Seifert matrices of braids with applications to isotopy and signatures.

Chris Palmer

December 15, 2015

Let β be a braid with closure $\widehat{\beta}$ a link. Collins developed an algorithm to find the Seifert matrix of the canonical Seifert surface Σ of $\widehat{\beta}$ constructed by Seifert's algorithm. Motivated by Collins' algorithm and a construction of Ghys, we define a 1-dimensional simplicial complex $K(\beta)$ and a bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ such that there is an inclusion $K(\beta) \hookrightarrow \Sigma$ which is a homotopy equivalence inducing an isomorphism $H_1(\Sigma;\mathbb{Z}) \cong H_1(K(\beta);\mathbb{Z})$ such that $[\lambda_\beta] : H_1(K(\beta);\mathbb{Z}) \times H_1(K(\beta);\mathbb{Z}) \to \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$ is the Seifert form of Σ. We show that this chain level model is additive under the concatenation of braids and then verify that this model is chain equivalent to Banchoff's combinatorial model for the linking number of two space polygons and Ranicki's surgery theoretic model for a chain level Seifert pairing. We then define the chain level Seifert pair (λ_β, d_β) of a braid β and equivalence relations, called A and $\widehat{A}$-equivalence. Two n-strand braids are isotopic if and only if their chain level Seifert pairs are A-equivalent and this yields a universal representation of the braid group. Two n-strand braids have isotopic link closures in the solid torus $D^2 \times S^1$ if and only if their chain level Seifert pairs are $\widehat{A}$-equivalent and this yields a representation of the braid group modulo conjugacy. We use the first representation to express the ω-signature of a braid β in terms of the chain level Seifert pair (λ_β, d_β).

1 Introduction

Let β be a braid with closure $\widehat{\beta}$ a link. The canonical Seifert surface of β constructed by Seifert's algorithm resolves each crossing of $\widehat{\beta}$

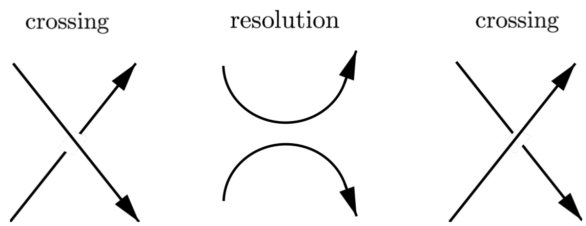


Figure 1: Resolving an overcrossing and an undercrossing.

to produce a collection of disjoint, oriented, simple, planar circles called Seifert circles. Each Seifert circle bounds a planar disc and we may push the planar disks vertically to make them disjoint. Attaching a twisted band between the Seifert circles for each resolution of a crossing, with the twist matching the type of the crossing, then produces the canonical Seifert surface of $\widehat{\beta}$, which is a closed orientable surface of genus $g \geq 0$ with boundary $\widehat{\beta}$.

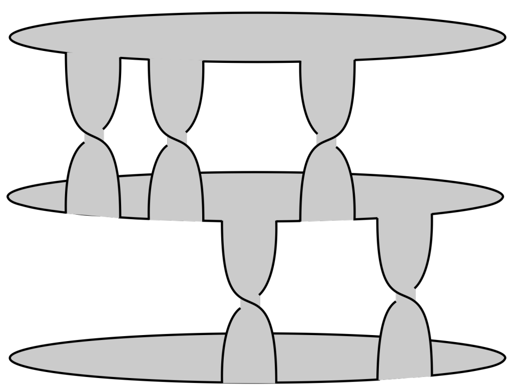


Figure 2: A Seifert surface produced by Seifert's algorithm.

Choosing an ordered basis $\{[\gamma_i]\}_{i=1}^{2g}$ of $H_1(\Sigma;\mathbb{Z})$, with each basis homology class $[\gamma_i]$ represented by simple, closed curve $\gamma_i \subset \Sigma$, we may push each γ_i in the positive normal direction to produce a simple closed curve γ_i^+ which lies in $S^3 - \Sigma$. The Seifert form $V : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$ is the bilinear form determined on the basis homology classes by the linking numbers $\mathrm{Lk}(\gamma_i, \gamma_j^+)$.

Two n-strand braids β, β' may be concatenated to produce an n-strand $\beta\beta'$. The effect of the concatenation of braids is a gluing of Seifert surfaces along parts of their boundaries. The Mayer-Vietoris sequence then provides an obstruction for the Seifert form to be additive under the concatenation of braids. This suggests that one could try to find a chain level Seifert form, expressed in terms of partial linking numbers, which is additive on the chain level under the concatenation of braids and descends to the Seifert form on the homology level.

Banchoff [3] gave a combinatorial linking formula for two disjoint space polygons expressed in terms of partial linking numbers of pairs of edges as follows. Let $X = \{X_0, X_1, \ldots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \ldots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$. For a unit vector $\xi \in S^2$ let $p_\xi : \mathbb{R}^3 \to P$ denote the projection map from $\mathbb{R}^3$ onto the plane P orthogonal to ξ. A vector $\xi \in S^2$ in called general for X and Y if the projections $p_\xi(X), p_\xi(Y) \subset \mathbb{R}^2$ are in general position. For a vector $\xi \in S^2$ which is general for X and Y, define $C_{i,j}(X,Y,\xi)$ to be the sign of $P_\xi(Y_{j+1} - Y_j) \times P_\xi(X_{i+1} - X_i).(\overline{X_i} - \overline{Y_j})$ if there are interior points $\overline{X_i}$ of the edge $X_i X_{i+1}$ and $\overline{Y_j}$ of the edge $Y_j Y_{j+1}$ such that $p_\xi(\overline{X_i}) = p_\xi(\overline{Y_j})$ and define $C_{i,j}(X,Y,\xi)$ to be zero otherwise.

The linking number of two space polygons is then expressible as the sum of the partial linking numbers of all edge pairs.

Theorem [3, p.1176-1177] For disjoint polygonal knots $X, Y \subset \mathbb{R}^3$ the value

$$C(X,Y,\xi) = \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} C_{i,j}(X,Y,\xi) \in \mathbb{Z}$$

is independent of the choice of general vector $\xi \in S^2$. The linking number of the polygonal knots determined by X and Y is the average value of $C(X,Y,\xi)$, that is

$$\mathrm{Lk}(X,Y) = \frac{1}{4\pi} \int_{\xi \in S^2} C(X,Y,\xi) d\omega = \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} \int_{\xi \in S^2} C_{i,j}(X,Y,\xi) d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. Moreover this integral may be expressed in terms of dihedral angles of tetrahedra.

The closure of an n-strand braid with ℓ-crossings arises as the trace of ℓ 0-surgeries on a disjoint union of n circles. Ranicki [12] applied the algebraic theory of surgery to the geometric surgeries to obtain a chain level formula which is defined inductively in terms of Seifert graphs. The Seifert graph of a braid β is the 1-dimensional CW-complex $X(\beta)$ constructed from the canonical Seifert surface of β by collapsing each Seifert disc to a point and collapsing each twisted band to its core. If β is an n-strand braid with ℓ-crossings then the Seifert graph $X(\beta)$ has ℓ 1-cells and n 0-cells and has a cellular chain complex of the form

$$d : C_1(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^\ell \to C_0(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^n$$

where d is a signed incidence matrix. If β' is another n-strand braid with ℓ' crossings then the Seifert graph $X(\beta')$ has a cellular chain complex of the form

$$d' : C_1(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^{\ell'} \to C_0(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The Seifert graph of the concatenated braid $\beta\beta'$ is a CW-complex which can be formed from the Seifert graphs of β, β' by identifying the 0-cells so that $X(\beta\beta')$ has $(\ell+\ell')$ 1-cells, n 0-cells and a cellular chain complex of the form

$$d'' = \begin{pmatrix} d & d' \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^\ell \oplus \mathbb{Z}^{\ell'} \to C_0(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

Ranicki defined the canonical generalised Seifert matrices of the elementary regular n-strand braids σ_i, σ_i^{-1} to be the 1×1 matrices

$$\psi_{\sigma_i} = \begin{pmatrix} 1 \end{pmatrix}, \quad \psi_{\sigma_i^{-1}} = \begin{pmatrix} -1 \end{pmatrix}$$

and inductively defined the generalised Seifert matrix of the concatenated braid $\beta\beta'$ to be the matrix

$$\psi_{\beta\beta'} = \begin{pmatrix} \psi_\beta & -d^*\chi d' \\ 0 & \psi_{\beta'} \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

where χ is the lower triangular $n \times n$ matrix with ones below the diagonal.

Theorem [12, p.37-38] Let β, β' be regular n-strand braids. The generalised Seifert matrix

$$\psi_{\beta\beta'} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

induces the Seifert form of $\beta\beta'$

$$\psi_{\beta\beta'} : H_1(X(\beta\beta');\mathbb{Z}) \times H_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}.$$

Motivated by the space polygon linking formula of Banchoff [3] and the surgery-theoretic linking formula of Ranicki [12] we construct a new chain level Seifert form. Following a suggestion of Étienne Ghys, to each braid β we associate a 1-dimensional simplicial complex $K(\beta)$ called a fence. The fence of an elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the oriented 1-dimensional simplicial complex $K(\beta)$ with $2n$ 0-simplices and $(n+1)$ 1-simplices as shown below

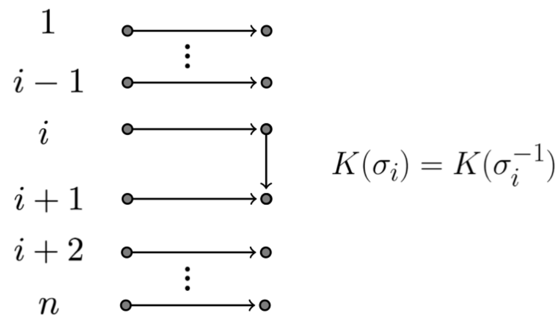


Figure 3: The fences associated to the elementary n-strand braids $\sigma_i^{\pm 1}$.

The fence of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ with ℓ crossings is the concatenation of the fences of the elementary braids from left to right and there is a natural embedding of the fence of β into the canonical Seifert surface of β.

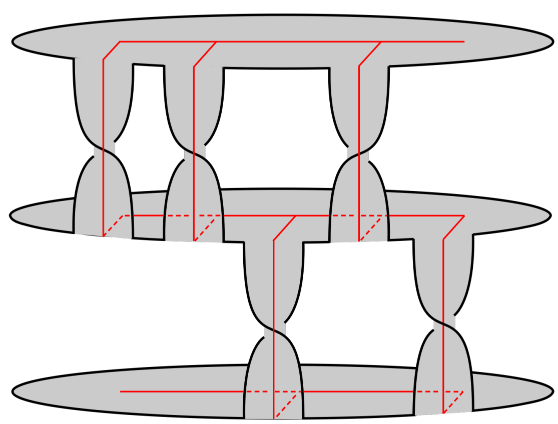


Figure 4: The embedding of the fence $K(\beta)$ in to the canonical Seifert surface for $\widehat{\beta}$.

By examining how a fence links with itself when it is pushed in the positive normal direction to the canonical Seifert surface

we can associate to each fence a $\mathbb{Z}[\frac{1}{2}]$-valued bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ which encodes partial self-linking information. This descends to the Seifert form of β on the homology level:

Theorem 6. The embedding $K(\beta) \hookrightarrow \Sigma$ is a homotopy equivalence inducing

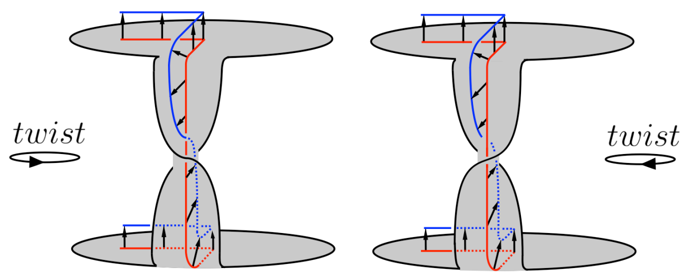


Figure 5: Pushing part of the fences in the positive normal direction.

an isomorphism $H_1(K(\beta);\mathbb{Z}) \cong H_1(\Sigma;\mathbb{Z})$ with a commutative diagram

$$\begin{array}{ccc} H_1(K(\beta);\mathbb{Z}) \times H_1(K(\beta);\mathbb{Z}) & \xrightarrow{[\lambda_\beta]} & \mathbb{Z} \subset \mathbb{Z}[\tfrac{1}{2}] \\ \downarrow \cong & \nearrow_{V} & \\ H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) & & \end{array}$$

Moreover, this chain level Seifert form is additive under the concatenation of braids:

Theorem 7. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a braid where each β_i is an elementary braid. The chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ can be represented by a block diagonal matrix

$$\begin{pmatrix} \lambda_{\beta_1} & 0 & \ldots & 0 \\ 0 & \lambda_{\beta_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_{\beta_\ell} \end{pmatrix}$$

We then compare our model to Banchoff's and Ranicki's. Our model has the advantage that the partial linking numbers are $\mathbb{Z}[\frac{1}{2}]$-valued and not $\mathbb{R}$-valued as in Banchoff's model. Moreover, the concatenation behaviour in our model is additive and gives an instant chain level Seifert form whereas Ranicki's model is inductively defined.

Propositions 5, 6. Our model is chain equivalent to Banchoff's combinatorial model for the linking number of two space polygons and chain equivalent to Ranicki's surgery-theoretic chain level Seifert pairing model.

We give two applications of this chain level Seifert form to the isotopy of braids and to the signature of braids.

Two n-strand braids β, β' are isotopic if β can be continuously deformed to β' through a family of n-strand braids. Isotopy is an equivalence relation on the set of n-strand braids and the set of isotopy classes form a group B_n called the n-strand braid group. Artin [2] showed that there is a presentation of the braid group B_n with generators the elementary n-strand braids $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations of the form

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1, \quad \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1.$$

We define the chain level Seifert pair (λ_β, d_β) of a braid β and two equivalence relations, called A and $\widehat{A}$-equivalence, such that:

Propositions 8, 10. The A-equivalence class of the chain level Seifert pair of an n-strand braid β is a complete isotopy invariant. The $\widehat{A}$-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant of the closure $\widehat{\beta}$ inside the solid torus.

The A-equivalence relation yields a universal representation of the braid group and the $\widehat{A}$-equivalence relation yields a representation of the braid group modulo conjugacy:

Theorems 10, 11. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and denote by B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection such that words $\beta, \beta' \in F_n$ differ by the relations in the braid group if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are A-equivalent. Moreover two words $\beta, \beta' \in B_n$ are conjugate if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are $\widehat{A}$-equivalent. This induces an isomorphism of groups

$$(\lambda, d) : B_n \to \frac{\{\text{chain level Seifert pairs}\}}{A - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

and descends to a bijection

$$(\lambda, d) : \frac{B_n}{\text{conjugacy}} \to \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A} - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

such that there is a commutative diagram

$$\begin{array}{ccc}
F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\
\downarrow & & \downarrow \\
B_n & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{A\text{–equivalence}} \\
\downarrow & & \downarrow \\
\dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{–equivalence}}
\end{array}$$

For a unit complex number $\omega \neq 1$ the ω-signature of a braid β with Seifert matrix V is the signature $\sigma_\omega(\beta)$ of the hermitian form $(H_1(\Sigma;\mathbb{Z}), (1-\omega)V + (1-\overline{\omega})V^t)$. We can express the ω-signature of a braid in terms of its chain level Seifert pair:

Theorem 12. Let β be a braid with chain level Seifert pair (λ_β, d_β) and let $\omega \neq 1$ be a unit complex number. The ω-signature of β is the signature of the hermitian pair

$$\left(C^1(K(\beta);\mathbb{C}) \oplus C_0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix} \right)$$

so that

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix} \right).$$

This paper is organised as follows.

In sections 2 to 4 we introduce the basic operations one can perform on braids such as concatenation, taking the closure, performing an isotopy and constructing a Seifert form from a canonical Seifert surface.

In section 5 we define the 1-dimensional simplicial complex $K(\beta)$ and the $\mathbb{Z}[\frac{1}{2}]$-valued bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. We show that there is an embedding of $K(\beta) \hookrightarrow \Sigma$ with image a deformation retract of the canonical Seifert surface Σ of $\widehat{\beta}$ constructed by Seifert's algorithm. In section 6 we examine how the image $K(\beta)$ is pushed along the positive normal to the Seifert surface to show that λ_β descends to the Seifert form on the homology level. In section 7 we show that the bilinear form λ_β is additive under the concatenation of braids and in section 8 we compare our chain level Seifert form to the space polygon linking model of Banchoff and the surgery-theoretic Seifert form of Ranicki.

In section 9 we define the A and $\widehat{A}$-equivalence relations and use the chain level Seifert pair (λ_β, d_β) of a braid β to produce a representation of the braid group and of the braid group modulo conjugacy and in section 10 we construct a chain level formula for the ω-signature of a braid.

2 Links and linking numbers

Definition 1. An *n-component link* is an embedding $L : \sqcup_n S^1 \hookrightarrow S^3$ of n disjoint, piecewise smooth, simple, closed curves. A *knot* is a one-component link. Let $P \subset \mathbb{R}^3$ be a 2-dimensional subspace of $\mathbb{R}^3$ and let $p : \mathbb{R}^3 \to P$ be the orthogonal projection map onto P. We say that $p : \mathbb{R}^3 \to \mathbb{R}$ is a *regular projection* of a link L if for each $x \in P$ the intersection $p^{-1}(x) \cap L$ consists of at most two points, in which case the *link diagram* is the image $p(L) \subset P$ with the over and under crossings recorded. An *oriented link* is a link for which each connected component has been given an orientation and this is recorded on a link diagram by a choice of arrow on each component of the link diagram. Two links L, L' are *ambient isotopic* if there is a homotopy of orientation preserving homeomorphisms $f_t : \sqcup_n S^1 \hookrightarrow S^3$ with $(0 \leq t \leq 1)$such that f_0 is the identity and $f_1(L) = L'$.

We will abuse the terminology in the standard way, with the word 'link' sometimes referring to the embedding and sometimes referring to the image of the embedding.

Example 1. Regular projections of an oriented trefoil knot and oriented Hopf link.

The linking number of two knots is an important numerical invariant in knot theory and may be defined in any of the following ways.

Definition 2. Let J, K be two disjoint oriented knots in S^3.

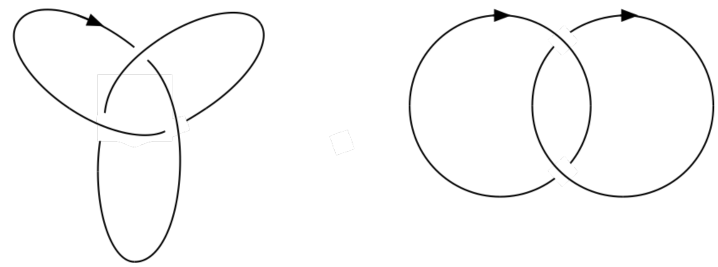


Figure 6: Projections of the trefoil knot and Hopf link.

(i) The knot $J \subset S^3 - K$ induces a homology class $[J] \in H_1(S^3 - K)$ in the complement of K. Since $H_1(S^3 - K) \cong \mathbb{Z}$ is infinite cyclic, fixing a generator $\gamma \in H_1(S^3 - K)$ we may write $[J] = n\gamma$ with $\mathrm{Lk}_1(J,K) = n \in \mathbb{Z}$.

(ii) Let $p : \mathbb{R}^3 \to P$ be a regular projection of the link $J \sqcup K \subset \mathbb{R}^3$. The linking number is half the sum of the signed crossings $\mathrm{Lk}_1(J,K) = \frac{1}{2}\sum_{x \in p(J)\cap p(K)} \epsilon_x \in \mathbb{Z}$ where each crossing $x \in p(J) \cap p(K)$ is assigned a sign $\epsilon_x = \pm 1$ as follows

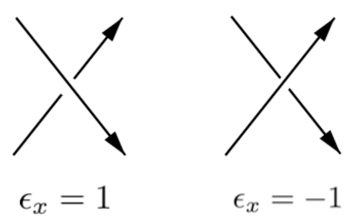


Figure 7: The signs associated to an overcrossing and an undercrossing.

(iii) The knots J and K induce 1-cycles $J, K \in C_1(S^3;\mathbb{Z})$. Since $H_1(S^3;\mathbb{Z}) = 0$ we may choose a 2-chain $\Sigma \in C_2(S^3;\mathbb{Z})$ such that $\partial\Sigma = J$. The cap product $\Sigma \cap K \in C_0(S^3;\mathbb{Z})$ is a 0-cycle which induces a well defined homology class $[\Sigma \cap K] \in H_0(S^3;\mathbb{Z}) \cong \mathbb{Z}$ which determines $\mathrm{Lk}_2(J,K) \in \mathbb{Z}$.

(iv) Orienting $S^1 \times S^1$ and S^2, the linking number $\mathrm{Lk}_2(J,K) \in \mathbb{Z}$ is the degree of the Gauss map

$$f : S^1 \times S^1 \to S^2; \quad f(u,v) = \frac{J(u) - K(v)}{\|J(u) - K(v)\|}.$$

(v) The linking number $\mathrm{Lk}_3(J, K)$ is the Gauss integral

$$\frac{1}{4\pi}\int_J\int_K \frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z'-z)(dxdy'-dydx')}{[(x'-x)^2+(y'-y)^2+(z'-z)^2]^{3/2}} \in \mathbb{Z}$$

Theorem 1. [15, p.132-135]. The above definitions of linking numbers agree (up to sign) and the linking number is an ambient isotopy invariant.

3 Seifert surfaces and Seifert matrices of links

Definition 3. A *Seifert surface* for an oriented link L is a compact oriented surface $\Sigma \subset S^3$ with oriented boundary $\partial\Sigma = L$ such that the normal bundle $\nu_{\Sigma\subset S^3}$ is trivial.

Seifert's algorithm [16] produces a Seifert surface for an oriented link L in the following way. Fix a regular projection of L and resolve each crossing as shown below.

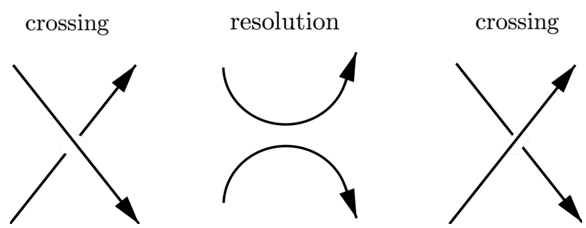


Figure 8: Resolving an overcrossing and an undercrossing.

Doing so produces a collection of disjoint, oriented, simple, planar circles called Seifert circles. Each Seifert circle bounds a planar disc. If some of the discs are not disjoint, because the corresponding Seifert circles are nested, we may push some the discs in a direction perpendicular to the plane to make them disjoint. We then attach a twisted band between the Seifert circles for each resolution of crossing with the twist matching the type of the crossing.

Example 2. Seifert surfaces for an oriented trefoil knot and oriented Hopf link constructed by Seifert's algorithm. We have labelled the Seifert circles to keep track of them when we move the discs they bound.

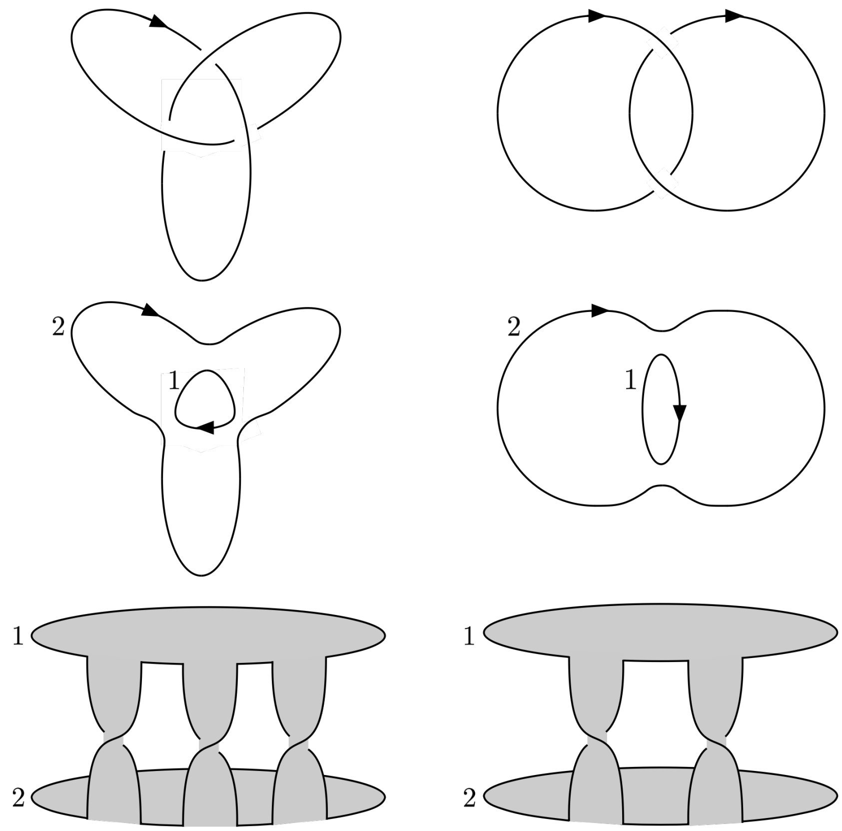


Figure 9: Seifert's algorithm performed on a trefoil knot and Hopf link.

A link has many regular projections so the Seifert surfaces constructed by Seifert's algorithm are highly non-unique. A Seifert surface for a link is however unique up to a certain relation called S-equivalence.

Definition 4. Two compact surfaces with boundary $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$ are *S-equivalent* if $(\Sigma_2, \partial\Sigma_2)$ can be obtained from $(\Sigma_1, \partial\Sigma_1)$ by a combination of ambient isotopy and adding or subtracting finitely many handles by ambient surgery.

Theorem 2. [9, Lemma 5.2.4] Any two Seifert surfaces of a link L are S-equivalent.

Let L be an oriented link with Seifert surface Σ of genus g. Then $H_1(\Sigma;\mathbb{Z})$ is a f.g. free abelian group of rank $2g$. Choose a basis $\{[\gamma_i]\}_{i=1}^{2g}$ of $H_1(\Sigma;\mathbb{Z})$ with each basis homology classes $[\gamma_i]$ represented by simple, closed curve $\gamma_i \subset \Sigma$. Use the triviality of the normal bundle $\nu_{\Sigma \subset S^3}$ to define a small bi-collar $\Sigma \times [-1,1]$ of $\Sigma \subset S^3$ and for each $1 \leqslant i \leqslant n$ define $\gamma_i^+ = \gamma_i \times \{1\} \subset \Sigma \times [-1,1]$ to be the simple, closed curve in S^3 obtained by pushing γ_i in the positive normal direction to Σ.

Definition 5. The *Seifert matrix* of Σ with respect to this bi-collar and this choice of basis is the $2g \times 2g$ matrix V defined by

$$V_{i,j} = \mathrm{Lk}(\gamma_i, \gamma_j^+), \quad (1 \leqslant i, j \leqslant 2g)$$

and the *Seifert form* of Σ is the bilinear form

$$V : H_1(\Sigma; \mathbb{Z}) \times H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}.$$

The ambiguity in the choice of Seifert surface for a link means that the Seifert matrix of a link is only unique up to an algebraic S-equivalence relation.

Definition 6. Two $n \times n$ integral matrices are *S-equivalent* if one can be transformed into the other by a finite sequence of the following operations:

(i) $V \mapsto PVP^t$ with P integral and unimodular.

(ii) $V \mapsto \left(\begin{array}{c|cc} V & \xi & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$

(iii) $V \mapsto \left(\begin{array}{c|cc} V & 0 & 0 \\ \hline \xi & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$.

Theorem 3. [11, Theorem 3.1] The S-equivalence class of the Seifert matrix of a link is an isotopy invariant.

In sections 5 and 6 we will develop a chain level lift of the Seifert matrix for a link which can be expressed as the closure of a braid. In section 9 we will develop equivalence relations, called A- and $\widehat{A}$-equivalence, such that the A-equivalence class of the chain level lift is an isotopy invariant of the braid and the $\widehat{A}$-equivalence class of the chain level lift is an isotopy invariant of the closure of the braid.

4 Regular braids, geometric braids and closures

We are particularly interested in those links which can be written as the closure of a braid.

Definition 7. For $1 \leqslant i \leqslant n-1$ the *elementary n-strand braid* σ_i is the n-strand braid of polygonal arcs with a single crossing of strand i over strand $i+1$ and no crossings between any other pairs of adjacent stands and the *elementary n-strand braid* σ_i^{-1} is the n-strand braid with a single crossing of strand i under strand $i+1$ and no crossings between any other pairs of adjacent strands. The *trivial n-strand braid* 1 is the n-strand braid of polygonal arcs with no crossings.

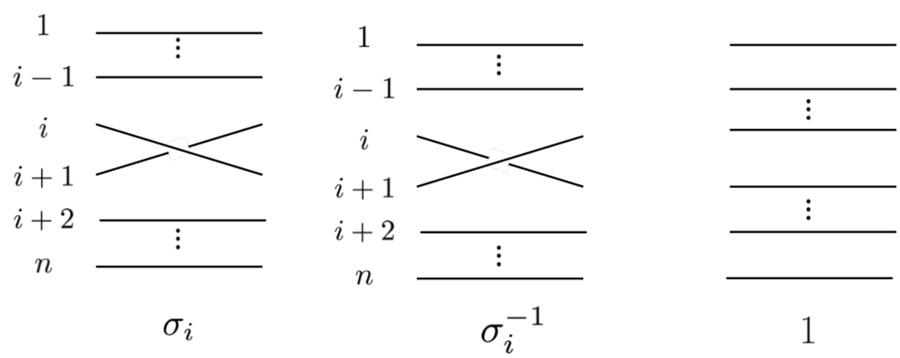


Figure 10: The elementary n-strand braids.

A *regular n-strand braid* $\beta = \beta_1\beta_2\ldots\beta_\ell$ is the concatenation from left to right of finitely many elementary n-strand braids and trivial n-strand braids.

Regular braids are combinatorial models for geometric braids.

Definition 8. Let $n \geqslant 1$. A *geometric n-strand braid* β with permutation $\sigma \in S_n$ of the set $\{1,2,\ldots,n\}$ is an embedding

$$\beta : \{1,2,\ldots,n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]; \quad (k,t) \mapsto \beta(k,t)$$

such that

$$\beta(k,0) = (k,0,0) \in \mathbb{R}^2 \times \{0\} \qquad (1 \leqslant k \leqslant n)$$
$$\beta(k,1) = (\sigma(k),0,1) \in \mathbb{R}^2 \times \{1\} \quad (1 \leqslant k \leqslant n)$$

and each composition

$$[0,1] \xrightarrow{\beta(k,-)} \mathbb{R}^2 \times [0,1] \xrightarrow{\text{projection}} [0,1] \quad (1 \leqslant k \leqslant n)$$

is a homeomorphism.

Example 3. A geometric 4-strand braid with permutation $\sigma = (123)(4) \in S_4$

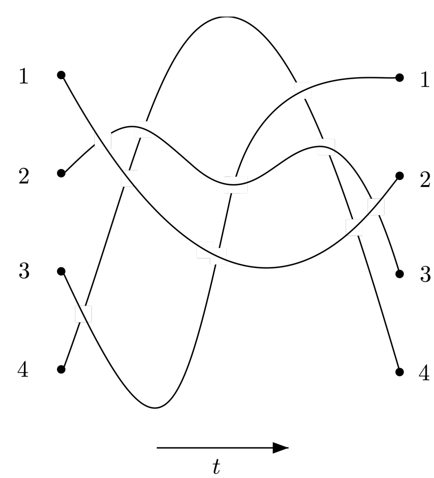


Figure 11: A 4-strand braid.

Definition 9. The *concatenation* of geometric n-strand braids β with permutation $\sigma \in S_n$ and β' with permutation $\sigma' \in S_n$ is the geometric n-strand braid

$$\beta\beta' : \{1, 2, \dots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1]$$

with permutation $\sigma\sigma' \in S_n$ defined by

$$\beta\beta'(k,t) = \begin{cases} \beta'(k,2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(k,2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Definition 10. Two geometric n-strand braids β, β' are *isotopic* if there exists a family of geometric n-strand braids

$$\beta_s : \{1, 2, \dots, n\} \times [0,1] \hookrightarrow \mathbb{R}^2 \times [0,1] \quad (s \in [0,1])$$

such that $\beta_0 = \beta$ and $\beta_1 = \beta'$ and each function

$$\{1, 2, \dots, n\} \times [0,1] \times [0,1] \to \mathbb{R}^2 \times [0,1]; \quad (k,t,s) \mapsto \beta_s(k,t) \quad (1 \leqslant k \leqslant n)$$

is continuous.

Lemma 1. Isotopy of geometric n-strand braids is an equivalence relation. The set of isotopy classes of geometric n-strand braids is a group with:

(i) The composition of the isotopy classes $[\beta], [\beta']$ of geometric n-strand braids β, β' equal to the isotopy class $[\beta\beta']$ of the geometric n-strand braid $\beta\beta'$.

(ii) The identity element equal to the isotopy class of the geometric n-strand braid

$$\{1, 2, \ldots, n\} \times [0, 1] \hookrightarrow \mathbb{R}^2 \times [0, 1]; \quad (k, t) \mapsto (k, 0, t)$$

(iii) The inverse of the isotopy class $[\beta]$ of a geometric n-strand braid β

$$\beta : \{1, 2, \ldots, n\} \times [0, 1] \hookrightarrow \mathbb{R}^2, \quad (k, t) \mapsto \beta(k, t)$$

equal to the isotopy class of the geometric n-strand braid

$$\{1, 2, \ldots, n\} \times [0, 1] \hookrightarrow \mathbb{R}^2, \quad (k, t) \mapsto \beta(k, 1 - t).$$

Regular braids can be used to give a presentation of the braid group.

Theorem 4. [2] Each geometric n-strand braid is isotopic to a regular n-strand braid so that the braid group B_n of isotopy classes of geometric n-strand braids has a presentation

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i - j| \geqslant 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i - j| = 1\rangle.$$

In particular, two geometric n-strand braids β, β' are isotopic if and only if they are isotopic to regular n-strand braids determined by braid words β, β' from the alphabet $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_{n-1}^{-1}\}$ such that β' can be obtained from β by applying finitely many of the relations

(i) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$

(ii) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geqslant 2$

(iii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i - j| = 1$.

Every braid β determines a link $\widehat{\beta}$ by a closure operation.

Proposition 1. [7, p.18] Let $U \subset \mathbb{R}^2$ be an open disc containing the set of points $\{(1, 0), (2, 0), \ldots, (n, 0)\}$.

(i) Any geometric n-strand braid

$$\beta : \{1, 2, \dots, n\} \times [0, 1] \hookrightarrow \mathbb{R}^2 \times [0, 1]$$

is isotopic to a geometric n-strand braid

$$\beta' : \{1, 2, \dots, n\} \times [0, 1] \hookrightarrow U \times [0, 1] \hookrightarrow \mathbb{R}^2 \times [0, 1]$$

with image contained in $U \times [0, 1]$.

(ii) Any two geometric n-strand braids which are isotopic in $\mathbb{R}^2 \times [0, 1]$ and have image in $U \times [0, 1]$ are isotopic in $U \times [0, 1]$.

(iii) The quotient map

$$D^2 \times [0, 1] \to D^2 \times S^1 = \frac{D^2 \times [0, 1]}{(x, 0) \sim (x, 1)}$$

sends a geometric n-strand braid β' contained in $U \times [0, 1] \subset D^2 \times [0, 1] \subset \mathbb{R}^2 \times [0, 1]$ to a canonically oriented link $\widehat{\beta}$ contained in $U \times S^1 \subset D^2 \times S^1$.

(iv) Given a geometric n-strand braid β, the isotopy class of the link $\widehat{\beta}$ in $D^2 \times S^1$ relative to the boundary $S^1 \times S^1$ depends only on the isotopy class of β.

Definition 11. The *closure* of a regular n-strand braid β is the isotopy class of the link $\widehat{\beta}$ formed from any geometric n-strand braid isotopic to the regular n-strand braid β.

Proposition 1 ensures that the closure operation is well-defined. It is often convenient to picture the closure of a braid, which is oriented from left to right, as follows

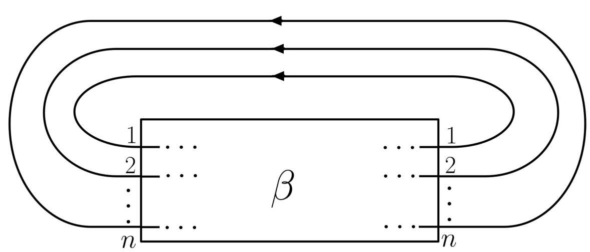


Figure 12: The closure of a braid.

Theorem 5. [1] Every oriented link in S^3 is isotopic to the closure of a regular braid.

The choice of such a braid is highly non-canonical. However, by Markov's theorem [10] any two such braids (with the same braid axis) differ only by a braid isotopy and a finite number of braid stabilisations and destabilisations.

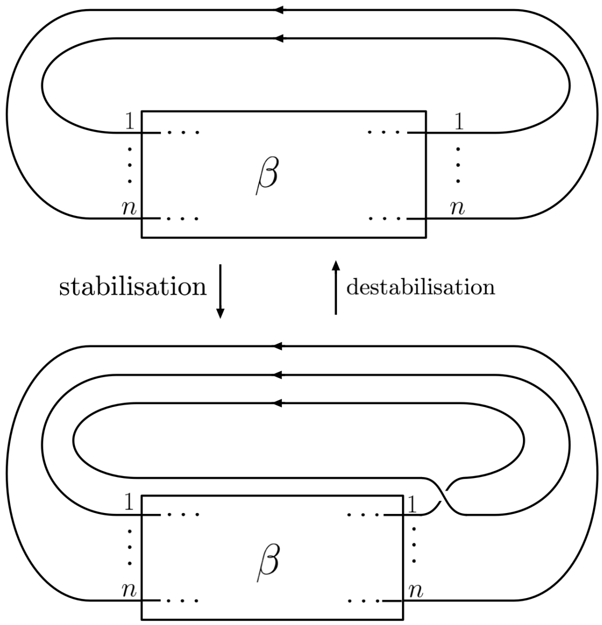


Figure 13: The stabilisation and destabilisation operations.

5 Pushing fences

Definition 12. The *fence* of the elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the oriented 1-dimensional simplicial complex $K(\beta)$ with $2n$ 0-simplices and $(n+1)$ 1-simplices as shown below. The *fence* of the trivial n-strand braid 1 is the 0-dimensional simplicial complex $K(1)$ with n 0-simplices as shown below.

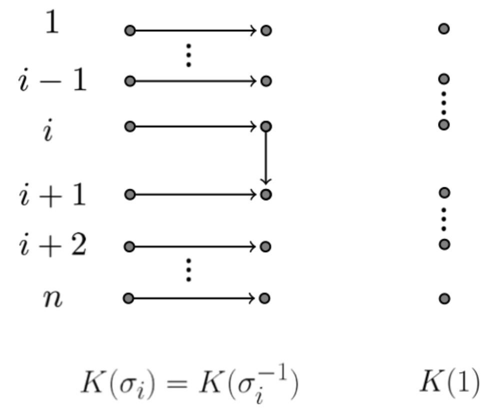


Figure 14: The fences associated to the elementary braids $\sigma_i^{\pm 1}$ and the trivial braid.

The *fence* of a regular n-strand braid $\beta = \beta_1\beta_2\ldots\beta_\ell$ is the concatenation of the fences $K(\beta_1), K(\beta_2), \ldots, K(\beta_\ell)$ from left to right so that $K(\beta_1\beta_2\ldots\beta_\ell) = \cup_{i=1}^{\ell} K(\beta_i)$ where $K(\beta_i)$ intersects $K(\beta_{i+1})$ in the right hand vertex set of $K(\beta_i)$ and the left hand vertex set of $K(\beta_{i+1})$.

Example 4. The 3-strand braid $\beta = \sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$ has the fence

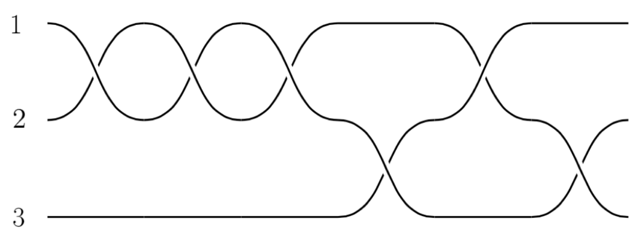


Figure 15: The braid $\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$.

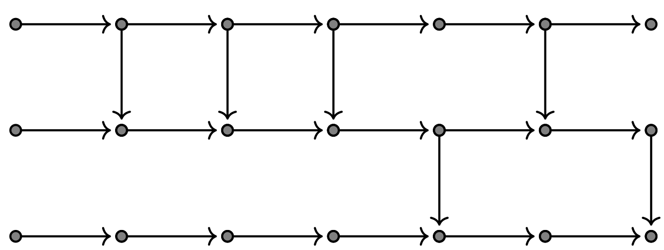


Figure 16: The fence $K(\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2)$.

Proposition 2. For a regular braid β with closure $\widehat{\beta}$ let Σ be the canonical Seifert surface of $\widehat{\beta}$ constructed by Seifert's algorithm. There is an inclusion $K(\beta) \hookrightarrow \Sigma$ which is a homotopy equivalence.

Proof. Suppose that $\beta = \beta_1\beta_2\ldots\beta_\ell$ is a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid . The orientation of the n strands of the braid from left to right induces an orientation of the link $\widehat{\beta}$ in a natural way. Seifert's algorithm resolves the ℓ crossings of $\hat{\beta}$ to produce n Seifert circles. The Seifert circles may be labelled $1, 2, \ldots, n$, stacked one below the other with 1 at the top and n at the bottom and then filled in with discs. For each $1 \leqslant k \leqslant \ell$ we then attach a twisted band between the Seifert circles corresponding the crossing encoded by β_k. The order in which the bands are attached from left to right is determined by the order in the braid word $\beta_1\beta_2\ldots\beta_\ell$.

Firstly suppose that Σ is connected. A deformation retraction of Σ onto an embedding of $K(\beta)$ is obtained by pushing the left and right most parts of the discs to meet the ends of $K(\beta)$ and then contracting each of the twisted bands to its central vertical core and contracting each of the discs to a part of its horizontal diameter. The inclusion $K(\beta) \hookrightarrow \Sigma$ is a homotopy inverse. The reader should try to visualise this in the case $\beta = \sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2$.

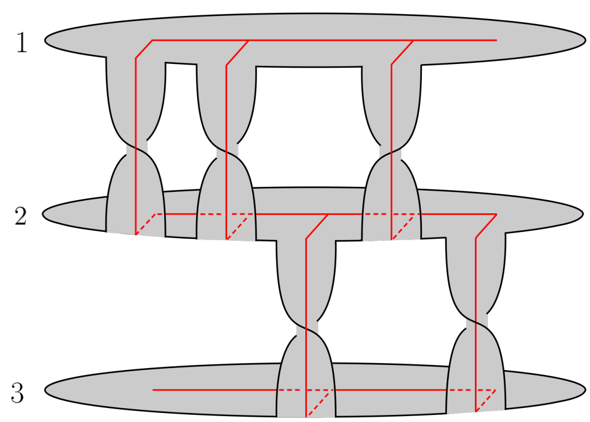


Figure 17: The inclusion of $K(\beta)$ into Σ.

Now suppose that $\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \ldots \sqcup \Sigma_k$ is disconnected with k connected components. It is then possible to write $\beta = \beta'_1 \sqcup \beta'_2 \sqcup \ldots \sqcup \beta'_k$ for sub-braids $\beta'_i \subset \beta$ such that Σ_i is the connected Seifert surface for the closure of the braid β'_i. Similarly we may write $K(\beta) = K(\beta'_1) \sqcup K(\beta'_2) \sqcup \ldots K(\beta'_k)$. It follows from the connected case that the inclusion $K(\beta_i) \hookrightarrow \Sigma_i$ is a homotopy equivalence and the inclusion $K(\beta) \hookrightarrow \Sigma$ is a homotopy equivalence. □

Definition 13. For a regular n-strand braid β with a fence $K(\beta)$ define a bilinear form $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ with values on the basis oriented 1-simplices as follows:

$$\lambda_\beta(x,y) = \begin{cases} -\frac{1}{2} & \text{if if } x = y = \downarrow = \text{a vertical simplex corresponding to a } \sigma_i \\ \frac{1}{2} & \text{if } x = y = \downarrow = \text{a vertical simplex corresponding to a } \sigma_i^{-1} \\ \frac{1}{2} & \text{if } (x,y) = (\rightarrow,\downarrow) \text{ are adjacent simplices meeting like} \\ \frac{1}{2} & \text{if } (x,y) = (\downarrow,\rightarrow) \text{ are adjacent simplices meeting like} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5. If the 1-simplices in the fence $K(\beta)$ from Example 1 are labelled as follows

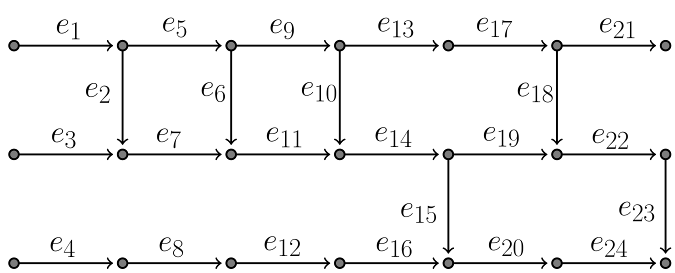


Figure 18: Labelled 1-simplices in the fence $K(\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2)$.

then $C_1(K(\beta);\mathbb{Z}) = \mathbb{Z}\langle e_1, e_2, \dots, e_{24}\rangle$ and for pairs of basis elements $(x,y) \in \{e_1, e_2 \dots, e_{24}\}^2$ we have

$$\lambda_\beta(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x,y) \in \{(e_1,e_2),(e_2,e_3),(e_5,e_6),(e_6,e_7)(e_9,e_{10}),(e_{10},e_{11}),(e_{14},e_{15}), \\ & (e_{15},e_{15}),(e_{15},e_{16}),(e_{17},e_{18}),(e_{18},e_{19}),(e_{22},e_{23}),(e_{23},e_{24})\} \\ -\frac{1}{2} & \text{if } (x,y) \in \{(e_2,e_2),(e_6,e_6),(e_{10},e_{10}),(e_{18},e_{18}),(e_{23},e_{23})\} \\ 0 & \text{otherwise.} \end{cases}$$

The motivation for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is as follows. Let $\beta = \sigma_i$ be the elementary n-strand braid with a single crossing of strand i over strand $i+1$. The Seifert surface for $\hat{\beta}$ consists of a disjoint union of n disks, stacked one above the other with a single twisted band attached from disc i to disc $i+1$. Smooth the corners of Σ and choose the positive normal direction to the smoothed Seifert surface to be in the upwards direction. Let K_i be the embedded part of K between disc i and disc $i+1$. If K_i^+ is obtained by pushing K_i in the direction of the positive normal, then 'reversing' the embeddings produces disjoint simplicial complexes with crossings of the following type

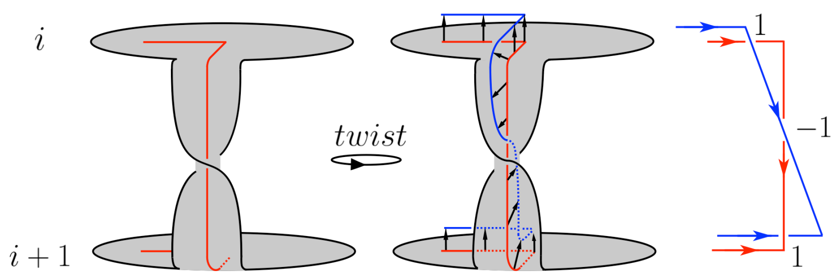


Figure 19: Pushing K_i in the normal direction.

The twist in the diagram refers to the direction of the twist in the attached band and the resulting twist of the positive normal vector to the Seifert surface along the vertical part of the red curve. In the case $\beta = \sigma_i^{-1}$ we obtain crossings of the type

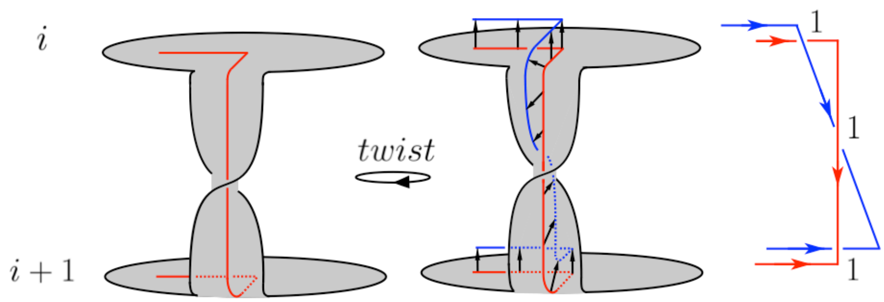


Figure 20: Pushing K_i in the normal direction.

Recall that from Definition 2 that the linking number of the components of a two component oriented link may be computed as one half of the sum of the signed crossings between one component and the other. The crossings above define a pairing $\lambda : C_1(K_i;\mathbb{Z}) \times C_1(K_i^+;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. Since K_i and K_i^+

are simplicially isomorphic we may equivalently think of this as a pairing $\lambda : C_1(K_i;\mathbb{Z}) \times C_1(K_i;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ which is given by

$$\lambda(x,y) = \begin{cases} -\frac{1}{2} & \text{if if } x = y = \downarrow = \text{ a vertical simplex corresponding to a } \sigma_i \\ \frac{1}{2} & \text{if } x = y = \downarrow = \text{ a vertical simplex corresponding to a } \sigma_i^{-1} \\ \frac{1}{2} & \text{if } (x,y) = (\to,\downarrow) \text{ are adjacent simplices meeting like } \begin{smallmatrix} \bullet\!\!\to & \bullet \\ & \downarrow \\ & \bullet \end{smallmatrix} \\ \frac{1}{2} & \text{if } (x,y) = (\downarrow,\to) \text{ are adjacent simplices meeting like } \begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet\!\!\to & \bullet \end{smallmatrix} \\ 0 & \text{otherwise.} \end{cases}$$

6 Descending to homology

We now show that the chain level formula gives the Seifert form on the homology level.

Theorem 6. Let β be a braid with Seifert surface Σ constructed by Seifert's algorithm and Seifert form $V : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$. If K is the fence of β then inclusion $K \hookrightarrow \Sigma$ induces an isomorphism $H_1(K;\mathbb{Z}) \cong H_1(\Sigma;\mathbb{Z})$ with a commutative diagram

$$\begin{array}{ccc} H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) & \xrightarrow{[\lambda]} & \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}] \\ \downarrow \cong & \nearrow V & \\ H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) & & \end{array}$$

Proof. Suppose that $\beta = \beta_1\beta_2 \dots \beta_\ell$ is a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid. Proposition 2 implies that there is an inclusion $K \hookrightarrow \Sigma$ which is a homotopy equivalence and hence $H(K;\mathbb{Z}) \cong H(\Sigma;\mathbb{Z})$. Suppose that Σ has k connected components. For $1 \leqslant i \leqslant n-1$ let l_i denote the number of crossings between strand i and strand $i+1$. By [5, Lemma 3.1] we may write

$$b_1(K;\mathbb{Z}) = b_1(\Sigma;\mathbb{Z}) = \sum_{i=1}^{n-1}(l_i - 1) = l - k + n$$

and Collins shows that there is a basis of $H_1(\Sigma;\mathbb{Z})$ with one basis element for each pair of consecutive crossings between adjacent strands. More explicitly, a pair of consecutive crossings between strand i and strand $i+1$

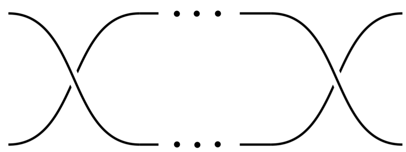


Figure 21: A pair of consecutive crossings between strand i and strand $i+1$.

determines a 1-cycle, shown in red below as an embedded polygonal circle oriented in the clockwise direction, in the part of Σ which is created by attaching to two Seifert disc two twisted bands corresponding to the two crossings between the same strands.

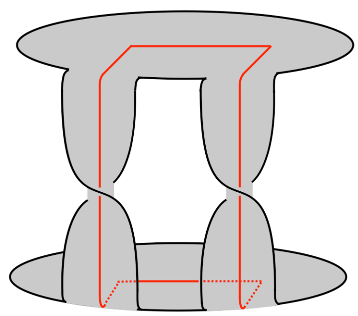


Figure 22: The 1-cycle.

The cycles may be labelled $c_1, c_2, \ldots, c_{\ell-n+k} \in Z_1(\Sigma;\mathbb{Z})$ according to their positions from left to right along the braid diagram. The set of homology classes $[c_1], [c_2], \ldots, [c_{\ell-n+k}]$ is then a basis for $H_1(\Sigma)$. The cycles $c_1, c_2, \ldots, c_{\ell-n+k} \in Z_1(\Sigma;\mathbb{Z})$ induce cycles $c'_1, c'_2, \ldots, c'_{\ell-n+k} \in Z_1(K;\mathbb{Z})$ giving a basis $[c'_1], [c'_2], \ldots, [c'_{\ell-n+k}]$ of $H_1(K;\mathbb{Z})$. The homology class $[c'_i] \in H_1(K;\mathbb{Z})$ maps to the homology class $[c_i] \in H_1(\Sigma;\mathbb{Z})$ under the isomorphism $H_1(K;\mathbb{Z}) \cong H_1(\Sigma;\mathbb{Z})$ induced by the inclusion $K \hookrightarrow \Sigma$. If c_j^+ is the push of the cycle c_j in the positive normal to Σ then it suffices to show that $\lambda(c'_i, c'_j) = Lk(c_i, c_j^+)$ for $1 \leqslant i, j \leqslant \ell - n + k$. Note that since the linking number $Lk(c_i, c_j^+)$ is always an integer and $H_1(K;\mathbb{Z})$ is a free abelian group this implies that $\lambda : H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ factors through a map $H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) \to \mathbb{Z}$. The proof now proceeds by cases.

Diagonal Entries: Suppose that $i = j$. The diagram below shows c_i in red and its push off c_i^+ in blue

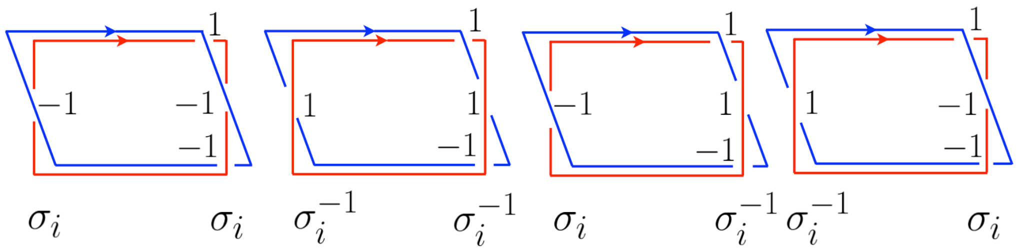


Figure 23: The cycle c_i and its pushoff c_i^+.

so that the self-linking numbers are given by

$$Lk(c_i, c_i^+) = \begin{cases} -1 & \text{if both crossings correspond to a } \sigma_i \\ 1 & \text{if both crossings correspond to a } \sigma_i^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

The cycle $c_i \in Z_1(\Sigma;\mathbb{Z})$ corresponds to a cycle $c_i' \in Z_1(K;\mathbb{Z})$ which may be written as $c_i' = -e_1 + (\sum_{p=2}^{s+2} e_p) - (\sum_{p=s+3}^{2s+2} e_p)$ as in the diagram

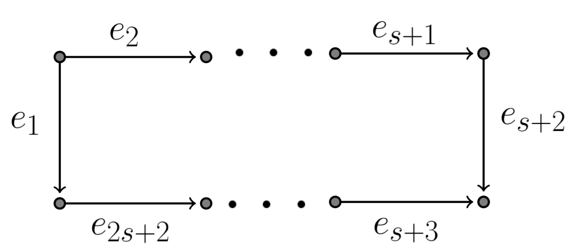


Figure 24: Labelled 1-simplices in the cycle c_i'.

It follows that

$$\begin{aligned} \lambda(c_i', c_i') &= \lambda(-e_1, -e_1) + \lambda(e_{s+1}, e_{s+2}) + \lambda(e_{s+2}, e_{s+2}) + \lambda(e_{s+2}, -e_{s+3}) \\ &= \lambda(e_1, e_1) + \frac{1}{2} + \lambda(e_{s+2}, e_{s+2}) - \frac{1}{2} \\ &= \lambda(e_1, e_1) + \lambda(e_{s+2}, e_{s+2}) \end{aligned}$$

and hence

$$\lambda(c_i', c_i') = \begin{cases} -1 & \text{if both crossings correspond to a } \sigma_i \\ 1 & \text{if both crossings correspond to a } \sigma_i^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Non-Diagonal Entries: Suppose that $1 \leqslant i < j \leqslant \ell - n + k$. . Let the cycle c_i' be written as in the diagonal case and let the cycle c_j' be written as $c_j' = -f_1 + (\sum_{q=2}^{t+2} f_q) - (\sum_{i=t+3}^{2t+2} f_q)$ as in the diagram

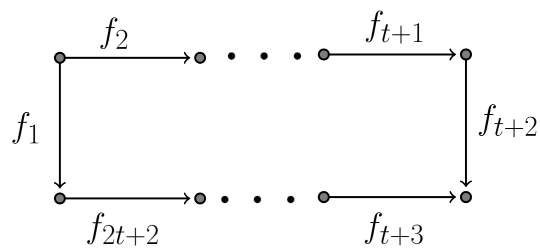


Figure 25: Labelled 1-simplices in the cycle c_j'.

Let $E = \{e_p\}_{p=1}^{2s+2}$ and $F = \{f_q\}_{q=1}^{2t+2}$. It is enough to consider the five cases of the relative positions of the cycles as in [5, Section 3.3]:

1. Either $E \cap F = \{e_{s'}, e_{s'+1}, \ldots, e_{s''}\} = \{f_{t+3}, f_{t+4}, \ldots, f_{2t+2}\}$ for some $2 < s' < s'' < s+1$ with $e_{s'} = f_{2t+2}$ and $e_{s''} = f_{t+3}$ as in or $E \cap F =$

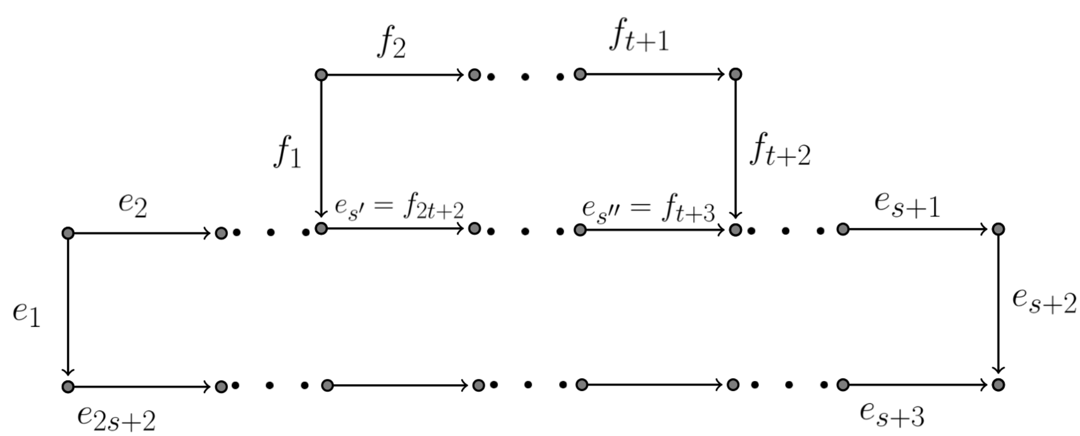


Figure 26: The cycles c_i' and c_j'.

$\{e_2, e_3, \ldots, e_{s+1}\} = \{f_{t'}, f_{t'+1}, \ldots, f_{t''}\}$ for some $t+3 < t' < t'' < 2t+2$ with $e_2 = f_{t''}$ and $e_{s+1} = f_{t'}$ as in

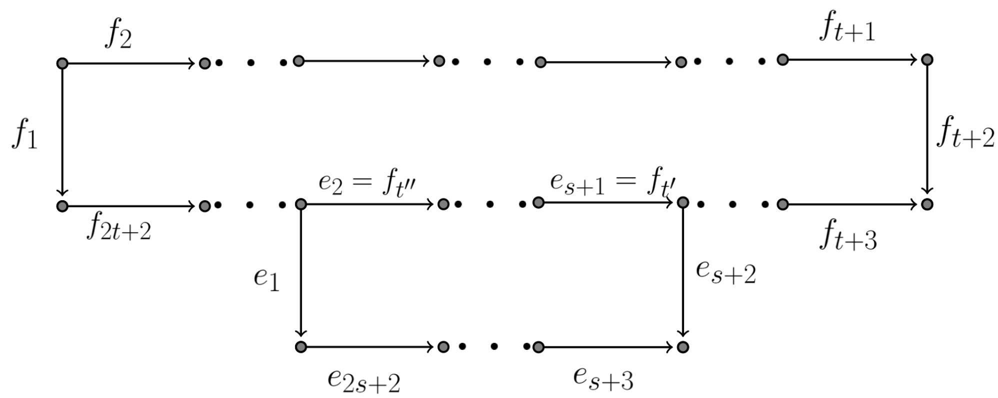


Figure 27: The cycles c'_i and c'_j.

so that in the first case

$$\lambda(c'_i, c'_j) = 0$$
$$\lambda(c'_j, c'_i) = \lambda(-f_1, e_{s'-1}) + \lambda(f_{t+2}, e_{s''}) = -\frac{1}{2} + \frac{1}{2} = 0$$

and in the second case

$$\lambda(c'_i, c'_j) = 0$$
$$\lambda(c'_j, c'_i) = \lambda(-f_{t''+1}, -e_1) + \lambda(-f_{t'}, e_{s+2}) = \frac{1}{2} - \frac{1}{2} = 0.$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second) case by

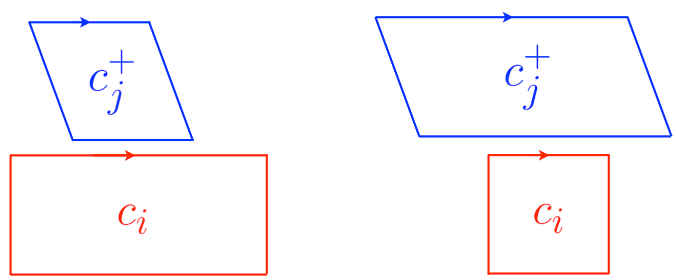


Figure 28: The cycles c_j^+ and c_i.

and in either case $\mathrm{Lk}(c_i, c_j^+) = 0$. The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second) case by and in either case

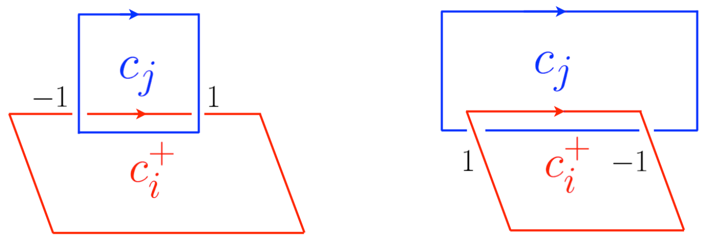


Figure 29: The cycles c_i^+ and c_j.

$\mathrm{Lk}(c_j, c_i^+) = 0$.

2. In this case E and F are disjoint as in

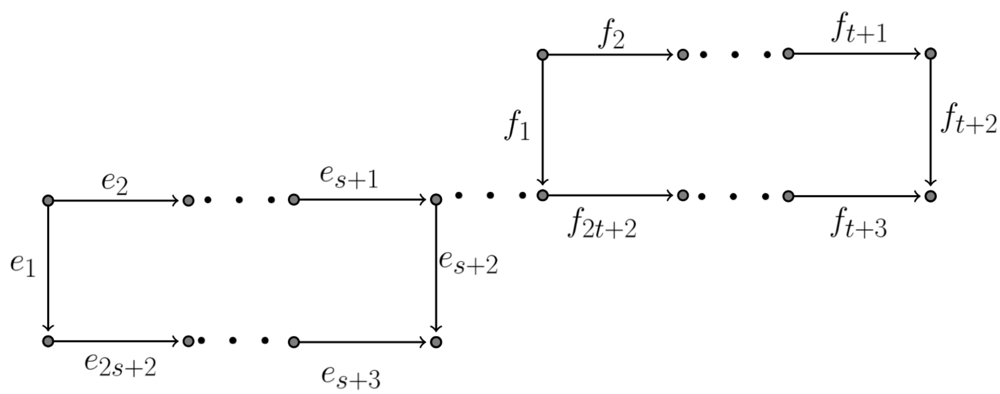
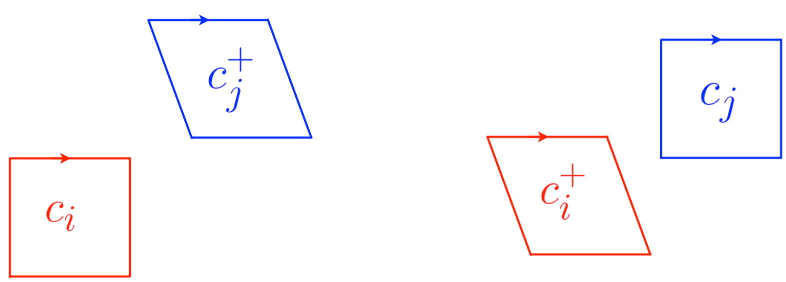


Figure 30: The cycles c_i' and c_j'.

so it is immediate that $\lambda(c_i', c_j') = \lambda(c_j', c_i') = 0$. The push-off c_j^+ of c_j in relation to c_i (respectively the push-off c_i^+ of c_i in relation to c_j) is given by



so that $\mathrm{Lk}(c_j, c_i^+) = \mathrm{Lk}(c_i^+, c_j) = 0$.

3. In this case $E \cap F = \{e_{s+2}\} = \{f_1\}$ as in

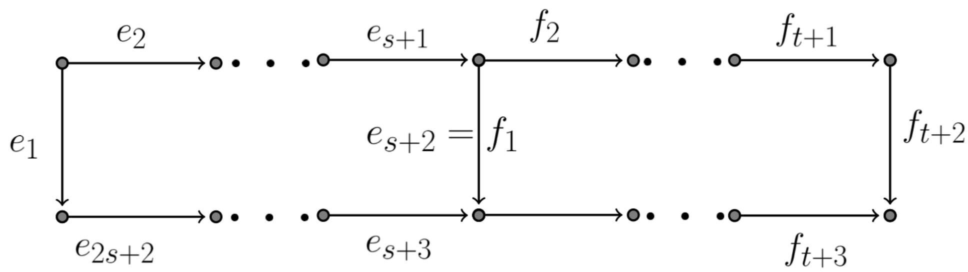


Figure 31: The cycles c_i' and c_j'.

and it follows that

$$\lambda(c_i', c_j') = \lambda(e_{s+1}, -f_1) + \lambda(e_{s+2}, -f_1) = -\frac{1}{2} - \lambda(f_1, f_1)$$

and hence

$$\lambda(c_i', c_j') = \begin{cases} 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ -1 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second) case by

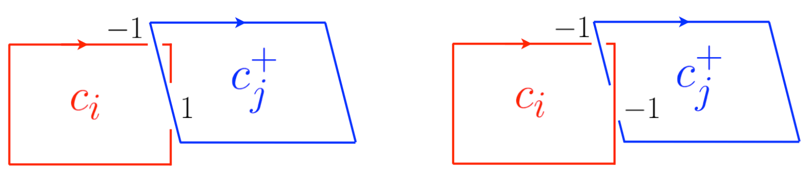


Figure 32: The cycles c_j^+ and c_i.

so that

$$\mathrm{Lk}(c_i, c_j^+) = \begin{cases} 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ -1 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

Similarly

$$\lambda(c_j', c_i') = \lambda(-f_1, e_{s+2}) + \lambda(-f_1, -e_{s+3}) = -\lambda(f_1, f_1) + \frac{1}{2}$$

and hence

$$\lambda(c'_j, c'_i) = \begin{cases} 1 & \text{if } f_1 \text{ corresponds to a } \sigma_i \\ 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second case) by

Figure 33: The cycles c_i^+ and c_j.

so that

$$\text{Lk}(c_j, c_i^+) = \begin{cases} 1 & \text{if } f_1 \text{corresponds to a } \sigma_i \\ 0 & \text{if } f_1 \text{ corresponds to a } \sigma_i^{-1}. \end{cases}$$

4. In this case E and F are disjoint as in

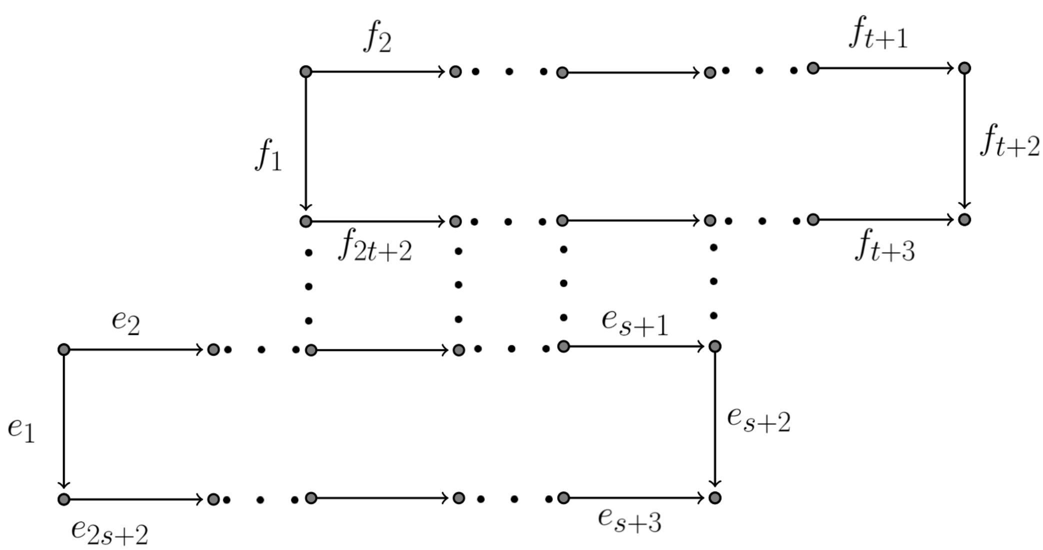


Figure 34: The cycles c'_i and c'_j.

so it is immediate that $\lambda(c_i', c_j') = \lambda(c_j', c_i') = 0$. The push-off c_j^+ in relation to c_i and c_i^+ in relation to c_j are given by the similar figures as in case 2 and it follows that $\text{Lk}(c_j, c_i^+) = \text{Lk}(c_i^+, c_j) = 0$.

5. Either $E \cap F = \{e_{s+3}, e_{s+4}, \ldots, e_{s'}\} = \{f_2, f_3, \ldots, f_{t'}\}$ for some $s+3 \leqslant s' < 2s+2$ and $2 \leqslant t' < t+1$ with $e_{s'} = f_2$ and $e_{s+3} = f_{t'}$ as in

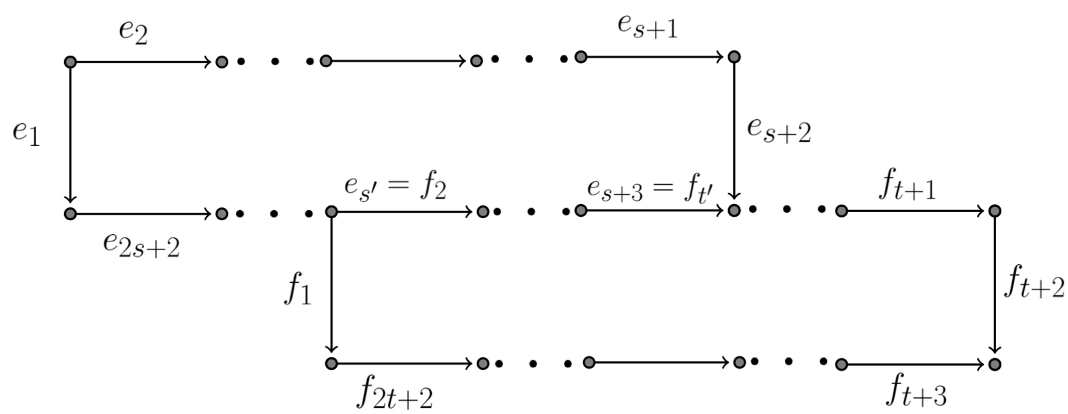


Figure 35: The cycles c_i' and c_j'.

or $E \cap F = \{e_{s'}, e_{s'+1}, \ldots, e_{s+1}\} = \{f_{t'}, f_{t'+1}, \ldots, f_{2t+2}\}$ for some $2 < s' \leqslant s+1$ and some $t+3 < t' \leqslant 2t+2$ with $e_{s'} = f_{2t+2}$ and $e_{s+1} = f_{t'}$ as in

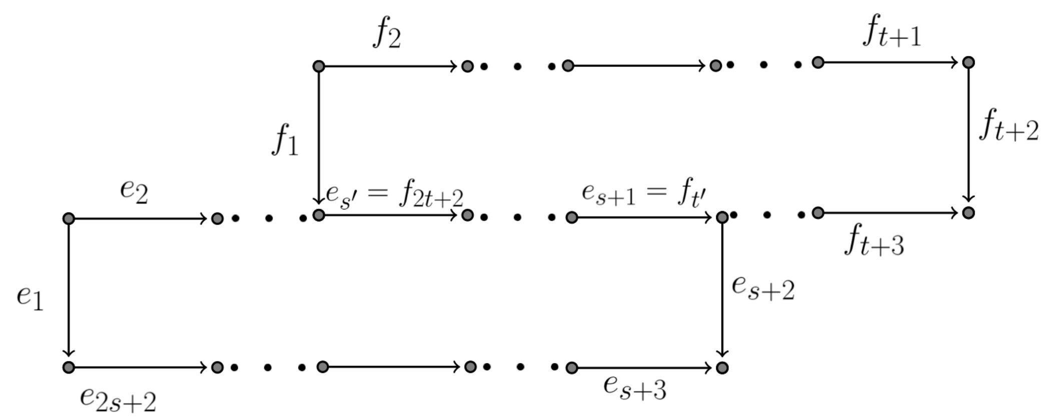


Figure 36: The cycles c_i' and c_j'.

In the first case

$$\lambda(c_i', c_j') = \lambda(e_{s+2}, f_{t'-1}) + \lambda(-e_{s'+1}, -f_1) = \frac{1}{2} + \frac{1}{2} = 1$$
$$\lambda(c_j', c_i') = -\lambda(f_{t'}, e_{s+2}) = 0.$$

and in the second case

$$\lambda(c_i', c_j') = 0$$
$$\lambda(c_j', c_i') = \lambda(-f_1, e_{s'-1}) + \lambda(-f_{t'}, e_{s+2}) = -\frac{1}{2} - \frac{1}{2} = -1$$

The push-off c_j^+ of c_j in relation to c_i is given in the first (respectively second case) by so that

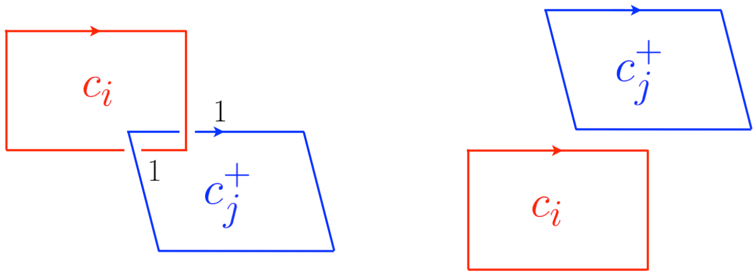


Figure 37: The cycles c_j^+ and c_i.

$$\mathrm{Lk}(c_i, c_j^+) = \begin{cases} 1 & \text{in the first case} \\ 0 & \text{in the second case.} \end{cases}$$

The push-off c_i^+ of c_i in relation to c_j is given in the first (respectively second) case by so that

$$\mathrm{Lk}(c_j, c_i^+) = \begin{cases} 0 & \text{in the first case} \\ -1 & \text{in the second case.} \end{cases}$$

□

This motivates the following definition.

Definition 14. The *chain level Seifert pair* of a regular n-strand braid β is the pair

$$(\lambda_\beta, d_\beta) = (\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}], d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z}))$$

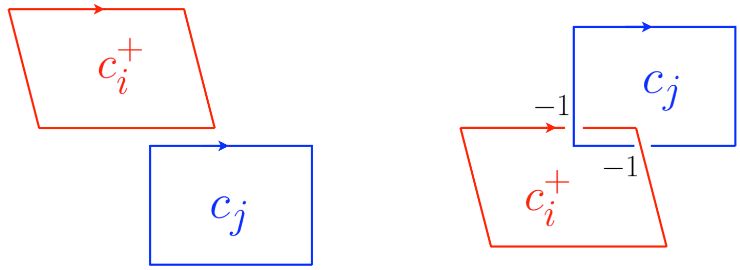


Figure 38: The cycles c_i^+ and c_j.

Corollary 1. A regular n-strand braid β with chain level Seifert pair (λ_β, d_β) has Seifert form

$$\lambda_\beta : \ker(d_\beta) \times \ker(d_\beta) \to \mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}].$$

Proof. The fence $K(\beta)$ is a 1-dimensional simplicial complex and hence

$$H_1(\Sigma) \cong H_1(K(\beta)) = \ker(d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z}))$$

□

Example 6. The 2-strand braid $\beta = \sigma_1\sigma_1\sigma_1$ with closure $\hat{\beta}$ the trefoil knot has the fence

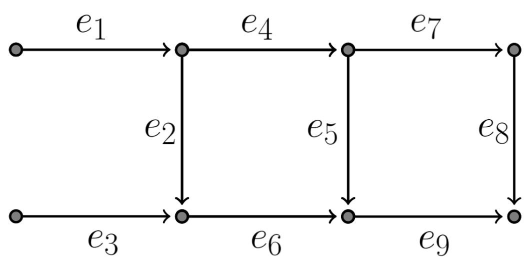


Figure 39: The fence $K(\sigma_1\sigma_1\sigma_1)$.

so that $C_1(K(\beta);\mathbb{Z})$ is a free abelian group of rank 9 with a basis $\{e_1, e_2, \dots, e_9\}$. The bilinear pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is represented with respect to the ordered basis $(e_1, e_2, \dots, e_9)$ by the upper triangular matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\gamma = e_4 + e_5 - e_6 - e_2$ and $\delta = e_7 + e_8 - e_9 - e_5$ then

$$H_1(K(\beta);\mathbb{Z}) = \ker(d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

is a free abelian group of rank 2 with a basis $\{\gamma, \delta\}$. One then checks that the Seifert matrix with respect to the ordered basis (γ, δ) of $H_1(K(\beta);\mathbb{Z})$ is given by

$$\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

as usual.

7 The effect of concatenation

We now examine the effect of the concatenation of braids on Seifert surfaces and fences to obtain an inductive formula for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$. We first construct the Seifert surface of a closure of a braid in a way which mirrors more closely the decomposition of a braid into a concatenation of elementary braids.

Definition 15. The *open Seifert surface* $\Sigma_{\sigma_i^{\pm 1}}$ of the elementary n-strand braid $\sigma_i^{\pm 1}$ with a single crossing between strand i and strand $i+1$ is the disjoint union of a single twisted band and $n-1$ line segments, stacked vertically one above the other, as shown below

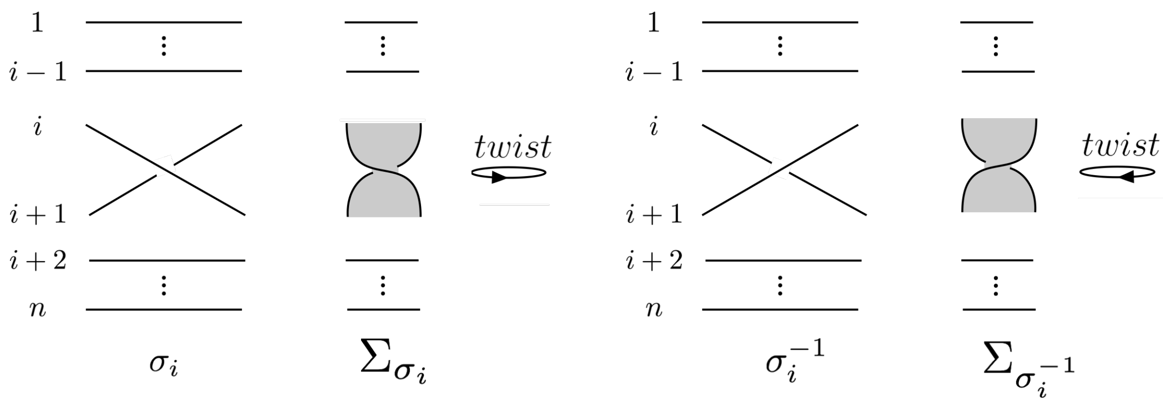


Figure 40: The open Seifert surfaces associated to the elementary braids $\sigma_i^{\pm 1}$.

The *open Seifert surface* Σ_β of a regular n-strand braid $\beta = \beta_1\beta_2\dots\beta_\ell$ is the concatenation of the open Seifert surfaces $\Sigma_{\beta_1}, \Sigma_{\beta_2}, \dots, \Sigma_{\beta_\ell}$ from left to right so that $\Sigma_\beta = \cup_{i=1}^{\ell}\Sigma_{\beta_i}$ where Σ_{β_i} intersects $\Sigma_{\beta_{i+1}}$ in the right hand part of Σ_{β_i} and the left hand part of $\Sigma_{\beta_{i+1}}$ as shown below

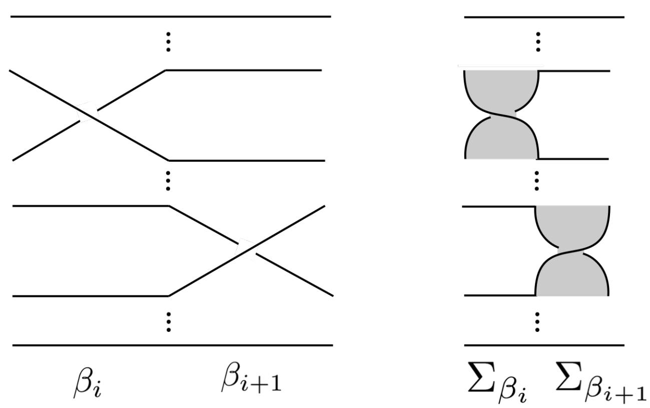


Figure 41: The concatenation of open Seifert surfaces associated to two adjacent elementary braids.

The *closure* $\widehat{\Sigma}_\beta$ of the open Seifert surface of a regular n-strand braid $\beta = \beta_1\beta_2\dots\beta_\ell$ is the union of the open Seifert surface Σ_β with n horizontal discs as shown in the diagram below

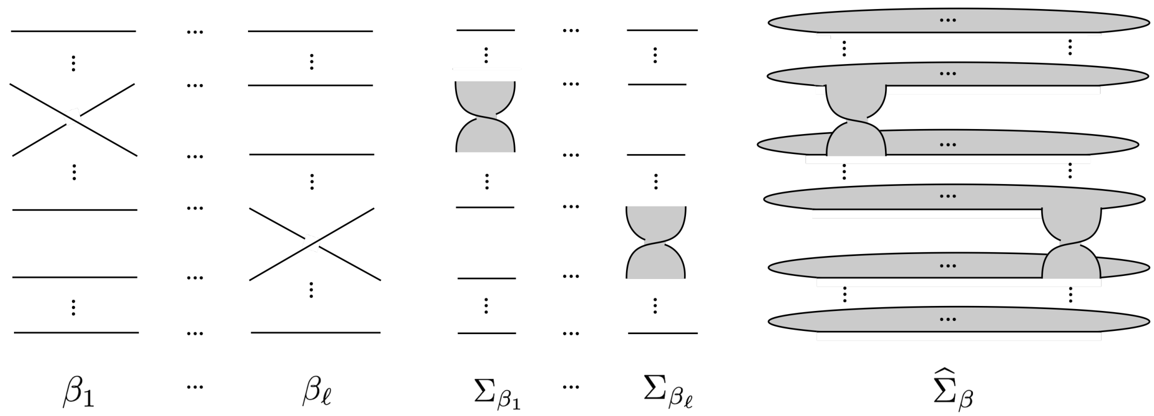


Figure 42: The closure of an open Seifert surface.

Proposition 3. Let β be a regular n-strand braid. The closure of the open Seifert surface for β is the Seifert surface for the closure of β constructed by Seifert's algorithm, that is $\widehat{\Sigma}_\beta = \Sigma_{\widehat{\beta}}$.

Proof. By induction on the length of the braid. □

In order to obtain an inductive formula for the chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ we first consider the effect of concatenating a braid β with an elementary braid β_i.

Proposition 4. Let β be a regular n-strand braid with ℓ crossings and fence K. Let β_i be an elementary n-strand braid with a single crossing between strand i and strand $i+1$. Define an $(n+1)\times(n+1)$-matrix λ_{β_i} by

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i \\ \frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i^{-1} \\ \frac{1}{2} & \text{if } j = i \text{ and } k = i+1 \\ \frac{1}{2} & \text{if } j = i+1 \text{ and } k = i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Then the chain level Seifert pairing for $\beta\beta_i$ is represented by the matrix

$$\lambda_{\beta\beta_i} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta_i} \end{pmatrix}.$$

Proof. The fence $K(\beta_i)$ is a simplicial complex with n 0-simplices, n-horizontal simplices and a single vertical 1-simplex as shown below. With respect to the ordered basis $(f_1, f_2 \ldots, f_{n+1})$, the pairing $\lambda_{\beta_i} : C_1(K(\beta_i);\mathbb{Z})\times C_1(K(\beta_i);\mathbb{Z}) \to \mathbb{R}$ is represented by the $(n+1)\times(n+1)$-matrix λ_{β_i} with

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i \\ \frac{1}{2} & \text{if } j = k = i \text{ and } \beta_i = \sigma_i^{-1} \\ \frac{1}{2} & \text{if } j = i \text{ and } k = i+1 \\ \frac{1}{2} & \text{if } j = i+1 \text{ and } k = i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that we have a matrix representation $\lambda_\beta : C_1(K(\beta);\mathbb{Z})\times C_1(K(\beta);\mathbb{Z}) \to \mathbb{R}$ with respected to an ordered basis $\mathbf{e}_{K(\beta)}$ of $C_1(K(\beta);\mathbb{Z})$. The fence $K(\beta\beta_i)$ of $\beta\beta_i$ is obtained from $K(\beta)$ by the fence $K(\beta_i)$ as follows

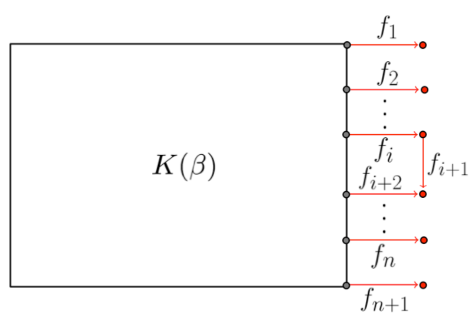


Figure 43: The fences of β, β_i and $\beta\beta_i$.

Here the red simplices are simplices added to $K(\beta)$ and the 1-simplices of $K(\beta)$ and $K(\beta_i)$ are disjoint. This gives an ordered basis $\mathbf{e}_{K(\beta\beta_i)} = (\mathbf{e}_k, f_1, f_2, \ldots, f_{n+1})$ of $C_1(K(\beta\beta_i);\mathbb{Z})$ and it follows that with respect to $\mathbf{e}_{K(\beta\beta_i)}$ that the pairing $\lambda_{\beta\beta_i} : C_1(K(\beta\beta_i);\mathbb{Z}) \times C_1(K(\beta\beta_i);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ is represented by the block diagonal matrix

$$\lambda_{\beta\beta_i} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta_i} \end{pmatrix}$$

as required.

□

Theorem 7. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a regular n-strand braid with ℓ crossings, where each β_i is an elementary n-strand braid with a single crossing between strand j_i and j_{i+1}. The chain level pairing $\lambda_\beta : C_1(K(\beta);\mathbb{Z})\times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ can be represented by a block diagonal matrix

$$\begin{pmatrix} \lambda_{\beta_1} & 0 & \dots & 0 \\ 0 & \lambda_{\beta_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_{\beta_\ell} \end{pmatrix}$$

where

$$(\lambda_{\beta_i})_{j,k} = \begin{cases} -\frac{1}{2} & \text{if } j = k = j_i \text{ and } \beta_i = \sigma_{j_i} \\ \frac{1}{2} & \text{if } j = k = j_i \text{ and } \beta_i = \sigma_{j_i}^{-1} \\ \frac{1}{2} & \text{if } j = j_i \text{ and } k = j_i + 1 \\ \frac{1}{2} & \text{if } j = j_i + 1 \text{ and } k = j_i + 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the definition of the concatenation of fences we may write $K(\beta) = \cup_{i=1}^{\ell} K(\beta_i)$. Since $K(\beta_i)$ intersects $K(\beta_{i+1})$ in a set of 0-simplices then $C_1(K(\beta);\mathbb{Z}) = \oplus_{i=1}^{\ell} C_1(K(\beta_i);\mathbb{Z})$. The proof follows induction on the number ℓ of crossings in the braid with the concatenation formula from Proposition 4. □

8 Comparison with other models

We now show that this model of a chain level Seifert pairing is chain equivalent to Banchoff's formula for the linking number of two space polygons and Ranicki's surgery-theoretic chain level linking formula.

Motivated by the Gauss map in Definition 2, Banchoff [3] gave a combinatorial linking formula for two disjoint space polygons expressed in terms of the partial linking numbers of pairs of edges as follows.

Definition 16. Let $X = \{X_0, X_1, \dots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \dots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$.

(i) For a unit vector $\xi \in S^2$ let $p_\xi : \mathbb{R}^3 \to P$ denote the projection map from $\mathbb{R}^3$ onto the plane P orthogonal to ξ. A vector $\xi \in S^2$ in called *general* for X and Y if the projections $p_\xi(X), p_\xi(Y) \subset \mathbb{R}^2$ are in general position.

(ii) For a vector $\xi \in S^2$ which is general for X and Y, define $C_{i,j}(X,Y,\xi)$ to be the sign of $P_\xi(Y_{j+1} - Y_j) \times P_\xi(X_{i+1} - X_i).(\overline{X_i} - \overline{Y_j})$ if there are interior points $\overline{X_i}$ of the edge X_iX_{i+1} and $\overline{Y_j}$ of the edge Y_jY_{j+1} such that $p_\xi(\overline{X_i}) = p_\xi(\overline{Y_j})$ and define $C_{i,j}(X,Y,\xi)$ to be zero otherwise

The linking number of two space polygons is then expressible as the sum of partial linking numbers of all edge pairs.

Theorem 8. [3, p.1176-1177] For disjoint polygonal knots $X, Y \subset \mathbb{R}^3$ the value

$$C(X,Y,\xi) = \sum_{\substack{0\leqslant i\leqslant m-1\\0\leqslant j\leqslant n-1}} C_{i,j}(X,Y,\xi) \in \mathbb{Z}$$

is independent of the choice of general vector $\xi \in S^2$. The linking number of the polygonal knots determined by X and Y is the average value of $C(X,Y,\xi)$, that is

$$\mathrm{Lk}(X,Y) = \frac{1}{4\pi}\int_{\xi\in S^2} C(X,Y,\xi)d\omega = \frac{1}{4\pi}\sum_{\substack{0\leqslant i\leqslant m-1\\0\leqslant j\leqslant n-1}}\int_{\xi\in S^2} C_{i,j}(X,Y,\xi)d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. Moreover this integral may be expressed in terms of dihedral angles of tetrahedra.

Ranicki gave an alternative chain level formula in terms of the Seifert graph. The Seifert graph of a braid records which strands of the braid cross but not whether the crossings and over-crossings or under-crossings.

Definition 17. The *Seifert graph* of a braid β is the 1-dimensional CW-complex $X(\beta)$ constructed from the canonical Seifert surface of β by collapsing each Seifert disc to a point and collapsing each twisted band to its core.

If β is an n-strand braid with ℓ-crossings then the Seifert graph $X(\beta')$ has ℓ 1-cells and n 0-cells and has a cellular chain complex of the form

$$d : C_1(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^\ell \to C_0(X(\beta);\mathbb{Z}) \cong \mathbb{Z}^n.$$

If β' is another n-strand braid with ℓ' crossings then the Seifert graph $X(\beta')$ has a cellular chain complex of the form

$$d' : C_1(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^{\ell'} \to C_0(X(\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The Seifert graph of the concatenated braid $\beta\beta'$ is a CW-complex which can be formed from the Seifert graphs of β and β' by identifying the 0-cells in pairs so that $X(\beta\beta')$ has $(\ell+\ell')$ 1-cells, n 0-cells and a cellular chain complex of the form

$$d'' = \begin{pmatrix} d & d' \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^\ell \oplus \mathbb{Z}^{\ell'} \to C_0(X(\beta\beta');\mathbb{Z}) \cong \mathbb{Z}^n.$$

The closure of an n-strand geometric braid with ℓ-crossings arises as the trace of ℓ 0-surgeries on a disjoint union of n circles. Ranicki applied the algebraic theory of surgery to the geometric surgeries to obtain a formula which is defined inductively.

Definition 18.

(i) The *canonical generalised Seifert matrices* of the elementary regular n-strand braids σ_i, σ_i^{-1} are the 1×1 matrices

$$\psi_{\sigma_i} = \begin{pmatrix} 1 \end{pmatrix}, \quad \psi_{\sigma_i^{-1}} = \begin{pmatrix} -1 \end{pmatrix}.$$

(ii) Let β, β' be regular n-strand braids and let χ be the lower triangular $n \times n$ matrix with ones below the diagonal. The *generalised Seifert matrix* for the concatenated braid $\beta\beta'$ is the inductively defined matrix

$$\psi_{\beta\beta'} = \begin{pmatrix} \psi_\beta & -d^*\chi d' \\ 0 & \psi_{\beta'} \end{pmatrix} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

Theorem 9. [12, p.37-38] Let β, β' be regular n-strand braids. The generalised Seifert matrix

$$\psi_{\beta\beta'} : C_1(X(\beta\beta');\mathbb{Z}) \times C_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

induces the Seifert form of $\beta\beta'$

$$\psi_{\beta\beta'} : H_1(X(\beta\beta');\mathbb{Z}) \times H_1(X(\beta\beta');\mathbb{Z}) \to \mathbb{Z}$$

on the homology level.

The equivalences of Banchoff's and Ranicki's models to the model we developed are both established via the following lemma.

Lemma 2. Let C and D be $\mathbb{Z}$-module chain complexes with C finitely generated free and concentrated in dimensions 0 and 1 and D concentrated in dimensions 1 and 2. If $H_0(C)$ is torsion free then the morphism

$$H_0(\mathrm{Hom}_{\mathbb{Z}}(C,D)) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(C), H_1(D)); \quad f \mapsto f_*$$

is an isomorphism, that is any two chain maps $f, g : C \to D$ are chain homotopic if and only if $f_* = g_* : H_1(C) \to H_1(D)$.

Proof. Any $\mathbb{Z}$-module homomorphism $f : C_1 \to D_1$ fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & C_1 & \xrightarrow{d_C} & C_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & D_2 & \xrightarrow[d_D]{} & D_1 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

so that there is an one-to-one correspondence between chain maps $f : C \to D$ and $\mathbb{Z}$-module homomorphisms $f : C_1 \to D_1$. It is then enough to show that if $f, f' : C \to D$ are chain maps satisfying $f_* = f'_* : H_*(C) \to H_*(D)$ then there is chain homotopy $\Delta : f \simeq f' : C \to D$. We can clearly choose $\Delta_r = 0 : C_r \to D_{r+1}$ if $r \neq 0, 1$ and it then suffices to construct $\mathbb{Z}$-module homomorphisms $\Delta_0 : C_0 \to D_1$ and $\Delta_1 : C_1 \to D_2$ such that $f - f' = \Delta_0 d_C + d_D \Delta_1 : C_1 \to D_1$.

The $\mathbb{Z}$-module $\mathrm{im}(d_C) \subset C_0$ is a submodule of a finitely generated free $\mathbb{Z}$-module and hence is also finitely generated free. Choose a basis $\{x_i\}_{i=1}^m$ of $\mathrm{im}(d_C)$ and for each x_i choose a point $z_i \in C_1$ such that $d_C(z_i) = x_i$. The short exact sequence

$$0 \to \mathrm{im}(d_C) \to C_0 \to H_0(C) \to 0$$

splits since $H_0(C)$ is f.g. free and hence there is an isomorphism $C_0 \cong \mathrm{im}(d_C) \oplus H_0(C)$. The $\mathbb{Z}$-module homomorphism $g : \mathrm{im}(d_C) \to D_1$ defined by $g(x_i) = (f - f')(z_i)$ induces a $\mathbb{Z}$-module homomorphism

$$\Delta_0 = (g, 0) : \mathrm{im}(d_C) \oplus H_0(C) \to D_1.$$

The $\mathbb{Z}$-module homomorphism $s : \mathrm{im}(d_C) \to C_1$ defined by $s(x_i) = z_i$ satisfies $d_C s = \mathrm{id}_{\mathrm{im}(d_C)}$ and hence provides a splitting of the short exact sequence

$$0 \to \ker(d_C) \to C_1 \to \mathrm{im}(d_C) \to 0$$

and induces an isomorphism $\mathrm{im}(d_C) \oplus \ker(d_C) \to C_1$. The $\mathbb{Z}$-module $\ker(d_C) \subset C_1$ is also finitely generated free and so choose a basis $\{y_j\}_{j=1}^n$ of $\ker(d_C)$. By assumption

$$(f - f')_* = 0 : H_1(C) = \ker(d_C) \to H_1(D) = \frac{D_1}{\mathrm{im}(d_D)}$$

and hence for each basis element y_j we may choose an element $w_j \in D_2$ such that $(f - f')(y_j) = d_D(w_j)$. The $\mathbb{Z}$-module homomorphism $f : \ker(d_C) \to D_2$

defined by $f(y_j) = w_j$ induces a $\mathbb{Z}$-module homomorphism $\Delta_1 = (0, f) : \mathrm{im}(d_C) \oplus \ker(d_C) \to D_2$. For element $c = (\sum_{i=1}^m \lambda_i z_i, \sum_{j=1}^n \mu_j y_j) \in C_1$ it follows that

$$\Delta_0 d_C(c) = \Delta_0(\sum_{i=1}^m \lambda_i x_i, 0) = \sum_{i=1}^m \lambda_i g(x_i) = \sum_{i=1}^m (f - f')(z_i) = (f - f')(\sum_{i=1}^m \lambda_i z_i)$$

and

$$\begin{aligned} d_D \Delta_1(c) = d_D f(\sum_{j=1}^n \mu_j y_j) = \sum_{j=1}^n \mu_j d_D f(y_j) = \sum_{j=1}^n \mu_j d_D(w_j) = \sum_{j=1}^n \mu_j (f - f')(y_j) \\ = (f - f')(\sum_{j=1}^n \mu_j y_j) \end{aligned}$$

and hence $(\Delta_0 d_C + d_D \Delta_1)(c) = (f - f')(c)$ as required. □

Proposition 5. Our model is chain homotopy equivalent to Banchoff's model.

Proof. Let $X = \{X_0, X_1, \ldots, X_{m-1}\}$ respectively $Y = \{Y_0, Y_1, \ldots, Y_{n-1}\}$ be a set of points in general position in $\mathbb{R}^3$. The set of vertices X respectively Y determines an oriented one-dimensional simplicial complex X respectively Y in $\mathbb{R}^3$ with positively oriented edges $\{e_i = X_i X_{i+1} | 0 \leqslant i \leqslant m-1\}$ respectively $\{f_j = Y_j Y_{j+1} | 0 \leqslant j \leqslant n-1\}$ where $X_m = X_0$ respectively $Y_n = Y_0$. By [3, p.1176-1177] the linking number of the space polygons X and Y is given by

$$\mathrm{Lk}(X, Y) = \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega \in \mathbb{Z}$$

where ω is the volume form on S^2. For basis elements e_i, f_j the associated integral $\frac{1}{4\pi} \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega$ is in general a real number and not an integer. Banchoff's formula induces a bilinear pairing

$$\begin{aligned} \mu : C_1(X; \mathbb{Z}) \times C_1(Y; \mathbb{Z}) \to \mathbb{R} \\ \left(\sum_{i=0}^{m-1} a_i e_i, \sum_{j=0}^{n-1} b_j f_j \right) \mapsto \frac{1}{4\pi} \sum_{\substack{0 \leqslant i \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}} a_i b_j \int_{\xi \in S^2} C_{i,j}(X, Y, \xi) d\omega. \end{aligned}$$

which has adjoint a $\mathbb{Z}$-module homomorphism

$$\mu : C_1(X; \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_1(Y; \mathbb{Z}), \mathbb{R}) = C^1(Y; \mathbb{R}).$$

Since X and Y are 1-dimensional simplicial complexes this is the same as a chain map $\mu : C_*(X;\mathbb{Z}) \to C^{2-*}(Y;\mathbb{R})$ by Lemma 2.

Now consider the special case where $X = K$ and $Y = K^+$ where $K = K(\beta)$ is the fence for a braid β and K^+ is its push off in the positive normal direction. This yields a chain map $\mu : C_*(K;\mathbb{Z}) \to C^{2-*}(K^+;\mathbb{R})$. Recall that the simplicial complexes K^+ and K are simplicially isomorphic and the bilinear form $\lambda : C_1(K;\mathbb{Z}) \times C_1(K;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$ may be considered as a bilinear form $\lambda : C_1(K;\mathbb{Z}) \times C_1(K^+;\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}] \subset \mathbb{R}$. As above, this yields a chain map $\lambda : C_*(K;\mathbb{Z}) \to C^{2-*}(K^+;\mathbb{R})$. Both λ and μ compute linking numbers when we pass to homology, that is

$$[\lambda] = [\mu] : H_*(K;\mathbb{Z}) \to H^{2-*}(K^+;\mathbb{Z}) \to H^{2-*}(K^+;\mathbb{R}).$$

By the universal coefficients theorem there is an isomorphism

$$H^{2-*}(K^+;\mathbb{R}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R})$$

and the inclusion $\mathbb{Z} \subset \mathbb{R}$ induces a monomorphism

$$\operatorname{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R}).$$

It follows that there is a factorisation

$$[\lambda] = [\mu] : H_*(K;\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{Z}) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_{2-*}(K^+;\mathbb{Z}),\mathbb{R})$$

with both maps admitting the same factorisation through $\operatorname{Hom}_{\mathbb{Z}}(H^{2-*}(K^+;\mathbb{Z}),\mathbb{Z})$. By Lemma 2 there is a chain homotopy $\lambda \simeq \mu : C_*(K;\mathbb{Z}) \to C^{2-*}(K^+;\mathbb{R})$ so that the models are the same up to chain homotopy. □

Our model has the advantage over Banchoff's in that the averaged partial linking numbers are $\mathbb{Z}[\frac{1}{2}]$-valued and not $\mathbb{R}$-valued.

Proposition 6. Our model is chain homotopy equivalent to Ranicki's model.

Proof. Let β be a braid with Seifert graph X and fence K. We work with the opposite orientations to Ranicki, so the differential $d : C_1(X;\mathbb{Z}) \to C_0(X;\mathbb{Z})$ is the negative of the differential Ranicki uses. This does not effect the definition of a generalised Seifert matrix [12, p.37-38]. Ranicki also chooses the opposite positive normal direction when defining linking numbers. This implies that the canonical generalised Seifert 1×1 for the elementary n-strands braids σ_i and σ_i^{-1} are defined in our situation by $\psi_{\sigma_i} = \begin{pmatrix} -1 \end{pmatrix}$ and

$\psi_{\sigma_i^{-1}} = \begin{pmatrix} 1 \end{pmatrix}$.

The Seifert graph $X = X(\beta)$ can be produced from the fence $K = K(\beta)$ by individually collapsing each horizontal row of simplices to a point so that the quotient map $q : K \to X$ is a homotopy equivalence. The chain map $q : C(K;\mathbb{Z}) \to C(X;\mathbb{Z})$ of cellular chain complexes is then a chain homotopy equivalence. The diagram

$$\begin{array}{ccc} H_1(K;\mathbb{Z}) \times H_1(K;\mathbb{Z}) & \xrightarrow{[\lambda]} & \mathbb{Z} \subset \mathbb{R} \\ {\scriptstyle q_* \times q_*}\Big\downarrow \cong & \nearrow {\scriptstyle [\psi]} & \\ H_1(X;\mathbb{Z}) \times H_1(X;\mathbb{Z}) & & \end{array}$$

is commutative since both $[\lambda]$ and $[\mu]$ compute the Seifert matrix of the Seifert surface of the link $\hat{\beta}$. This is implies that

$$[\lambda] = [q^{-1}\psi q] : H_*(K;\mathbb{Z}) \to H^{2-*}(K;\mathbb{Z}) \hookrightarrow H^{2-*}(K;\mathbb{R})$$

where as before the injection $H^{2-*}(K;\mathbb{Z}) \hookrightarrow H^{2-*}(K;\mathbb{R})$ is induced by the inclusion $\mathbb{Z} \subset \mathbb{R}$ and the universal coefficients theorem. By Lemma 2 there is a chain homotopy

$$\lambda \simeq q^{-1}\psi q : C_*(K;\mathbb{Z}) \to C^{2-*}(K;\mathbb{R})$$

giving a chain homotopy equivalence to Ranicki's model. □

Our model has the advantage over Ranicki's model in that the concatenation behaviour is additive and gives an instant chain level Seifert form whereas Ranicki's model is inductively defined.

9 Isotopy of braids and their closures

We first examine the effect of isotopy on the chain level Seifert pair (λ_β, d_β), firstly by an isotopy of β and secondly by an isotopy of its closure $\widehat{\beta}$ in the solid torus $D^2 \times S^1$.

Definition 19. Two square matrices with entries in $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$ are *A-equivalent* if one can be transformed into the other by a finite sequence of *A-operations* defined as follows:

(i) $$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| \geqslant 2$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| \geqslant 2$$

(ii) $$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_j} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| = 1$$

$$\begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_j^{-1}} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\sigma_j^{-1}} & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix} \text{ with } |i-j| = 1$$

(iii) $$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i} & 0 \\ 0 & 0 & 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i^{-1}} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda_{\sigma_i} & 0 & 0 \\ 0 & 0 & \lambda_{\sigma_i^{-1}} & 0 \\ 0 & 0 & 0 & B \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We now examine the effect of an A-operation on the chain level Seifert pair (λ_β, d_β) of a braid β. Once we write β as a concatenation of elementary braids then the effect of an A-operation on λ_β is clear from Theorem 7. It then remains examine the effect of an A-operation on the differential d_β. We first give a matrix representation for the differential d_{β_i} of an elementary n-strand braid and then examine the effect of concatenation on the differential.

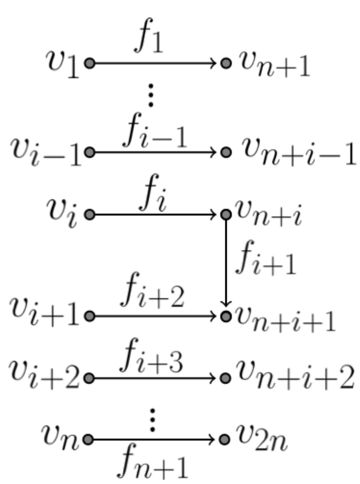
Lemma 3. The elementary n-strand braid β_i has a fence $K(\beta_i)$ with differential

$$d_{\beta_i} : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i);\mathbb{Z})$$

represented by the $(n+1) \times 2n$ matrix

$$(d_{\beta_i})_{j,k} = \begin{cases} 1 & \text{if } 1 \leqslant k \leqslant i \text{ and } j = n+k \\ -1 & \text{if } 1 \leqslant k \leqslant i \text{ and } j = i \\ 1 & \text{if } k = i+1 \text{ and } j = n+i+1 \\ -1 & \text{if } k = i+1 \text{ and } j = n+i \\ 1 & \text{if } i+2 \leqslant k \leqslant n+1 \text{ and } j = n+k-1 \\ -1 & \text{if } i+2 \leqslant k \leqslant n+1 \text{ and } j = n+k-2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is the representation with respect to the ordered bases $(f_1, f_2, \ldots, f_{n+1})$ of $C_1(K(\beta_i);\mathbb{Z})$ and $(v_1, v_2, \ldots, v_{2n})$ of $C_0(K(\beta_i);\mathbb{Z})$ as shown below



□

Lemma 4. Let $\beta_i\beta_j$ be the concatenation of two elementary n-strand braids β_i and β_j with fences $K(\beta_i)$ and $K(\beta_j)$.

(i) The decompositions

$$K(\beta_i) = (K(\beta_i) \smallsetminus K(\beta_j)) \sqcup (K(\beta_i) \cap K(\beta_j))$$
$$K(\beta_j) = (K(\beta_i) \cap K(\beta_j)) \sqcup (K(\beta_j) \smallsetminus K(\beta_i))$$

imply that the differentials

$$d_\beta : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i);\mathbb{Z})$$
$$d_{\beta'} : C_1(K(\beta_j);\mathbb{Z}) \to C_0(K(\beta_j);\mathbb{Z})$$

may be written as

$$\begin{pmatrix} d'_{\beta_i} \\ d''_{\beta_i} \end{pmatrix} : C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z})$$
$$\begin{pmatrix} d'_{\beta_j} \\ d''_{\beta_j} \end{pmatrix} : C_1(K(\beta_j);\mathbb{Z}) \to C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z})$$

where $(n+1) \times 2n$-matrix representation of d_{β_i} from Lemma 3 induces $(n+1) \times n$-matrix representations of d'_{β_i} and d''_{β_i} with

$$(d'_{\beta_i})_{k,l} = (d_{\beta_i})_{k,l}, \quad (d''_{\beta_i})_{k,l} = (d_{\beta_i})_{n+k,l} \quad (1 \leqslant k \leqslant n, 1 \leqslant l \leqslant n+1)$$

and similarly for d_{β_j} and $d'_{\beta_j}, d''_{\beta_j}$.

(ii) The decomposition

$$K(\beta_i) \cup K(\beta_j) = (K(\beta_i) \smallsetminus K(\beta_j)) \sqcup (K(\beta_i) \cap K(\beta_j)) \sqcup (K(\beta_j) \smallsetminus K(\beta_i))$$

implies that the regular n-strand braid with two crossings $\beta_i \beta_j$ has a fence $K(\beta_i \beta_j)$ with differential

$$d_{\beta_i \beta_j} : C_1(K(\beta_i \beta_j);\mathbb{Z}) \to C_0(K(\beta_i \beta_j);\mathbb{Z})$$

has a block decomposition

$$\begin{pmatrix} d'_{\beta_i} & 0 \\ d''_{\beta_i} & d'_{\beta_j} \\ 0 & d''_{\beta_i} \end{pmatrix} : C_1(K(\beta_i);\mathbb{Z}) \oplus C_1(K(\beta'_j);\mathbb{Z}) \to$$
$$C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}).$$

Proof. The simplicial complexes $K(\beta_i), K(\beta_j) \subset K(\beta_i\beta_j)$ intersect in a 0-dimensional simplicial complex so that

$$\begin{aligned}
C_0(K(\beta_i);\mathbb{Z}) &= C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \\
C_0(K(\beta_j);\mathbb{Z}) &= C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}) \\
C_0(K(\beta_i\beta_j);\mathbb{Z}) &= C_0(K(\beta_i) \smallsetminus K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_j);\mathbb{Z}) \oplus C_0(K(\beta_j) \smallsetminus K(\beta_i);\mathbb{Z}) \\
C_1(K(\beta_i\beta_j);\mathbb{Z}) &= C_1(K(\beta_i);\mathbb{Z}) \oplus C_1(K(\beta_j);\mathbb{Z})
\end{aligned}$$

from which the decomposition of the differentials is clear. □

This decomposition may be extended to a concatenation of elementary braids.

Proposition 7. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid. The decompositions

$$\begin{aligned}
K(\beta_i) &= (K(\beta_i) \smallsetminus K(\beta_{i+1})) \sqcup (K(\beta_i) \cap K(\beta_{i+1})) \\
K(\beta_{i+1}) &= (K(\beta_i) \cap K(\beta_{i+1})) \sqcup (K(\beta_{i+1}) \smallsetminus K(\beta_i))
\end{aligned}$$

imply that the differentials

$$\begin{aligned}
d_{\beta_i} &: C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i);\mathbb{Z}) \\
d_{\beta_{i+1}} &: C_1(K(\beta_{i+1});\mathbb{Z}) \to C_0(K(\beta_{i+1});\mathbb{Z})
\end{aligned}$$

may be written as

$$\begin{aligned}
\begin{pmatrix} d'_{\beta_i} \\ d''_{\beta_i} \end{pmatrix} &: C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_i) \smallsetminus K(\beta_{i+1});\mathbb{Z}) \oplus C_0(K(\beta_i) \cap K(\beta_{i+1});\mathbb{Z}) \\
\begin{pmatrix} d'_{\beta_{i+1}} \\ d''_{\beta_{i+1}} \end{pmatrix} &: C_1(K(\beta_{i+1});\mathbb{Z}) \to C_0(K(\beta_i) \cap K(\beta_{i+1});\mathbb{Z}) \oplus C_0(K(\beta_{i+1}) \smallsetminus K(\beta_i);\mathbb{Z}).
\end{aligned}$$

The decomposition

$$K(\beta) = \cup_{i=1}^{\ell} K(\beta_i) = (K(\beta_1) \smallsetminus K(\beta_2)) \sqcup (\sqcup_{i=1}^{\ell-1}(K(\beta_i) \cap K(\beta_{i+1}))) \sqcup (K(\beta_\ell) \smallsetminus K(\beta_{\ell-1}))$$

implies β has a fence $K(\beta)$ with differential

$$d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

which has a block decomposition

$$\begin{pmatrix} d'_{\beta_1} & 0 & \dots & 0 & 0 \\ d''_{\beta_1} & d'_{\beta_2} & \dots & 0 & 0 \\ 0 & d''_{\beta_2} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & d'_{\beta_{\ell-1}} & 0 \\ 0 & 0 & \dots & d''_{\beta_{\ell-1}} & d'_{\beta_\ell} \\ 0 & 0 & \dots & 0 & d''_{\beta_\ell} \end{pmatrix}:$$

$$\oplus_{i=1}^{n} C_1(K(\beta_i);\mathbb{Z}) \to C_0(K(\beta_1) \smallsetminus K(\beta_2)) \oplus (\oplus_{i=1}^{\ell-1} C_0(K(\beta_i) \cap K(\beta_{i+1});\mathbb{Z})) \oplus C_0(K(\beta_\ell) \smallsetminus K(\beta_{\ell-1})).$$

Proof. Follows by induction on ℓ with the base case $\ell = 2$ given by Lemma 4 and the equality

$$C_0(K(\beta_i) \smallsetminus K(\beta_{i+1});\mathbb{Z}) = C_0(K(\beta_{i-1}) \cap K(\beta_i);\mathbb{Z}) \quad (2 \leqslant i \leqslant \ell - 1).$$

□

Corollary 2. The elementary n-strand braid relations

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geqslant 2$

(ii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i - j| = 1$

(iii) $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1$

have the effect of replacing the differentials

(i) $$\begin{pmatrix} d'_{\sigma_i} & 0 \\ d''_{\sigma_i} & d'_{\sigma_j} \\ 0 & d''_{\sigma_j} \end{pmatrix} : C_1(K(\sigma_i\sigma_j);\mathbb{Z}) \to C_0(K(\sigma_i\sigma_j);\mathbb{Z})$$

respectively

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} \\ 0 & d''_{\sigma_j^{-1}} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z})$$

(ii) $$\begin{pmatrix} d'_{\sigma_i} & 0 & 0 \\ d''_{\sigma_i} & d'_{\sigma_j} & 0 \\ 0 & d''_{\sigma_j} & d'_{\sigma_i} \\ 0 & 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_i\sigma_j\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_i\sigma_j\sigma_i);\mathbb{Z})$$

respectively

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} & 0 \\ 0 & d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} \\ 0 & 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z})$$

(iii) $$\begin{pmatrix} d'_{\sigma_i} & 0 \\ d''_{\sigma_i} & d'_{\sigma_i^{-1}} \\ 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_i\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_i\sigma_i^{-1});\mathbb{Z})$$

and

$$\begin{pmatrix} d'_{\sigma_i^{-1}} & 0 \\ d''_{\sigma_i^{-1}} & d'_{\sigma_i} \\ 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_i^{-1}\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_i^{-1}\sigma_i);\mathbb{Z})$$

by the differentials

(i) $$\begin{pmatrix} d'_{\sigma_j} & 0 \\ d''_{\sigma_j} & d'_{\sigma_i} \\ 0 & d''_{\sigma_i} \end{pmatrix} : C_1(K(\sigma_j\sigma_i);\mathbb{Z}) \to C_0(K(\sigma_j\sigma_i);\mathbb{Z})$$

respectively

$$\begin{pmatrix} d'_{\sigma_j^{-1}} & 0 \\ d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} \\ 0 & d''_{\sigma_i^{-1}} \end{pmatrix} : C_1(K(\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z}) \to C_0(K(\sigma_j^{-1}\sigma_i^{-1});\mathbb{Z})$$

(ii) $$\begin{pmatrix} d'_{\sigma_j} & 0 & 0 \\ d''_{\sigma_j} & d'_{\sigma_i} & 0 \\ 0 & d''_{\sigma_i} & d'_{\sigma_j} \\ 0 & 0 & d''_{\sigma_j} \end{pmatrix} : C_1(K(\sigma_j\sigma_i\sigma_j);\mathbb{Z}) \to C_0(K(\sigma_j\sigma_i\sigma_j);\mathbb{Z})$$

respectively

$$\begin{pmatrix} d'_{\sigma_j^{-1}} & 0 & 0 \\ d''_{\sigma_j^{-1}} & d'_{\sigma_i^{-1}} & 0 \\ 0 & d''_{\sigma_i^{-1}} & d'_{\sigma_j^{-1}} \\ 0 & 0 & d''_{\sigma_j^{-1}} \end{pmatrix} : C_1(K(\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z}) \to C_0(K(\sigma_j^{-1}\sigma_i^{-1}\sigma_j^{-1});\mathbb{Z})$$

(iii) $0 : C_1(K(1);\mathbb{Z}) = 0 \to C_0(K(1);\mathbb{Z})$

Definition 20. Let β and β' be regular n-strand braids. The chain level Seifert pairs (λ_β, d_β) and $(\lambda_{\beta'}, d_{\beta'})$ are A-*equivalent* if there exists a finite sequence of A-operations which transforms both λ_β to $\lambda_{\beta'}$ and d_β to $d_{\beta'}$.

Proposition 8. The A-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant.

Proof. Two geometric n-strand braids β, β' are isotopic if and only if they are isotopic to regular n-strand braids determined by braid words β, β' from the alphabet $\{\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$ such that β' can be obtained from β by applying finitely many of the relations

(i) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geqslant 2$

(ii) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for $|i - j| = 1$

(iii) $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$

and their inverses. By Theorem 7 and Proposition 7, these relations and their inverses correspond to transformations (i)-(iii) in the definition of A-equivalence of a chain level Seifert pair. □

The isotopy invariance of the A-equivalence class of the chain level Seifert pair of a braid yields a universal representation of the braid group.

Theorem 10. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ and denote by B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection which respects the concatenation of braid words such that words $\beta, \beta' \in F_n$ differ by the relations in the braid group if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are A-equivalent. This induces a well defined bijection

$$(\lambda, d) : B_n \to \frac{\{\text{chain level Seifert pairs}\}}{A - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

which is group homomorphism and which determines a commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\ \downarrow & & \downarrow \\ B_n & \xrightarrow[\cong]{(\lambda,d)} & \frac{\{\text{chain level Seifert pairs}\}}{A\text{-equivalence}} \end{array}$$

where the vertical maps are quotient maps.

Proof. This follows from Corollary 2 and Proposition 8. □

Example 7. Let β be the regular 4-strand braid with 8-crossings represented by the braid word $\beta = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1$. The sequence of isotopies

$$\begin{aligned}\sigma_1\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 &= \sigma_3\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 \\ &= \sigma_3\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_1 \\ &= \sigma_3\sigma_2\sigma_3^{-1}\sigma_1 \\ &= \sigma_2\sigma_3\sigma_3^{-1}\sigma_1 \\ &= \sigma_2\sigma_1\end{aligned}$$

arising from applying the relations of the braid group B_4, implies that the chain level Seifert pairing

$$\lambda_\beta : C_1(K(\beta);\mathbb{Z}) \times C_1(K(\beta);\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$$

and differential

$$d_\beta : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

is A-equivalent to the chain level pairing

$$\lambda_{\sigma_2\sigma_1} = \begin{pmatrix} \lambda_{\sigma_2} & 0 \\ 0 & \lambda_{\sigma_1} \end{pmatrix}$$
$$:(C_1(K(\sigma_2);\mathbb{Z}) \oplus C_1(K(\sigma_1);\mathbb{Z})) \times (C_1(K(\sigma_2);\mathbb{Z}) \oplus C_1(K(\sigma_1);\mathbb{Z}));\mathbb{Z}) \to \mathbb{Z}[\frac{1}{2}]$$

and differential

$$d_{\sigma_2\sigma_1} : C_1(K(\sigma_2\sigma_1);\mathbb{Z}) \to C_0(K(\sigma_2\sigma_1);\mathbb{Z})$$

We now construct a second equivalence relation which corresponds to isotopy of the closure of a braid inside in the solid torus.

Definition 21. Two square real matrices with entries in $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$ are $\widehat{A}$-*equivalent* if one can be transformed into the other by a finite sequence of $\widehat{A}$-*operations* defined as follows:

(i) A-operations

(ii) $A \mapsto \begin{pmatrix} \lambda_\alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda_{\alpha^{-1}} \end{pmatrix}$ for α an elementary n-strand braid

(iii) $\begin{pmatrix} \lambda_\alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda_{\alpha^{-1}} \end{pmatrix} \mapsto A$ for α an elementary n-strand braid

The $\widehat{A}$-operations have the following effect on the differential of a fence.

Proposition 9. Let $\beta = \beta_1\beta_2\ldots\beta_\ell$ be a regular n-strand braid with ℓ crossings where each β_i is an elementary n-strand braid and let α be an elementary n-strand braid. The conjugacy transformation $\beta \in B_n \mapsto \alpha\beta\alpha^{-1}$ is such that if the fence $K(\beta)$ has differential represented by the block matrix as in Proposition 7

$$d_\beta = \begin{pmatrix} d'_{\beta_1} & 0 & \ldots & 0 & 0 \\ d''_{\beta_1} & d'_{\beta_2} & \ldots & 0 & 0 \\ 0 & d''_{\beta_2} & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & d'_{\beta_{\ell-1}} & 0 \\ 0 & 0 & \ldots & d''_{\beta_{\ell-1}} & d'_{\beta_\ell} \\ 0 & 0 & \ldots & 0 & d''_{\beta_\ell} \end{pmatrix} : C_1(K(\beta);\mathbb{Z}) \to C_0(K(\beta);\mathbb{Z})$$

then the fence $K(\alpha\beta\alpha^{-1})$ has differential represented by the block matrix

$$d_{\alpha\beta\alpha^{-1}} = \begin{pmatrix} d'_{\alpha} & 0 & 0 & \ldots & 0 & 0 \\ d''_{\alpha} & d'_{\beta_1} & 0 & \ldots & 0 & 0 \\ 0 & d''_{\beta_1} & d'_{\beta_2} & \ldots & 0 & 0 \\ 0 & 0 & d''_{\beta_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & d'_{\beta_\ell} & 0 \\ 0 & 0 & 0 & \ldots & d''_{\beta_\ell} & d'_{\alpha^{-1}} \\ 0 & 0 & 0 & \ldots & 0 & d''_{\alpha^{-1}} \end{pmatrix} : C_1(K(\alpha\beta\alpha^{-1});\mathbb{Z}) \to C_0(K(\alpha\beta\alpha^{-1});\mathbb{Z})$$

Proof. Follows from Proposition 7. □

Definition 22. Let β, β' be regular n-strand braids. The chain level Seifert pairs (λ_β, d_β) and $(\lambda_{\beta'}, d_{\beta'})$ are $\widehat{A}$-*equivalent* if there exists a finite sequence of $\widehat{A}$-operations which transforms both λ_β to $\lambda_{\beta'}$ and d_β to $d_{\beta'}$.

Proposition 10. The $\widehat{A}$-equivalence class of the chain level Seifert pair of an n-strand geometric braid β is an isotopy invariant of the closure $\widehat{\beta}$ inside the solid torus.

Proof. By [7, Theorem 2.1] for any regular n-strand braids $\beta, \beta' \in B_n$, the closed braids $\widehat{\beta}, \widehat{\beta'}$ are isotopic in the solid torus if and only if β and β' are conjugate in B_n. The proof is then similar to the proof of Proposition 8 but now with the conjugacy of elements in the braid group corresponding to operations (ii) and (iii) in the definition of $\widehat{A}$-equivalence. □

The isotopy invariance of the $\widehat{A}$-equivalence class of the chain level Seifert pair of a braid yields a representation of the quotient of the braid group by the conjugacy relation.

Theorem 11. Let $n \geqslant 2$ and denote by F_n the free group on the set of elementary n-strand braids $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and by let B_n denote the braid group. The map

$$(\lambda, d) : F_n \to \{\text{chain level Seifert pairs}\}, \quad \beta \mapsto (\lambda_\beta, d_\beta)$$

is a bijection such that conjugate words $\beta, \beta' \in B_n$ have chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ which are $\widehat{A}$-equivalent. This induces a well-defined bijection

$$(\lambda, d) : \frac{B_n}{\text{conjugacy}} \to \frac{\{\text{chain level Seifert pairs}\}}{\widehat{A} - \text{equivalence}}, \quad [\beta] \mapsto [(\lambda_\beta, d_\beta)]$$

and determines a commutative diagram

$$\begin{array}{ccc} B_n & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{A\text{–equivalence}} \\ \downarrow & & \downarrow \\ \dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{–equivalence}} \end{array}$$

Moreover, words $\beta, \beta' \in F_n$ differ by the relations in the braid group plus conjugacy if and only if the chain level Seifert pairs $(\lambda_\beta, d_\beta), (\lambda_{\beta'}, d_\beta)$ are $\widehat{A}$-equivalent so that there is commutative diagram

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\ \downarrow & & \downarrow \\ \dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{–equivalence}} \end{array}$$

which factors as

$$\begin{array}{ccc}
F_n & \xrightarrow[\cong]{(\lambda,d)} & \{\text{chain level Seifert pairs}\} \\
\downarrow & & \downarrow \\
B_n & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{A\text{–equivalence}} \\
\downarrow & & \downarrow \\
\dfrac{B_n}{\text{conjugacy}} & \xrightarrow[\cong]{(\lambda,d)} & \dfrac{\{\text{chain level Seifert pairs}\}}{\widehat{A}\text{–equivalence}}
\end{array}.$$

Proof. Follows from Theorem 10 and Proposition 10. □

10 ω-signatures of braids

We now use the chain level Seifert pair (λ_β, d_β) of a braid β to give a chain level combinatorial formula for the ω-signature of a braid.

Definition 23. If L is an oriented link with Seifert matrix V then the *signature* of L is the signature $\sigma(L)$ of the symmetric form $(H_1(\Sigma;\mathbb{Z}), V + V^t)$. For a unit complex number $\omega \neq 1$ the ω-*signature* of L is the signature $\sigma_\omega(L)$ of the hermitian form $(H_1(\Sigma;\mathbb{C}), (1-\omega)V + (1-\overline{\omega})V^t)$.

The -1-signature of an oriented link is the same as its signature.

Proposition 11. [15, p.219] For an oriented link L and a unit complex number $\omega \neq 1$ the value $\sigma_\omega(L)$ does not depend on the choice of Seifert surface for L.

The signature of a link may also be interpreted as the signature of a 4-manifold with boundary.

Proposition 12. [8] Let $L \subset S^3$ be a link with Seifert surface $\Sigma \subset S^3 = \partial D^4$. Keeping the boundary of Σ fixed in S^3, push Σ inside D^4 to form a new surface Σ' with boundary L. If W is the two-fold branched cover of D^4 branched along Σ' then W is an oriented 4-manifold with boundary such that ∂W is a 2-fold cover of S^3 branched over L. Moreover, there exists a choice of basis such that the intersection form on $H_2(W)$ is represented by the matrix $V + V^t$ so that $\sigma(L) = \sigma(W)$.

Definition 24. If β is braid and if $\omega \neq 1$ is a unit complex number then the ω-*signature* of β is the ω-signature $\sigma_\omega(\beta)$ of the oriented link $\widehat{\beta}$.

Example 8. From Example 6 the 2-strand braid $\beta = \sigma_1\sigma_1\sigma_1$ with closure $\widehat{\beta}$ the trefoil knot has Seifert matrix V and symmetrisation $V + V^t$ given by

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad V + V^t = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

so that $\sigma(\beta) = -2$.

Theorem 12. Let β be a braid with chain level Seifert pair (λ_β, d_β) and let $\omega \neq 1$ be a unit complex number. The ω-signature of β may be expressed on the chain level as the signature of the hermitian form

$$\left(C_1(K(\beta);\mathbb{C}) \oplus C^0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right)$$

so that

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right).$$

Proof. The $\mathbb{C}$-coefficients chain level Seifert pair $\lambda_\beta : C_1(K(\beta);\mathbb{C}) \times C_1(K(\beta);\mathbb{C}) \to \mathbb{C}$ determines a commutative diagram

$$\begin{array}{ccc} 0 & \xrightarrow{\quad 0 \quad} & C^0(K(\beta);\mathbb{C}) \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle d_\beta^*} \\ C_1(K(\beta);\mathbb{C}) & \xrightarrow{(1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t} & C^1(K(\beta);\mathbb{C}) \\ {\scriptstyle d_\beta}\big\downarrow & & \big\downarrow \\ C_0(K(\beta);\mathbb{C}) & \xrightarrow{\quad 0 \quad} & 0 \end{array} .$$

The algebraic lemma of [14, p.26] implies that the signature of the hermitian form

$$\left(H_1(K(\beta);\mathbb{C}), (1-\omega)V + (1-\overline{\omega})V^t\right)$$

is equal to the signature of the hermitian form

$$\left(C_1(K(\beta);\mathbb{C}) \oplus C^0(K(\beta);\mathbb{C}), \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right)$$

and hence

$$\sigma_\omega(\beta) = \sigma\left(\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix}\right).$$

□

This chain level formula shows that the signature of a braid is not additive under the concatenation of braids.

Corollary 3. Let $\omega \neq 1$ be a unit complex number. The ω-signature concatenation defect

$$\sigma_\omega(\beta\beta') - \sigma_\omega(\beta) - \sigma_\omega(\beta')$$

is equal to the difference in signature between the block matrix

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & d_{\beta'}'' \\ 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t \end{pmatrix}$$

and the block matrix

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}'' \\ 0 & 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}$$

where we have decomposed the differential of $K(\beta\beta')$ in terms of the differentials of $K(\beta)$ and $K(\beta')$ as in Proposition 7.

Proof. By Proposition 4 the chain level Seifert pairing for the concatenation $\beta\beta'$ is represented by the block diagonal matrix

$$\lambda_{\beta\beta'} = \begin{pmatrix} \lambda_\beta & 0 \\ 0 & \lambda_{\beta'} \end{pmatrix}$$

so that there is an equality of block matrices

$$\begin{pmatrix} (1-\omega)\lambda_{\beta\beta'} + (1-\overline{\omega})\lambda_{\beta\beta'}^t & d_{\beta\beta'}^t \\ d_{\beta\beta'} & 0 \end{pmatrix} = \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & 0 & d_\beta'^t & d_\beta''^t & 0 \\ 0 & \lambda_{\beta'} + \lambda_{\beta'}{}^t & 0 & d_{\beta'}'^t & d_{\beta'}''^t \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & d_{\beta'}' & 0 & 0 & 0 \\ 0 & d_{\beta'}'' & 0 & 0 & 0 \end{pmatrix}.$$

One can then perform identical row and column exchanges to find a congruence

$$\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & 0 & d_\beta'^t & d_\beta''^t & 0 \\ 0 & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}{}^t & 0 & d_{\beta'}'^t & d_{\beta'}''^t \\ d_\beta' & 0 & 0 & 0 & 0 \\ d_\beta'' & d_{\beta'}' & 0 & 0 & 0 \\ 0 & d_{\beta'}'' & 0 & 0 & 0 \end{pmatrix}$$

$$\simeq \begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & \\ d_\beta'' & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t \end{pmatrix}$$

so that by Theorem 12

$$\sigma_\omega(\beta\beta') = \sigma\begin{pmatrix} \lambda_{\beta\beta'} + \lambda_{\beta\beta'}^t & d_{\beta\beta'}^t \\ d_{\beta\beta'} & 0 \end{pmatrix} = \sigma\begin{pmatrix} (1-\omega)\lambda_\beta + (1-\overline{\omega})\lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}.$$

On the other hand, there is an equality and congruence of block matrices

$$\begin{pmatrix} (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t & d_{\beta'}^t \\ d_{\beta'} & 0 \end{pmatrix} = \begin{pmatrix} (1-\omega)\lambda_{\beta'} + (1-\overline{\omega})\lambda_{\beta'}^t & d_{\beta'}'^t & d_{\beta'}''^t \\ d_{\beta'}' & 0 & 0 \\ d_{\beta'}'' & 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & d_{\beta'}' \\ 0 & 0 & d_{\beta'}' \\ d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}$$

so that

$$\begin{aligned} \sigma_\omega(\beta) + \sigma_\omega(\beta') &= \sigma_\omega\begin{pmatrix} \lambda_\beta + \lambda_\beta^t & d_\beta^t \\ d_\beta & 0 \end{pmatrix} + \sigma\begin{pmatrix} \lambda_{\beta'} + \lambda_{\beta'}^t & d_{\beta'}^t \\ d_{\beta'} & 0 \end{pmatrix} \\ &= \sigma\begin{pmatrix} \lambda_\beta + \lambda_\beta^t & d_\beta^t & 0 & 0 \\ d_\beta & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\beta'} + \lambda_{\beta'}^t & d_{\beta'}^t \\ 0 & 0 & d_{\beta'} & 0 \end{pmatrix} \\ &= \sigma\begin{pmatrix} \lambda_\beta + \lambda_\beta^t & d_\beta'^t & d_\beta''^t & 0 & 0 & 0 \\ d_\beta' & 0 & 0 & 0 & 0 & 0 \\ d_\beta'' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & 0 & 0 & d_{\beta'}' \\ 0 & 0 & 0 & d_{\beta'}'^t & d_{\beta'}''^t & \lambda_{\beta'} + \lambda_{\beta'}^t \end{pmatrix}. \end{aligned}$$

□

11 An open question

One would wish to find an elementary closed form expression for the ω-signature concatenation defect, but this is not possible in general. In the spirit of [8] Gambaudo and Ghys [6] constructed from n-strand braids β, β' an oriented, compact, connected 4-manifold $M(\beta, \beta')$ of signature zero in such that way that $M(\beta, \beta')$ can be obtained by glueing three oriented 4-manifold manifolds $C(\beta), C(\beta'), C(\beta\beta')$ with signatures which satisfy

$$\sigma(C(\beta)) = \sigma(\beta), \quad \sigma(C(\beta')) = \sigma(\beta'), \quad \sigma(C(\beta\beta')) = \sigma(\beta\beta').$$

They extended this to an equivariant version for branched cyclic covers where there is an action of $\mathbb{Z}_k$ on $M(\beta, \beta'), C(\beta), C(\beta')$, $C(\beta\beta')$ which respects the decomposition of $M(\beta\beta')$ and used an equivariant version of Wall's non-additivity theorem for the signature [17] to express the ω-signature concatenation defect in terms of the Meyer cocycle and the Burau-Squier hermitian representation of the braid group $\mathcal{B}_\omega : B_\infty \to \mathbf{Sp}(\infty, \mathbb{R})$. Bourrigan [4] gave a different proof using infinite cyclic covers.

Theorem 13. ([6, Theorem A], [4, Chapter V]). Let $\omega \neq 1$ be a root of unity. The ω-signature of the concatenated braid $\beta\beta'$ is related to the ω-signature of the braids β, β' by

$$\sigma_\omega(\beta\beta') = \sigma_\omega(\beta) - \sigma_\omega(\beta') - Meyer(\mathcal{B}_\omega(\beta), \mathcal{B}_\omega(\beta')).$$

This suggests the following:

Open question: Is it possible to use the chain level Seifert pair (λ_β, d_β) of a braid and the L-theory techniques of [13] to express the ω-signature concatenation defect in terms of an L-theoretic analogue of the Meyer cocycle?

References

[1] J.W. Alexander. A lemma on systems of knotted curves. *Proc. Nat. Acad. Sci. USA.*, 9:93–95, 1923.

[2] E. Artin. Braids and permutations. *Ann. of Math. (2)*, 48:643–649, 1947.

[3] T. Banchoff. Self linking numbers of space polygons. *Indiana Univ. Math. J.*, 25(12):1171–1188, 1976.

[4] Maxime Bourrigan. Quasimorphismes sur les groupes de tresses et forme de blanchfield, 2013. `https://tel.archives-ouvertes.fr/tel-00872081`.

[5] J. Collins. An algorithm for computing the Seifert matrix of a link from a braid representation, 2012. `http://www.maths.ed.ac.uk/ jcollins/SeifertMatrix/SeifertMatrix.pdf`.

[6] Jean-Marc Gambaudo and Étienne Ghys. Braids and signatures. *Bull. Soc. Math. France*, 133(4):541–579, 2005.

[7] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247 of *Graduate Texts in Mathematics*. Springer, New York, 2008. With the graphical assistance of Olivier Dodane.

[8] Louis H. Kauffman and Laurence R. Taylor. Signature of links. *Trans. Amer. Math. Soc.*, 216:351–365, 1976.

[9] Akio Kawauchi. *A survey of knot theory*. Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[10] A. Markov. Über die freie äquivalenz geschlossener zöpfe. *Matematiceskij sbornik*, 43(1):73–78.

[11] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[12] A. Ranicki. Braids and their Seifert surfaces, 2014. `http://www.maths.ed.ac.uk/ aar/slides/maynooth.pdf`.

[13] Andrew Ranicki. *High-dimensional knot theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998. Algebraic surgery in codimension 2, With an appendix by Elmar Winkelnkemper.

[14] Andrew Ranicki and Dennis Sullivan. A semi-local combinatorial formula for the signature of a $4k$-manifold. *J. Differential Geometry*, 11(1):23–29, 1976.

[15] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[16] H. Seifert. Über das Geschlecht von Knoten. *Math. Ann.*, 110(1):571–592, 1935.

[17] C. T. C. Wall. Non-additivity of the signature. *Invent. Math.*, 7:269–274, 1969.