to show that both are generated by p-elementary induction. This reduces the problem of describing $\mathrm{Ker}(\eta_2)$ to the p-group case, which is handled in [4, Section 4], Similarly, this reduces the problem of finding generators for $\mathrm{Ker}[K_2^{top}(A\pi) \to K_2^*(A\pi)]$ to the case where π is a p-group. It should be noted that neither $K_2^{top}(A\pi)$ nor $K_2^*(A\pi)$ is generated by p-elementary induction.

REFERENCES

1. A. Dress, Introduction and structure theorems for orthogonal representations of finite groups, Annals of Math. 102 (1975) 291 - 325

2. R. Oliver, SK_1 for finite group rings: II, Math. Scand. 47 (1980), 195 - 231

3. R. Oliver, SK_1 for finite group rings III, Lecture Notes in Math., vol. 854, Springer Verlag (1981), 299 - 337

4. R. Oliver, Lower bounds for $K_2^{top}(\hat{\mathbb{Z}}_p\pi)$ and $K_2(\mathbb{Z}\pi)$ (to appear)

5. C. T. C. Wall, On the classification of hermetian forms, III: Complete semilocal rings, Invent. Math. 19 (1973), 59 - 71

A RELATION BETWEEN WITT GROUPS AND ZERO-CYCLES IN A REGULAR RING

William Pardon
Department of Mathematics
Duke University
Durham, NC 27706

Let R be a commutative ring with unit, suppose $1/2 \in R$, and le $W(R)$ denote the Witt group of R. This is the Grothendieck group of isometry classes of nonsingular symmetric bilinear froms $\phi: P \times P \to$] where P is finitely-generated projective, modulo the subgroup generated by hyperbolic forms. (It is denoted $W_0^1(R)$ or $W_0^1(CM_0(R))$ in the body of the paper.) The main results of this paper are the follo ing theorems, A and B.

Theorem A. Let A be a regular Noetherian domain containing 1, and let K be its fraction field.

(i) If $\dim A \leq 3$, then the natural map $W(A) \to W(K)$ is injective.

(ii) If $\dim A \leq 4$ and A is local, then $W(A) \to W(K)$ is injective.

It was shown in [P4] that, if in part (ii) of Theorem A it is al assumed that A is essentially of finite type over a field, then the dimension restriction can be removed. This was first proved (for loc ization at a closed point) by Ojanguren ([O2]). He has also proved Theorem A(i) independently and by different methods in [O1].

The question of injectivity of $W(A) \to W(K)$ when A is local, seems first to have been raised by Grothendieck in a more general for which also includes, for example, the theorem of Auslander and Goldma on the injectivity of Brauer groups, $Br(A) \to Br(K)$. An excellent su very of this and related problems has been written by Colliot-Thélène ([C1]; cf. also [CRW] and [K2].)

There is a heuristic, if somewhat technical reason for the upper bound on the dimension of the rings appearing in Theorem A. It is th if something becomes trivial at the generic point (of $Spec(A)$), then it ought to lift back to codimension one. It turns out that the resu ing thing is trivial at all generic points of its support, so lifts back to codimension two. The process can be continued until codimens four is reached, at whose generic points symmetric bilinear forms app again. These may be non-hyperbolic (i.e., non-zero in a Witt group), impeding further progress.

This informal explanation is made precise in (2.4). Although I don't know what happens in Theorem A(ii) for arbitrary 5-dimensional regular local rings, the co-ordinate ring A of the real variety $S^2 \times S^2$ is an example for which $W(A) \to W(K)$ is not injective ([KOS]), showing Theorem A(i) is false in dimension four. The following theorem implies there aren't any such four-dimensional examples over $\mathbb{C}$ and that the group of zero-cycles mod rational equivalence, $A_0(-)$, accounts for the examples over $\mathbb{R}$ (at least modulo odd torsion). For its full statement we need the Witt group $W^{-1}(A)$ of skew-symmetric forms (denoted $W_0^{-1}(A)$ or $W_0^{-1}(CM_0(A))$ in the body of the paper). Its definition is the obvious modification of that of $W(A)$.

Theorem B. (i) Let A be a regular Noetherian domain, of dimension four and of finite type over a field k, char $k \neq 2$. If k is algebraically or real closed, then there is a surjection of abelian groups,

$$A_0(\mathrm{Spec}\ A) \otimes \mathbb{Z}/2 \to \ker(W(A) \to W(K)) \otimes \mathbb{Z}/2$$

If the group $C_3(A) \otimes \mathbb{Z}/2 = 0$ (defined in [CF]), then this is an isomorphism.

(ii) If A is as in (i) but is 2-dimensional, then there is a surjection

$$A_0(\mathrm{Spec}\ A) \otimes \mathbb{Z}/2 \to W^{-1}(A) \otimes \mathbb{Z}/2$$

If $\mathrm{Pic}(A) \otimes \mathbb{Z}/2 = 0$, then this is an isomorphism.

For instance, if $k = \mathbb{C}$ then $A_0(\mathrm{Spec}\ A)$ is divisible, so $A_0(\mathrm{Spec}\ A) \otimes \mathbb{Z}/2 = 0$. On the other hand, according to [CI], $A_0(\mathrm{Spec}\ A) \otimes \mathbb{Z}/2 = (\mathbb{Z}/2)^r$ when $k = \mathbb{R}$; here r is (in some cases) the number of compact topological components of the real algebraic 4-manifold defined by A. (This is the explanation for the above example where A is the co-ordinate ring of $S^2 \times S^2$.) A precise statement can be found in (3.2).

A result like Theorem B, connecting Witt groups to geometric invariants, is the motivation for a program to study quadratic forms on rings of dimension higher than one (the one-dimensional case being that of classical arithmetic interest). Indeed, the techniques of this paper can be globalized to apply to forms on schemes, for which the foundations have been laid in [K3]. For example, using the methods here, it can be shown that $W(X)$ is a birational invariant of smooth projective surfaces X over C. This was suggested by Colliot-Thélène and Sansuc in [CS]. The papers [C2], [C3] and [CS] present another approach to the study of quadratic forms over geometric rings.

In a more algebraic vein, it is shown in §4 below that a construction of Serre and Horrocks (used by Horrocks to find indecomposable

bundles on $\mathbb{P}^3$) is closely related to the surjection in part (ii) Theorem B. The consequence ((4.5)) is a fairly explicit set of gene tors of $W^{-1}(A) \otimes \mathbb{Z}/2$.

As has already been suggested, the method of proof of Theorem A to use localization sequences, comparing what happens in adjacent cc dimensions. A template both for this and Theorem B exists already i [O], following ideas of Gersten [G] and Claborn-Fossum [CF]. Howeve the technical details are necessarily different. For instance, one must first find a suitable value group V_p for forms defined on moc supported in codimension p. And not every such M can support a r singular form $M \times M \to V_p$: M is at least reflexive in the sense that the natural map $M \to \mathrm{Hom}(M, \mathrm{Hom}(M, V_p))$ is an isomorphism. (I was le to some of these ideas by surgery theory and, to find a suitable V_p by [Ba].)

In §1 we make choices of the value groups V_p and of the categ of codimension p modules M which support non-singular forms. (I [P4] a more refined choice is used in a proof of the "Gersten conjec ture" for Witt groups over rings of geometric type.). We then defir the Witt groups ((1.7) and (1.12f)) and give some of their propertie

In §2 we define the maps in the localization sequences ((2.1)) prove Theorem A(i) in (2.3)(a), Theorem A(ii) in (2.5). The definit are needed for the proof of Theorem B and the discussion of §4, but more technical parts of the proof of exactness are deferred to §§5-8

In §3 Theorem B is proved and the necessary computations of A_0 $\otimes \mathbb{Z}/2$ cited. In §4 the connection between the Serre-Horrocks construction and Theorem B is made. It is also pointed out how Theorem (ii) answers a question of Kustin and Miller about algebra structure on resolutions of codimension four Gorenstein ideals in a regular lc ring.

§5 is devoted to a proof of the "Dévissage" theorem for Witt gr In §§6-8 the notion of a Poincaré complex is introduced and its rela to the Witt group is carried far enough for the applications in §8 (exactness in the localization sequences of (2.1)). This theory is due ess tially to Ranicki and the exposition attempts to familiarize the rea with the considerable simplification in [R1] and [R2] brought about the assumption that 1/2 is in the ground ring. (I am informed tha a program using Poincaré complexes and localization sequences to prc Theorem A(i) is being carried out jointly by Barge, Sansuc and Vogel

Here are the chapter headings.

Conventions

In this paper, dim refers to Krull dimension. If A is a ring and $p \subseteq A$ is a prime ideal, then ht p:= dim A_p; for any ideal I, ht I = min{ht$p$ | $p \supseteq$ I}. If M is an A-module, dim M:= max{dimA/p | $p \in$ Supp(M)} and ht M = min{ht p | $p \in$ Supp(M)}. (There is the usual inconsistency between the two possible definitions of ht I, I an ideal; this should cause no confusion.)

If M is an A-module, an M-regular sequence $\{x_1,\dots,x_n\} \subseteq A$ satisfies (i) $(x_1,\dots,x_n)M \neq M$ and (ii) x_i is not a zero-divisor on $M/(x_1,\dots,x_{i-1})M$, $i = 1,\dots,n$. ([K, p. 84]). If M is an A-module and $I \subseteq A$ an ideal, then $\mathrm{depth}_I(M)$ = max{n | there is an M-regular sequence $\{x_1,\dots,x_n\} \subseteq I$}; depth M means $\mathrm{depth}_A(M)$. A <u>Cohen-Macaulay ring</u>, or CM ring, is finite-dimensional Macaulay in the sense of [N , p. 82]: it is Noetherian, finite-dimensional, and dim A_m = dim A = depth A_m for all maximal $m \subseteq A$. A <u>CM-module</u> M satisfies dim M_p = depth M_p, for all primes p, or is the zero module.

Finally we make the following convention: All rings will be commutative with unit, contain 1/2, and will be CM.

§1. <u>Witt groups in the category of Cohen-Macaulay modules</u>.

Let A be a commutative CM ring of Krull dimension n.

(1.1) <u>Definition</u>. For each $p \geq 0$, $CM_p(A)$ denotes the categor of finitely-generated CM-modules of height p, together with the zero module: If $M \neq 0$, then ht(M) = p and for every $p \in$ Supp(M), depth(M_p) = ht(p) - ht(M_p) = dim M_p.

When A is contextually specified we write CM_p for $CM_p(A)$.

Here are some simple examples.

(1.2) <u>Proposition</u>. a) If $M \in CM_p$, it is height-unmixed. b) $M \in CM_n$ if and only if M has finite length. c) If $\{x_1,\dots,x_p\} \subseteq A$ is an A-regular sequence, then $A/(x_1,\dots,x_p) \in CM_p$. d) Let $M_1 \rightarrowtail M_2 \twoheadrightarrow M_3$ be a short exact sequence of finitely-generated A-modul

(i) If $M_1, M_3 \in CM_p$, then $M_2 \in CM_p$

(ii) If $M_2, M_3 \in CM_p$, then $M_1 \in CM_p$.

(iii) If $M_1, M_2 \in CM_p$ and ht $M_3 > p$, then $M_3 \in CM_{p+1}$.

<u>Proof</u>: a) [K , Thm. 141]. b) is clear from our assumption that dim $A_M = n$ for all maximal M. c) [M , Thm. 30(i)]. d) [K , p. 10: Ex. 14].

(1.3) From now on let A be a Gorenstein ring of dimension n such that dim $A_M = n$ for all maximal M. Let

$$0 \to A \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} 0$$

be the minimal injective resolution of A over itself, so that $E_k = \coprod_{\mathrm{ht}\, q = k} E(A/q) = \coprod_{\mathrm{ht}\, q = k} E(A_q/q A_q)$ where E denotes injective hull (See [B , §1]). Let

$$V_k(A) = \ker d_k, \quad 0 \leq k \leq n .$$

Two useful properties of the injective resolution are recorded here for later use. They follow from [B, (2.2)] and [B, (2.5)], respectively.

(1.4) For each prime $p \subseteq A$ and k, $0 \leq k \leq n$, there are isomorphisms $E_k(A)_p \xrightarrow{\cong} E_k(A_p)$ commuting with the d_i's hence

$$V_k(A)_p \xrightarrow{\cong} V_k(A_p).$$

(1.5) If $x \in A$ is a non-zero-divisor on A then there are isomorphisms $E_k(A/(x)) \overset{\approx}{\to} (0:x) \subseteq E_{k+1}(A)$ commuting with the d_i's, $0 \leq k \leq n-1$, hence

$$V_k(A/(x)) \overset{\approx}{\to} (0:x) \subseteq V_{k+1}(A).$$

The next result states some simple, mostly well-known properties of CM_p. Note in particular that the non-singularity asserted in part a) is essentially the same as the isomorphism

$$M \simeq \mathrm{Ext}^p(\mathrm{Ext}^p(M,A)A)$$

proved in [AB,(4.35)].

(1.6) Proposition. a) If $M \in CM_p$, then $\mathrm{Hom}(M,V_p) \in CM_p$, $\mathrm{Ass}(M) = \mathrm{Ass}(\mathrm{Hom}(M,V_p))$ and the natural pairing

$$\nu\colon \mathrm{Hom}(M,V_p) \times M \to V_p, \quad \nu(f,m) = f(m)$$

is non-singular (both adjoints are isomorphisms).

b) If $N \in CM_{p+1}$, then there are $M_0, M_1 \in CM_p$ and a short exact sequence

$$M_1 \overset{\alpha}{\rightarrowtail} M_0 \twoheadrightarrow N.$$

Given any such exact sequence, there is a short exact sequence

$$M_0^\wedge \overset{\alpha^\wedge}{\rightarrowtail} M_1^\wedge \twoheadrightarrow N^\sim,$$

where $(-)^\wedge := \mathrm{Hom}(-,V_p)$ and $(-)^\sim = \mathrm{Hom}(-,V_{p+1})$. M_0 may be assumed to be of the form $(A/(x_1,\ldots,x_p))^m$ where $\{x_1,\ldots,x_p\}$ is an A-regular sequence.

c) If $M_1 \overset{\alpha}{\rightarrowtail} M_2 \overset{\beta}{\twoheadrightarrow} M_3$ is short exact in CM_p,

$$M_3^\wedge \overset{\beta^\wedge}{\rightarrowtail} M_2^\wedge \overset{\alpha^\wedge}{\twoheadrightarrow} M_1^\wedge$$

is short exact in CM_p, where $M_i^\wedge = \mathrm{Hom}(M_i,V_p)$.

Proof: We begin with b). Let $p \geq 0$ and $N \in CM_{p+1}$. Then since $\mathrm{ht}(\mathrm{Ann}(N)) = p + 1$, there is by [K , Thm. 136] an A-regular sequence $\{x_1,\ldots,x_{p+1}\} \subseteq \mathrm{Ann}(N)$. Thus, there is a surjection j for some m, with kernel M_1,

$$M_1 \rightarrowtail (R/(x_1,\ldots,x_p))^m \overset{j}{\twoheadrightarrow} N.$$

Then $M_0 := (R/(x_1,\ldots,x_p))^m \in CM_p$ by (1.2) and $M_1 \in CM_p$ by [K , Ex. 14, p. 103].

From $V_p \rightarrowtail E_p \twoheadrightarrow V_{p+1}$ we obtain the exact sequence

$$\mathrm{Hom}(N,E_p) \to \mathrm{Hom}(N,V_{p+1}) \to \mathrm{Ext}^1(N,V_p) \to \mathrm{Ext}^1(N,E_p).$$

Since E_p is injective, $\mathrm{Ext}^1(N,E_p) = 0$; on the other hand, $\mathrm{Ass}\,\mathrm{Hom}(N,E_p) = \mathrm{Ass}(E_p) \cap \mathrm{Supp}(N) = \emptyset$. Hence

$$\mathrm{Hom}(N,V_{p+1}) \overset{\approx}{\to} \mathrm{Ext}^1(N,V_p).$$

Next, from $M_1 \rightarrowtail M_0 \twoheadrightarrow N$ we obtain the exact sequence

$$\mathrm{Hom}(N,V_p) \to \mathrm{Hom}(M_0,V_p) \to \mathrm{Hom}(M_1,V_p) \to \mathrm{Ext}^1(N,V_p) \to \mathrm{Ext}^1(M_0,V_p).$$

As above, $\mathrm{Hom}(N,V_p) = 0$. From $V_{p-1} \rightarrowtail E_{p-1} \twoheadrightarrow V_p$ we get $\mathrm{Ext}^1(M_0,V_p) \simeq \mathrm{Ext}^2(M_0,V_{p-1})$; continuing in this way, $\mathrm{Ext}^1(M_0,V_p) \simeq \mathrm{Ext}^{p+1}(M_0,A)$, which vanishes by [K , Thm. 217].

The proof of a) is by induction on p. By [AB, (4.12),(3.8)], if $M \in CM_0$, then $M \overset{\approx}{\to} M^{**}$ $(M^* := \mathrm{Hom}(M,A))$ and $\mathrm{Ext}^i(M^*,A) = 0$, $i > 0$. By [K , p. 163, Ex. 7] $M^* \in CM_0$. This proves a) in case $p = 0$.

Let $p \geq 0$, $\mathrm{Hom}(-,V_p) = (\ )^\wedge$, $\mathrm{Hom}(-,V_{p+1}) = (-)^\sim$, and $N \in CM_{p+1}$. We have $\mathrm{Ass}(N^\sim) = \mathrm{Supp}(N) \cap \mathrm{Ass}(V_{p+1}) = \mathrm{Ass}(N)$, since N is unmixed. Let $M_1 \rightarrowtail M_0 \twoheadrightarrow N$ be a resolution as in b). Then $M_0^\wedge \rightarrowtail M_1^\wedge \twoheadrightarrow N^\sim$ is also exact and $M_0^\wedge, M_1^\wedge \in CM_p$ by induction. So $N^\sim \in CM_{p+1}$ by (1.2)(d).

Finally, consider the diagram of exact sequences

$$\begin{array}{ccccc} M_1 & \rightarrowtail & M_0 & \twoheadrightarrow & N \\ \downarrow\simeq & & \downarrow\simeq & & \\ M_1^{\wedge\wedge} & \rightarrowtail & M_0^{\wedge\wedge} & \twoheadrightarrow & N^{\sim\sim} \end{array}$$

It follows that there is an isomorphism $N \overset{\approx}{\to} N^{\sim\sim}$ which can be shown to be the right adjoint of ν.

c) is immediate from the chain of isomorphisms $\mathrm{Ext}^1(M_3,V_p) \simeq \mathrm{Ext}^2(M_3,V_{p-1}) \simeq \ldots \simeq \mathrm{Ext}^{p+1}(M_3,A) = 0$ by [K , Thm. 215].

(1.7) Now let

$$\Omega^\lambda(CM_p), \quad \lambda = \pm 1$$

denote the category of pairs (M,ϕ) where $M \in CM_p$ and $\phi: M \times M \to V_p$ is a λ-symmetric, A-bilinear form which is non-singular, $\mathrm{Ad}\phi: M \overset{\cong}{\to} \mathrm{Hom}(M,V_p)$. For instance if $N \in CM_p$ and $N^{\wedge} := \mathrm{Hom}(N,V_p)$ then the <u>hyperbolic form</u> $(N \oplus N^{\wedge},\phi_h) \in Q^{\lambda}(CM_p)$ is defined by $\phi_h|N \times N \equiv 0 \equiv \phi_h|N^{\wedge} \times N^{\wedge}$, and $\phi_h|N^{\wedge} \times N$ is the natural pairing ν; it is non-singular by (1.6)(a). More generally, (M,ϕ) is called a <u>lagrangian</u> if there is $N \subseteq M$ such that $\phi|N \times N \equiv 0$, the induced pairing $N \times (M/N) \to V_p$ is non-singular (both adjoints are isomorphisms) and $N, M/N \in CM_p$. N will be called a <u>sublagrangian</u>.

One may add isometry classes of objects in $Q^{\lambda}(CM_p)$ by orthogonal sum and obtain an abelian semi-group. The corresponding Grothendieck group, modulo the subgroup generated by lagrangians is denoted

$$W_0(CM_p).$$

For instance if A is regular, then $CM_0(A)$ is the category of finitely-generated projectives, and we set

(1.8) $$W_0^{\lambda}(CM_0(A)) = W_0^{\lambda}(A),$$

the Witt group referred to in the introduction.

Here is a result describing the typical equivalence of elements in $W_0^{\lambda}(CM_p)$.

(1.9) <u>Proposition</u>. Let $(M,\phi) \in Q^{\lambda}(CM_p)$, $L \subseteq M$, $L \subseteq L^{\perp}$ be given such that M/L and $M/L^{\perp} \in CM_p$. Then $L^{\perp}/L \in CM_p$, and there is an induced form $(L^{\perp}/L,\psi) \in Q^{\lambda}(CM_p)$ such that in $W_0^{\lambda}(CM_p)$.

$$[L^{\perp}/L,\psi] = [M,\phi]$$

<u>Proof</u>: The same as [P1, (3.4),(3.5)].

Next comes the corresponding K_1-functor, which compares two ways of making something in $Q^{\lambda}(CM_p)$ equal zero in $W_0^{\lambda}(CM_p)$.

(1.10) A <u>λ-formation</u> in CM_p is a triple (K,H,Δ), where $K,H \in CM_p$ and $\Delta: K \to H \oplus H^{\wedge}$ is an injection whose image is a sublagrangian of the hyperbolic form $(H \oplus H^{\wedge},\rho_h)$. For instance, if $\rho: K \to K^{\wedge}$ satisfies $\rho + \lambda\rho^{\wedge} = 0$, then $(K,K^{\wedge},(\rho,1))$ is a λ-formation called a <u>graph formation</u>. The collection of λ-formations is denoted

$$F^{\lambda}(CM_p), \quad \lambda = \pm 1$$

(K,H,Δ) and (K',H',Δ') are <u>isomorphic</u> if there are isomorphisms

$A: K \to K'$ and $B: H \to H'$ so that

$$\begin{array}{ccc} K & \overset{\Delta}{\to} & H \oplus H^{\wedge} \\ \downarrow A & & \downarrow B \oplus B^{\wedge -1} \\ K' & \overset{\Delta'}{\to} & H' \oplus H'^{\wedge} \end{array}$$

commutes. The (orthogonal) <u>sum</u> of these λ-formations is $(K \oplus K', H \oplus H', \Delta \oplus \Delta')$; and the zero-formation has $K = (0) = H$.

(1.11) Here are two useful properties of λ-formations (K,H,Δ): If $\Delta = (\alpha,\gamma): K \to H \oplus H^{\wedge}$ then $\gamma^{\wedge}\alpha + \lambda\alpha^{\wedge}\gamma = 0$; and the hyperbolic form on $H \oplus H^{\wedge}$ induces an isomorphism $H \oplus H^{\wedge}/\mathrm{im}(\Delta) \overset{\cong}{\to} K^{\wedge}$, by definition of sublagrangian.

(1.12) Next define two operations on $(K,H,\Delta) \in F^{\lambda}(CM_p)$ as follows.

a) Let $(\varepsilon) = (L \rightarrowtail H_1 \twoheadrightarrow H)$ be an extension in CM_p and K_1 the pullback in

$$\begin{array}{ccccc} L & \rightarrowtail & K_1 & \overset{j_1}{\twoheadrightarrow} & K \\ \downarrow = & & \downarrow \alpha_1 & & \downarrow \alpha \\ L & \rightarrowtail & H_1 & \overset{j}{\twoheadrightarrow} & H \end{array}$$

where $\Delta = (\alpha,\gamma)$. Let $\gamma_1 := j^{\wedge}\gamma j_1$ and $\Delta_1 = (\alpha_1,\gamma_1): K_1 \to H_1 \oplus H_1^{\wedge}$. Then $(K_1,H_1,\Delta_1) \in F^{\lambda}(CM_p)$, and its isomorphism class depends only on that of (K,H,Δ) and the extension (ε). It is denoted

$$\sigma_{\varepsilon}(K,H,\Delta)$$

and is called the <u>stabilization</u> of (K,H,Δ) by (ε).

b) Let $\psi: H \times H \to V_p$ be a $(-\lambda)$-symmetric A-bilinear form and set $\Delta_{\psi} = (\alpha, \gamma + (\mathrm{Ad}\,\psi)\alpha)$. Then $(K,H,\Delta_{\psi}) \in F^{\lambda}(CM_p)$ and is denoted

$$\chi(H,\psi)(K,H,\Delta).$$

It is said to be <u>isometric</u> to (K,H,Δ).

We define

$$W_1^{\lambda}(CM_p)$$

to be the Grothendieck group on isomorphism classes of elements of $F^{\lambda}(CM_p)$, modulo the subgroup generated by elements of the form

$(K,H,\Delta) - \chi(H,\psi)\cdot(K,H,\Delta), (K,H,\Delta) - \sigma_\varepsilon(K,H,\Delta)$, and $(K,K^\wedge,(\rho,1))$.

(1.13) Remarks. a) This is Ranicki's definition [ ], except that e roles of H and $H^\wedge$ are reversed. This makes no difference be- use of (1.19) below. b) When $p = 0$ and A is regular, then CM_0 the category of finitely generated projectives and we write in this se

$$W_1^\lambda(CM_0(A)) = W_1^\lambda(A)$$

$\tilde{K}_0(A) = 0$, then $W_1^\lambda(A)$ is the commutator quotient of the (stabil- ed) group of isometries of $(A^\infty + A^{\wedge\infty}, \phi_h)$ modulo hyperbolic rota- ons (cf. [P1, 1.21 (b)] and matrices w_n^λ ([P1, 1.21 (a)]).

We need to discuss the above relations in $W_1^\lambda(CM_p)$. Clearly, $H, -\psi)\cdot\{\chi(H,\psi)\cdot(K,H,\Delta)\} = (K,H,\Delta)$. We next show how to invert eration (1.12)(b), as well.

(1.14) Given $(K_1,H_1,\Delta_1) \in F^\lambda(CM_p)$, suppose there is $L \in CM_p$ d an inclusion $i_1: L \to K_1$ such that $\gamma_1 i_1 = 0$, where $\Delta_1 = (\alpha_1,\gamma_1)$, d $\operatorname{cok}(i_1), \operatorname{cok}(\alpha_1 i_1) \in CM_p$. Then $\alpha_1 i_1$ is injective (since Δ_1), and we set $K = K_1/i_1(L)$, $H = H_1/\alpha_1 i_1(L)$, and $\alpha: K \to H$ equal to e induced map. As $\Delta_1(L) \subseteq H_1$ we have $\Delta_1(L)^\perp \supseteq H_1^\perp = H_1$. From the perbolic form on $H_1 \oplus H_1^\wedge$ we get a form

$$H \times (\Delta_1(L)^\perp \cap H_1^\wedge) \to V_p$$

ich induces an isomorphism

$$\Delta_1(L)^\perp \cap H_1^\wedge \xrightarrow{\simeq} H^\wedge.$$

nce $\operatorname{im}(\Delta_1) = \operatorname{im}(\Delta_1)^\perp$ and $\gamma_1(L) = 0$, γ_1 induces a map $K \to \Delta_1(L)^\perp$ $H_1^\wedge$, which, composed with the above isomorphism, gives $\gamma: K \to H^\wedge$. ting $\Delta = (\alpha,\gamma)$ it is easily checked that $(K,H,\Delta) \in F^\lambda(CM_p)$, the stabilization of (K_1,H_1,Δ_1) by L. The reason for this terminology the following.

(1.15) Proposition. a) Let (ε) denote the extension $L \overset{\alpha_1 i_1}{\rightarrowtail} H_1$ I in the construction (1.14) above. Then $\sigma_\varepsilon(K,H,\Delta) \simeq (K_1,H_1,\Delta_1)$.

b) Given $(K,H,\Delta) \in F^\lambda(CM_p)$ and an extension $(\varepsilon) = (L \rightarrowtail H_1 \twoheadrightarrow H)$, destabilization of $\sigma_\varepsilon(K,H,\Delta)$ by L is (K,H,Δ).

Thus the operations (1.12)(a) and (b) are reflexive and symmetric. we call the equivalence relation generated by isomorphism plus these rations stable isometry, then $W_1^\lambda(CM_p)$ is the Grothendieck group on

stable isometry classes of λ-formations, modulo the subgroup generated by graph formations.

The next proposition allows us to represent elements of $W_1^\lambda(CM_p)$ by formations instead of by their formal differences. It is analogous to the fact that if $(M,\phi) \in Q^\lambda(CM_p)$, then $(M,\phi) \oplus (M,-\phi)$ is hyper- bolic.

(1.16) Proposition. Let $(K,H,(\alpha,\gamma)) \in F^\lambda(CM_p)$ be given. Then the formation

$$(K,H,(\alpha,\gamma)) \oplus (K,H^\wedge,(\gamma,-\lambda\alpha))$$

is stably isometric to a graph formation

$$(K,K^\wedge,(\rho,1)).$$

Proof: Let $\theta_1 = (K_1,H_1,(\alpha_1,\gamma_1))$, $\theta_2 = (K_2,H_2^\wedge,(\gamma_2,-\lambda\alpha_2))$ where $K_i = K$, $H_i = H$, $\alpha_i = \alpha$, $\gamma_i = \gamma$. Then $\theta_1 \oplus \theta_2 = (K_1 \oplus K_2, H_1 \oplus H_2^\wedge,$ $(\alpha_1 \oplus \gamma_2, \gamma_1 \oplus -\lambda\alpha_2))$. Define the $(-\lambda)$-symmetric form

$$\psi: (H_1 \oplus H_2^\wedge) \times (H_1 \oplus H_2^\wedge) \to V_p$$

to be the standard hyperbolic form $(H_1 = H_2)$. Then

$$(1.17)\ \chi(H_1\oplus H_2^\wedge,-\psi)\cdot(\theta_1\oplus\theta_2) = (K_1\oplus K_2, H_1\oplus H_2^\wedge, (\alpha_1\oplus\gamma_2, (\gamma_1-\gamma_2)\oplus\lambda(\alpha_1-\alpha_2)))\ .$$

Let $i: K \to K_1 \oplus K_2$ be the diagonal inclusion.

Then if we denote the right side of (1.17) by $(M,N,(\beta,\delta))$, $\delta i = 0$, βi is injective, and $\operatorname{cok}(i), \operatorname{cok}(\beta i) \in CM_p$. By the construc- tion of (1.14) we may destabilize. It is now easily shown that if $(G,L,(\rho,\tau))$ is the resultant λ-formation, τ is an isomorphism and $G \simeq K$. Such a formation is isomorphic to a graph formation of the re- quired form.

Besides allowing us to represent elements of $W_1^\lambda(CM_p)$ by actual formations, the last result shows that the following two operations on a formation (K,H,Δ) do not change its class in $W_1^\lambda(CM_p)$.

(1.18) a) Given $(\varepsilon) = (L \to H_1^\wedge \to H^\wedge)$, let K_1 and γ_1 be de- fined by pullback in

$$\begin{array}{ccccc} L & \rightarrowtail & K_1 & \overset{\ell_1}{\twoheadrightarrow} & K \\ \downarrow = & & \downarrow \gamma_1 & & \downarrow \gamma \\ L & \rightarrowtail & H_1^\wedge & \overset{\ell}{\twoheadrightarrow} & H^\wedge \end{array}$$

and set $\alpha_1 := \ell^\wedge \alpha \ell_1$, $\Delta_1 := (\alpha_1, \gamma_1): K_1 \to H_1 \oplus H_1^\wedge$. Then $(K_1, H_1, \Delta_1) \in F^\lambda(CM_p)$ and is denoted ${}_\varepsilon\sigma(K,H,\Delta)$.

(1.18) b) Let $\psi: H^\wedge \times H^\wedge \to V_p$ be a $(-\lambda)$-symmetric form and set ${}_\psi\Delta = (\alpha + (\mathrm{Ad}\ \psi)\gamma, \gamma)$. Then $(K,H,{}_\psi\Delta) \in F^\lambda(CM_p)$ and is denoted $\chi(H^\wedge,\psi) \cdot (K,H,\Delta)$.

(1.19) <u>Proposition</u>. Let (K,H,Δ) be a λ-formation. Then elements of the form $(K,H,\Delta) - {}_\varepsilon\sigma(K,H,\Delta)$ and $(K,H,\Delta) - \chi(H^\wedge,\psi) \cdot (K,H,\Delta)$ in $W_1^\lambda(CM_p)$ are trivial.

It is now easy to deduce that the definition of $W_1^\lambda(CM_p)$ given above agrees with that of [P1, (1.34)]. (Note that $(K,K^\wedge,(\rho,1)) = \chi(K^\wedge,\rho) \cdot \{{}_\varepsilon\sigma(0,0,(0,0))\}$.) We will freely use this fact in what follows.

The idea of a formation is to compare two lagrangian structures on a non-singular form. However, somewhat awkwardly, one structure is actually hyperbolic, while the other is not in general so. This causes some serious technical difficulties (cf. (2.23)), which we will not address here. In one case it is useful to know that a lagrangian structure is always hyperbolic.

(1.20) <u>Proposition</u>. Let $(M,\phi) \in \Omega^{-1}(CM_n(A))$ where $n = \dim A$ and $1/2 \in A$. If $L \subseteq M$ is a sublagrangian, then there is an isometry $(L \oplus L^\wedge, \phi_h) \xrightarrow{\simeq} (M,\phi)$ extending the inclusion of L in M.

<u>Proof</u>: Suppose we have $K \subseteq M$ with $K \subseteq K^\perp$, $K \cap L = (0)$ and $\ell(K) + \ell(L) < \ell(M)$. Then we first show it is possible to choose $y \in M - (K + L)$ so that $\phi(y,K) \equiv 0$.

Choose any $y \in M - (K + L)$. Define $\phi_y: K \to E_n = V_n$ by $\phi_y(k) = \phi(y,k)$. Since E_n is injective and $i: K \to M/L$ is injective, there is a homomorphism $\phi_y': M/L \to E_n$ so that $\phi_y = \phi_y' \circ i$. Since L is a sublagrangian, $L \simeq \mathrm{Hom}(M/L,E_n)$ so there is $\ell \in L$ such that

$$\phi(\ell,x) = \phi_y'(x), \quad x \in M/L .$$

Then $y - \ell \in M - (K + L)$ and $\phi(y - \ell, K) \equiv 0$.

If $K' = K + (y)$, we may repeat the above procedure (on K' and L) until $\ell(K) + \ell(L) = \ell(M)$, in which case $K \oplus L = M$. Since $K \subseteq K^\perp$ and $L \subseteq L^\perp$ it is clear that M is hyperbolic.

Finally we give a simpler description of zero in $W_1^\lambda(CM_p)$.

(1.21) <u>Proposition</u>. Let $(K,H,\Delta) \in F^\lambda(CM_p)$. Then (K,H,Δ) represents zero in $W_1^\lambda(CM_p)$ if and only if it is stably isometric to a graph formation.

<u>Proof</u>: See (8. 1).

§2. <u>Localization sequences</u>.

This section provides the localization sequences and states the dévissage theorem needed for the applications.

(2.1) <u>Theorem</u>. Let A be a regular Noetherian domain of dimension ≤ 4 with fraction field K. Then there are exact sequences where $CM_i = CM_i(A)$, $\lambda = \pm 1$, and K_i^λ is induced by $\coprod(\otimes A_p)$,

$$\text{a)}: \quad W_1^\lambda(K) \xrightarrow{L_1^\lambda} W_1^\lambda(CM_1) \xrightarrow{D_1^\lambda} W_0^\lambda(A) \xrightarrow{K_0^\lambda} W_0^\lambda(K)$$

$$\text{b)}: \quad \coprod_{\mathrm{ht}\ p=1} W_0^{-\lambda}(CM_1(A_p)) \xrightarrow{L_0^{-\lambda}} W_0^{-\lambda}(CM_2) \xrightarrow{D_0^{-\lambda}} W_1^\lambda(CM_1) \xrightarrow{K_1^\lambda} \coprod_{\mathrm{ht}\ p=1} W_1^\lambda(CM_1(A_p))$$

$$\text{c)}: \quad \coprod_{\mathrm{ht}\ p=2} W_1^{-1}(CM_2(A_p)) \xrightarrow{L_1^{-1}} W_1^{-1}(CM_3) \xrightarrow{D_1^{-1}} W_0^{-1}(CM_2) \xrightarrow{K_0^{-1}} \coprod_{\mathrm{ht}\ p=2} W_0^{-1}(CM_2(A_p))$$

$$\text{d)}: \quad \coprod_{\mathrm{ht}\ p=3} W_0^{1}(CM_3(A_p)) \xrightarrow{L_0^{1}} W_o^{1}(CM_4) \xrightarrow{D_0^{1}} W_1^{-1}(CM_3) \xrightarrow{K_1^{-1}} \coprod_{\mathrm{ht}\ p=3} W_1^{-1}(CM_3(A_p)) .$$

The proof of the theorem is given later in this section.

Let (R,M) be a local Gorenstein ring of dimension n. Since $k(M)$ is isomorphic as an R-module to $\{e \in E_n(R) = E(R/M) \mid Me = 0\}$ ([SV,4.24]), we may choose an imbedding $k(M) \to E_n$ and use the inclusion of the category of $k(M)$-vector spaces into $CM_n(R)$ to produce a map

$$W_i^\lambda(k(M)) \to W_i^\lambda(CM_n(R)).$$

(2.2) <u>Theorem</u>. (Dévissage) Let (R,M) be a local Gorenstein ring of dimension n with residue class field $k(M)$. Then there is an isomorphism ($i = 0,1$ and $\lambda = \pm 1$)

$$W_i^\lambda(k(M)) \xrightarrow{\simeq} W_i^\lambda(CM_n(R)).$$

<u>Proof</u>: §5.

It is well-known that $W_1^\lambda(k(M)) = 0 = W_0^{-1}(k(M))$, so we have the following consequences of the two theorems:

(2.3) <u>Corollary</u>. a) If A is a regular Noetherian domain of dimension ≤ 4 then there are isomorphisms

$$\ker(W_0^1(A) \to W_0^1(K)) \simeq W_1^1(CM_1)$$

$$\simeq W_0^{-1}(CM_2)$$

and set $\alpha_1 := \ell^{\wedge} \alpha \ell_1$, $\Delta_1 := (\alpha_1, \gamma_1): K_1 \to H_1 \oplus H_1^{\wedge}$. Then $(K_1, H_1, \Delta_1) \in F^{\lambda}(CM_p)$ and is denoted ${}_{\varepsilon}\sigma(K,H,\Delta)$.

(1.18) b) Let $\psi: H^{\wedge} \times H^{\wedge} \to V_p$ be a $(-\lambda)$-symmetric form and set $\psi^{\Delta} = (\alpha + (\mathrm{Ad}\ \psi)\gamma, \gamma)$. Then $(K,H,{}_{\psi}\Delta) \in F^{\lambda}(CM_p)$ and is denoted $\chi(H^{\wedge},\psi) \cdot (K,H,\Delta)$.

(1.19) Proposition. Let (K,H,Δ) be a λ-formation. Then elements of the form $(K,H,\Delta) - {}_{\varepsilon}\sigma(K,H,\Delta)$ and $(K,H,\Delta) - \chi(H^{\wedge},\psi) \cdot (K,H,\Delta)$ in $W_1^{\lambda}(CM_p)$ are trivial.

It is now easy to deduce that the definition of $W_1^{\lambda}(CM_p)$ given above agrees with that of [Pl, (1.34)]. (Note that $(K,K^{\wedge},(\rho, 1)) = \chi(K^{\wedge},\rho) \cdot \{{}_{\varepsilon}\sigma(0,0,(0,0)).)$ We will freely use this fact in what follows.

The idea of a formation is to compare two lagrangian structures on a non-singular form. However, somewhat awkwardly, one structure is actually hyperbolic, while the other is not in general so. This causes some serious technical difficulties (cf. (2.23)), which we will not address here. In one case it is useful to know that a lagrangian structure is always hyperbolic.

(1.20) Proposition. Let $(M,\phi) \in Q^{-1}(CM_n(A))$ where $n = \dim A$ and $1/2 \in A$. If $L \subseteq M$ is a sublagrangian, then there is an isometry $(L \oplus L^{\wedge},\phi_h) \xrightarrow{\simeq} (M,\phi)$ extending the inclusion of L in M.

Proof: Suppose we have $K \subseteq M$ with $K \subseteq K^{\perp}$, $K \cap L = (0)$ and $\ell(K) + \ell(L) < \ell(M)$. Then we first show it is possible to choose $y \in M - (K + L)$ so that $\phi(y,K) \equiv 0$.

Choose any $y \in M - (K + L)$. Define $\phi_y: K \to E_n = V_n$ by $\phi_y(k) = \phi(y,k)$. Since E_n is injective and $i: K \to M/L$ is injective, there is a homomorphism $\phi_y': M/L \to E_n$ so that $\phi_y = \phi_y' \circ i$. Since L is a sublagrangian, $L \simeq \mathrm{Hom}(M/L,E_n)$ so there is $\ell \in L$ such that

$$\phi(\ell,x) = \phi_y'(x), \quad x \in M/L .$$

Then $y - \ell \in M - (K + L)$ and $\phi(y - \ell,K) \equiv 0$.

If $K' = K + (y)$, we may repeat the above procedure (on K' and L) until $\ell(K) + \ell(L) = \ell(M)$, in which case $K \oplus L = M$. Since $K \subseteq K^{\perp}$ and $L \subseteq L^{\perp}$ it is clear that M is hyperbolic.

Finally we give a simpler description of zero in $W_1^{\lambda}(CM_p)$.

(1.21) Proposition. Let $(K,H,\Delta) \in F^{\lambda}(CM_p)$. Then (K,H,Δ) represents zero in $W_1^{\lambda}(CM_p)$ if and only if it is stably isometric to a graph formation.

Proof: See (8. 1).

§2. Localization sequences.

This section provides the localization sequences and states the dévissage theorem needed for the applications.

(2.1) Theorem. Let A be a regular Noetherian domain of dimension ≤ 4 with fraction field K. Then there are exact sequences where $CM_i = CM_i(A)$, $\lambda = \pm 1$, and K_i^{λ} is induced by $\coprod(\otimes A_p)$,

a): $W_1^{\lambda}(K) \xrightarrow{L_1^{\lambda}} W_1^{\lambda}(CM_1) \xrightarrow{D_1^{\lambda}} W_0^{\lambda}(A) \xrightarrow{K_0^{\lambda}} W_0^{\lambda}(K)$

b): $\coprod_{\mathrm{ht}\,p=1} W_0^{-\lambda}(CM_1(A_p)) \xrightarrow{L_0^{-\lambda}} W_0^{-\lambda}(CM_2) \xrightarrow{D_0^{-\lambda}} W_1^{\lambda}(CM_1) \xrightarrow{K_1^{\lambda}} \coprod_{\mathrm{ht}\,p=1} W_1^{\lambda}(CM_1(A_p))$

c): $\coprod_{\mathrm{ht}\,p=2} W_1^{-1}(CM_2(A_p)) \xrightarrow{L_1^{-1}} W_1^{-1}(CM_3) \xrightarrow{D_1^{-1}} W_0^{-1}(CM_2) \xrightarrow{K_0^{-1}} \coprod_{\mathrm{ht}\,p=2} W_0^{-1}(CM_2(A_p))$

d): $\coprod_{\mathrm{ht}\,p=3} W_0^{1}(CM_3(A_p)) \xrightarrow{L_0^{1}} W_0^{1}(CM_4) \xrightarrow{D_0^{1}} W_1^{-1}(CM_3) \xrightarrow{K_1^{-1}} \coprod_{\mathrm{ht}\,p=3} W_1^{-1}(CM_3(A_p))$.

The proof of the theorem is given later in this section.

Let (R,M) be a local Gorenstein ring of dimension n. Since $k(M)$ is isomorphic as an R-module to $\{e \in E_n(R) = E(R/M) \mid Me = 0\}$ ([SV,4.24]), we may choose an imbedding $k(M) \to E_n$ and use the inclusion of the category of $k(M)$-vector spaces into $CM_n(R)$ to produce a map

$$W_i^{\lambda}(k(M)) \to W_i^{\lambda}(CM_n(R)).$$

(2.2) Theorem. (Dévissage) Let (R,M) be a local Gorenstein ring of dimension n with residue class field $k(M)$. Then there is an isomorphism $(i = 0,1$ and $\lambda = \pm 1)$

$$W_i^{\lambda}(k(M)) \xrightarrow{\simeq} W_i^{\lambda}(CM_n(R)).$$

Proof: §5.

It is well-known that $W_1^{\lambda}(k(M)) = 0 = W_0^{-1}(k(M))$, so we have the following consequences of the two theorems:

(2.3) Corollary. a) If A is a regular Noetherian domain of dimension ≤ 4 then there are isomorphisms

$$\ker(W_0^1(A) \to W_0^1(K)) \simeq W_1^1(CM_1)$$
$$\simeq W_0^{-1}(CM_2)$$

$$\simeq W_1^{-1}(CM_3)$$

$$\simeq \operatorname{cok}\Big(\coprod_{\mathrm{ht}\,p=3} W_0^1(CM_3(A_p)) \xrightarrow{L_0^1} \coprod_{\mathrm{ht}\,p=4} W_0^1(CM_4(A_p))\Big) .$$

In particular $\ker(W_0^1(A) \to W_0^1(K)) = 0$ if $\dim A \leq 3$.

b) With A as above but $\dim A \leq 2$, there are isomorphisms

$$W_0^{-1}(A) \simeq W_1^{-1}(CM_1) \simeq \operatorname{cok}\Big(\coprod_{\mathrm{ht}\,p=1} W_0^1(CM_1(A_p)) \to \coprod_{\mathrm{ht}\,p=2} W_0^1(CM_2(A_p))\Big)$$

(2.4) Remark. The corollary furnishes one reason for the restrictions, $\dim A \leq 4$ or $\dim A \leq 2$: the obstructions to increasing the codimension p of the quadratic objects $W_i^\lambda(CM_p)$ all vanish until an element of W_0^1 is reached ($p = 4$ in a), $p = 2$ in b)). This seems to be a kind of internal periodicity. The other reason has to do with the problem of finding CM-modules (cf. (2.11)).

But it is easy now to show that the codimension four obstruction vanishes if we localize. (This presages the connection with cycles in §3.)

(2.5) Theorem. Let (R,M) be a regular local ring of dimension ≤ 4. Then $W_0^1(R) \to W_0^1(K)$ is injective.

Remark: Since R is local the conclusion is equivalent to the assertion that two symmetric forms $\phi,\psi\colon R^n \times R^n \to R$ are isometric if and only if their extensions $\phi \otimes K$ and $\psi \otimes K$ are.

Proof: From (2.3) it is sufficient to show that the composition

$$W_0^1(CM_4) \xrightarrow{\mathcal{D}_0^1} W_1^{-1}(CM_3) \xrightarrow{\mathcal{D}_1^{-1}} W_0^{-1}(CM_2)$$

is trivial.

Choose two regular parameters $x,y \in M$, so that $\bar{R} := R/(x,y)$ is regular local, 2-dimensional, and has maximal ideal $\bar{M} := M/(x,y)$. By (1.5) there are isomorphisms $E_i(\bar{R}) \simeq (0:(x,y)) \subseteq E_{i+2}(R)$, $i = 0,1,2$, commuting with the differentials of $E_*(\bar{R})$ and $E_*(R)$. Consequently there are commutative diagrams, $i = 0,1,2$,

$$\begin{array}{ccccc} V_i(\bar{R}) & \rightarrowtail & E_i(\bar{R}) & \twoheadrightarrow & V_{i+1}(\bar{R}) \\ \downarrow & & \downarrow & & \downarrow \\ V_{i+2}(R) & \rightarrowtail & E_{i+2}(R) & \twoheadrightarrow & V_{i+3}(R) \end{array}$$

Also there are obvious inclusions

$$CM_i(\bar{R}) \to CM_{i+2}(R), \quad i = 0,1,2$$

and

$$CM_i(\bar{R}_{\bar{p}}) \to CM_{i+2}(R_p), \quad i = 0,1,2$$

where $\mathrm{ht}\, p = i + 2$, and $(x,y) \subseteq p$.

With these facts and the definitions of the maps $\mathcal{D}_0^1$ and $\mathcal{D}_1^{-1}$ given below, it is immediate that there is a commutative diagram

$$\begin{array}{ccccc} W_0^1(CM_2(\bar{R})) & \xrightarrow{\mathcal{D}_0^1} & W_1^{-1}(CM_1(\bar{R})) & \xrightarrow{\mathcal{D}_1^{-1}} & W_0^{-1}(CM_0(\bar{R})) \\ \downarrow & & \downarrow & & \downarrow \\ W_0^1(CM_4(R)) & \xrightarrow{\mathcal{D}_0^1} & W_1^{-1}(CM_3(R)) & \xrightarrow{\mathcal{D}_1^{-1}} & W_0^{-1}(CM_2(R)) \end{array}$$

But because $\dim \bar{R} = 2$ and $\dim R = 4$, dévissage says each of the extreme left terms is isomorphic to $W_0^1(k(M))$ and the left vertical is an isomorphism. And $W_0^{-1}(CM_0(\bar{R})) = W_0^{-1}(\bar{R})$ (cf. (1.8)) vanishes. Hence the composition along the bottom is trivial, as claimed.

Proof (of (2.1)). The first exact sequence is part of the theorem in [P1]. Most of the remaining work just generalizes this, so the more routine details will be omitted. We begin with the definitions of the maps.

(2.6) Definition of $\mathcal{D}_0^\lambda\colon W_0^\lambda(CM_{p+1}) \to W_1^{-\lambda}(CM_p)$.

Given $(T,\mu) \in \Omega^\lambda(CM_{p+1})$, [P3, 3.12(b)] shows how to produce $(K,H,\Delta) \in F^{-\lambda}(CM_p)$:

a) Choose a resolution of T, $K \overset{\alpha}{\rightarrowtail} H \overset{j}{\twoheadrightarrow} T$ where $K,H \in CM_p$.

b) Lift μ to τ,

$$\begin{array}{ccc} H \times H & \xrightarrow{\tau} & E_p \\ \downarrow j \times j & & \downarrow d_p \\ T \times T & \xrightarrow{\mu} & V_{p+1} \end{array}$$

c) Define γ by the commutativity of the diagram

$$\begin{array}{ccc} K & \xrightarrow{\gamma} & H\hat{} := \mathrm{Hom}(H,V_p) \\ \downarrow \alpha & & \downarrow \\ H & \xrightarrow{\mathrm{Ad}\tau} & \mathrm{Hom}(H,E_p) \end{array}$$

and set $\Delta = (\alpha,\gamma)$.

The proof of [P3, (3.19)] shows that the class of $(K,H,\Delta) \in W_1^{-\lambda}(CM_p)$ depends only on the isometry class of (T,μ). Moreover, since the construction is additive (sending orthogonal sums to orthogonal sums), we need only check it sends lagrangians to zero in $W_1^{-\lambda}(CM_p)$. Indeed, [P3, (3.16)(b)] shows that if (T,μ) is a lagrangian, then there is a choice of $\alpha\colon K \to H$ and τ so that (K,H,Δ) has $\Delta = (\alpha,\gamma)$ where γ is invertible. This is isomorphic to a graph formation, so (K,H,Δ) represents zero in $W_1^{-\lambda}(CM_p)$.

(2.7) <u>Definition of</u> $L_0^{\lambda}\colon \coprod_{ht\,p=p} W_0^{\lambda}(CM_p(A_p)) \to W_0^{\lambda}(CM_{p+1}(A))$ in case

a) $p = 0$ and A is a Gorenstein domain.

b) $p = 1$ and A is Gorenstein and locally factorial.

c) $p = \dim A - 1$ or $\dim A - 2$ and A is Gorenstein.

For the duration of this section we write, when R is local CM of dimension p,

$$CM_p(R) = F(R),$$

the category of modules of finite length.

(2.8) <u>Definition</u>. Let $p \geq 0$ and $F = \coprod F_i$, where $F_i \in F(A_{q_i})$ and $q_i \in \operatorname{Spec} A$ has height p, $i = 1,\dots,n$. A <u>CM_p-lattice</u> $L \subseteq F$ is an A-module $L \in CM_p(A)$ such that $L_{q_i} = F_i$ for each i. If $(F_i,\tau_i) \in Q^{\lambda}(F(A_{q_i}))$, then $L \subseteq F$ is called an <u>integral lattice</u> if it is a CM_p-lattice and $\tau(L \times L) \subseteq V_p (\subseteq E_p = \coprod_{ht\,q=p} E_p(A_q))$, where $\tau = \coprod \tau_i$.

(2.9) <u>Proposition</u>. Let $(F_i,\tau_i) \in Q^{\lambda}(F(A_{q_i}))$ where $ht_{q_i} = p$, $i = 1,\dots,n$. Then under the conditions a), b), c) above there is an integral lattice $L \subseteq F := \coprod F_i$.

<u>Proof</u>: We may assume $n = 1$ and set $q = q_1$. To begin with we need a CM_p-lattice in F. For $p = 0$ this is obvious. If $p = 1$, we use the fact that A_q is a DVR to assume $F = A_q/(q A_q)^t$, $t \geq 1$. In this case we take $L = A/q^{(t)}$, where $q^{(t)}$ denotes the t-th symbolic power. L belongs to CM_1 because A is locally factorial.

Finally, if $p = \dim A - 1$ or $\dim A - 2$, then we need the following result, essentially due to Hochster. A proof is given in [P4].

(2.10) <u>Lemma</u>. Let M be a finitely-generated R-module, $ht\, M = p$, where R is local Gorenstein and $p = \dim R - 1$ or $\dim R - 2$. Then $\operatorname{Hom}(M,V_p) \in CM_p(R)$.

From this lemma, it follows that, choosing any finitely-generated $L \subseteq F$ where $L_q = F$, $\operatorname{Hom}(L,V_p)$ will be everywhere locally CM. This

is also true of $\operatorname{Hom}(\operatorname{Hom}(L,V_p),V_p) \subseteq \operatorname{Hom}(\operatorname{Hom}(F,E_p),E_p) \simeq F$, so we have our lattice.

Now suppose L is a CM_p-lattice in F for any $p \geq 0$. Let $\{\ell_1,\dots,\ell_s\}$ generate L and let $\{p_1,\dots,p_m\}$ be a finite set of height $p + 1$ primes such that $d_p\tau(\ell_i,\ell_j) \subseteq \coprod_{k=1}^{m} E(A/p_k)$, for some fixed i and j. Let $d_p\tau(\ell_i,\ell_j) = \Sigma r_k$, $r_k \in E(A/p_k)$. By [SV, (4.23)] there is an integer n_k so that $p_k^{n_k} r_k = 0$, $k = 1,\dots,m$. Each $p_k \supsetneq q$ by [SV, 4.21] so let $x_k \in p_k - q$. Then if $n = \max(n_i)$ and $z = \Pi x_i^n$, we get $zr_k = 0$ for all k. Since $z \notin q$, z acts isomorphically on F. Consequently, zL is still a CM_p-lattice and $d_p\tau(z\ell_i,z\ell_j) = 0$, so $\tau(z\ell_i,z\ell_j) \in V_p$. Continuing in this way (for all i and j) completes the proof.

(2.11) <u>Remark</u>. The existence of CM_p-lattices is essentially equivalent to the existence of small CM-modules ([Ho, Conj E'''']) about which very little is known if $p \geq 3$. It is perhaps only accidential, but if A is regular, then (2.9) applies for all p if and only if $\dim A \leq 4$, which is precisely the range of dimensions for which our quadratic form techniques are successful (see (2.4)).

(2.12) <u>Definition</u>. Let $(F_i,\tau_i) \in Q^{\lambda}(F(A_{q_i}))$ be given, $ht\,q_i = p$, $i = 1,\dots,n$. Set $F = \coprod F_i$, $\tau = \coprod \tau_i\colon F \times F \to E_p$. Suppose $L \subseteq F$ is an integral lattice. The <u>dual lattice</u> (to L) is

$$L' = \{f \in F \mid \tau(f,L) \subseteq V_p\} .$$

Since L is integral, $L \subseteq L'$. Since τ is non-singular, there is an isomorphism

$$(2.13)\qquad \begin{aligned} L' &\xrightarrow{\simeq} \operatorname{Hom}(L,V_p) \\ \ell' &\to \{\ell \to \tau(\ell',\ell)\} \end{aligned}$$

so $L' \in CM_p$ by (1.6). Using the fact that $L_{q_i} = L'_{q_i} = F$ for each i, it follows from (1.2)(d) that $L'/L \in CM_{p+1}$. Set $L'/L = M$ and define $\phi\colon M \times M \to V_{p+1}$ by

$$(2.14)\qquad \phi(j\ell_1,j\ell_2) = d_p\,\tau(\ell_1,\ell_2)$$

where $j\colon L' \to M$ is the quotient map.

(2.15) <u>Remark</u>. The composition $L \to L' \xrightarrow{\simeq} \operatorname{Hom}(L,V_p)$ is $Ad(\tau|L \times L)$. Thus the map γ constructed in (2.6)(c) is the identity (modulo the identification $L \equiv L^{\wedge\wedge}$). Conversely, if for a given $(M,\phi) \in Q^\lambda(CM_{p+1})$ one gets γ equal to the identity in the construction of $\mathcal{D}_0^\lambda$ above, then it is straightforward to verify there is a nonsingular form (F,τ) and an integral lattice $L \subseteq F$ so that $M \simeq L'/L$ and τ defines ϕ by the formula (2.14).

Returning to the construction of L_0^λ, one proves that $(M,\phi) \in Q^\lambda(CM_{p+1})$ as in [P3, p. 354]; and that the class of (M,ϕ) in $W_0^\lambda(CM_{p+1})$ depends only on the isometry class of (F,τ) (not on the choice of lattice L) as in [P3, 3.3].

Since the construction preserves (orthogonal) sums, it remains to show that lagrangians are sent to lagrangians. The case $p = 0$ is done in [P3, §3]. For the remaining cases we need

(2.16) Lemma. Let $i\colon N \to G$ be a CM_p-lattice where G is a finite length A_q-module and ht $q = p$. Then there is an injection of A-modules whose image is a CM_p-lattice, $i_*\colon \mathrm{Hom}(N,V_p) \to \mathrm{Hom}(G,E_p)$, such that $(i_*f)(i(n)) = f(n)$, where $f \in \mathrm{Hom}(N,V_p)$ and $n \in N$.

Proof: Use injectivity of E_p to fill in:

$$\begin{array}{ccc} N & \to & G \\ \downarrow & & \downarrow i_*f \\ V_p & \to & E_p \end{array}$$

This defines i_*f and imbeds $\mathrm{Hom}(N,V_p)$ in $\mathrm{Hom}(G,E_p)$ because inclusion induces isomorphisms $N_q \xrightarrow{\simeq} G$ and $(V_p)_q \xrightarrow{\simeq} E(A/q)$.

Now for any p, if $\lambda = -1$, then by (1.20), a lagrangian (F,τ) is hyperbolic: $(F,\tau) \simeq (G \oplus \overline{G}, \phi_h)$, where $\overline{G} = \mathrm{Hom}(G,E_p)$. Setting $(-)^\wedge = \mathrm{Hom}(-,V_p)$, we use the lemma to find a CM_p-lattice $L = N \oplus i_*(N^\wedge)$ in F such that $\tau|L \times L$ is the skew-symmetric hyperbolic form on $(N \oplus N^\wedge) \times (N \oplus N^\wedge)$. Hence $L = L'$, so the construction produces the zero form in $Q^\lambda(CM_{p+1})$.

In case $p = 1$ and $\lambda = 1$, $R := A_q$ is a DVR. In this case a lagrangian over $CM_1(R)$ is an sum of hyperbolics and "unary" forms, $(R/(t^{2m}),r)$ where t is a uniformizer and $r \in (R/t^{2m})^\times$. (This is an exercise.) The form $(R/(t^{2m}),r)$ has a sublagrangian $t^m(R/(t^{2m}) \subseteq R/(t^{2m})$. Since the hyperbolic case was treated above, we may assume $(F,\tau) = (R/(t^{2m}),r)$.

Choose an integral lattice $A/q^{(2m)} \to R/(t^{2m})$ (compare the proof of (2.9)). Define K to be the kernel of the natural surjection $A/q^{(2m)} \to A/q^{(m)}$. Then $K \in CM_1(A)$ by (1.2) and $K_q = t^mF$, the lagrangian of (F,τ).

Consider the diagram, where $(-) = \mathrm{Hom}(-,V_p)$,

$$\begin{array}{ccccc} K & \rightarrowtail & A/q^{(2m)} & \twoheadrightarrow & A/q^{(m)} \\ \downarrow & & \downarrow & & \downarrow \\ (A/q^{(m)})^\wedge & \rightarrowtail & (A/q^{(2m)})^\wedge & \twoheadrightarrow & K \\ \downarrow & & \downarrow & & \downarrow \\ N & \rightarrowtail & M & \twoheadrightarrow & M/N \end{array}$$

in which the second row is the dual of the first (short exact by (1.6)) the upper row of verticals is induced from $\tau|A/q^{(2m)} \times A/q^{(2m)}$ and the fact that $\tau|K \times K = 0$; and the last row is the resulting sequence of cokernels. Since (F,τ) is non-singular, the upper verticals are injective, making the bottom row a short exact sequence in CM_{p+1} (cf. (1.2)(d)).

Now $(A/q^{(2m)})^\wedge$ is identified with the dual lattice by (2.13), so $M \in CM_{p+1}$ supports the form ϕ constructed above. By (1.6), $M/N \simeq N^\wedge$. In fact $\phi|N \times N \equiv 0$ and it is easily checked that ϕ induces the latter isomorphism. Thus, (M,ϕ) is a lagrangian, as required.

Finally, assume $p = \dim A - 2$ or $\dim A - 1$, and that $(F,\tau) \in Q^\lambda(F(A_q))$ (ht $q = p$) is a lagrangian with sublagrangian G. Choose an integral lattice $L \subseteq F$ and set $I = \mathrm{im}\{L \to F \to F/G\}$. Then $\mathrm{Ass}(I)$ $\{q\}$ and so for every maximal ideal M of A, depth $I_M \geq 1$. Since $L \in CM_p(A)$, it follows from (1.2)(d) that $N := \ker(L \to I) \in CM_p(A)$ if $p = \dim A - 1$. If $p = \dim A - 2$, we again get $N \in CM_p(A)$, by [K, Ex. 14, p. 103].

Now $N \subseteq N^\perp$ since $N \subseteq G$, so there is a commutative diagram of exact sequences

$$\begin{array}{ccccc} N & \rightarrowtail & L & \twoheadrightarrow & I \\ \downarrow & & \downarrow & & \downarrow \\ I^\wedge & \rightarrowtail & L' & \to & N^\wedge \end{array}$$

where the verticals are induced by $\mathrm{Ad}(\tau|L \times L)$, the bottom line is the dual of the top one, L' is identified with $L^\wedge$ by (2.13), but $L' \to N^\wedge$ need not be surjective. ($\mathrm{Ext}^1(I,V_p) \neq 0$ if $p = \dim A - 2$ and depth $I_M = 1$ for some maximal $M \in \mathrm{Supp}(I)$.) However, if $p = \dim A -$ the bottom line is exact by (1.6) and if $p = \dim A - 2$, it is exact at all primes of height $\dim A - 1$. Thus, taking cokernels of the vertical maps we are finished (as in the case $p = 1$ above), if $p = \dim A - 1$. If $p = \dim A - 2$, the sequence of cokernels

$$S \rightarrowtail L'/L \to T$$

is exact at all primes of height dim A − 1. Hence to complete the proof, it suffices to apply the following lemma to the class of $(L'/L,\phi)$ in $W_0^\lambda(CM_{n-1}(A))$.

(2.17) Lemma. Let A be Gorenstein of dimension n. Then

$$W_0^\lambda(CM_{n-1}(A)) \to \coprod_{\mathrm{ht}\, q=n-1} W_0^\lambda(CM_{n-1}(A_q))$$

is injective.

Proof: Let $(M,\phi) \in Q^\lambda(CM_{n-1}(A))$ be a lagrangian at each $q \in$ Ass(M). Consider the collection L of submodules $N \subseteq M$ such that $N \subseteq N^\perp$ and N_q is a sublagrangian of (M_q,ϕ_q) for each $q \in \mathrm{Ass}(M)$. L is not empty: if K_q is a sublagrangian of M_q and $i: M \to \coprod M_q$ ($q \in \mathrm{Ass}(M)$) is the canonical imbedding, then $i^{-1}(\coprod K_q)$ is such an N. Since A is noetherian and M is finitely-generated, L has a maximal element N_0. Then $N_0 = N_0^\perp$: if $x \in N_0 - N_0$, $n_1,n_2 \in N_0$ and $a_1,a_2 \in A$, then

$$\phi(n_1 + a_1x, n_2 + a_2x) = a_1 a_2 \phi(x,x).$$

If this is zero, then $(N_0 + Ax) \subseteq (N_0 + Ax)^\perp$, contradicting the maximality of N_0. But $(N_0)_q = ((N_0)_q)^\perp$ for each $q \in \mathrm{Ass}(M)$ so $\phi_q(x,x) = 0$, hence $\phi(x,x) = 0$. Write N for N_0.

Now consider the commutative diagram

$$\begin{array}{ccc} & N & \\ {\scriptstyle\kappa}\swarrow & & \searrow{\scriptstyle\alpha} \\ N^{\wedge\wedge} & \xrightarrow{\beta^\wedge} & (M/N)^\wedge \end{array}$$

where κ is the canonical map and α,β are induced by ϕ. Ad ϕ is injective, so is α, which means κ is. Since $N = N^\perp$, β is injective; this implies M/N is height-unmixed (because $N^\wedge$ is). But this means N and $M/N \in CM_{n-1}(A)$ because for $\mathcal{M}$ maximal, $N_{\mathcal{M}}$ (or $(M/N)_{\mathcal{M}}$) can have depth zero if and only if $\mathcal{M} \in \mathrm{Ass}(N_{\mathcal{M}})$ (or $\mathrm{Ass}(M/N)_{\mathcal{M}}$). Thus κ is an isomorphism by (1.6)(a) and $\beta^\wedge$ is surjective because β was injective. This means α and β are isomorphisms, so $N \subseteq M$ is a sublagrangian.

(2.18) Definition of $D_1^\lambda: W_1^\lambda(CM_{p+1}) \to W_0^\lambda(CM_p)$.

Let $(K,H,\Delta) \in F^\lambda(CM_{p+1})$ and let

(2.19) $$L \overset{\alpha}{\rightarrowtail} N \overset{j}{\twoheadrightarrow} H$$

be a short exact sequence with $L,N \in CM_p$ (cf. (1.6)). Dualize (2.19) to $N^\wedge \overset{\alpha^\wedge}{\rightarrowtail} L^\wedge \to H^\sim$ using (1.6) and its notation. Add the sequences, getting a resolution of $H \oplus H^\sim$,

(2.20) $$L \oplus N^\wedge \overset{\alpha \oplus \lambda\alpha^\wedge}{\rightarrowtail} N \oplus L^\wedge \overset{k}{\twoheadrightarrow} H \oplus H^\sim$$

and set $M = k^{-1}(\Delta(K))$; clearly $M \in CM_p$.

Let $Q = \mathrm{Ass}(L) = \mathrm{Ass}(N)$ and let $(-)_Q$ denote localization at the set of primes in Q. Using the isomorphisms and inclusions

$$L_Q \overset{\alpha_Q}{\cong} N_Q \supseteq N$$

(2.21) and

$$\mathrm{Hom}(L,V_p) \subseteq \mathrm{Hom}(L,V_p)_Q \cong \mathrm{Hom}(L_Q,E_p)$$

define $\tau: (N \oplus L^\wedge) \times (N \oplus L^\wedge) \to E_p$ by

$$\tau(n_1 \oplus f_1, n_2 \oplus f_2) = f_2(\alpha_Q^{-1}(n_1)) + \lambda f_1(\alpha_Q^{-1}(n_2))$$

where $n_i \in N$ and $f_i \in L^\wedge$. Then $((N \oplus L^\wedge)_q,\tau_q) \in Q^\lambda(CM_p(A_q))$ for all $q \in Q$ and the diagram

(2.22) $$\begin{array}{ccc} (N \oplus L^\wedge) \times (N \oplus L^\wedge) & \xrightarrow{\tau} & E_p \\ \downarrow k \times k & & \downarrow d_p \\ (H \oplus H^\sim) \times (H \oplus H^\sim) & \xrightarrow{\phi_h} & V_{p+1} \end{array}$$

commutes. It follows that $\tau(M \times M) \subseteq V_p$ so $M \subseteq N \oplus L^\wedge \subseteq (N \oplus L)_Q$ is an integral lattice in $((N \oplus L^\wedge)_Q,\tau_Q)$.

Set $\psi = \tau|M \times M$. Since τ_Q is non-singular, in order to show $\psi: M \times M \to V_p$ is non-singular, it suffices to show $M = M'$, the dual lattice. Let $m \in M'$, so that $\tau(M,m) \in V_p$. Then $m \in N \oplus L^\wedge \subseteq (N \oplus L^\wedge)_Q$ because $(\alpha \oplus \lambda\alpha^\wedge)(L \oplus N^\wedge) \subseteq M$ and $\{(\alpha \oplus \lambda\alpha^\wedge)(L \oplus N^\wedge)\}' = N \oplus L^\wedge$. By (2.22) we get $\phi_h(\Delta(K),k(m)) = 0$. But since $\Delta(K) = \Delta(K)^\perp$, $k(m) \in \Delta(K)$; hence $m \in M$. Since $M \subseteq M'$, this proves equality.

Thus we conclude $(M,\psi) \in Q^\lambda(CM_p(A))$ and denote it by $I((K,H,\Delta);R)$, where R is the resolution (2.19). We next show the class of $I(K,H,\Delta);R)$ in $W_0^\lambda(CM_p)$ is independent of R.

Let $R_1 = (L_1 \overset{\alpha_1}{\to} N_1 \overset{j_1}{\to} H)$ be another resolution and form the pullback P,

$$\begin{array}{ccccc} & & L_1 & = & L_1 \\ & & \downarrow{\scriptstyle i_1} & \searrow{\scriptstyle \ell_1} & \downarrow{\scriptstyle \alpha_1} \\ L & \overset{i}{\rightarrowtail} & P & \twoheadrightarrow & N_1 \\ \| & & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle j_1} \\ L & \rightarrowtail & N & \overset{j}{\twoheadrightarrow} & H \end{array}$$

From this we get a commutative diagram with exact rows

$$\begin{array}{ccccc} L_1 & \overset{\alpha_1}{\rightarrowtail} & N_1 & \overset{j_1}{\twoheadrightarrow} & H \\ \uparrow{\scriptstyle pr_2} & & \uparrow{\scriptstyle \ell_1} & & \uparrow = \\ L \oplus L_1 & \overset{i_1+i_2}{\rightarrowtail} & P & \overset{\ell j = \ell_1 j_1}{\twoheadrightarrow} & H \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \ell} & & \downarrow = \\ L & \overset{\alpha}{\rightarrowtail} & N & \overset{j}{\twoheadrightarrow} & H \end{array}$$

Hence it is sufficient to compare resolutions connected by

$$\begin{array}{ccccc} L_1 & \overset{\alpha_1}{\rightarrowtail} & N_1 & \overset{j_1}{\twoheadrightarrow} & H \\ \downarrow & & \downarrow{\scriptstyle \ell} & & \downarrow = \\ L & \overset{\alpha}{\rightarrowtail} & N & \twoheadrightarrow & H \end{array}$$

Let $(M_1,\psi_1) = I((K,H,\Delta);R_1)$. Then $\ker \ell \subseteq M_1$, $\ker \ell \subseteq (\ker \ell)^{\perp}$ and the induced form on $(\ker \ell)^{\perp}/\ker \ell$ is isometric to (M,ψ). Hence by (1. 9) $[M,\psi] = [M_1,\psi_1]$ in $W_0^{\lambda}(CM_p)$, as claimed.

Next we claim that $I((K,H,\Delta);R)$ is isometric to $I(\chi(H,\psi) \cdot (K,H,\Delta);R)$ where $\psi\colon H \times H \to V_p$ is any $(-\lambda)$-symmetric form. This is shown following the argument in [Pl, p. 376].

Finally, there are choices R and R' for any extension (ε) of H so that $I((K,H,\Delta);R)$ is isometric to $I(\sigma_{\varepsilon}(K,H,\Delta);R')$. Indeed if $R = (L \to N \to H)$ and $(\varepsilon) = (I \to H_1 \to H)$, then from the pull-back diagram

$$\begin{array}{ccccc} & & L & \overset{=}{\to} & L \\ & & \downarrow & & \downarrow \\ I & \rightarrowtail & P & \twoheadrightarrow & N \\ \downarrow = & & \downarrow & & \downarrow \\ I & \rightarrowtail & H_1 & \twoheadrightarrow & H \end{array}$$

we take $R' = (L \to P \to H_1)$. Details are left to the reader (cf. [Pl, p. 376]).

(2.23) <u>Definition of</u> $L_1^{-1}\colon \coprod_{ht\,p=p} W_1^{-1}(CM_p(A_p)) \to W_1^{-1}(CM_{p+1})$

It is sufficient to define $L_1^{-1}|W_1^{-1}(CM_p(A_p))$ for each p of height p. Given an element $(F,G,\Gamma) \in F^{-1}(CM_p(A_p))$, we have seen in (1.20) that if $\Gamma = (\mu,\nu)\colon F \to G \oplus \overline{G}$, $\overline{G} := \mathrm{Hom}(G,E_p)$, then Γ can be extended to an isometry

$$A = \begin{pmatrix} \mu & \sigma \\ \nu & \rho \end{pmatrix}\colon F \oplus \overline{F} \to G \oplus \overline{G},$$

of the hyperbolic forms on $F \oplus \overline{F}$ and $G \oplus \overline{G}$. It is easy to verify that

(2.24) $$\begin{pmatrix} \overline{\rho} & \overline{\sigma} \\ \nu & \mu \end{pmatrix}\begin{pmatrix} \mu & \sigma \\ \nu & \rho \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

the identity map of $F \oplus \overline{F}$.

Now choose a CM_p-lattice

$$N \oplus N^{\wedge} \subseteq G \oplus \overline{G}$$

where $N \subseteq G$, $N^{\wedge} = \mathrm{Hom}(N,V_p)$ and $N^{\wedge} \subseteq \overline{G}$ according to (2.16). Similarly, choose a CM_p-lattice

$$L \oplus L^{\wedge} \subseteq F \oplus \overline{F},$$

this time so that

$$A(L \oplus L^{\wedge}) \subseteq N \oplus N^{\wedge}.$$

By restricting the entries of the above matrix for A to L and $L^{\wedge}$ we set

$$B = A|L \oplus L^{\wedge} = \begin{pmatrix} m & s \\ n & r \end{pmatrix}\colon L \oplus L^{\wedge} \to N \oplus N^{\wedge}.$$

Let $B^{\wedge}\colon N \oplus N^{\wedge} \to L \oplus L^{\wedge}$ be given by

$$\begin{pmatrix} r^{\wedge} & s^{\wedge} \\ n^{\wedge} & m^{\wedge} \end{pmatrix}$$

Then by (2.24),

$$B^{\wedge}B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{\wedge} \end{pmatrix}\colon L \oplus L^{\wedge} \to L \oplus L^{\wedge}$$

Let $H = \mathrm{cok}\,\zeta (\in CM_{p+1}$ by (1.2)), so that $H^{\sim} (:= \mathrm{Hom}(H,V_{p+1})) = \mathrm{cok}\,\zeta^{\wedge}$ by (1.6). Let $K = \mathrm{cok}\,B$ and set

$$\Delta\colon K \to H \oplus H^{\sim}$$

equal to the map $\text{cok } B \to \text{cok } B^{\wedge}B$ induced by $B^{\wedge}$. Then $\text{im}(\Delta)$ is a sublagrangian of the skew-symmetric hyperbolic form on $H \oplus H^{\sim}$ so $(K,H,\Delta) \in F^{-1}(CM_{p+1})$. Further details, including well-definedness, are an easy generalization of [P1,§6].

(2.25) Exactness of the sequences (2.1)(a)-(d): The exactness of the first sequence is part of the theorem of [P1]. The exactness of c) follows easily from [P1,§8]. This will use the fact that lagrangians are hyperbolic ((1.20)), so that given the above construction of the maps involved, exactness is another straightforward generalization.

Given the Remark (2.15), exactness at $W_0^{\lambda}(CM_p)$ in b) and d) is immediate from (1.21). It follows from the constructions in [P3, §3] that a λ-formation (K,H,Δ), with $\Delta = (\alpha,\gamma)$ and α injective, represents an element in $\text{im } \mathcal{D}_0^{-\lambda}$. Thus exactness at $W_1^{\lambda}(CM_1)$ in b) follows from (8. 4).

Given what we now have, it remains to prove $W_0^1(CM_4) \to W_1^{-1}(CM_3)$ is surjective in d). (The analogous fact in b) was proved using the fact that A is a DVR at all height one primes.). Since we now also know that c) is exact and the extreme terms are zero, it is sufficient to prove that the composite

(2.26) $$W_0^1(CM_4) \to W_1^{-1}(CM_3) \to W_0^{-1}(CM_2)$$

is surjective. This is what was proved in [P3,§3] except that in place of the sequence $\{4,3,2\}$ of heights, the descent was from height 2 to height 0. The surjectivity of (2.26) is a straightforward variation.

§3. The Witt group and A_0.

To state the result we first recall the definition of the Chow group of cycles mod rational equivalence ([Ch], [Fu]). Let X be an n-dimensional scheme of finite type over a field. Let X_k denote the set of generic points of irreducible closed subsets of dimension k and Z_k the free abelian group on X_k. Let $x \in X_k$, $y \in X_{k+1}$, $x \in \overline{\{y\}}$, $f \in k(y)^{\times}$. Then $f = g/h$, $g,h \in \mathcal{O}_{Y,x}$, where Y is the integral subscheme corresponding to y. Define a homomorphism

$$b_y\colon k(y)^{\times} \to X_k$$

by

$$b_y(f) = \sum_{x\in\overline{\{y\}}} \text{length}(\mathcal{O}_{Y,x}/g\mathcal{O}_{Y,x})x - \sum_{x\in\overline{\{y\}}} \text{length}(\mathcal{O}_{Y,x}/h\mathcal{O}_{Y,x})x .$$

Finally, set

$$b_{k+1} := \coprod b_y\colon \coprod_{y\in X_{k+1}} k(y)^{\times} \to Z_k$$

and

$$A_k(X) := \text{cok } b_{k+1},$$

the k-th Chow group.

Next is the definition of "higher class groups" from [CF]. Given an A-regular sequence $\{x_1,\dots,x_{n-k}\}$, let

$$d(x_1,\dots,x_{n-k}) = \sum_{\text{ht}\,q=n-k} \text{length}(A_q/(x_1,\dots,x_{n-k})) \cdot q \in Z_k$$

where $Z_k = Z_k(\text{Spec } A)$, the free abelian group on height $n-k$ primes. Let $R_k \subseteq Z_k$ be the subgroup generated by expressions $d(x_1,\dots,x_{n-k})$, for all A-sequences $\{x_1,\dots,x_{n-k}\}$, and let

$$C_k = Z_k/R_k .$$

(This is denoted C_{n-k} in [CF]). For instance C_{n-1} is the class group of A, when A is a normal domain. If A is regular local and essentially of finite type over a field, then it is a consequence of truth of the Gersten conjecture in this case that the groups $W_*(A)$ of [CF] are trivial. Consequently, $C_*(A) \equiv 0$ also, by [CF,3.4].

(3.1) Theorem. Let A be a regular domain of finite type over a field k which is either real or algebraically closed. Let K be the fraction field of A and let $X = \text{Spec } A$. Then

a) if $\dim A = 4$, there is a surjection

$$A_0(X) \otimes \mathbb{Z}/2 \to \ker(W_0^1(A) \to W_0^1(K)) \otimes \mathbb{Z}/2$$

b) if $\dim A = 2$, there is a surjection

$$A_0(X) \otimes \mathbb{Z}/2 \to W_0^{-1}(A) \otimes \mathbb{Z}/2$$

$C_1(A) \otimes \mathbb{Z}/2 = 0$ in either case, then the corresponding map is an omorphism.

In some cases the left side of the surjection of a) and b) is own.

(3.2) Proposition. Let A be as in (3.1). Then

a) if $k = \bar{k}$, $A_0(X) \otimes \mathbb{Z}/2 = 0$;

b) if $k = \mathbb{R}$, and X is an open subscheme of $\bar{X}$, an integral heme proper over $\mathbb{R}$ where $X(\mathbb{R}) = \bar{X}(\mathbb{R})$, then $A_0(X) \otimes \mathbb{Z}/2 =$ $\mathbb{Z}/2)^r$ where r is the number of connected components of $X(\mathbb{R})$, in ıe Euclidean topology of $X(\mathbb{R})$.

(3.3) Remark. a) Suppose $A = \mathbb{R}[x_1,\ldots,x_n]/I$, A is a regular ımain and there is $f \in I$ such that $V(f_d) = \{0\}$ or $\emptyset$, where f_d ; the homogeneous part of f of highest degree. (Here $V(f_d)$ is ıe set of real points x such that $f_d(x) = 0$.) Then the conditions ı (3.2)(b) are satisfied on setting $X = \operatorname{Spec} A$ and $\bar{X}$ = the closure : X in $\mathbb{P}^n_{\mathbb{R}}$.

b) It is possible to make a statement in (3.2)(b) for k any ːal closed field (by defining "algebraic components"--cf. [DK]); and ır the case where $\bar{X}(\mathbb{R}) \supsetneq X(\mathbb{R})$. These are left to the reader.

Proof of (3.1). What will be shown is the following more general ıct, which is sufficient by (2.1).

(3.4) Proposition. Let A be an n-dimensional regular Noetherian ımain of finite type over k, where k is either real or algebraically .osed. Let $X = \operatorname{Spec} A$ and identify $CM_n(A)$ with the category of ıdules of finite length. Then there is a surjection

$$A_0(X) \otimes \mathbb{Z}/2 \to \operatorname{cok}(L: \coprod_{\operatorname{ht}p=n-1} W_0^1(CM_{n-1}(A_p))$$

$$\to \coprod_{\operatorname{ht}p=n} W_0^1(CM_n(A_p))) \otimes \mathbb{Z}/2.$$

ıere $L = L_0^1$ is defined in (2.7). If $C_1(A) \otimes \mathbb{Z}/2 = 0$, then this ɜ an isomorphism.

Notice that (if $C_1(A) \otimes \mathbb{Z}/2 = 0$) the proposition is consistent

with Bloch's formula ([Q]) and a Gersten resolution for $W(A)$ (cf. [P4]).

To simplify notation, whenever $\operatorname{ht} p = p$, we replace $CM_p(A_p)$ by $F(A_p)$, the category of A_p-modules of finite length. (This uses (1.2).) Also $W(-)$ will always mean $W_0^1(-)$. If $(M,\phi) \in Q^\lambda(F(A_p))$ is a lagrangian, then the length of M is clearly even. Thus there is a homomorphism

$$\ell_p: W(F(A_p)) \to \mathbb{Z}/2$$

given by $\ell_p(M,\phi)$ = length of M, mod 2; its composition with the dévissage isomorphism is the usual rank homomorphism

$$\mathrm{rk}: W(k(p)) \to \mathbb{Z}/2 \ .$$

The proof of the proposition comes down to the following three lemmas.

(3.5) Lemma. Assume $C_1(A) = 0$. Then for each prime $p \subseteq A$ of height $n-1$, there is an element $\coprod (M_q,\phi_q) \in \coprod_{\operatorname{ht}q=n-1} W(F(A_q))$ such that

$$\ell_q(M_q,\phi_q) = \begin{cases} 1, & q = p \\ 0, & q \neq p \end{cases}$$

and

$$L(\coprod (M_q,\phi_q)) = 0.$$

(3.6) Lemma. Let $IW(F(A_p)) = \ker \ell_p$. Then there is a commutative diagram

$$\begin{array}{ccc}
\coprod_{\operatorname{ht}q=n-1} IW(F(A_q)) & \xrightarrow{L} & \coprod_{\operatorname{ht}M=n} W(F(A_M)) \\
\simeq \downarrow \text{dévissage} & & \simeq \downarrow \text{dévissage} \\
\coprod_{\operatorname{ht}q=n-1} IW(k(q)) & & \coprod_{\operatorname{ht}M=n} W(k(M)) \\
\downarrow \text{dis} & & \downarrow \text{rk} \\
\coprod_{\operatorname{ht}q=n-1} k(q)^\times/k(q)^{\times 2} & \xrightarrow{b_1 \otimes \mathbb{Z}/2} & \coprod_{\operatorname{ht}M=n} \mathbb{Z}/2
\end{array}$$

where dis and rk are the discriminant and rank mod 2, respectively.

(3.7) Lemma. For k either real or algebraically closed, there are isomorphisms

a) $\mathrm{rk}: W(k(M)) \otimes \mathbb{Z}/2 \xrightarrow{\simeq} \mathbb{Z}/2$,

where M is maximal in A, and

b) $I \otimes \mathbb{Z}/2 \stackrel{\simeq}{\to} I/I^2 \otimes \mathbb{Z}/2 \stackrel{\simeq}{\to} k(q)^x/k(q)^{x^2}$ where $I = IW(k(q))$, ht $q = n-1$ and the second isomorphism is induced by dis.

Assuming the three lemmas, the proof of (3.4) is completed as follows. Lemma (3.5) says that, if $C_1(A) = 0$, then im L = (im $L| \coprod_{htq=n-1} IW(F(A_q)))$ and Lemmas (3.6) and (3.7) then imply that, mod 2, this is im$(b_1 \otimes \mathbb{Z}/2)$; taken by themselves, (3.6) and (3.7) give the surjection in (3.4).

Proof of (3.5). Let p be a fixed height $n-1$ prime. Then the hypothesis $C_1(A) \otimes \mathbb{Z}/2 = 0$ means there are A-regular sequences

$$X_i = \{x_{i,1},\dots,x_{i,n-1}\}, \quad i = 1,\dots,k$$

such that for any height n-1 prime q,

$$\sum_i \text{length}(A_q/(X_i)_q) \equiv \begin{cases} 1 & q = p \\ 0, & q \neq p \end{cases}, \quad \text{mod } 2$$

By the reasoning for (2.5), there is a homomorphism

$$W(CM_0(A/(Z)) \to W(CM_{n-1}(A))$$

for any A-sequence Z with n-1 elements. For each $i = 1,\dots,k$, let $\alpha_i \in W(CM_{n-1}(A))$ be the image of $<1> \in W(CM_0(A/(X_i)))$.
Set

$$\phi = K(\Sigma\, \alpha_i)$$

where $K\colon W(CM_{n-1}(A)) \to \coprod_{htq=n-1} W(F(A_q))$ is from (2.1). It is an easy consequence of the definition of L in (2.7), that the composition $L \circ K = 0$. This means the desired element is a representative of ϕ.

Proof of (3.6). For any maximal ideal M of A, localization at M induces the vertical homomorphisms in the commutative diagram (cf. (1.4)) where $R = A_M$,

$$\begin{array}{ccc} \coprod_{htq=n-1} W(F(A_q)) & \stackrel{L}{\to} & \coprod_{htq=n} W(F(A_q)) \\ \downarrow & & \downarrow \\ \coprod_{\substack{htq=n-1 \\ q\subseteq M}} W(F(R_q)) & \stackrel{L}{\to} & W(F(R)). \end{array}$$

Note that $A_q = R_q$ if $q \subseteq M$ (a slight abuse of notation) and that the verticals are the consequent projections. Thus, to prove (3.6),

we may replace A with its localization R. To compute $L|W(F(R_p))$ for p of height $n-1$, the idea is to reduce to the 1-dimensional case by passing to R/p. However, since R/p is not, in general, Gorenstein, this does not seem to be possible. (But if dim R = 2, then R/p is Gorenstein and the following argument can be greatly simplified.) Instead we work with a complete intersection ideal in p, whose quotient is Gorenstein.

So fix p, ht $p = n-1$. Let $\{x_1,\dots,x_{n-1}\} \subseteq p$ be a regular sequence with

(3.8) $$(x_1,\dots,x_{n-1})_p = p\, R_p,$$

and set

$$S = R/(x_1,\dots,x_{n-1}),$$

a one-dimensional Gorenstein ring. From (1.5) it is easy to construct a commutative diagram

(3.9) $$\begin{array}{ccccc} S & \rightarrowtail & E_0(S) & \twoheadrightarrow & E_1(S) \\ \downarrow & & \downarrow & & \downarrow \\ V_{n-1}(R) & \rightarrowtail & E_{n-1}(R) & \twoheadrightarrow & V_n(R) = E_n(R) \end{array}$$

There are inclusions

$$CM_i(S) \to CM_{i+n-1}(R), \quad i = 0,1$$

and

(3.10) $$F(S_{\bar{\pi}}) \stackrel{=}{\to} F(R_\pi), \quad \bar{\pi} = \pi/(x_1,\dots,x_{n-1}).$$

From these facts we get an exact commutative diagram of localization sequences

(3.11) $$\begin{array}{ccccc} W(CM_0(S)) & \to & \coprod_{ht\bar{q}=0} W(F(S_{\bar{q}})) & \xrightarrow{L(S)} & W(F(S)) \\ \downarrow & & \downarrow & & \downarrow \\ W(CM_{n-1}(R)) & \to & \coprod_{htq=n-1} W(F(R_q)) & \xrightarrow{L = L(R)} & W(F(R)) \end{array}$$

(The exactness of the top sequence can be extracted from [P1]). Using dévissage the right vertical is an isomorphism; it is similarly seen that the middle vertical is an inclusion to the summands corresponding to q containing $(x_1,\dots,x_{n-1})$.

Now if Σ is the set of non-zero-divisors of S, there is a canonical map $S_\Sigma \to E_0(S)$ extending $S \to E_0(S)$, hence also an induced map $S_\Sigma/S \to E_1(S)$ of S-modules. Since

$$S \rightarrowtail S_\Sigma \twoheadrightarrow S_\Sigma/S$$

is also a minimal injective resolution of S over itself (cf. [B, 6.2]) these maps are isomorphisms making

$$\begin{array}{ccccc} S & \rightarrowtail & S_\Sigma & \twoheadrightarrow & S_\Sigma/S \\ \downarrow = & & \downarrow \simeq & & \downarrow \simeq \\ S & \rightarrowtail & E_0(S) & \twoheadrightarrow & E_1(S) \end{array}$$

commutative. We also have $E_0(S) = \coprod E(S/\bar{q})$ and (clearly) $S_\Sigma \overset{\simeq}{\to} \coprod S_{\bar{q}}$ where the sums are over the minimal primes $\bar{q}$ of S. Hence there are canonical isomorphisms

(3.12) $$S_{\bar{q}} \overset{\simeq}{\to} E(S/\bar{q}) ,$$

the localizations of $S \to E_0(S)$ at the minimal primes $\bar{q} \subseteq S$.

Next let

$$(0) = Q_1 \cap \dots \cap Q_r$$

be the minimal primary decomposition of the zero ideal in S, with associated primes $\bar{q}_1, \dots, \bar{q}_r$. Let

$$\bar{q}_1 = \bar{p} := p/(x_1, \dots, x_{n-1}) .$$

Then by (3.8),

(3.12) $$Q_1 = \bar{q}_1, \quad \text{so} \quad S_{\bar{q}_1} = k(p) .$$

Since the decomposition is unmixed,

$$Q_i = \ker(S \to S_{\bar{q}_i}), \quad i = 1, \dots, r.$$

From this there are canonical factorizations of $S \to S_{\bar{q}_i}$,

(3.13) $$\begin{array}{ccc} S & \to & S_{\bar{q}_i} \\ & \searrow \; \swarrow & \\ & S/Q_i & \end{array}$$

We are now prepared to prove (3.6) commutes (when A is local).

It is well-known that $IW(k(p))$ $(= IW(F(R_p)))$ is generated by binary forms $\langle f,-1 \rangle$, $f \in k(p) = k(\bar{p})$. Clearing its denominator we may take f to be in the image of $\bigcap_{i \neq 1} Q_i - \bar{q}_1$ in $k(p)$; similarly replace -1 by minus the square of an element of $\bigcap_{i \neq 1} Q_i - \bar{q}_1$. The result is a symmetric bilinear form

$$\phi_1 : (S/\bar{q}_1)^2 \times (S/\bar{q}_1)^2 \to S_{\bar{q}_1}$$

such that

$$\operatorname{im} \phi_1 \subseteq \operatorname{im}((\bigcap_{i \neq 1} Q_i - \bar{q}_1) \to S_{\bar{q}_1})$$

and

$$\phi_1 \otimes S_{\bar{q}_1} \simeq \langle f,-1 \rangle .$$

Similarly, for each $j = 2, \dots, r$, choose

$$\phi_j : (S/Q_j)^2 \times (S/Q_j)^2 \to S_{\bar{q}_j}$$

with

$$\operatorname{im} \phi_j \subseteq \operatorname{im}((\bigcap_{i \neq j} Q_i - \bar{q}_j) \to S_{\bar{q}_j}$$

and

$$\phi_j \otimes S_{\bar{q}_j} \simeq \langle f_j, -f_j \rangle$$

for some $f_j \in S_{\bar{q}_j}$. (This will use the fact that $\bigcap_{i \neq j} Q_i - \bar{q}_j \neq \emptyset$.)

Choose liftings $a \in \bigcap_{i \neq 1} Q_i - \bar{q}_1$ for f and $a_j \in \bigcap_{i \neq j} Q_i - \bar{q}_j$ for f_j. Then

$$b := a + a_2 + \dots + a_r, \quad b' := a' - a_2 \dots - a_r$$

are not zero-divisors in S. Indeed, if so, then $b \in \cup \bar{q}_j$, so $b \in \bar{q}_1$ (say). But for each $j \neq 1$, $a_j \in Q_1 \subseteq \bar{q}_1$, which implies $a \in \bar{q}_1$, contradicting the choice of a.

Now let $\psi : S^2 \times S^2 \to S$ have matrix $\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}$. Then

$$\psi \otimes S_{\bar{q}_i} \simeq \begin{cases} \begin{pmatrix} f & 0 \\ 0 & -1 \end{pmatrix}, & i = 1 \\ \begin{pmatrix} f_i & 0 \\ 0 & -f_i \end{pmatrix}, & i \neq 1 , \end{cases}$$

because $a_j \in \bigcap_{k \neq j} Q_k$ localizes to zero in $S_{\bar{q}_i}$ if $i \neq j$. Referring to (2.8), this means we have found a lattice (S^2, ψ) for the element of $\coprod_{\operatorname{ht} \bar{q} = 0} W(F(S_{\bar{q}}))$ represented by $\langle f,-1 \rangle \in W(k(\bar{q})) = W(F(S_{\bar{q}}))$.

Referring now to the diagram (3.11), the image $L(S)\langle f,-1 \rangle \in W(F(S))$ is supported on the S-module

$$\operatorname{cok}(\operatorname{Ad} \psi : S^2 \to S^2) ;$$

this is immediate from the definition (2.7) of $L(S)$. Since we have seen that the right vertical in the diagram is an isomorphism and the center one induces $W(F(S_{\bar{q}_1})) \overset{\simeq}{\to} W(F(R_p))$, we will be done if we can show that

$$\text{length}(\text{cok}(\text{Ad } \psi)) \equiv b_1(\bar{a}) \mod 2$$

here $\bar{a}$ is the image of a in $S/\bar{q}_1 = R/p$.

By a formula of Grothendieck ([Gr, IV.21.10.17.8]),

$$(3.14) \quad \ell_S(\text{cok Ad } \psi) = \Sigma\, \ell_{S_{\bar{q}_i}}(S_{\bar{q}_i})\,\ell_{S/\bar{q}_i}(\text{cok}((\text{Ad } \psi) \otimes S/\bar{q}_i))$$

here the subscript on ℓ indicates the local ring with respect to hich length ℓ is being computed. Now for each $i > 1$,

$$\psi \otimes S/\bar{q}_i = \begin{pmatrix} \bar{a}_i & 0 \\ 0 & -\bar{a}_i \end{pmatrix}$$

here $\bar{a}_i \neq 0$ and is the image of a_i in $S/\bar{q}_i$ (because $a_j \in Q_j \subseteq$ $_j$ for $j \neq i$); also, by construction, $\bar{a}'$ is a square in the domain $/\bar{q}_1 = S/\bar{p}$. Finally, $S_{\bar{q}_1} = k(\bar{q}_1)$, so $\ell_{S_{\bar{q}_1}}(S_{\bar{q}_1}) = 1$. Thus, from 3.14) we get, mod 2,

$$\ell_S(\text{cok Ad } \psi) \equiv \ell_{S/\bar{p}}(\text{cok}(\bar{a}\colon S/\bar{p} \to S/\bar{p}))$$

ince the latter is $b_1(\bar{a})$, the proof of (3.6) is complete.

Proof of (3.7). If k is algebraically closed, $\text{rk}\colon W(k) \xrightarrow{\approx} \mathbb{Z}/2$ nd if k is real closed, $W(k) \xrightarrow{\approx} \mathbb{Z}$ by the signature. Since rank nd signature are congruent mod two and since $k(M)$ is either real or lgebraically closed, we get a). The second follows from [L] and Kl, §11] and uses the fact that if $\text{ht}\, p = n - 1$, then $k(p)$ has ranscendence degree one over k.

Proof of (3.2). If $k = \bar{k}$, then we claim $A_0(X)$ is divisible; his is sufficient for a). Each closed point of X is in the image f a proper map from a smooth affine curve C. Covariance of A_0 for roper maps ([Fu, 1.9]) reduces the claim to the case $X = C$. Complete to a smooth projective curve $\bar{C}$; the existence of a Jacobian for implies that the group of zero-cycles of degree zero, $\tilde{A}_0(\bar{C})$, is ivisible. Now from the exact sequence of [Fu, 1.9],

$$A_0(\text{pts.}) \to A_0(\bar{C}) \twoheadrightarrow A_0(C),$$

o $A_0(C)$ also divisible since $A_0(\bar{C}) \cong \tilde{A}_0(\bar{C}) \oplus \mathbb{Z}$.

For (3.2)(b), tensor [Fu, 1.9] with $\mathbb{Z}/2$ to get an exact sequence

$$A_0(\bar{X} - X) \otimes \mathbb{Z}/2 \xrightarrow{i} A_0(\bar{X}) \otimes \mathbb{Z}/2 \twoheadrightarrow A_0(X) \otimes \mathbb{Z}/2 .$$

rom [CI, 3.2], it follows that there is an isomorphism

$$\theta\colon A_0(\bar{X}) \otimes \mathbb{Z}/2 \xrightarrow{\approx} (\mathbb{Z}/2)^r .$$

(The argument in [CI, (3.2)(i)] requires that $\bar{X}/\mathbb{R}$ be smooth in order to prove $\tilde{A}_0(X_{\mathbb{C}})$ is divisible using Bertini's theorem. This hypothesis is avoided with the argument in the $k = \bar{k}$ case above.) In fact, looking at (3.2)(i) and the proof of (3.1) in [CI], θ is induced by the natural map $Z_0(\bar{X}) \to (\mathbb{Z}/2)^r$ which sends complex points to zero and a real point in the i-th component of $\bar{X}(\mathbb{R})$ to the i-th standard basis vector of $(\mathbb{Z}/2)^r$. But since we are assuming $(\bar{X} - X)(\mathbb{R}) = \emptyset$, $\theta \circ i = 0$. Hence $i = 0$, which means $A_0(X) \otimes \mathbb{Z}/2 \xrightarrow{\approx} (\mathbb{Z}/2)^r$.

(I learned the above argument for the $k = \bar{k}$ case from V. Srinivas. The reference for (3.2)(b) and a correction of its attempted proof I owe to J.-L. Colliot-Thélène.)

Examples: skew-symmetric forms on surfaces and algebra structures on resolutions.

We begin by summarizing (without proofs) Ferrand's [F, §1] descrip- of a construction due to Serre and Horrocks. Throughout, I is a ıt two ideal of an n-dimensional regular domain A such that $\in CM_{n-2}(A)$.

Suppose given an isomorphism $\eta: A/I \to Ext^2(A/I,A)$; then there is xtension (ε) in $Ext^1(I,A)$ corresponding to $\eta(1)$ under the ɔrphism

$$Ext^2(A/I,A) \simeq Ext^1(I,A).$$

$$(\varepsilon) = (A \overset{s'}{\rightarrowtail} E \overset{s}{\twoheadrightarrow} I)$$
)

E is projective and there is an isomorphism $\sigma: \Lambda^2 E \overset{\simeq}{\to} A$ defined $s'\sigma(x \wedge y) = s(x)y - s(y)x$. Thus, setting $h: E \to A$ equal to s .owed by the inclusion of I into A, the top line in

$$\begin{array}{ccccccc} A & \overset{s'}{\rightarrowtail} & E & \overset{h}{\to} & A & \twoheadrightarrow & A/I \\ \downarrow & & \downarrow Ad\phi \downarrow & & \eta' \downarrow & & \searrow \eta \\ \bar{A} & \overset{\bar{h}}{\rightarrowtail} & \bar{E} & \to & \bar{A} & \twoheadrightarrow & Ext^1(I,A) \simeq Ext^2(A/I,A) \end{array}$$
:)

(using σ') a Koszul resolution of A/I; the maps $A \to \bar{A}(= Hom(A,A))$ the obvious ones; ϕ is the non-singular skew-symmetric form $E \to \Lambda^2 E \overset{\sigma}{\to} A$; the bottom line is the Ext-sequence of (ε); η' is ɹced by the verticals on its left; and the whole diagram is commu- ive.

Conversely, given rank 2 A-projective E and $h: E \to A$ a regular ection (at any prime containing $im(h)$ the image under h of a is of E is a regular sequence of A) such that $\Lambda^2 E \simeq A$, we have Koszul resolution of A/I, which can be dualized to reproduce (4.2) hence also the isomorphism $\eta: A/I \overset{\simeq}{\to} Ext^2(A/I,A)$.

The two constructions are inverse in an appropriate sense.

Using the canonical isomorphism $Ext^2(A/I,A) \simeq Hom(A/I,V_2)$, we regard the initial data in the first construction as coming from element $(A/I,\mu) \in Q^1(CM_2(A))$, and the resulting pair (E,ϕ) as an ment of $Q^{-1}(CM_0(A))$.

(4.3) Proposition. The diagram

$$\begin{array}{ccccc} Q^1(CM_2(A)) & & \overset{SH}{\longrightarrow} & & Q^{-1}(CM_0(A)) \\ \downarrow & & & & \mathcal{D}_1^{-1} \downarrow \\ W_0^1(CM_2(A)) & \overset{\mathcal{D}_0^1}{\to} & W_1^{-1}(CM_1(A)) & \overset{\mathcal{D}_1^{-1}}{\to} & W_0^{-1}(CM_0(A)) \end{array}$$

commutes, where SH is the Serre-Horrocks construction.

Proof: We recall from (2.6) and (2.18) the definition of the maps $\mathcal{D}_0^1$ and $\mathcal{D}_1^{-1}$. Given $(A/I,\mu) \in Q^1(CM_2(A))$ choose $a \in I$, $a \neq 0$, hence a surjection $j: A/aA \to A/I$ and lift μ to τ in

$$\begin{array}{ccc} A/aA \times A/aA & \overset{\tau}{\to} & E_1 \\ \downarrow j \times j & & \downarrow d_1 \\ A/I \times A/I & \overset{\mu}{\to} & V_2 \end{array}$$

Setting $A/aA = H$, $I/aA = K$, $(inc: I/aA \to A/aA) = \alpha$ and defining γ by the commutativity of

$$\begin{array}{ccc} I/aA & \overset{\alpha}{\to} & A/aA \\ \downarrow \gamma & & \downarrow Ad\ \tau \\ H\hat{} =: Hom(A/aA,V_1) & \to & Hom(A/aA,E_1), \end{array}$$

the triple $(K,H,(\alpha,\gamma)) \in F^{-1}(CM_1(A))$ represents $\mathcal{D}_0^1(A/I,\mu)$.

Now let $J: A \oplus A \to H \oplus H\hat{} \simeq A/aA \oplus A/aA$ be the natural surjection and set

(4.4) $$E' = J^{-1}(im(\alpha,\gamma)) \subseteq A \oplus A$$

Then E' is A-projective, and there is a non-singular skew-symmetric form $\phi': E' \times E' \to A$ such that $\mathcal{D}_1^{-1}[K,H,(\alpha,\gamma)] = [E',\phi'] \in W_0^1(A)$.

Now consider the commutative diagram, with exact rows and columns,

$$\begin{array}{ccccc} Hom(A/I,V_2) & & & & \\ \downarrow & & & & \\ Hom(A,V_2) & \leftarrow & Hom(A,E_1) & \leftarrow & Hom(A,V_1) \\ \downarrow & & \downarrow & & \downarrow \\ Hom(I,V_2) & \leftarrow & Hom(I,E_1) & \leftarrow & Hom(I,V_1) \\ & & & & \downarrow \\ & & & & Ext^1(A/I,V_1) \end{array}$$

Let $\mu_0 = (Ad\ \mu)(1) \in Hom(A/I,V_2)$ and let $\bar{\mu}_0$ be its image in $Hom(A,V_2)$. View $(Ad\ \tau)(1)$ as a map $A \to E_1$; it then maps to $\bar{\mu}_0$ and so its image in $Hom(I,E_1)$ lifts back to a unique element $f \in Hom(I,V_1)$, where

$$\begin{array}{ccc} I & \overset{f}{\to} & V_1 \\ \downarrow p & & \uparrow \\ I/aA & \overset{\gamma}{\to} & \mathrm{Hom}(A/aA,V_1) \end{array}$$

ommutes. Further, the image of f under $\mathrm{Hom}(I,V_1) \to \mathrm{Ext}^1(A/I,V_1)$ quals the image of μ_0 under $\mathrm{Hom}(A/I,V_2) \overset{\simeq}{\to} \mathrm{Ext}^1(A/I,V_1)$. From this, he commutativity of

$$\begin{array}{ccccc} & & \mathrm{Hom}(I,V_1) & \to & \mathrm{Ext}^1(I,A) \\ & & \downarrow & & \downarrow\simeq \\ \mathrm{Hom}(A/I,V_2) & \overset{\simeq}{\to} & \mathrm{Ext}^1(A/I,V_1) & \overset{\simeq}{\to} & \mathrm{Ext}^2(A/I,A) \ , \\ \mu_0 & & \longmapsto & & \eta(1) \end{array}$$

nd the definition of $\mathrm{Hom}(I,V_1) \to \mathrm{Ext}^1(I,A)$ we see that the extension ε) corresponding to $\eta(1)$ (or to μ_0) in the Serre-Horrocks con- truction is the top line in the pull-back

$$\begin{array}{ccccc} A & \rightarrowtail & E & \overset{s}{\twoheadrightarrow} & I \\ \downarrow = & & \downarrow & & \downarrow f \\ A = V_0 & \rightarrowtail & E_0 & \twoheadrightarrow & V_1 \end{array}$$

rom the commutative diagram

$$\begin{array}{ccccccc} A & \rightarrowtail & E_0 & \twoheadrightarrow & V_1 & \overset{f}{\longleftarrow} & \\ \updownarrow = & & \uparrow & & \uparrow & & \\ A & \rightarrowtail & A[a^{-1}] & \twoheadrightarrow & (0:a) = \mathrm{Hom}(A/aA,V_1) & \overset{\gamma p}{\longleftarrow} & I \\ \uparrow = & & \uparrow xa^{-1} & & \uparrow = & & \\ A & \overset{xa}{\rightarrowtail} & A & \twoheadrightarrow & A/aA & & \end{array}$$

e see that (ε) is likewise the top line of the pull-back in

$$\begin{array}{ccccc} A & \rightarrowtail & E & \overset{s}{\twoheadrightarrow} & I \\ \downarrow = & & \downarrow & & \downarrow \gamma p \\ A & \overset{a}{\rightarrowtail} & A & \twoheadrightarrow & \mathrm{Hom}(A/aA,V_1) \end{array}$$

etting h equal to s followed by the inclusion of I into A, we et a map $E \to A \oplus A$ and it is immediate that E' (from (4.4)) and define the same submodule of $A \oplus A$. One shows routinely that $E',\phi') = (E,\phi) := SH(A/I,\eta)$.

As a corollary of (4.3) and (3.1) we get the following description f $W_0^{-1}(A) \otimes \mathbb{Z}/2$, where A is a smooth 2-dimensional real affine k- lgebra satisfying the conditions of (3.2)(b). Let $I_1,\ldots,I_r \subseteq A$ be deals representing a basis of $A_0(\mathrm{Spec}\ A) \otimes \mathbb{Z}/2$. These can be

taken to be the maximal ideals of real points in the r topological components of $(\mathrm{Spec}\ A)(\mathbb{R})$. Then the unary forms $\langle 1\rangle : A/I_i \times A/I_i \to \mathbb{R} \to E(A/I_i)$ give elements (ε_i) in $\mathrm{Ext}^1(I_i,A)$ as above, hence rank 2 projectives $P_1,\ldots,P_r$.

(4.5) <u>Proposition</u>. The skew-symmetric forms

$$P_i \times P_i \to \Lambda^2 P_i \simeq A$$

generate $W_0^{-1}(A) \otimes \mathbb{Z}/2$ as a $\mathbb{Z}/2$-vector space; they form a basis if $\mathrm{Pic}(A) \otimes \mathbb{Z}/2 = 0$.

(4.6) <u>Remark</u>. I don't know whether class group assumptions are necessary here or in (3.1). Observe also that no class group assumption is needed for $\Lambda^2 P_i$ to be free: by construction, the P_i come from codimension 2 in the Grothendieck filtration of $\tilde{K}_0$.

(4.6) We next give an example of the kind of forms predicted by the Serre-Horrocks construction in dimensions 2 and 4. However, the connection between the 4-dimensional example and the Koszul resolution will not be justified.

Let $A_n = \mathbb{R}[x_0,\ldots,x_n]/(\Sigma\, x_i^2 - 1)$ be the real co-ordinate ring of the n-sphere. According to [F] the upper left $2n \times 2n$ block of the following matrix defines an endomorphism $\alpha_n\colon (A_n)^{2n} \to (A_n)^{2n}$ whose kernel (or cokernel) generates $\tilde{K}_0(A_n)$, $n = 1,2$, or 4.

$$\begin{pmatrix} 1-x_0 & -x_1 & -x_2 & 0 & -x_3 & 0 & 0 & -x_4 \\ -x_1 & 1+x_0 & 0 & -x_2 & 0 & -x_3 & x_4 & 0 \\ -x_2 & 0 & 1+x_0 & x_1 & 0 & -x_4 & -x_3 & 0 \\ 0 & -x_2 & x_1 & 1-x_0 & x_4 & 0 & 0 & -x_3 \\ -x_3 & 0 & 0 & x_4 & 1+x_0 & x_1 & x_2 & 0 \\ 0 & -x_3 & -x_4 & 0 & x_1 & 1-x_0 & 0 & x_2 \\ 0 & x_4 & -x_3 & 0 & x_2 & 0 & 1-x_0 & -x_1 \\ -x_4 & 0 & 0 & -x_3 & 0 & x_2 & -x_1 & 1+x_0 \end{pmatrix}$$

Let $P_n = \ker \alpha_n$. Define $h\colon P_n \to A_n$ by $(b_1,\ldots,b_{2n}) \to b_1$. I claim h is a regular cosection $(n = 1,2,$ or $4)$ and will verify it when $n = 4$. To simplify notation, drop the subscript n's. First $(1 + x_0, 1 - x_0)$ is the unit ideal so $\mathrm{Spec}(A) = \mathrm{Spec}(A_{1+x_0}) \cup \mathrm{Spec}(A_{1-x_0})$. Then $(b_1,\ldots,b_8) \in P_{1-x_0}$ can be written

$$\Big(\frac{x_1}{1-x_0}b_2 + \frac{x_2}{1-x_0}b_3 + \frac{x_3}{1-x_0}b_5 + \frac{x_4}{1-x_0}b_8, b_2, b_3,$$
$$\frac{x_2}{1-x_0}b_2 - \frac{x_1}{1-x_0}b_3 - \frac{x_4}{1-x_0}b_5 + \frac{x_3}{1-x_0}b_8, b_5, \ldots\Big)$$

Hence a basis for P_{1-x_0} can be gotten by setting successively each of b_2, b_3, b_5 and b_8 equal to 1 and the others equal to zero. The image under h of this basis is $\left\{\frac{x_1}{1-x_0}, \frac{x_2}{1-x_0}, \frac{x_3}{1-x_0}, \frac{x_4}{1-x_0}\right\}$, a regular sequence in A_{1-x_0}. The ideal it generates has zero set equal to $(-1,0,0,0,0) \in S^4 - \{(1,0,0,0,0)\} = (\mathrm{Spec}\, A_{1-x_0})(\mathbb{R})$. Inverting $1 + x_0$ and carrying out the same procedure, the image of a basis turns out to be the unit ideal. Thus h is a regular cosection and $V(\mathrm{im}(h)) = \{(-1,0,0,0,0)\}$, a real point which generates $A_0(S^4) \otimes \mathbb{Z}/2 = \mathbb{Z}/2$.

Now consider the form

$$\phi\colon \Lambda^2 P \times \Lambda^2 P \to \Lambda^4 P \overset{e}{\underset{\simeq}{\to}} A$$

where $\phi(x \wedge y, z \wedge w) = e(x \wedge y \wedge z \wedge w)$ and e is some isomorphism. The claim is that the class of $(\Lambda^2 P, \phi) \in Q^1(A)$ in $W_0^1(A)$ is the non-zero element of $\ker(W_0^1(A) \to W_0^1(K)) \otimes \mathbb{Z}/2$ (corresponding to the above generator of $A_0(S^4) \otimes \mathbb{Z}/2$). Going through the same construction, but using instead the co-ordinate ring A_2 of S^2, the class of $(P,\phi) \in Q^{-1}(A_2)$ (where P is a rank 2 projective) is the non-trivial element of $W_0^{-1}(A_2)$ by (4.5), since A_2 is a UFD. But in the 4-dimensional case, it is not known (to me) whether $C_1(A_4) \otimes \mathbb{Z}/2 = 0$, so it may be of some interest to show that the above element of $\ker(W_0^1(A) \to W_0^1(K)) \otimes \mathbb{Z}/2$ is non-trivial. (That the form $(\Lambda^2 P, \phi)$ over A_4 becomes hyperbolic over the fraction field is left to the reader.)

Kere is a sketch of the non-triviality of $[\Lambda^2 P, \phi]$ in $W_0^1(A)$, found with the help of M. Ojanguren.

First, Fossum [Ibid.] showed that the usual Serre-Swan construction gives a surjection

$$\tilde{K}_0(A) \overset{\simeq}{\to} \widetilde{KO}(S^4) \ ,$$

so we can use the computation of Adams operations on the generator of $\widetilde{KO}(S^4)$ in [Hu, p. 175] together with a simple computation to show

(4.7) $$[\Lambda^2 \xi] - [\varepsilon^6] = 2g$$

where ξ is the vector bundle corresponding to P, ε^6 is the 6-dimensional trivial bundle and g is the generator of $\widetilde{KO}(S^4)$. Now view $(\Lambda^2 P, \phi)$ as an element of the topological Witt group [MH, App. 1].

$$W(S^4) \overset{\simeq}{\to} KO(S^4)$$

where the isomorphism sends a "bundle of forms" η to $[\eta_+] - [\eta_-]$, and $\eta_+ (\eta_-)$ is the positive- (negative-) definite subbundle. Hence if $(\Lambda^2 P, \phi)$ is hyperbolic, then the corresponding topological bundle of forms is also: hence it is zero in $W(S^4)$. The isomorphism above thus says $[(\Lambda^2\xi)_+] = [(\Lambda^2\xi)_-]$ in $KO(S^4)$. Combining this with (4.7) we get

(4.8) $$g = [(\Lambda^2\xi)_+] - [\varepsilon^3] \quad \text{in} \quad \widetilde{KO}(S^4).$$

But $\pi_4(BO_3) \to \pi_4(BO)$ is not surjective (it is $\mathbb{Z} \overset{\times 2}{\to} \mathbb{Z}$ by [Hu, 5.1211] so $[(\Lambda^2\xi)_+] - [\varepsilon^3]$ cannot generate $\widetilde{KO}(S^4)$, contradicting (4.8).

(4.9) We now make a connection between the results of §2 (specifically, (2. 5)) and the constructions in §6. This involves the theory of algebra structures on resolutions from [KM] and [BE].

Let A be a regular local ring of dimension n and I an ideal such that A/I is Gorenstein.

Let

$$F = (F_p \overset{d_p}{\to} F_{p-1} \to \quad \overset{d_1}{\to} F_0 = A \to A/I)$$

be a free resolution of A/I. A differential graded commutative algebra structure on F is a multiplication $F \otimes F \to F$, $F_i F_j \subseteq F_{i+j}$, such that if $\deg x_\ell = \ell$,

(i) $d(x_i x_j) = d(x_i)x_j + (-1)^i x_i \, dx_j$,

(ii) $x_i x_j = (-1)^{ij} x_j x_i$.

By [BE, (1.1), (1.5)] each such F supports an algebra structure (not in general associative), and if F is minimal, the multiplication $\sum_{i+j=p} F_i \otimes F_j \to F_p \simeq A$ induces isomorphisms $\sigma_k\colon F_k \to F_{p-k}^*$, $0 \le k \le p$ such that

a) $\sigma_{k-1} d_{k-1} = (-1)^{k+1} d_{p-k+1}^* \sigma_k$

and by (ii) above one also has

b) $\sigma_k^* = (-1)^{k(p-k)} \sigma_{p-k}$.

Now F has only one non-zero homology group, $H_0(F) = A/I$; hence, since A/I is Gorenstein, the only non-zero cohomology group is $H^p(F) \simeq \mathrm{Ext}^p(A/I,A) \simeq \mathrm{Hom}(A/I, V_p)$. This means that if we define $\phi_r = \sigma_{p-r}^{-1}$, $\psi(F,d,\phi)$ satisfies (6. 1), and so defines a Poincare complex $(F,d,\{\psi\}) \in P_p^1(M_p)$, where $\psi + T_1\psi = \phi$ (cf. (6.2ff)).

According to [KM, (1.5)], when $p = 2q$ is even, the map $\sigma_q\colon F_q \to F_q^*$ defines a non-singular symmetric bilinear form $s\colon F_q \times F_q \to A$ which becomes hyperbolic at the generic point. By (2. 5) if $n = p = 4$ (F_q, s) is also hyperbolic, making the algebra structure in [KM, Thm.

1.1] canonical in a sense.

Thus, if one believes the claim of (7.1), then the algebra structures above (when $n = p = 4$) are null-cobordant and this fact is equivalent to the algebra structures being canonical.

§5. Dévissage.

The purpose of this section is to prove Theorem (2. 2). To simplify notation, we write $F(R)$ for the category of R-modules of finite length, where R is local Gorenstein. If $\dim R = n$, then this is $CM_n(R)$ by (1.2).

Begin with surjectivity. Let $(M,\phi) \in \Omega^\lambda(F(R))$ and let ℓ be the smallest integer for which $m^\ell M = 0$. If $\ell = 1$, there is nothing to prove, so we suppose $\ell > 1$, and set $M_i = m^i M$.

Claim. $M_{\ell-1} \subseteq (M_{\ell-1})^\perp$ and the naturally induced form $\phi': M_{\ell-1}^\perp/M_{\ell-1} \times M_{\ell-1}^\perp/M_{\ell-1} \to E := E_n(R)$ is non-singular.

Assuming this, it follows from (1.9) that $[M_{\ell-1}^\perp/M_{\ell-1},\phi'] = [M,\phi]$ in $W_0^\lambda(F(R))$. But since $M_{\ell-1}^\perp/M_{\ell-1} \subseteq M/M_{\ell-1}$, $m^{\ell-1}(M_{\ell-1}^\perp/M_{\ell-1}) = 0$ so induction on ℓ finishes the proof of surjectivity.

To prove the first statement of the claim, let $m_1,\ldots,m_k$ generate M and let $a,b \in m^{\ell-1}M$. Then $a = \Sigma\, p_i m_i$, $b = \Sigma\, q_j m_j$ where $p_i, q_j \in m^{\ell-1}$. Then $\phi(a,b) = \sum_{i,j} \phi(p_i m_i, q_j m_j) = \sum_{i,j} \phi(p_i q_j m_i, m_j) = 0$ since $q_i p_j \in m^{2(\ell-1)}$ and $2(\ell - 1) \geq \ell$. To prove the second statement, we need to know that

$$K = K^{\perp\perp}$$

for any $K \subseteq M$. Clearly $K \subseteq K^{\perp\perp}$. There is an obvious bilinear form $K^\perp \times M/K \to E$ inducing an isomorphism $K^\perp \xrightarrow{\cong} \mathrm{Hom}(M/K,E)$. Since $\ell(M/K) = \ell(\mathrm{Hom}(M/K,E))$ ([B, (2.1)iv] where ℓ denotes length, it follows that $\ell(K^\perp) + \ell(K) = \ell(M)$. Similiarly, $\ell(K^{\perp\perp}) + \ell(K^\perp) = \ell(M)$, so $\ell(K) = \ell(K^{\perp\perp})$. Since $K \subseteq K^{\perp\perp}$, $K = K^{\perp\perp}$. Applying this to $K = M_{\ell-1}$ gives the injectivity of Ad ϕ'; surjectivity follows from this and [ibid.]. Hence ϕ' is non-singular and the claim is proved.

Next suppose given $(k(m)^n,\phi)$ whose class vanishes in $W_0^\lambda(F(R))$. This means there are lagrangians (M_1,γ_1), $(M_2,\gamma_2) \in \Omega^\lambda(F(R))$ and an isometry

$$(k(m)^n,\phi) \oplus (M_1,\gamma_1) \cong (M_2,\gamma_2) .$$

Using the technique of the claim above, we may assume $mM_1 = 0$, but it remains to show (M_1,γ_1) and (M_2,γ_2) are still lagrangians. It is evidently sufficient to show that if (M,γ) is a lagrangian and $L \subseteq M$ satisfies $L \subseteq L^\perp$, then the induced form on $L^\perp/L$ is a lagrangian. The following stronger statement suffices for this.

(5.1) Lemma. Let (M,γ) be a lagrangian and let $L \subseteq M$ satisfy $L \subseteq L^\perp$. Then if $K' \supset L$ is maximal such that $K' \subset (K')^\perp$, then

$K' = (K')^{\perp}$ and K' is a sublagrangian.

<u>Proof</u>: Let K be a sublagrangian of (M,γ), and set $J = K \cap K'$. Then in $J^{\perp}/J$, $K'/J \subseteq (K'/J)^{\perp}$; and from [P1, (3.5)] it follows that K' is a sublagrangian of M if and only if K'/J is a sublagrangian in $J^{\perp}/J$. Hence, working inductively on $\ell(M)$ we may assume $K' \cap K = (0)$. We have $K \cong (M/K)^{\wedge}$, so $\ell(K) = \ell(M/K)^{\wedge}) = \ell(M/K) = \ell(M) - \ell(K)$ where $(M/K)^{\wedge} := \mathrm{Hom}(M/K,E)$. Hence $\ell(M) = 2\ell(K)$. Since $K'^{\perp\perp} = K'$, the naturally induced form $K' \times (M/K'^{\perp}) \to E$ is nonsingular, so $\ell(K') = \ell((M/K'^{\perp})\;) = \ell(M/K'^{\perp}) = \ell(M) - \ell(K'^{\perp})$.

If $\ell(K') < \ell(K)$, then $\ell(K'^{\perp}) > \ell(M) - \ell(K)$, which means $K'^{\perp} \to M \to M/K$ is not injective, so that $K'^{\perp} \cap K \neq 0$. Then since $K' \cap K = (0)$, $K'^{\perp} - K'$ contains an isotropic element, which contradicts maximality of K'. Thus $\ell(K') = \ell(K) = 1/2\,\ell(M)$, so $\ell(K') = \ell(K'^{\perp})$. Since $K' \subseteq K'^{\perp}$, $K' = K'^{\perp}$ and K' is a sublagrangian.

<u>$i = 1$</u>. It is well-known that $W_1^{\lambda}(k(m)) = 0$ (see, e.g., [P2, (4.1)] where our W_1^{λ} is denoted L_{λ}.). Hence surjectivity is enough here.

Let $[M,N,(\alpha,\gamma)] \in F^{\lambda}(F(R))$ be given. We denote by $\phi_h\colon (N \oplus N^{\wedge}) \times (N \oplus N^{\wedge}) \to E$ the standard λ-symmetric hyperbolic form. By stablizing ((1.12(a))) using the extension

$$\ker p \rightarrowtail (R/m^n)^{\ell} \overset{p}{\twoheadrightarrow} N\ ,$$

(for some n,ℓ and surjection p), we assume $N = (R/m^n)^{\ell}$. The point of the proof is to reduce n to 1, inside the class of $[M,N,(\alpha,\gamma)] \in W_1^{\lambda}(F(R))$.

We begin with some observations. First, for any k, $0 \le k \le n$, and with respect to the natural pairing $\nu\colon N^{\wedge} \times N \to E$

$$(m^{n-k}N^{\wedge})^{\perp} = m^{k}N$$

(5.2) and

$$m^{n-k}N^{\wedge} = (m^{k}N)^{\perp}$$

Second, if $\{m_{ij} \mid 1 \le i \le r,\ 1 \le j \le \ell\}$ is any $k(m)$-basis of $m^{n-1}N$ $(r = \dim_{k(m)} m^{n-1}/m^n)$ then there is a subset $\{f_{ij} \mid 1 \le i \le r,\ 1 \le j \le \ell\} \subseteq N^{\wedge}$ such that

$$f_{i'j'}(m_{ij}) = \delta_{ii'}\delta_{jj'} \in k(m) \subseteq E$$

and $\{f_{ij}\}$ is a basis of $N^{\wedge}$ mod m. To prove this use (5.2) to get a nonsingular pairing

$$N^{\wedge}/mN^{\wedge} \times m^{n-1}N \to k(m) \subseteq E.$$

The set $\{f_{ij}\}$ is thus a lifting to $N^{\wedge}$ of a dual basis in $N^{\wedge}/mN^{\wedge}$.

Now let $\{p_i \mid 1 \le i \le r\}$ be a $k(m)$-basis of $m^{n-1}/m^n \subseteq R/m^n$ and $\{b_1,\dots,b_{\ell}\}$, an R/m^n-basis of N. Then we may take $m_{ij} = p_i b_j$ in the discussion above. Note that

$$(5.3)\qquad \delta_{ii'}\delta_{jj'} = f_{i'j'}(m_{ij}) = p_i\, f_{i'j'}(b_j)$$

In particular, for each j, each of

$$(5.4)\qquad p_i\, f_{ij} = p_k\, f_{kj} \neq 0,\quad 1 \le i,\ k \le r$$

is a basis for the 1-dimensional $k(m)$-vector space it generates (because $m^{n-1}(R/m^n)^{\wedge} \cong \mathrm{Hom}(R/mR,E) = k(m)$ and $N/mN \cong (R/mR)^{\ell}$ has for a basis the classes of $b_1,\dots,b_{\ell}$ mod m).

<u>Step 1</u>. is to show we may modify $(M,N,(\alpha,\gamma))$ within its class in $W_1^{\lambda}(F(R))$ so that

$$(5.5)\qquad m^{n-1}\alpha(M) = 0.$$

If this is not already so, choose $y \in M$ with $m^{n-1}\alpha(y) \neq 0$. Then $\alpha(y) \notin mN$, so by Nakayama's lemma, we may take $\alpha(y)$ to be part of an R/m^n-basis of N, say $\{\alpha(y) = b_1, b_2, \dots, b_{\ell}\}$. Clearly, y generates a summand of M isomorphic to R/m^n.

Suppose $\gamma(y) = \sum_{i=1}^{\ell} h_i \in N^{\wedge}$, where h_j is characterized by: $h_j(b_k) \neq 0$ only if $j = k$. Define $r\colon N \to N^{\wedge}$ by $r(b_1) = 1/2\, h_1 + \sum_{i>1} h_i$ and $r(b_i) = 0$, $i > 2$. Set

$$\rho = r - \lambda r^{\wedge}\colon N \to N^{\wedge}\quad (N \equiv N^{\wedge\wedge})\ .$$

Then ρ is $(-\lambda)$-symmetric and a computation shows

$$\rho(b_1) = \begin{cases} \Sigma\, h_i, & \lambda = -1 \\ \sum_{i>1} h_i, & \lambda = 1. \end{cases}$$

When $\lambda = 1$, $0 = \phi_h(\alpha(y) + \gamma(y), \alpha(y) + \gamma(y)) = 2h_1(b_1)$ so $h_1 = 0$. Thus, in general (i.e., $\lambda = \pm 1$), $\rho(b_1) = \gamma(b_1)$, which means $(\gamma - \rho\alpha)(b_1) = 0$. After an operation of type (2.11)(b) we may thus destablize using the submodule $R_y \subseteq M$ (even splitting off a formation $(R/m^n, R/m^n, (1,0))$). Repeating the process if necessary, we eventually get (5.5).

<u>Step 2</u>. is to show $(M,N,(\alpha,\gamma))$ can be modified within its class in $W_1^{\lambda}(F(R))$ so that

(5.6) $$m^{n-1}\gamma(M) = 0,$$

keeping (5.5) also. Again if this is not so, choose $y \in M$ with $\gamma(y) = \sum_{i=1}^{\ell} h_i$, where h_i is characterized as above and where, say, $ph_1 \neq 0$ for some $p \in m^{n-1}$.

Suppose first that

(5.7) $$p\,h_i = 0, \quad i > 1.$$

Then since $\alpha(py) = p\alpha(y) = 0$ by (5.5), we may destabilize using the submodule $R(py) \subseteq M$. (Notice that only the first terms of the direct sum decompositions of N and $N^\wedge$ are changed in this process.) In general, we can get (5.7) by replacing b_j with

$$b_j' = b_j - \frac{p\,h_j(b_j)}{p\,h_1(b_1)}\,b_1, \qquad j > 1$$

and setting $b_1' = b_1$. Then in terms of this new basis, if $j > 1$,

$$p\gamma(y)(b_j') = \sum_i ph_i(b_j') = ph_j(b_j) - \frac{ph_j(b_j)}{ph_1(b_1)}\,ph_1(b_1) = 0,$$

so (5.7) is satisfied.

Assuming the result still has $m^{n-1}\gamma(M) \neq 0$, we continue the process; this time the first term ph_1 of $p\gamma(y)$ must be zero for all $p \in m^{n-1}$ because of (5.4). This completes Step 2, and shows we may assume

(5.8) $$m^{n-1}\alpha(M) = 0 = m^{n-1}\gamma(M).$$

From this it follows that

$$m^{n-1}(N + N^\wedge) \subseteq (\alpha,\gamma)(M).$$

For $\nu(\gamma(M), m^{n-1}N) = \nu(m^{n-1}\gamma(M), N) \equiv 0$; similarly, $\nu(m^{n-1}N^\wedge, \alpha(M)) \equiv 0$. Consequently

$$\phi_h((\alpha,\gamma)M, m^{n-1}(N + N^\wedge)) \equiv 0$$

so that

$$m^{n-1}(N + N^\wedge) \subseteq (\alpha,\gamma)(M)^{\perp} = (\alpha,\gamma)(M) .$$

Now let $L' = \{m \in M \mid (\alpha,\gamma)(m) \in m^{n-1}(N + N^\wedge)\}$. Then $(\alpha,\gamma)|L'\colon L' \xrightarrow{\cong} m^{n-1}(N + N^\wedge)$, an isomorphism of $k(m)$-vector spaces. Consider $\gamma|L'\colon L' \to m^{n-1}N^\wedge$. This is surjective with kernel L, say,

and $\alpha|L\colon L \xrightarrow{\cong} m^{n-1}N$. Evidently, destabilizing with $L \subseteq M$ leaves a formation $(M', N', (\alpha',\gamma'))$ with $m^{n-1}N' = 0$. Now we may begin the process over again, replacing n with $n - 1$ at the beginning of th proof then performing Steps 1 and 2.

This completes the proof.

In one case we need to "de-localize" these results.

(5.9) <u>Corollary</u>. Let A be a locally factorial Gorenstein ring Then each element of $W_1^{\lambda}(CM_1(A))$ has a representative $(K,H,(\alpha,\gamma))$ such that if $\operatorname{ht} q = 1$, then $(K_q, H_q, (\alpha_q,\gamma_q))$ is isomorphic to the orthogonal sum of formations of the form

$$(R/(x^t) \oplus R/(x^{m-t}), R/(x^m), ((0,x^t),(x^{m-t},0)))$$

where $R = A_q$, $(x) = qA_q$ and $0 \le t \le m$. If $\lambda = -1$, then we can take $t = m$.

<u>Proof</u>: There is a surjection $p\colon (A/q_1^{(m)} \cap \ldots \cap q_r^{(m)})^n \to H$, fo some $m,n > 0$ and height one primes $q_1,\ldots,q_r$. Since A is locall factorial, $A/q_1^{(m)} \cap \ldots \cap q_r^{(m)} \in CM_1(A)$, so $\ker p \in CM_1(A)$. After stabilization, we may therefore assume

$$H_q = (R/(t^m))^n, \quad R = A_q$$

for each $q \in \operatorname{Ass}(H)$ and $(t) = qA_q$.

Let $\lambda = 1$ and set $q = q_1$. We will carry out "Step 1" of the proof of Dévissage for $i = 1$ above, but over A. Namely, recall th γ_q was first changed to $\gamma_q + \rho\alpha_q$, where $\rho\colon H_q \to H_q^\wedge$ is the adjoint of a skew-symmetric form. To be able to lift this operation back to we change (α_q,γ_q) to $(s\alpha_q, s^{-1}\gamma_q)$, $s \in A - q$, an isomorphism of th formation. Now ρ becomes $s^2\rho$. For suitable s, $s^2\rho$ is the loca zation of the adjoint $H \to H^\wedge$ of a skew-symmetric form. If s is further chosen in $q_2^{(m)} \cap \ldots \cap q_r^{(m)}$, then at every other height one prime, there is no change in $(K,H,(\alpha,\gamma))$.

The effect of this modification is to give a subformation $(R/(t^m), R/(t^m), (1,0))$ of $(K_q,H_q,(\alpha_q,\gamma_q))$. Further operations will be carried out below to actually split off such a formation. We will assume these operations have also been delocalized (as above, up to isomorphism of the local formation $(K_q,H_q,(\alpha_q,\gamma_q))$). The fact that $(R/(t^m))^\wedge \cong R/(t^m)$ (because $\dim R = 1$) means we can carry out modifications in place of Step 2 exactly like those above from Step 1: only the roles of α and γ are reversed.

We next want to see that the subformations $(R/(t^m), R/(t^m), (1,0)$ and $(R/(t^m), (R/(t^m))^\wedge, (0,1))$ of $(K_q,H_q,(\alpha_q,\gamma_q))$ $(q \in \operatorname{Ass}(H))$ can be split off. This will be carried out more generally below, for subformations where $0 \le t \le m$ in the statement of (5.9).

We assume we have completed Steps 1 and 2, so that

(5.10) $$t^m K_q = 0, \quad q \in \mathrm{Ass}(H), \quad (t) = qA_q .$$

Let $(K_q, H_q, (\alpha_q, \gamma_q)) = (M,N,(\beta,\delta))$. Suppose $m \in M$ has minimal annihilator, (x^t), say. Either $\mathrm{Ann}(\beta(m))$ or $\mathrm{Ann}(\delta(m)) = (x^t)$. If it is $\beta(m)$, we may assume

(5.11) $$\beta(m) = x^{m-t} b_1 \in N$$

where $\{b_1, \dots, b_n\}$ is an $R/(x^m)$-basis for N. Let $\delta(m) = \Sigma\, r_i b_i^*$, where $\{b_1^*, \dots, b_n^*\}$ is the dual basis. Since $x^t \delta(m) = 0$, $x^{m-t} | r_i$. Define $r \colon N \to N^\wedge$ by

$$r(b_j) = \begin{cases} \Sigma\, r_i\, x^{t-m}\, b_i^*, & j = 1 \\ 0, & j > 1 . \end{cases}$$

Then if we set $\rho = r - r^\wedge$, ρ is skew-symmetric and $\rho(b_1) = \sum_{i>2} r_i x^{t-m} b_i^*$; so after an operation of type (2.11)(b) we get

$$\delta(m) = r_1 b_1^*$$

By considering the expression $\phi_h(\beta(m) + \delta(m), \beta(m) + \delta(m)) = 0$ we find that $x^t | r_1$.

Let $G = \{m = m_1, m_2, \dots, m_n\}$ be a minimal set of generators of M. Since $\mathrm{Ann}(m) \subseteq \mathrm{Ann}(m_j)$, the coefficient of b_1 in the expression of $\beta(m_j)$ is divisible by x^{m-t} (or is zero). Hence by adding appropriate multiples of m to the m_j, $j \ge 2$, we may assume

(5.12) $$\beta(m_j) = \sum_{i \ge 2} s_{ij} b_i, \quad j = 2, \dots, n .$$

Now there is $m' \in M$ such that $\beta(m') = 0$ and $\delta(m') = x^t b_1^*$ because $\phi_h(\beta(m_j) + \delta(m_j), x^t b_1^*) = 0$, for all j by (5.11) and (5.12). So $x^t b_1^* \in [\mathrm{im}(\beta,\delta)]^\perp = \mathrm{im}(\beta,\delta)$. Replacing $m = m_1$ with $m - r_1 x^{-t} m'$, we get

(5.13)
$$\beta(m - r_1 x^{-t} m') = x^{m-t} b_1, \quad \beta(m') = 0$$
$$\delta(m - r_1 x^{-t} m') = 0, \quad \delta(m') = x^t b_1^* .$$

It is easy to see that we may take m' to be part of a minimal generat-

ing set (along with $m - r_1 x^{-t} m'$); say, $m_1 = m - r_1 x^{-t} m'$ and $m_2 = m'$ in G above.

By considering the expression $\phi_h(\beta(m_1) + \delta(m_1), \beta(m_j) + \delta(m_j)) = 0$, $j \ge 3$, we see that the coefficient of b_1^* in the expression for $\delta(m_j)$ is divisible by x^t, $j \ge 3$. Again, by adding appropriate multiples of m_2 to the m_j, $j \ge 3$, we can take

(5.14) $$\delta(m_j) = \sum_{i \ge 2} r_{ij} b_i^*, \quad j \ge 3 .$$

Putting (5.12), (5.13) and (5.14) together, we have shown $(M,N,(\beta,\delta))$ has an orthogonal summand $(R/(x^t) \oplus R/(x^{m-t}), R/(x^m), ((0,x^{m-t}),(x^t,0)))$ as desired.

The proof of (5.9) when $\lambda = -1$ is much easier. Given $(M,N,(\beta,\delta)) \in F^{-1}(F(R))$ we know from (1.20) that $\mathrm{im}(\beta,\delta)$ is a summand of $N \oplus N^\wedge \cong (R/(x^m))^{2n}$, so that $M \cong (R/(x^m))^n$. Following the construction of L_1^{-1} in (2.7), we can thus extend $(M,N,(\beta,\delta))$ to an element

(5.15) $$A = \begin{pmatrix} \beta & \mu \\ \delta & \nu \end{pmatrix} \in Sp_{2n}(R/(x^m))$$

where $Sp_{2n}(-)$ is the symplectic group. Now by [P2,(3.12),(3.14)], there is a symmetric $\rho \colon (R/(x^m))^n \to (R/(x^m))^n$ such that $\beta + \rho\delta$ is invertible. Then after a basis change (in N), we can take $\beta = \mathrm{id}$ in (5.15). Hence an operation of type (1.18)(b) using $-\delta$ gives $\delta = 0$ in (5.15).

§6. Poincaré complexes.

The aim of the next three sections is to prove (1.21). The proof is rather long because in this section we first introduce another definition of $W_1^\lambda(CM_p)$, in terms of chain complexes. This is a natural thing to do, since, for example when $p = 0$ and A is local, (1.21) is a consequence of Sharpe's normal form for elements of the unitary Steinberg group. This in turn is motivated by the geometric description of odd-dimensional surgery theory, so the chain complexes are taking the place of manifolds. The idea of using chain complexes to prove (1.21) comes from Ranicki's work ([R1] and [R2]), on which we will rely heavily. For the convenience of the reader, we recall some of the definitions, making explicit the simplifications that come from the assumption, $1/2 \in A$.

Let (C,d) be an n-dimensional chain complex of finitely-generated projective A-modules, C_i, and let $\phi_r: C^r \to C_{n-r}$ be a sequence of homomorphisms, $0 \leq r \leq n$, where $C^r := \text{Hom}(C_r,A)$. Then (C,d,ϕ) is a Poincaré complex if, for $0 \leq r \leq n$,

$$(6.1)\qquad \begin{aligned} &\text{a)}\quad d\phi_r = (-1)^{n-r}\,\phi_{r+1}\,d^* \\ &\text{b)}\quad \phi_r = \lambda(-1)^{r(n-r)}\,\phi^*_{n-r} \\ &\text{c)}\quad \phi_r \text{ induces } H^r(C) \xrightarrow{\cong} H_{n-r}(C) \end{aligned}$$

where $\phi^* = \text{Hom}(\phi,A)$ and $d^* = \text{Hom}(d,A)$.

It is useful to reformulate this. Given finitely-generated projective A-modules P and Q, there is an isomorphism

$$\begin{aligned} \text{Hom}(P,A) \otimes Q &\xrightarrow{\cong} \text{Hom}(P,Q) \\ f \otimes q &\to \{p \to f(p)q\} \end{aligned}$$

Taking this and $C \cong C^{**}$ as identifications, let $C \otimes C$ be the chain complex with

$$(C \otimes C)_k = \sum_{i+j=k} \text{Hom}(C^i,C_j)\ ,$$

and differential

$$\delta(\phi: C^i \to C_j) = (C^i \xrightarrow{\phi} C_j \xrightarrow{d} C_{j-1}) + (-1)^j(C^{i-1} \xrightarrow{d^*} C^i \xrightarrow{\phi} C_j)$$

Thus, the collection of maps $\{\phi_i\}$ satisfying a) and b) above give rise to a "λ-symmetric cycle" in $H_n(C \otimes C)$. In [R1] and [R2] the collection of such maps is denoted ϕ_0, and there is also postulated a sequence of higher chain operators $\phi_k \in (C \otimes C)_{n+k}$, $k \geq 1$, (not the

ϕ_k above!) satisfying certain conditions [R1, p. 104]. All these chain operators together define a cocycle in a certain $\mathbb{Z}/2$-hypercohomology group, $Q^n(C,\lambda)$. Since $1/2 \in A$, this simplifies as follows.

Let $T_\lambda: C \otimes C \to C \otimes C$ be the chain map defined, for $f \in \text{Hom}(C^i,C_j) \subseteq (C \otimes C)_{i+j}$, by

$$(6.2)\qquad T_\lambda(f) = (-1)^{ij}\lambda f^* \in \text{Hom}(C^j,C_i)\ .$$

Since $1/2 \in A$, the sequence

$$(6.3)\ \ldots \xrightarrow{(1-T_\lambda)_*} H_n(C \otimes C) \xrightarrow{(1+T_\lambda)_*} H_n(C \otimes C) \xrightarrow{(1-T_\lambda)_*} H_n(C \otimes C) \to \ldots$$

is exact, so $(1 + T_\lambda)_*$ induces

$$(6.4)\qquad H_n(C \otimes C)/\text{im}(1 - T_\lambda)_* \xrightarrow{\cong} \text{im}(1 + T_\lambda)_* \subseteq H_n(C \otimes C)\ .$$

This is the isomorphism $Q_n(C,\lambda) \xrightarrow{\cong} Q^n(C,\lambda)$ of [R1, pp. 102-3] and we here adopt the notation

$$(6.5)\qquad Q_n(C,\lambda) := H_n(C \otimes C)/\text{im}(1 - T_\lambda)_*\ .$$

We view the homomorphisms ϕ_r satisfying (6.1)(a) and (b) as elements of $\text{im}(1 + T_\lambda)_*$, but in practice we work with a sequence of antecedents (under (6.4)) $\{\psi_r\} \in Q_n(C,\lambda)$.

(6.6) Definition. Let A be a commutative ring with $1/2 \in A$. A λ-symmetric chain complex over A is a triple $(C,d,\{\psi\})$ where (C,d) is a chain complex of finitely-generated projective A-modules and $\{\psi\} \in Q_k(C,\lambda)$. It is called n-dimensional if (C,d) is homotopy equivalent to a complex (D,d') with $D_i = 0$, $i > n$ or $i < 0$, and $\{\psi\} \in Q_n(C,\lambda)$. It is λ-Poincaré if, in addition, the maps $\phi_r := \psi_r - (-1)^{r(n-r)}\lambda\psi^*_{n-r}$ induce isomorphisms $H^r(C) \xrightarrow{\cong} H_{n-r}(C)$, $0 \leq r \leq n$.

Let (C,d_C) and (D,d_D) be chain complexes and $f: (C,d_C) \to (D,d_D)$, a chain map. Define the mapping cone of f, $(C(f),d_{C(f)})$ by $C(f)_r = D_r \oplus C_{r-1}$ and

$$(d_{C(f)})_r = \begin{pmatrix} d_D & (-1)^{r-1}f \\ 0 & d_C \end{pmatrix} : D_r \oplus C_{r-1} \to D_{r-1} \oplus C_{r-2}\ .$$

If C and D are n- and (n+1)-dimensional respectively, then $C(f)$ is (n+1)-dimensional. Let $(C(f) \hat{\otimes} C(f),\hat{d})$ be the chain complex with

(6.7) $$(C(f) \hat{\otimes} C(f))_k = \sum_{i+j=k} \mathrm{Hom}(D^i, D_j) \oplus \sum_{r+s=k-1} \mathrm{Hom}(C^r, C_s)$$

and, for $(\delta\psi_i, \psi_r) \in \mathrm{Hom}(D_i, D_j) \oplus \mathrm{Hom}(C^r, C_s)$

(6.8) $$\hat{d}(\delta\psi_i, \psi_r) = d_D\ \delta\psi_i + (-1)^j (\delta\psi)_{i+1} d_D^* + (-1)^n\ f\ \psi_i\ f^*$$

In analogy with what has been done above, we define

(6.9) $$\Omega_m(f,\lambda) = H_m(C(f) \hat{\otimes} C(f))/\mathrm{im}(1 - T_\lambda)_*.$$

(6.10) Definition. Let A be a ring with $1/2 \in A$. A λ-symmetric pair over A is a pair $(f: C \to D, \{\delta\psi, \psi\})$ where $f: C \to D$ is a chain map of chain complexes of finitely-generated projective A-modules and $\{\delta\psi, \psi\} \in \Omega_m(f,\lambda)$, for some m. It is $(n+1)$-dimensional if C and D are n- and $(n+1)$-dimensional, respectively, and $\{\delta\psi, \psi\} \in Q_{n+1}(f,\lambda)$. Finally, it is relative λ-Poincaré of dimension $n + 1$ if, in addition, the pair $(\delta\phi, \phi) := (\delta\psi + T_\lambda(\delta\psi), \psi + T_\lambda\psi)$ induces isomorphisms,

(6.11) $$H^r(C(f)) \xrightarrow{\cong} H_{n+1-r}(D)$$
$$(x,y) \to f\phi x + (\delta\phi)y$$

where $x \in C^{r-1}$, $y \in D^r$.

In this case the λ-symmetric complex $(C, d_C, \{\psi\})$ is λ-Poincaré of dimension n, and is called the boundary of $(f: C \to D, \{\delta\psi, \psi\})$.

Let M_p denote the category of A-modules M such that $\mathrm{ht}\ M \geq p$, together with the zero module. Then

$$P_n^\lambda(M_p)$$

denotes the collection of n-dimensional λ-Poincaré complexes $(C,d,\{\psi\})$ such that $H_*(C) \in M_p$. A cobordism (resp. M_p-cobordism) between $(C,d,\{\psi\})$ and $(C',d',\{\psi'\}) \in P_n^\lambda(M_p)$ is an $(n+1)$-dimensional λ-Poincaré pair $(f \oplus f': C \oplus C' \to D, \{\delta\psi, \psi \oplus \psi'\})$ with boundary $(C \oplus C', d \oplus d', \{\psi \oplus \psi'\})$ (resp. such that $H_*(D) \in M_p$).

(6.12) Definition-Proposition [Rl,(3.2)] M_p-cobordism is an equivalence relation on $P_n^\lambda(M_p)$. The cobordism classes form an abelian group

$$\Omega_n^\lambda(M_p)$$

with addition induced by direct sum of complexes.

(6.13) Next let $(f: C \to D, \{\delta\psi, \psi\})$ be an $(n+1)$-dimensional λ-symmetric pair whose boundary $(C,d,\{\psi\})$ is λ-Poincaré. From this an

n-dimensional Poincare complex $(C', d', \{\psi'\})$ is constructed as follow
Let $C_r' = C_r \oplus D_{r+1} \oplus D^{n-r+1}$

$$d_{C'} = \begin{pmatrix} d_C & 0 & (-1)^{n+1}(1 + T_\lambda)\psi f^* \\ (-1)^r f & d_D & (-1)^r(1 + T_\lambda)\delta\psi \\ 0 & 0 & (-1)^r\ d_D^* \end{pmatrix},$$

$$\psi' = \begin{pmatrix} \psi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(6.14) Proposition. With the above notation, if $(C,d,\{\psi\}) \in P_n^\lambda(M_p)$ and $H_*(D) \in M_p$ then $(C',d',\{\psi'\}) \in P_n^\lambda(M_p)$ and is M_p-cobordant to $(C,d,\{\psi\})$.

Proof: [Rl, (4.1)(ii)] The cobordism is $((g,g'): C \oplus C' \to D')$, $\{0, \psi \oplus \psi'\})$ where

$$d_{D'} = \begin{pmatrix} d_C & (-1)^{n+1}(1+T_\lambda)\psi f^* \\ 0 & (-1)^r\ d_D^* \end{pmatrix} : D_r' = C_r \oplus D^{n-r+1} \to D_{r-1}' = C_{r-1} \oplus D^{n-r+2}$$

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C_r \to D_r' = C_r \oplus D^{n-r+1}$$

and

$$g' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : C_r' = C_r \oplus D_{r+1} \oplus D^{n-r+1} \to D_r'.$$

We next need to assemble some facts about chain complexes (C,d) for which $H_*(C) \in M_p$. We begin with a quotation of the Acyclicity Lemma of Peskine and Szpiro [PS, Lemme 1.8].

(6.15) Lemma. Let R be local Noetherian and let

$$0 \to L_s \to L_{s-1} \to \dots \to L_1 \to L_0 \to 0$$

be a finite complex of finitely-generated R-modules. Suppose that fo
every $i > 0$,

1) depth $L_i \geq i$ and

2) depth $H_i(L) = 0$ or $H_i(L) = 0$.

Then $H_i(L_*) = 0$, $i \geq 1$.

Using this result we may restrict the homology of certain comple

(6.16) Proposition. Let A be a CM ring and let

$$0 \to C_{p+k} \xrightarrow{d_{p+k}} C_{p+k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

be a chain complex of finitely-generated projective A-modules for which $H_*(C) \in \mathcal{M}_p$. Then $H_i(C) = 0$, $i > k$.

<u>Proof</u>: Applying the Acyclicity Lemma to the complex $C_{p+k} \to \cdots \to C_{k+1} \to \operatorname{im} d_{k+1}$ at all height p primes shows $H_i(C) \in \mathcal{M}_{p+1}$ for each $i > k$. Localizing at all height $p+1$ primes shows $H_i(C) \in \mathcal{M}_{p+2}$; eventually, $H_i(C) = 0$, $i > k$.

We also need a kind of universal coefficient theorem.

(6.17) <u>Proposition</u>. Let A be a CM ring and let

$$0 \to C_{p+k} \xrightarrow{d_{p+k}} C_{p+k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

be a chain complex of finitely-generated projective A-modules such that $H_*(C) \in \mathcal{M}_p$ and $p \geq k$. Then there is an isomorphism

$$H^p(C) \cong \operatorname{Ext}^p(H_0(C), A)$$

<u>Remark</u>: Since ht $H_0(C) \geq p$ the arguments used in (1.6) show $\operatorname{Ext}^p(H_0(C),A) \cong \operatorname{Hom}(H_0(C), V_p)$.

<u>Proof</u>: When $k = 0$, the previous proposition says C_* is a resolution of $H_0(C)$, so the result is immediate from the definition of Ext. Hence we assume $k > 0$.

If $p = 1$ (hence $k = 1$, since $p \geq k > 0$), then $\operatorname{Ext}^1(H_0(C),A) \cong \operatorname{cok}(C^0 \to (\operatorname{im} d_1)^*)$. From the exactness of $(\operatorname{im} d_1)^* \rightarrowtail C^1 \to (\ker d_1)^*$ and the fact that $(\ker d_1)^* \rightarrowtail C^2$ (dualize the exact sequence $C_2 \to \ker d_1 \twoheadrightarrow H_1(C)$), we find that

$$(\operatorname{im} d_1)^* \rightarrowtail C^1 \xrightarrow{d_2^*} C^2$$

is exact. This means $\operatorname{cok}(C^0 \to (\operatorname{im} d_1)^*) \cong H^1(C)$ as required.

Finally, if $p \geq 2$, then $\operatorname{Ext}^p(H_0(C),A) \cong \operatorname{Ext}^{p-1}(\operatorname{im} d_1, A) \cong \operatorname{Ext}^{p-2}(\ker d_1, A) \cong \cdots \cong \operatorname{Ext}^1(\operatorname{im} d_{p-1}, A) \cong \operatorname{cok}(C^{p-1} \to (\ker d_{p-1})^*)$. But $(\ker d_{p-1})^* = (\operatorname{im} d_p)^*$ because $H^*_{p-1} = 0 = \operatorname{Ext}^1(H_{p-1},A)$ (the latter because $\operatorname{Ext}^1(H_{p-1},A) \cong \operatorname{Hom}(H_{p-1},V_1)$ and ht $H_{p-1} \geq p \geq 2$). As above we find that $\operatorname{cok}(C^{p-1} \to (\operatorname{im} d_p)^*) \cong H^p(C)$ so the proof is done.

Finally, we need a spectral sequence describing representatives in $H_n(C \otimes C)$ of the duality maps $\{\psi\} \in Q_n(C,\lambda)$ (cf. (6.5)) where (C,d) is an n-dimensional λ-symmetric complex. Let $F_s(C \otimes C) := \sum_{i<s} \operatorname{Hom}(C^i, C)$, a subcomplex of $C \otimes C$. The corresponding spectral

sequence has

$$E^2_{s,t} = H_s(\operatorname{Hom}(C^*, H_t(C)))$$

with H_s being computed using the differential on the chain complex $E_* = \operatorname{Hom}(C^*, H_t(C))$ induced from d^* on C^*. It converges to the associated graded groups of the filtration

$$F_s H_{s+t}(C \otimes C) := \operatorname{im}[H_{s+t}(F_s(C \otimes C)) \to H_{s+t}(C \otimes C)].$$

Note that in the special case where $H_i(C) = 0$, $i > 0$, and $H_0(C) \in CM_n$, so that $C^0 \xrightarrow{d_1^*} C^1 \to \cdots \to C^n \twoheadrightarrow H^n(C)$ is exact,

$$H_s(\operatorname{Hom}(C, H_t(C))) = \operatorname{Ext}^{n-s}(H^n(C), H_t(C)) \tag{6.18}$$

$$= \begin{cases} \operatorname{Ext}^{n-s}(H^n(C), H_0(C)), & t = 0 \\ 0, & t \neq 0 \end{cases}$$

When $s + t = n$, this is the well-known fact that chain homotopy classes of chain maps $C^* \to C$ are in 1-1 correspondence with the homomorphisms $H^n(C) \to H_0(C)$ they induce.

§7. Poincaré complexes and Witt groups.

In this section it is assumed that A is an n-dimensional regular domain. As always, $1/2 \in A$ and CM_p means $CM_p(A)$. We prove the connection between stable isometry classes in $F^\lambda(CM_p)$ (cf. (1.15)) and homotopy classes in $P^\lambda_{p+1}(CM_p)$, $p \geq 0$. There is an analogous (and more apprent) one-to-one correspondence between isometry classes in $\Omega^\lambda(CM_p)$ and homotopy classes $P^\lambda_p(M_p)$. These one-to-one correspondences give rise to isomorphisms

(7.1)
$$W_0^\lambda(CM_p) \cong \Omega_p^\lambda(M_p)$$
$$W_1^\lambda(CM_p) \cong \Omega_p^\lambda(M_{p+1}) ,$$

which might be thought of as a replacement for the "resolution theorem" in algebraic K-theory. We prove only a connection between $F^\lambda(CM_p)$ and $\Omega_p^\lambda(M_{p+1})$ since it is all that is needed here.

(7.2) Proposition. Let A be a regular Noetherian domain, $1/2 \in A$. For each $p \geq 0$, there is a map of sets

$$A: \begin{Bmatrix} \text{stable isometry classes} \\ \text{in } F^\lambda(CM_p) \end{Bmatrix} \to \begin{Bmatrix} \text{homotopy classes} \\ \text{in } P^\lambda(M_p) \end{Bmatrix}$$

Proof: Let $(K,H,(\alpha,\gamma))$ be given. Choose a complex R of dimension p such that

$$H_i(R) = \begin{cases} H^\wedge, & i = 0 \\ 0 , & i \neq 0 \end{cases}$$

where $(-)^\wedge$ means $\mathrm{Hom}(-,V_p)$. Since A is Cohen-Macaulay, $\mathrm{depth}_{\mathrm{Ann}(M)}(A) = p$ ([K, Thm. 136]), so by [M, p. 103, Prop.] $H_i(R) = 0$, $i < p$, while $H^p(R) = \mathrm{Ext}^p(H^\wedge,A) \cong \mathrm{Hom}(H^\wedge,V_p) = H^{\wedge\wedge} \cong H$. Hence, setting

$$R_i^t = \mathrm{Hom}(R_{p-i},A),$$

R^t is a resolution of H. Set

$$D = R \oplus R^t$$

and define $\psi_i: D^i \to D_{p-i}$ by

$$R^i \oplus (R^t)^i \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} R_{p-i} \oplus (R^t)_{p-i} .$$

Then $(D,\{\psi\}) \in P_p(M_p)$. (The differential in D is excluded here from the notation.)

Next let D' be a resolution of $K^\wedge$; as above, $H^p(D') = K^{\wedge\wedge} \cong K$ and $H^i(D') = 0$, $i \neq p$. By general principles, there is a chain map $f': D \to D'$, unique up to homotopy, such that

$$(\alpha,\gamma) = H^p(f'): H^p(D') \to H^p(D).$$

Let $f' = (g,h)$, where $g: R \to D'$, $h: R^t \to D$. Then (referring to (7.3) below), $f'_\%\{\psi\} := \{f'\psi f'^*\} = \{hg^*\} = \{hg^* - 1/2(1 - T_\lambda)(hg^*)\} = \{1/2(1 + T_\lambda)hg^*\}$, which is trivial in $\Omega_p(D',\lambda)$ since it induces the zero map $\alpha^\wedge\gamma + \lambda\gamma^\wedge\alpha: K = H^p(D') \to H_0(D') = K^\wedge$ (cf. (1.11)). Now from [R1,] we have the exact sequence

(7.3)
$$\Omega_{p+1}(D',\lambda) \to \Omega_{p+1}(f',\lambda) \to \Omega_p(D,\lambda) \xrightarrow{f'_\%} \Omega_p(D',\lambda)$$

But $\Omega_{p+1}(D',\lambda) = 0$ since (6.18) shows $H_{p+1}(D' \otimes D') = 0$. This means there is a unique $\{\delta\psi,\psi\} \in \Omega_{p+1}(f',\lambda)$ giving a λ-symmetric complex

(7.4)
$$(f': D \to D',\{\delta\psi,\psi\})$$

unique up to a homotopy equivalence inducing the identity on the boundary $(D,\{\psi\})$. It is easily checked to be λ-Poincaré of dimension $p+1$ with $H_*(D) \in M_p$.

Similarly, set $D'' = R^t$ and let $f'': D \to D''$ be projection to the second factor, so that $H^p(f''): H^p(D'') \to H^p(D)$ is inclusion $H^\wedge \to H \oplus H^\wedge$ to the second factor. Once again there is a relative λ-Poincaré complex,

$$(f'': D \to D'',\{\delta\psi'',\psi\})$$

well-defined up to homotopy equivalence inducing the identity on the boundary $(D,\{\psi\})$.

Referring to [R1, p. 135], let

(7.5)
$$(C = D' \cup_D D'',d_C,\{\eta\})$$

be the glueing of D' and D'' along D. Evidently, C is a $(p+1)$-dimensional complex with $H_*(C) \in M_p$. Since D' and D'' were relative λ-Poincaré with common boundary D, $(C,d_C,\{\eta\}) \in P^\lambda_{p+1}(M_p)$. This is $A(K,H,(\alpha,\gamma))$.

For the usual reasons, the homotopy type of C (as defined in [R1,p. 140]) is independent of the resolutions R and D' of $H^\wedge$ and $K^\wedge$; as observed above the classes $\{\delta\psi',\psi\} \in \Omega_{p+1}(f',\lambda)$ and $\{\delta\psi'',\psi\} \in$

$Q_{p+1}(f'',\lambda)$ are also well-defined. Hence the homotopy type of $(C,d_C,\{\eta\}) \in P^{\lambda}_{p+1}(M_p)$ depends only on $(K,H,(\alpha,\gamma))$.

We next need to show invariance of homotopy type after

1) stabilization of $(K,H,(\alpha,\gamma))$ (1.12(a))
2) isometry of $(K,H,(\alpha,\gamma))$ (1.12(b)).

For 1) let $(K_1,H_1(\alpha_1,\gamma_1))$ be a stabilization of $(K,H,(\alpha,\gamma))$ so that we have, in particular, a commutative diagram

$$\text{(7.6)}\qquad \begin{array}{ccc} L & = & L \\ \downarrow & & \downarrow \\ K_1 & \xrightarrow{\alpha_1} & H_1 \\ \downarrow & & \downarrow \\ K & \xrightarrow{\alpha} & H \end{array}$$

Correspondingly, there is a homotopy commutative diagram of chain complexes

$$\begin{array}{ccc} D_1' & \xleftarrow{g_1} & R_1 \\ \uparrow \ell & & \uparrow k \\ D' & \xleftarrow{g} & R \end{array}$$

realizing the bottom square of (7.6) as its H^p. By construction of C,

$$C_r = D_r' \oplus D_{r-1} \oplus D_r'', \quad \text{and}$$

$$d_C = \begin{pmatrix} d_{D'} & (-1)^{r-1}f' & 0 \\ 0 & d_D & 0 \\ 0 & (-1)^{r-1}f'' & d_{D''} \end{pmatrix} : D_r' \oplus D_{r-1} \oplus D_r'' \to D_{r-1}' \oplus D_{r-2} \oplus D_{r-1}''.$$

If $C_1 := D_1' \cup_{D_1} D_1''$ is the chain complex constructed for $(K_1,H_1,(\alpha_1,\gamma_1))$ define a chain map $S: C \to C_1$ by

$$C_r = D_r' \oplus R_{r-1} \oplus R_{r-1}^t \oplus D_r'' \xrightarrow{(\ell\oplus k\oplus 0\oplus 0)} (D_1')_r \oplus (R_1)_{r-1} \oplus (R_1^t)_{r-1} \oplus (D_1'')_r.$$

It is not difficult to verify that S is a homotopy equivalence in $P_p^{\lambda}(M_{p+1})$.

For part 2), the isometry of $(H \oplus H^{\wedge},\phi_h)$

$$\begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix} : H \oplus H^{\wedge} \to H \oplus H^{\wedge}$$

is induced as H^p of a map of the λ-symmetric complex

$$R^t \oplus R \xrightarrow{G=\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}} R^t \oplus R = D$$

where $k: R^t \to R$ satisfies $(1 + T_{\lambda})k = 0$. It is easily verified that

a) the glueing C_1 of D' and D'' along D obtained from $f'G: D \to D'$ and $f'': D \to D''$ is a λ-Poincaré complex constructed from $(K,H,(\alpha,\gamma + \rho\alpha))$ (by A), and

b) C_1 is homotopic to C by a map restricting to the identity on D' and D'', and to K on D.

Finally, it is now easy to show that if the formations $(K,H,(\alpha,\gamma))$ and $(K',H',(\alpha',\gamma'))$ are isomorphic, then their corresponding Poincare complexes are homotopy equivalent (for any choice of the resolutions R and D').

Next we define the inverse to A.

(7.7) <u>Proposition</u>. Let A be a regular noetherian domain containing $1/2$. Then for each $p \geq 0$ there is a map of sets

$$B: \left\{\begin{matrix}\text{homotopy classes} \\ \text{in } P^{\lambda}_{p+1}(M_p)\end{matrix}\right\} \to \left\{\begin{matrix}\text{stably isometry classes} \\ \text{in } F^{\lambda}(CM_p)\end{matrix}\right\}$$

<u>Proof:</u> Let $(C,d,\{\eta\}) \in P^{\lambda}_{p+1}(M_p)$ be given. By (6.16) we have

$$H_i(C) = \begin{cases} \in M_p, & i = 0,1 \\ 0 \ , & i \neq 0,1 \ . \end{cases}$$

We first show how to find a chain map

$$f: C \to D$$

where $D = (D_{p+1} \to \dots \to D_1 \to 0)$ satisfies

$$H^i(D) = \begin{cases} H \in CM_{p+1}, & i = p+1 \\ 0 \ , & i \neq p+1 \ , \end{cases}$$

and

$$\text{(7.8)} \qquad H^{p+1}(f): H^{p+1}(D) \to H^{p+1}(C)$$

is surjective. To do this we need to define $\bar{f}: H \to H^{p+1}(C)$ and fill in the verticals in

$$\begin{array}{ccccccccc} \dots \to & D^{p-1} & \xrightarrow{\delta} & D^p & \to & D^{p+1} & \twoheadrightarrow & H & \\ & \downarrow f^{p-1} & & \downarrow f^p & & \downarrow f^{p+1} & & \downarrow \bar{f} := H^{p+1}(f) & \\ \dots \to & C^{p-1} & \xrightarrow{d_p^*} & C^p & \xrightarrow{d_{p+1}^*} & C^{p+1} & \twoheadrightarrow & H^{p+1}(C) & . \end{array}$$

For this, let $I = \text{Ann } H^*(C)$, an ideal of height $> p$ by assumption, and choose a regular sequence $x_1,\dots,x_p \in I$ (cf. [K,Thm.136]). Let D^* be the associated (sum of) Koszul resolutions of $H:=(A/(x_1,\dots,x_p))^n$ where $H^{p+1}(C)$ has n generators. Then $\bar{f}$ is defined by a choice of such generators and f^{p+1} is any lifting of $\bar{f}$. Since the bottom sequence is exact at C^{p+1}, f^{p+1} can be lifted to f^p. However, it may not (á priori) be possible to fill in f^{p-1} because C^* is not in general exact at C^p. To remedy this, we change D^* by replacing the regular sequence $\{x_1,\dots,x_p\}$ by $\{x_1^2,\dots,x_p^2\}$ (still a regular sequence in I by [K, p.103]), thus multiplying the values of f^p on generators of D^p by elements of I. This keeps commutativity of the second square. Now in any case, $d^*_{p+1}\cdot f^p\cdot\delta = 0$ so that, since $IH^p(C) = 0$, the above change in f^p (and δ) means $\mathrm{im}(f^p\delta)\subseteq \mathrm{im}\, d^*_p$. Thus, f^{p-1} can be chosen to make the left square commute. Completing the rest of the diagram is immediate because $H^i(C) = 0$, $i < p$.

Consider now the exact sequence from [R1,3.1] (cf. (7.3))

(7.9) $$Q_{p+2}(D,\lambda) \to Q_{p+2}(f,\lambda) \to Q_{p+1}(C,\lambda) \xrightarrow{f_\%} Q_{p+1}(D,\lambda)$$

According to (6.18), $H_{p+1}(D\otimes D)\cong \mathrm{Ext}^1(H,H^\wedge)$ and the construction and naturality of the spectral sequence giving this result shows that $f_\%\{\eta\}$ is represented by the composition $D^p \xrightarrow{f^p} C^p \to C_1 \xrightarrow{f_1} D_1 \to H_1(D):=H$, viewed as a cycle in the chain complex $\mathrm{Hom}(D^*,H^\wedge)$. Recall, however, that in the construction of f above we arranged that $\mathrm{im}\, f^p \subseteq I(C_p)$. Hence the cocycle above is actually zero, so $f_\%\{\eta\} = 0$. Thus, we obtain a class $\{\delta\eta,\eta\}\in Q_{p+2}(f,\lambda)$ and hence a λ-symmetric pair

$$(f\colon C\to D,\{\delta\eta,\eta\})$$

with $H_*(D)\in \mathcal{M}_p$.

Now do algebraic surgery on f ([R1, p. 145] or (6.14)) getting $(C',d_{C'},\{\eta'\})$, where $d_{C'}$ and η' are described in [R1, p. 145] or (6.14).

Let $C(f)_{+1}$ denote the mapping cone of f, $C(f)$, shifted in degree,

$$(C(f)_{+1})_r = C(f)_{r+1}\ .$$

Let D' be the complex defined by

$$(D')_r := D^{p+2-r},\quad d_{D'} := d^*_D.$$

Then we have a short exact sequence of complexes

(7.10) $$C(f)_{+1} \overset{i}{\rightarrowtail} C' \overset{f'}{\twoheadrightarrow} D'$$

where i is the inclusion of the first two factors and f' is the projection to the third. From this we get a canonical chain homotopy equivalence

$$C(f') \xrightarrow{\simeq} C(f)$$

which we use to identify $H_*(f)$ with $H_*(f')$.

We are now ready to define the λ-formation $(K,H,(\alpha,\gamma))$ associated to the λ-Poincaré complex $(C,d_C,\{\eta\})$ by $\mathcal{B}$. Namely, set

- $K = H^{p+1}(C(f)) \equiv H^{p+1}(C(f'))$
- $H = H^{p+1}(D)$
- $\alpha\colon H^{p+1}(C(f)) \to H^{p+1}(D)$ is the map $H^{p+1}(j)$, where $j\colon D\to C(f)$.
- γ is the composition $H^{p+1}(C(f)) := H^p(C(f)_{+1}) \xrightarrow{\delta} H^{p+1}(D') := H_1(D) \cong H^\wedge$, where δ is the boundary in the cohomology sequence of (7.10).

Specifically, straightforward computations show that α is induced by

$$(a,b) \to b,$$

and γ by

$$(a,b) \to \lambda f\,\phi(a) + \lambda(\delta\phi)b \in D_1$$

where $a\in C^p$, $b\in D^{p+1}$, $\phi = (1+T_\lambda)\eta$ and $\delta\phi = (1+T_\lambda)\delta\eta$.

(7.11) Remark. a) The formula for γ induces (up to sign) the duality maps

$$H^r(C(f)) \to H_{p+2-r}(D)$$

for the λ-symmetric complex $(f\colon C\to D,\{\delta\eta,\eta\})$ (cf. (6.11)). Consequently, the latter is λ-Poincaré if and only if γ is an isomorphism.

b) The formula for γ also shows that $\mathcal{B}$ applied $(C,d,\{-\eta\})$ gives $(K,H,(\alpha,-\gamma))$.

Before we show $(K,H,(\alpha,\gamma))$ above is actually a formation, we sketch how it depends on the choices in its construction. Fix the chain map $f\colon C\to D$. Then if $\{\delta'\eta,\eta\}\in Q_{p+2}(f)$ is another element mapping to $\{\eta\}$ in (7.9), then there is $\{\tau\}\in Q_{p+2}(D,\lambda)$ such that $\tau = \delta'\eta - \delta\eta\colon D^*\to D$. If $\rho\colon H\to H^\wedge$ is the map on homology induced by τ and (α',γ') are the maps produced from the choice $\{\delta'\eta,\eta\}$, then $\rho = -\lambda\rho^\wedge$, $\alpha' = \alpha$ and $\gamma' = \gamma + \rho\alpha$. Thus $(K,H,(\alpha,\gamma))$ and $(K,H,(\alpha',\gamma'))$ are isometric ((1.12b)). If we choose another chain map $g\colon C\to E$ where E satisfies the conditions D did and $H^{p+1}(g) = H^{p+1}(f)$ then the resultant formation changes by an isomorphism. If another surjection $H'\to H^{p+1}(C)$ is realized by H^{p+1} of a chain map, it is not difficult to show that the resultant formation changes by stabilizations ((1.12(a)) and/or destabilization ((1.14)).

We now show $(K,H,(\alpha,\gamma))$ is a λ-formation, beginning with the injectivity of

$$(\alpha,\gamma): H^{p+1}(C(f)) \to H^{p+1}(D) \oplus H^{p+1}(D') := H^{p+1}(D) \oplus H_1(D)$$

First the following observations:

i) (α,γ) is given on the chain level by

$$C^p \oplus D^{p+1} \to D^{p+1} \oplus D_1$$

$$(a,b) \to (b,\lambda f\phi(a) + \lambda\delta\phi(b))$$

ii) $(a,b) \in C^p \oplus D^{p+1}$ is a cocycle if $d_C^*a + (-1)^{p+1}f^*b = 0$;

iii) (a,b) is a coboundary if, for some $(a',b') \in C^{p-1} \oplus D^p$, $(a,b) = (d_C^*a' + (-1)^p f^*b', d_D^*b')$;

iv) (Co)cycle conditions on D^{p+1} and D_1 on the right side of (i) are automatic;

v) the right side of (i) is a coboundary if there are $x \in D^p$, $y \in D_2$ such that $(d_D^*x, d_D y) = (b,\lambda f\phi(a) + \lambda(\delta\phi)b)$.

Assume then that (in (i)) $(b,\lambda f\phi a + \lambda(\delta\phi)b) = (d_D^*x, d_D y)$. By (iii) it suffices to find a' such that

$$d_C^*a' = (-1)^{p+1}f^*x + a$$

(taking $x = b'$). But $f\phi$ induces an injection $H^p(C) \to H_1(D) = H^\wedge$ (because $H_2(C(f)) = 0$), so it suffices to show $(-1)^{p+1}f\phi f^*x + f\phi a$ is a boundary.

Now from the definition of $(f: C \to D,\{\delta\eta,\eta\})$

$$\begin{aligned}(-1)^{p+1}f\phi f^*x &= -d_D(\delta\phi)x - (-1)(\delta\phi)d_D^*x \\ &\sim (\delta\phi)d_D^*x \\ &= (\delta\phi)b\end{aligned}$$

where $\sim$ means "homologous." By assumption, $d_D y = \lambda f\phi a + \lambda(\delta\phi)b$, so

$$f\phi a \sim -(\delta\phi)b$$

Hence

$$\begin{aligned}&(-1)^{p+1}f\phi f^*x + f\phi a \\ &\sim (\delta\phi)b + f\phi a \\ &\sim (\delta\phi)b - (\delta\phi)b = 0.\end{aligned}$$

Next we show $\mathrm{im}(\alpha,\gamma)$ is totally isotropic. This is equivalent to $\gamma^\wedge\alpha + \lambda\alpha^\wedge\gamma = 0$. We have by (i) above that $\alpha^\wedge\gamma$ is induced by

$$C(f)^{p+1} := C^p \oplus D^{p+1} \to D_1 \to C_0 \oplus D_1 := C(f)_1$$

$$(a,b) \to \lambda f\phi a + \lambda(\delta\phi)b \to (0,\lambda f\phi a + \lambda(\delta\phi)b)$$

and $\gamma^\wedge\alpha$ by

$$C^p \oplus D^{p+1} \to D^{p+1} \to C_0 \oplus D_1$$

$$(a,b) \to b \to ((-1)^{p+1}\phi f^*b,(-\delta\phi)b)$$

Hence $\gamma^\wedge\alpha + \lambda\alpha^\wedge\gamma$ is induced by

$$(a,b) \to ((-1)^{p+1}\phi f^*b, f\phi a).$$

Since (a,b) is a cocycle in $C(f)^{p+1}$,

$$(-1)^{p+1}\phi f^*b = -\phi d_C^*a = -d_C\phi a$$

so

$$d_{C(f)}(-\phi a,0) = (-d_C\phi a, f\phi a) = ((-1)^{p+1}\phi f^*b, f\phi a)$$

Thus $\gamma^\wedge\alpha + \lambda\alpha^\wedge\gamma = 0$, as claimed.

It is finally necessary to show that $\mathrm{im}(\alpha,\gamma) \subseteq H \oplus H^\wedge$ is a sublagrangian. This is in the spirit of the above and is left to the reader.

The next result compares the functions in (7.2) and (7.7).

(7.12) <u>Proposition</u>. The composition $\mathcal{B}\mathcal{A}$ is the identity.

<u>Proof:</u> Start with a formation $(K,H,(\alpha,\gamma)) \in F^\lambda(\mathcal{CM}_p)$ and apply $\mathcal{A}$ to get a $(p+1)$-dimensional λ-Poincaré complex C. We will show there is an obvious choice of the chain map $f: C \to D$ in construction $\mathcal{B}$ so that if $(K',H',(\alpha',\gamma'))$ is the resulting formation, then $K = K'$, $H = H'$, $\alpha = \alpha'$ and $\gamma = \gamma'$.

To begin, let $D = R \oplus R^t$ and D' be chosen as in (7.2) with

$$H^p(D) = H^p(R) \oplus H^p(R^t) = H \oplus H^\wedge$$

$$H^p(D') = K$$

and $f' = (g,h): D = R \oplus R^t \to D'$ such that

$$H^p(f) = (\alpha,\gamma).$$

Let $f'' = pr_2: D = R \oplus R^t \to R^t := D''$ and glue to get (7.5),

$$(C = D' \cup_D D', d_C, \{\eta\}) .$$

Now in the construction of $\mathcal{B}$, define $p: C \to C(f'')$ by

$$C_r = D_r' \oplus D_{r-1} \oplus D_r'' \to D_{r-1} \oplus D_r'' := C(f'')_r$$

$$(a,b,c) \to (b,c)$$

The following are easily verified:

(i) there are inverse chain homotopy equivalences $R \xrightarrow{\mu} C(f'')_{+1}$ and $C(f'')_{+1} \xrightarrow{\nu} R$, the first induced from the short exact sequence of chain complexes

$$R \rightarrowtail D \overset{f''}{\twoheadrightarrow} R^t = D''$$

and defined by $x \to (x,0,0)$ in

$$R_r \to (C(f'')_{+1})_r = D_r \oplus D''_{r+1} = R_r \oplus R_r^t \oplus D''_{r+1} .$$

The second is $(x,y,z) \to x$ in $(C(f'')_{+1})_r \to R_r$.

(ii) There is a short exact sequence of chain complexes $D' \to C \xrightarrow{p} C(f'')$ (inclusion to the first summand and projection to the last two), hence a chain homotopy equivalence $\kappa: D' \to C(p)_{+1}$ induced by $x \to (x,0,0,0)$ in

$$D_r' \to (C(p)_{+1})_r := C_r \oplus C(f'')_{r+1} = D_r' \oplus D_{r-1} \oplus D_r' \oplus C(f'')_{r+1}$$

(iii) $(p: C \to C(f''),\{0,\eta\})$ is a λ-symmetric complex.

We now complete the construction in $\mathcal{B}$ by setting $H' = H^{p+1}(C(f''))$, $K' = H^{p+1}(C(p))$, $\alpha' = H^{p+1}(j)$ where j is the canonical map $j: C(f'') \to C(p)$ and $\gamma': H^{p+1}(C(p)) \to H_1(C(f''))$ is the map induced by the duality map in the λ-symmetric complex of (iii) above.

Using i) and ii) we identify K with K' by $H^p(\kappa)$ and H with H' by $H^p(\mu)$. It is easily checked that the diagrams

$$\begin{array}{ccc} H^{p+1}(C(p)) & \xrightarrow{\alpha'} & H^{p+1}(C(f'')) \\ \downarrow \cong & & \downarrow \cong \\ H^p(D') & \xrightarrow{\alpha} & H^p(R) \end{array}$$

and

$$\begin{array}{ccc} H^{p+1}(C(p)) & \xrightarrow{\gamma'} & H_1(C(f'')) \\ \downarrow \cong & & \downarrow \cong \\ H^p(D') & \xrightarrow{\gamma} & H^p(R^t) \equiv H_0(R) \end{array}$$

commute. This completes the proof of (7.12).

(7.13) We next show that the function A above actually induces a homomorphism of abelian groups

$$W_1^\lambda(CM_p) \to \Omega^\lambda_{p+1}(M_p)$$

Since A in (7.2) clearly preserves sum, it is sufficient to show that the Poincare complex (C,d_C,η) obtained from the graph formation $(H^\wedge,H,(\alpha,1))$ is M_p-cobordant to one for which $H_* \equiv 0$. By the proof of (7.12) there are natural choices so that $\mathcal{B}(C,d_C,\eta)$ gives γ an isomorphism; but in (7.11)(a) we observed that in the construction of $\mathcal{B}$, $(f: C \to D,\{\delta\eta,\eta\})$ is λ-Poincaré if and only if γ is an isomorphism. Now an easy calculation shows that the result $(C',d',\{\eta'\})$ of algebraic surgery on $(f: C \to D,\{\delta\eta,\eta\})$ yields $H_*(C') \equiv 0$ (cf. (6.14)). Since $(C',d',\{\eta'\})$ is M_p-cobordant to $(C,d_C,\{\eta\})$, the latter represents zero in $\Omega^\lambda_{p+1}(M_p)$.

Now in the construction of $\mathcal{B}$, define $p\colon C \to C(f'')$ by

$$C_r = D'_r \oplus D_{r-1} \oplus D''_r \to D_{r-1} \oplus D''_r := C(f'')_r$$

$$(a,b,c) \to (b,c)$$

The following are easily verified:

(i) there are inverse chain homotopy equivalences $R \overset{\mu}{\to} C(f'')_{+1}$ and $C(f'')_{+1} \overset{\nu}{\to} R$, the first induced from the short exact sequence of chain complexes

$$R \rightarrowtail D \overset{f''}{\twoheadrightarrow} R^t = D''$$

and defined by $x \to (x,0,0)$ in

$$R_r \to (C(f'')_{+1})_r = D_r \oplus D''_{r+1} = R_r \oplus R^t_r \oplus D''_{r+1} .$$

The second is $(x,y,z) \to x$ in $(C(f'')_{+1})_r \to R_r$.

(ii) There is a short exact sequence of chain complexes $D' \to C \overset{p}{\to} C(f'')$ (inclusion to the first summand and projection to the last two), hence a chain homotopy equivalence $\kappa\colon D' \to C(p)_{+1}$ induced by $x \to (x,0,0,0)$ in

$$D'_r \to (C(p)_{+1})_r := C_r \oplus C(f'')_{r+1} = D'_r \oplus D_{r-1} \oplus D'_r \oplus C(f'')_{r+1}$$

(iii) $(p\colon C \to C(f''),\{0,\eta\})$ is a λ-symmetric complex.

We now complete the construction in $\mathcal{B}$ by setting $H' = H^{p+1}(C(f''))$, $K' = H^{p+1}(C(p))$, $\alpha' = H^{p+1}(j)$ where j is the canonical map $j\colon C(f'') \to C(p)$ and $\gamma'\colon H^{p+1}(C(p)) \to H_1(C(f''))$ is the map induced by the duality map in the λ-symmetric complex of (iii) above.

Using i) and ii) we identify K with K' by $H^p(\kappa)$ and H with H' by $H^p(\mu)$. It is easily checked that the diagrams

$$\begin{array}{ccc} H^{p+1}(C(p)) & \overset{\alpha'}{\to} & H^{p+1}(C(f'')) \\ \downarrow \cong & & \downarrow \cong \\ H^p(D') & \overset{\alpha}{\to} & H^p(R) \end{array}$$

and

$$\begin{array}{ccc} H^{p+1}(C(p)) & \overset{\gamma'}{\to} & H_1(C(f'')) \\ \downarrow \cong & & \downarrow \cong \\ H^p(D') & \overset{\gamma}{\to} & H^p(R^t) \equiv H_0(R) \end{array}$$

commute. This completes the proof of (7.12).

(7.13) We next show that the function A above actually induces a homomorphism of abelian groups

$$W^\lambda_1(CM_p) \to \Omega^\lambda_{p+1}(M_p)$$

Since A in (7.2) clearly preserves sum, it is sufficient to show that the Poincare complex (C,d_C,η) obtained from the graph formation $(H^\wedge,H,(\alpha,1))$ is M_p-cobordant to one for which $H_* \equiv 0$. By the proof of (7.12) there are natural choices so that $\mathcal{B}(C,d_C,\eta)$ gives γ an isomorphism; but in (7.11)(a) we observed that in the construction of $\mathcal{B}$, $(f\colon C \to D,\{\delta\eta,\eta\})$ is λ-Poincaré if and only if γ is an isomorphism. Now an easy calculation shows that the result $(C',d',\{\eta'\})$ of algebraic surgery on $(f\colon C \to D,\{\delta\eta,\eta\})$ yields $H_*(C') \equiv 0$ (cf. (6.14)). Since $(C',d',\{\eta'\})$ is M_p-cobordant to $(C,d_C,\{\eta\})$, the latter represents zero in $\Omega^\lambda_{p+1}(M_p)$.

§8. Two applications of the theory of Poincaré complexes to Witt groups

The first application is the proof of (1.21).

(8.1) Proposition. Let A be a regular domain. Then a λ-formation $(K,H,(\alpha,\gamma)) \in F^{\lambda}(CM_p)$ represents zero in $W_1^{\lambda}(CM_p)$ if and only if it is stably isometric to a graph formation.

Proof: One implication is clear. Conversely, suppose $[K,H,(\alpha,\gamma)] = 0$ in $W_1^{\lambda}(CM_p)$. Then the λ-Poincaré complex $(C,d,\{\eta\})$ associated to it (by (7.2)) is trivial in $\Omega_{p+1}^{\lambda}(M_p)$ by (7.13). If $(g: C \to D,\{\delta\eta,\eta\})$ is the null M_p-cobordism (a λ-Poincaré complex), then by (6.16) the only possible non-zero cohomology groups in the exact sequence of $g: C \to D$ appear in

$$(8.2)\quad H^p(g) \to H^p(D) \to H^p(C) \to H^{p+1}(g) \to H^{p+1}(D) \to H^{p+1}(C) \to H^{p+2}(g) \to H^{p+2}(D).$$

Now, proceeding as in (7.7), find a λ-symmetric complex $(h: D \to E, \{\delta\phi,\delta\eta\})$ such that the homomorphism in degree $p + 2$ is the identity, $H^{p+2}(E) \in CM_p$ and E is a sum of Koszul resolutions (cf. the construction of f at the beginning of the proof of (7.7)). Doing algebraic surgery using h ([R1, p. 145] or (6.14)) yields a λ-symmetric complex $(D',d_{D'},\{\delta'\eta\})$ which has boundary homotopy equivalent to $(C,d_C,\{\eta\})$ (by [R1, (4.1)(i)]) and for which $H^{p+2}(D') = 0 = H_0(D')$. (Compare [R2, pp. 335-6].

Hence (using (7.12)) we can assume in (8.2) that $H^{p+2}(D) = 0 = H_0(D)$. But by (6.17), $H^p \cong \mathrm{Ext}^p(H_0(D),A) = 0$ and by duality $H^{p+2}(g) \cong H_0(D) = 0$. Hence the remaining terms in (8.2) are

$$H^p(C) \rightarrowtail H^{p+1}(g) \to H^{p+1}(D) \twoheadrightarrow H^{p+1}(C) .$$

In particular the surjectivity on H^{p+1} induced by g qualifies g for use in the construction of $\mathcal{B}$ in (7.7) applied to C. By (7.11)(a), the resultant λ-formation has γ an isomorphism and by (7.12) it is stably isometric to $(K,H,(\alpha,\gamma))$.

The next result amounts to exactness of (2.1)(b) at $W_1^{\lambda}(CM_1)$. For its proof we need a special case $(p = 1)$ of the result referred to in (7.1),

$$(8.3)\qquad A_*: W_1^{\lambda}(CM_1) \xrightarrow{\cong} \Omega_2^{\lambda}(M_1) .$$

This is due to Ranicki ([R2, p. 359]).

(8.4) Proposition. Let A be a regular Noetherian domain containing 1/2. Then every element $[K,H,(\alpha,\gamma)] \in W_1^{\lambda}(CM_1)$ admits a representative for which α is injective.

Proof: Let $(C,d,\{\eta\}) \in P_2^{\lambda}(M_1)$ be associated to $(K,H,(\alpha,\gamma))$ by (7.2). Suppose it is possible to find a λ-symmetric pair $(f: C \to D, \{\delta\eta,\eta\})$ such that

(8.5)
- $H_i(D) = 0, \quad i \neq 2$
- $H^2(f): H^2(D) \to H^2(C)$ is an isomorphism at all height one primes

Then $H^1(C) \xrightarrow{\cong} H^2(f)$, which easily implies $(f: C \to D,\{\delta\eta,\eta\})$ is λ-Poincaré. By (7.11)(a), surgery on f (cf. (6.14)) yields a complex $(C',d',\{\eta'\})$ which is M_1-cobordant to $(C,d,\{\eta\})$ and whose associated λ-formation (in (7.7)) $(K',H',(\alpha',\gamma'))$ has γ' an isomorphism at all height one primes.

Now (7.11)(b), together with (8.3) and the fact that $(C',d',\{\eta'\})$ is inverse to $(C',d',\{-\eta'\})$ in $\Omega_1^{\lambda}(M_1)$, show that $[K',H',(\alpha',-\gamma')]$ is inverse to $[K',H',(\alpha',\gamma')]$ in $W_1^{\lambda}(CM_1)$. But by (1.16) the latter is also inverse to $[K',H',(\gamma',-\lambda\alpha')]$. Hence in $W_1^{\lambda}(CM_1)$, $[K',H',(\alpha',\gamma')] = [K',H',(\gamma',\lambda\alpha')]$, so we are done.

It remains to provide (8.5). Let q be a height one prime of A and let $R = A_q$, $(x) = qA_q$. By (5.9), $(K_q,H_q,(\alpha,\gamma)_q)$ is isomorphic to a sum of formations of the form

$$(R/(x^t) \oplus R/(x^{m-t}),R/(x^m),((0,x^t),(x^{m-t},0))$$

where $0 \le t \le m$. Evidently, each such formation can be destabilized to $(R/(x^t),R/(x^t),(0,1))$. So if $(\bar{C},\bar{d},\bar{\eta}) \in P_2^{\lambda}(M_1(R))$ is a formation constructed from $(K_q,H_q,(\alpha_q,\gamma_q))$ by (7.2), there is a homotopy equivalence

$$h: (C,d,\{\eta\})_q \xrightarrow{\cong} (\bar{C},\bar{d},\{\bar{\eta}\})$$

Moreover, since $\alpha = 0$ in $(R/(x^t),R/(x^t),(0,1))$ we can extract from the construction in (7.7) (cf. also proof of (7.12)) a λ-symmetric complex over R,

$$(\bar{g}: \bar{C} \to \bar{D},(\delta\bar{\eta},\bar{\eta}))$$

such that $H^2(\bar{g}): H^2(\bar{D}) \xrightarrow{\cong} H^2(\bar{C})$. Setting $\bar{f} = \bar{g}h$, we get a λ-symmetric complex

$$(\bar{f}: C_q \to \bar{D},\{\bar{\delta}\eta,\eta_q\})$$

with

$$H^2(\bar{f}): H^2(\bar{D}) \xrightarrow{\cong} H^2(C_q) .$$

Now if $H^2(\bar{D}) = (R/(x^t))^m$, let $M = (A/q^{(t)})^m$, where $q^{(t)}$ denotes the t-th symbolic power. Then $M \in CM_1$ because A is locally factorial. Hence we may construct a chain complex of projective A-modules

$$E = (E_2 \to E_1)$$

with

$$H_i(E) = \begin{cases} M, & i = 1 \\ 0, & i = 2 \end{cases}$$

and from the natural inclusion $M \to (R/(x^t))^m$, a commutative diagram,

$$\begin{array}{ccccc} E_2 & \to & E_1 & \to & M \\ \downarrow & & \downarrow & & \downarrow \\ \bar{D}_2 & \to & \bar{D}_1 & \to & (R/(x^t))^m \end{array}$$

Thus the corresponding map of complexes $E \to \bar{D}$ is a homotopy equivalence at q. Finally, multiplying $\bar{f}$ and $\bar{\delta}\eta$ by some element of $A - q$ gives $\bar{f}(C) \subseteq E$ and $\bar{\delta}\eta(E^*) \subseteq E$, hence a λ-symmetric complex

(8.6) $$(f\colon C \to E, \{\delta\eta, \eta\})$$

where $f = \bar{f}|C$, $\delta\eta = \bar{\delta}\eta|E^*$ and $H^2(f_q)\colon H^2(E)_q \xrightarrow{\cong} H^2(C)_q$. Taking the sum of the complexes (8.6) for each height one q gives the desired λ-symmetric complex in (8.5).

Bibliography

[AB] M. Auslander and M. Bridger, Stable Module Theory, Mem. A.M.S. 94 (1969).

[Ba] J. Barge, Formes Hermitiennes sur les $\mathbb{Z}[X,X^{-1}]$-modules, Bull. Soc. Math. France, Mem. 48 (1976), 7-10.

[B] H. Bass, On the ubiquity of Gorenstein rings, Math. Zeit. 82 (1963), 8-28.

[BE] D. Buchsbaum and D. Eisenbud, Algebra structures on finite free resolutions, Am. J. Math. 99 (1977), 447-485.

[CF] L. Claborn and R. Fossum, Generalizations of the notion of class group, Ill. J. Math. 12 (1968), 228-253.

[Ch] C. Chevalley, Anneaux de Chow et applications, Séminaire C. Chevalley, 2^{e}année, Sec. Math. Paris (1958).

[C1] J.-L. Colliot-Thélène, Formes quadratiques sur les anneaux semi-locaux réguliers, Mém. Soc. Math. France 59 (1979), 13-31.

[C2] ______________, Formes quadratiques multiplicatives et variétés

[C3] ______________, Formes quadratiques multiplicatives et variétés algébriques: deux compléments, Bull. Math. Soc. France 108 (1980), 213-227.

[CI] ______________ and F. Ischebeck, L'équivalence rationelle sur les cycles de dimension zéro des variétés algébriques réeles, C. R. Acad. Sc. Paris 292 (1981), 723-725.

[CS] ______________ and J.-J. Sansuc, Fibrés quadratiques et composantes connexes réeles, Math. Ann. 244 (1979), 105-134.

[CRW] T. Craven, A. Rosenberg, and R. Ware, The map of the Witt ring of a domain into the Witt ring of its field of fractions, Proc. A. M.S. 51 (1975), 25-30.

[DK] H. Delfs and M. Knebusch, Semi-algebraic topology over a real closed field I: Paths and components in the set of rational points of analegraic variety, Math. Zeit. 177 (1981), 107-129.

[F] D. Ferrand, Construction des fibrés de rang deux, Asterisque 36, 149-169.

[Fo] R. Fossum, Vector bundles over spheres are algebraic, Inv. Math. 8 (1969), 222-225.

[Fu] W. Fulton, Rational equivalence on singular varieties, Publ. Math. I.H.E.S. 45 (1975), 147-164.

[G] S. Gersten, Some exact sequences in the higher K-theory of rings, Springer Lecture Notes No. 341 (1973), 211-244.

[Gr] A. Grothendieck, Éléments de geometrie algébrique, Pub. Math. I.H.E.S. 32 (1967).

[Ho] M. Hochster, Cohen-Macaulay modules, Lecture Notes in Math. No. 311, Springer Verlag (New York, Heidelberg).

[Hu] D. Husemoller, Fibre Bundles, McGraw-Hill (1966).

[K] I. Kaplansky, Commutative Rings (Rev. Ed.), University of Chicago Press, (1974).

[K1] M. Knebusch, On algebraic curves over real closed fields II, Math. Zeit. 151 (1976), 189-205.

[K2] ______________, Specialization of quadratic and symmetric bilinear forms, and a norm theorem, Acta Arith. 24 (1973), 279-299.

[K3] ______________, Symmetric bilinear forms over algebraic varieties, Conf. on quadratic forms (Kingston 1976), Queens Papers in Pure and Applied Math. 46 (1977), 103-283.

[KOS] M. A. Knus, M. Ojanguren and R. Sridharan, Quadratic forms and Azumaya algebras, Journal fur die reine und angewandte Math. 303/304 (1978), 231-248.

[KM] A. Kustin and M. Miller, Algebra structures on minimal resolutions of Gorenstein rings, In: Commutative algebra: analytic methods, Marcel Dekker (1982), 45-66.

[L] T.-Y. Lam, The algebraic theory of quadratic forms, Benjamin

[M] H. Matsumura, Commutative Algebra (Sec. Ed.), Benjamin/Cummings (1980), Reading.

[MH] J. Milnor and D. Husemoller, Symmetric Bilinear Forms, Springer Verlag (1973), New York-Heidelberg.

[N] M. Nagata, Local Rings, Interscience (1962), New York-London.

[O1] M. Ojanguren, A splitting theorem for quadratic forms, Comment. Math. Helv. 57 (1982), 145-157.

[O2] ______________, Quadratic forms over regular rings, J. Indian Math. Soc., to appear.

[P1] W. Pardon, The exact sequence of a localization for Witt groups, Lecture Notes in Math., No. 551 (1976), 336-379.

[P2] ______________, An invariant determining the Witt class of a unitary transformation over a semi-simple ring, J. Alg. 44 (1977), 394-410.

[P3] ______________, A "Gersten conjecture" for Witt groups, Springer Lecture Notes in Math. No. 966 (Proceedings of Conference on Alg. K-Theory Oberwolfach, 1980).

[P4] ______________, A relation between the Witt group of a regular local ring and the Witt groups of its residue class fields, preprint.

[PS] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Pub. Math. I.H.E.S. 42 (1973), 47-119.

[Q] D. Quillen, Higher algebraic K-Theory I, Springer Lecture Notes in Math. No. 341 (1973), 85-147.

[R1] A. Ranicki, The algebraic theory of surgery I: Foundations, Proc. L.M.S. 40 (1980), 87-192.

[R2] ______________, Exact sequences in the algebraic theory of surgery, Math. Notes, Princeton University Press (1981), Princeton.

[SV] D. W. Sharpe and P. Vamós, Injective Modules, Cambridge University Press (1972), Cambridge.

AN UPPER BOUND FOR ALGEBRAIC K-THEORY

Victor Snaith*

§1: INTRODUCTION

The title of this paper coincides with that of section IV.3 of [Sn 1]. In that section I gave an upper bound, in terms of unitary K-homology, for the algebraic K-theory (mod ℓ^ν) of a ring-*after the latter has been inflicted with Bott periodicity*. In this paper I will improve the results of [Sn 1, §IV.3] in two ways:

(i) We remove the condition that the ring we consider should have ℓ^ν-th roots of unity.

(ii) We show that the indecomposable quotient of the upper bound of [Sn 1] is a (better) upper bound.

The upper bound is sometimes "easier" to compute. For example, when A is a finite ring, the upper bound is determined by the representation theory of GL_nA $(n \geq 1)$. See [Sn 1, §IV.3].

Recall [B;G-Q;Q1] that the algebraic K-groups of a ring A, with unit, may be defined as

$$K_iA = [S^i, BGLA^+] = \pi_i(BGLA^+), \quad (i \geq 1)$$

and

$$K_i(A;Z/n) = [S^{i-1} \cup_n e^i, BGLA^+] = \pi_i(BGLA^+;Z/n), \quad (i \geq 2)$$

where $BGLA^+$ is obtained from the classifying space of the infinite general linear group of A by applying the "plus"-construction relative to the commutator subgroup of GLA. One can extend these groups to lower dimensions, although we will not need them, in such a way as to make

$$K_*A = \bigoplus_{i \geq 0} K_iA$$

and

$$K_*(A;Z/n) = \bigoplus_{i \geq 0} K_i(A;Z/n)$$

into graded rings (provided that $16|n$ if n is even and $9|n$ if $3|n$). In fact we will require only a (graded) multiplication to exist on $K_*(A;Z/n)$ for which it

---

1980 Mathematics Subject Classification. 18F25, 55N15, 55Q10, 55P47, 55P42.

*Research was partially supported by grant #A4633 of the Natural Sciences and Engineering Research Council of Canada.