6_{1,5} \xrightarrow{i_{7*}} \pi^6_{1,6} \xrightarrow{p_{7*}} \pi_7(S^6) \to,$$

i.e.

$$\to Z_2 \to Z_\infty \to \pi^6_{1,6} \to Z_2 \to 0,$$

by § 9.4 (*d*), and since p_{7*} is onto $\pi_7(S^6)$ by (*b*). Further, since it is impossible to map a finite group essentially into an infinite cyclic one, $i_{7*}^{-1}(0) = 0$. Thus $\pi^6_{1,6}$ *is an extension of* Z_∞ *by* Z_2, *as, by Theorem* 4.2 (*a*), *is* $\pi^5_{2,5}$. Note that Δ_* is trivial, whence p_{8*} is onto.

In order to determine the extension we consider the diagram

$$\begin{array}{ccccccccc} \to \pi_7(S^2) & \xrightarrow{i_{3,4*}} & \pi^5_{2,5} & \xrightarrow{p_{7,4*}} & \pi^4_{3,4} & \xrightarrow{\Delta_*} & \pi_6(S^2) \to \\ \uparrow i_{3,0*} & & \uparrow i_{6,1*} & & \uparrow i_{6,1*} & & \uparrow i_{3,0*} \\ \to \pi_7(S^2) & \xrightarrow{i_{3,3*}} & \pi^5_{2,4} & \xrightarrow{p_{6,3*}} & \pi^4_{3,3} & \xrightarrow{\Delta_*} & \pi_6(S^2) \to, \end{array}$$

where the horizontal sequences are associated with the fibrings $V_{7,5}/S^2 \to V_{7,4}$ and $V_{6,4}/S^2 \to V_{6,3}$. By § 2.1 the diagram is commutative and the $i_{3,0*}$ are isomorphisms. Also, by the last paragraph and by § 8.4 (*e*), the $i_{6,1*}$ are monomorphisms. Further, we have from § 9.4 (*d*) that $i_{3,3*}\pi_7(S^2) = 0$, whence also $i_{3,4*}\pi_7(S^2) = 0$. Thus, with the results of §§ 7.2 (*f*), 8.4 (*e*), and 9.4 (*d*), the diagram becomes

$$\begin{array}{ccccccc} 0 \to & \pi^5_{2,5} & \to & Z_4 + Z_\infty & \to & Z_{12} \to \\ & \uparrow & & \uparrow & & \uparrow \\ 0 \to & Z_\infty & \to & Z_4 + Z_\infty & \to & Z_{12} \to. \end{array}$$

Now let a' be a generator of order four in $\pi^4_{3,3}$, and a be $i_{6,1*}a'$. Then we have from § 9.4 (*d*) that $\Delta_* 2a' = 6\{h_{2,6}\}$. Hence

$$\Delta_* 2a = \Delta_* i_{6,1*} 2a' = i_{3,0*} \Delta_* 2a' = i_{3,0*} 6\{h_{2,6}\}.$$

Thus

$$\Delta_* 2a = 6\{h_{2,6}\} \neq 0.$$

Now, if $\pi^5_{2,5}$ were $Z_\infty + Z_2$, $2a$ would be in $p_{7,4*}\pi^5_{2,5}$ since $2a$ is the only element of order two in $\pi^4_{3,4}$ and so must be the image of the element of order two in $\pi^5_{2,5}$. Thus $\Delta_* 2a$ would have to be zero, and we have just proved the contrary. Hence

$$\pi^5_{2,5} = Z_\infty, \qquad \pi^6_{1,6} = Z_\infty,$$

each generated by an element a such that $p_{7,1*}a = \{h_{6,7}\}$.

(*d*) When $p = 7$, (E) gives

$$\xrightarrow{p_{9*}} \pi_9(S^6) \xrightarrow{\Delta_*} \pi^7_{1,5} \xrightarrow{i_{8*}} \pi^7_{1,6} \xrightarrow{p_{8*}} \pi_8(S^6) \to,$$

i.e.

$$\to Z_{24} \to Z_{24} \to \pi^7_{1,6} \to Z_2 \to 0,$$

by § 9.4 (*e*), and since p_{8*} is onto $\pi_8(S^6)$ by (*c*). Further,

$$i^{-1}_{8*}(0) = t_{7,6*}\pi_8(S^5),$$

which is generated by $h^*_{5,8}\{t_{7,6}\}$. But

$$p_{6,1*}h^*_{5,8}\{t_{7,6}\} = h^*_{5,8}p_{6,1*}\{t_{7,6}\} = h^*_{5,8}2\{h_{5,5}\}, \quad \text{by } (a).$$

Hence $i^{-1}_{8*}(0) = 2\pi^7_{1,5}$, whence $\pi^7_{1,6}$ *is an extension of* Z_2 *by* Z_2. Note that, since $\Delta_* Z_{24} = 2(Z_{24})$, the image of p_{9*} is the Z_2 subgroup generated by $12\{h_{6,9}\}$.

To determine the extension, consider the portion of the sequence associated with the fibring $V_{7,6}/V_{3,2} \to V_{7,4}$ which is of the form

$$\to \pi^7_{1,6} \xrightarrow{p_{7,4*}} \pi^5_{3,4} \to \pi^6_{1,2} \to,$$

i.e.

$$\to \pi^7_{1,6} \to Z_2 + Z_2 + Z_2 \to Z_2 \to,$$

by § 8.4 (*g*) and 5.3 (*b*). But we have just seen that $\pi^7_{1,6}$ has four elements, whence $p_{7,4*}$ is a monomorphism. Hence

$$\pi^7_{1,6} = Z_2 + Z_2,$$

generated by $i_{6,1*}p^{-1}_{6,1*}\{h_{5,8}\}$ and a such that $p_{7,1*}a = \{h_{6,8}\}$. Thus, by Theorem 4.2 (*a*) we also have that

$$\pi^6_{2,5} = Z_2 + Z_2,$$

generated by $i_{6,1*}p^{-1}_{6,1*}\{h_{5,8}\}$ and a such that $p_{7,1*}a = \{h_{6,8}\}$.

10.6. $k \equiv 7 \pmod 8$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.

$$\to Z_\infty \to Z_8 + Z_\infty \to \pi^4_{k,6} \to 0,$$

by § 9.2 (*b*). Also $i^{-1}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$. But, by § 2.3 (*b*),

$$p_{k+5,1*}\{t_{k+6,6}\} = 2\{h_{k+4,k+4}\},$$

whence it follows that $\{t_{k+6,6}\}$ generates an infinite summand of $\pi^4_{k,5}$. Hence

$$\pi^4_{k,6} = Z_8,$$

generated by $i_{k+4,2*}\,a$, where $p_{k+4,1*}\,a = \{h_{k+3,k+4}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^5_{k,6} \to 0,$$

by § 9.2 (*c*), and since p_{k+5*} is trivial by (*a*). We know from Theorem 4.2 (*b*) that $\pi^5_{k,6} \approx \pi^4_{k+1,5} = Z_2 + Z_2$, but to determine the generators we have to evaluate $h^*_{k+4,k+5}\{t_{k+6,6}\}$ which generates $i^{-1}_{k+5*}(0)$. To do this consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{k+5}(S^k) & \to & \pi^5_{k,5} & \xrightarrow{p_{k+5,4*}} & \pi^4_{k+1,4} & \to & \pi_{k+4}(S^k) \to \\
 & & \uparrow {\scriptstyle i_{k+4,1*}} & & \uparrow {\scriptstyle i_{k+4,1*}} & & \\
\to \pi_{k+5}(S^k) & \to & \pi^5_{k,4} & \xrightarrow{p_{k+4,3*}} & \pi^4_{k+1,3} & \to & \pi_{k+4}(S^k) \to \\
 & & {\scriptstyle p_{k+4,1*}} \searrow & & \swarrow {\scriptstyle p_{k+4,1*}} & & \\
 & & & \pi_{k+5}(S^{k+3}) & & &
\end{array}$$

in which the horizontal sequences are associated with the fibrings $V_{k+5,5}/S^k \to V_{k+5,4}$ and $V_{k+4,4}/S^k \to V_{k+4,3}$. By §§ 9.2 (*c*) and 8.1 the $i_{k+4,1*}$ are monomorphisms. With the results of §§ 9.2 (*c*), 8.4 (*f*), 8.1, and 7.31 (*d*), the diagram becomes

$$\begin{array}{ccccccc}
0 \to & Z_2 + Z_2 + Z_2 & \to & Z_2 + Z_2 + Z_2 & \to 0 \\
 & \uparrow & & \uparrow & \\
0 \to & Z_2 + Z_2 & \to & Z_2 + Z_2 & \to 0 \\
 & \searrow & & \swarrow & \\
 & & Z_2 & &
\end{array}$$

Now we have from § 9.61 (*b*) that

$$h^*_{k+4,k+5}\{t_{k+6,5}\} = i_{k+4,1*}\,b',$$

where
$$p_{k+4,1*}\,b' = \{h_{k+3,k+5}\}.$$

Choose $b \in \pi^5_{k,4}$ such that $p_{k+4,3*}\,b = b'$. Then

$$p_{k+4,1*}\,b = p_{k+4,1*}\,p_{k+4,3*}\,b = p_{k+4,1*}\,b' = \{h_{k+3\,k+5}\},$$

and

$$\begin{aligned} p_{k+5,4*}(h^*_{k+4,k+5}\{t_{k+6,6}\} - i_{k+4,1*}\,b) &= h^*_{k+4,k+5}\,p_{k+5,4*}\{t_{k+6,6}\} - i_{k+4,1*}\,p_{k+4,3*}\,b \\ &= h^*_{k+4,k+5}\{t_{k+6,5}\} - i_{k+4,1*}\,b' \\ &= 0. \end{aligned}$$

Thus

$$h^*_{k+4,k+5}\{t_{k+6,6}\} = i_{k+4,1*}b, \quad \text{where } p_{k+4,1*}b = \{h_{k+3,k+5}\}.$$

Hence

$$\pi^5_{k,6} = Z_2 + Z_2,$$

generated by $i_{k+3,3*}p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$, and $i_{k+5,1*}a$, where

$$p_{k+5,1*}a = \{h_{k+4,k+5}\}.$$

Note that $\Delta^{-1}_*(0) = 0$, whence p_{k+6*} is trivial.

10.7. $k \equiv 3 \pmod 8$.

Our first task is to calculate $\{t_{k+6,6}\} \in \pi^4_{k,5}$. We have that

$$t_{k+6,6}|S^{k+3} = i_{k+4,1}t_{k+5,5}$$

by § 2.3 (*b*), and that

$$\{t_{k+5,5}\} = 0$$

by § 9.1. Thus we can extend $i_{k+4,1}t_{k+5,5}$ over the hemisphere E^{k+4}_+ of S^{k+4}, and, since $t_{k+5,5}$ is a symmetric map (2.3*a*), we can extend it symmetrically over E^{k+4}_-. Denote this extension by

$$g: S^{k+4} \to i_{k+4,1}(V_{k+4,4}) \subset V_{k+5,5}.$$

Now we use construction 'Q^r' of § 6, with $r = k+4$, $X = V_{k+5,5}$, $f_1 = t_{k+6,6}$, and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\}+\{f_2\} = \{t_{k+6,6}\}+\{g\}.$$

Hence

$$p_{k+5,1*}2\{h\} = p_{k+5,1*}\{t_{k+6,6}\}+p_{k+5,1*}\{g\} = 2\{h_{k+4,k+4}\}$$

by § 2.3 (*b*) and since $\{g\} \in i_{k+4,1*}\pi^4_{k,4}$. Thus

$$p_{k+5,1*}\{h\} = \{h_{k+4,k+4}\}$$

and

$$\{h\} = \{\mathrm{p}h_{k+4,k+4}\}+i_{k+4,1*}w \quad (w \in \pi^4_{k,4}).$$

Further, if we consider $\{g\}$ for the moment as in $\pi^4_{k,4}$, we see that $p_{k+4,1}g: S^{k+4} \to S^{k+3}$ is a symmetric map such that, in the notation of § 2.3 (*c*),

$$p_{k+4,1}gu^{-1}_{k+3} = p_{k+4,1}t_{k+5,5}u^{-1}_{k+3}: P^{k+3} \to S^{k+3},$$

which is essential by § 2.3 (*c*): whence $p_{k+4,1}g$ is essential by Theorem 6.1.

Thus, when $k \geqslant 11$, $\{g\}$ generates $i_{k+4,1*}\pi^4_{k,4}$ [see §§ 8.4 (*d*) and 9.1], and

$$\{t_{k+6,6}\} = 2\{h\}-\{g\}$$

$$= 2\{\mathrm{p}h_{k+4,k+4}\}+2i_{k+4,1*}w-\{g\},$$

i.e.

$$\{t_{k+6,6}\} = i_{k+4,1*}a+2\mathrm{p}_*\{h_{k+4,4}\} \quad \textit{when } k \geqslant 11,$$

where a generates $\pi^4_{k,4}$.

When $k = 3$, we have from § 8.4 (*e*) that $\pi^4_{3,4} = Z_4 + Z_\infty$. If a generates the Z_4 summand and b an infinite cyclic one, then it follows from the fact that $p_{7,1}g$ is essential that $\{g\}$ is of the form

$$\{g\} = i_{7,1*}[\alpha a+(2\beta+1)b].$$

Thus

$$\begin{aligned}\{t_{9,6}\} &= 2\{h\}-\{g\}\\ &= 2\mathrm{p}_*\{h_{7,7}\}+i_{7,1*}[2\alpha' a+2\beta' b-\alpha a-(2\beta+1)b]\\ &= 2\mathrm{p}_*\{h_{7,7}\}+i_{7,1*}[\sigma a-(2\rho+1)b]\\ &= 2\mathrm{p}_*\{h_{7,7}\}-i_{7,1*}(2\rho+1)(b\mp\sigma a).\end{aligned}$$

Hence

$$\{t_{9,6}\} = i_{7,1*}(2\rho+1)c+2\mathrm{p}_*\{h_{7,7}\},$$

where c generates an infinite summand of $\pi^4_{3,4}$.

(*a*) When $p = 4$ and $k \geqslant 11$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.

$$\to Z_\infty \to Z_8+Z_\infty \to \pi^4_{k,6} \to 0,$$

by §§ 9.1 and 8.4 (*d*). But $i^{-1}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$: that is, by $i_{k+4,1*}a+2\mathrm{p}_*\{h_{k+4,k+4}\}$ in the above notation. Thus

$$\pi^4_{k,6} = Z_{16} \quad (k \geqslant 11),$$

generated by $\{i_{k+5,1}\,\mathrm{p}h_{k+4,k+4}\}$. Note that p_{k+5*} is trivial.

(*b*) When $p = 4$ and $k = 3$, (E) gives

$$\xrightarrow{p_{8*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^4_{3,5} \xrightarrow{i_{7*}} \pi^4_{3,6} \to \pi_7(S^8),$$

i.e.

$$\to Z_\infty \to Z_4+Z_\infty+Z_\infty \to \pi^4_{3,6} \to 0,$$

by §§ 9.1 and 8.4 (*e*). But $i^{-1}_{7*}(0)$ is generated by $\{t_{9,6}\}$, i.e. by

$$[i_{7,1*}(2\rho+1)c+2\mathrm{p}_*\{h_{7,7}\}]$$

in the above notation. Thus

$$\pi^4_{3,6} = Z_4+Z_\infty,$$

generated by $i_{6,3*}a$, where a is of order four and $p_{6,1*}a = \{h_{5,7}\}$, and $[\{i_{8,1}\,\mathrm{p}h_{7,7}\}+\rho i_{7,2*}c]$, where c generates an infinite summand of $\pi^4_{3,4}$, i.e. $p_{7,1*}c = \{h_{6,7}\}$. Note that p_{8*} is trivial.

(*c*) When $k = 5$ and $k \geqslant 11$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.

$$\to Z_2 \to Z_2+Z_2+Z_2 \to \pi^5_{k,6} \to 0,$$

by §§ 9.1 and 8.4 (*f*), and since p_{k+5*} is trivial by (*a*). Further

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}),$$

which is generated by

$$h^*_{k+4,k+5}[i_{k+4,1*}a+2\mathrm{p}_*\{h_{k+4,k+4}\}] = i_{k+4,1*}h^*_{k+4,k+5}a.$$

But

$$p_{k+4,1*}h^*_{k+4,k+5}a = h^*_{k+4,k+5}p_{k+4,1*}a = h^*_{k+4,k+5}\{h_{k+3,k+4}\}$$
$$= \{h_{k+3,k+5}\}.$$

Thus we see from § 8.4 (f) that $i^{-1}_{k+5*}(0)$ is a Z_2 summand, and that

$$\pi^5_{k,6} = Z_2+Z_2,$$

generated by $i_{k+3,3*}p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$, and $\{i_{k+5,1}\,\mathrm{p}h_{k+4,k+5}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{k+6*} is trivial.

(d) When $p = 5$ and $k = 3$, (E) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^5_{3,5} \xrightarrow{i_{8*}} \pi^5_{3,6} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e. $$\to Z_2 \to Z_2+Z_2+Z_2+Z_2 \to \pi^5_{3,6} \to 0,$$

by §§ 9.1 and 8.4 (g), and since p_{8*} is trivial by (b). Further

$$i^{-1}_{8*}(0) = t_{9,6*}\pi_8(S^7),$$

which is generated by

$$h^*_{7,8}[i_{7,1*}c+2(\rho i_{7,1*}c+\mathrm{p}_*\{h_{7,7}\})] = i_{7,1*}h^*_{7,8}c.$$

But $$p_{7,1*}h^*_{7,8}c = h^*_{7,8}p_{7,1*}c = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Thus we see from § 8.4 (g) that $i^{-1}_{8*}(0)$ is a Z_2 summand, and that

$$\pi^5_{3,6} = Z_2+Z_2+Z_2,$$

generated by $i_{5,4*}p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$, $i_{6,3*}a$ where $p_{6,1*}a = \{h_{5,8}\}$, and $\{i_{8,1}\,\mathrm{p}h_{7,8}\}$. Note that p_{9*} is again trivial.

11. Calculation of $\pi^p_{k,7}$

We consider the fibring $V_{k+7,7}/V_{k+6,6} \to S^{k+6}$, and examine the sequence

$$\text{(F)} \qquad \to \pi_{k+p+1}(S^{k+6}) \xrightarrow{\Delta_*} \pi^p_{k,6} \xrightarrow{i_{k+p*}} \pi^p_{k,7} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+6}) \to.$$

11.1. $k \equiv 1 \pmod 8$.

In this case there is a six-field on S^{k+6} (1), and so the fibring admits a cross-section p. Hence Theorem 1.1 gives that

$$\pi^p_{k,7} = i_*\pi^p_{k,6}+\mathrm{p}_*\pi_{k+p}(S^{k+6}).$$

Using the values of $\pi^p_{k,6}$ as calculated in §§ 10.4 and 10.5, we obtain the values shown in the tables for $\pi^p_{k,7}$ when $k \equiv 1 \pmod 8$. Those of $\pi^p_{2,6}$ are then obtained by Theorem 4.2 (a).

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+7,7}\} = 0 \quad \textit{for } k \equiv 1 \pmod 8.$$

11.2. $k \equiv 5 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e. $$\to Z_\infty \to Z_2 \to \pi^5_{k,7} \to 0,$$

by § 10.4 (b). Also $i^{-1}_{k+5*}(0) \neq 0$, for otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a six-field on S^{k+6}, which is impossible by Theorem 1.1 of (3). Thus $i^{-1}_{k+5*}(0) = \pi^5_{k,6}$, and

$$\pi^5_{k,7} = 0.$$

Note that the image of Δ_* is of order two, whence it follows that the image of p_{k+6*} is the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$. Further, since $\{t_{k+7,7}\}$ generates $i^{-1}_{k+5*}(0)$, we have that

$$\{t_{k+7,7}\} = \{i_{k+3,3}\, \mathrm{p} h_{k+2,k+5}\}.$$

11.3. $k \equiv 7 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_2 \to \pi^5_{k,7} \to 0,$$

by § 10.6 (b). Also $i^{-1}_{k+5*}(0) \neq 0$ since otherwise this again would imply a six-field on S^{k+6}, which is impossible by Theorem 1.1 in (3). But by exactness $i^{-1}_{k+5*}(0)$ is cyclic, whence $$\pi^5_{k,7} = Z_2.$$

Note that the image of p_{k+6*} is again the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$.

In order to determine the generator of $\pi^5_{k,7}$ we evaluate $\{t_{k+6,6}\}$ which generates $i^{-1}_{k+5*}(0)$. For this we consider the section of the sequence associated with the fibring $V_{k+6,6}/V_{k+3,3} \to V_{k+6,3}$ which is of the form

$$\to \pi^5_{k,3} \xrightarrow{i_{k+3,3*}} \pi^5_{k,6} \xrightarrow{p_{k+6,3*}} \pi^2_{k+3,3} \to.$$

Then we have by § 2.3 (b) that

$$p_{k+6,3*}\{t_{k+7,7}\} = \{t_{k+7,4}\} = \{i_{k+5,1}\, \mathrm{p} h_{k+4,k+5}\}$$

by § 8.2 (a). Thus $\{t_{k+7,7}\} \notin i_{k+3,3*}\pi^5_{k,3}$, whence

$$\{t_{k+7,7}\} = i_{k+5,1*}\, a, \quad \textit{where} \quad p_{k+5,1*}\, a = \{h_{k+4,k+5}\}.$$

Thus $\pi^5_{k,7}$ is generated by $i_{k+3,4*}\, p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$.

11.4. $k \equiv 3 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_2 \to \pi^5_{k,7} \to 0 \qquad \text{when } k \geqslant 11,$$

and $$\to Z_\infty \to Z_2 + Z_2 + Z_2 \to \pi^5_{3,7} \to 0 \quad \text{when } k = 3,$$

by §§ 10.7 (c) and (d). Again $i_{k+5*}^{-1}(0) \neq 0$, since otherwise we should again have an impossible six-field on S^{k+6}, and again it is cyclic. Hence
$$\pi_{k,7}^5 = Z_2 \quad when\ k \geqslant 11,\ and \quad \pi_{3,7}^5 = Z_2 + Z_2.$$
Note that the image of p_{k+6*} is again the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$.

To determine the generators of $\pi_{k,7}^5$ we evaluate $\{t_{k+7,7}\}$ by examining the sequence
$$\to \pi_{k,3}^5 \xrightarrow{i_{k+3,3*}} \pi_{k,6}^5 \xrightarrow{p_{k+6,3*}} \pi_{k+3,3}^2 \to$$
which is associated with the fibring $V_{k+6,6}/V_{k+3,3} \to V_{k+6,3}$. From § 2.3 (b) we have that
$$p_{k+6,3*}\{t_{k+7,7}\} = \{t_{k+7,4}\}$$
$$= \{i_{k+5,1}\, \mathrm{p}h_{k+4,k+5}\}, \quad \text{by § 8.2 (a).}$$
Thus
$$\{t_{k+7,7}\} \notin i_{k+3,3*}\,\pi_{k,3}^5,$$
whence
$$\{t_{k+7,7}\} = \{i_{k+5,1}\, \mathrm{p}h_{k+4,k+5}\} + w, \quad where\ w \in i_{k+3,3*}\,\pi_{k,3}^5.$$
Thus $i_{k+3,4*}\,p_{k+3,1*}^{-1}\{h_{k+2,k+5}\}$ generates $\pi_{k,7}^5$ when $k \geqslant 11$; and $\pi_{3,7}^5$ is generated by $i_{5,5*}\,p_{5,1*}^{-1}\mathfrak{E}\{h_{3,7}\}$ and $i_{6,4*}\,a$, where $p_{8,1*}\,a = \{h_{5,8}\}$.

11.5. $k \equiv 0 \pmod 4$.

When $p = 5$, (F) gives
$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi_{k,6}^5 \xrightarrow{i_{k+5*}} \pi_{k,7}^5 \to \pi_{k+5}(S^{k+6}),$$
i.e.
$$\to Z_\infty \to Z_\infty \to \pi_{k,7}^5 \to 0 \qquad \text{when } k > 4,$$
and
$$\to Z_\infty \to Z_2 + Z_2 + Z_2 + Z_\infty \to \pi_{4,7}^5 \to 0 \quad \text{when } k = 4,$$
by § 10.3 (b). Also $i_{k+5*}^{-1}(0)$ is generated by $\{t_{k+7,7}\}$ and, by § 2.3 (b),
$$p_{k+6,1*}\{t_{k+7,7}\} = 2\{h_{k+5,k+5}\}.$$
Thus $i_{k+5*}^{-1}(0)$ is a Z_∞ summand of $\pi_{k,6}^5$, whence
$$\pi_{k,7}^5 = 0 \quad when\ k > 4,\ and \quad \pi_{4,7}^5 = Z_2 + Z_2 + Z_2,$$
generated by $\{i_{5,6}\,\mathfrak{E}h_{3,8}\}$, $\{i_{5,6}\,\bar{p}h_{7,9}\}$ and $\{i_{6,5}\,\mathrm{p}h_{5,9}\}$. Note that in both cases $\Delta_*^{-1}(0) = 0$, whence p_{k+6*} is trivial.

11.6. $k \equiv 6 \pmod 8$.

When $p = 5$, (F) gives
$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi_{k,6}^5 \xrightarrow{i_{k+5*}} \pi_{k,7}^5 \to \pi_{k+5}(S^{k+6}),$$
i.e.
$$\to Z_\infty \to \pi_{k,5}^5 + Z_\infty \to \pi_{k,7}^5 \to 0,$$
by § 10.2 (b). But $i_{k+5*}^{-1}(0)$ is generated by $\{t_{k+7,7}\}$, and, by § 2.3 (b), we have that
$$p_{k+6,1*}\{t_{k+7,7}\} = 2\{h_{k+5,k+5}\},$$

whence it follows that $\{t_{k+7,7}\}$ generates the Z_∞ summand displayed above. Thus $\pi^5_{k,7} \approx \pi^5_{k,5}$, i.e.

$$\pi^5_{k,7} = Z_8 \quad \textit{when } k > 6,$$

generated by $i_{k+5,2*}\,a$, where $p_{k+5,1*}\,a = \{h_{k+4,k+5}\}$; and

$$\pi^5_{6,7} \textit{ is either } Z_\infty + Z_8 \textit{ or } Z_4 + Z_\infty,$$

generated by $\{i_{7,6}\,h_{6,11}\}$ and $i_{11,2*}\,a$, where $8a = 0$ and $p_{11,1*}\,a = \{h_{10,11}\}$; or by $i_{10,3*}\,a$, where $4a = 0$ and $p_{10,1*}\,a = \{h_{9,11}\}$, and $i_{11,2*}\,b$ where $p_{11,1*}\,b = \{h_{10,11}\}$.

Note that in all cases p_{k+6*} is trivial.

11.7. $k \equiv 2 \pmod 8$ *and* $\geqslant 10$.

Our first task is to calculate $\{t_{k+7,7}\}$ in $\pi^5_{k,6} = Z_8 + Z_\infty$, by §§ 10.1 and 9.5 (*c*). We have from § 2.3 (*b*) that

$$t_{k+7,7} \mid S^{k+4} = i_{k+5,1}\,t_{k+6,6},$$

and from § 10.1 that $\{t_{k+6,6}\} = 0$. Thus we can extend $i_{k+5,1}\,t_{k+6,6}$ over the hemisphere E^{k+5}_+ of S^{k+5}, and, since $t_{k+6,6}$ is a symmetric map (2.3 *a*), we can extend it symmetrically over E^{k+5}_-. Denote this extension by

$$g : S^{k+5} \to i_{k+5,1}\,V_{k+5,5} \subset V_{k+6,6}.$$

Now we use construction 'Q^r' of § 6, with $r = k+4$, $X = V_{k+6,6}$, $f_1 = t_{k+7,7}$ and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\} + \{f_2\} = \{t_{k+7,7}\} + \{g\}.$$

Hence,
$$p_{k+6,1*}\,2\{h\} = p_{k+6,1*}\{t_{k+6,6}\} + p_{k+6,1*}\{g\}$$
$$= 2\{h_{k+5,k+5}\}$$

by § 2.3 (*b*) and since $\{g\} \in i_{k+5,1*}\,\pi^5_{k,5}$. Thus

$$p_{k+6,1*}\{h\} = \{h_{k+5,k+5}\}$$

and
$$\{h\} = \{\mathrm{p}h_{k+5,k+5}\} + i_{k+5,1*}\,w, \quad \textit{where } w \in \pi^5_{k,5}.$$

Further, if we consider $\{g\}$ for the moment as in $\pi^5_{k,5}$, we see that $p_{k+5,1}\,g : S^{k+5} \to S^{k+4}$ is a symmetric map such that, in the notation of § 2.3 (*c*),

$$p_{k+5,1}\,g u^{-1}_{k+4} = p_{k+5,1}\,t_{k+6,6}\,u^{-1}_{k+4} : P^{k+4} \to S^{k+4},$$

which is essential by § 2.3 (*c*): whence $p_{k+5,1}\,g$ is essential by Theorem 6.1. Thus $\{g\}$ generates $i_{k+5,1*}\,\pi^5_{k,5}$ (see §§ 9.5 (*c*) and 10.1). But

$$\{t_{k+7,7}\} = 2\{h\} - \{g\}$$
$$= 2\{\mathrm{p}h_{k+5,k+5}\} + 2i_{k+5,1*}\,w - \{g\},$$

i.e.
$$\{t_{k+7,7}\} = i_{k+5,1*}\,a + 2\mathrm{p}_*\{h_{k+5,k+5}\}, \quad \textit{where } a \textit{ generates } \pi^5_{k,5}.$$

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e. $$\to Z_\infty \to Z_8 + Z_\infty \to \pi^5_{k,7} \to 0,$$

by §§ 10.1 and 9.5 (c). Since $i^{-1}_{k+5*}(0)$ is generated by $\{t_{k+7,7}\}$ evaluated above,

$$\pi^5_{k,7} = Z_{16},$$

generated by $\{i_{k+6,1}\, \mathrm{p}h_{k+5,k+5}\}$. Note that p_{k+6*} is trivial.

12. Calculation of $\pi^p_{1,8}$

We consider the fibring $V_{9,8}/V_{8,7} \to S^8$ and examine the sequence

(G) $$\to \pi_{p+2}(S^8) \xrightarrow{\Delta_*} \pi^p_{1,7} \xrightarrow{i_{p+1*}} \pi^p_{1,8} \xrightarrow{p_{p+1*}} \pi_{p+1}(S^8) \to.$$

Our first task is to calculate $\{t_{9,8}\}$ in $\pi^6_{1,7} = Z_\infty + Z_\infty$ by §§ 11.1 and 10 5 (c). Since $\{t_{8,7}\} = 0$ by § 11.1, the method used is word for word that used in § 11.7 above and yields the result that

$$\{t_{9,8}\} = i_{7,1*}(2\lambda+1)a + 2\mathrm{p}_*\{h_{7,7}\}, \quad \textit{where } a \textit{ generates } \pi^6_{1,6}.$$

(a) When $p = 6$, (G) gives

$$\xrightarrow{p_{9*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^6_{1,7} \xrightarrow{i_{7*}} \pi^6_{1,8} \to \pi_7(S^8),$$

i.e. $$\to Z_\infty \to Z_\infty + Z_\infty \to \pi^6_{1,8} \to 0,$$

by §§ 11.1 and 10.5 (c). Also $i^{-1}_{7*}(0)$ is generated by $\{t_{9,8}\}$ evaluated above. Thus

$$\pi^6_{1,8} = Z_\infty,$$

generated by $$[\{i_{8,1}\, \mathrm{p}h_{7,7}\} + \lambda i_{7,2*}\, a],$$

where $p_{7,1*}\, a = \{h_{6,7}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{8*} is trivial. We also have, by Theorem 4.2 (a), that

$$\pi^5_{2,7} = Z_\infty,$$

generated by $$[\{i_{8,1}\, \mathrm{p}h_{7,7}\} + \lambda i_{7,2*}\, a],$$

where $p_{7,1*}\, a = \{h_{6,7}\}$.

(b) When $p = 7$, (G) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^7_{1,7} \xrightarrow{i_{8*}} \pi^7_{1,8} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e. $$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^7_{1,8} \to 0,$$

by §§ 11.1 and 10.5 (d), and since p_{8*} is trivial by (a). Also $i^{-1}_{8*}(0)$ is generated by

$$h^*_{7,8}\{t_{9,8}\} = h^*_{7,8}[i_{7,1*}\, a + 2(i_{7,1*}\, \lambda a + \mathrm{p}_*\{h_{7,7}\})]$$
$$= i_{7,1*}\, h^*_{7,8}\, a.$$

But $$p_{7,1*}\, h^*_{7,8}\, a = h^*_{7,8}\, p_{7,1*}\, a = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Hence $i_{8*}^{-1}(0)$ is a Z_2 summand, whence we have that

$$\pi_{1,8}^7 = Z_2 + Z_2,$$

generated by $i_{6,3*}\, p_{6,1*}^{-1}\{h_{5,8}\}$ and $\{i_{8,1}\, \mathrm{p} h_{7,8}\}$. Note that p_{9*} again is trivial. As before, we have by Theorem 4.2 (*a*) that

$$\pi_{2,7}^6 = Z_2 + Z_2,$$

generated by $i_{6,3*}\, p_{6,1*}^{-1}\{h_{5,8}\}$ and $\{i_{8,1} \mathrm{p} h_{7,8}\}$.

13. Calculation of $\pi_{1,9}^7$

We consider the fibring $V_{10,9}/V_{9,8} \to S^9$, and examine

$$\xrightarrow{p_{9*}} \pi_9(S^9) \xrightarrow{\Delta_*} \pi_{1,8}^7 \xrightarrow{i_{8*}} \pi_{1,9}^7 \to \pi_8(S^9) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,9}^7 \to 0,$$

by § 12 (*b*). To find $i_{8*}^{-1}(0)$ we have to evaluate its generator $\{t_{10,9}\}$. To do this consider the sequence

$$\to \pi_{1,5}^7 \xrightarrow{i_{6,3*}} \pi_{1,8}^7 \xrightarrow{p_{9,3*}} \pi_{6,3}^2 \to$$

which is associated with the fibring $V_{9,8}/V_{6,5} \to V_{9,3}$. Then we have from § 2.3 (*b*) that

$$p_{9,3*}\{t_{10,9}\} = \{t_{10,4}\} = \{i_{8,1}\, \mathrm{p} h_{7,8}\}, \quad \text{by § 8.2 } (a).$$

Hence
$$\{t_{10,9}\} \notin i_{6,3*}\, \pi_{1,5}^7,$$

i.e.
$$\{t_{10,9}\} = \{i_{8,1}\, \mathrm{p} h_{7,8}\} + w, \quad \textit{where } w \in i_{6,3*}\, \pi_{1,5}^7.$$

Thus $i_{8*}^{-1}(0)$ is a Z_2 subgroup, and so

$$\pi_{1,9}^7 = Z_2,$$

generated by $i_{6,4*}\, p_{6,1*}^{-1}\{h_{5,8}\}$. Note that, since the image of Δ_* is of order two, it follows that the image of p_{9*} is the Z_∞ subgroup generated by $2\{h_{9,9}\}$. Again Theorem 4.2 (*a*) gives

$$\pi_{2,8}^6 = Z_2,$$

generated by $i_{6,4*}\, p_{6,1*}^{-1}\{h_{5,8}\}$.

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