A REMARKABLE SERIES OF ORTHOGONAL FUNCTIONS (I)

By R. E. A. C. PALEY.

[Received 29 April, 1931.—Read 14 May, 1931.]

1. We define a sequence of functions $\phi_0(t)$, $\phi_1(t)$, ..., $\phi_n(t)$, ... by the following conditions:

$$\phi_0(t) = 1 \quad (0 \leqslant t < \tfrac{1}{2}),$$

$$\phi_0(t) = -1 \quad (\tfrac{1}{2} \leqslant t < 1),$$

$$\phi_0(t+1) = \phi_0(t),$$

$$\phi_n(t) = \phi_0(2^n t) \quad (n = 1, 2, \ldots).$$

We call the functions $\phi_n(t)$ Rademacher's† functions.

By means of Rademacher's functions, we define a new system as follows. Let $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}$; then we write $\psi_0(t) = 1$ and

$$\psi_n(t) = \phi_{n_1}(t)\,\phi_{n_2}(t) \ldots \phi_{n_\nu}(t) \quad (n > 0). \tag{1.1}$$

This system has already been obtained in a different way by Walsh‡. Walsh has proved that the equation (1.1) defines a normalized set of orthogonal functions for the interval (0, 1). Every function $f(t)$ absolutely integrable in the interval (0, 1) can be expanded (quite formally) by means of the functions $\psi_n(t)$ in the form of a series

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t), \tag{1.2}$$

where the numbers c_n are defined by means of the equation

$$c_n = \int_0^1 f(t)\,\psi_n(t)\,dt. \tag{1.21}$$

† See Rademacher, **9**; Khintchine, **6**; Paley and Zygmund, **8**; Kaczmarz and Steinhaus, **5**.
‡ Walsh, **12**.

Walsh established a connection between the series (1.2) and the corresponding expansion in terms of another series of orthogonal functions. These are the Haar† system of orthogonal functions $\{\chi\}$, which are defined in the following way. We write

$$\chi_0(t) = 1,$$
$$\chi_1(t) = 1 \quad (0 \leqslant t < \tfrac{1}{2}),$$
$$\chi_1(t) = -1 \quad (\tfrac{1}{2} \leqslant t < 1),$$
$$\chi_2^{(1)}(t) = \sqrt{2}, \quad \chi_2^{(2)}(t) = 0 \quad (0 \leqslant t < \tfrac{1}{4}),$$
$$\chi_2^{(1)}(t) = -\sqrt{2}, \quad \chi_2^{(2)}(t) = 0 \quad (\tfrac{1}{4} \leqslant t < \tfrac{1}{2}),$$
$$\chi_2^{(1)}(t) = 0, \quad \chi_2^{(2)}(t) = \sqrt{2} \quad (\tfrac{1}{2} \leqslant t < \tfrac{3}{4}),$$
$$\chi_2^{(1)}(t) = 0, \quad \chi_2^{(2)}(t) = -\sqrt{2} \quad (\tfrac{3}{4} \leqslant t < 1).$$

We divide the interval into 2^n equal sub-intervals, and denote by $i_n^{(m)}$ the interval $(m-1)/2^n \leqslant t < m/2^n$. Then we define the n-th group of functions $\{\chi\}$ in the following way:

$$\chi_n^{(m)}(t) = \sqrt{2^{n-1}} \text{ in the interval } i_n^{(2m-1)},$$
$$= -\sqrt{2^{n-1}} \text{ in the interval } i_n^{(2m)},$$
$$= 0 \text{ elsewhere} \quad (m = 1, 2, \ldots, 2^{n-1}).$$

Walsh proves that, if the function $f(t)$ is expanded by means of the functions $\{\chi\}$ in a series

$$f(t) \sim c_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} c_n^{(m)} \chi_n^{(m)}(t), \tag{1.3}$$

then the 2^n-th partial sums of the two series (1.2), (1.3) are equal. He is able to deduce a number of properties of the series (1.2) from the corresponding properties of the Haar orthogonal system. In a more recent paper‡ Kaczmarz uses the same method to obtain further properties of the system $\{\psi\}$. Most of the theorems which we obtain in this paper follow from a recent theorem of Hardy and Littlewood§ on averages.

We begin by giving a fresh proof that the system $\{\psi\}$ forms a normalized orthogonal system for the interval (0, 1); in other words that

$$\int_0^1 \psi_m(t)\,\psi_n(t) = 0 \quad (m \neq n), \tag{1.31}$$
$$= 1 \quad (m = n).$$

† Haar, **1**.

‡ Kaczmarz, **4**.

§ Hardy and Littlewood, **2**.

The second result is immediate. For the first we have only to observe† that

(1.4) $$\int_0^1 \phi_{n_1}^{\alpha_1}\phi_{n_2}^{\alpha_2}\ldots\phi_{n_\nu}^{\alpha_\nu}\,dt$$

vanishes, unless the numbers $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ are all even, in which case the integral is equal to 1.

The system is also complete‡. In fact, the functions

$$\psi_m(t) \quad (0 \leqslant m \leqslant 2^n-1)$$

are all constant in the intervals

$$i_n^{(\nu)} \equiv (\nu-1)\,2^{-n} \leqslant t < \nu 2^{-n} \quad (\nu = 1, 2, \ldots, 2^n).$$

Suppose that $f(t) \in L$ is a function for which

(1.5) $$\int_0^1 f(t)\,\psi_m(t) = 0 \quad (0 \leqslant m \leqslant 2^n-1).$$

We write $$I_n^{(\nu)} = \int_{(\nu-1)\,2^{-n}}^{\nu 2^{-n}} f(t)\,dt \quad (\nu = 1, 2, \ldots, 2^n).$$

Then (1.5) gives

(1.6) $$\int_0^1 f(t)\,\psi_n(t) = \sum_{\nu=1}^{2^n} I_n^{(\nu)}\,\psi_m(i_n^{(\nu)}) = 0.$$

The equation (1.6) is satisfied for $0 \leqslant m \leqslant 2^n-1$. Now, in virtue of (1.31), the determinant

$$|\psi_m(i_n^{(\nu)})| \quad (0 \leqslant m \leqslant 2^n-1,\ 1 \leqslant \nu \leqslant 2^n),$$

does not vanish. Thus the numbers $\psi_m(i_n^{(\nu)})$ are linearly independent, and it follows from (1.6) that

(1.7) $$I_n^{(\nu)} = \int_{(\nu-1)\,2^{-n}}^{\nu 2^{-n}} f(t)\,dt = 0 \quad (\nu = 1, 2, \ldots, 2^n).$$

Now suppose that (1.5) is satisfied for all m, and let $F(t)$ denote the continuous function

$$\int_0^t f(\theta)\,d\theta.$$

† See Paley and Zygmund, **8**, 340–341.

‡ See also Kaczmarz, **4**, 190-191.

Then we have, by (1.7),

$$F\{(\nu-1)\,2^{-n}\} = F(\nu 2^{-n}) \quad (\nu = 1, 2, \ldots, 2^n, \; n = 1, 2, \ldots),$$

and so $F(t)$ is a constant. It follows that $f(t)$ is equivalent to zero. Thus the functions $\psi_n(t)$ form a normal complete orthogonal system.

2. Walsh† observes that the behaviour of the functions (1.1) is in many respects similar to that of the trigonometrical functions. In particular, he discusses this similarity in the case of convergence at a point, and of uniqueness theorems. It will appear in this paper that in other directions this similarity is even more striking.

We begin with some more or less elementary theorems. These are the consequences of Hardy and Littlewood's maximal theorem. We first prove a generalization of Kaczmarz's result, that if $f(t) \in L$ and $s_n(t)$ denotes the n-th partial sum‡ of its expansion in terms of the system $\{\psi\}$, then $s_{2^n}(t) \to f(t)$ for almost all t. Kaczmarz's proof deduces this result from the corresponding theorem for Haar's functions§. If $f(t) \in L^k$, where $k > 1$, we are able to prove a rather stronger result for both systems of functions by an entirely different method.

We go on to consider some results of a more difficult nature. Let $f_n(t)$ denote the difference $s_{2^{n+1}} - s_{2^n}$. We show that

$$(2.1) \qquad B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k}\,dt \leqslant \int_0^1 |f(t)|^k\,dt \leqslant B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k}\,dt \quad (1 < k < \infty),$$

B_k denoting a constant which depends only on k (it may denote different constants in different contexts). The corresponding result for Fourier series is true‖, but the proof for Walsh-Kaczmarz functions is considerably easier. From the result (2.1) we now deduce the $\{\psi\}$ analogue of M. Riesz's well known result¶

$$\int_0^1 |s_n(t)|^k\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (1 < k < \infty).$$

---

† Walsh, 12.

‡ By the n-th partial sum we mean

$$\sum_{m=0}^{n-1} \psi_m(t) \int_0^1 f(\theta)\,\psi_m(\theta)\,d\theta.$$

§ Haar, 1.

‖ Littlewood and Paley, 7.

¶ M. Riesz, 10.

We next consider a generalization of the result (2.1). Suppose that $\{\lambda_n\}$ denotes an increasing lacunary sequence, that is to say a sequence for which $\lambda_{n+1}/\lambda_n \geqslant q > 1$. We denote by $\delta_n(t)$ the difference

$$s_{\lambda_n}(t) - s_{\lambda_{n-1}}(t).$$

Then

$$(2.2) \qquad B_{k,\delta} \int_0^1 \{\Sigma\, \delta_n^2\}^{\frac{1}{2}k}\, dt \leqslant \int_0^1 |f(t)|^k\, dt \leqslant B_{k,\delta} \int_0^1 \{\Sigma\, \delta_n^2\}^{\frac{1}{2}k}\, dt \quad (1 < k < \infty).$$

The constants $B_{k,\delta}$ depend only on k and δ. From (2.2) we may obtain a generalization of the theorem

$$s_{2^n} \to f \quad \text{p.p. in } (0, 1).$$

In fact, if $\{\lambda_n\}$ denotes a lacunary sequence of the type already discussed, then

$$(2.3) \qquad s_{\lambda_n}(t) \to f(t)$$

for almost all t in (0, 1). The results (2.2), (2.3) have already† been established for Fourier series.

3. We begin with the following lemma, due to Hardy and Littlewood‡. We denote by B, here and in the sequel, a positive absolute constant, and by B_k, B_δ, ... constants which depend only on k, δ, ... (B, B_k, B_δ, ... may denote different constants in different contexts).

LEMMA 1. *Let $f(t)$ be a function absolutely integrable in* (0, 1). *Let $F(t)$ denote the upper bound*

$$F(t) = \sup \frac{1}{2x} \int_{t-x}^{t+x} |f(\theta)|\, d\theta.$$

Then

$$\int_0^1 F^k(t)\, dt \leqslant B_k \int_0^1 |f(t)|^k\, dt \quad (k > 1);$$

$$\int_0^1 F(t)\, dt \leqslant B \int_0^1 |f(t)| \overset{+}{\log} |f(t)|\, dt + B,$$

the right-hand side in each case being supposed to exist.

The function $F(t)$ is what Hardy and Littlewood call the maximum average of f at the point t. They show that numerous functions common in analysis, such, for example, as the Cesàro mean of positive order of the

† Littlewood and Paley, 7.
‡ Hardy and Littlewood, 2.

Fourier series of f at the point t, are maximized by $F(t)$. For the functions $\{\psi\}$ we have

LEMMA 2. *With the notation of the last lemma*

$$|s_{2^n}(t)| \leqslant 2F(t).$$

We have
$$s_{2^n}(t) = \sum_{m=0}^{2^n-1} \psi_m(t) \int_0^1 f(\theta)\,\psi_m(\theta)\,d\theta$$
$$= \int_0^1 f(\theta) \left[\sum_{m=0}^{2^n-1} \psi_m(t)\,\psi_m(\theta)\right] d\theta.$$

The kernel $\sum_{m=0}^{2^n-1} \psi_m(t)\,\psi_m(\theta)$ is identically equal to

$$\prod_{m=0}^{n-1} \{1+\phi_m(t)\,\phi_m(\theta)\}, \tag{3.1}$$

and the expression (3.1) vanishes except in an interval of length 2^{-n} enclosing the point t. In this interval it takes the value 2^n. Thus

$$|s_{2^n}(t)| \leqslant 2^n \int_{t-2^{-n}}^{t+2^{-n}} |f(\theta)|\,d\theta \leqslant 2F(t).$$

This proves the lemma.

From the last two lemmas we deduce the following theorem.

THEOREM I. *Let $n(t)$ denote an integer which varies arbitrarily with t. Then, with the previous notation,*

$$\int_0^1 |s_{2^{n(t)}}(t)|^k\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt \quad (k>1); \tag{3.2}$$

$$\int_0^1 |s_{2^{n(t)}}(t)|\,dt \leqslant B \int_0^1 |f(t)| \overset{+}{\log} |f(t)|\,dt + B;$$

$$|s_{2^{n(t)}}(t)| \leqslant \max |f(t)|,$$

the right-hand side in each case being supposed to exist.

Suppose that $2^{l-1} \leqslant n < 2^l-1$, and that $n = 2^{l-1}+(m-1)$. We write $\chi_n(t) = \chi_l^{(m)}(t)$, and rewrite the series (1.3) as

$$\sum_{n=0}^{\infty} c_n\,\chi_n(t).$$

If t is fixed we observe that exactly one of the functions

$$\chi_n(t) \quad (2^{l-1} \leqslant n \leqslant 2^l-1,\ l \text{ fixed})$$

differs from zero.

From Theorem I we may easily deduce the following result.

THEOREM II. *Let $n(t)$ denote an integer which varies arbitrarily with t. Let $s'_{n(t)}(t)$ denote the $n(t)$-th partial sum of the expansion of $f(t)$ by means of the Haar system of orthogonal functions. Then*

$$\int_0^1 |s'_{n(t)}(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt \quad (k > 1);$$

$$\int_0^1 |s'_{n(t)}(t)|\, dt \leqslant B \int_0^1 |f(t)| \overset{+}{\log} |f(t)|\, dt + B;$$

$$|s'_{n(t)}| \leqslant \max f(t),$$

the right-hand side in each case being supposed to exist.

Suppose that $2^{m(t)} < n(t) \leqslant 2^{m(t)+1}$. Let

$$f(t) \sim \sum_{n=0}^{\infty} c_n' \chi_n(t)$$

denote the Haar expansion of $f(t)$. Then, since at most one of the terms

$$c_n' \chi_n(t) \quad (2^{m(t)} \leqslant n < 2^{m(t)+1})$$

can differ from zero, it follows that $s'_{n(t)}(t)$ is equal to either $s'_{2^{m(t)}}(t)$ or $s'_{2^{m(t)}+1}(t)$. But since†

$$s'_{2^{m(t)}}(t) = s_{2^{m(t)}}(t),$$

the required result follows at once from Theorem I.

THEOREM III. *Let n denote a fixed positive integer. Then*

$$\int_0^1 |s_{2^n}(t)|\, dt \leqslant \int_0^1 |f(t)|\, dt,$$

the right-hand side being supposed to exist.

Let $g(t)$ denote an arbitrary function of unit modulus, and let $\gamma_{2^n}(t)$ denote the 2^n-th partial sum of the $\{\psi\}$ expansion of $g(t)$. Then, in virtue of Theorem I, we have $|\gamma_{2^n}(t)| \leqslant 1$. Let

$$f(t) \sim \sum_{m=0}^{\infty} a_m \psi_m(t), \quad g(t) \sim \sum_{m=0}^{\infty} b_m \psi_m(t).$$

† Kaczmarz, 4, 191.

Then, since $s_{2^n}(t)$ is a polynomial, we may use (1.21) to obtain

$$\int_0^1 s_{2^n}(t)\, g(t)\, dt = \sum_{m=0}^{2^n-1} a_m b_m = \int_0^1 \gamma_{2^n}(t) f(t)\, dt \leqslant \int_0^1 |f(t)|\, dt,$$

from which the desired result follows.

The following result has been established by Kaczmarz†.

THEOREM IV. *If $f(t) \epsilon L$, then, for almost all t in* (0, 1),

$$s_{2^n}(t) \to f(t).$$

4. For the theorem which follows we need the following two lemmas. These, with Hardy and Littlewood's Lemma 1, are the key lemmas to the theory. Lemma 3 is the averaging lemma, and expresses the fact that functions of the form (4.1) are of more or less uniform modulus in the interval (0, 1). Thus the functions $\phi_m(t)$ are averaging functions. Lemma 4 we use for interpolation. If we have established a result of the form (3.2) for two values of k, we may often apply the lemma to give a similar result for intermediate values of k. We use the lemma exclusively in this paper with $\gamma = \alpha$, and generally one of the two cases from which we start is that with $\gamma = \alpha = \frac{1}{2}$. Sometimes the other is that with $\gamma = \alpha = 0$, and in this case the expression (4.2) is to be interpreted as meaning

$$\text{Effective maximum } |T(f)|/\text{Effective maximum } |f|.$$

LEMMA 3. *Let $\Phi(t)$ denote the function*

$$\sum_{m=0}^{\infty} c_m \phi_m(t). \tag{4.1}$$

Then, if $0 \leqslant r < \infty$,

$$B_r \left\{ \sum_{m=0}^{\infty} c_m^2 \right\}^{\frac{1}{2}r} \leqslant \int_0^1 |\Phi(t)|^r\, dt \leqslant B_r \left\{ \sum_{m=0}^{\infty} c_m^2 \right\}^{\frac{1}{2}r}.$$

This result is due to Khintchine‡.

LEMMA 4. *Let f be a function defined in a field λ, and let $T(f)$ be a function defined in a field μ, whose values depend on the values of f in λ. Then $T(f)$ is described as a linear operation of the class $L_\lambda{}^\alpha$, if*

(i) *the operation is distributive, that is to say, for arbitary constants p and q we have*

$$T(pf_1 + qf_2) = pT(f_1) + qT(f_2);$$

† Kaczmarz, **4**, Theorem II.

‡ Khintchine, **6**; see also Paley and Zygmund, **8**.

(ii) *there exists a constant M such that*

$$| T(f) | \leqslant M \left\{ \int | f |^a d\lambda \right\}^{1/a}.$$

Let $M_{\alpha, \gamma}$ denote the upper bound for varying f of the ratio

$$\left\{ \int | T(f) |^c d\mu \right\}^{1/c} \Big/ \left\{ \int | f |^a d\lambda \right\}^{1/a}, \tag{4.2}$$

where $\alpha = 1/a$, $\gamma = 1/c$. Let (α, γ) describe a segment of a straight line, situated in the triangle

$$0 \leqslant \alpha \leqslant 1, \quad 0 \leqslant \gamma \leqslant 1, \quad \gamma \leqslant \alpha.$$

Then $\log M_{\alpha, \gamma}$ *is a convex function of the points of the segment.*

The result is due to M. Riesz†.

5. We now proceed to the proof of the following theorem.

THEOREM V. *Let $f(t) \epsilon L$ and have the $\{\psi\}$ expansion*

$$f(t) \sim \sum_{m=0}^{\infty} c_m \psi_m(t).$$

Let $f_n(t)$ denote the partial sum

$$f_n(t) = \sum_{m=2^n}^{2^{n+1}-1} c_m \psi_m(t) \quad (n = 0, 1, 2, \ldots). \tag{5.01}$$

For simplicity let $c_0 = 0$. Then, for $1 < k < \infty$,

$$B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k} dt \leqslant \int_0^1 | f(t) |^k dt \leqslant B_k \int_0^1 \{\Sigma f_n^2(t)\}^{\frac{1}{2}k} dt, \tag{5.1}$$

whenever either member exists.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a set of arbitrary unit factors. Let

$$f^*(t) = \sum_{n=0}^{\infty} \epsilon_n f_n(t).$$

Then

$$B_k \int_0^1 | f^*(t) |^k dt \leqslant \int_0^1 | f(t) |^k dt \leqslant B_k \int_0^1 | f^*(t) |^k dt \quad (1 < k < \infty). \tag{5.2}$$

The constants B_k are independent of the choice of the numbers ϵ_n.

---

† M. Riesz, 11.

It is convenient to divide the proof up into a number of parts, which we state separately as lemmas.

LEMMA 5. *The assertions* (5.1), (5.2) *are equivalent.*

That (5.2) follows from (5.1) is immediate. For the opposite result we observe that, if (5.2) is true, and if $f_\theta^*(t)$ denotes the function

$$f_\theta^*(t) = \sum_{n=0}^{\infty} \phi_n(\theta) f_n(t),$$

$\phi_n(\theta)$ being Rademacher's functions, then, for all θ,

$$B_k \int_0^1 |f_\theta^*(t)|^k \, dt \leqslant \int_0^1 |f(t)|^k \, dt \leqslant B_k \int_0^1 |f_\theta^*(t)|^k \, dt \quad (1 < k < \infty).$$

Integration with respect to θ gives

$$B_k \int_0^1 dt \int_0^1 |f_\theta^*(t)|^k \, d\theta \leqslant \int_0^1 |f(t)|^k \, dt \leqslant B_k \int_0^1 dt \int_0^1 |f_\theta^*(t)|^k \, dt \quad (1 < k < \infty),$$

and an application of Lemma 3 gives the required result.

LEMMA 6. *Let* $m_1 > \max(m_2, m_3, \ldots, m_q)$. *Then*

$$\int_0^1 f_{m_1} f_{m_2} \ldots f_{m_q} \, dt = 0.$$

We observe that, when we expand $f_{m_2}, f_{m_3}, \ldots, f_{m_q}$ by means of (5.01), and then express the separate terms by means of Rademacher's functions, using the equation (1.1), then none of the terms contains $\phi_{m_1}(t)$ as a factor. Thus, when the product $f_{m_2} f_{m_3} \ldots f_{m_q}$ is expressed by means of Rademacher's functions, none of the terms contains $\phi_{m_1}(t)$ as a factor. On the other hand, in the expansion of f_{m_1} by means of Rademacher's functions, all of the terms contain $\phi_{m_1}(t)$ as a factor. The result of the lemma then follows from (1.4)

LEMMA 7. *For* $q \geqslant 2$,

$$\sum_{n=0}^{\infty} \int_0^1 |f_n|^q \, dt \leqslant \int_0^1 |f(t)|^q \, dt.$$

We observe first that

$$f_n(t) = \int_0^1 f(\theta) \, \phi_n(t) \, \phi_n(\theta) \prod_{m=0}^{n-1} \{1 + \phi_m(t) \, \phi_m(\theta)\} \, d\theta,$$

and thus

$$\max_{n,t} |f_n(t)| \leqslant \max |f(t)|.$$

Also, from elementary considerations,

$$\left\{\sum_{n=0}^{\infty}\int_0^1 |f_n|^2 dt\right\}^{\frac{1}{2}} = \left\{\int_0^1 |f(t)|^2 dt\right\}^{\frac{1}{2}}.$$

We now apply Lemma 4 for the line $\alpha = \gamma$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, λ, μ denoting respectively the fields

$$(0 \leqslant t \leqslant 1) \quad \text{and} \quad n = 0, 1, 2, \ldots \quad (0 \leqslant t \leqslant 1).$$

Then the logarithm of the upper bound $M_{1/q}$ of

$$\left\{\sum_{n=0}^{\infty}\int_0^1 |f_n|^q dt\right\}^{1/q} \Big/ \left\{\int_0^1 |f(t)|^q dt\right\}^{1/q}$$

is a convex function of $1/q$ for $2 \leqslant q \leqslant \infty$. Since $M_0 \leqslant 1$, $M_{\frac{1}{2}} \leqslant 1$, it follows that $M_{1/q} \leqslant 1$ $(2 \leqslant q \leqslant \infty)$, and the desired result follows.

LEMMA 8. *The conclusions of Theorem V are true when $k = 2\nu$ is an even integer.*

Let $F_n(t) = s_{2^n}(t)$ denote the partial sum

$$F_n(t) = \sum_{m=0}^{n-1} f_m(t).$$

We have

$$(5.3) \quad \int_0^1 \{F_{n+1}^k - F_n^k\} dt = \int_0^1 \left\{k f_n F_n^{k-1} + \frac{k(k-1)}{2} f_n^2 F_n^{k-2} + \ldots + f_n^k\right\} dt$$
$$= \int_0^1 \left\{\frac{k(k-1)}{2} f_n^2 F_n^{k-2} + \ldots + f_n^k\right\} dt,$$

in virtue of Lemma 6. Now, by Hölder's inequality, we have, for $3 \leqslant \mu \leqslant k-1$,

$$\left|\int_0^1 f_n^{\mu} F_n^{k-\mu} dt\right| \leqslant \left\{\int_0^1 f_n^2 F_n^{k-2} dt\right\}^{\vartheta} \left\{\int_0^1 f_n^k dt\right\}^{1-\vartheta},$$

where $\vartheta = \vartheta(\mu)$ lies between 0 and 1. It follows that

$$\left|\int_0^1 f_n^{\mu} F_n^{k-\mu} dt\right| \leqslant \int_0^1 f_n^2 F_n^{k-2} dt + \int_0^1 f_n^k dt.$$

Substitution in (5.3) gives

$$\left|\int_0^1 \{F_{n+1}^k - F_n^k\} dt\right| \leqslant B_k \left[\int_0^1 f_n^2 F_n^{k-2} dt + \int_0^1 f_n^k dt\right].$$

Summing from $n=0$ to $n=N$ we get

$$\int_0^1 F_{N+1}^k\,dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^{N} f_n^2\right\} \max_{0\leqslant n\leqslant N} \{F_n^{k-2}\}\,dt + B_k \int_0^1 \sum_{n=0}^{\infty} f_n^k\,dt$$

$$\leqslant B_k \left[\int_0^1 \left\{\sum_{n=0}^{N} f_n^2\right\}^{\frac{1}{2}k} dt\right]^{2/k} \left[\int_0^1 \max_{0\leqslant n\leqslant N} \{F_n^k\}\,dt\right]^{(k-2)/k} + B_k \left[\int_0^1 \left\{\sum_{n=0}^{N} f_n^2\right\}^{\frac{1}{2}k}\right],$$

by Hölder's inequality. Now, by Theorem I,

$$\int_0^1 \max_{0\leqslant n\leqslant N} \{F_n^k\}\,dt \leqslant B_k \int_0^1 F_{N+1}^k\,dt,$$

and thus it is not difficult to deduce that

$$\int_0^1 F_{N+1}^k\,dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^{N} f_n^2\right\}^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 \left\{\sum_{n=0}^{\infty} f_n^2\right\}^{\frac{1}{2}k} dt.$$

The same argument will show that, if $N_1 < N_2$,

$$\int_0^1 \{F_{N_2}(t) - F_{N_1}(t)\}^k\,dt \leqslant B_k \int_0^1 \left\{\sum_{n=N_1+1}^{N_2} f_n^2\right\}^{\frac{1}{2}k} dt,$$

and thus

$$\lim_{N_1\to\infty,\ N_2\to\infty} \int_0^1 \{F_{N_2}(t) - F_{N_1}(t)\}^k\,dt = 0.$$

Thus F_{N+1} tends strongly† with index k to some limit function $F(t)$, and

$$\int_0^1 F^k(t)\,dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^{\infty} f_n^2\right\}^{\frac{1}{2}k} dt,$$

the right-hand side being supposed to exist. Also‡

$$\int_0^1 F(t)\,\psi_n(t)\,dt = \lim_{N\to\infty} \int_0^1 F_N(t)\,\psi_n(t)\,dt = c_n,$$

so that $F(t)$ is in fact identical with $f(t)$. Thus

$$\int_0^1 f^k(t)\,dt \leqslant B_k \int_0^1 \left\{\sum_{n=0}^{\infty} f_n^2\right\}^{\frac{1}{2}k} dt.$$

† See, *e.g.*, Hobson, 3, 254.

‡ See, *e.g.*, Hobson, 3, 251, equation (1).

We now go on to prove the opposite inequality. Consider the integral

$$\int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_N^2\, dt,$$

where $\nu = \frac{1}{2}k$, and $N-1 \geqslant n_1 > n_2 > n_3 > \dots > n_{\nu-1}$. It may be written

$$\int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_{n_1+1}^2\, dt + \sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n^2\, dt$$

$$+2 \sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n F_{n_1+1} + 2 \sum_{\substack{n \neq m \\ n_1 < n,\, m < N-1}}\sum \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n f_m\, dt.$$

The last two summations vanish in virtue of Lemma 6. It follows that

$$\sum_{n=n_1+1}^{N-1} \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 f_n^2\, dt \leqslant \int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_{\nu-1}}^2 F_N^2\, dt.$$

Summing over all the possible combinations of the numbers

$$n_1,\ n_2,\ \dots,\ n_{\nu-1},$$

for which $\max(n_1, n_2, \dots, n_{\nu-1}) < N-1$, we get

$$\sum_{\substack{n_1, n_2, n_{\nu-1}, n_\nu \\ \text{all different} \\ \max(n_1, n_2, \dots, n_\nu) \leqslant N-1}} \left\{\int_0^1 f_{n_1}^2 f_{n_2}^2 \dots f_{n_\nu}^2\, dt\right\} \leqslant \int_0^1 F_N^2 \left\{\sum_{n=0}^{N-1} f_n^2\right\}^{k-2} dt. \tag{5.4}$$

Also Lemma 7 gives

$$\sum_{n=0}^{N-1} \left\{\int_0^1 f_n^k\, dt\right\} \leqslant B_k \int_0^1 F_N^k\, dt.$$

Now, if S denotes the summation on the left-hand side of (5.4), it is not difficult to see that

$$\int_0^1 \left\{\sum_{n=0}^{N-1} f_n^2\right\}^{\frac{1}{2}k} dt \leqslant B_k \left[S + \sum_{n=0}^{N-1} \int_0^1 f_n^k\, dt\right]$$

$$\leqslant B_k \left[\int_0^1 F_N^2 \left\{\sum_{n=0}^{N-1} f_n^2\right\}^{k-2} dt + \int_0^1 F_N^k\, dt\right]. \tag{5.5}$$

From Hölder's inequality again, the right-hand side of (5.5) does not exceed

$$B_k \left[\left\{\int_0^1 F_N^k\, dt\right\}^{2/k} \left[\int_0^1 \left\{\sum_{n=0}^{N-1} f_n^2\right\}^{\frac{1}{2}k} dt\right]^{(k-2)/k} + \int_0^1 F_N^k\, dt\right],$$

and thus

$$\int_0^1 \left\{\sum_{n-1}^{N-1} f_n^2\right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 F_N^k\, dt \leqslant B_k \int_0^1 f^k(t)\, dt,$$

from which the desired result follows.

We have proved (5.1) in the case when k is an even integer, and (5.2) follows by Lemma 5.

LEMMA 9. *The conclusions of Theorem V are true for* $k \geqslant 2$.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a fixed sequence of unit factors. Let F_N^* denote the function

$$F_N^* = \sum_{n=0}^{N-1} \epsilon_n f_n(t).$$

Then, if k is an even integer, we have

$$(5.6) \qquad \left\{\int_0^1 |F_N^*(t)|^k dt\right\}^{1/k} \leqslant B_k \left\{\int_0^1 |F_N(t)|^k dt\right\}^{1/k} \leqslant B_k \left\{\int_0^1 |f(t)|^k dt\right\}^{1/k},$$

by Theorem I. The conditions of Lemma 4 are satisfied, and so we may use that lemma to interpolate between $k = 2$ and any arbitrary even integer; hence the inequality (5.6) is true for all $k \geqslant 2$. Thus F_N^* tends strongly†, with index k, to a limit function f^*, whose Walsh-Kaczmarz series is obtained by expanding $\Sigma\, \epsilon_n f_n(t)$, and

$$\int_0^1 |f^*(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt.$$

Since f is obtained from f^* in the same way as f_1 is obtained from f, we have also

$$\int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 |f^*(t)|^k dt.$$

Thus (5.2) is satisfied for the particular sequence of ϵ considered, and consequently for all sequences ϵ. In virtue of Lemma 5, (5.1) is also true for $k \geqslant 2$.

LEMMA 10. *The conclusions of Theorem V are true for* $1 < k \leqslant 2$.

† See Hobson, **3**, 254, and the argument of Lemma 8.

Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots$ denote a fixed sequence of unit factors. Let $g(t) \in L^k$, where k' is defined by the equation

$$\frac{1}{k}+\frac{1}{k'}=1.$$

Then, if g^* is formed from g in the same way as f^* is obtained from f, we have

$$\int_0^1 f_N^*(t)\, g(t)\, dt = \int_0^1 g_N^*(t) f(t)\, dt$$
$$\leqslant \left\{\int_0^1 |g_N^*(t)|^{k'} dt\right\}^{1/k'} \left\{\int_0^1 |f(t)|^k dt\right\}^{1/k},$$

by Hölder's inequality, whenever the right-hand side exists. Using (5.6),

$$\int_0^1 f_N^*(t)\, g(t)\, dt \leqslant B_k \left\{\int_0^1 |g(t)|^{k'} dt\right\}^{1/k'} \left\{\int_0^1 |f(t)|^k dt\right\}^{1/k}. \tag{5.7}$$

Since (5.7) is satisfied for all choices of $g(t)$, it follows that

$$\int_0^1 |f_N^*(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt,$$

where B_k is independent of N, and of the sequence $\epsilon_0, \epsilon_1, \ldots$. Thus, arguing as in the last lemma, we may show that $f_N^*(t)$ tends strongly with index k to $f^*(t)$, and that (5.2), and consequently also (5.1), is satisfied for $1 < k \leqslant 2$.

Lemmas 9 and 10 together give the result of Theorem V.

6. From Theorem V we deduce the following result, analogous to M. Riesz's well-known result† for Fourier series.

THEOREM VI. *Let n denote a fixed positive integer. We suppose that* $1 < k < \infty$. *Then*

$$\int_0^1 |s_n(t)|^k dt \leqslant B_k \int_0^1 |f(t)|^k dt.$$

We have

$$s_n(t) = \int_0^1 f(\theta) \sum_{m=0}^{n-1} \psi_m(t)\, \psi_m(\theta)\, d\theta. \tag{6.1}$$

† M. Riesz, 10.

Suppose that $n = 2^{n_1}+2^{n_2}+\ldots+2^{n_\lambda}$. We write

$$g(\theta) = f(\theta)\,\phi_{n_1}(\theta)\,\phi_{n_2}(\theta)\ldots\phi_{n_\lambda}(\theta) = f(\theta)\,\psi_n(\theta).$$

Suppose that
$$g(\theta) \sim \sum_{m=0}^{\infty} c_m'\,\psi_m(\theta),$$

and write
$$g_n(\theta) = \sum_{m=2^n}^{2^{n+1}-1} c_m'\,\psi_m(\theta) \quad (n = 0, 1, 2, \ldots).$$

From (6.1) we have

$$s_n(t)\,\psi_n(t) = \int_0^1 f(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)\,d\theta$$

$$= \int_0^1 g(\theta)\,\psi_n(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)\,d\theta.$$

Now the kernel

$$\psi_n(\theta)\,\psi_n(t) \sum_{m=0}^{n-1} \psi_m(t)\,\psi_m(\theta)$$

is identically equal to

$$\sum_{m=2^{n_\lambda}}^{2^{n_\lambda+1}-1} \psi_m(t)\,\psi_m(\theta) + \sum_{m=2^{n_{\lambda-1}}}^{2^{n_{\lambda-1}+1}-1} \psi_m(t)\,\psi_m(\theta) + \ldots + \sum_{m=2^{n_1}}^{2^{n_1+1}-1} \psi_m(t)\,\psi_m(\theta).$$

It follows that

$$s_n(t)\,\psi_n(t) = g_{n_\lambda}(t) + g_{n_{\lambda-1}}(t) + \ldots + g_{n_1}(t).$$

Hence, using Theorem V twice,

$$\int_0^1 |s_n(t)|^k\,dt = \int_0^1 |g_{n_\lambda}(t) + g_{n_{\lambda-1}}(t) + \ldots + g_{n_1}(t)|^k\,dt$$

$$= B_k \int_0^1 \left\{ \sum_{\nu=1}^{\lambda} g_{n_\nu}^2 \right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 \left\{ \sum_{n=0}^{\infty} g_n{}^2 \right\}^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 |g(t)|^k\,dt = B_k \int_0^1 |f(t)|^k\,dt,$$

the desired result.

7. We now prove the following lemmas.

LEMMA 11. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} a_{m,n}\,\phi_m(t)\,\phi_n(t).$$

Then $$B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \leqslant \int_0^1 |f(t)|^r\,dt \leqslant B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \quad (0 \leqslant r < \infty).$$

We first prove the lemma in the case when $r = 2\nu$ is an even integer. We observe that $f_n(t)$, defined in the usual way, is

$$\phi_n(t) \sum_{m=0}^{n-1} a_{m,n}\,\phi_m(t).$$

Now, using Theorem V, we have

$$\int_0^1 |f(t)|^r\,dt \leqslant B_r \int_0^1 \{\Sigma f_n{}^2\}^{\frac{1}{2}r}\,dt = B_r \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_\nu=0}^{\infty} \int_0^1 f_{n_1}^2 f_{n_2}^2 \cdots f_{n_\nu}^2\,dt.$$

Now $$\int_0^1 f_{n_1}^2 f_{n_2}^2 \cdots f_{n_\nu}^2\,dt \leqslant \left\{\int_0^1 f_{n_1}^r\right\}^{1/\nu} \left\{\int_0^1 f_{n_2}^r\right\}^{1/\nu} \cdots \left\{\int_0^1 f_{n_\nu}^r\right\}^{1/\nu}$$

$$\leqslant B_r \left\{\sum_{m=0}^{n_1-1} a_{m,n_1}^2\right\} \left\{\sum_{m=0}^{n_2-1} a_{m,n_2}^2\right\} \cdots \left\{\sum_{m=0}^{n_\nu-1} a_{m,n_\nu}^2\right\}.$$

It follows that

(7 . 1) $$\int_0^1 |f(t)|^r\,dt \leqslant B_r\{\Sigma a_{m,n}^2\}^{\nu}$$

whenever r is an even integer. Since

(7 . 2) $$\left\{\int_0^1 |f(t)|^r\,dt\right\}^{1/r}$$

increases with r, the result (7 . 1) is true for all values of r.

To prove the opposite inequality for $r \leqslant 2$ we need only combine the result (7 . 1) with the equation

$$\int_0^1 f^2(t)\,dt = \Sigma a_{m,n}^2,$$

using Hausdorff's theorem on the convexity of means. For $r > 2$, we have again to observe that the mean (7 . 2) increases with r.

LEMMA 12. *Let*

$$f(t, t') \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{m,n}\,\phi_m(t)\,\phi_n(t').$$

Then $$B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \leqslant \int_0^1\int_0^1 |f(t, t')|^r\,dt\,dt' \leqslant B_r\{\Sigma a_{m,n}^2\}^{\frac{1}{2}r} \quad (0 \leqslant r < \infty).$$

The proof is similar to that of the last lemma.

8. THEOREM VII. *Let $f(t)$ denote the function*

$$f(t) \sim \sum_{m=0}^{\infty} c_m \psi_m(t).$$

Let f_{n_1, n_2} $(n_1 > n_2)$ denote the polynomial

$$f_{n_1, n_2}(t) = \sum_{m=2^{n_1}+2^{n_2}}^{2^{n_1}+2^{n_2+1}-1} c_m \psi_m(t).$$

Then, assuming for simplicity that $c_{2^n} = 0$ $(n = 0, 1, 2, \ldots)$, $c_0 = 0$,

$$(8.1) \qquad B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt \quad (1 < k < \infty),$$

whenever either member exists.

Let $\{\epsilon_{n_1, n_2}\}$ denote a double sequence of arbitrary unit constants. Let

$$f^*(t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \epsilon_{n_1, n_2} f_{n_1, n_2}(t).$$

Then, assuming for simplicity that $c_{2^n} = 0$ $(n = 0, 1, 2, \ldots)$, $c_0 = 0$,

$$(8.2) \qquad B_k \int_0^1 |f^*(t)|^k dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 |f^*(t)|^k dt \quad (1 < k < \infty),$$

whenever either member exists.

We write

$$f_n(t) = \sum_{m=2^n}^{2^{n+1}-1} c_m \psi_m(t) = \phi_n(t) \gamma_n(t);$$

$$f_{n_1, n_2}(t) = \phi_{n_1}(t) \phi_{n_2}(t) \gamma_{n_1, n_2}(t).$$

The result (5.1) may be expressed in the form

$$(8.3) \qquad \int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \left| \sum_{n=0}^{\infty} \gamma_n(t) \phi_n(t_1) \right| dt_1,$$

with the opposite inequality. Applying Theorem V again, we get

$$\int_0^1 \left| \sum_{n=0}^{\infty} \gamma_n(t) \phi_n(t_1) \right|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \left| \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \gamma_{n_1, n_2}(t) \phi_{n_1}(t_1) \phi_{n_2}(t_2) \right|^k dt_2.$$

Thus (8.3) gives

$$\int_0^1 |f(t)|^k dt \leqslant B_k \int_0^1 dt \int_0^1 \int_0^1 \left| \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1-1} \gamma_{n_1, n_2}(t) \phi_{n_1}(t_1) \phi_{n_2}(t) \right|^k dt_1 dt_2$$

$$\leqslant B_k \int_0^1 \{\Sigma \gamma_{n_1, n_2}^2\}^{\frac{1}{2}k} dt = B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt,$$

by Lemma 12. Similarly we get

$$\int_0^1 |f(t)|^k dt \geqslant B_k \int_0^1 \{\Sigma f_{n_1, n_2}^2\}^{\frac{1}{2}k} dt.$$

This proves (8.1), and the extension to (8.2) is immediate.

The reader will see how, by further inductive stages, we may obtain further results analogous to those of Lemmas 11 and 12 and Theorem VII.

9. We now prove a generalization of Theorem V in a slightly different direction.

THEOREM VIII. *Let $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ denote an increasing sequence of positive integers for which $\lambda_{n+1}/\lambda_n \geqslant q > 1$. Let $\delta_n(t)$ denote the difference*

$$\delta_n(t) = s_{\lambda_n}(t) - s_{\lambda_{n-1}}(t) \quad (\lambda_{-1} = 0).$$

Then

$$(9.1) \qquad B_{k,q} \int_0^1 \{\Sigma \delta_n^2\}^{\frac{1}{2}k} dt \leqslant \int_0^1 |f(t)|^k dt \leqslant B_{k,q} \int_0^1 \{\Sigma \delta_n^2\}^{\frac{1}{2}k} dt \quad (1 < k < \infty),$$

whenever either member exists.

We first prove the following lemma.

LEMMA 13. *Let $2^n \leqslant \lambda_n < 2^{n+1}$. Let $\rho_n(t)$ denote the partial sum $s_{\lambda_n}(t) - s_{2^n}(t)$. Then*

$$\int_0^1 \{\Sigma \rho_n^2\}^{\frac{1}{2}k} \leqslant B_k \int_0^1 |f(t)|^k dt \quad (1 < k < \infty).$$

We may assume without loss of generality that $c_0 = 0$, and that λ_n is actually greater than 2^n. Let

$$\lambda_n - 2^n = 2^{\nu_1(n)} + 2^{\nu_2(n)} + \ldots + 2^{\nu_\mu(n)} \quad [\mu = \mu(n)].$$

We define f_n, γ_n as previously. We write

$$\gamma_n^* = \gamma_n \phi_{\nu_1(n)} \phi_{\nu_2(n)} \cdots \phi_{\nu_\mu(n)}; \quad f_n^* = \phi_n \gamma_n^*,$$

and suppose that f_n^* is split up into partial sums

$$(9.2) \qquad f_n^* = \sum_{\nu=0}^{n-1} f_{n,\nu}^* + c_{2^n}^* \phi_n$$

in the usual way. As in Theorem VI, we observe that

$$\rho_n(t)\,\phi_{\nu_1(n)}(t)\,\phi_{\nu_2(n)}(t)\ldots\phi_{\nu_\mu(n)}(t) = f^*_{n,\nu_\mu}(t)+f^*_{n,\nu_{\mu-1}}(t)+\ldots+f^*_{n,\nu_1(n)}(t).$$

Thus

$$(9.3)\quad \int_0^1 \{\Sigma\rho_n{}^2\}^{\frac{1}{2}k}\,dt = \int_0^1 \{\Sigma[f^*_{n,\nu_\mu}+f^*_{n,\nu_{\mu-1}}+\ldots+f^*_{n,\nu_1}]^2\}^{\frac{1}{2}k}\,dt$$

$$\leqslant B_k\int_0^1\left|\sum_n\{f^*_{n,\nu_\mu(n)}+f^*_{n,\nu_{\mu-1}(n)}+\ldots+f^*_{n,\nu_1(n)}\}\right|^k dt,$$

by Theorem V. Thus, by Theorem VII, the expression (9.3) does not exceed

$$B_k\int_0^1\left[\sum_n\{f^{*2}_{n,\nu_\mu(n)}+f^{*2}_{n,\nu_{\mu-1}(n)}+\ldots+f^{*2}_{n,\nu_1(n)}\}\right]^{\frac{1}{2}k} dt$$

$$(9.4)\qquad \leqslant B_k\int_0^1\left\{\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1-1} f^{*2}_{n_1,n_2}\right\}^{\frac{1}{2}k} dt.$$

Using Theorem VII again, we see that the expression (9.4) does not exceed

$$(9.5)\qquad B_k\int_0^1\left|f^*(t)-\sum_{n=0}^{\infty} c^*_{2^n}\,\phi_n(t)\right|^k dt,$$

where f^* denotes the function $\Sigma f_n{}^*(t)$. Now, by Theorem V,

$$\int_0^1 |f^*(t)|^k\,dt \leqslant B_k\int_0^1\{\Sigma f_n^{*2}\}^{\frac{1}{2}k}\,dt$$

$$9.6)\qquad = B_k\int_0^1\{\Sigma f_n{}^2\}^{\frac{1}{2}k}\,dt \leqslant B_k\int_0^1 |f(t)|^k\,dt.$$

Also, if $h(t)=\Sigma d_n\phi_n(t)$ $(\Sigma d_n{}^2=1)$, then†, by Lemma 3,

$$\Sigma c^*_{2^n} d_n = \int_0^1 f^*(t)\,h(t)\,dt$$

$$\leqslant\left\{\int_0^1 |f^*(t)|^k\,dt\right\}^{1/k}\left\{\int_0^1 |h(t)|^{k'}dt\right\}^{1/k'}$$

$$\leqslant B_k\left\{\int_0^1 |f^*(t)|^k\,dt\right\}^{1/k} \leqslant B_k\left\{\int_0^1 |f(t)|^k\,dt\right\}^{1/k}.$$

† The argument is due to Zygmund. See Zygmund, 13, Theorem G (106).

It follows that

$$\Sigma c_{2^n}^{*2} \leqslant B_k \left\{ \int_0^1 |f(t)|^k \, dt \right\}^{2/k},$$

and thus

$$\int_0^1 \left| \sum_{n=0}^{\infty} c_{2^n}^{*} \phi_n(t) \right|^k dt \leqslant B_k \{\Sigma c_{2^n}^{*2}\}^{\frac{1}{2}k} \leqslant B_k \int_0^1 |f(t)|^k \, dt.$$

Combining this with (9.5), (9.6), we obtain the required result.

We can now prove the first half of (9.1). Suppose that $q^G > 2$. Then the number of partial sums $\delta_n(t)$ which are contained partially or entirely in the summation

$$\sum_{m=2^n}^{2^{n+1}-1} c_n \psi_n(t)$$

cannot exceed $G+1$. We divide up the set $\{\delta_n\}$ into two subsets, and write

$$\delta_n(t) = \delta_n'(t) + \delta_n''(t),$$

where $\delta_n'(t) = \delta_n(t)$, except when the range $(\lambda_{n-1}, \lambda_n)$ contains a power of 2, in which case $\delta_n'(t)$ vanishes. By Lemma 13,

$$\int_0^1 \left| \sum_{n=0}^{\infty} \delta_{(G+1)n+\sigma}'^2(t) \right|^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 |f(t)|^k \, dt \quad (\sigma = 0, 1, \ldots, G),$$

and thus

$$(9.7) \qquad \int_0^1 \left| \sum_{n=0}^{\infty} \delta_n'^2(t) \right|^{\frac{1}{2}k} dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k \, dt.$$

Also, if we define $\tau(n)$, $\sigma(n)$ for (not necessarily all) values of n by means of the inequalities

$$2^{n-1} \leqslant \lambda_{\tau(n)-1} < 2^n, \quad 2^{n+\sigma(n)} < \lambda_{\tau(n)} \leqslant 2^{n+\sigma(n)+1},$$

then we have

$$\int_0^1 \left| \sum_{n=0}^{\infty} \delta_n''^2(t) \right|^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 \left| \sum_{n=0}^{\infty} (s_{\lambda_{\tau(n)}} - s_{2^{n+\sigma(n)}})^2 + (s_{2^{n+\sigma(n)}} - s_{2^n})^2 + (s_{2^n} - s_{\lambda_{\tau(n)-1}})^2 \right|^{\frac{1}{2}k} dt$$

$$\leqslant B_k \int_0^1 |f(t)|^k \, dt.$$

Combining the last result with (9.7), we obtain

$$B_{k,\delta}\int_0^1 \{\Sigma\,\delta_n^{\,2}\}^{\frac{1}{2}k}\,dt \leqslant \int_0^1 |f(t)|^k\,dt.$$

For the proof of the opposite inequality we write

$$f^{(\sigma)}(t) = \sum_{n=0}^{\infty} \epsilon_{(2G+2)\,n+\sigma}\,\delta_{(2G+2)\,n+\sigma}(t) \quad (\sigma = 0,\,1,\,\ldots,\,2G+1),$$

and observe that at most one of the partial sums

$$\delta_{(2G+2)\,n+\sigma}(t) \quad (\sigma \text{ fixed})$$

has points in common with a given interval $(2^\nu,\,2^{\nu+1})$. Thus we may apply Theorem V, and without much difficulty we obtain

$$\int_0^1 |f^{(\sigma)}(t)|^k\,dt \leqslant B_k\int_0^1 \left\{\sum_{n=0}^{\infty} \delta^2_{(2G+2)n+\sigma}\right\}^{\frac{1}{2}k} dt \quad (\sigma = 0,\,1,\,\ldots,\,2G+1).$$

Combining these results, we obtain

$$\int_0^1 |f(t)|^k\,dt \leqslant B_{k,\delta}\int_0^1 \left\{\sum_{n=0}^{\infty} \delta_n^{\,2}\right\}^{\frac{1}{2}k} dt.$$

This completes the proof of the theorem.

10. From Theorem VIII we may obtain the following result which is the generalization of Theorem I.

THEOREM IX. *Let $\lambda_0,\,\lambda_1,\,\ldots,\,\lambda_n,\,\ldots$ denote an increasing sequence of positive integers for which $\lambda_{n+1}/\lambda_n \geqslant q > 1$. Let $n(t)$ denote an arbitrary positive integer which may vary with t. Then*

$$(10.1) \qquad \int_0^1 |s_{\lambda_{n(t)}}(t)|^k\,dt \leqslant B_{k,q}\int_0^1 |f(t)|^k\,dt \quad (1 < k < \infty),$$

whenever the right-hand member exists.

Let N denote a fixed positive integer. Let $f(t)$ denote a fixed function of class L^k $(1 < k < \infty)$. Then the integral

$$(10.2) \qquad \int_0^1 \sup_{n \geqslant N} |f(t) - s_{\lambda_n}(t)|^k\,dt$$

tends to zero as $N \to \infty$, so that

$$s_{\lambda_n}(t) \to f(t) \quad (f \in L^k)$$

for almost all t.

We prove the assertion (10.1) first for $q = 2$. Then no two members of the sequence $\{\lambda_n\}$ can be in the same interval $2^\nu \leqslant \lambda_n < 2^{\nu+1}$. We suppose that

$$2^{\nu(n)} \leqslant \lambda_n < 2^{\nu(n)+1}.$$

Then

$$\int_0^1 |s_{\lambda_{n(t)}}(t) - s_{2^{\nu[n(t)]}}(t)|^k \, dt \leqslant \int_0^1 \left\{ \sum_{n=0}^{\infty} (s_{\lambda_n}(t) - s_{2^{\nu(n)}})^2 \right\}^{\frac{1}{2}k} dt \leqslant B_k \int_0^1 |f(t)|^k \, dt,$$

by Theorem VIII. Also, by Theorem I,

$$\int_0^1 |s_{2^{\nu[n(t)]}}|^k \, dt \leqslant B_k \int_0^1 |f(t)|^k \, dt.$$

Thus
$$\int_0^1 |s_{\lambda_{n(t)}}(t)|^k \, dt \leqslant B_k \int_0^1 |f(t)|^k \, dt.$$

If $q < 2$, then we may divide the sequence $\{\lambda_n\}$ into a finite number of subsequences

$$\{\lambda_{Gn+\sigma}\} \quad (\sigma = 0, 1, \ldots, G-1) \quad G = G(\delta),$$

for each of which the theorem is satisfied. It follows at once that the theorem is satisfied for the sequence $\{\lambda_n\}$, but with a constant which now depends on G as well as on k.

Suppose that $\lambda_N > 2^\tau$. Then, by (10.1), we have

$$\int_0^1 \sup_{n \geqslant N} |f(t) - s_{\lambda_n}(t)|^k \, dt \leqslant B_{k,\delta} \int_0^1 \left| f(t) - \sum_{m=0}^{2^\tau - 1} c_m \psi_m(t) \right|^k dt$$

$$= B_{k,\delta} \int_0^1 \left| \sum_{n=\tau}^{\infty} f_n(t) \right|^k dt$$

$$\leqslant B_{k,\delta} \int_0^1 \left\{ \sum_{n=\tau}^{\infty} f_n^2(t) \right\}^{\frac{1}{2}k} dt,$$

by Theorem V; the last expression tends to zero in virtue of the existence of

$$\int_0^1 \left\{ \sum_{n=0}^{\infty} f_n^2(t) \right\}^{\frac{1}{2}k} dt.$$

The last assertion, that, for almost all t in $(0, 1)$,

$$s_{\lambda_n}(t) \to f(t),$$

follows at once from (10.2).

References.

1. A. Haar, "Zur Theorie der orthogonalen Funkionenstysteme", *Math. Annalen*, 69 (1910), 331–371.
2. G. H. Hardy and J. E. Littlewood, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.
3. E. W. Hobson, *Theory of functions of a real variable*, 2 (1926).
4. S. Kaczmarz, "Über ein Orthogonalsystem", *Comptes rendus du premier congrès des math. des pays slaves* (Varsovie, 1929), 189–192.
5. S. Kaczmarz and H. Steinhaus, "Le système orthogonal de M. Rademacher", *Studia Math.*, 2 (1930), 231–247.
6. A. Khintchine, "Über dyadische Brüche", *Math. Zeitschrift*, 18 (1923), 109–116.
7. J. E. Littlewood and R. E. A. C. Paley, "Theorems on Fourier series and power series", *Journal London Math. Soc.*, 6 (1931), 230–233.
8. R. E. A. C. Paley and A. Zygmund, "On some series of functions (1)", *Proc. Camb. Phil. Soc.*, 26 (1930), 337–357.
9. H. Rademacher, "Einige Sätze über Reihen von allgemeinen Orthogonalfunctionen", *Math. Annalen*, 87 (1922), 112–138.
10. M. Riesz, "Sur les fonctions conjuguées", *Math. Zeitschrift*, 27 (1927), 218–244.
11. M. Riesz, "Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires", *Acta Math.*, 49 (1926), 465–497.
12. J. L. Walsh, "A closed set of normal orthogonal functions", *American J. of Math.*, 45 (1923), 5–24.
13. A. Zygmund, "On the convergence of lacunary trigonometric series", *Fundamenta Math.*, 16 (1930), 90–107.

A REMARKABLE SERIES OF ORTHOGONAL FUNCTIONS (II)

By R. E. A. C. PALEY.

[Received April 29, 1931.—Read May 14, 1931.]

11. We begin this half of the paper by considering the behaviour of Cesàro means of Walsh-Kaczmarz series. Let $f(t) \in L^k$ $(1 < k < \infty)$. Then, for almost all t, the $\{\psi\}$ series of f is strongly summable $(C, 1/k+\delta)$, and is summable (C, δ) to sum $f(t)$. In fact, let

$$(11.1) \qquad \tau_n^{(\eta)}(t) = \frac{\sum_{m=0}^{n-1} \left(1-\frac{m}{n}\right)^{\eta-1} |s_m(t)|}{\sum_{m=0}^{n-1} \left(1-\frac{m}{n}\right)^{\eta-1}} \qquad (\eta > 0)$$

denote the strong Rieszian mean, and $\sigma_n^{(\eta)}$ the weak Rieszian mean of order η. Then†, for all positive δ, we have

$$(11.2) \qquad \int_0^1 \sup_{1 \leqslant n < \infty} \{\tau_n^{(1/k+\delta)}(t)\}^k \, dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k \, dt,$$

$$(11.3) \qquad \int_0^1 \sup_{1 \leqslant n < \infty} |\sigma_n^{(\delta)}(t)|^k \, dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k \, dt.$$

We next consider convergence factors of $\{\psi\}$ series. If $1 < k \leqslant 2$, then $\log^{-1/k}(n+2)$ is almost everywhere a convergence factor of the $\{\psi\}$ series of $f(t) \in L^k$. In fact, if

$$s_n^*(t) = \sum_{m=0}^{n-1} c_m \log^{-1/k}(m+2)\, \psi_m(t),$$

c_m being the ψ coefficients of $f(t)$, then we have

$$\int_0^1 |s_{n(t)}^*(t)|^k \, dt \leqslant B_k \int_0^1 |f(t)|^k \, dt,$$

† The Fourier analogues of both theorems are known. That of (11.2) seems never to have been stated explicitly. For the analogue of (11.3), see Hardy and Littlewood, 3.

where $n(t)$ varies arbitrarily with t. If $f(t) \epsilon L$, then $\log^{-1}(n+2)$ is almost everywhere a convergence factor of the $\{\psi\}$ series of f. In each case the Fourier series analogue has already been established†.

12. We next consider the ψ-analogue of an inequality of Hardy and Littlewood‡, of which proofs have also been given by Gabriel and F. Riesz§. Let $c_0, c_1, \ldots, c_n, \ldots$ denote a bounded set of real numbers, and let $c_0^*, c_1^*, \ldots, c_n^*, \ldots$ denote the set $|c_0|, |c_1|, \ldots, |c_n|, \ldots$ rearranged in descending order of magnitude. Let

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t), \quad F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t).$$

We show that, if q is an even integer, then

$$\int_0^1 f^q(t)\,dt \leqslant \int_0^1 F^q(t)\,dt, \tag{12.1}$$

whenever the right-hand side exists. From the result (12.1) we may obtain results analogous to those which Hardy and Littlewood|| deduce for Fourier series. The arguments are almost identical.

13. We begin by stating without proof the following theorem for strong means.

THEOREM X. *Let $\tau_n^{(\eta)}$ denote the strong Rieszian mean* (11.1) *of order η of the ψ series of $f(t)$. Then, for positive δ,*

$$\int_0^1 \sup_{1 \leqslant n < \infty} \{\tau_n^{(1/k+\delta)}(t)\}^k\,dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k\,dt \quad (k > 1),$$

the right-hand side being supposed to exist.

For Fourier series more is known. In fact, we may substitute δ for $1/k+\delta$ in the above theorem. The result has never been explicitly stated, but follows at once from the results of the papers Hardy and Littlewood, **3** and **6**. It seems likely that a similar result is also true for ψ-series, but I have not yet succeeded in obtaining one.

† Littlewood and Paley, **7**; Hardy, **2**.

‡ Hardy and Littlewood, **4**.

§ Gabriel, **1**, F. Riesz, **9**.

|| Hardy and Littlewood, **5**.

For the weak Rieszian mean we obtain a stronger result, analogous to Hardy and Littlewood's theorem† for Fourier series.

THEOREM XI. *Let $\sigma_n^{(\eta)}$ denote the weak Rieszian mean*

$$\sigma_n^{(\eta)}(t) = \sum_{m=0}^{n-1} \left(1-\frac{m}{n}\right)^{\eta} c_m \psi_m(t) \quad (\eta > 0).$$

Then, for positive δ,

$$\int_0^1 \sup_{1 \leqslant n < \infty} |\sigma_n^{(\delta)}(t)|^k \, dt \leqslant B_{k,\delta} \int_0^1 |f(t)|^k \, dt \quad (k > 1),$$

the right-hand side being supposed to exist.

We write

$$H(t) = \sup_{1 \leqslant n < \infty} |\sigma_n(t)|.$$

Putting $\delta = 1/k'$ in Theorem X, we observe that

$$(13.1) \qquad \int_0^1 H^k(t)\, dt \leqslant B_k \int_0^1 |f(t)|^k \, dt.$$

Let

$$n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}.$$

Then

$$(13.2) \qquad \sum_{m=0}^{2^{n_1}-1} c_m \psi_m(t) \left(1-\frac{m}{n}\right)^{\delta} = \int_0^1 f(\theta) \left[\sum_{m=0}^{2^{n_1}-1} \psi_m(t)\psi_m(\theta)\left(1-\frac{m}{n}\right)^{\delta}\right] d\theta.$$

If we write $\qquad f'(\theta) = f(\theta)\,\phi_0(\theta)\,\phi_1(\theta) = \phi_{n_1-1}(\theta),$

the last integral is identically equal to

$$\phi_0(t)\,\phi_1(t)\ldots\phi_{n_1-1}(t) \int_0^1 f'(\theta) \left[\sum_{m=0}^{2^{n_1}-1} \psi_m(t)\psi_m(\theta)\left(\frac{n-2^{n_1}+1+m}{n}\right)^{\delta}\right] d\theta.$$

If now c_m', s_m', σ_m', $H'(t)$ correspond to $f'(\theta)$ in the same way as c_m, s_m, σ_m, $H(t)$ correspond to $f(\theta)$, then the expression (13.2) has the same modulus as

$$(13.3) \qquad n^{-\delta} \sum_{m=0}^{2^{n_1}-1} c_m' \psi_m(t)\,(n-2^{n_1}+1+m)^{\delta}$$

$$= n^{-\delta}\left[\sum_{m=1}^{2^{n_1}-1} s_m'(t)\Delta\{(n-2^{n_1}+1+m)^{\delta}\} + s_{2^{n_1}}(t)\, n^{\delta}\right],$$

† Hardy and Littlewood, 3.

where $\Delta\{(n-2^{n_1}+1+m)^\delta\}$ denotes $(n-2^{n_1}+m)^\delta-(n-2^{n_1}+1+m)^\delta$. The second member of (13.3) is less in modulus than $2F(t)$ in virtue of Lemma 2, where $F(t)$ is the maximum average of $|f|$ at the point t. Using Abel's transformation again, we may write the first as

$$(13.4)\quad n^{-\delta}\Bigg[\sum_{m=1}^{2^{n_1}-2} m\sigma_m'(t)\,\Delta_2\{(n-2^{n_1}+1+m)^\delta\} + (2^{n_1}-1)\,\sigma'_{2^{n_1}-1}(t)\{(n-1)^\delta-n^\delta\}\Bigg],$$

$\Delta_2\{(n-2^{n_1}+1+m)^\delta\}$ denoting the double difference

$$(n-2^{n_1}+m)^\delta-2(n-2^{n_1}+1+m)^\delta+(n-2^{n_1}+2+m)^\delta.$$

Observing that $\qquad \Delta_2\{(n-2^{n_1}+1+m)^\delta\}=O(m^{\delta-2}),$

we see that the expression (13.4) is of the form

$$O(n^{-\delta})\left[\sum_{m=1}^{2^{n_1}-2} m^{\delta-1}H'(t)+2^{n_1}n^{\delta-1}H'(t)\right]=O\{H'(t)\}.$$

Thus we get

$$\left|\sum_{m=0}^{2^{n_1}-1} c_m\psi_m(t)\left(1-\frac{m}{n}\right)^\delta\right|\leqslant B_\delta\{H'(t)+F(t)\}.$$

Now

$$(13.5)\quad \sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1} c_m\psi_m(t)\left(1-\frac{m}{n}\right)^\delta = \int_0^1 f(\theta)\left[\sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1}\psi_m(t)\,\psi_m(\theta)\left(1-\frac{m}{n}\right)^\delta\right]d\theta.$$

If we write $\qquad f''(\theta)=f(\theta)\,\phi_{n_1}(\theta)\,\phi_0(\theta)\,\phi_1(\theta)\ldots\phi_{n_2-1}(\theta),$

the last integral is identically

$$\phi_{n_1}(t)\,\phi_0(t)\,\phi_1(t)\ldots\phi_{n_2-1}(t)\int_0^1 f''(\theta)\left[\sum_{m=0}^{2^{n_2}-1}\psi_m(t)\,\psi_m(\theta)\left(\frac{n-2^{n_1}-2^{n_2}+1+m}{n}\right)^\delta\right]d\theta.$$

Arguing as before, we obtain

$$\left|\sum_{m=2^{n_1}}^{2^{n_1}+2^{n_2}-1} c_m\psi_m(t)\left(1-\frac{m}{n}\right)^\delta\right|\leqslant B_\delta\left(\frac{n-2^{n_1}}{n}\right)^\delta\{H''(t)+F(t)\}$$
$$\leqslant 2^{-\delta}B_\delta\{H''(t)+F(t)\},$$

where $H''(t)$ is obtained from $f''(\theta)$ in the same way as $H(t)$ is obtained from $f(\theta)$. In the same way,

$$\left|\sum_{m=2^{n_1}+2^{n_2}}^{2^{n_1}+2^{n_2}+2^{n_3}-1} c_m \psi_m(t)\left(1-\frac{m}{n}\right)^{\delta}\right| \leqslant 2^{-2\delta} B_\delta\{H'''(t)+F(t)\},$$

and so on. Combining these results we get

$$|\sigma_n^{(\delta)}(t)| \leqslant B_\delta\{F(t)+H'(t)+2^{-\delta}H''(t)+2^{-2\delta}H'''(t)+\ldots\}.$$

Thus Minkowski's inequality gives

$$|\sigma_n^{(\delta)}(t)|^k \leqslant B_\delta^k(1+1+2^{-\delta}+2^{-2\delta}+\ldots)^{k-1}$$
$$\times\{F^k(t)+H'^k(t)+2^{-\delta}H''^k(t)+2^{-2\delta}H'''^k(t)+\ldots\},$$

and the desired result follows at once by (13.1),

It is not difficult to extend the proofs of the last two theorems to prove that, if $f(t) \in L^k$ $(k > 1)$, then, for almost all values of t, the $\{\psi\}$ series of $f(t)$ is strongly summable $(C, 1/k+\delta)$, and summable (C, δ) for all positive δ.

14. We may prove the following theorem on convergence factors of $\{\psi\}$ series.

THEOREM XII. *Let $f(t) \in L^k$ $(1 < k \leqslant 2)$, and let*

$$s_n^*(t) = \sum_{m=0}^{n-1} \frac{c_m \psi_m(t)}{\log^{1/k}(m+2)}.$$

Let $n(t)$ denote an integer which varies arbitrarily with t. Then

$$\int_0^1 |s_{n(t)}^*(t)|^k\,dt \leqslant B_k \int_0^1 |f(t)|^k\,dt,$$

where the constant B_k is independent of the choice of $n(t)$.

For almost all t, $\log^{-1/k}(n+2)$ is a convergence factor for the series $\Sigma c_n \psi_n(t)$.

The proof is almost identical with that of the corresponding result† for Fourier series.

15. When we come to investigate what happens to the above theorems in the case $k = 1$, we obtain weaker results. We state without proof the following two theorems.

† Littlewood and Paley, 7.

THEOREM XIII. *Let n denote a fixed positive integer. Then, with the notation of Theorem XI,*

$$\int_0^1 |\sigma_n^{(\delta)}(t)|\,dt \leqslant B_\delta \int_0^1 |f(t)|\,dt,$$

the right-hand side being supposed existent. The constant B_δ is independent of the choice of n or f(t), and depends only on δ.

THEOREM XIV. *Let*

$$(15.1) \qquad f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t) \in L.$$

Then, for almost all values of t, $\log^{-1}(n+2)$ *is a convergence factor for the series* (15.1).

The last result is the $\{\psi\}$ analogue of Hardy's well-known theorem† for Fourier series.

16. We now go on to prove the $\{\psi\}$ analogue of Hardy and Littlewood's‡ "star" theorem. We first have to consider what exactly the analogue is. We introduce a fresh notation. Suppose that

$$(16.1) \qquad \psi_{n_1}(t)\,\psi_{n_2}(t)\,\psi_{n_3}(t)\ldots\psi_{n_\nu}(t) = \psi_m(t) \quad (0 \leqslant t < 1);$$

then we say that

$$(n_1, n_2, n_3, \ldots, n_\nu) \equiv m.$$

If the left-hand side of (16.1) is identically 1, then we have§

$$(n_1, n_2, n_3, \ldots, n_\nu) \equiv 0.$$

† Hardy, **2**.

‡ Hardy and Littlewood, **4**.

§ The game of Nim is played with matches according to the following rules. A number of matches is arranged in three heaps on a table. Two players move alternately. At each turn the player must remove one or more matches, all from the same heap. He is at liberty to decide how many matches he will remove, and from which heap. The player who removes the last match loses the game. We call (l, m, n) a winning combination, if, by leaving his opponent with heaps of l, m, and n matches, a player is in a position to force a win. The criterion that (l, m, n) should be a winning combination is either

$$(l, m, n) \equiv 0, \qquad \max(l, m, n) > 1,$$

or

$$l+m+n \equiv 1 \pmod 2, \quad \max(l, m, n) = 1.$$

The last formula may be extended to give winning combinations if the game is played with any number of heaps. For an analysis of the game see, *e.g.*, Rouse Ball, **10**.

Suppose that $a_0, a_1, \ldots, a_\lambda$ are $\lambda+1$ non-negative numbers. Suppose further that $a_0^* \geqslant a_1^* \geqslant a_2^* \ldots \geqslant a_\lambda^*$ are the same numbers rearranged in descending order of magnitude. We call this arrangement the *standard* arrangement of the a_r and say that the a_r^* are arranged *in type P*. We prove the following theorem.

THEOREM XV. *The sum*

$$S = \sum_{(r, s, t, \ldots) \equiv 0} a_r b_s c_t \ldots$$

is greatest when the a's, b's, c's, ... are all arranged in type P, i.e.

$$S = \Sigma a_r b_s c_t \ldots \leqslant S^* = \Sigma a_r^* b_s^* c_t^* \ldots .$$

We first need

LEMMA 14. *It is sufficient to prove the theorem in the case when all of the numbers a_r, b_s, c_t are either* 1 *or* 0.

The proof of the lemma is exactly similar to that of the analogous lemma for Fourier series, proved by Hardy and Littlewood in their paper†.

We proceed now to the proof of the theorem. We may suppose without loss of generality that the numbers of members of the sets $\{a\}$, $\{b\}$, $\{c\}$, ... are all equal, and that this number is of the form 2^ν, where ν is a positive integer. Assuming, in virtue of Lemma 14, that all the numbers a, b, c, ... are either 1 or 0, we give an inductive proof. We show that, if the theorem is assumed to be true for a given value of ν, then it may be deduced for higher values of ν.

We first establish the theorem for $\nu = 1$. Let N denote the number of sets $\{a\}$, $\{b\}$, $\{c\}$, ... which have two non-vanishing members. We assume, without loss of generality, that each set has at least one non-vanishing member. Then, if $N = 0$, we have $S \leqslant 1$, $S^* = 1$, from which the result follows. If $N > 0$, we may assume, without loss of generality, that the set $\{a\}$ contains two non-vanishing members. Now let s, t, ... denote any set of numbers for which $b_s = c_t = \ldots = 1$. It is not difficult to see that $(s, t, \ldots) \equiv 0$ or 1, and thus‡ the corresponding member of the set $\{a\}$ is also non-vanishing. Thus we have

$$S = \sum_{(r, s, t, \ldots) \equiv 0} a_r b_s c_t \ldots = 2^{N-1}.$$

Similarly $S^* \doteq 2^{N-1}$. This proves the required result for $\nu = 1$.

17. The proof of the result for $\nu \geqslant 2$ is best represented diagrammatically. We assume that the result has already been established for $\nu - 1$. We

† Hardy and Littlewood, 4, 108.

‡ Clearly, if $(s, t, \ldots) \equiv r$, then $(r, s, t, \ldots) \equiv 0$.

consider first a new type of rearrangement, defined as follows. We are given 2^ν numbers $a_0, a_1, \ldots, a_{2^\nu-1}$, all equal to 1 or 0. We arrange the first $2^{\nu-1}$ numbers $a_0, a_1, \ldots, a_{2^{\nu-1}-1}$ in descending order, and we arrange the last $2^{\nu-1}$ numbers $a_{2^{\nu-1}}, a_{2^{\nu-1}+1}, \ldots, a_{2^\nu-1}$ also in descending order. We say that the numbers $\{a\}$ have been rearranged in type Q_ν. We prove the following lemma.

LEMMA 15. *The sum S is not decreased by rearrangement in type Q_ν, it being assumed that Theorem XV has already been established for sets $\{a\}, \{b\}, \{c\}, \ldots$ which contain each not more than $2^{\nu-1}$ members.*

Let

$$\sum_{n=0}^{2^\nu-1} a_n \psi_n(\theta) = f(\theta) = f_0(\theta) + \phi_{\nu-1}(\theta) f_1(\theta), \tag{17.1}$$

where $f_0(\theta)$ denotes the first $2^{\nu-1}$ terms of the polynomial (17.1) and $\phi_{\nu-1}(\theta) f_1(\theta)$ denotes the remainder. Then $f_1(\theta)$ is a polynomial of the same type as $f_0(\theta)$, containing ψ's only of suffix not exceeding $2^{\nu-1}-1$. We write similarly

$$\sum_{n=0}^{2^\nu-1} b_n \psi_n(\theta) = g(\theta) = g_0(\theta) + \phi_{\nu-1}(\theta) g_1(\theta);$$

$$\sum_{n=0}^{2^\nu-1} c_n \psi_n(\theta) = h(\theta) = h_0(\theta) + \phi_{\nu-1}(\theta) h_1(\theta);$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Then

$$S = \sum_{(r, s, t, \ldots) \equiv 0} a_r b_s c_t \ldots = \int_0^1 f(\theta) g(\theta) h(\theta) \ldots d\theta$$

$$= \int_0^1 \{f_0(\theta) + \phi_{\nu-1}(\theta) f_1(\theta)\} \{g_0(\theta) + \phi_{\nu-1}(\theta) g_1(\theta)\} \ldots d\theta$$

$$= \sum_{(\rho, \sigma, \tau, \ldots) \equiv 0} \int_0^1 f_\rho(\theta) g_\sigma(\theta) h_\tau(\theta) \ldots d\theta. \tag{17.2}$$

Since the theorem is supposed to have been established for sets which contain each less than $2^{\nu-1}$ members, it follows that each term of the sum (17.2) is not decreased when we rearrange the terms of $f_\rho, g_\sigma, h_\tau, \ldots$ in star order. But this is exactly equivalent to the rearrangement Q_ν, and the result of Lemma 15 follows.

The possible distributions of the non-zero elements of any given one of the sets $\{a\}, \{b\}, \{c\}, \ldots$, after a rearrangement Q_ν, are shown in Fig. (i).

The thick lines denote that part of the range $(0, 2^{\nu}-1)$ which contains non-zero elements. The range is divided into four equal quarters.

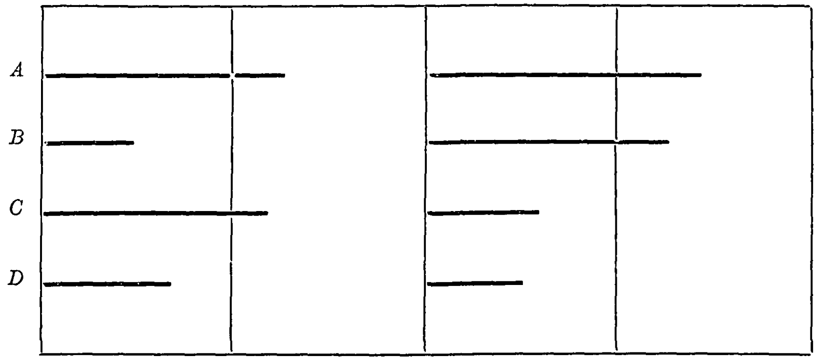

Fig. (i).

We now interchange $\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)$, with the consequent alterations in the order of the functions ψ_n, so that

(17.3) $$\psi_n(\theta) = \phi_{n_1}(\theta)\,\phi_{n_2}(\theta) \ldots \phi_{\nu-1}(\theta)$$

is changed into

$$\psi_{n'}(\theta) = \phi_{n_1}(\theta)\,\phi_{n_2}(\theta) \ldots \phi_{\nu-2}(\theta).$$

The relation between n and n' is (1, 1) and has the effect of interchanging the second and third quarters in Fig. (i). We observe that the relations

$$(r, s, t, \ldots) \equiv 0, \quad (r', s', t', \ldots) \equiv 0$$

are equivalent. It follows that S is unaltered by the change which we have carried out. The new possible distributions of the non-zero elements are shown in Fig. (ii).

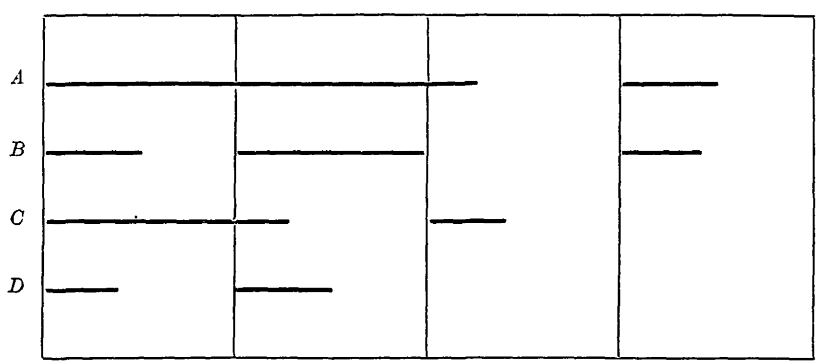

Fig. (ii).

We now apply the operation Q_ν a second time. This cannot decrease S. The new distributions of the non-zero elements are shown in Fig. (iii).

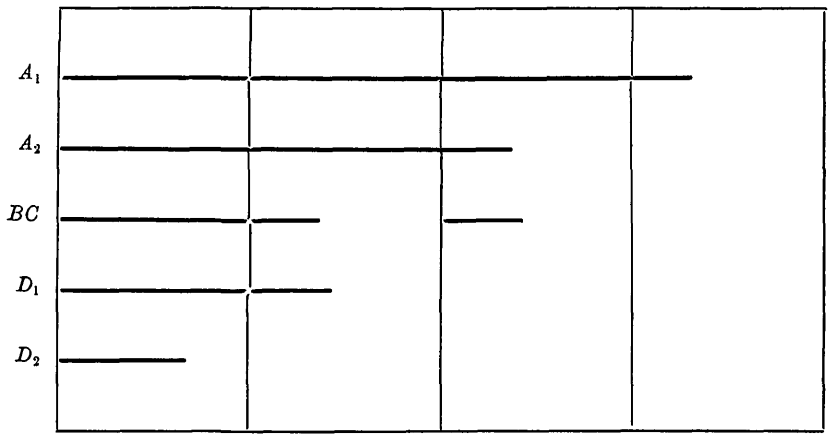

Fig. (iii).

The next step is to change $\phi_{\nu-2}(\theta)$, whenever it occurs in the expansion (17.3), into $\phi_{\nu-2}(\theta)\,\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)\,\phi_{\nu-1}(\theta)$ into $\phi_{\nu-2}(\theta)$, leaving $\phi_{\nu-1}(\theta)$ unchanged when it occurs without $\phi_{\nu-2}(\theta)$. This again does not change S. The result is shown in Fig. (iv).

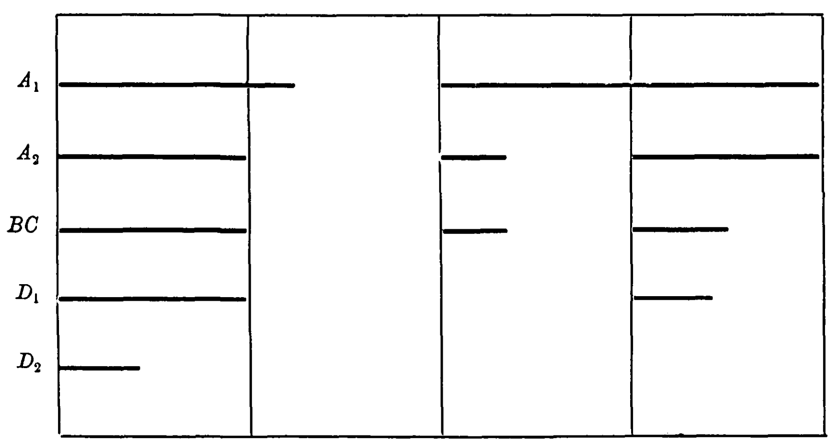

Fig. (iv).

A fresh application of Q_ν gives the distributions of Fig. (v).

We now interchange $\phi_{\nu-1}(\theta)$ and $\phi_{\nu-2}(\theta)$ once more, which can be done without altering S. The possible distributions of the non-zero elements are shown in Fig. (vi).

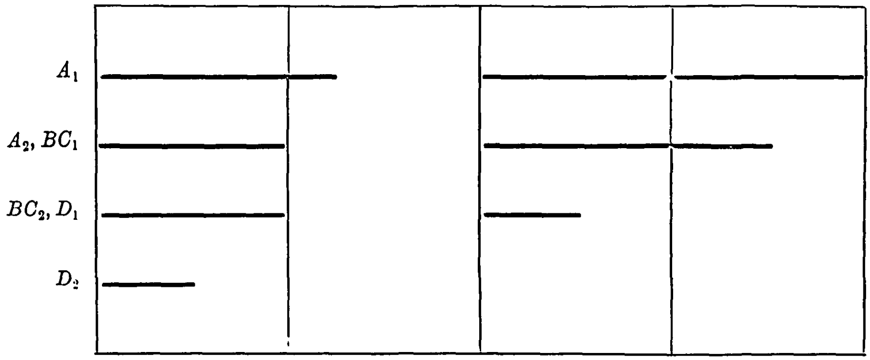

Fig. (v).

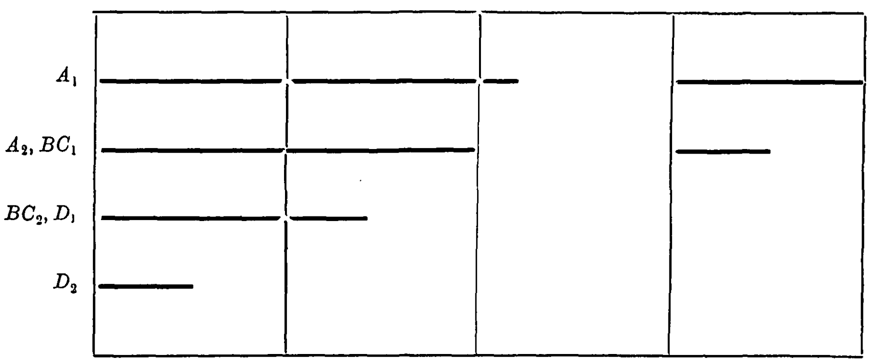

Fig. (vi).

Finally, another application of the operation Q_ν gives the standard star rearrangement of the elements. Each of the operations carried out has had the effect of increasing S or of leaving it unchanged. It follows that $S \leqslant S^*$, when the number of elements does not exceed 2^ν. The desired result now follows.

18. If we put $a_n = b_n = c_n = \ldots$ in the above theorem we obtain the following theorem, which is the $\{\psi\}$ analogue of Hardy and Littlewood's theorem† for Fourier series. Clearly, by allowing the moduli of the coefficients to vary, we do not increase the q-th mean of $f(t)$ when q is an even integer.

† Hardy and Littlewood, 5.

THEOREM XVI. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t),$$

where $\Sigma c_n^2 < \infty$, *and let*

$$F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t),$$

where the members $c_0^* \geqslant c_1^* \geqslant c_2^* \geqslant \ldots$ *denote the set* $\{|c_n|\}$ *rearranged in descending order of magnitude. If* q *is an even integer, then*

$$\int_0^1 f^q(t)\,dt \leqslant \int_0^1 F^q(t)\,dt,$$

whenever the right-hand side exists.

We need the following lemmas. The first is known†.

LEMMA 16. *Let* $\vartheta_n(t)$ $(n = 0, 1, \ldots)$ *denote a set of normalized orthogonal functions, all less in modulus than some constant* λ. *Let*

$$1 < p \leqslant 2 \leqslant q < \infty.$$

Let‡

$$f(t) \sim \sum_{m=0}^{\infty} c_m \vartheta_m(t).$$

Then

$$\sum_{m=0}^{\infty} |c_m|^p (m+1)^{p-2} \leqslant B_{p,\lambda} \int_0^1 |f(t)|^p\,dt, \tag{18.1}$$

$$\int_0^1 |f(t)|^q\,dt \leqslant B_{q,\lambda} \sum_{m=0}^{\infty} |c_m|^q (m+1)^{q-2}, \tag{18.2}$$

the right-hand side in each case being supposed existent.

In the particular case where $\vartheta_n(t)$ is identified with $\psi_n(t)$ and the coefficients c_n are in starred order, we can assert more than Lemma 16. We have, in fact,

LEMMA 17. *Let* $1 < k < \infty$, *and*

$$F(t) \sim \sum_{m=0}^{\infty} c_m^* \psi_m(t),$$

† See Paley, 8.

‡ That is to say, we write

$$c_m = \int_0^1 f(t)\,\vartheta_m(t)\,dt. \tag{i}$$

The inequality (18.2) is then to be interpreted as meaning that when the sum on the right-hand side is finite, then an $f(t)$ exists satisfying (i) and (18.2).

where the coefficients $c_n{}^*$ *are positive and decreasing. Then*

$$(18.3) \qquad B_k \sum_{m=0}^{\infty} c_m^{*k}(m+1)^{k-2} \leqslant \int_0^1 |F(t)|^k\,dt \leqslant B_k \sum_{m=0}^{\infty} c_m^{*k}(m+1)^{k-2},$$

whenever either member exists†.

We consider separately the cases $k<2$ and $k>2$. When $k<2$, the first of the inequalities (18.3) follows from (18.1). For the second we observe by Theorem V that, if

$$F_n(t) = \sum_{m=2^{n-1}}^{2^n-1} c_m{}^* \psi_m(t),$$

then

$$\int_0^1 |F(t)|^k\,dt \leqslant B_k \int_0^1 \left\{c_0{}^2 + \sum_{n=1}^{\infty} F_n{}^2(t)\right\}^{\frac{1}{2}k} dt$$

$$\leqslant B_k \left\{|c_0|^k + \sum_{n=1}^{\infty} \int_0^1 |F_n(t)|^k\,dt\right\}.$$

For

$$\int_0^1 |F_n(t)^k\,dt$$

we observe first that, for $0<t<2^{-n}$, we have $|F_n(t)| \leqslant 2^{n-1} c^*_{2^{n-1}}$. Also it may easily be verified‡ that

$$\left|\sum_{m=0}^{M} \psi_m(t)\right| \leqslant \frac{B}{t},$$

and, since the coefficients $c_m{}^*$ are decreasing, Abel's transformation formula at once gives

$$\left|\sum_{m=2^{n-1}}^{2^n-1} c_m{}^* \psi_m(t)\right| \leqslant \frac{Bc^*_{2^{n-1}}}{t}.$$

Thus

$$\int_0^1 |F_n(t)|^k\,dt \leqslant \int_0^{2^{-n}} + \int_{2^{-n}}^1$$

$$\leqslant B_k c^{*k}_{2^{n-1}} 2^{(k-1)n} \leqslant B_k \sum_{m=2^{n-2}}^{2^{n-1}-1} c_m^{*k}(m+1)^{k-2},$$

from which the desired result follows.

† For the interpretation of (18.3) see the preceding foot-note.

‡ In fact, if $2^{-(N+1)} \leqslant t < 2^{-N}$, then $\phi_N(t) = -1$, and thus, for arbitrary integral l,

$$\sum_{m=0}^{l2^{N+1}-1} \psi_m(t) = 0,$$

since, when expressed by means of Rademacher's functions, the left-hand side is divisible by $1+\phi_N(t)$.

For $k > 2$, the second of the inequalities (18.3) follows from (18.2). For the first we observe that, by Theorem V,

$$\int_0^1 |F_n(t)|^k dt \geqslant B_k \int_0^1 \left\{c_0^2 + \sum_{n=1}^{\infty} F_n^2(t)\right\}^{\frac{1}{2}k} dt$$

$$\geqslant B_k \left\{|c_0|^k + \sum_{n=1}^{\infty} \int_0^1 |F_n(t)|^k dt\right\}.$$

Also

$$\int_0^1 |F_n(t)|^k dt \geqslant \int_0^{2^{-n}} |F_n(t)|^k dt$$

$$\geqslant 2^{-n} \left(\sum_{m=2^{n-1}}^{2^n-1} c_m^*\right)^k \geqslant B_k\, 2^{(k-1)n} c^{*k}_{2^n-1}$$

$$\geqslant B_k \sum_{m=2^n}^{2^{n+1}-1} c_m^{*k} (m+1)^{k-2},$$

from which the desired result follows.

We may now continue the argument in exactly the same way as in Hardy and Littlewood's investigations, and we obtain results analogous to those which they obtain for Fourier series. Of these the most important are given in the theorems which follow, which also result immediately from Lemmas 16 and 17.

THEOREM XVII. *Let*

$$f(t) \sim \sum_{n=0}^{\infty} c_n \psi_n(t); \quad F(t) \sim \sum_{n=0}^{\infty} c_n^* \psi_n(t),$$

where $c_0^*, c_1^*, c_2^*, \ldots$ *denote the set* $\{|c_n|\}$ *rearranged in descending order. If* $2 \leqslant q < \infty$, *we have*

$$\int_0^1 |f(t)|^q dt \leqslant B_q \int_0^1 |F(t)|^q dt,$$

whenever the right-hand side exists. B_q *is a constant which depends only on* q.

THEOREM XVIII. *With the notation of the last theorem we have, for* $1 < p \leqslant 2$,

$$\int_0^1 |F(t)|^p dt \leqslant B_p \int_0^1 |f(t)|^p dt,$$

whenever the right-hand side exists. B_p *depends only on* p.

References.

1. R. M. Gabriel, "An additional proof of a theorem on rearrangements", *Journal London Math. Soc.*, 3 (1928), 134–136.
2. G. H. Hardy, "On the summability of Fourier series", *Proc. London Math. Soc.* (2), 12 (1913), 365–372.
3. G. H. Hardy and J. E. Littlewood, "A maximal theorem with function-theoretic applications", *Acta Math.*, 54 (1930), 81–116.
4. G. H. Hardy and J. E. Littlewood, "Notes on the theory of series (VIII): An inequality", *Journal London Math. Soc.*, 3 (1928), 105–110.
5. G. H. Hardy and J. E. Littlewood, "Notes on the theory of series (XIII): Some new properties of Fourier constants", *Journal London Math. Soc.*, 6 (1931), 3–9.
6. G. H. Hardy and J. E. Littlewood, "Notes on the theory of series (IV): On the strong summability of Fourier series", *Proc. London Math. Soc.* (2), 26 (1926), 273–286.
7. J. E. Littlewood and R. E. A. C. Paley, "Theorems on Fourier series and power series", *Journal London Math. Soc.*, 6 (1931), 230-233.
8. R. E. A. C. Paley, "Some theorems on orthogonal functions (1)", *Studia Math.*, 3 (1931), 226–238.
9. F. Riesz, "Sur une inégalité intégrale", *Journal London Math. Soc.*, 5 (1930), 162–168.
10. W. W. Rouse Ball, *Mathematical recreations and essays*, 10 ed. (1922), 21.