THEOREM 3. *On a nonsingular algebraic variety, c_k^d is the dimension of the Q-submodule of $H^d(M_n, Q)$ generated by cohomology classes dual to $(2n - d)$-dimensional rational cycles formed by intersecting $(2n + 2k - d)$-dimensional rational cycles with $(n - k)$-dimensional algebraic subvarieties of M_n.*

Thus the index c_{n-k}^{2n-d} in a sense gives the dimension of the set of d-dimensional cycles lying on k-dimensional algebraic subvarieties of M_n. The proof of this result follows from a theorem of Lefschetz,[2] which asserts that a $(2n - 2)$-dimensional cycle is represented by a divisor (effective or not) if and only if its dual cohomology class is represented by a differential form of type (1, 1), and from a theorem of Severi,[4] which asserts that an irreducible algebraic subvariety $V_k \subset M_n$ is the complete intersection of $r - k$ divisors (effective or not) on M_n.

[1] A. Comessatti, "Sugl'indici di singolarità a più dimensioni delle varietà abeliane," *Rend. Seminar. mat. Univ. Padova*, Vol. 5, 1934.

[2] W. V. D. Hodge, *Harmonic Integrals* (Cambridge, 1952).

[3] This corresponds to the Riemann conditions on Ω.

[4] F. Severi, *Serie, sistemi d'equivalenza e corrispondenza algebriche sulla varietà algebriche* (Rome, 1942).

ON DEHN'S LEMMA AND THE ASPHERICITY OF KNOTS

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Everything in this note will be considered from the *semilinear* point of view; i.e., any 3-manifold will be considered with a fixed triangulation (this is permissible according to Moise's work[1]), any curve or line will be considered as polygonal, any surface as polyhedral, and so on.

The following theorem was first considered by Dehn,[2] but it was pointed out by Kneser[3] that Dehn's proof contains a gap.

DEHN'S LEMMA. *Let M be a 3-manifold, compact or not, with boundary which may be empty, and in M let D be a 2-cell with self-intersections (singularities), having as boundary the simple closed polygonal curve C and such that there exists a closed neighborhood of C in D which is an annulus (i.e., no point of C is singular). Then there exists a 2-cell D_0 with boundary C, semilinearly imbedded in M.*

Johansson[4] proved that, if Dehn's lemma holds for all orientable 3-manifolds, it also holds for all nonorientable ones. We prove that Dehn's lemma holds for all *orientable* 3-manifolds, and by a modification of our method we also prove the following theorem.

SPHERE THEOREM. *Let M be an orientable 3-manifold, compact or not, with boundary which may be empty, such that $\pi_2(M) \neq 0$, and which can be topologically imbedded in a 3-manifold N, having the following property: The first homology group of any nontrivial (but not necessarily proper) subgroup of $\pi_1(N)$, has an element of infinite order (note in particular that this holds if $\pi_1(N) = 1$). Then there exists a 2-sphere S semilinearly imbedded in M, such that S is not homotopic to zero in M.*

From these two theorems follow others, which may be stated easily if we introduce the following notions: Let F be a nonempty proper closed subset in S^3. We say that F is *geometrically splittable* if there is a 2-sphere $S^2 \subset S^3 - F$ such that both components of $S^3 - S^2$ contain points of F. We say that F is *algebraically splittable* if $\pi_1(S^3 - F)$ is the free product of two groups, each of which is nontrivial.

THEOREM 1. *Let U be a nonempty proper open connected subset of the 3-sphere S^3. Then U is aspherical if and only if $S^3 - U$ is not geometrically splittable.*

The above theorem provides us with a solution of a problem of Whitehead.[5] From Theorem 1 follows easily the following:

COROLLARY 1. *If F is a nonempty proper closed connected subset of S^3, then each component of $S^3 - F$ is aspherical.*

The above corollary provides us with a solution of a problem of Eilenberg.[6] An immediate consequence of Corollary 1 is the following:

COROLLARY 2. *If F is a connected graph or knot, then $S^3 - F$ is aspherical.*

The following theorem solves completely a problem initiated by Higman.[7]

THEOREM 2. *Let K be a link in S^3. The following three statements are equivalent:*

(*i*) *$S^3 - K$ is not aspherical.*
(*ii*) *K is geometrically splittable.*
(*iii*) *K is algebraically splittable.*

Let K be a *knot* in S^3. According to Dehn (*op. cit.*, Satz 2, p. 158), Specker,[8] Papakyriakopoulos,[9] and the above Corollary 2, the following theorem holds:

THEOREM 3. (*i*) *K is unknotted if and only if $\pi_1(S^3 - K)$ is free cyclic.* (*ii*) *The number of ends of $\pi_1(S^3 - K)$ is either 1 or 2.* (*iii*) *$\pi_1(S^3 - K)$ has 1 end if K is knotted and 2 ends if K is unknotted.*

The following statement is known as *Hopf's conjecture.*

THEOREM 4. *If U is an open connected subset of the 3-sphere, then $\pi_1(U)$ has no element of finite order.*

The proof is based on the sphere theorem and the following simple consequence of a theorem due to P. A. Smith:[10] The fundamental group of an aspherical polyhedron (of finite dimension) has no element of finite order.

A Sketch of the Proof of Dehn's Lemma.—By a *Dehn disk* we mean a 2-cell, which may have singularities, but not on its boundary. We suppose that M is *orientable* and that D has no *branch points* (see Whitehead[11]). Let G be the *inverse image* of D under f, where G is a 2-cell. There is a finite set J of *triples* (J', J'', ψ), where J', J'' are closed curves on G, called the *J-curves*, and ψ: $J' \rightarrow J''$ is a "nice map" such that $f(r) = f\psi(r)$, for any point $r \in J'$, i.e., $f(J') = f(J'')$ is a *double line*[11] of D. We emphasize that, according to Johansson,[12] $J' \neq J''$ because M is orientable. So we have a map f: $(G, J) \rightarrow D$ called a *realized diagram.* We denote by $t(D)$ and $d(D)$ the number of *triple points*[11] and double lines of D, respectively. The couple $(t(D), d(D))$ is called the *complexity* of D.

Let $V \subset M$ be a 3-manifold with boundary such that int V is a neighborhood of $D - C$, $C \subset$ bdry V, and V is very small and "very nice," so that each face of bdry V is "parallel" to a face of D. Such a V is called a *prismatic neighborhood* of D in M. We observe that D is a deformation retract of V. Let p_*: $M_* \rightarrow V$ be the universal covering of V, let D_* be a Dehn disk covering D just once, and let

$q_* = p_*|D_*$. Finally, let V_* be a prismatic neighborhood of D_* in M_*. So we have the following diagram:

$$
\begin{tikzcd}
M_* \arrow[dr, "p_*"'] & \supset V_* \supset & D_* \arrow[d, "q_*"] & (G, J_*) \arrow[l, "f_*"'] \\
M \supset & V \supset & D & (G, J) \arrow[l, "f"']
\end{tikzcd}
$$

called an *elementary tower* over $D \subset V \subset M$, where f_* is the lifted map, and $f_*: (G, J_*) \rightarrow D_*$ is a realized diagram.

If V is *not* simply connected, then there is a covering translation τ of $p_*: M_* \rightarrow V$, such that $D_* \cap \tau^{-1}(D_*)$ is not empty, and so it consists of a finite number of closed curves T_{*i}, $i = s(\tau, 1), \ldots, s(\tau, d(\tau))$. These curves for all possible τ's are called the T_*-*curves*. Then

$$d(D) = d(D_*) + \frac{1}{2} \sum d(\tau),$$

where the sum runs over all covering translations $\tau \neq 1$ (note that if $D_* \cap \tau^{-1}(D_*) = \varnothing$, then naturally $d(\tau) = 0$). Hence $d(D_*) < d(D)$. We emphasize that, in the special case where D_* is a 2-*cell*, $q_*: (D_*, T_*) \rightarrow D$ is a *realized diagram*, where T_* is now the set of all triples (T', T'', τ) such that $T' \in D_* \cap \tau^{-1}(D_*)$ and $T'' = \tau(T') \in \tau(D_*) \cap D_*$.

If V is *simply connected*, then each of the components of bdry V is a 2-*sphere*, according to a result due to Seifert.[31] In this case $d(D_*) = d(D)$.

The diagram

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
(m) \qquad
\begin{tikzcd}
M_m \arrow[dr, "p_m"'] & \supset V_m \supset & D_m \arrow[d, "q_m"] & (G, J_m) \arrow[l, "f_m"'] \\
M_{m-1} \supset & V_{m-1} \supset & D_{m-1} & (G, J_{m-1}) \arrow[l, "f_{m-1}"']
\end{tikzcd}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$
(1) \qquad
\begin{tikzcd}
M_1 \arrow[dr, "p_1"'] & \supset V_1 \supset & D_1 \arrow[d, "q_1"] & (G, J_1) \arrow[l, "f_1"'] \\
M \supset & V \supset & D & (G, J) \arrow[l, "f"']
\end{tikzcd}
$$

is called a *tower* over $D \subset M$ and is defined as follows: The diagram (m) is an elementary tower over $D_{m-1} \subset V_{m-1} \subset M_{m-1}$, for $m = 1, 2, \ldots$, where $M_0 = M$, $V_0 = V$, $D_0 = D$, $f_0 = f$, $J_0 = J$. Then

$$d(D_0) \geqq d(D_1) \geqq d(D_2) \geqq \ldots \geqq 0,$$

and there exists a number $n \geqq 0$, such that $d(D_i) > d(D_{i+1})$, for $i < n$, and $d(D_j) = d(D_{j+1})$ for $j \geqq n$, i.e., V_n is *simply connected* but V_i is *not*. This number n is called the *height* of the tower. The following two cases are possible: (1) $d(D_n) > 0$, i.e., D_n is not a 2-cell; (2) $d(D_n) = 0$, i.e., D_n is a 2-cell.

In case (1) V_n is simply connected, and so bdry V_n is composed of 2-spheres, according to the result of Seifert mentioned above. Let D_n' be one of the 2-cells bounded by C_n, on the component of bdry V_n containing the boundary C_n of D_n. Then $D' = p_1 \ldots p_n(D_n') \subset M$ is a Dehn disk with boundary C and "roughly speaking" having the following property: Either $t(D') < t(D)$, or $t(D') = t(D)$ and $d(D') < d(D)$, i.e., we may say that *complexity of* D' < *complexity of* D.

In case (2) we prove that there exists a triple (J', J'', ψ) of f: $(G, J) \to D$ such that J' and J'' are *disjoint simple* closed curves. The proof of this is rather algebraic and makes use of the fact that q_n: $(D_n, T_n) \to D_{n-1}$ is a realized diagram, as we have observed above, and that each element of T_n is a triple (T', T'', τ), where τ is a covering translation of p_n: $M_n \to V_{n-1}$. Then by a *cut* ("Umschaltung";[14] note that this is the only case in which we can apply Dehn's process without any danger [see Dehn, *op. cit.* p. 150, B, and Kneser, *op. cit.*, p. 260]) of D along $J = f(J') = f(J'')$ we obtain a new Dehn disk $D' \subset M$, with boundary C and such that either $t(D') < t(D)$ or $t(D') = t(D)$ and $d(D') < d(D)$, i.e., we may say that *complexity of* D' < *complexity of* D.

In the same way we obtain from D' a new Dehn disk $D'' \subset M$ with boundary C such that complexity of D'' < complexity of D', and so on. Thus, after a finite number of repetitions of the above process, we finally obtain a Dehn disk in M with boundary C and complexity (0, 0), i.e., we obtain a 2-cell in M with boundary C.

As far as the proof of the sphere theorem is concerned, we restrict ourself to the remark that the method of proof makes use of the above process, standard Hurewicz theorems, and the Poincaré duality theorem.

[1] E. E. Moise, *Ann. Math.*, **56**, 96–114, 1952, and **59**, 159–170, 1954.

[2] M. Dehn, *Math. Ann.*, **69**, 147, 1910.

[3] H. Kneser, *Jahresber. Deut. math. Verein.*, **38**, 260, 1929.

[4] I. Johansson, *Math. Ann.*, **115**, 658–669, 1938, sec. 2.

[5] J. H. C. Whitehead, *Fundamenta math.*, **32**, 161, 1939.

[6] S. Eilenberg, *Fundamenta math.*, **28**, 241, 1937.

[7] G. Higman, *Quart. J. Math.* (Oxford), **29**, 117–122, 1948.

[8] E. Specker, *Comm. Math. Helv.*, **23**, 329, 1949.

[9] C. D. Papakyriakopoulos, *Ann. Math.*, **62**, 298, 1955, Corollary 3.

[10] W. Hurewicz, *Proc. Acad. Sci. Amsterdam*, **39**, 216, 1936.

[11] J. H. C. Whitehead, *J. London Math. Soc.*, **12**, 63–71, 1937, first footnote on p. 66.

[12] I. Johansson, *Math. Ann.*, **110**, 319, 1935, 2. Satz.

[13] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Satz IV, 223, 1934.

[14] J. H. C. Whitehead, *J. London Math. Soc.*, **12**, 63–71, 1937, second footnote on p. 66.