A "Gersten Conjecture" for Witt groups

by

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Let $A$ be a commutative domain with $\frac{1}{2} \in A$, $K$ its fraction field, and $W(A)$ the Witt group of non-singular symmetric bilinear forms on projective modules over $A$. In [8] and [11], the question of the injectivity of the homomorphism $W(A) \to W(K)$ was studied, under the assumption that $A$ be regular local. A technique is outlined here and carried out in detail for one case, which answers this question affirmatively if the Krull dimension of $A$ is $\leq 4$. The method introduces analogues for Witt groups of localization, dévissage, and resolution theorems in algebraic K-theory; one consequence is a "Gersten Conjecture" [9] for Witt groups, which considerably generalizes the original question, but remains mostly conjectural. (C.F. Addendum following (3.19)).

For $A$ a local Gorenstein ring, a suitable category $M_p(A)$ of A-modules supported in codimension $p$ is introduced in §1, so that Witt groups $W_i^\lambda(M_p)$ $(i = 0,1; \lambda = \pm 1)$ fitting into localization sequences may be defined. This plus dévissage gives rise in §2 to the "Gersten Conjecture" and to conjectured relations with other results inspired by the same ideas. In §3, the methods of §1 are carried out in detail to prove $W(A) \to W(K)$ is injective if $A$ is a 2-dimensional regular domain.

It is assumed throughout that $A$ is a commutative noetherian ring containing $\frac{1}{2}$.

## §1. Localization and Dévissage for Witt group functors.

Let $(A, m)$ be a local Gorenstein ring of Krull dimension $n$. Let

$$A \xrightarrow{d_0} E_0 \longrightarrow E_1 \to \cdots \xrightarrow{d_n} E_n$$

be the minimal injective resolution of $A$ over itself, so that $E_k = \coprod_{ht(q)=k} E(A/q) = \coprod_{ht(q)=k} E(A_q/qA_q)$, where $E$ = injective hull (see [2, §1].). Let $M_p(A)$ denote the category of A-modules $M$ such that $\dim(M) = n - p = \text{depth}\ (M)$ ($M$ has "codimension $p$"); such modules are called Cohen-Macauley in [7], and, if $A$ is regular local, are perfect in the sense of [10,p. 126]). When $A$ is specified we write $\mathcal{M}_p(A) = \mathcal{M}_p$.

Let $V_p = \operatorname{im} d_p$. Using the fact that each $M \in \mathcal{M}_p$ admits a length one resolution by objects of $\mathcal{M}_{p-1}$ $(p \geq 1)$, one shows by induction that the natural A-bilinear pairing

$$\nu: \ \operatorname{Hom}(M,V_p) \times M \to V_p, \ \nu(f,m) = f(m)$$

is non-singular (i.e., that $M \cong \operatorname{Hom}(\operatorname{Hom}(M,V_p),V_p))$. (Alternatively, one may use [1,4.35] and the easy fact that $\operatorname{Hom}(M,V_p) \cong \operatorname{Ext}^p(M,A)$.) Thus, in the category $Q^\lambda\mathcal{M}_p$, whose objects $(N,\phi)$, called $\lambda$-forms in $\mathcal{M}_p$, are non-singular, $\lambda$-symmetric A-bilinear pairings

$$\phi: \ N \times N \to V_p \quad (N \in \mathcal{M}_p, \quad \lambda = \pm 1) ,$$

there is a natural notion of hyperbolic form: $(N,\phi)$ is called hyperbolic if $N = \operatorname{Hom}(M,V_p) \oplus M$ and $\phi(f_1 + m_1, f_2 + m_2) = f_1(m_2) + \lambda f_2(m_1)$; more generally, $(N,\phi)$ is called a lagrangian if there is $M \subseteq N$, where $M \in \mathcal{M}_p$, $\phi|M \times M \equiv 0$ and the induced bilinear form $M \times (N/M) \to V_p$ is non-singular; $M$ is called a sublagrangian. (In [15], the term (sub) kernel was used for (sub) lagrangian ).

One defines Witt groups

$$W_i^\lambda(\mathcal{M}_p) \qquad (\lambda = \pm 1, \quad i = 0,1)$$

as follows. One may add isomorphism classes of objects in $Q^\lambda\mathcal{M}_p$ by orthogonal sum and, in this way, form an abelian semigroup. The Grothendieck group modulo the subgroup generated by lagrangians is denoted $W_0^\lambda(\mathcal{M}_p)$.

Define a $\lambda$-formation in $\mathcal{M}_p$ to be a triple $(M_1,M_2,\Delta)$ where $M_i \in \mathcal{M}_p$ and $\Delta: \ M_1 \to \operatorname{Hom}(M_2,V_p) \oplus M_2$ is an injection whose image is a sublagrangian of the $\lambda$-symmetric hyperbolic form on $\operatorname{Hom}(M_2,V_p) \oplus M_2$. One defines the zero formation and the orthogonal sum of two $\lambda$-formations in the obvious way, so that the set of isomorphism classes of $\lambda$-formations is an abelian semigroup. An equivalence relation on this semigroup is defined in [15,1.34], by replacing the category $\mathcal{D}_F^1$ used there by $\mathcal{M}_p$ and removing the quadratic form $\kappa$ (because $\frac{1}{2} \in A$ is assumed here). The equivalence classes form a group, $W_1^\lambda(\mathcal{M}_p)$.

Theorem 1 (Localization) Let A be an n-dimensional local Gorenstein ring. For p = 0,1,n - 2 or n - 1, there is a long exact sequence ($M_p = M_p(A)$),

$$\begin{aligned} \cdots &\to W_0^{\lambda}(M_p) \to \coprod_q W_0^{\lambda}(M_p(A_q)) \to W_0^{\lambda}(M_{p+1}) \\ &\to W_1^{-\lambda}(M_p) \to \coprod_q W_1^{-\lambda}(M_p(A_q)) \to W_1^{-\lambda}(M_{p+1}) \\ &\to W_0^{-\lambda}(M_p) \to \coprod_q W_0^{-\lambda}(M_p(A_q)) \to \cdots \end{aligned}$$

where the direct sums $\coprod$ extend over all primes $q$ of height p. (I conjecture that Theorem 1 holds for all p.)

The theorem should be compared to [16, Thm. 5] where the subscript shifts are different, and the category $M_p$ is less restrictive. If p=0, Theorem 1 is closely related to the theorem of [15]: when A is regular, $V_0 = A$, $E_0$ = fraction field of A = K, $V_1 = K/A$, $M_0$ = category of projectives, and $M_1$ = category of A-torsion modules which have a short free resolution. It is likely that the theorem holds without the restriction that A be local.

Note that when p = ht $q$, $M_p(A_q) = A_q$ - modules of finite length.

Theorem 2 (Dévissage) If $(A,m)$ is local Gorenstein with residue field $k(m)$, and $F_A$ = category of finite length A-modules, then $F_A = M_{\dim A}(A)$ and there are inclusions $F_{k(M)} \to F_A$, $k(m) \to E(k(m))$ which induce isomorphisms

$$W_i^{\lambda}(F_{k(m)}) \overset{\sim}{\to} W_i^{\lambda}(F_A) .$$

Remarks a) $W_i^{\lambda}(F_{k(m)}) = W_i^{\lambda}(k(m)) = 0$ if $i = 1$; or if $i = 0$ and $\lambda = -1$. b) To my knowledge, no "resolution theorem" for the functors $W_*^{\lambda}(A)$ has appeared. A suitable version is furnished by the theory of [17]. For example, one may show that to any object $(M,\phi) \in Q^{\lambda}(M_p)$ can be associated a chain complex $C_*$ with "Poincaré duality" pairings $C_k \otimes C_{p-k} \to A$ (induced by $\phi$) such that $H_i(C_*) = 0$ $i \neq 0$ and $H_0(C) = M$. It is necessary to use this theory to prove exactness in Theorem 1 at $W_1^{-\lambda}(M_p)$.

## §2. Conjectures

It is now possible to make conjectures analogous to those of [9,§7].

<u>Conjecture A</u> If $(A,m)$ is an n-dimensional regular local ring with fraction field $K$, then there is an exact sequence

$$W_0^1(A) \xrightarrow{D_0} W_0^1(K) \xrightarrow{D_1} \coprod_{ht(q)=1} W_0^1(k(q)) \to \cdots \xrightarrow{D_n} W_0^1(k(M))$$

where $D_0$ is induced by $\otimes K$, $D_i$ $(i \geq 1)$ is the composition of maps from the $p = i - 1$ and $\iota$ versions of Theorem 1,

$$\coprod_{ht(q)=i-1} W_0^1(M_{i-1}(A_q)) \to W_0^1(M_i) \to \coprod_{ht(q)=i} W_0^1(M_i(A_q))$$

and $W_0^1(F_{A_q}) \cong W_0^1(k(q))$ by Theorem 2.

The following holds if $p = 0$ and would imply Conjecture A.

<u>Conjecture B</u> If $A$ is regular local, then

$$W_1^\lambda(M_p) = 0 = W_0^{-1}(M_p), \quad \lambda = \pm 1, \quad p \geq 0.$$

The injectivity of $D_0$ in $A$ is the issue in [8] and, as pointed out in [5], is equivalent to a special case of a conjecture of Grothendieck; it is equivalent to $W_1^1(M_1) = 0$. It is well-known to hold if $\dim A = 1$ and is shown in [8] to hold if $A$ is complete, or if $A$ is a polynomial ring over a field. In [5] forms of low rank over $A$ which become hyperbolic over $K$ are shown to be hyperbolic. Since this paper was written, Ojanguren [14] has given a remarkably simple proof that $D_0$ is injective, in case $A$ is regular local and essentially of finite type over an infinite perfect field (see also Addendum following (3.19) below.

Assuming Theorems 1 and 2 all vertical and horizontal sequences are exact in the following diagram:

$$\begin{array}{ccccccc}
 & & W_0^1(M_4) & & & & \\
 & & \downarrow & & & & \\
 & & W_1^{-1}(M_3) & \to & W_0^{-1}(M_2) & \to & \coprod_{htq=2} W_0^{-1}(k(q)) \\
 & & \downarrow & & \downarrow & & \\
\coprod_{htq=3} & & W_1^{-1}(k(q)) & & W_1^1(M_1) & \to & W_0^1(A) \to W_0^1(K) \\
 & & & & \downarrow & & \\
 & & & & \coprod_{htq=1} W_1^1(k(q)) & & .
\end{array}$$

Given an element of $W_0^1(A)$ which vanishes in $W_0^1(K)$, a diagram chase using the remark following Theorem 2 gives the following result if $\dim A \leq 3$.

<u>Theorem 4</u> If $A$ is regular local and $\dim A \leq 4$, then $W_0^1(A) \to W_0^1(K)$ is injective.

The 4-dimensional case requires the machinery of [17] and will not be treated here. If $\dim A \leq 3$, Theorem 4 holds without the assumption that $A$ be local; complete details are given for the 2-dimensional case in §3, following the argument for Theorem 4 above.

When $A$ is regular, the negative K-groups of [3, XII, §7], $K_{-i}(A)$, are trivial, $i \geq 1$. In one case this alone suffices to guarantee $D_0$ is injective.

<u>Proposition</u> Let $A$ be a one-dimensional integral k-algebra, k a field, $\mathrm{char}(k) \neq 2$, $K$ = fraction field of $A$. Let $\bar{A}$ be the integral closure of $A$ in $K$, $\mathcal{C}$ = the conductor of $\bar{A}$ in $A$, and suppose $W_0^1(A/\mathcal{C}) \to W_0^1(\bar{A}/\mathcal{C})$ is injective. Then there is an exact sequence

$$K_{-1}(A) \otimes \mathbb{Z}/2 \to W_0^1(A) \xrightarrow{D_0} W_0^1(K).$$

The hypotheses are all satisfied if $A = k[x,y]/(y^2 - x^2(x+1))$ (node); here $K_{-1}(A) = \mathbb{Z}$ and $\ker D_0 = \mathbb{Z}/2$. Also, if $A = k[x,y]/(y^2 - x^3)$ (cusp), $K_{-1}(A) = 0$ but $A$ is not regular. Other examples of a less trivial nature, where $\dim A > 1$, are given in [8] to show the necessity of the regularity assumption. However, the hope that the vanishing of $K_{-*}(A)$ for higher dimensional $A$ might imply $D_0$ is injective must take into account the example in [12, §5] of an 8-dimensional regular ring (not local) for which $\ker D_0 \neq 0$. The example disappears if the ring is localized since it depends on the non-triviality of $\tilde{K}_0$. In fact, as Ojanguren has pointed out to me, one may use the same idea as [12] to produce a non-trivial element in $\ker D_0$ when $A = B \otimes_{\mathbb{R}} B$, $B = \mathbb{R}[X,Y,Z]/(X^2 + Y^2 + Z^2 - 1)$.

The lower sequence in the following conjecture was constructed and proved exact in [4].

<u>Conjecture C</u> Let $(A, m)$ be a regular local ring. There is a commutative ladder for each $r > 0$,

$$\begin{array}{ccccccc}
W_0^1(A) & \to & W_0^1(K) & \to & \coprod\limits_{ht(q)=1} W_0^1(k(q)) & \to \cdots \to & \coprod\limits_{htq=r} W_0^1(k(q)) \\
\downarrow w_r & & \downarrow w_r & & \downarrow w_{r-1} & & \downarrow w_0 \\
H^r_{\acute{e}t}(A;\mathbb{Z}/2) & \to & H^r_{\acute{e}t}(K;\mathbb{Z}/2) & \to & \coprod\limits_{ht(q)=1} H^{r-1}_{\acute{e}t}(k(q);\mathbb{Z}/2) & \to \cdots \to & \coprod\limits_{ht(q)=r} H^0_{\acute{e}t}(k(q);\mathbb{Z}/2)
\end{array}$$

where $w_i$ = ith Stiefel-Whitney class defined in [6] or [18]; $w_0$ may be identified with the rank mod 2, $w_1$ with the discriminant, and $w_2$ with the Hasse-Witt invariant. This is the relation of Conjecture A to classical invariants of quadratic forms.

## §3. $D_0$ is injective for 2-dimensional regular domains

Although the complete proof of Theorem 1 will not be given here, the ideas involved are essentially the same as those in [15]. To illustrate this, the injectivity of $D_0$ will be proved in a special case, following the method of Theorem 4. The proof begins with a device which simplifies the first step (obtaining $(K,H,\Delta)$ in (3.1)) and which does not work for 3-dimensional rings. This is the reason for restricting to the 2-dimensional case. Since $A$ is assumed regular, but not local $M_p(A)$ will now denote the category of A-modules $M$ whose homological dimension = grade (M) = p. (If $A$ is local, this is the same as the definition given in §1).

Theorem 5: Let $A$ be a regular Noetherian domain of dimension 2 with $\frac{1}{2} \in A$. Then $W_0^1(A) \to W_0^1(K)$ is injective, where $K$ = fraction field of $A$.

Proof: Let $\phi: N \times N \to A$ be a non-singular symmetric bilinear form where $N$ is projective and $\phi \otimes K$ is hyperbolic. Choose $L \subseteq N$ so that $(L \otimes K)^{\perp} = L \otimes K \subseteq N \otimes K$. Then $L \subseteq L^{\perp}$. If $x \in L^{\perp} - L$, $(L + Ax) \subseteq (L + Ax)^{\perp}$; thus, there exists $M \subseteq N$ with $M = M^{\perp}$. We first show $M$ is projective.

Let $\overline{P} = \mathrm{Hom}(P,A)$ for any A-module $P$. There is a commutative diagram

$$\begin{tikzcd}
 & M \arrow[dl, "\kappa"'] \arrow[dr, "\alpha"] & \\
\overline{\overline{M}} \arrow[rr, "\beta"] & & \overline{N/M}
\end{tikzcd}$$

where $\kappa$ is the canonical map and $\alpha: M \to \overline{N/M}$, $\beta: N/M \to \overline{M}$ are induced from the adjoint of $\phi$. Since $M = M^{\perp}$, $\alpha$ is an isomorphism and

$\beta$ is injective; $\bar\beta$ is injective because $\bar{\bar{M}}$ is torsion free and $\bar\beta \otimes K$ is an isomorphism. Thus $\kappa$ is an isomorphism.

This implies $M$ is projective, for given any partial projective resolution of $\bar M$,

$$Q_1 \xrightarrow{f} Q_0 \twoheadrightarrow \bar M ,$$

we may dualize it to the exact sequence

$$\bar{\bar{M}} \rightarrowtail \bar Q_0 \xrightarrow{\bar f} \bar Q_1 \twoheadrightarrow \operatorname{cok} \bar f \ ;$$

since $A$ has global homological dimension 2, $\bar{\bar{M}} = M$ is projective.

Next, following [15, §7] we construct a $(+1)$-formation $(K,H,\Delta)$ representing an element of $W_1^1(M_1)$ $(= W_1^1(K/A)$ in [15]) whose image in $W_0^1(M_0)$ $(= W_0^1(A))$ under the map of Theorem 1 for $p = 0$ (= the exact sequence [15, 2.1]) is $[N,\phi]$. To do this we first choose an $A$-submodule $L \subseteq N$ such that $L \otimes K = N \otimes K$, $L \cong P + \bar P$ ($P$ is projective), and the form $\psi := \phi|L \times L$ has $\mathrm{Ad}(\psi)\colon P \oplus \bar P \to \overline{P \oplus \bar P} = \bar P \oplus P$ given by a $(2 \times 2)$-matrix $\begin{pmatrix} 0 & \bar h \\ h & 0 \end{pmatrix}$ where $h \in \mathrm{Hom}(P,P)$. (Here $\psi$ is a hyperbolic approximation to $\phi$, although it need not be non-singular).

Since $N/M$ is torsion free, the natural map $\kappa\colon N/M \to \overline{\overline{N/M}}$ is injective; let $t \in A$ be chosen so that $t(\operatorname{cok}(\kappa)) = 0$. Then there is a injection $\sigma\colon \overline{\overline{N/M}} \to N/M$ so that $\kappa\sigma = tI_{\overline{\overline{N/M}}}$. Let $j\colon N \twoheadrightarrow N/M$ be the projection and define

$$L := j^{-1}\,\sigma(\overline{\overline{N/M}}) .$$

Then $M \subseteq L$ and there is short exact sequence $M \rightarrowtail L \twoheadrightarrow \overline{\overline{N/M}}$ where $\overline{\overline{N/M}}$ is identified with $\sigma(\overline{\overline{N/M}})$. As $\overline{\overline{N/M}} \cong \bar M$ is projective, $L \cong M \oplus \bar M$. One may verify that, for a suitable choice of splitting of $L \twoheadrightarrow \overline{\overline{N/M}}$, $\psi = \phi|L \times L$ has $\mathrm{Ad}\,\psi = \begin{pmatrix} 0 & tI \\ tI & 0 \end{pmatrix}$. (This uses $\frac{1}{2} \in A$.)

Let $L' = \{n \in N \otimes K \mid \phi(n,L) \subseteq A\}$, the dual lattice of $L$; then $\phi$ induces an isomorphism $\ell\colon L' \cong \bar L$, and composing this with the inclusion $L \to L'$ yields $\mathrm{Ad}\,\psi\colon L \to \bar L$. Thus, letting $i\colon L \to N$ be the inclusion and $k := \ell \circ (N \to L')$, we have a commutative triangle

$$\begin{tikzcd}
P \oplus \bar P = L \arrow[rr, "{\mathrm{Ad}\,\psi = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}}"] \arrow[dr, "i"'] & & \bar L = P \oplus \bar P \\
& N \arrow[ur, "k"'] &
\end{tikzcd}$$

Set $K := \mathrm{cok}(i)$ and $H := \mathrm{cok}(t\colon P \to P)$; then setting

$$H^{\wedge} := \mathrm{Hom}(H,V_1) \qquad (V_1 = K/A)$$

$H^{\wedge} \cong \mathrm{cok}(t\colon \overline{P} \to \overline{P})$ (see [15, 1.4] or (3.8)) and $\mathrm{cok}(\mathrm{Ad}\,\psi) = H \oplus H^{\wedge}$. Let $\Delta\colon K \to H \oplus H^{\wedge}$ be the induced map (covered by $k$). By the argument of [15, 7.3], we obtain the following.

(3.1) <u>Lemma</u> $(K,H,\Delta)$ is a $(+1)$-formation in $M_1$. (Note that these are precisely the objects of [15, 1.32], after omitting the form $\kappa$ - since $\frac{1}{2} \in A$ - and allowing projective resolutions for $K$ and $H$ in place of free ones.)

Next we generalize the method of [15, 1.17, 1.18] to produce a non-singular skew-symmetric form $\mu\colon T \times T \to V_2$, where $T \in M_2$. ($[T,\mu] \in W_0^{-1}(M_2)$ is the inverse image of $[K,H,\Delta]$ under the map $W_0^{-1}(M_2) \to W_1^1(M_1) := W_1^1(K/A)$ of Theorem 1 for $p = 1$; see the proof of Theorem 4.)

(3.2) <u>Lemma</u> Suppose given $K,H \in M_p$ and $(\alpha,\gamma)\colon K \to H \oplus H^{\wedge}$ where $H^{\wedge} := \mathrm{Hom}(H,V_p)$, $\alpha$ is injective, $\mathrm{cok}\,\alpha \in M_{p+1}$, and $\alpha^{\wedge}\gamma = \lambda(\alpha^{\wedge}\gamma)^{\wedge}$. Then there is a unique $\lambda$-symmetric bilinear form $\tau\colon H \times H \to E_p$ such that

(3.3)
$$\begin{array}{ccc} K & \xrightarrow{\alpha} & H \\ \downarrow\gamma & & \downarrow \mathrm{Ad}\tau \\ H^{\wedge} & \xrightarrow{i_*} & \mathrm{Hom}(H,E_p) \end{array}$$
commutes,

where $i_*$ is induced by $V_p \to E_p$.

b) Conversely, given a bilinear $\lambda$-symmetric form $\tau\colon H \times H \to E_p$ and $\alpha\colon K \to H$ with $\mathrm{cok}\,\alpha \in M_{p+1}$, such that $(d_p)_* \cdot \mathrm{Ad}\tau \cdot \alpha = 0$ (where $(d_p)_*\colon \mathrm{Hom}(H,E_p) \to \mathrm{Hom}(H,E_{p+1})$ is induced by $d_p\colon E_p \to E_{p+1}$) there is a unique map $\gamma\colon K \to H^{\wedge}$ making (3.3) commute and satisfying $\alpha^{\wedge}\gamma = \lambda(\alpha^{\wedge}\gamma)^{\wedge}$.

<u>Proof</u>. Part b) follows from the exactness of $H^{\wedge} = \mathrm{Hom}(H,V_p) \xrightarrow{i_*} \mathrm{Hom}(H,E_p) \xrightarrow{(d_p)_*} \mathrm{Hom}(H,E_{p+1})$ and the $\lambda$-symmetry of $\tau$.

To prove a), let $\mathrm{Ass}(H) = \{p_1,\ldots,p_n\}$; by [10, Thm. 175], $\mathrm{ht}(p_i) = p$, $i = 1,\ldots,n$. Thus,

$$\mathrm{Hom}(H,E_p) = \coprod_1^n \mathrm{Hom}(H,E(A/p_i)) = \coprod_1^n \mathrm{Hom}(H_{p_i},E(A/p_i)) ,$$

the second inequality because $E(A/p_i)$ is an $A_{p_i}$-module ([19, Prop.5.6]).

Hence the desired $\tau$ must split as an orthogonal sum $\tau = \coprod \tau_i \colon H_{p_i} \times H_{p_i} \to E(A/p_i)$, $p_i \in \mathrm{Ass}(H)$. Since $\alpha_{p_i} \colon K_{p_i} \xrightarrow{\cong} H_{p_i}$ (because $\mathrm{cok}(\alpha) \in M_{p+1}$) define

$$\mathrm{Ad}\tau_i := (i_*)_{p_i} \gamma_{p_i} (\alpha_{p_i})^{-1} .$$

It is easily shown that $\tau$ is $\lambda$-symmetric.

(3.4) <u>Remark</u> The proof shows that $\alpha$ induces an isomorphism $\mathrm{Hom}(H,E_p) \xrightarrow{\cong} \mathrm{Hom}(K,E_p)$.

(3.5) <u>Lemma</u> a) Given $\tau\colon H \times H \to E_p$ and $\alpha\colon K \to H$ satisfying the hypotheses of (3.2)(b), there is a $\lambda$-symmetric form $\mu\colon T \times T \to V_{p+1}$, where $T := \mathrm{cok}(\alpha)$ such that

$$(3.6)\qquad \begin{array}{ccc} H \times H & \xrightarrow{\tau} & E_p \\ \downarrow & & \downarrow d_p \\ T \times T & \xrightarrow{\mu} & V_{p+1} \end{array} \qquad \text{commutes.}$$

b) Conversely, given a $\lambda$-symmetric bilinear form $\mu\colon T \times T \to V_{p+1}$, $T \in M_{p+1}$, there is a resolution $K \overset{\alpha}{\rightarrowtail} H \twoheadrightarrow T$ with $K,H \in M_p$ and a $\lambda$-symmetric form $\tau\colon H \times H \to E_p$ making (3.6) commute; in particular, $\tau$ and $\alpha$ satisfy the hypotheses of (3.2)(b).

<u>Proof</u>. Part a) is immediate.

Let $\{t_1,\ldots,t_n\}$ generate $T$. Suppose $\mu(t_i,t_j) = v_{ij} \in V_{p+1}$ and let $d_p(e_{ij}) = v_{ij}$, $e_{ij} \in E_p$. Since $v_{ij} = \lambda v_{ji}$, we may assume $e_{ij} = \lambda e_{ji}$. Let $V_{ij} = \{p \mid v_{ij}$ has non-zero component in $E(A/p) \subseteq E_{p+1}\}$, and $E_{ij} = \{q \mid e_{ij}$ has non-zero component in $E(A/q) \subseteq E_p\}$. Since $d_p e_{ij} = v_{ij}$, each $q \in E_{ij}$ is contained in some $p \in V_{ij}$ (see [19, Prop. 4.21]).

Because $\mu$ is bilinear, $\mathrm{Ann}(t_i) \cup \mathrm{Ann}(t_j) \subseteq \mathrm{Ann}(v_{ij}) \subseteq \{p \mid p \in V_{ij}\}$. Thus, $\cup \mathrm{Ass}(At_i) \subseteq V := \cup V_{ij}$. From this it follows that there is an integer $\ell$ such that

$$\Big(\prod_{p \in V} p^{\ell}\Big)(At_i) = 0, \quad i = 1,\ldots,n ,$$

and hence a surjection

$$j\colon (A/\prod_{q\in E} q^{\ell})^n \to T$$

where $E = \bigcup E_{ij}$, and, if $g_1,\dots,g_n$ are the obvious generators of $(A/\Pi\, q^{\ell})^n$, $g_i \to t_i$. Using [19, Prop. 4.23], and increasing $\ell$ if necessary, there is a $\lambda$-symmetric bilinear form $\tau\colon (A/\Pi\, q^{\ell})^n \times (A/\Pi\, q^{\ell})^n \to E_p$ defined by $\tau(g_i,g_j) = e_{ij}$. Now taking $H = (A/\Pi\, q^{\ell})^n$ makes (3.6) commute; but the choice of $H$ must be modified as follows to insure $H \in M_p$.

If $I \subseteq A$ is any ideal and $Z(I) := \{\, p \mid p$ is prime, $p \supseteq I\}$, then $Z(\Pi\, q^{\ell}) = Z(\cap q)$. Since $Z(I)$ and $\mathrm{Ass}(I)$ have the same minimal elements for any $I$, and $\mathrm{Ass}(\cap\{q \mid q \in E\}) = E$ (by definition), $\mathrm{ht}(\Pi q^{\ell}) = p$. Since $A$ is regular, hence Gorenstein, there is an A-sequence $\{a_1,\dots,a_p\} \subseteq \Pi\, q^{\ell}$ [10, Thm. 136], hence a surjection

$$(A/(a_1,\dots,a_p))^n \twoheadrightarrow (A/\Pi\, q^{\ell})^n. \tag{3.7}$$

The Koszul complex and [2, §1] shows $A/(a_1,\dots,a_p) \in M_p$. Compose (3.7) with $\tau$ and $j$ constructed above, and set $K = \ker((A/(a_1,\dots,a_p))^n \to T)$. By [10,4-1], h.d. $K = p$; and $\mathrm{Ass}(K) \subseteq \mathrm{Ass}(H)$, which consists of height $p$ primes. Thus $K \in M_p$ and the proof is complete.

(3.8) Given a short exact sequence $K \overset{\alpha}{\rightarrowtail} H \twoheadrightarrow T$ with $K,H \in M_p$, $T \in M_{p+1}$, we obtain a short exact sequence

$$H^\wedge = \mathrm{Hom}(H,V_p) \overset{\alpha^\wedge}{\rightarrowtail} K^\wedge \twoheadrightarrow \mathrm{Ext}^1(T,V_p).$$

From the exact sequence $V_p \overset{i}{\rightarrowtail} E_p \overset{d_p}{\twoheadrightarrow} V_{p+1}$ and the fact that $E_p$ is injective, there is an isomorphism $\mathrm{Ext}^1(T,V_p) \underset{\overline{1}}{\cong} \mathrm{Hom}(T,V_{p+1})$. We need to make this isomorphism explicit, viewing $\mathrm{Ext}^1(T,V_p)$ as $\mathrm{cok}(\alpha^\wedge)$. Given $f\colon K \to V_p$, there is a unique $\ell\colon H \to E_p$ such that $\ell\alpha = if$ (see (3.4)). Then $d_p\, \ell\alpha = d_p\, i\alpha = 0$, so there is a unique $e\colon T \to V_{p+1}$ such that

$$ej = d_p\, \ell\ . \tag{3.9}$$

(3.10) <u>Lemma</u> With the notation of (3.5) and the identification of (3.8), there is a commutative diagram

(3.11)
$$\begin{array}{ccc} K & \xrightarrow{\lambda\gamma} & H^\wedge \\ \downarrow\alpha & & \downarrow\alpha^\wedge \\ H & \xrightarrow{\gamma^\wedge} & K^\wedge \\ \downarrow j & & \downarrow \\ T & \xrightarrow{Ad\mu} & T^\sim := \mathrm{Hom}(T,V_{p+1}) \ . \end{array}$$

Proof: It is only necessary to prove the bottom square commutes. Given $h \in H$, $\gamma^\wedge(h)$ is the homomorphism $f\colon K \to V_p$ defined by $f(k) = [\gamma(k)](h)$, for all $k \in K$. Then $if(k) = i[\gamma(k)](h) \overset{(3.3)}{=\!=} [((Ad\tau)\alpha)(k)](h)$. Thus the map $\ell\colon H \to E_p$ produced in (3.8) is $(Ad\tau)(h)\colon H \to E_p$. Since for any $h \in H$, and $t' := j(h')$, $\mu(t,t') = d_p\,\tau(h,h')$ (by assumption), the equation (3.9) completes the proof.

Returning to the proof of Theorem 5 we have a (+1)-formation $(K,H,\Delta)$ in $M_1$; setting $\Delta = (\alpha,\gamma)$, $\alpha\colon K \to H$, $\gamma\colon K \to H^\wedge$, Lemmas (3.2)(a) and (3.5)(a) produce a (-1)-symmetric bilinear form $\mu\colon T \times T \to V_2$. The next two lemmas show it is a nonsingular lagrangian.

(3.12) Lemma a) Given a $(-\lambda)$-formation $(K,H,\Delta)$ in $M_p$, let $\mu\colon T \times T \to V_{p+1}$ be the $\lambda$-symmetric bilinear form constructed from (3.2)(a) and (3.5)(a), where $\Delta = (\alpha,\gamma)\colon K \to H + H^\wedge$. Then $\mu$ is non-singular.

b) Conversely, given $\mu\colon T \times T \to V_{p+1}$, $\lambda$-symmetric, bilinear and nonsingular, the triple $(K,H,\Delta)$, $\Delta = (\alpha,\gamma)$, produced by (3.2)(b) and (3.5)(b) is a $(-\lambda)$-formation in $M_p$.

Proof: Putting (3.2), (3.5) and (3.10) together, it suffices to show that in the commutative diagram

(3.13)
$$\begin{array}{ccc} K & \xrightarrow{\lambda\gamma} & H^\wedge \\ \downarrow\alpha & & \downarrow\alpha^\wedge \\ H & \xrightarrow{\gamma^\wedge} & K^\wedge \\ \downarrow j & & \downarrow \\ T & \xrightarrow{Ad\,\mu} & T^\sim = \mathrm{Hom}(T,V_{p+1}) \end{array}$$

$\alpha$ and $\alpha^\wedge$ induce isomorphisms

(3.14) $$\ker(\lambda\gamma) \xrightarrow{\cong} \ker(\gamma^\wedge), \quad \mathrm{cok}(\lambda\gamma) \xrightarrow{\cong} \mathrm{cok}(\gamma^\wedge)$$

if and only if $(K,H,\Delta)$ is a $(-\lambda)$-formation in $M_p$; the latter holds if and only if $\mathrm{im}(\Delta)$ is a sublagrangian, since $\gamma^\wedge\alpha = \lambda\alpha^\wedge\gamma$ implies

implies $im(\Delta) \subseteq im(\Delta)^{\perp}$; and $im(\Delta)$ is a sublagrangian if and only if the adjoint of the natural $(-\lambda)$-symmetric hyperbolic form $\phi$ on $H + H^{\wedge}$ induces $k\colon H \oplus H^{\wedge}/im(\Delta) \xrightarrow{\cong} im(\Delta)^{\wedge}$. But from the commutativity of

$$\begin{tikzcd} K \arrow[r, tail, "(\alpha,\gamma)"] & H \oplus H^{\wedge} \arrow[r, two heads] \arrow[d, "\cong"', "Ad = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}"] & H \oplus H^{\wedge}/im(\Delta) \arrow[d, "k"] \\ & H^{\wedge} \oplus H \arrow[r, two heads, "(\alpha^{\wedge},\gamma^{\wedge})"] & K^{\wedge} \end{tikzcd},$$

$k$ is an isomorphism if and only if

$$K \xrightarrow{(-\lambda\gamma,\alpha)} H \oplus H^{\wedge} \xrightarrow{(\alpha^{\wedge},\gamma^{\wedge})} K$$

is short exact. This in turn is equivalent to (3.14).

In case $A$ is 2-dimensional, each object $(T,\mu) \in Q^{\lambda}(M_2)$ = the category of nonsingular $\lambda$-symmetric bilinear forms in $M_2$, splits orthogonally into the sum $\oplus\,(T_m,\phi_m)$ of its localizations at all maximal ideals $m$ of $A$, and each $T_m$ is of finite length over $A_m$. We next prove a special case of Theorem 2 ($\lambda = -1$, $i = 0$).

(3.15) Lemma If $(T,\mu)$ is a nonsingular $(-1)$-symmetric bilinear form in $M_p$, $p = \dim A$, then each $(T_m,\phi_m)$, and hence $(T,\phi)$, is a lagrangian.

Proof: The proof is by induction on the length $\ell(T_m)$ of $T_m$.

Set $T = T_m$, $\mu = \mu_m$, $A = A_m$, and $E = E(k(m))$. Then if $K \subseteq T$, $K^{\perp\perp} = K$. For it is clear that $K \subseteq K^{\perp\perp}$ and that $\mu$ induces $K^{\perp} \xrightarrow{\cong} \mathrm{Hom}(T/K,E)$; by [3, (2.1)(iv)], $\ell(\mathrm{Hom}(T/K,E)) = \ell(T/K)$, and hence $\ell(K^{\perp}) + \ell(K) = \ell(T)$. Similarly, $\ell(K^{\perp\perp}) + \ell(K^{\perp}) = \ell(T)$, so $\ell(K^{\perp\perp}) = \ell(K)$. This together with $K \subseteq K^{\perp\perp}$ implies $K = K^{\perp\perp}$.

Now if $t \in T$, the submodule $K := At \subseteq T$ satisfies $K \subseteq K^{\perp}$ because $\mu(t,t) = -\mu(t,t)$ and $\frac{1}{2} \in k(m) \subseteq E$. The naturally induced $(-1)$-symmetric form on $K^{\perp}/K$, $(K^{\perp}/K,\mu')$ has $\ell(K^{\perp}/K) < \ell(T)$. Moreover, both it and the naturally induced $K \times (T/K^{\perp}) \to E$ are nonsingular: this is immediate from $K = K^{\perp\perp}$. By induction and the argument of [15, 3.5] the proof is complete.

Thus, the $(-1)$-symmetric form $(T,\mu)$ produced in (3.12) is a lagrangian. We next use (3.5)(b) to find a special resolution $M \overset{\nu}{\rightarrowtail} L \twoheadrightarrow T$, $L,M \in M_1$ and $\sigma\colon L \times L \to E_1$ covering $\mu$ as in (3.6).

(3.16) Lemma Let $(T,\mu)$ be a $\lambda$-symmetric lagrangian in $M_{p+1}$.

Then there is a resolution $I \oplus J^{\wedge} \overset{\nu}{\rightarrowtail} J \oplus I^{\wedge} \twoheadrightarrow T$ with $I, J \in M_p$ and $\nu = \begin{pmatrix} \varepsilon & \delta \\ 0 & \varepsilon^{\wedge} \end{pmatrix}$, $\varepsilon\colon I \to J$, $\delta\colon J^{\wedge} \to J$; and a $\lambda$-symmetric form $\sigma\colon (J \oplus I^{\wedge}) \times (J \oplus I^{\wedge}) \to E_p$ covering $\mu$ as in (3.6), with $(\mathrm{Ad}\sigma) \cdot \nu\colon I \oplus J^{\wedge} \to \mathrm{Hom}(J \oplus I^{\wedge}, E_p)$ equal to

$$I \oplus J^{\wedge} \overset{\kappa}{\to} J^{\wedge} \oplus I = (J \oplus I^{\wedge})^{\wedge} \xrightarrow{i_*} \mathrm{Hom}(J \oplus I^{\wedge}, E_p)$$

where $\kappa = \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix}$ and $i_*$ is induced by $i\colon V_p \to E_p$. (Thus, $(I \oplus J^{\wedge}, J \oplus I^{\wedge}, (\nu, \kappa))$ is a $(-\lambda)$-formation in $M_p$, by (3.12)(b)).

Proof: Let $S = S^{\perp} \subseteq T$ be a sublagrangian. The adjoint of $\mu$ induces an isomorphism $T/S \overset{\cong}{\to} S^{\wedge}$, hence an exact sequence

$$S \rightarrowtail T \twoheadrightarrow S^{\sim} := \mathrm{Hom}(S, V_{p+1}) \ .$$

Choose generators $s_1', \ldots, s_m'$ for $S^{\sim}$ and lift them back to elements $s_1, \ldots, s_m \in T$. Carrying out the procedure of the proof of (3.5)(b), find an A-sequence $a_1, \ldots, a_p \in A$, a map $(A/(a_1, \ldots, a_p))^m \overset{k}{\twoheadrightarrow} T$ and a $\lambda$-symmetric form $\tau\colon (A/(a_1, \ldots, a_p))^m \times (A/(a_1, \ldots, a_p))^m \to E_p$ such that $d_p\,\tau(b, b') = \mu(kb, kb')$ for all $b, b' \in (A/(a_1, \ldots, a_p))^m$. Let $I^{\wedge} = (A/(a_1, \ldots, a_p))^m$ and $J^{\wedge} \xrightarrow{\alpha^{\wedge}} I^{\wedge}$ be the kernel of the composition $I^{\wedge} \overset{k}{\to} T \twoheadrightarrow S^{\sim}$. (We are using the fact (§1) that $M^{\wedge\wedge} = M$ for all $M \in M_p$ and $\beta^{\wedge\wedge} = \beta$ for all maps in $M_p$.) Thus from $J^{\wedge} \xrightarrow{\alpha^{\wedge}} I^{\wedge} \twoheadrightarrow S^{\sim}$ we obtain $I \overset{\alpha}{\to} J \twoheadrightarrow S$ as in (3.8), since $S \cong S^{\sim\sim}$ canonically for all $S \in M_{p+1}$. In the usual way we obtain $\nu$ as required.

To prove $\sigma$ can be chosen as claimed, let $r$ be the unique map making

$$\begin{tikzcd}
I^{\wedge} = \mathrm{Hom}(I, V_p) \arrow[rr, "r"] \arrow[dr, "i_*"'] & & \mathrm{Hom}(J, E_p) \arrow[dl, "\cong (3.4)"] \\
& \mathrm{Hom}(I, E_p) &
\end{tikzcd}$$

commute, and let $\rho\colon I^{\wedge} \times J \to E_p$ satisfy $\mathrm{Ad}\,\rho = r$; define $\rho^{\wedge}\colon J \times I^{\wedge} \to E_p$ by $\rho^{\wedge}(a, b) = \lambda\ \rho(b, a)$, $b \in I^{\wedge}$, $a \in J$. We claim $\sigma = \begin{pmatrix} 0 & \rho \\ \rho^{\wedge} & \tau \end{pmatrix}$ covers $\mu$ in the sense of (3.6). It suffices to show

$$(3.17) \qquad \begin{tikzcd}
I^{\wedge} \times J \arrow[r, "\rho"] \arrow[d] & E_p \arrow[d, two heads, "dp"] \\
S^{\sim} \times S \arrow[r] & V_{p+1}
\end{tikzcd} \qquad \text{commutes,}$$

where $S^{\sim} \times S \to V_{p+1}$ is the natural pairing. Let $f \in I^{\wedge}$, $a \in J$. If $e: S \to V_{p+1}$ is the image of $f$ by the construction of (3.8) and $\ell: J \to E_p$ satisfies (3.9), then $e(ja) = d_p \ell(a)$; but by construction of $\rho$, $\ell(a) = \rho(f,a)$, so (3.17) commutes. Evidently, $(\mathrm{Ad}\ \rho) \cdot \varepsilon^{\wedge} := r\varepsilon^{\wedge}: J^{\wedge} \to \mathrm{Hom}(J,E_p)$ is $i_*$, so the proof of (3.16) is complete.

To summarize, we have shown that the $(+1)$-formations $(K,H,\Delta)$ of (3.1) and $(I \oplus J^{\wedge}, J \oplus I^{\wedge}, (\nu,\kappa))$ of (3.16) induce the $(-1)$-symmetric form $(T,\mu)$, in the sense of (3.5); moreover, $\kappa$ is an isomorphism.

(3.18) Lemma Any $(-\lambda)$-formation $\theta = (F,F^{\wedge},(\nu,\kappa))$ in $M_p$, for which $\kappa$ is an isomorphism, represents zero in $W_1^{-\lambda}(M_p)$ $(F^{\wedge} := \mathrm{Hom}(F,V_p))$.

Proof: Add to $\theta$ the formation $(F^{\wedge},F,(0,1))$ where $0: F^{\wedge} \to F$ is the zero map and $1: F^{\wedge} \to F^{\wedge}$ is the identity; denote the new formation $\theta'$, and note $[\theta] = [\theta']$ in $W_1^{-\lambda}(M_p)$ by [15,1.34(ii)]. Now $\theta'$ has the form $(K',H',\Delta')$ where $H' = H_1 \oplus H_1^{\wedge}$. Letting $w^{\lambda}: H' \times H' \to V_1$ denote the standard $\lambda$-symmetric hyperbolic form, the identity of [15,4.9] shows we may assume that if $\Delta' = (\alpha',\gamma')$, then $\alpha': K' \to H'$ is an isomorphism. Such a formation is easily seen to represent zero in $W_1^{-\lambda}(M_p)$.

Since $[K,H,\Delta] \to [N,\phi]$ under the map $W_1^1(M_1) \to W_0^1(M_0) := W_0^1(A) := W(A)$ the following lemma completes the proof of Theorem 5.

(3.19) Lemma. The formations $(K,H,\Delta)$ of (3.1) and $(I \oplus J^{\wedge}, J \oplus I^{\wedge}, (\nu,\kappa))$ of (3.16) represent the same element (hence zero by (3.18)) in $W_1^1(M_1)$.

Proof: Set $\theta = (K,H,\Delta)$ and $\theta_1 = (K_1,H_1,\Delta_1) := (I \oplus J^{\wedge}, J \oplus I^{\wedge}, (\nu,\kappa))$; let $\Delta = (\alpha,\gamma)$, $\Delta_1 = (\alpha_1,\gamma_1)$. We first show we may assume $K = K_1$, $H = H_1$ and $\alpha = \alpha_1$.

Let $H_2$ be the pullback of $H_1 \twoheadrightarrow T \twoheadleftarrow H$, so that there is a commutative diagram of short exact sequences

$$\begin{tikzcd} & K \arrow[r, equal] \arrow[d, tail, "i"] & K \arrow[d, tail, "\alpha"] \\ K_1 \arrow[r, tail, "i_1"] \arrow[d, equal] & H_2 \arrow[r, two heads, "k"] \arrow[d, two heads, "k_1"] & H \arrow[d, two heads, "j"] \\ K_1 \arrow[r, tail, "\alpha"] & H_1 \arrow[r, two heads, "j_1"] & T \end{tikzcd}$$

Setting $\alpha_2 = i_1 \oplus i\colon K_1 \oplus K \to H_2$, it follows that

$$\begin{array}{ccc} K_2 := K_1 \oplus K & \xrightarrow{p_1} & K_1 \\ \downarrow \alpha_2 & & \downarrow \alpha_1 \\ H_2 & \xrightarrow{k_1} & H_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} K_2 := K_1 \oplus K & \xrightarrow{p_1} & K \\ \downarrow \alpha_2 & & \downarrow \alpha \\ H_2 & \xrightarrow{k} & H \end{array}$$

are pullback diagrams. Hence, denoting the short exact sequences

$$E := (K_1 \xrightarrow{i_1} H_2 \xrightarrow{k} H)$$
$$E_1 := (K \xrightarrow{i} H_2 \xrightarrow{k_1} H_1)$$

and using the operation [15, 1.34(i)], we obtain new formations $\sigma_E\theta$ and $\sigma_{E_1}\theta_1$, with $[\sigma_E\theta] = [\theta]$, $[\sigma_{E_1}\theta_1] = [\theta_1]$ in $W_1^1(M_1)$ and $\sigma_E\theta = (K_2,H_2,(\alpha_2,\gamma_2))$, and $\sigma_{E_1}\theta_1 = (K_2,H_2,(\alpha_2,\gamma_2'))$ for some homomorphisms $\gamma_2$, $\gamma_2'\colon K_2 \to \hat{H_2}$.

Finally, we claim that there is a (-1)-symmetric bilinear form $\rho\colon H_2 \times H_2 \to V_1$ such that $\gamma_2 - \gamma_2' = (\mathrm{Ad}\,\rho)\alpha_2$. By [15, 1.34(iii)], this will complete the proof of (3.19). To construct $\rho$, observe first that there is a commutative diagram

$$\begin{array}{ccc} K_2 & \xrightarrow{-\beta} & \hat{H_2} \\ \downarrow \alpha_2 & & \downarrow \hat{\alpha_2} \\ H_2 & \xrightarrow{\beta^\wedge} & \hat{K_2} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\mathrm{Ad}\mu} & T^\sim = \mathrm{Hom}(T,V_2) \end{array}$$

where $\beta = \gamma_2$ or $\gamma_2'$ (See (3.13)). From this is follows that there is a unique $\delta\colon H_2 \to \hat{H_2}$ such that $\hat{\alpha_2}\delta = \hat{\gamma_2} - \gamma_2'^\wedge$; hence $\hat{\alpha_2}\delta\alpha_2 = \gamma_2\alpha_2 - \gamma_2'^\wedge\alpha_2$. Dualizing, $\hat{\alpha_2}\delta^\wedge\alpha_2 = (\hat{\alpha_2}\delta\alpha_2)^\wedge = (\hat{\gamma_2}\alpha_2 - \gamma_2'^\wedge\alpha_2)^\wedge = -\hat{\gamma_2}\alpha_2 - (-\gamma_2'^\wedge\alpha_2) = -\hat{\alpha_2}\delta\alpha_2$. Since $\alpha_2$ and $\hat{\alpha_2}$ are injective, $\delta = -\delta^\wedge$. Define $\rho$ so that $\mathrm{Ad}\,\rho = \delta^\wedge$.

Addendum (February, 1981) By the method of [16,5.11], I have proved exactness in Conjecture A if $\dim A \leq 4$ and that $D_0$ is injective (without dimension restriction), both results assuming $A$ is essentially of finite type over a field of characteristic $\neq 2$.

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