COHOMOLOGY OF SYMPLECTIC GROUPS AND MEYER'S SIGNATURE THEOREM

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ABSTRACT. Meyer showed that the signature of a closed oriented surface bundle over a surface is a multiple of 4, and can be computed using an element of $H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z})$. Denoting by $1 \to \mathbb{Z} \to \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} \to \mathsf{Sp}(2g,\mathbb{Z}) \to 1$ the pullback of the universal cover of $\mathsf{Sp}(2g,\mathbb{R})$, Deligne proved that every finite index subgroup of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ contains $2\mathbb{Z}$. As a consequence, a class in the second cohomology of any finite quotient of $\mathsf{Sp}(2g,\mathbb{Z})$ can at most enable us to compute the signature of a surface bundle modulo 8. We show that this is in fact possible and investigate the smallest quotient of $\mathsf{Sp}(2g,\mathbb{Z})$ that contains this information. This quotient $\mathfrak{H}$ is a non-split extension of $\mathsf{Sp}(2g,2)$ by an elementary abelian group of order 2^{2g+1}. There is a central extension $1 \to \mathbb{Z}/2 \to \tilde{\mathfrak{H}} \to \mathfrak{H} \to 1$, and $\tilde{\mathfrak{H}}$ appears as a quotient of the metaplectic double cover $\mathsf{Mp}(2g,\mathbb{Z}) = \widetilde{\mathsf{Sp}(2g,\mathbb{Z})}/2\mathbb{Z}$. It is an extension of $\mathsf{Sp}(2g,2)$ by an almost extraspecial group of order 2^{2g+2}, and has a faithful irreducible complex representation of dimension 2^g. Provided $g \geqslant 4$, $\widetilde{\mathfrak{H}}$ is the universal central extension of $\mathfrak{H}$. Putting all this together, we provide a recipe for computing the signature modulo 8, and indicate some consequences.

1. INTRODUCTION

Let $\Sigma_g \to M \to \Sigma_h$ be an oriented surface bundle over a surface. This is determined by a homotopy class of maps $\Sigma_h \to B\mathsf{Aut}^+(\Sigma_g)$. If $g \geqslant 2$ then the connected components of $\mathsf{Aut}^+(\Sigma_g)$ are contractible (Hamstrom [17]), and $\pi_0 \mathsf{Aut}^+(\Sigma_g) = \Gamma_g$ is the (orientation preserving) mapping class group of Σ_g. So $B\mathsf{Aut}^+(\Sigma_g) \simeq B\Gamma_g$, and the bundle is classified by a homotopy class of maps $\Sigma_h \to B\Gamma_g$, or equivalently by the monodromy homomorphism

$$\pi_1(\Sigma_h) = \langle a_1, b_1, \dots, a_h, b_h \mid [a_1,b_1]\dots[a_h,b_h] = 1\rangle \to \Gamma_g.$$

Now Γ_g acts on $H^1(\Sigma_g,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ preserving the symplectic form given by cup product into $H^2(\Sigma_g,\mathbb{Z}) \cong \mathbb{Z}$. So we have a map $\Gamma_g \to \mathsf{Sp}(2g,\mathbb{Z})$, which is surjective. Composing, we obtain a map

$$\chi\colon \pi_1(\Sigma_h) \to \Gamma_g \to \mathsf{Sp}(2g,\mathbb{Z}),$$

and an induced map in cohomology

$$\chi^*\colon H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z}) \to H^2(\pi_1(\Sigma_h),\mathbb{Z}).$$

Meyer [22] constructed a 2-cocycle τ on $\mathsf{Sp}(2g,\mathbb{Z})$ and proved that

$$\text{signature}(M) = \langle \chi^*[\tau], [\Sigma_h]\rangle,$$

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and that this is divisible by 4. In fact, for $g \geqslant 3$, we have

$$H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$$

and $[\tau]$ corresponds to 4 under a suitably chosen isomorphism. For $g \geqslant 3$, $\mathsf{Sp}(2g,\mathbb{Z})$ is perfect, so has a universal central extension.

Denote by $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ the central extension obtained by pulling back the universal cover of $\mathsf{Sp}(2g,\mathbb{R})$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{R})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{R}) & \longrightarrow & 1
\end{array}$$

Then for $g \geqslant 4$ the group $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is the universal central extension of $\mathsf{Sp}(2g,\mathbb{Z})$, while for $g = 3$ there is an extra copy of $\mathbb{Z}/2$ coming from the fact that $\mathsf{Sp}(6,2)$ has an exceptional double cover (see Lemma 6.11). Note also that the centre of $\mathsf{Sp}(2g,\mathbb{Z})$ has order two. The centre of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is twice as big as the subgroup $\mathbb{Z}$ displayed above; it is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$ if g is even, and $\mathbb{Z}$ if g is odd.

Theorem 1.1 (Deligne [5])**.** *The group $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is not residually finite. Every subgroup of finite index contains the subgroup $2\mathbb{Z}$.*

To rephrase Deligne's theorem, every finite quotient of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is in fact a finite quotient of the metaplectic double cover $\mathsf{Mp}(2g,\mathbb{Z})$ of $\mathsf{Sp}(2g,\mathbb{Z})$ defined by

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 2\mathbb{Z} & = & 2\mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathsf{Mp}(2g,\mathbb{Z}) & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

A consequence of the theorem of Deligne is that if we compose χ with the map to a finite quotient of $\mathsf{Sp}(2g,\mathbb{Z})$, we lose information about the signature; the best we can hope to do is compute the signature modulo eight.

Our purpose in this paper is to produce a normal subgroup $\mathfrak{K} \trianglelefteq \mathsf{Sp}(2g,\mathbb{Z})$ with finite quotient $\mathfrak{H} = \mathsf{Sp}(2g,\mathbb{Z})/\mathfrak{K}$ of shape $2^{2g+1} \cdot \mathsf{Sp}(2g,2)$ (see §5.2 of the Introduction to the Atlas [4] for notation describing group extensions), and a double cover $\tilde{\mathfrak{H}}$ of $\mathfrak{H}$ which inflates to the metaplectic double cover $\mathsf{Mp}(2g,\mathbb{Z})$ of $\mathsf{Sp}(2g,\mathbb{Z})$. The group $\tilde{\mathfrak{H}}$ has a 2^g dimensional faithful irreducible representation over $\mathbb{Q}[\mathrm{i}]$ which we shall investigate in a subsequent paper [1].

2. The subgroup $\mathfrak{K} \leqslant \mathsf{Sp}(2g,\mathbb{Z})$ and the main theorem

Denote by J the $2g \times 2g$ matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Regarding J as a symplectic form, we have

$$\mathsf{Sp}(2g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X \mid X^t J X = J \right\}.$$

Since $J^{-1} = -J$, a matrix is symplectic if and only if its transpose is symplectic. Writing out the above condition explicitly, a matrix is symplectic if and only if

(i) AB^t and CD^t are symmetric, and $AD^t - BC^t = I$, or equivalently

(ii) A^tC and B^tD are symmetric, and $A^tD - C^tB = I$.

We write $\mathsf{Sp}(2g,2)$ for the matrices satisfying the same conditions over $\mathbb{F}_2$, and note that reduction modulo two $\mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{Sp}(2g,2)$ is surjective (Newman and Smart [24]).

We write $\Gamma(2g,N) \leqslant \mathsf{Sp}(2g,\mathbb{Z})$ for the *principal congruence subgroup* consisting of symplectic matrices which are congruent to the identity modulo N. We write $\Gamma(2g,N,2N)$ for the *Igusa subgroup* [20] of $\Gamma(2g,N)$ consisting of the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where the entries of $\mathsf{Diag}(AB^t)$ and $\mathsf{Diag}(CD^t)$ are divisible by $2N$, or equivalently where the entries of $\mathsf{Diag}(A^tC)$ and $\mathsf{Diag}(B^tD)$ are divisible by $2N$. If $N = 1$, this is the *theta subgroup*, also known as the *symplectic quadratic group*, and denoted $\mathsf{Sp}^{\mathsf{q}}(2g,\mathbb{Z})$. It is the inverse image in $\mathsf{Sp}(2g,\mathbb{Z})$ of the orthogonal subgroup $\mathsf{O}^+(2g,2) \leqslant \mathsf{Sp}(2g,2)$.

Definition 2.1. We write $\mathfrak{K}$ for the subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$ consisting of matrices

$$\begin{pmatrix} I + 2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I + 2\mathrm{d} \end{pmatrix} \in \mathsf{Sp}(2g,\mathbb{Z})$$

satisfying:

(i) The vectors of diagonal entries $\mathsf{Diag}(\mathrm{b})$ and $\mathsf{Diag}(\mathrm{c})$ are even, and

(ii) the trace $\mathsf{Tr}(a)$ is even.

Thus we have $\Gamma(2g,4) \leqslant \mathfrak{K} \leqslant \Gamma(2g,2)$ and $|\Gamma(2g,2) : \mathfrak{K}| = 2^{2g+1}$. The interpretation of the subgroup $\mathfrak{K}$ is that it is the inverse image in $\mathsf{Sp}(2g,\mathbb{Z})$ of the largest subspace of $\Gamma(2g,2)/\Gamma(2g,4)$ on which the quadratic form in part (iv) of the theorem below is identically zero.

Our main theorem is as follows. We assume that $g \geqslant 4$ for the purpose of simplifying the statements. In an appendix we include statements for all values of g. The main difference for low values of g is that the cohomology of $\mathsf{Sp}(2g,2)$ in degrees one and two contributes some further annoying complications.

Theorem 2.2. *Let $g \geqslant 4$.*

(i) *$\mathfrak{K}$ is a normal subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$. We write $\mathfrak{H}$ for the quotient $\mathsf{Sp}(2g,\mathbb{Z})/\mathfrak{K}$.*

(ii) *The quotient $\Gamma(2g,2)/\mathfrak{K} \leqslant \mathfrak{H}$ is an elementary abelian 2-group $(\mathbb{Z}/2)^{2g+1}$.*

(iii) *The extension*
$$1 \to (\mathbb{Z}/2)^{2g+1} \to \mathfrak{H} \to \mathsf{Sp}(2g, 2) \to 1$$
does not split.

(iv) *The group* $(\mathbb{Z}/2)^{2g+1}$ *supports an invariant quadratic form* $\mathfrak{q}$ *given by*
$$\mathfrak{q}\begin{pmatrix} I + 2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I + 2\mathrm{d} \end{pmatrix} = \mathsf{Tr}(\mathrm{a}) + \langle \mathsf{Diag}(\mathrm{b}), \mathsf{Diag}(\mathrm{c}) \rangle$$
(see Remark 5.3 for definition of the pointy brackets here).

(v) *The action of* $\mathsf{Sp}(2g, 2)$ *on* $(\mathbb{Z}/2)^{2g+1}$ *described by the extension in* (iii) *gives the exceptional isomorphism* $\mathsf{Sp}(2g, 2) \cong \mathsf{O}(2g+1, 2)$*, the orthogonal group of the quadratic form* $\mathfrak{q}$*.*

(vi) *We have* $H^2(\mathfrak{H}, \mathbb{Z}/2) \cong \mathbb{Z}/2$*, and an associated central extension*
$$1 \to \mathbb{Z}/2 \to \tilde{\mathfrak{H}} \to \mathfrak{H} \to 1.$$

(vii) *For* $n \geqslant 2$*, the inflation map* $H^2(\mathfrak{H}, \mathbb{Z}/2) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}/2^n), \mathbb{Z}/2)$ *is an isomorphism.*

(viii) *The non-zero element of* $H^2(\mathsf{Sp}(2g, \mathbb{Z})/\mathfrak{K}, \mathbb{Z}/2)$ *inflates to the reduction modulo two of* $\frac{1}{4}[\tau]$ *as an element of* $H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}/2)$*.*

(ix) *Restricting the central extension of* $\mathfrak{H}$ *to the subgroup* $\Gamma(2g, 2)/\mathfrak{K}$ *gives an almost extraspecial group* $2^{1+(2g+1)} \leqslant \tilde{\mathfrak{H}}$*.*

The proof of this theorem occupies the rest of the paper.

3. Extraspecial and almost extraspecial groups

For background on extraspecial and almost extraspecial groups, we refer to §I.5.5 of Gorenstein [13] and §III.13 of Huppert [19], as well as the papers of Bouc and Mazza [2], Carlson and Thévenaz [3], Glasby [10], Griess [14], Hall and Higman [15], Lam and Smith [21], Quillen [26], Schmid [31], Stancu [32], and the letter from Isaacs to Diaconis reproduced in the appendix of Diaconis [7].

The cohomology ring $H^*((\mathbb{Z}/2)^n, \mathbb{Z}/2)$ is a polynomial ring in generators $z_1, \dots, z_n$ of degree one. Thus
$$H^1((\mathbb{Z}/2)^n, \mathbb{Z}/2) = \mathsf{Hom}((\mathbb{Z}/2)^n, \mathbb{Z}/2)$$
is an n dimensional vector space spanned by the linear forms $z_1, \dots, z_n$. An element of degree two is therefore a quadratic form $\mathfrak{q}$ on $(\mathbb{Z}/2)^n$. Letting $\mathfrak{b}$ be the associated bilinear form $(\mathbb{Z}/2)^n \times (\mathbb{Z}/2)^n \to \mathbb{Z}/2$, we have
$$\mathfrak{q}(x + y) = \mathfrak{q}(x) + \mathfrak{q}(y) + \mathfrak{b}(x, y).$$

In the corresponding central extension
$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^n \to 1$$
the role played by $\mathfrak{q}$ and $\mathfrak{b}$ is as follows. If x and y are elements of $(\mathbb{Z}/2)^n$, choose preimages $\hat{x}$ and $\hat{y}$ in E. Then as elements of the central $\mathbb{Z}/2$, we have $\hat{x}^2 = \mathfrak{q}(x)$ and $[\hat{x}, \hat{y}] = \mathfrak{b}(x, y)$.

Definition 3.1. We say that a quadratic form $\mathfrak{q}$ is *non-singular* if the radical $\mathfrak{b}^\perp$ of the associated bilinear form $\mathfrak{b}$ is $\{0\}$, and *non-degenerate* if $\mathfrak{b}^\perp \cap \mathfrak{q}^{-1}(0) = \{0\}$.

If $\mathfrak{q}$ is non-singular then $n = 2g$ is even; in this case there are two isomorphism classes of quadratic forms, distinguished by the Arf invariant. The corresponding groups E defined by the central extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^{2g} \to 1$$

are called *extraspecial* 2*-groups*, and are characterised by the properties

$$\Phi(E) = [E, E] = Z(E) \cong \mathbb{Z}/2.$$

The two isomorphism classes of extraspecial groups are denoted 2_+^{1+2g} (Arf invariant zero) and 2_-^{1+2g} (Arf invariant one).

If $\mathfrak{q}$ is singular but non-degenerate then $n = 2g + 1$ is odd; in this case there is one isomorphism class of quadratic forms. The corresponding groups E defined by the central extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^{2g+1} \to 1$$

are called *almost extraspecial groups*. The central product of $\mathbb{Z}/4$ with an extraspecial group of either isomorphism type of order 2^{1+2g} gives the almost extraspecial group of order $2^{1+(2g+1)}$.

If G is a group, we write $\mathsf{Aut}(G)$ for the group of automorphisms of G, $\mathsf{Out}(G)$ for the group of outer automorphisms, and $\mathsf{Inn}(G)$ for the group of inner automorphisms. These fit into short exact sequences

$$1 \to Z(G) \to G \to \mathsf{Inn}(G) \to 1$$
$$1 \to \mathsf{Inn}(G) \to \mathsf{Aut}(G) \to \mathsf{Out}(G) \to 1.$$

Writing the automorphism groups of the extraspecial and almost extraspecial groups as extensions of the outer by the inner automorphisms in this way, we have sequences

$$1 \to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2_+^{1+2g}) \to \mathsf{O}^+(2g, 2) \to 1$$
$$1 \to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2_-^{1+2g}) \to \mathsf{O}^-(2g, 2) \to 1$$
$$1 \to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2^{1+(2g+1)}) \to \mathsf{Sp}(2g, 2) \times \mathbb{Z}/2 \to 1. \tag{3.2}$$

which do not split provided $g \geqslant 4$. It is the last case that is of interest to us: in this case the extra factor of $\mathbb{Z}/2$ in the outer automorphism group $\mathsf{Out}(E)$ acts by inverting the central element of order four, and for $g \geqslant 3$ the derived subgroup $\mathsf{Out}(E)'$ is $\mathsf{Sp}(2g, 2)$.

It was proved by Griess [14] using representation theory, that in each case, there is an extension of the extraspecial group by its outer automorphism group, and of the almost extraspecial group by the subgroup of index two in its outer automorphism group.

We are interested in the almost extraspecial case. In this case, what Griess proved (part (b) of Theorem 5 of [14]) is that there is a group which he denotes H_0 of shape $2^{1+(2g+1)}\mathsf{Sp}(2g, 2)$, with the following properties. The normal 2-subgroup $O_2(H_0)$ is the almost extraspecial group $2^{1+(2g+1)}$, and the quotient $H_0/Z(H_0)$ is isomorphic to the subgroup of index two in $\mathsf{Aut}(2^{1+(2g+1)})$.

Dempwolff [6] proved that for $g \geqslant 2$ there is a unique isomorphism class of non-split extensions of $\mathsf{Sp}(2g, 2)$ by an elementary abelian group $(\mathbb{Z}/2)^{2g}$ with non-trivial action. We shall combine the results of Griess and Dempwolff to show that the group $\mathfrak{H}$ of Theorem 2.2

is isomorphic to the quotient of Griess' group H_0 by the central subgroup of order two. This in turn allows us to compute $H^2(\mathfrak{H}, \mathbb{Z}/2)$ and relate it to $H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z})$.

There is another approach to this, which we describe in a separate paper [1]. This avoids the use of the theorems of Griess and Dempwolff, replacing them with a computation showing that the group $\tilde{\mathfrak{H}}$ has a Curtis–Tits–Steinberg type presentation. This approach is closely related to the action of $\tilde{\mathfrak{H}}$ on a certain 2^g dimensional space of theta functions, and shows that the following defines a projective representation $\sigma\colon \mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])/\{\pm I\}$ with kernel $\mathfrak{K}$, and then induces a 2^g dimensional representation $\tilde{\mathfrak{H}} \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])$.

The underlying vector space for the representation has as a basis the vectors e_w for $w \in \{0,1\}^n$. In the following matrices, we regard $\det A$, which is really an element of $(\mathbb{Z}/4)^\times = \{1,-1\}$, as being either $+1$ or -1 in $\mathbb{C}$, and $\sqrt{\det A}$ is either 1 or i.

$$\sigma\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \colon e_w \mapsto \mathrm{i}^{w^tBw} e_w$$
$$\sigma\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \colon e_w \mapsto \sqrt{\det A}\; e_{(A^t)^{-1}w}$$
$$\sigma\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \colon e_w \mapsto \frac{1}{(1-\mathrm{i})^g}\sum_{w'}(-1)^{w^tw'}e_{w'}.$$

Note that $\mathsf{Sp}(2g,\mathbb{Z})$ is generated by these elements, but it is not at all obvious that the relations in $\mathsf{Sp}(2g,\mathbb{Z})$ hold up to sign for the linear transformations listed here; this is proved in [1]. Note also that in the first formula above, the matrix B may be interpreted as having diagonal entries in $\mathbb{Z}/4$ and off-diagonal entries in $\mathbb{Z}/2$, so that it represents a quadratic form on $(\mathbb{Z}/2)^g$, taking values in $\mathbb{Z}/4$.

Further references for the representation described here include Funar and Pitsch [9], Glasby [10], Gocho [11, 12], Nebe, Rains and Sloane [23], Runge [27, 28, 29], and Tsushima [35].

4. Signature modulo eight

Given an oriented surface bundle over a surface $\Sigma_g \to M \to \Sigma_h$, recall that we had an associated map $\chi\colon \pi_1(\Sigma_h) \to \mathsf{Sp}(2g,\mathbb{Z})$. Composing with $\sigma\colon \mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])/\{\pm I\}$, we obtain a map

$$\phi\colon \pi_1(\Sigma_h) = \langle a_1, b_1, \dots, a_h, b_h \mid [a_1,b_1]\dots[a_h,b_h] = 1\rangle \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])/\{\pm I\}.$$

Now the commutators $[\phi(a_i),\phi(b_i)]$ are well defined in $\mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])$, since changing the sign on $\phi(a_i)$ or $\phi(b_i)$ changes the sign twice in the commutator. Since the product of the commutators is in the kernel of ϕ, we have

$$[\phi(a_1),\phi(b_1)]\dots[\phi(a_h),\phi(b_h)] = \pm I \in \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}]).$$

Theorem 4.1. *We have*

$$[\phi(a_1),\phi(b_1)]\dots[\phi(a_h),\phi(b_h)] = \begin{cases} I & \text{iff signature}(M) \equiv 0 \pmod 8 \\ -I & \text{iff signature}(M) \equiv 4 \pmod 8). \end{cases}$$

Remarks 4.2. (1) As a method of computation, this theorem is not very useful, because of the large size of the matrices involved. Endo [8] provided a much more efficient and purely algebraic method for computing the signature, and not just modulo eight. On the other hand, there are consequences of the theorem that are not very apparent from the point of view of Endo's method.

(2) The following is a consequence of the theta function point of view, and will be discussed in a separate paper [1]. Let $\mathsf{Sp}^{\mathfrak{q}}(2g,\mathbb{Z})$ be the theta subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$. If the image of χ lies in $\mathsf{Sp}^{\mathfrak{q}}(2g,\mathbb{Z})$ then we have signature$(M) \equiv 0 \pmod 8$. In particular, this holds if the action of $\pi_1(\Sigma_h)$ on $H^1(\Sigma_g,\mathbb{Z}/2)$ is trivial. This proves a special case of the Klaus–Teichner conjecture; see the introduction to [16] for details.

(3) Consider next the subgroup consisting of the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{Sp}(2g,\mathbb{Z})$ such that the entries of C are even, and of $\mathsf{Diag}(C)$ are divisible by four. If the image of χ lies in this subgroup then again we have signature$(M) \equiv 0 \pmod 8$. This will be proved in [1].

5. Symplectic groups and their Lie algebras

Let R be a commutative ring, and $\mathsf{Sp}(2g,R)$ be the symplectic group of dimension $2g$ over R. Explicitly, this consists of matrices X with entries in R, and satisfying $X^tJX = J$, where J is the symplectic form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I is a $g \times g$ identity matrix. Denoting by V_R a free R-module of rank g, and setting

$$W_R = V_R^* = \mathsf{Hom}_R(V_R, R),$$

the matrices X act on $U_R = V_R \oplus W_R$, preserving the skew-symmetric bilinear form

$$\langle \quad , \quad \rangle \colon U_R \times U_R \to R$$

given by

$$\langle (v,w),(v',w') \rangle = w'(v) - w(v').$$

For the action of matrices in $\mathsf{Sp}(2g,R)$, we regard (v,w) as a column vector of length $2g$ with entries in R. The skew-symmetric bilinear form induces an isomorphism from U_R to U_R^* sending u to $\langle u, \quad \rangle$. If $R = \mathbb{F}_2$, we shall write U, V and W instead of $U_{\mathbb{F}_2}$, $V_{\mathbb{F}_2}$ and $W_{\mathbb{F}_2}$.

The Lie algebra $\mathfrak{sp}(2g,R)$ consists of matrices Y with entries in R, and satisfying

$$JY + Y^tJ = 0.$$

Thus

$$Y = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & -\mathrm{a}^t \end{pmatrix}$$

where b and c are symmetric. To say that b is symmetric is to say that as an element of

$$\mathsf{Hom}_R(W_R, V_R) \cong V_R \otimes_R V_R$$

it is invariant under the transposition swapping the two tensor factors. Thus b is an element of the divided square $D^2(V_R)$ (which may not be identified with the symmetric square $S^2(U_R)$ unless 2 happens to be invertible in R, which will not be the case for us). Similarly, we have $\mathrm{c} \in D^2(W_R)$ and

$$\mathrm{a} \in \mathsf{Hom}_R(V_R, V_R) \cong V_R \otimes_R W_R.$$

Putting this together, we see that

$$Y \in D^2(V_R) \oplus D^2(W_R) \oplus (V_R \otimes_R W_R) \cong D^2(U_R).$$

Thus, as a module for $\mathsf{Sp}(2g, R)$, we have identified the Lie algebra $\mathfrak{sp}(2g, R)$ with the divided square of the natural module. More abstractly, if $u \in U_R$ then the symmetric tensor $u \otimes u$ is identified with the endomorphism sending x to $\langle u, x\rangle u$. Polarising, this identifies $u \otimes u' + u' \otimes u$ with the endomorphism of U_R sending x to $\langle u, x\rangle u' + \langle u', x\rangle u$. We have therefore proved the following.

Theorem 5.1. *For any commutative ring R, we have isomorphisms*

$$\mathfrak{sp}(2g, R) \cong D^2(U_R) \cong R^{g(2g+1)}.$$

The first isomorphism is an isomorphism of $\mathsf{Sp}(2g, R)$-modules, while the second is an isomorphism of R-modules.

We are interested in the group $\mathsf{Sp}(2g, \mathbb{Z}/4)$. This sits in a short exact sequence

$$1 \to \mathfrak{sp}(2g, \mathbb{F}_2) \to \mathsf{Sp}(2g, \mathbb{Z}/4) \to \mathsf{Sp}(2g, 2) \to 1.$$

The elementary abelian 2-subgroup is identified with $\mathfrak{sp}(2g, \mathbb{F}_2)$, and consists of the matrices $I + 2Y$ with $Y \in \mathfrak{sp}(2g, \mathbb{F}_2)$. These have the form

$$\begin{pmatrix} I + 2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I - 2\mathrm{a}^t \end{pmatrix}$$

with b and c symmetric. We have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^2(U) & \longrightarrow & D^2(U) & \longrightarrow & U & \longrightarrow & 0 \\ & & & & \downarrow \cong & & & & \\ & & & & \mathfrak{sp}(2g, \mathbb{F}_2) & & & & \end{array}$$

where $\Lambda^2(U)$ is spanned by elements of the form $u \otimes u' + u' \otimes u$. As a submodule of $\mathfrak{sp}(2g, \mathbb{F}_2)$, this consists of the matrices where $\mathsf{Diag}(\mathrm{b}) = \mathsf{Diag}(\mathrm{c}) = 0$. The quotient U corresponds to the diagonal entries in b and c. Thus the above short exact sequence can be thought of as a short exact sequence of groups

$$1 \to \Gamma(2g, 2, 4)/\Gamma(2g, 4) \to \Gamma(2g, 2)/\Gamma(2g, 4) \to \Gamma(2g, 2)/\Gamma(2g, 2, 4) \to 1.$$

More generally, we have short exact sequences

$$1 \to \mathfrak{sp}(2g, \mathbb{F}_2) \to \mathsf{Sp}(2g, \mathbb{Z}/2^{n+1}) \to \mathsf{Sp}(2g, \mathbb{Z}/2^n) \to 1$$

and

$$1 \to \Gamma(2g, 2^n, 2^{n+1})/\Gamma(2g, 2^{n+1}) \to \Gamma(2g, 2^n)/\Gamma(2g, 2^{n+1}) \to \Gamma(2g, 2^n)/\Gamma(2g, 2^n, 2^{n+1}) \to 1.$$

Proposition 5.2. *As modules over $\mathsf{Sp}(2g, 2)$, for $g \geqslant 1$ and $n \geqslant 1$ we have*

$$\Gamma(2g, 2^n)/\Gamma(2g, 2^n, 2^{n+1}) \cong U, \qquad \Gamma(2g, 2^n, 2^{n+1})/\Gamma(2g, 2^{n+1}) \cong \Lambda^2(U).$$

Now the symplectic form on U gives us a map $\Lambda^2(U) \to \mathbb{F}_2$, which sends $u \otimes u' + u' \otimes u$ to $\langle u, u' \rangle$. We write Y for the kernel of this map, and we write Z for $D^2(U)/Y$, an $\mathbb{F}_2$-vector space of dimension $2g+1$. Putting these together, we have the following diagram of modules.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & Y & = & Y & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda^2(U) & \longrightarrow & D^2(U) & \longrightarrow & U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{F}_2 & \longrightarrow & Z & \longrightarrow & U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

We claim that the symplectic form on U lifts to a non-degenerate orthogonal form on Z, invariant under $\mathsf{Sp}(2g, 2)$. The quadratic form $Z \to \mathbb{F}_2$ is given by

$$\mathfrak{q}(u \otimes u) = 0, \quad \mathfrak{q}(u \otimes u' + u' \otimes u) = \langle u, u' \rangle,$$

and the associated symmetric bilinear form is

$$\begin{gathered} \mathfrak{b}(u \otimes u, u' \otimes u') = \langle u, u' \rangle, \\ \mathfrak{b}(u \otimes u' + u' \otimes u, u'' \otimes u'') = 0, \\ \mathfrak{b}(u \otimes u' + u' \otimes u, u'' \otimes u''' + u''' \otimes u'') = 0. \end{gathered}$$

A priori, these are a quadratic form and associated bilinear form on $D^2(U)$. But they clearly vanish identically on Y, and define a non-degenerate but singular quadratic form and associated bilinear form on Z. These are invariant under $\mathsf{Sp}(2g, 2)$, which is therefore the orthogonal group on $Z \cong \mathbb{F}_2{}^{2g+1}$, displaying the isomorphism

$$\mathsf{Sp}(2g, 2) \cong \mathsf{O}(2g+1, 2).$$

Remark 5.3. Translating back from $D^2(U)$ to $\mathfrak{sp}(2g, \mathbb{F}_2)$, the quadratic and bilinear form are given as follows:

$$\begin{gathered} \mathfrak{q}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & -\mathrm{a}^t \end{pmatrix} = \mathsf{Tr}(\mathrm{a}) + \langle \mathsf{Diag}(\mathrm{b}), \mathsf{Diag}(\mathrm{c}) \rangle \\ \mathfrak{b}\left(\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & -\mathrm{a}^t \end{pmatrix}, \begin{pmatrix} \mathrm{a}' & \mathrm{b}' \\ \mathrm{c}' & -\mathrm{a}'^t \end{pmatrix} \right) = \langle \mathsf{Diag}(\mathrm{b}), \mathsf{Diag}(\mathrm{c}') \rangle + \langle \mathsf{Diag}(\mathrm{b}'), \mathsf{Diag}(\mathrm{c}) \rangle. \end{gathered}$$

Here, the pointy brackets denote the standard inner product on $\mathbb{F}_2{}^g$ given by multiplying corresponding coordinates and summing.

The normal subgroup $\mathfrak{K}$ described in Section 2 is the inverse image of

$$Y \leqslant \mathfrak{sp}(2g, \mathbb{F}_2) \leqslant \mathsf{Sp}(2g, \mathbb{Z}/4)$$

under the quotient map $\mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{Sp}(2g,\mathbb{Z}/4)$. Thus there is a short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g,2) \to 1$$

and the subgroup $Z \cong (\mathbb{Z}/2)^{2g+1}$ may be viewed as the orthogonal module ${\mathbb{F}_2}^{2g+1}$ for $\mathsf{Sp}(2g,2)$ via conjugation.

Remark 5.4. The submodule structure of the $\mathsf{Sp}(2g,2)$-modules $\Lambda^2(U)$ of dimension $g(2g-1)$ and $D^2(U) \cong \mathfrak{sp}(2g,\mathbb{F}_2)$ of dimension $g(2g+1)$ can be described explicitly as follows (see also Hiss [18]). There is a map $\Lambda^2(U) \to \mathbb{F}_2$ corresponding to the symplectic form, given by

$$u \otimes u' + u' \otimes u \mapsto \langle u, u' \rangle.$$

There is a dual map $\mathbb{F}_2 \to \Lambda^2(U)$ coming from the fact that the representation $\Lambda^2(U)$ is self-dual. In terms of the natural bases $v_1, \dots, v_g$ of V and $w_1, \dots, w_g$ of W, this is given by

$$1 \mapsto \sum_i (v_i \otimes w_i + w_i \otimes v_i).$$

If $g = 1$ then $\Lambda^2(U) \cong \mathbb{F}_2$ is one dimensional, $Y = 0$, and $Z = D^2(U)$ decomposes as a direct sum $\mathbb{F}_2 \oplus U$.

If $g \geqslant 2$ is even then the composite $\mathbb{F}_2 \to \Lambda^2(U) \to \mathbb{F}_2$ is zero, and the quotient of the kernel by the image is a simple module S of dimension $g(2g-1)-2$. Thus $\Lambda^2(U)$ is uniserial (i.e., it has a unique composition series) with composition factors $\mathbb{F}_2$, S, $\mathbb{F}_2$.

If $g \geqslant 3$ is odd, then the composite is non-zero, and $\Lambda^2(U)$ decomposes as a direct sum of a trivial module $\mathbb{F}_2$ and a simple module S of dimension $g(2g-1)-1$.

In both cases with $g \geqslant 2$, $D^2(U)$ has $\Lambda^2(U)$ as its unique maximal submodule. We can therefore draw diagrams for the structure of $D^2(U) \cong \mathfrak{sp}(2g,\mathbb{F}_2)$ as follows.

$$g=1: \quad \mathbb{F}_2 \oplus U \qquad\qquad g \geqslant 2 \text{ even}: \begin{array}{c} U \\ | \\ \mathbb{F}_2 \\ | \\ S \\ | \\ \mathbb{F}_2 \end{array} \qquad\qquad g \geqslant 3 \text{ odd}: \begin{array}{ccc} & U & \\ / & & \backslash \\ \mathbb{F}_2 & & S \end{array}$$

For $g \geqslant 2$, the quotient Z of $D^2(U)$ has structure

$$\begin{array}{c} U \\ | \\ \mathbb{F}_2 \end{array}$$

and this is the orthogonal module for $\mathsf{Sp}(2g,2) \cong \mathsf{O}(2g+1,2)$. The submodule Y is S for $g \geqslant 3$ odd, it is a non-split extension

$$0 \to \mathbb{F}_2 \to Y \to S \to 0$$

for $g \geqslant 2$ even, and $Y = 0$ for $g = 1$.

Lemma 5.5. (i) *For $g \geqslant 1$ we have $H_0(\mathsf{Sp}(2g,2), Y) = 0$ and $H_0(\mathsf{Sp}(2g,2), U) = 0$.*

(ii) *For $g \geqslant 2$ we have $H_0(\mathsf{Sp}(2g,2), Z) = 0$ and $H_0(\mathsf{Sp}(2g,2), \mathfrak{sp}(2g,\mathbb{F}_2)) = 0$.*

Proof. This follows immediately from the structure of Y, U, Z, and $\mathfrak{sp}(2g, \mathbb{F}_2)$ as $\mathsf{Sp}(2g, 2)$-modules given in the above remark, since these modules admit no non-trivial homomorphisms to $\mathbb{F}_2$ with trivial action. ☐

6. Computations in degree two homology and cohomology

Lemma 6.1. (i) *We have $H_2(\mathsf{Sp}(2g, 2)) = 0$ for $g \geqslant 4$, and $H_2(\mathsf{Sp}(6, 2)) \cong \mathbb{Z}/2$.*
(ii) *We have $H^2(\mathsf{Sp}(2g, 2), \mathbb{Z}/2) = 0$ for $g \geqslant 4$ and $H^2(\mathsf{Sp}(6, 2), \mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. (i) This is computed in the paper of Steinberg [34].
(ii) This follows from the universal coefficient theorem, since $\mathsf{Sp}(2g, 2)$ is perfect for $g \geqslant 3$. ☐

Lemma 6.2. *For $n \geqslant 1$ and $g \geqslant 2$ we have $H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), H_1(\Gamma(2g, 2^n))) = 0$.*

Proof. Proposition 10.1 of Sato [30] computes $H_1(\Gamma(2g, N))$, the abelianisation of $\Gamma(2g, N)$. Namely, the derived subgroup is $\Gamma(2g, N^2)$ if N is odd, and $\Gamma(2g, N^2, 2N^2)$ if N is even.
Taking $N = 2^n$, it gives

$$H_1(\Gamma(2g, 2^n)) \cong \Gamma(2g, 2^n)/\Gamma(2g, 2^{2n}, 2^{2n+1}).$$

As modules over $\mathsf{Sp}(2g, \mathbb{Z}/2^n)$ we have

$$\Gamma(2g, 2^n)/\Gamma(2g, 2^{2n}) \cong \mathfrak{sp}(2g, \mathbb{Z}/2^n)$$
$$\Gamma(2g, 2^{2n})/\Gamma(2g, 2^{2n}, 2^{2n+1}) \cong U$$

(cf. Section 5). This gives us a short exact sequence

$$0 \to U \to H_1(\Gamma(2g, 2^n)) \to \mathfrak{sp}(2g, \mathbb{Z}/2^n) \to 0. \tag{6.3}$$

We also have short exact sequences

$$0 \to \mathfrak{sp}(2g, \mathbb{F}_2) \to \mathfrak{sp}(2g, \mathbb{Z}/2^n) \to \mathfrak{sp}(2g, \mathbb{Z}/2^{n-1}) \to 0.$$

By Lemma 5.5 (ii), for $g \geqslant 2$ we have

$$H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), \mathfrak{sp}(2g, \mathbb{F}_2)) = H_0(\mathsf{Sp}(2g, 2), \mathfrak{sp}(2g, \mathbb{F}_2)) = 0$$

and so by induction on n and right exactness of H_0, we have

$$H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), \mathfrak{sp}(2g, \mathbb{Z}/2^n)) = 0.$$

Finally, by Lemma 5.5 (i) we have

$$H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), U) = H_0(\mathsf{Sp}(2g, 2), U) = 0.$$

Therefore, using right exactness of H_0 on the sequence (6.3), the lemma is proved. ☐

Proposition 6.4. *For $n \geqslant 1$ and $g \geqslant 2$,*
(i) *the map $H_2(\mathsf{Sp}(2g, \mathbb{Z})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n))$ is surjective, and*
(ii) *the map $H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^{n+1})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n))$ is surjective.*

Proof. (i) The short exact sequence

$$1 \to \Gamma(2g, 2^n) \to \mathsf{Sp}(2g, \mathbb{Z}) \to \mathsf{Sp}(2g, \mathbb{Z}/2^n) \to 1$$

gives rise to a five term sequence in homology

$$H_2(\mathsf{Sp}(2g, \mathbb{Z})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n)) \to H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), H_1(\Gamma(2g, 2^n)) \\ \to H_1(\mathsf{Sp}(2g, \mathbb{Z})) \to H_1(\mathsf{Sp}(2g, \mathbb{Z}/2^n)) \to 0.$$

The proposition therefore follows immediately from Lemma 6.2.

(ii) This is similar, using the observation that $H_1(\Gamma(2g, 2^n)/\Gamma(2g, 2^{n+1})) \cong \mathfrak{sp}(2g, \mathbb{F}_2)$, so that by Lemma 5.5 (ii) we have

$$H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), H_1(\Gamma(2g, 2^n)/\Gamma(2g, 2^{n+1}))) = 0.$$ □

Corollary 6.5. *For $n \geqslant 1$ and $g \geqslant 3$, the map*

$$H^2(\mathsf{Sp}(2g, \mathbb{Z}/2^n), A) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}), A)$$

is injective for any abelian group of coefficients A with trivial action.

Proof. This follows directly from Proposition 6.4 together with the universal coefficient theorem for cohomology, as the groups $\mathsf{Sp}(2g, \mathbb{Z})$ and $\mathsf{Sp}(2g, \mathbb{Z}/2^n)$ are perfect for $g \geqslant 3$. □

Proposition 6.6. *For $g \geqslant 2$, the maps $H_2(\mathsf{Sp}(2g, \mathbb{Z}/4)) \to H_2(\mathfrak{H}) \to H_2(\mathsf{Sp}(2g, 2))$ are surjective.*

Proof. For the first map, we use the five term sequence for the short exact sequence

$$1 \to Y \to \mathsf{Sp}(2g, \mathbb{Z}/4) \to \mathfrak{H} \to 1,$$

and the computation

$$H_0(\mathfrak{H}, H_1(Y)) = H_0(\mathsf{Sp}(2g, 2), Y) = 0$$

given in Lemma 5.5. Note that Y is an elementary abelian 2-group, so $H_1(Y) \cong Y$.

The computation for the second map is similar, using the short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g, 2) \to 1$$

and the computation $H_0(\mathsf{Sp}(2g, 2), Z) = 0$ given in Lemma 5.5. □

Corollary 6.7. *For $g \geqslant 3$, the inflation map $H^2(\mathfrak{H}, A) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}/4), A)$ is injective for any abelian group of coefficients A with trivial action.*

Proof. This follows directly from Proposition 6.6 and the universal coefficient theorem for cohomology, as the groups $\mathsf{Sp}(2g, \mathbb{Z}/4)$ are perfect for $g \geqslant 3$, hence all their quotients are perfect as well. □

Proposition 6.8. *For $g \geqslant 4$ the group $\mathfrak{H} = \mathsf{Sp}(2g, \mathbb{Z})/\mathfrak{K} \cong \mathsf{Sp}(2g, \mathbb{Z}/4)/Y$ is isomorphic to the quotient $\bar{H}_0$ of the group H_0 of Griess, described in Section 3, by its central subgroup of order two.*

Proof. Examine the extension

$$1 \to \Gamma(2g,2)/\Gamma(2g,2,4) \to \mathsf{Sp}(2g,\mathbb{Z}/4)/\Gamma(2g,2,4) \to \mathsf{Sp}(2g,2) \to 1. \tag{6.9}$$

This is non-split, since the element of order two in $\mathsf{Sp}(2g,2)$ which swaps the first basis vectors of L and L^* and fixes the remaining basis vectors does not lift to an element of order two in $\mathsf{Sp}(2g,\mathbb{Z}/4)/\Gamma(2g,2,4)$.

Let E be the almost extraspecial group $O_2(H_0)$ of shape $2^{1+(2g+1)}$. The action of $\mathsf{Sp}(2g,2)$ on $\Gamma(2g,2)/\Gamma(2g,2,4) \cong U$ is the same as the action of $\mathsf{Out}(E)'$ on $\mathsf{Inn}(E)$ (see (3.2)), namely the natural symplectic module. It follows from the main theorem of Dempwolff [6] that

$$H^2(\mathsf{Sp}(2g,2),\Gamma(2g,2)/\Gamma(2g,2,4))$$

is one dimensional. Therefore $\mathsf{Sp}(2g,\mathbb{Z}/4)/\Gamma(2g,2,4)$ is isomorphic to the group $\mathsf{Aut}(E)'$.

Since $\Gamma(2g,4) \subseteq \mathfrak{K} \subseteq \Gamma(2g,2,4)$ it follows that the short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g,2) \to 1$$

also does not split. We have $\Gamma(2g,2,4)/\mathfrak{K} \cong \mathbb{Z}/2$, and since $g \geqslant 4$, by Lemma 6.1 we have $H^2(\mathsf{Sp}(2g,2),\mathbb{Z}/2) = 0$. So $H^2(\mathsf{Sp}(2g,2),\Gamma(2g,2,4)/Y) = 0$, and hence $H^2(\mathsf{Sp}(2g,2),Z)$ is at most one dimensional. Since we have a non-split extension (6.9), it is exactly one dimensional. The modules $E/[E,E]$ and Z for $\mathsf{Sp}(2g,2)$ are both isomorphic to the natural orthogonal module of dimension $2g+1$, so it follows that $\mathfrak{H}$ is isomorphic to $\bar{H}_0$. □

Remark 6.10. In the case $g = 3$, Proposition 6.8 is still true, but needs a bit more work. The group $H^2(\mathsf{Sp}(6,2),\mathbb{Z}/2)$ is one dimensional by Lemma 6.1, and we are left with the nasty possibility that $\mathfrak{H} = \mathsf{Sp}(6,\mathbb{Z}/4)/Y$ is isomorphic to a quotient of the pullback of $\bar{H}_0 \to \mathsf{Sp}(6,2)$ and $\widetilde{\mathsf{Sp}(6,2)} \to \mathsf{Sp}(6,2)$ by the diagonal central element of order two. In order to prove that $\mathfrak{H}$ is really isomorphic to $\bar{H}_0$ and not this other group, it suffices to construct a matrix representation of a double cover of $\mathfrak{H}$ of dimension eight. Explicit matrices for this representation were given in Section 3. On the other hand, the smallest faithful irreducible complex representation in the case of the other possibility has dimension 64. It is worth noticing, though, that it does not matter which possibility is true, if we just wish to prove the next theorem.

Lemma 6.11. *We have $H_2(\mathsf{Sp}(2g,\mathbb{Z})) = \mathbb{Z}$ for $g \geqslant 4$ and $\mathbb{Z} \oplus \mathbb{Z}/2$ for $g = 3$.*

Proof. See for example Stein [33], Theorem 2.2 for $g = 3$ and Theorem 5.3 and Remark 5 following Corollary 5.5 for $g \geqslant 4$. See also Theorem 5.1 of Putman [25] for $g \geqslant 4$. □

Theorem 6.12. *For $g \geqslant 3$ we have $H_1(\mathfrak{H}) = 0$. For $g \geqslant 4$ we have $H_2(\mathfrak{H}) = \mathbb{Z}/2$, and for $g = 3$ we have $H_2(\mathfrak{H}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The map*

$$H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n)) \to H_2(\mathfrak{H})$$

is an isomorphism for $n \geqslant 2$.

Proof. The computation of the abelianisation $H_1(\mathfrak{H})$ is straightforward. It follows from Propositions 6.4 and 6.6 that for $n \geqslant 2$ the maps

$$H_2(\mathsf{Sp}(2g,\mathbb{Z})) \to H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n)) \to H_2(\mathsf{Sp}(2g,\mathbb{Z}/4)) \to H_2(\mathfrak{H}) \to H_2(\mathsf{Sp}(2g,2)) \tag{6.13}$$

are surjective, and from Theorem 1.1 that the kernel of the first map contains every element divisible by two. Consulting Lemma 6.11, we see that $H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n))$ and $H_2(\mathfrak{H})$ are quotients of the groups given.

By Proposition 6.8, there is a non-trivial element of $H_2(\mathfrak{H},\mathbb{Z}/2)$ which is killed by the map to $H_2(\mathsf{Sp}(2g,2))$. Namely, the central extension $\tilde{\mathfrak{H}} \to \mathfrak{H}$ is not inflated from $\mathsf{Sp}(2g,2)$ because the kernel of $\tilde{\mathfrak{H}} \to \mathsf{Sp}(2g,2)$ is the non-abelian group E.

Comparing the value of $H_2(\mathsf{Sp}(2g,\mathbb{Z}))$ given in Lemma 6.11 with the value of $H_2(\mathsf{Sp}(2g,2))$ given in Lemma 6.1, the theorem follows. ☐

Corollary 6.14. *The map*

$$H^2(\mathfrak{H},A) \to H^2(\mathsf{Sp}(2g,\mathbb{Z}/2^n),A)$$

is an isomorphism for $g \geqslant 3$, $n \geqslant 2$, and any abelian group of coefficients A with trivial action.

Proof. This follows from Theorem 6.12 and the universal coefficient theorem. ☐

Summary of homology and cohomology groups

Values for $g \geqslant 4 \begin{cases} \text{even} \\ \text{odd} \end{cases}$ are as follows:

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_g	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8$	$\mathbb{Z}/2$
$\mathsf{Sp}(2g,\mathbb{Z})$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2g,\mathbb{Z})$	0	$\begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 \\ \mathbb{Z} \end{cases}$	$\mathbb{Z}$	$\begin{cases} \mathbb{Z}/8\oplus\mathbb{Z}/2 \\ \mathbb{Z}/8 \end{cases}$	$\begin{cases} \mathbb{Z}/2\oplus\mathbb{Z}/2 \\ \mathbb{Z}/2 \end{cases}$
$\mathsf{Sp}(2g,\mathbb{Z}/4)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2g,\mathbb{Z}/4)$	0	$\begin{cases} \mathbb{Z}/2\oplus\mathbb{Z}/2 \\ \mathbb{Z}/4 \end{cases}$	0	$\begin{cases} \mathbb{Z}/2\oplus\mathbb{Z}/2 \\ \mathbb{Z}/4 \end{cases}$	$\begin{cases} \mathbb{Z}/2\oplus\mathbb{Z}/2 \\ \mathbb{Z}/2 \end{cases}$
$\mathfrak{H}=\mathsf{Sp}(2g,\mathbb{Z}/4)/Y$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(2g,2)$	0	0	0	0	0

Values for $g=3$:

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_3	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathsf{Sp}(6,\mathbb{Z})$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$P\mathsf{Sp}(6,\mathbb{Z})$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(6,\mathbb{Z}/4)$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$P\mathsf{Sp}(6,\mathbb{Z}/4)$	0	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathfrak{H}=\mathsf{Sp}(6,\mathbb{Z}/4)/Y$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathsf{Sp}(6,2)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Values for $g = 2$:

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_2	$\mathbb{Z}/10$	$\mathbb{Z}/2$	$\mathbb{Z}/10$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
$\mathsf{Sp}(4,\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/8\oplus(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$
$P\mathsf{Sp}(4,\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^4$
$\mathsf{Sp}(4,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
$P\mathsf{Sp}(4,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^4$
$\mathfrak{H}=\mathsf{Sp}(4,\mathbb{Z}/4)/Y$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
$\mathsf{Sp}(4,2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$

Values for $g = 1$:

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_1	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$	$\mathbb{Z}/4$	$\mathbb{Z}/2$
$\mathsf{Sp}(2,\mathbb{Z})$	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$	$\mathbb{Z}/4$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2,\mathbb{Z})$	$\mathbb{Z}/6$	0	$\mathbb{Z}/6$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(2,\mathbb{Z}/4)$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/4\oplus\mathbb{Z}/2$
$P\mathsf{Sp}(2,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
$\mathfrak{H}=\mathsf{Sp}(2,\mathbb{Z}/4)/Y$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/4\oplus\mathbb{Z}/2$
$\mathsf{Sp}(2,2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

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