THE EXACT SEQUENCE OF A LOCALIZATION FOR WITT GROUPS

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§0 Introduction

It is often useful to compare the value of a functor defined on a "global" ring to its values on the "local" components of the ring. For example, the Hasse-Minkowski theorem [L,6.3.1] states that a quadratic form over a global field F is isotropic if and only if it is isotropic over its completions at the "places" of F. A straightforward consequence of this deep theorem is a local-global comparison of the above type, where the functor W of a ring A, $W(A)$, is a stable Grothendieck group on isometry classes of quadratic forms over A:

(0.1): There is an injection $W(F) \to \prod_{\mathfrak{p}} W(F_{\mathfrak{p}})$ where F is a global field and $F_{\mathfrak{p}}$ is its completion at the place (or prime, possibly infinite) $\mathfrak{p}$.

There are two observations to be made here. First, if $\mathfrak{p}$ is a finite prime, F is an algebraic number field and A is the ring of integers in F, then $W(F_{\mathfrak{p}}) \cong W(A/\mathfrak{p})$. $A/\mathfrak{p}$ is a finite field, so $W(A/\mathfrak{p})$ is easy to compute. Second, even though the components $W(F_{\mathfrak{p}}) \cong W(A/\mathfrak{p})$ of (0.1) give a classical, tractable list of invariants for elements of $W(F)$, (0.1) does not reveal which elements of $\prod_{\mathfrak{p}} W(F_{\mathfrak{p}})$ arise from global forms. That is, (0.1) does

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not compute $W(F)$, even in the simple case $F = \mathbb{Q}$. It turns out that if $F = \mathbb{Q}$, all finite collections of local invariants are realized, so that $W(\mathbb{Q}) \cong \coprod_p W(\mathbb{Z}/p\mathbb{Z}) + W(\mathbb{R})$. Since $W(\mathbb{R}) \cong \mathbb{Z} \cong W(\mathbb{Z})$ (via the signature), this fact may be expressed through a split exact sequence [MH, IV.2.1]

$$(0.2) \qquad W(\mathbb{Z}) \rightarrowtail W(\mathbb{Q}) \twoheadrightarrow \coprod_p W(\mathbb{Z}/p\mathbb{Z})$$

which amounts to the desired computation. It turns out that (0.2) generalizes to (see [MH, IV.3.4])

$$(0.3) \qquad W(A) \rightarrowtail W(F) \longrightarrow \coprod_{\mathfrak{p} \text{ finite}} W(A/\mathfrak{p}) \twoheadrightarrow \mathcal{C}/\mathcal{C}^2$$

where $\mathcal{C}$ is the ideal class group of A. Taking into account Quillen's localization sequence for algebraic K_i-groups (for low dimensions [S, 8.4]) and the fact that there exists a comparable sequence of Witt groups W_i, it is reasonable to expect that (0.3) extends to a "long exact" sequence. Theorem (2.1) of this paper exhibits the localization sequence for Witt groups.

The above remarks are hindsight since (2.1) was first proved (in the special case where A is the integral group ring of a finite group) using surgery theory; the algebraic constructions in this paper are thus geometrically "realizable". Further, in the applications of (2.1) (or (0.3)) one calculates $W_i^\lambda(A)$ (or $W(A)$) from knowledge of the rest of the exact sequence. Max Karoubi [K] has independently produced a part of this exact sequence (2.1), but some of his arguments required conditions on the ring A in (2.1) (e.g., $1/2 \in A$ or A Dedekind), which made the applications I had in mind inaccessible. Theorem (2.1) contains strong (but removeable) restrictions on the localized ring B, but is general enough to make calculations of Witt groups and surgery obstruction groups. These could not appear here due to space limitations and will be in another paper (see [P2]). Andrew Ranicki has also announced a version of (2.1) in [R'].

I have followed the definitions and notational conventions of Bass' foundational paper [B] as closely as possible. In an effort to make the paper self-contained, §1 integrates into a summary of background material from [B] the basic definitions and facts concerning quadratic forms in the category of torsion modules with short free resolution. A statement of the main theorem (2.1) and a summary of the rest of the paper is the content of §2.

Proposition (1.17) and the idea of an "integral lattice" ((1.10),(3.1)) are the basic tools in the proof of (2.1). I am aware that (2.1) can be proved much more generally, but it seemed that in making the most general formulation the paper would be too long and technical and that the prominence of these two ideas would be obscured.

Notational Conventions

$+$	means	direct sum
$\rightarrowtail$	"	1 - 1 homomorphism
$\twoheadrightarrow$	"	onto homomorphism
[*]	"	bibliographical reference to *
(*)	"	reference to (*) in this paper.

If M_i and N_j are modules, and $f_{ij}: M_i \to N_j$ are module homomorphisms, $1 \leq i \leq n$, $1 \leq j \leq m$, then the matrix (f_{ji}) denotes the obvious homomorphism $M_1 + M_2 + \ldots + M_n \to N_1 + N_2 + \ldots + N_m$.

§1 The category $\mathfrak{D}^1_F$; basic definitions and results for quadratic forms

(1.1) The localization. Let A be a ring-with-involution containing 1, where the involution is denoted "$\overline{}$": $\overline{a + b} = \bar{a} + \bar{b}$, $\overline{ab} = \bar{b}\bar{a}$, $\bar{1} = 1$, for all $a,b \in A$. All A-modules will be right A-modules unless otherwise specified. Let $\Sigma \subset A$ be a central multiplicative subset containing 1 such that Σ contains no zero-divisors ($as = 0 \Rightarrow a = 0$, $a \in A$, $s \in \Sigma$), $\bar{s} = s$ for all $s \in \Sigma$, and the ring of quotients $B = A_\Sigma$ is an Artin, semi-simple, A-injective ring containing 1/2. Thus the localization map ℓ: $A \to B$ is an injection of rings-with-involution and $- \otimes_A B$ is exact. All A-modules considered in this paper, with the exception of B and B/A, will be finitely generated.

Let V be a two-sided A-module (in practice, V = A,B, or B/A) and M an A-module. If $\underline{M^*_V}$ denotes $\mathrm{Hom}_A(M,V)$ then M^*_V is in a natural way a left A-module, but is always taken to be a right A-module by setting $(fa)(m) = \bar{a}f(m)$ for $a \in A$, $f \in M^*_V$, and $m \in M$. If h: $M \to N$ is an A-module homomorphism, there is a natural homomorphism h^*_V: $N^*_V \to M^*_V$. If V = A, then set $M^*_V = \bar{M}$ and $h^*_V = \bar{h}$; we make the same conventions for B-modules M and set $M^*_B = \bar{M}$, $h^*_B = \bar{h}$. A homomorphism μ: $F \to G$ between free A-modules (B-modules) with chosen bases will be identified (when convenient) with its matrix, also denoted μ; conversely, any matrix μ will be identified with a homomorphism of free modules. If we write $F = A^n$ ($F = B^n$) we mean F has a chosen basis. The induced homomorphism $\bar{\mu}$: $\bar{G} \to \bar{F}$ has matrix $\bar{\mu}$ [B,I.2.7], the conjugate transpose of the matrix μ. If V = B/A, set $M^*_V = M^\wedge$ and $h^*_V = h^\wedge$, where M is an A-module and h: $M \to N$ an A-module homomorphism.

(1.2) Σ-torsion. An element m in the A-module M is said to be Σ-torsion or a torsion element if there is $s \in \Sigma$ such that $ms = 0$. Since Σ is central, the subset $tM \subseteq M$ consisting of torsion elements is an A-submodule, the torsion submodule. M is called a torsion module or torsion if $tM = M$. A standard argument [c, 0.6.1] shows that the sequence

$$(1.3) \qquad tM \rightarrowtail M \longrightarrow M \otimes_A B$$

is exact. Hence M is a torsion module if and only if $M \otimes_A B = 0$.

Let $\mathfrak{D}_F^1$ be the category of torsion modules having short free resolution; hence $S \in \mathfrak{D}_F^1$ if $S \otimes B = 0$ and there is an exact sequence $A^n \rightarrowtail A^n \twoheadrightarrow S$. Here is a fundamental proposition.

(1.4) Proposition: If M is a torsion module with short free resolution $F \overset{\mu}{\rightarrowtail} G \overset{j}{\twoheadrightarrow} M$, there is a natural isomorphism

$$k\colon \mathrm{Ext}_A^1(M,A) \cong M^\wedge$$

and hence an exact sequence $\bar{G} \overset{\bar{\mu}}{\rightarrowtail} \bar{F} \twoheadrightarrow M^\wedge$.

Proof: From the short free resolution, we obtain the exact sequence

$$\bar{G} \overset{\bar{\mu}}{\rightarrowtail} \bar{F} \overset{j'}{\twoheadrightarrow} \mathrm{Ext}_A^1(M,A)$$

since G is A-free and $tM = M$ implies $\bar{M} = 0$. Identifying $\mathrm{Ext}_A^1(M,A)$ with $\mathrm{cok}(\bar{\mu})$, define k as follows. Let $f \in \bar{F}$. The bottom horizontal map can be defined to make the following diagram commute because B is A-injective

$$\begin{array}{ccc} F & \xrightarrow{f} & A \\ \downarrow{\scriptstyle \mu} & & \downarrow \\ G & \xrightarrow{f(\mu^{-1})} & B \end{array} .$$

The map $f(\mu^{-1})$ is unique because $\mu \otimes B$ is an isomorphism. If $m \in M$ and $x \in \mathrm{Ext}^1_A(M,A) \cong \mathrm{cok}(\bar{\mu})$ define k by the formula

$$(1.5) \qquad k(x)(m) = r \cdot f(\mu^{-1})(j^{-1}(m))$$

where $r: B \to B/A$ is the quotient map, $j'(f) = x$, and $j^{-1}(m) \in G$ is any element for which $j(j^{-1}(m)) = m$. Thus,

$$(1.6) \qquad \tilde{j}(f)(m) = r \cdot f(\mu^{-1})(j^{-1}(m)).$$

Details are left to the reader.

The following is well-known:

(1.7) <u>Proposition</u>: Let $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} S)$ and $(R') = (A^m \overset{\alpha'}{\rightarrowtail} A^m \overset{j'}{\twoheadrightarrow} S)$ be short free resolutions of $S \in \mathfrak{D}^1_F$. Then there are non-negative integers k, ℓ with $n + k = m + \ell$ and automorphisms μ, μ': $A^{m+\ell} \to A^{m+\ell}$ such that $\mu(\alpha + I_k) = (\alpha' + I_\ell)\mu'$ and $(j,0_k) = (j',0_\ell)\mu$, where $(j,0_k)$: $A^{n+k} \to S$ is j on the first n factors and zero on the rest; $(j',0_\ell)$ is defined analogously.

(1.8) <u>Examples</u>: (a) Let π be a finite group, $A = \mathbb{Z}[\pi]$, $B = \mathbb{Q}[\pi]$, $\overline{\Sigma n_g g} = \Sigma n_g g^{-1}$, $n_g \in \mathbb{Z}$ or $\mathbb{Q}$, $g \in \pi$. (b) Let B be a number field with $\mathbb{Z}_2 \subseteq \mathrm{Gal}(B/\mathbb{Q})$ furnishing the involution (or B could have trivial involution), and A = the ring of integers. More generally, B could be a semi-simple algebra-with-involution, $1/2 \in B$, and A any order taken to itself by the involution. (c) Let $A = \mathbb{Z}[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$, "$\overline{}$" induced by $\bar{t}_i = t_i^{-1}$, $B = \mathbb{Q}(t_1,\ldots,t_n)$. (d) Making (c) more exotic, let Φ be a Bieberbach group. This means there is an extension $\mathbb{Z}^n \rightarrowtail \Phi \twoheadrightarrow \pi$, where $\mathbb{Z}^n$ is free abelian, normal and maximal abelian in Φ, and π is finite. Φ is "classified" by $\theta \in H^2(\pi;\mathbb{Z}^n)$ where $\mathbb{Z}^n$ has a π-action arising from the extension. From θ construct, by extension of coefficients, $\theta' \in H^2(\pi; \mathbb{Q}(t_1,\ldots,t_n)^{\cdot})$ determining a division ring B such that $\mathbb{Z}[\phi]$ imbeds in B as a localization.

(1.9) <u>The setting for quadratic forms</u>: We will recall from [B] the context and definitions of unitary K-theory. Many of our definitions will be stated only for A-modules, but when they make sense, they are understood to apply to B-modules as well. Let (A,λ,Λ) be a unitary ring with involution ([B,I.4.1]) where we require special values for λ and Λ. Thus, A is a ring-with-involution (1.1), $\lambda = \pm 1$, and $\Lambda = \underline{S_\lambda(A)} = \{a + \lambda\bar{a} \mid a \in A\}$. If $\alpha \in M_n(A)$, the set of $(n\times n)$-matrices over A, $\bar{\alpha}$ denotes its conjugate transpose, thus making $M_n(A)$ a ring-with-involution. Set $\underline{S_\lambda(A^n)} = S_\lambda(M_n(A)) = \{\alpha \in M_n(A) \mid \alpha = \lambda\bar{\alpha},\ \alpha_{ii} \in S_\lambda(A)\}$. If $1/2 \in A$, $S_\lambda(A) = \{a \in A \mid a = \lambda\bar{a}\}$.

(1.10) <u>Forms</u>: If M,N are A-modules and $V = A$ or $V = B/A$, a function $g\colon N \times M \to V$ is a <u>sesquilinear form</u> if it is biadditive and satisfies $g(na,mb) = \bar{a}g(n,m)b$, for all $a,b \in A$, $n \in N$, $m \in M$. If $L \subseteq N$ we define its <u>orthogonal complement</u>, $\underline{L^\perp}$ to be $\{m \in M \mid g(L,m) = 0\}$. The <u>natural form</u> $\langle\ ,\ \rangle_M\colon M_V^* \times M \to V$, defined by $\langle f,m\rangle_M = f(m)$, where $f \in M_V^*$, $m \in M$, is a sesquilinear form. If $h\colon M \to N$ is a homomorphism, $\langle h_V^* f,m\rangle_M = \langle f,hm\rangle_N$. ([B,I.2.1].) A sesquilinear form $g\colon N \times M \to V$ is <u>nonsingular</u> if the <u>adjoints</u>

$$(1.11)\qquad \begin{array}{ll} {}_g d\colon N \longrightarrow M_V^*, & \langle {}_g d(n),m\rangle = g(n,m) \\ d_g\colon M \longrightarrow N_V^*, & \langle d_g(m),n\rangle = \overline{g(n,m)} \end{array}$$

are isomorphisms. If $N = M$ is A-free with basis $\{e_1,\ldots,e_n\}$, the map $d_g: A^n \to (A^n)^*_V$ has matrix $(g(e_i,e_j))$ ([B,I.2.7]). We sometimes use the matrix $(g(e_i,e_j))$ to denote d_g; on the other hand $_g d$ is more convenient for some purposes since no "$\overline{\quad}$" appears in its definition. From the natural form we obtain the adjoint map $d_M = d_{\langle\ ,\ \rangle_M}: M \to (M^*_V)^*_V$. M is called <u>V-reflexive</u> if d_M is an isomorphism. If M is A-projective (B-projective) then it is A-reflexive (B-reflexive); if $M \in \mathfrak{N}^1_F$, then by (1.4) it is B/A-reflexive. If M and N are V-reflexive then [B,I.2.4] we need only verify one of the conditions (1.11) for nonsingularity.

An A-submodule $L \subset B^n$ is a <u>lattice</u> if the inclusion induces $L \otimes B \cong B^n$. If $g: B^n \times B^n \to B$ is sesquilinear and $L \subset B^n$ is a lattice, then L inherits a sesquilinear form $g_L: L \times L \to B$. The <u>dual lattice</u> <u>L</u>$' = \{x \in B^n \mid g(x,L) \subseteq A\}$. If g is nonsingular, then the induced sesquilinear form $\ell: L' \times L \to A$ is nonsingular and there is a natural identification of L' with $\bar{L}$. If L is A-free with basis $\{e_1,\ldots,e_n\}$ then there is a "dual basis" $\{e^*_1,\ldots,e^*_n\}$ such that $\ell(e^*_i,e_j) = \delta_{ij}$. If L has a basis, it will be assumed L' has the dual basis.

(1.12) <u>λ-Forms</u>: If a sesquilinear form $g: M \times M \to V$ satisfies $g(m,m') = \lambda\overline{g(m',m)}$, for all $m,m' \in M$, it is called <u>λ-hermitian</u>. It is <u>even λ-hermitian</u> if, in addition, $g(m,m) \in S_\lambda(A)$, for all $m \in M$. If M is free and $V = A$, g is even if and only if $d_g \in S_\lambda(A^n)$. Now let $r: B \to B/A$ and $r'_\lambda: B \to B/S_\lambda(A)$ be the projections.

(1.12a) If $V = A$, M is A-free and $q: M \to A/S_{-\lambda}(A)$ is a function, the triple (M,g,q) is a <u>λ-form</u> if $g: M \times M \to A$ is λ-hermitian and for all $m,m' \in M$, $a \in A$,

(i) $q(ma) = \bar{a}q(m)a$

(ii) $q(m+m') - q(m) - q(m') = r'_{-\lambda}g(m,m')$

(iii) $q(m) + \lambda\overline{q(m')} = r'_{-\lambda}g(m,m)$.

$K \subseteq M$ is <u>totally isotropic</u> if $g \mid K \times K \equiv 0 \equiv q \mid K$. <u>$F_0^\lambda(A)$</u> denotes the set of isometry classes of nonsingular λ-forms (M,g,q) where M is free of <u>even rank</u>; $F_0^\lambda(A)$ is an abelian semigroup under <u>orthogonal sum</u>, denoted "$\perp$". We define λ-forms (B^n,g,q) in a similar way.

<u>Remark</u>: Given λ-hermitian g: $M \times M \to A$, M projective, the condition "$g(m,m) \in S_\lambda(A)$ for all $m \in M$" guarantees the existence of a function q with the properties above [B,I.3.4]. In our definition we require a specific choice of q. The reader should compare (1.12(a)) with the setting of [B,I,4.4]. If $1/2 \in A$, g determines q.

(1.12b): If $V = B/A$, $M \in \mathfrak{D}_F^1$, φ: $M \times M \to B/A$ is λ-hermitian, and ψ: $M \to B/S_\lambda(A)$ is a function, the triple (M,φ,ψ) is called a <u>λ-form</u> if for all $m,m' \in M$, $a \in A$

(i) $\psi(ma) = \bar{a}\psi(m)a$

(ii) $\psi(m+m') - \psi(m) - \psi(m') = \varphi(m,m') + \varphi(m',m)$

(the right-hand side is well-defined in $B/S_\lambda(A)$)

(iii) $r\psi(m) = \varphi(m,m)$

$K \subseteq M$ is called <u>totally isotropic</u> if $\varphi \mid K \times K \equiv 0 \equiv \psi \mid K$. <u>$F_0^\lambda(B/A)$</u> denotes the set of isometry classes of nonsingular λ-forms (M,φ,ψ), $M \in \mathfrak{D}_F^1$; $F_0^\lambda(B/A)$ is an abelian semigroup under the operation of <u>orthogonal sum</u>, denoted "$\perp$".

<u>Remark</u>: (<u>i</u>) We have defined "λ-form" in two quite different ways; the context makes clear which of (1.12a) or (1.12b) we mean and

in any case we always use Latin characters f,g,h,q,p, etc. in (M,g,q) for (1.12a), and Greek characters in [1.12b). (<u>ii</u>) We are unable to show that an intrinsic condition like "$\varphi(m,m) \in S_\lambda(B/A)$" allows us to conclude the existence of ψ as an analogous condition does in the category of projectives (remark after (1.12a)). This is what keeps us from a neat formalism for λ-forms over $\mathfrak{D}_F^1$ analogous to that in [B,I.4.4] for projectives; (1.17) and (1.18) might be thought of as a substitute.

(1.13) <u>$W_0^\lambda(A)$, $W_0^\lambda(B)$, and $W_0^\lambda(B/A)$</u>: (<u>a</u>) Let P be A-free. The λ-form $(P + \bar{P}, g_h, q_h)$ is <u>hyperbolic</u>, and is denoted <u>$\mathcal{H}(P)$</u> if P and $\bar{P}$ are totally isotopic and $g_h \mid \bar{P} \times P$ is the natural form. The quotient of the Grothendieck group on $F_0^\lambda(A)$ by the subgroup generated by $\mathcal{H}(A^n)$, $n \in \mathbb{Z}^+$, is denoted <u>$W_0^\lambda(A)$</u>, the <u>Witt group</u>. We define $W_0^\lambda(B)$ similarly.

(<u>b</u>) Let $S \in \mathfrak{D}_F^1$. The λ-form <u>$\mathcal{H}(S)$</u> $= (S + S^\wedge, \varphi_h, \psi_h)$ is <u>hyperbolic</u> if S and $S^\wedge$ are totally isotropic and $\varphi_h \mid S^\wedge \times S$ is the natural form. If $T \in \mathfrak{D}_F^1$, (T,φ,ψ) is a λ-form, and $K \subset T$, $K \in \mathfrak{D}_F^1$, is a totally isotropic submodule, (T,φ,ψ) is a <u>kernel</u> and K a <u>subkernel</u> if the induced sesquilinear form $(T/K) \times K \to B/A$ is nonsingular. Clearly $\mathcal{H}(S)$ is a kernel with subkernels S and $S^\wedge$. <u>$W_0^\lambda(B/A)$</u> is the Grothendieck group on $F_0^\lambda(B/A)$ modulo the subgroup generated by kernels.

(1.14) Before we define the W_1^λ-functors, we make constructions (1.17) and (1.18) fundamental to this paper.

(1.15) <u>Lemma</u> [C2,1.4]: Let (S,φ,ψ) be a λ-form. For each $t \in S$ there is $b \in B$ such that $r'b = \psi(t)$ and $b = \lambda\,\bar{b}$, where

$r'\colon B \to B/S_\lambda(A)$.

<u>Proof</u>: By (1.12b(i)) $\psi(-t) = \psi(t)$ and by (1.12b(ii)) $\psi(t+(-t)) - \psi(t) - \psi(-t) = -(\varphi(t,t) + \lambda\overline{\varphi(t,t)})$ so $2\psi(t) = \varphi(t,t) + \lambda\overline{\varphi(t,t)}$. Hence if $b,b' \in B$ are such that $r'b = \psi(t)$ and $rb' = \varphi(t,t)$, $r\colon B \to B/A$, then $2b = b' + \lambda\bar{b}' + c$, $c \in S_\lambda(A)$. Hence $b = 1/2(b' + \lambda\bar{b}') + 1/2c$ so $b = \lambda\bar{b}$.

(1.16) <u>Corollary</u>: $\psi(t) = \lambda\overline{\psi(t)}$ for each $t \in S$.

With (S,φ,ψ) as above and $A^n \overset{\mu}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} S$ a resolution of S, let $\{e_1,\ldots,e_n\}$ be a basis for A^n and choose (by (1.15)) $\tau_{ik} \in B$, $1 \le i,k \le n$, such that $\varphi(je_i,je_k) = r\tau_{ik}$, $\psi(je_i) = r'\tau_{ii}$ and $\tau_{ik} = \lambda\overline{\tau_{ki}}$. We use $\tau = (\tau_{ik}) \in M_n(B)$ to define in the obvious way a λ-hermitian form, also denoted τ,

$$\tau\colon A^n \times A^n \longrightarrow B.$$

τ is called a <u>covering</u> of (S,φ,ψ) with respect to the resolution $A^n \overset{\mu}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} S$. This gives part (a) of

(1.17) <u>Proposition</u>: Given a λ-form (S,φ,ψ), $S \in \mathfrak{D}_F^1$, and any resolution $A^n \overset{\mu}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} S$, we may find a λ-hermitian form $\tau\colon A^n \times A^n \to B$ such that if $r\colon B \to B/A$ and $r'\colon B \to B/S_\lambda(A)$ are the projections and $m,m' \in A^n$,

(a) $\qquad \varphi(j(m),j(m')) = r\tau(m,m')$

and $\qquad \psi(j(m)) = r'\tau(m,m)$.

(b) If τ also denotes the matrix of $\tau\colon A^n \times A^n \to B$, then $\bar{\mu}\tau \in M_n(A)$, $\bar{\mu}\tau\mu \in S_\lambda(A)$ and the diagram

$$\begin{array}{ccc} A^n & \xrightarrow{\;\bar{\mu}\tau\;} & \bar{A}^n \\ \downarrow j & & \downarrow \tilde{j} \\ T & \xrightarrow[\;d_\varphi\;]{} & T^\wedge \end{array}$$

commutes ((1.6) for $\tilde{j}$).

Proof (of (b)): Consider the sesquilinear form $g\colon A^n \times A^n \to B$ where $g(m,n) = \tau(\mu(m),n)$, $m,n \in A^n$. Since $r\cdot g(m,n) = r\cdot\tau(\mu(m),n) = \varphi(j\mu(m),n)$ and $j\mu = 0$, g takes values in A. We claim $d_g = \bar{\mu}\cdot d_\tau$; this gives $\bar{\mu}\tau \in M_n(A)$ since the matrix of $d_\tau\colon A^n \to (A^n)^*_B$ is (τ_{ij}) by (1.10). Let $m,n \in A^n$: $(\bar{\mu}d_\tau(m))(n) = \langle\bar{\mu}d_\tau(m),n\rangle_{B^n} = \langle d_\tau(m),\mu(n)\rangle_{B^n}$ $= \overline{\tau(\mu(n),m)} = d_g(m)(n)$, which verifies the claim. Similarly, if we consider the λ-hermitian form $f\colon A^n \times A^n \to A$, where $f(m,n) = \tau(\mu(m),\mu(n))$ we find its matrix is $\bar{\mu}\tau\mu$; it is in $S_\lambda(A^n)$ because $r'f(m,m) = r'\tau(\mu(m),\mu(m)) = \psi(j\mu(m)) = 0$. Finally, let $m \in A^n$, $t \in S$, and $j^{-1}(t) \in A^n$ any element of A^n such that $j(j^{-1}(t)) = t$. Then $\{\tilde{j}(\bar{\mu}d_\tau(m))\}(t) = r\langle\bar{\mu}d_\tau(m),\mu^{-1}(j^{-1}(t))\rangle_{B^n} = r\langle d_\tau(m),j^{-1}(t)\rangle_{B^n}$ $= \overline{r\tau(j^{-1}(t),m)} = \overline{\varphi(t,jm)} = d_\varphi(jm)(t)$. This completes the proof.

The following "converse" is left to the reader.

(1.18) Proposition: Let $S \in \mathfrak{D}^1_F$ have short free resolution $A^n \overset{\mu}{\rightarrowtail} A^n \twoheadrightarrow S$ and let $\tau\colon A^n \times A^n \to B$ be a λ-hermitian form such that $\bar{\mu}\tau \in M_n(A)$ and $\bar{\mu}\tau\mu \in S_\lambda(A^n)$. Then the equations of (1.17a) define a λ-form (S,φ,ψ).

(1.19) The unitary group $U^\lambda_{2n}(A)$: The set of isometries of the hyperbolic λ-form $\mathcal{H}(A^n) = (A^n + \bar{A}^n, g_h, q_h)$ (1.13) form a group, $U^\lambda_{2n}(A)$, whose elements σ may be written ([B,II.4.1.2])

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}(A)$$

where

(1.20):

(i) $\alpha,\beta,\gamma,\delta \in M_n(A)$

(ii) $\bar{\delta}\alpha + \lambda\bar{\beta}\gamma = I_n$, the $(n\times n)$ identity matrix

(iii) $\bar{\beta}\delta$, $\bar{\gamma}\alpha \in S_{-\lambda}(A^n)$ ((1.9) for $S_{-\lambda}(A^n)$).

If $\sigma \in U^{\lambda}_{2n}(A)$ as above and $\sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ set

$$\sigma \perp \sigma' = \begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha' & 0 & \beta' \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma' & 0 & \delta' \end{bmatrix} \in U^{\lambda}_{2(m+n)}(A).$$

If $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in U^{\lambda}_{2n}$, we obtain a sequence of <u>stabilization</u> homomorphisms $U^{\lambda}_{2n}(A) \to U^{\lambda}_{2(n+1)}(A)$, $\sigma \mapsto \sigma \perp I_2$. Letting $n \to \infty$ we obtain $U^{\lambda}(A)$, whose abelianization is denoted <u>$KU^{\lambda}_1(A)$</u>.

(1.21): Here are some special elements of $U^{\lambda}_{2n}(A)$.

(a) $w^{\lambda}_1 = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \in U^{\lambda}_2(A)$; $w^{\lambda}_n = w^{\lambda}_{n-1} \perp w^{\lambda}_1$.

(b) The homomorphism $H: GL_n(A) \to U^{\lambda}_{2n}(A)$ sends $\alpha \in GL_n(A)$ to $H(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha}^{-1} \end{pmatrix}$. This stabilizes to $H: GL(A) \to U^{\lambda}(A)$, whose abelianization is a homomorphism $H_1: K_1(A) \to KU^{\lambda}_1(A)$. <u>$W^{\lambda}_1(A)$</u> is the cokernel of H_1.

(c) Let $\rho,\tau \in S_{-\lambda}(A^n)$ and set $X_+(\rho) = \begin{pmatrix} I_n & 0 \\ \rho & I_n \end{pmatrix}$ $X_-(\tau) = \begin{pmatrix} I_n & 0 \\ \tau & I_n \end{pmatrix} \in U^{\lambda}_{2n}(A)$. If $\langle Z\rangle$ denotes the smallest subgroup containing Z, then ([B,II.5.2,II.4.1.3])

(1.22) $KU^{\lambda}_1(A) = U^{\lambda}(A)/\langle\{X_+(\rho),X_-(\tau) \mid \rho,\tau \in S_{-\lambda}(A^n), n \in \mathbb{Z}^+\}\rangle$

(1.23) $W_1^\lambda(A)=U^\lambda(A)/\langle\{X_+(\rho),X_-(\tau),H(\alpha) \mid \rho,\tau\in S_{-\lambda}(A^n),\alpha\in GL_n(A),n\in\mathbb{Z}^+\}\rangle$.

(1.24) The matrices α,γ (1.20) define an injective map (α,γ): $A^n \to A^n + \bar{A}^n$, whose image has a complementary summand equal to $\mathrm{im}((\beta,\delta)\colon \bar{A}^n \to A^n + \bar{A}^n)$. The relation $\bar{\gamma}\alpha \in S_{-\lambda}(A^n)$ in (1.20) is equivalent to the condition that g_h and q_h annihilate $\mathrm{im}(\alpha,\gamma)$. Indeed, let a sesquilinear form $[\ ,\ ]\colon A^n \times A^n \to A$ be defined by

$$[x,y] = g_h(\gamma(x),\alpha(y)), \qquad x,y \in A^n. \tag{1.25}$$

Then ${[\ ,\ ]}^d = \bar{\gamma}\alpha$, so $\bar{\gamma}\alpha \in S_{-\lambda}(A^n)$ if and only if $[\ ,\ ]$ is an even $(-\lambda)$-hermitian form. But $g_h(\mathrm{im}(\alpha,\gamma)) \equiv 0$ if and only if $g_h(\gamma(x),\alpha(y)) + g_h(\alpha(x),\gamma(y)) = 0$ for all x,y; and $q_h(\mathrm{im}(\alpha,\gamma)) \equiv 0$ if and only if $q_h(\alpha(x) + \gamma(x)) = g_h(\alpha(x),\gamma(x)) \in S_{-\lambda}(A)$. The condition $\bar{\beta}\delta \in S_{-\lambda}(A^n)$ may be interpreted analogously.

In a similar way the condition $\bar{\delta}\alpha + \lambda\bar{\beta}\gamma = I_n$ (1.20(ii)) means that $\mathrm{im}(\alpha,\gamma) + \mathrm{im}(\beta,\delta) = A^n + \bar{A}^n$ and that the form induced by g_h on $\bar{A}^n \times A^n$--identified with $\mathrm{im}((\beta,\delta)\colon \bar{A}^n \to A^n + \bar{A}^n) \times \mathrm{im}((\alpha,\gamma)\colon A^n \to A^n + \bar{A}^n)$-- is the natural form (1.10).

(1.26): Using the remarks in (1.25), suppose conversely that a split injection $(\alpha,\gamma)\colon A^n \to A^n + \bar{A}^n$ is given, with totally isotropic image. Then [B,I.3.10] shows we may find $(\beta,\delta)\colon \bar{A}^n \to A^n + \bar{A}^n$ such that $\mathrm{im}(\beta,\delta)$ is totally isotropic, $\mathrm{im}(\alpha,\gamma) + \mathrm{im}(\beta,\delta) = A^n + \bar{A}^n$, and $\mathrm{im}(\beta,\delta) \times \mathrm{im}(\alpha,\gamma)$ has induced on it the natural form. By (1.25), this means

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U_{2n}^\lambda(A).$$

In fact it is easy to show that (α,γ) determines the class of σ in

$W_1^\lambda(A)$. In this setting stabilization of σ, addition of w_1^λ, the right action of $X_+(\rho)$, $X_-(\tau)$ and $H(\epsilon)$ (1.21) on σ translate, respectively, to

(1.27):

(i) $(\alpha,\gamma) \perp I_2 = (\alpha + 1_A, \gamma + 0_A)$.

(ii) $(\alpha,\gamma) \perp w_1^\lambda = (\alpha + 0_A, \gamma + \lambda 1_A)$.

(iii) $(\alpha,\gamma)X_+(\rho) = (\alpha + \gamma\rho, \gamma)$.

(iv) $(\alpha,\gamma)X_-(\tau) = (\alpha, \gamma + \alpha\tau)$.

(v) $(\alpha,\gamma)H(\epsilon) = (\alpha\epsilon, \gamma\bar{\epsilon}^{-1})$.

The operations of (1.27) (i), (iii), (iv), (v) generate an equivalence relation on split injections (α,γ) with totally isotropic image, having a group of equivalence classes equal to $W_1^\lambda(A)$. (See [R].)

(1.28) $\underline{W_1^\lambda(B/A)}$: The purpose of the remarks in (1.24) and (1.26) was to motivate the following definitions. Suppose $H,K \in \mathfrak{D}_F^1$, $\mathcal{H}(H) = (H + H^\wedge, \varphi_h, \psi_h)$ is the hyperbolic λ-form and $\Delta: K \to H + H^\wedge$ is an injection with totally isotropic image. Then if $\Delta = (\xi,\zeta)$, $\xi: K \to H$, $\zeta: K \to H^\wedge$, define a $(-\lambda)$-hermitian form $[\ ,\ ]: K \times K \to B/A$ for $k,k' \in K$ by

$$(1.29) \qquad [k,k'] = \varphi_h(\zeta(k), \xi(k')) \qquad \text{(compare (1.25))}.$$

(1.30) Definition: A λ-formation is a 4-tuple (K,H,Δ,κ) where K and $H \in \mathfrak{D}_F^1$, $\Delta: K \to H + H^\wedge$ is an injection whose image is a subkernel of $\mathcal{H}(H)$ and $(K, [\ ,\ ], \kappa)$ is a $(-\lambda)$-form, where $[\ ,\ ]$ is given by (1.29). Let $F_1^\lambda(B/A)$ denote the set of isomorphism classes of λ-formations (isomorphisms induced by isomorphisms of K and H

preserving Δ and $\varkappa$). $F_1^\lambda(B/A)$ is an abelian semigroup in the obvious way, with zero element the λ-formation where $K = H = 0$, (the "zero formation").

(1.31) <u>Remarks</u>: (<u>a</u>) $[\ ,\ ]^d = \xi^\wedge\zeta$. Indeed, if $k,k' \in K$, then $\{\xi^\wedge\zeta(k)\}(k') = \langle \xi^\wedge\zeta(k),k'\rangle_K = \langle \zeta(k),\xi(k')\rangle_K = \varphi_h(\zeta(k),\xi(k'))$ $= \{[\ ,\ ]^d(k)\}(k')$. (<u>b</u>) The $(-\lambda)$-form $(K,[\ ,\ ],\varkappa)$ corresponds by (a), (1.24), and (1.25) to the condition $\bar{\alpha}_\gamma \in S_{-\lambda}(A^n)$ (1.20). We have required <u>additionally</u> in (1.30) a choice $\varkappa$ of splitting, corresponding to a choice of splitting for $\bar{\alpha}_\gamma$ (or a choice of q-form for $[\ ,\ ]$ of (1.25)--c.f. Remark (1.12a).) Thus $\varkappa: K \to B/S_{-\lambda}(A)$ is extra structure and is comparable to Sharpe's idea of the "split unitary group" ([Sh, §3]).

(1.32) <u>Operations on $F_1^\lambda(B/A)$</u>: Let $\theta = (K,H,\Delta,\varkappa) \in F_1^\lambda(B/A)$ be given and let E denote the short exact sequence of elements of $\mathfrak{D}_F^1$, (E) $= (J \rightarrowtail H_1 \twoheadrightarrow H)$. We define a λ-formation $\sigma_E\theta = (K_1,H_1,\Delta_1,\varkappa_1)$ as follows. Let K_1 be the pullback in the diagram

(1.33)
$$\begin{array}{ccccc} J & \rightarrowtail & K_1 & \overset{j_1}{\dashrightarrow\!\!\!\to} & K \\ \Big\| = & & \Big\downarrow \xi_1 & & \Big\downarrow \xi \\ J & \rightarrowtail & H_1 & \overset{j}{\twoheadrightarrow} & H \end{array} .$$

Define $\zeta_1 = j^\wedge\zeta j_1$, and $\varkappa_1 = \varkappa j_1$. Then one verifies that if $\Delta_1 = (\xi_1,\zeta_1)$, then $\sigma_E\theta = (K_1,H_1,\Delta_1,\varkappa_1)$ is a λ-formation, well-defined up to isomorphism by θ and the isomorphism class of (E). In a similar way, given (E) $= (L \to H_1^\wedge \overset{\ell}{\to} H^\wedge)$ let K_1 be the pullback in

$$\begin{array}{ccccc} L & \rightarrowtail & K_1 & \overset{\ell_1}{\dashrightarrow} & K \\ \downarrow = & & \downarrow \zeta_1 & & \downarrow \zeta \\ L & \rightarrowtail & H_1^{\wedge} & \overset{\ell}{\twoheadrightarrow} & H^{\wedge} \end{array}$$

and set $\xi_1 = \ell^{\wedge}\cdot\xi\cdot\ell_1$, $\varkappa_1 = \varkappa\ell_1$ and $\Delta_1 = (\xi_1,\zeta_1)$. Then ${}_E\sigma\theta = (K_1,H_1,\Delta_1,\varkappa_1)$ is a λ-formation.

(b) If (H,φ,ψ) is a $(-\lambda)$-form, let $\chi_-(H,\varphi,\psi)\theta = (K,H,\Delta',\varkappa')$ where $\Delta' = (\xi,\zeta + d_\varphi\cdot\xi)$ and $\varkappa' = \varkappa + \psi\cdot\xi$. If $(H^{\wedge},\varphi,\psi)$ is a $(-\lambda)$-form, let $\chi_+(H^{\wedge},\varphi,\psi)\theta = (K,H,\Delta',\varkappa')$ where $\Delta' = (\xi + d_\varphi\cdot\zeta,\zeta)$ and $\varkappa' = \varkappa + \psi\cdot\zeta$.

(1.34) <u>Definition</u>: <u>$W_1^\lambda(B/A)$</u> is the semigroup $F_1^\lambda(B/A)$ modulo the equivalence relation, $\approx$, generated by the following four operations (notation as in (1.32 (a)(b)):

(i) $\theta \to \sigma_E\theta$, $(E) = (J \to H_1 \to H)$.

(ii) $\theta \to {}_E\sigma\theta$, $(E) = (L \to H_1^{\wedge} \to H)$.

(iii) $\theta \to \chi_-(H,\varphi,\psi)\theta$, (H,φ,ψ) a $(-\lambda)$-form.

(iv) $\theta \to \chi_+(H,\varphi,\psi)\theta$, $(H^{\wedge},\varphi,\psi)$ a $(-\lambda)$-form.

(1.35) <u>Remark</u>: $W_1^\lambda(B/A)$ is an abelian semigroup. The main theorem (2.1) implies it is a group, but we have no direct proof of this. The notation has been chosen so that the operations of (1.34) correspond to the following operations on $\sigma \in U_{2n}^\lambda(A)$ (see (1.27))

(i) $\sigma \to I_{2m} \perp \sigma$.

(ii) $\sigma \to w_{2m}^\lambda \perp \sigma$.

(iii) $\sigma \to X_-(\rho)\sigma$, $\rho \in S_{-\lambda}(A^n)$.

(iv) $\sigma \to X_+(\tau)\sigma$, $\tau \in S_{-\lambda}(A^n)$.

It is customary to call $W_1^\lambda(A)/\langle w_n, n \in \mathbb{Z}^+\rangle$ the Wall group, $L_\lambda^h(A)$.

Hence our $W_1^\lambda(B/A)$ is essentially a Wall group, not a Witt group.

§2 The main theorem

(2.1) Theorem: Let A be a ring-with-involution, B a ring of quotients as in (1.1). Then there is a long exact sequence of abelian groups, $\lambda = \pm 1$,

$$\cdots \longrightarrow W_1^\lambda(A) \xrightarrow{\mathcal{K}_1^\lambda} W_1^\lambda(B) \xrightarrow{\mathcal{L}_1^\lambda} W_1^\lambda(B/A) \xrightarrow{\mathcal{S}_1^\lambda} W_0^\lambda(A) \xrightarrow{\mathcal{K}_0^\lambda} W_0^\lambda(B)$$

$$\xrightarrow{\mathcal{L}_0^\lambda} W_0^\lambda(B/A) \xrightarrow{\mathcal{S}_0^\lambda} W_1^{-\lambda}(A) \xrightarrow{\mathcal{K}_1^{-\lambda}} W_1^{-\lambda}(B) \longrightarrow \cdots$$

The proof of this theorem occupies the rest of this paper. We first define $\mathcal{L}_0^\lambda$ and $\mathcal{S}_0^\lambda$ (§3,4) and prove exactness of the last five terms (§5). Then we define $\mathcal{L}_1^\lambda$ and $\mathcal{S}_1^\lambda$ (§6,7) and prove exactness of the first five terms (§8). The maps $\mathcal{K}_i^\lambda$, i = 0,1, are induced by tensoring with B ("change of rings," [B, I.6.3]). The theorem implies $W_1^\lambda(B/A)$ is a group (cf. (1.35)).

§3 $\mathcal{L}_0^\lambda$: $W_0^\lambda(B) \to W_0^\lambda(B/A)$

(3.1): This homomorphism is classical for example (1.8b). It is easily shown that, when A is the ring of integers in a number field B, $\mathcal{L}_0^\lambda$ is essentially the direct sum over all primes $\mathfrak{p} \subseteq A$ of the "second residue class map", denoted ∂_2 in [L], ψ_2 in [MH](c.f. (0.3))

Let a λ-form (B^n, g, q) be given (since $1/2 \in B$, g determines q, but we keep it in the notation for completeness). Find an "integral" lattice $L \subset B^n$; i.e., find $L \cong A^n$ such that $g(L \times L) \subseteq A$, $q(L) \subseteq A/S_{-\lambda}(A)$ (clear denominators in a matrix representation for g). If L' denotes the dual lattice (1.10), then $L \subseteq L'$ and if S denotes L'/L, $S \in \mathfrak{D}_F^1$. We construct a non-singular λ-form (S,φ,ψ) as follows. Since $g(L' \times L) = g(L \times L') \subseteq A$ and $q(L) \subseteq A/S_{-\lambda}(A)$, $g_{L'} = g|L'$ defines a λ-form (S,φ,ψ) (1.12b)

$$\varphi\colon (L'/L) \times (L'/L) \longrightarrow B/A, \quad \varphi(j\ell,j\ell') = rg(\ell,\ell') \tag{3.2}$$
$$\psi\colon L'/L \longrightarrow B/S_\lambda(A), \qquad \psi(j\ell) = r'g(\ell,\ell)$$

where $S = L'/L$, $j\colon L' \to L'/L$, $r\colon B \to B/A$, $r'\colon B \to B/S_\lambda(A)$, and $\ell,\ell' \in L'$. That ${}_\varphi d$ is injective follows from the definition of dual lattice. To see it is surjective, if $f \in S\hat{} = (L'/L)\hat{}$ there is $\tilde{f} \in (L')^*_B$ such that $r\tilde{f}(\ell) = fj(\ell)$, $\ell \in L'$. Since g is nonsingular, there is $\tilde{y} \in B^n$ such that ${}_g d(\tilde{y}) = \tilde{f}$. Since $\tilde{f}(L) \subseteq A$ and $\tilde{f}(L) = g(\tilde{y},L)$, we have $\tilde{y} \in L'$. By definition ${}_\varphi d(j\tilde{y}) = f$. Thus ${}_\varphi d$ is an isomorphism, so (S,φ,ψ) is nonsingular. Let $L_0^\lambda(B,g,q;L)$ denote the isometry class of $(S,\varphi,\psi) \in F_0^\lambda(B/A)$ and $\mathscr{L}_0^\lambda(B,g,q;L)$ its class in $W_0^\lambda(B/A)$.

(3.3) __Proposition__: $\mathscr{L}_0^\lambda(B^n,g,q;L)$ depends only on the class of (B,g,q) in $W_0^\lambda(B)$.

__Proof__: To show independence of $\mathscr{L}_0^\lambda(B^n,g,q;L)$ from L, it suffices to show $\mathscr{L}_0^\lambda(B^n,g,q;L) = \mathscr{L}_0^\lambda(B^n,g,q;I)$ where $I \subseteq L$, since any two integral lattices L and M contain a common sublattice I. From $I \subseteq L$ we obtain

$$I \subseteq L \subseteq L' \subseteq I'.$$

If $L_0^\lambda(B^n,g,q;I) = (T,\varphi_I,\psi_I)$ and $K = L/I$ then $K \in \mathfrak{D}_F^1$, $K \subseteq T$ and K is totally isotropic. By definition of dual lattice, $K^\perp = L'/I$ and under the identification $K^\perp/K = (L'/I)/(L/I) \cong L'/L$ we have

$\varphi_I | (K^{\perp}/K) \times (K^{\perp}/K) = \varphi$ and $\psi_I | (K^{\perp}/K) = \psi$. Hence to show $\mathcal{L}_0^{\lambda}(B^n,g,q;L) = \mathcal{L}_0^{\lambda}(B^n,g,q;I)$ it suffices to prove the following lemma.

(3.4) <u>Lemma</u>: Let a nonsingular λ-form (T,φ,ψ) be given, $T \in \mathfrak{D}_F^1$. Suppose there is $K \in \mathfrak{D}_F^1$, $K \subseteq T$, and K is totally isotropic. Then if $S = K^{\perp}/K$, the naturally induced λ-form (S,φ',ψ') equals (T,φ,ψ) in $W_0^{\lambda}(B/A)$, provided (S,φ',ψ') is nonsingular.

<u>Proof</u>: $(S,\varphi',\psi') \perp (S,(-\varphi'),(-\psi')) = 0$ in $W_0^{\lambda}(B/A)$ (the diagonal submodule is a subkernel) so it suffices to show $(T,\varphi,\psi) \perp (S,(-\varphi'),(-\psi')) = 0$ in $W_0^{\lambda}(B/A)$. Hence it suffices to take $K = K_1$ in the following sublemma.

(3.5): Suppose given $(S,\varphi,\psi) \in F_0^{\lambda}(B/A)$ and $K_1 \subset S$ totally isotropic such that the induced sesquilinear form $(S/K_1^{\perp}) \times K_1 \to B/A$ is nonsingular and the induced λ-form $(K_1^{\perp}/K_1,\varphi_1,\psi_1)$ is nonsingular and is a kernel. Then (S,φ,ψ) is a kernel (see (1.13b)).

<u>Proof</u>: Let K_2 be a subkernel for $(K_1^{\perp}/K_1,\varphi_1,\psi_1)$. Let J' be the pullback in

$$\begin{array}{ccccc} K_1 & \rightarrowtail & J' & \dashrightarrow\!\!\!\!\rightarrow & K_2 \\ \downarrow = & & \vdots & & \downarrow \\ K_1 & \rightarrowtail & K_1^{\perp} & \twoheadrightarrow & K_1^{\perp}/K_1 \end{array}$$

and let J be its image in $K_1^{\perp} \subseteq S$; it is easy to see J is totally isotropic. Further, the diagram shows that $K_1^{\perp}/J \cong (K_1^{\perp}/K_1)/K_2 \cong K_2^{\wedge}$. The sequence of injections $J \to K_1^{\perp} \to S$ gives rise to the short exact sequence of cokernels

$$K_2^{\wedge} \cong K_1^{\perp}/J \rightarrowtail S/J \twoheadrightarrow S/K_1^{\perp} \cong K_1^{\wedge},$$

which by construction is $\mathrm{Hom}_A(-,B/A)$ of the exact sequence $K_1 \to J \to K_2$. This shows $S/J \cong J^\wedge$ so the induced form $(S/J) \times J \to B/A$ is nonsingular. Hence J is a subkernel.

Continuing the proof of (3.3), suppose (B^n,g,q) is isometric to (B^n,g',q'). Then there is an automorphism $\alpha\colon B^n \to B^n$ such that $d_g = \bar{\alpha}d_{g'}\alpha$. If $L \subseteq B^n$ is integral with respect to g, then $I = \alpha(L)$ is integral with respect to g' and $\bar{\alpha}(I') = L'$. It follows easily that $L_0^\lambda(B^n,g,q;L) = L_0^\lambda(B^n,g',q';I)$. Using this we may take $\mathcal{H}(B^n)$ to be represented by $\mathcal{H}(A^n) \otimes B$ and clearly $\mathcal{L}_0^\lambda(\mathcal{H}(A^n) \otimes B) = 0$ (we may find a lattice $L = A^n + \bar{A}^n$ with $L = L'$). This completes the proof of (3.3) and shows we have a well-defined homomorphism $\mathcal{L}_0^\lambda\colon W_0^\lambda(B) \to W_0^\lambda(B/A)$.

§4 $\underline{\partial_0^\lambda\colon W_0^\lambda(B/A) \to W_1^{-\lambda}(A)}$

(4.1): Let a nonsingular λ-form (S,φ,ψ) be given. Choose a short free resolution for S, $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} S)$ and let $(R^\wedge) = (\bar{A}^n \overset{\bar{\alpha}}{\rightarrowtail} \bar{A}^n \overset{\tilde{j}}{\twoheadrightarrow} S^\wedge)$ be the dual resolution (1.4). Choose a λ-hermitian form $\tau\colon A^n \times A^n \to B$ satisfying the conditions of (1.17). Hence for $m,m' \in A^n$, and τ also denoting the matrix of $\tau\colon A^n \times A^n \to B$,

$$\text{(4.2)}\qquad \begin{aligned} &\varphi(j(m),j(m')) = r\tau(m,m'), && r\colon B \longrightarrow B/A \\ &\psi(j(m)) = r'\tau(m,m), && r'\colon B \longrightarrow B/S_\lambda(A) \\ &\bar{\alpha}\tau \in M_n(A), && \bar{\alpha}\tau\alpha \in S_\lambda(A^n) \\ &\tilde{j}\cdot\bar{\alpha}\tau = d_\varphi\cdot j\colon A^n \longrightarrow S^\wedge. && \end{aligned}$$

Setting $\gamma = \tau\alpha$, we have $\bar{\gamma}\alpha = \lambda\bar{\alpha}\tau\alpha \in S_\lambda(A^n)$; further, we claim $(\alpha,\gamma)\colon A^n \to A^n + \bar{A}^n$ is a split injection with totally isotropic image. Once we verify this the discussion of (1.26) shows how to construct

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U_{2n}^{-\lambda}(A), \tag{4.3}$$

which, as an element of $W_1^{-\lambda}(A)$, depends only on the pair (α,γ). Since α is injective, (α,γ) is injective. We will show that $t(\operatorname{cok}(\alpha,\gamma))$, the torsion submodule of $\operatorname{cok}(\alpha,\gamma)$, is zero and conclude that $\operatorname{cok}(\alpha,\gamma)$ is free. Hence (α,γ) is a split injection. Suppose $(e,f) \in A^n + \bar{A}^n$ and $(\alpha(m),\gamma(m)) = (e,f)s = (es,fs)$, for some $m \in M$, $s \in \Sigma$. Since $\gamma = \tau\alpha$, we get $\alpha(m) = es$ and $\tau\alpha(m) = fs$; hence $\tau(e) = f$. If $e \notin \operatorname{im}(\alpha)$, then $j(e) \neq 0$ so $d_\varphi j(e) = \tilde{j}\bar{\alpha}\tau(e) \neq 0$ since d_φ is bijective. But $\tau(e) = f \in \bar{A}^n$ and $\tilde{j}\alpha$ is the zero map. Thus $\alpha(m') = e$, for some $m' \in A^n$ and $\gamma(m')s = \tau\alpha(m')s = fs$, so $\gamma(m') = f$. Thus $(e,f) \in \operatorname{im}(\alpha,\gamma)$. Denote $J = \operatorname{im}(\alpha,\gamma)$, let $h\colon (A^n + \bar{A}^n) \times (A^n + \bar{A}^n) \to A$ denote the hyperbolic λ-hermitian form and let $g\colon J \times ((A^n + \bar{A}^n)/J) \to A$ denote the induced sesquilinear form. We have the exact sequence $J \overset{i}{\rightarrowtail} A^n + \bar{A}^n \overset{j}{\twoheadrightarrow} (A^n + \bar{A}^n)/J$, and the commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{\quad(\alpha,\gamma) = i\quad} & A^n + \bar{A}^n \\ \Big\downarrow {}^{g^d} & & \cong \Big\downarrow {}^{h^d} \\ (A^n + \bar{A}^n)/J & \xrightarrow{\quad\bar{j}\quad} & A^n + \bar{A}^n \end{array} .$$

Since h is nonsingular, ${}_g d$ is injective; clearly ${}_g d \otimes B$ is an isomorphism so $\operatorname{cok}({}_g d)$ is torsion (1.2). But $\operatorname{cok}(i)$ is torsion-free, so ${}_g d$ is an isomorphism. Thus $\operatorname{cok}(\alpha,\gamma) = (A^n + \bar{A}^n)/J \cong \bar{J} \cong \bar{A}^n$ is free, so J is a summand of $A^n + \bar{A}^n$ as desired.

We denote $I_0^\lambda(S,\varphi,\psi;R,\tau) = \sigma \in U_{2n}^{-\lambda}(A)$ (4.3) and let $\mathfrak{s}_0^\lambda(S,\varphi,\psi;R,\tau)$ be the class of σ in $W_1^{-\lambda}(A)$.

(4.4) Proposition: $\mathfrak{s}_0^\lambda(S,\varphi,\psi;R,\tau)$ is independent of the choices of R and τ.

Proof: Suppose first that $\tau': A^n \times A^n \to B$ also satisfies (4.2) with respect to the resolution (R), so that $\rho = \tau - \tau' \in S_\lambda(A^n)$. Then if $I_0^\lambda(S,\varphi,\psi;R,\tau') = \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$, we have $\alpha = \alpha'$ and $(\alpha,\gamma) = (\alpha,\tau\alpha)$ $= (\alpha,\tau'\alpha + \rho\alpha) = (\alpha,\tau'\alpha' + \rho\alpha') = (\alpha',\gamma' + \rho\alpha')$. Hence $I_0^\lambda(S,\varphi,\psi;R,\tau) = X_+(\rho)I_0^\lambda(S,\varphi,\psi;R,\tau')$ so $\mathfrak{s}_0^\lambda(S,\varphi,\psi;R,\tau) = \mathfrak{s}_0^\lambda(S,\varphi,\psi;R,\tau')$.

Stabilizing $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow S)$ to $(R + I_k) = (A^{n+k} \xrightarrow{\alpha+I_k} A^{n+k} \twoheadrightarrow S)$ we get $I_0^\lambda(S,\varphi,\psi;R + I_k, \tau \perp O_k) = I_0^\lambda(S,\varphi,\psi;R,\tau) \perp I_{2k}$ (stabilization (1.19)) where $O_k: A^k \times A^k \to B$ is the zero form. If $(R') = (A^m \overset{\alpha'}{\rightarrowtail} A^m \twoheadrightarrow S)$ is another resolution for S, then by (1.7) there are integers k, ℓ with $n + k = m + \ell$ and automorphisms $\mu,\mu': A^{n+k} \to A^{n+k}$ such that $(\alpha' + I_\ell)\mu' = \mu(\alpha + I_k)$. If $\tau': A^{m+\ell} \times A^{m+\ell} \to B$ is defined by $\tau'(m,m') = (\tau \perp O_k)(\mu^{-1}(m),\mu^{-1}(m'))$ we have conditions (4.2) with $\tau',\alpha' + I_\ell$ replacing τ,α and $\tau' = \bar{\mu}^{-1}(\tau + O_k)\mu^{-1}$. Thus, if $O_k: A^k \to A^k$ is the zero map, $(\alpha + I_k, \gamma + O_k) = (\alpha + I_k, (\tau + O_k)(\alpha + I_k))$ $= (\mu^{-1}(\alpha' + I_\ell)\mu', (\bar{\mu}\tau'\mu)(\mu^{-1}(\alpha' + I_\ell)\mu')) = (\mu^{-1}(\alpha' + I_\ell)\mu', \bar{\mu}\gamma'\mu')$ where $\gamma' = \tau'(\alpha' + I_\ell)$. Hence if $\tau'': A^m \times A^m \to B$ is any covering of (S,φ,ψ) with respect to (R') (i.e., satisfying (4.2)),

$I_0^\lambda(S,\varphi,\psi;R,\tau) \perp I_{2k} = I_0^\lambda(S,\varphi,\psi;R + I_k,\tau \perp O_k)$
$= H(\mu^{-1})I_0^\lambda(S,\varphi,\psi;R' + I_\ell,\tau')H(\mu')$
$= H(\mu^{-1})X_+(\tau' - (\tau'' \perp O_\ell))I_0^\lambda(S,\varphi,\psi;R' + I_\ell,\tau'' + O_\ell)H(\mu')$
$= H(\mu^{-1})X_+(\tau' - (\tau'' \perp O_\ell))(I_0^\lambda(S,\varphi,\psi;R',\tau'') \perp I_{2\ell})H(\mu')$. This completes the proof.

(4.5) <u>Corollary</u>: Given $I_0^\lambda(S,\varphi,\psi;R,\tau) \in U_{2n}^{-\lambda}(A)$, $\rho \in S_\lambda(A^n)$ and $\mu' \in GL_n(A)$, we may find (R,τ') and (R'',τ'') satisfying (4.2) such that $I_0^\lambda(S,\varphi,\psi;R'',\tau'')H(\mu') = I_0^\lambda(S,\varphi,\psi;R,\tau) = X_-(\rho)I_0^\lambda(S,\varphi,\psi;R,\tau')$.

<u>Proof</u>: This follows from the proof of (4.4).

(4.6): To complete the construction of $\mathfrak{s}_0^\lambda$, it remains only to show that if $(S,\varphi,\psi) = 0$ in $W_0^\lambda(B/A)$, then $\mathfrak{s}_0^\lambda(S,\varphi,\psi) = 0$. Thus we assume $(S,\varphi,\psi) \perp (U,\varphi',\psi') = (V,\varphi'',\psi'')$ where (U,φ',ψ') and (V,φ'',ψ'') are kernels. The following lemma suffices.

(4.7) <u>Lemma</u>: Let (S,φ,ψ) be a kernel. Then $\mathfrak{s}_0^\lambda(S,\varphi,\psi) = 0$.

<u>Proof</u>: By assumption there is $K \in \mathfrak{D}_F^1$ such that K is totally isotropic and there is a short exact sequence $K \rightarrowtail S \twoheadrightarrow S/K \cong K^\wedge$. If $A^n \overset{\mu}{\rightarrowtail} A^n \overset{j}{\twoheadrightarrow} K$ is a short free resolution for K, then (1.4) we have $\bar{A}^n \overset{\bar{\mu}}{\rightarrowtail} \bar{A}^n \overset{\tilde{j}}{\twoheadrightarrow} K^\wedge$ and consequently a short free resolution (R) for S:

$$(R) = (A^n + \bar{A}^n \overset{\begin{pmatrix}\mu & \theta \\ 0 & \bar{\mu}\end{pmatrix}}{\rightarrowtail} A^n + \bar{A}^n \twoheadrightarrow S + S^\wedge)$$

where $\theta: \bar{A}^n \to A^n$ is a homomorphism. We claim there is $\tau: (A^n + \bar{A}^n) \times (A^n + \bar{A}^n) \to B$ satisfying (4.2) with matrix

$$\tau = \begin{bmatrix} 0 & \lambda\bar{\mu}^{-1} \\ \mu^{-1} & \nu \end{bmatrix} .$$

Since $\varphi | K \times K \equiv 0$ and, under the identification $S/K \cong K^\wedge$, $\varphi | (S/K) \times K$ is the natural form, $\langle\ ,\ \rangle_K$, it suffices to show that $r\langle x,\mu^{-1}y\rangle_{B^n} = \langle \tilde{j}(x), j(y)\rangle_K$ where $x,y \in A^n$ and $r: B \to B/A$. But this follows directly from the definition (1.6) of $\tilde{j}$. So τ has the form claimed. Thus

$$I_0^\lambda(S,\varphi,\psi;R,\tau) = \left[\begin{array}{cc|c} \mu & \theta & \beta \\ 0 & \bar{\mu} & \\ \hline 0 & \lambda I_n & \delta \\ I_n & \eta & \end{array}\right] \in U_{4n}^{-\lambda}(A),$$

where $\eta = \mu^{-1}\theta + \nu\bar{\mu}$. Consider the element $w_1^\lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \in GL_2(A)$ (see 1.20--w_1^λ is not in $U_2^{-\lambda}(A)$) and the element $T_4^\lambda \in U_4^{-\lambda}(A)$.

(4.9) $$T_4^\lambda = X_-(-w_1^\lambda)X_+(\bar{w}_1^\lambda)H(w_1^\lambda)X_-(w_1^\lambda)$$

introduced by Sharpe in [Sh]. Then T_4^λ represents zero in $W_1^{-\lambda}(A)$ and

(4.10) $$T_4^\lambda = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix} \quad ; \quad \text{set } T_{4n}^\lambda = T_4^\lambda \perp \cdots \perp T_4^\lambda$$
(n terms)

Thus,

$$T_{4n}^\lambda L_0^\lambda(S,\varphi,\psi;R,\tau) = \left[\begin{array}{cc|c} 0 & \lambda I_n & \delta \\ I_n & \eta & \\ \hline \lambda\mu & \lambda\theta & \lambda\beta \\ 0 & \lambda\bar{\mu} & \end{array}\right]$$

The upper left-hand $(2n \times 2n)$ block is invertible so by [B,II.2.5(b)], (4.11) has the form $X_-(\gamma_1)H(\alpha)X_+(\gamma_2)$, hence represents zero in $W_1^{-\lambda}(A)$.

§5 Exactness of the last five terms of (2.1)

(5.1) Proposition: The following sequence of groups and homomorphisms is exact, where $\mathcal{K}_i^{\pm\lambda}$ is induced by $\otimes$ B:

$$W_0^\lambda(A) \xrightarrow{\mathcal{K}_0^\lambda} W_0^\lambda(B) \xrightarrow{\mathcal{L}_0^\lambda} W_0^\lambda(B/A) \xrightarrow{\partial_0^\lambda} W_1^{-\lambda}(A) \xrightarrow{\mathcal{K}_1^{-\lambda}} W_1^{-\lambda}(B).$$

Proof: (a) Exactness at $W_0^\lambda(B)$. It is clear that $\mathcal{L}_0^\lambda \mathcal{K}_0^\lambda = 0$. Let (B^n, g, q) be given with $L_0^\lambda(B^n, g, q; L) \perp (U, \varphi, \psi) = (V, \varphi, \psi)$ in $F_0^\lambda(B/A)$ (notation as in (3.1)) where (U, φ, ψ) and (V, φ', ψ') are kernel λ-forms. We need the following lemma.

(5.2) Lemma: Let (U, φ, ψ) be a kernel λ-form, $U \in \mathfrak{D}_F^1$. Then there is a hyperbolic λ-form (B^n, g, q) and an integral lattice $L \subset B^n$ such that $L_0^\lambda(B^n, g, q; L) = (U, \varphi, \psi)$.

Proof: In (4.7) we showed there was a resolution R of U and a covering τ of (U, φ, ψ) (4.2) so that

$$I_0^\lambda(U, \varphi, \psi; R, \tau) = \left[\begin{array}{cc|c} \mu & \theta & \beta \\ 0 & \bar{\mu} & \\ \hline 0 & \lambda I_n & \delta \\ I_n & \eta & \end{array}\right]$$

Hence $I_0^\lambda(U, \varphi, \psi; R, \tau) H\begin{pmatrix} -\eta & \lambda I_n \\ I_n & 0 \end{pmatrix} = \left[\begin{array}{cc|c} -\mu\eta + \theta & \lambda\mu & \beta \\ \bar{\mu} & 0 & \\ \hline I_n & 0 & \delta' \\ 0 & I_n & \end{array}\right], -\mu\eta + \theta = -\mu\nu\bar{\mu}$

$\in S_\lambda(A^n) = I_0^\lambda(U, \varphi, \psi; R', \tau')$, for some choice of R', τ' by (4.5). By definition this implies there is a λ-form $\theta = (L, g, q)$, $L = A^n \times \bar{A}^n$ where g has matrix $\begin{pmatrix} -\mu\nu\bar{\mu} & \lambda\mu \\ \bar{\mu} & 0 \end{pmatrix}$ such that $\theta \otimes B = (B^n + \bar{B}^n, g_B q_B)$ has integral lattice equal to $L \subset B^n + \bar{B}^n$. Thus $L_0^\lambda(B^n + \bar{B}^n, g_B, q_B; L) = (S, \varphi, \psi)$. The matrix for θ above implies $(B^n + \bar{B}^n, g_B, q_B)$ is hyperbolic, so the proof of (5.2) is complete.

Returning to exactness, (5.2) allows us to assume $L_0^\lambda(B^n, g, q; L) = (V, \varphi, \psi) =$ a kernel λ-form. We thus have a resolution $L \rightarrowtail L' \overset{j}{\twoheadrightarrow} V$ and $\tau = g|\,L'$ covering (V,φ,ψ) as in (4.2). By (5.2) again, we have a hyperbolic λ-form (B^n,h,p), an integral lattice $I \subset B^n$ with corresponding resolution $I \rightarrowtail I' \overset{k}{\twoheadrightarrow} V$, and a covering $\theta\colon I' \times I' \to B$ ($\theta = h|I'$) of (V,φ,ψ) satisfying (4.2). Let P be the pullback $\{(x,y) \in L' \times I' \mid j(x) = k(y)\}$,

$$\begin{array}{ccc} P & \dashrightarrow & I' \\ \Big\downarrow & & \Big\downarrow k \\ L' & \xrightarrow{\ j\ } & V \end{array} \quad .$$

Clearly P is stably free. Define f in a λ-form (P,f,s) by setting $f((x,y),(x',y')) = \tau(x,x') - \theta(y,y')$ where $(x,y),(x',y') \in P$. This f is clearly λ-hermitian and, since τ and θ each cover (S,φ,ψ), f takes values in A and $f((x,y),(x,y)) \in S_\lambda(A)$, for all $(x,y) \in P$. To define s, take P to be free with basis $\{e_1,\ldots,e_n\}$, $f(e_i,e_i) = a_i + \lambda\bar{a}_i \in S_\lambda(A)$, $a_i \in A$. Define $s(e_i) \equiv a_i \bmod S_{-\lambda}(A)$, and extend to $s\colon P \to A/S_{-\lambda}(A)$ using the conditions (1.12a).

By definition of P, there are split exact sequences

$$L \overset{i_1}{\rightarrowtail} P \twoheadrightarrow I', \qquad i_1(x) = (x,0)$$

and

$$I \overset{i_2}{\rightarrowtail} P \twoheadrightarrow L', \qquad i_2(y) = (0,y).$$

Clearly $f(L \times I) \equiv 0 \equiv f(I \times L)$ from which we obtain induced forms

$$L \times (P/I) \longrightarrow A \qquad \text{and} \qquad I \times (P/L) \longrightarrow A \quad .$$

Identifying P/I with L' and P/L with I' these become the nonsingular forms pairing a lattice with its dual lattice. It is easy to verify that this implies f is nonsingular. But since

$(L + I) \otimes B \xrightarrow{(i_1+i_2)\otimes B} P \otimes B$ is an isomorphism the construction of f shows $(P \otimes B, f \otimes B, s \otimes B)$ is isometric to $(B^n,g,q) \perp (B^n,h,p)$. Since (B^n,h,p) is hyperbolic $\mathcal{K}_0^\lambda((P,f,s)) = (B^n,g,q)$ in $W_0^\lambda(B)$.

(b) Given (B^n,g,q) representing an element of $W_0^\lambda(B)$, let $L \subset B^n$ be an integral lattice and let $L_0^\lambda(B^n,g,q;L) = (S,\varphi,\psi)$. Then $(R) = (L \overset{\alpha}{\rightarrowtail} L' \twoheadrightarrow S)$ gives a resolution and $\tau\colon L' \times L' \to B$ covers the λ-form (S,φ,ψ) as in (4.2), $\tau = g|L' \times L'$. Recalling that if L has basis $\{e_i\}$, we give L' the basis $\{e_i{*}\}$ satisfying $g(e_i{*},e_j) = \delta_{ij}$, it is easily verified that $\tau\alpha = I_n$ so that

$$I_0^\lambda(S,\varphi,\psi;R,\tau) = \begin{pmatrix} \alpha & \beta \\ I_n & \delta \end{pmatrix} = \sigma.$$

Since n is even (1.13a) we may multiply σ on the left by T_{2n}^λ as in (4.11) to see that σ represents zero in $W_1^{-\lambda}(A)$. Thus $\partial_0^\lambda \mathcal{L}_0^\lambda(B,g,q) = 0$.

Now suppose given $(S,\varphi,\psi) \in F_0^\lambda(B/A)$, a resolution $(R) = (A^{2n} \overset{\alpha}{\rightarrowtail} A^{2n} \twoheadrightarrow S)$, and a covering $\tau\colon A^{2n} \times A^{2n} \to B$ of (S,φ,ψ) (4.2) such that $I_0^\lambda(S,\varphi,\psi;R,\tau) = \sigma \in U_{4n}^{-\lambda}(A)$ and σ represents zero in $W_1^{-\lambda}(A)$. By [Sh, 5.5,5.6] there exist $\rho,\tau_1,\tau_2 \in S_\lambda(A^{2n})$, $\alpha \in GL_{2n}(A)$ such that

$$\sigma = X_-(\tau_1)T_{4n}^\lambda X_-(\tau_2)H(\alpha)X_+(\bar{\rho}),$$

where T_{4n}^λ is defined in (4.10). (Sharpe works in the "unitary Steinberg group", but his matrix calculations show that the above form is valid.) Hence $X_-(-\tau_1)\sigma X_+(-\bar{\rho})H(\alpha^{-1}) = T_{4n}X_-(\tau_2) = \begin{pmatrix} \tau_2 & * \\ -\lambda I_{2n} & * \end{pmatrix}$.

Right multiplication by $X_+(-\bar{\rho})$ does not change the first column of $(2n \times 2n)$ blocks in σ while left multiplication by $X_-(-\tau_1)$ and right multiplication by $H(\alpha^{-1})$ are realized by changes in R and

τ (4.5). Hence there are R', τ' so that $I_0^{\lambda}(S,\varphi,\psi;R',\tau') = \begin{pmatrix} \tau_2 & \beta \\ -\lambda I_{2n} & \delta \end{pmatrix}$. As we saw in the proof of (5.2), this means (S,φ,ψ) is in the image of L_0^{λ}.

(5.3) <u>Remark</u>: In proving that $\mathcal{S}_0^{\lambda}\mathcal{L}_0^{\lambda} = 0$, we needed n even in the λ-form (B^n,g,q). This is the only place we will use this condition. If $\lambda = 1$, it is unnecessary: if n is odd we may add $\mathcal{H}(B)$ to (B^n,g,q), observe that $[1] \perp [-1] \perp (B^n,g,q) \cong \mathcal{H}(B) \perp (B^n,g,q)$ (where [b] denotes the unary form on B with matrix (b)), that [1] is sent to zero by $\mathcal{S}_0^{\lambda}\mathcal{L}_0^{\lambda}$ and that $[-1] \perp [B^n,g,q]$ is a λ-form on B^{n+1}, n + 1 even. If $\lambda = -1$ and B has simple component acted upon trivially by the involution (e.g., $B = \mathbb{Q}\pi$, π finite), then each (B^n,g,q) has n even. The author does not know whether the assumption that n be even is necessary in general.

(c) Exactness at $W_1^{-\lambda}(A)$. If $(S,\varphi,\psi) \in F_0^{\lambda}(B/A)$, choose $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow S)$ and τ as usual so that $I_0^{\lambda}(S,\varphi,\psi;R,\tau) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \sigma$.

Since $\alpha \otimes B$ is invertible, we may apply [B.II.2.5b] to conclude $\mathcal{K}_1^{-\lambda}(\sigma) = 0$.

Next suppose $\sigma \in U_{2n}^{-\lambda}(A)$ is such that $\mathcal{K}_1^{-\lambda}(\sigma) = 0$ in $W_1^{-\lambda}(B)$. By stabilizing (if necessary) we may assume n is even. If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we showed in [P3,3.8] that $[\sigma] = 0$ in $W_1^{-\lambda}(B)$ implies there is $\tau \in S_{\lambda}(B^n)$ such that $\gamma + \tau\alpha \in GL_n(B)$; moreover, τ is required to be the adjoint of the orthogonal sum of a λ-form nonsingular on $\alpha(\ker \gamma)$ with the zero form on some complement of $\alpha(\ker \gamma)$ in A^n. Replacing τ by $a\tau\bar{a}$, where $a\tau\bar{a} \in S_{\lambda}(A^n)$, $a \in \Sigma$, the λ-form $a\tau\bar{a}$ still has the above mentioned properties. Hence we may find

$\tau \in S_\lambda(A^n)$ such that $\gamma + \tau\alpha \in GL_n(B)$. If $\sigma' = T^\lambda_{2n}X_-(\tau)\sigma$ and $\sigma' = \begin{pmatrix}\alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$ then α' is a unit in $M_n(B)$. Taking $\tau = \gamma'\alpha'^{-1}$ and $\mu = \alpha'$ in (1.18) we obtain a λ-form (S,φ,ψ) for which (by definition--see (4.2))

$$I_0^\lambda(S,\varphi,\psi;R,\tau) = \sigma',$$

where $(R) = (A^n \overset{\alpha'}{\rightarrowtail} A^n \twoheadrightarrow S)$. This completes the proof of (5.1).

§6 $\underline{\partial_1^\lambda: W_1^\lambda(B) \to W_1^\lambda(B/A)}$

(6.1): Let $\sigma = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in U^\lambda_{2n}(B)$ be given; we want to construct an element $(K,H,\Delta,\varkappa) \in F_1^\lambda(B/A)$. We may find $\nu,\eta \in M_n(A)$, invertible as elements of $M_n(B)$, such that $\alpha\nu$, $\gamma\nu$, $\beta\eta$, $\delta\eta \in M_n(A)$. Let

(6.2) $$\sigma(\nu,\eta) = \begin{pmatrix}\alpha\nu & \beta\eta \\ \gamma\nu & \delta\eta\end{pmatrix}, \qquad \sigma'(\nu,\eta) = \begin{pmatrix}\overline{\delta\eta} & \lambda\overline{\beta\eta} \\ \overline{\lambda\gamma\nu} & \overline{\alpha\nu}\end{pmatrix}$$

and consider the sequence of injections,

(6.3) $$A^n + \bar{A}^n \xrightarrow{\sigma(\nu,\eta)} A^n + \bar{A}^n \xrightarrow{\sigma'(\nu,\eta)} A^n + \bar{A}^n.$$

Denote the terms of (6.3) L,I,L', respectively, keeping in mind that each has a basis (it turns out below that L' is the dual lattice to L). We can thus consider the sequence of lattices in $B^n + \bar{B}^n$,

(6.4) $$L \subseteq I \subseteq L' \subset B^n + \bar{B}^n,$$

where $I \otimes B \subseteq B^n + \bar{B}^n$ is the <u>identity</u>. Hence it makes sense to endow $I = A^n + \bar{A}^n$ with the hyperbolic structure (1.13a) so that $I \otimes B = \mathcal{H}(A^n) \otimes B \to \mathcal{H}(B^n)$ is the identity. L and L' inherit λ-hermitian forms h_L and $h_{L'}$ using the injections $\sigma(\nu,\eta)$ and $\sigma'(\nu,\eta)$; $h_{L'}$ may take values in B. Calculation using the relations (1.20) defining σ as an element of $U^{\lambda}_{2n}(B)$ shows that h_L has matrix $\begin{pmatrix} 0 & \lambda\bar{\nu}\eta \\ \bar{\eta}\nu & 0 \end{pmatrix}$ and $h_{L'}$ has matrix $\begin{pmatrix} 0 & \lambda(\bar{\nu}\eta)^{-1} \\ (\bar{\eta}\nu)^{-1} & 0 \end{pmatrix}$, where inverses are taken in $GL_n(B)$. It follows that L and L' are dual lattices in $\mathcal{H}(B^n)$. If $D\colon A^n + \bar{A}^n \overset{\cong}{\to} \overline{A^n + \bar{A}^n}$ denotes the adjoint of the hyperbolic λ-form on $A^n + \bar{A}^n$, another calculation shows

(6.5) $$\overline{\sigma(\nu,\eta)}D = D\sigma'(\nu,\eta) \qquad \text{(see [B,II.1.2])}$$

and

(6.6) $$\sigma'(\nu,\eta)\sigma(\nu,\eta) = \begin{pmatrix} \bar{\eta}\nu & 0 \\ 0 & \bar{\nu}\eta \end{pmatrix}.$$

(6.7) <u>Construction of K, H and Δ in (K,H,Δ,κ):</u> Taking $\tau = \begin{pmatrix} 0 & \lambda(\bar{\nu}\eta)^{-1} \\ (\bar{\eta}\nu)^{-1} & 0 \end{pmatrix}$ (= the matrix of $h_{L'}\colon L' \times L' \to B$) and $\mu = \sigma'(\nu,\eta)\sigma(\nu,\eta)$ in (1.18) we obtain the hyperbolic form $\mathcal{H}(H) = (H + H^{\wedge}, \varphi_h, \psi_h)$, $H = \operatorname{cok}(\bar{\eta}\nu)$. If K denotes I/L, $\sigma'(\nu,\eta)$ induces an injection $\Delta\colon K \to H + H^{\wedge} = L'/L$. Explicitly, if $\Delta = (\xi,\zeta)$, $\xi\colon K \to H$, $\zeta\colon K \to H^{\wedge}$, then the diagrams

(6.8) $$\begin{array}{ccc} A^n + \bar{A}^n & & A^n \\ \downarrow{\scriptstyle \sigma'\sigma} & & \downarrow{\scriptstyle \bar{\eta}\nu} \\ I = A^n + \bar{A}^n & \xrightarrow{(\overline{\delta\eta},\, \lambda\overline{\beta\eta})} & A^n \\ \downarrow{\scriptstyle j_K} & & \downarrow{\scriptstyle j_H} \\ K & \xrightarrow{\ \xi\ } & H \end{array} \quad \text{and} \quad \begin{array}{ccc} A^n + \bar{A}^n & & \bar{A}^n \\ \downarrow{\scriptstyle \sigma'\sigma} & & \downarrow{\scriptstyle \bar{\nu}\eta} \\ I = A^n + \bar{A}^n & \xrightarrow{(\lambda\overline{\gamma\nu},\, \overline{\alpha\nu})} & \bar{A}^n \\ \downarrow{\scriptstyle j_K} & & \downarrow{\scriptstyle \tilde{j}_H} \\ K & \xrightarrow{\ \zeta\ } & H^{\wedge} \end{array}$$

commute.

(6.9) <u>Lemma:</u> $\operatorname{Im}(\Delta) \subset H + H^{\wedge}$ is a subkernel (1.13b).

<u>Proof:</u> To see $\operatorname{im}(\Delta)$ is totally isotropic, observe that the

λ-hermitian form $h_{L'}$ convering $\mathcal{H}(H)$ (in the sense of (1.17)) is A-valued and even (in fact hyperbolic) on I; hence, by construction, the forms φ_h and ψ_h restricted to the image of $I/L = K$ are identically zero in B/A and $B/S_\lambda(A)$. It remains to verify that the sesquilinear form g: $((H + H^\wedge)/K) \times K \to B/A$ induced by φ_h is nonsingular, where $K \equiv \text{im}(\Delta)$. There is a commutative diagram of exact sequences

$$\begin{array}{ccccc}
\text{cok}(\sigma(\nu,\eta)) & \rightarrowtail & \text{cok}(\sigma(\nu,\eta)\sigma'(\nu,\eta)) & \longrightarrow & \text{cok}(\sigma'(\nu,\eta)) \\
\downarrow = & & \downarrow = & & c \downarrow \cong \\
K & \overset{\Delta}{\rightarrowtail} & H + H^\wedge & \xrightarrow{\Delta^\wedge \cdot (\varphi_h)d} & K^\wedge \\
\downarrow = & & \downarrow = & & \downarrow g^d \\
K & \overset{\Delta}{\rightarrowtail} & H + H^\wedge & \twoheadrightarrow & (H + H^\wedge)/K
\end{array}$$

where c is an isomorphism from (6.5) and (1.4). $\Delta^\wedge \cdot (\varphi_h)d$ is a surjection by commutativity, so $_g d$ is an isomorphism.

(6.10) <u>Construction of $\varkappa$: $K \to B/S_{-\lambda}(A)$:</u> To construct and study the $(-\lambda)$-form $(K, [\ ,\ ], \varkappa)$ (1.30) we need an explicit expression for $[\ ,\ ]$: $K \times K \to B/A$, where $[k,\ell] = \varphi_h(\zeta(k),\xi(\ell))$. Let $k,\ell \in K$ and $x,y \in A^n + \bar{A}^n$, with $j_K(x) = k$, $j_K(y) = \ell$ (6.8); let $x = x_1 + x_2$, $y = y_1 + y_2$, $x_1,y_1 \in A^n$, $x_2,y_2 \in \bar{A}^n$. Then by construction and (6.8), if r: $B \to B/A$ is the projection,

$$(6.11)\quad [k,\ell] = \varphi_h(\zeta(k),\xi(\ell)) = rh_{L'}(\lambda\overline{\gamma\nu}(x_1)+\overline{\alpha\nu}(x_2),\overline{\delta\eta}(y_1)+\lambda\overline{\beta\eta}(y_2)).$$

Expanding the term on the right, we find that if τ: $(A^n + \bar{A}^n) \times (A^n + \bar{A}^n) \to B$ is the sesquilinear form with matrix

$$(6.12)\qquad \tau = \begin{pmatrix} \lambda\gamma\bar{\delta} & \gamma\bar{\beta} \\ \alpha\bar{\delta} & \lambda\alpha\bar{\beta} \end{pmatrix}$$

then $[k,\ell] = r\tau(x,y)$, where r,k,ℓ,x,y are as above. (τ is not λ-hermitian, so does not fit into the context of (1.17). But $\tau + \lambda\tau = \begin{pmatrix} 0 & I_n \\ \lambda I_n & 0 \end{pmatrix} \in M_{2n}(A)$ which is sufficient for τ to induce the $(-\lambda)$-hermitian form $[\ ,\ ]: K \times K \to B/A$ using the first equation of (1.17a).)

Let $k \in K$, $x = x_1 + x_2 \in A^n + \bar{A}^n$, $j_K(x) = k$. Define

$$(6.13)\quad \varkappa(k) = r'\tau(x,x) - q_h(x)$$

$$= r'h_{L'}(\lambda\overline{\gamma\nu}(x_1) + \overline{\alpha\nu}(x_2), \overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2)) - q_h(x)$$

where $r': B \to B/S_{-\lambda}(A)$, $\mathcal{H}(A^n) = (A^n + \bar{A}^n = I, g_h, q_h)$, $q_h(x) \in A/S_{-\lambda}(A) \subset B/S_{-\lambda}(A)$. Clearly $\varkappa(k) \equiv r\tau(x,x) \bmod A$ and $r\tau(x,x) = [k,k]$ by (6.11).so 1.12 b(iii) is satisfied; it is routine to verify the rest of (1.12b) so $[K, [\ ,\ ], \varkappa)$ is a$(-\lambda)$-form. Now define

$$(6.14)\qquad L_1^\lambda(\sigma;\nu,\eta) = (K,H,\Delta,\varkappa) \in F_1^\lambda(B/A)$$

as constructed in (6.7), (6.10), and (6.13), and let $\mathcal{L}_1^\lambda(\sigma;\nu,\eta)$ denote its class in $W_1^\lambda(B/A)$.

(6.15) <u>Proposition</u>: $\mathcal{L}_1^\lambda(\sigma;\nu,\eta)$ depends only on the class of σ in $W_1^\lambda(B)$.

<u>Proof</u>: We need to show independence of $\mathcal{L}_1^\lambda(\sigma;\nu,\eta)$ from the choices made in its construction. Denote $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

(i) <u>Choice of ν,η</u>. Given $\nu \in M_n(A)$, invertible as an element of $M_n(B)$, there is $\nu' \in M_n(A)$ and $s \in \Sigma$ such that $\nu\nu' = \nu'\nu = sI_n$. Hence it suffices to show that $\mathcal{L}_1^\lambda(\sigma;\nu\nu',\eta) = \mathcal{L}_1^\lambda(\sigma;\nu,\eta) = \mathcal{L}_1^\lambda(\sigma;\nu,\eta\eta')$. But it is easy to verify that if $(E) = (\mathrm{cok}(\nu') \rightarrowtail \mathrm{cok}(\bar{\eta}\nu\nu') \twoheadrightarrow \mathrm{cok}(\bar{\eta}\nu) = H)$, then (1.32) $L_1^\lambda(\sigma;\nu\nu',\eta) = \sigma_E L_1^\lambda(\sigma;\nu,\eta)$.

Similarly, $L_1^\lambda(\sigma;\nu,\eta\eta') = {}_E\sigma L_1^\lambda(\sigma;\nu,\eta)$ if $(E) = (\mathrm{cok}(\eta') \to \mathrm{cok}(\bar{\nu}\eta\eta') \to \mathrm{cok}(\bar{\nu}\eta) = H^\wedge)$. Thus we have $\mathcal{L}_1^\lambda: U_{2n}^\lambda(B) \to W_1^\lambda(B/A)$.

(ii) Stabilizing σ. Clearly $L_1^\lambda(\sigma;\nu,\eta) = L_1^\lambda(\sigma \perp I_2;\nu + I_1,\eta + I_1)$. Hence $\mathcal{L}_1^\lambda: U^\lambda(B) \to W_1^\lambda(B/A)$.

(iii) It remains to show that $\mathcal{L}_1^\sigma(\sigma;\nu,\eta) = \mathcal{L}_1^\lambda(\sigma E;\nu',\eta')$ for each $E \in X = \langle\{H(\epsilon),X_+(\rho),X_-(\tau) \mid \epsilon \in GL_n(B),\rho,\tau \in S_{-\lambda}(B^n),n \in \mathbb{Z}^+\}\rangle$ (see 1.23).

First let $E = H(\epsilon)$. Choose $\nu,\eta \in M_n(A)$, invertible in $M_n(B)$ such that $\epsilon\nu$, $\alpha\epsilon\nu$, $\gamma\epsilon\nu$, $\bar{\epsilon}^{-1}\eta$, $\beta\bar{\epsilon}^{-1}\eta$ and $\delta\bar{\epsilon}^{-1}\eta$ are in $M_n(A)$. Then $\sigma(\epsilon\nu,\bar{\epsilon}^{-1}\eta) = \{\sigma H(\epsilon)\}(\nu,\eta)$ so $L_1^\lambda(\sigma;\epsilon\nu,\bar{\epsilon}^{-1}\eta) = L_1^\lambda(\sigma H(\epsilon);\nu,\eta)$.

Next if $E = X_-(\rho)$, $\rho \in S_{-\lambda}(B^n)$, choose $a' \in \Sigma$ so that if $a = a'\bar{a}'$, then αa^2, γa^2, βa, δa, $\rho a \in M_n(A)$. Then

$$\{\sigma X_-(\rho)\}(a^2 I_n, aI_n) = \begin{bmatrix} \alpha a^2 + (\beta a)(\rho a) & \beta a \\ \gamma a^2 + (\delta a)(\rho a) & \delta a \end{bmatrix} .$$

We claim that $L_1^\lambda(\sigma X_-(\rho);\, a^2 I_n, aI_n) = \chi_-(\varphi,\psi)L_1^\lambda(\sigma;a^2 I_n,aI_n)$, where (H,φ,ψ) is induced as in (1.18) where we take $\mu = a^3 I_n$, $\tau = \lambda\bar{\rho}a^{-2}$. Since $\{\sigma X_-(\rho)\}(a^2 I_n,aI_n) = \{\sigma H(a^2)X_-(a^4\rho)\}(I_n,aI_n)$ this is a special case of the following lemma, where $\nu = a^2 I_n$, $\eta = aI_n$, $P = a\rho$.

(6.16) Lemma: Let $\sigma \in U_{2n}^\lambda(B)$, $\nu,\eta \in M_n(A)$, invertible in $M_n(B)$ as usual, and let $P\colon A^n \to \bar{A}^n$ be such that $\bar{P}\bar{\eta}\nu \in S_{-\lambda}(A^n)$ and the λ-form (H,φ,ψ) is induced as in (1.18) for $\mu = \bar{\eta}\nu$ and $\tau = \tau_H = \lambda(\bar{\nu}\eta)^{-1}\bar{P}$. Then $L_1^\lambda(\sigma H(\nu)X_-(\bar{\nu}\eta P),I_n,\bar{\nu}\eta) = \chi_-(\varphi,\psi)L_1^\lambda(\sigma;\nu,\eta)$.

Proof: We have

$$(6.17)\qquad \{\sigma H(\nu)X_-(\bar{\nu}\eta P)\}(I_n,\bar{\nu}\eta) = \begin{bmatrix}\alpha\nu+(\beta\eta)P & \beta\eta \\ \gamma\nu+(\delta\eta)P & \delta\eta\end{bmatrix} = \sigma(\nu,\eta)\cdot\begin{bmatrix}I_n & 0 \\ P & I_n\end{bmatrix}.$$

Denote $\sigma H(\nu)X_-(\bar{\nu}\eta P) = \theta$. Then a calculation shows

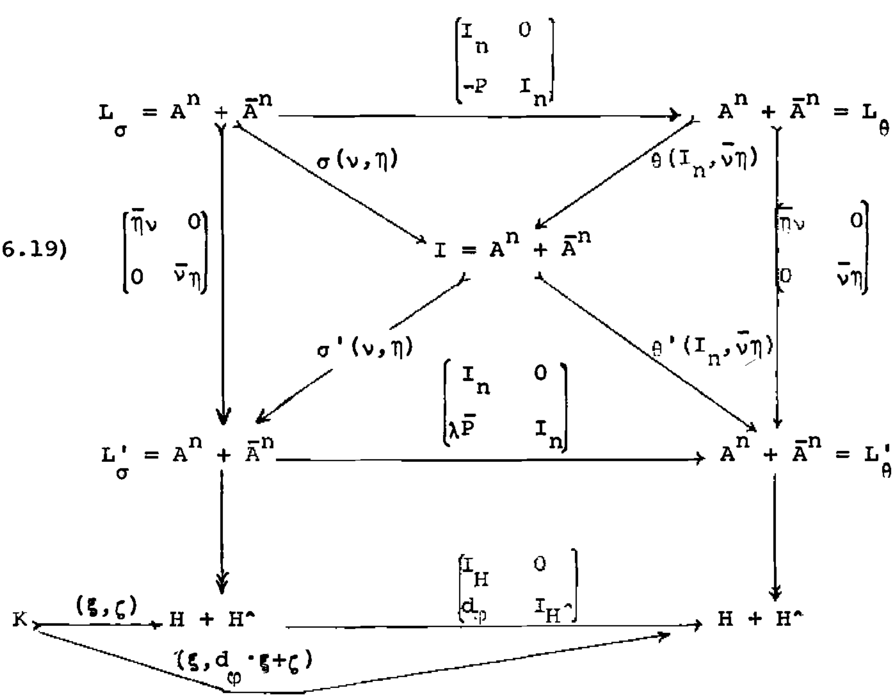
$$(6.18)\qquad \theta'(I_n,\bar{\nu}\eta)\theta(I_n,\bar{\nu}\eta) = \begin{pmatrix}\bar{\eta}\nu & 0 \\ -0 & \bar{\nu}\eta\end{pmatrix} = \sigma'(\nu,\eta)\sigma(\nu,\eta)$$

where

$$\theta'(I_n,\bar{\nu}\eta) = \begin{bmatrix}\overline{\delta\eta} & \lambda\overline{\beta\eta} \\ \lambda(\overline{\gamma\nu}+\bar{P}\,\overline{\delta\eta}) & \overline{\alpha\nu}+\bar{P}\,\overline{\beta\eta}\end{bmatrix}.$$

Denote by L_σ and L'_σ the lattices L' and L' obtained for $\sigma(\nu,\eta)$ in (6.4) and by L_θ, L'_θ the lattices constructed for $\theta(I_n,\bar{\nu}\eta)$. Let τ_σ and τ_θ be the corresponding sesquilinear forms (6.12). From (6.17) and (6.18) we obtain (using the same I for σ and θ) the following commutative diagram where the top two horizontal maps are isometries

(6.19)



If $L_1^\lambda(\theta; I_n, \bar{\nu}\eta) = (K', H, \Delta', \kappa')$ then (6.19) shows that $\mathrm{im}(L_\sigma) = \mathrm{im}(L_\theta) \subseteq I$ so that $K = K'$ and, using (6.8) and the bottom horizontal composite in (6.18), that $\Delta' = (\xi, d_\varphi\cdot\xi + \zeta)$.

Now let us compute κ'. Let $k \in K$, $x = x_1 + x_2 \in A^n + \bar{A}^n$, $j_K(x_1 + x_2) = k$. By definition (6.13)

$$\kappa'(k) = r'h_{L'_\theta}(\lambda(\overline{\gamma\nu+\delta\eta P})(x_1) + (\overline{\alpha\nu+\beta\eta P})(x_2), \overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2)) - q_h(x)$$

$$= r'\tau_\theta(x,x) - q_h(x).$$

Since

$$h_{L'_\sigma} = h_{L'_\theta},\ r'\tau_\theta(x,x) = r'h_{L'_\sigma}(\lambda\overline{\gamma\nu}(x_1) + \overline{\alpha\nu}(x_2), \overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2))$$

$$+ r'h_{L'_\sigma}(\lambda\bar{P}(\overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2)), \overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2)).$$

Denoting $\overline{\delta\eta}(x_1) + \lambda\overline{\beta\eta}(x_2) = z$, the last expression is $r'\tau_\sigma(x,x) + r'\langle\lambda\bar{P}z, (\bar{\eta}\nu)^{-1}z\rangle_{B^n} = r'\tau_\sigma(x,x) + r'\langle\lambda(\bar{\nu}\eta)^{-1}\bar{P}z, z\rangle_{B^n}$ $= r'\tau_\sigma(x,x) + r'\tau_H(z,z)$, since $\lambda(\bar{\nu}\eta)^{-1}\bar{P} = \tau_H$ by assumption. By (6.8) and the definition of τ_H and ψ, the last expression is $r'\tau_\sigma(x,x) + \psi(\xi(k))$. This completes the proof of (6.16) since, from above, $\kappa'(k) = r'\tau_\theta(x,x) - q_h(x) = r'\tau_\sigma(x,x) - q_h(x) + \psi(\xi(k))$ $= \kappa(k) + \psi(\xi(k))$.

Returning finally to the proof of (6.15), an argument similar to that just given shows $L_1^\lambda(\sigma X_+(\rho); aI_n, a^2 I_n) = \chi_+(\varphi,\psi)L_1^\lambda(\sigma; aI_n, a^2 I_n)$ for suitable $a \in \Sigma$ and $(-\lambda)$-form (H,φ,ψ). This completes the proof of (6.15) and shows we have a well-defined map $\mathscr{L}_1^\lambda\colon W_1^\lambda(B) \to W_1^\lambda(B/A)$ which is easily seen to be a homomorphism.

For §8 we will need the following proposition, which "reverses" (6.15).

(6.20) <u>Proposition</u>: Let $L_1^\lambda(\sigma;\nu,\eta) = \theta \approx \theta'$ in $F_1^\lambda(B/A)$, for some $\sigma \in U_{2n}^\lambda(B)$. Then there is a nonnegative integer r, $D \in \langle\{X_+(\rho), X_-(\tau), H(\epsilon) \mid \rho,\tau \in S_{-\lambda}(B^n),\ \epsilon \in GL_n(B),\ n \in \mathbb{Z}^+\}\rangle$, and $\nu',\eta' \in M_{n+r}(A)$ such that

$$L_1^\lambda((\sigma \perp I_{2r})D;\nu',\eta') = \theta'.$$

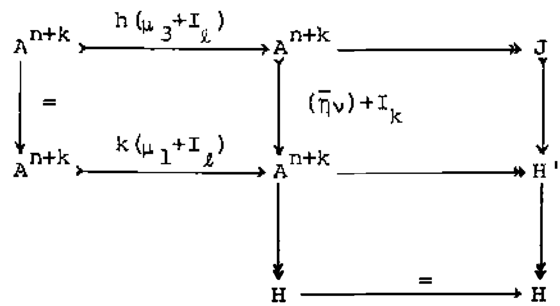
<u>Proof</u>: Suppose $\theta' = \sigma_E\theta$, where E is the extension $(E) = (J \overset{i}{\rightarrowtail} H \twoheadrightarrow H)$. By construction we have a resolution $A^n \overset{\bar{\eta}\nu}{\rightarrowtail} A^n \twoheadrightarrow H$. We need to find $\mu: A^n \rightarrowtail A^n$, $\mathrm{cok}(\mu) \cong J$, such that the sequence of cokernels,

$$\mathrm{cok}(\mu) \rightarrowtail \mathrm{cok}(\bar{\eta}\nu\mu) \twoheadrightarrow \mathrm{cok}(\bar{\eta}\nu)$$

is isomorphic to the extension E. Then it is clear that $\theta' = L_1^\lambda(\sigma;\nu\mu,\eta)$; actually, we work stably. Precisely, let $A^m \overset{\mu_1}{\rightarrowtail} A^m \overset{j}{\twoheadrightarrow} H'$ be any resolution of H' and let F be the pullback of j and i in

(6.21)
$$\begin{array}{ccccc} A^m & \overset{\mu_3}{\rightarrowtail} & F & \dashrightarrow\!\!\!\!\rightarrow & J \\ \big\downarrow = & & \big\downarrow \mu_2 & & \big\downarrow i \\ A^m & \overset{\mu_1}{\rightarrowtail} & A^m & \overset{j}{\twoheadrightarrow} & H' \end{array} .$$

Then F is stably free, so we may assume it is free, $F \cong A^m$. Since $\mathrm{cok}(\mu_2) \cong H$, by (1.7) there are integers k, ℓ with $m + \ell = n + k$ and automorphisms $k,h: A^{m+\ell} \to A^{n+k}$ such that $k(\mu_2 + I_\ell)h^{-1} = (\bar{\eta}\nu) + I_k$. Then we have the commutative diagram



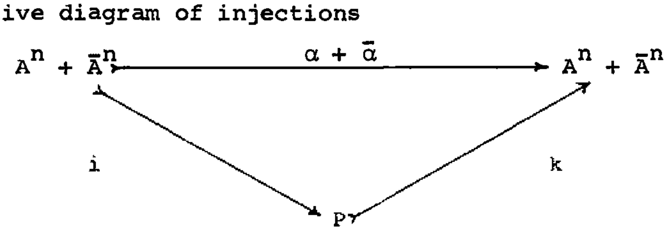
where the right vertical extension is isomorphic to E. Denoting $\mu = h(\mu_3 + I_\ell)$, we find $\theta' = \sigma_E\theta = L_1^\lambda((\sigma \perp I_{2\ell});(\nu + I_k)\mu,\eta + I_k)$. A similar argument applies in the cases $\sigma_E\theta' = \theta$, $\theta' = {}_E\sigma\theta$ and ${}_E\sigma\theta' = \theta$.

To complete the proof of (6.20) suppose $\theta' = \chi_-(\varphi,\psi)\theta$ and $L_1^\lambda(\sigma;\eta,\nu) = \theta$. Construct $P: A^n \to \bar{A}^n$ so that if in (1.17) we take $\mu = \bar{\eta}\nu$ and $\tau = \lambda(\bar{\nu}\eta)^{-1}\bar{P}$, the $(-\lambda)$-form (H,φ,ψ) is produced. Let $Q = \bar{\nu}\eta P$. Then $Q \in S_{-\lambda}(A^n)$ and by (6.16) $L_1^\lambda(\sigma H(\nu)X_-(Q);I_n,\bar{\nu}\eta)$ $= \chi_-(\varphi,\psi)L_1^\lambda(\sigma;\nu,\eta) = \theta'$. A similar argument applies in case $\theta' = \chi_+(\varphi,\psi)\theta$. Since $\chi_\pm(-\varphi,-\psi)\chi_\pm(\varphi,\psi)\theta = \theta$, the proof of (6.20) is complete.

§7 $\partial_1^\lambda: W_1^\lambda(B/A) \to W_0^\lambda(A)$

(7.1): Let $\theta = (K,H,\Delta,\kappa) \in F_1^\lambda(B/A)$ be given, let $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow H)$ be a short free resolution of H and consider the short exact sequence $A^n + \bar{A}^n \overset{\alpha+\bar{\alpha}}{\rightarrowtail} A^n + \bar{A}^n \overset{j}{\twoheadrightarrow} H + H^\wedge$. Setting $P = j^{-1}(K)$ we have the commutative diagram of injections

(7.2)



Let L' = the range of $\alpha + \bar{\alpha}$, identify the domain $A^n + \bar{A}^n$ of $\alpha + \bar{\alpha}$ with its image $L \subset L'$ and give $L (\equiv A^n + \bar{A}^n)$ the λ-hermitian form $h_L: L \times L \to A$ with matrix $\begin{pmatrix} 0 & \lambda\bar{\alpha} \\ \alpha & 0 \end{pmatrix}$. Then $L \otimes B$ inherits a (hyperbolic) λ-hermitian form and (as the notation already indicates) L' is the dual lattice to L in $L \otimes B$. Replacing (7.2) we have the sequence of lattices

(7.2)' $$L \subseteq P \subseteq L' \qquad \text{(compare (6.4))}$$

in which h_L induces a λ-hermitian form $g: P \times P \to B$ and $h_{L'}: L' \times L' \to B$, the latter having matrix $\begin{pmatrix} 0 & \lambda\bar{\alpha}^{-1} \\ \alpha^{-1} & 0 \end{pmatrix}$.

(7.3) <u>Proposition</u>: g takes values in A and is nonsingular.

<u>Proof</u>: The hyperbolic λ-form $(H + H^{\wedge}, \varphi_h, \psi_h)$ (in the definition of $\theta \in F_1^{\lambda}(B/A)$) is constructed by taking $\tau = h_{L'}$ and $\mu = \alpha + \bar{\alpha}$ in (1.18). Since the image of P under j in $H + H^{\wedge}$ is K, K is totally isotropic by assumption, and $g = h_{L'} \mid P \times P$, we have $g(P \times P) \subseteq A$. To show g is nonsingular it suffices to show that if P' is the dual lattice to P, then P = P'. Since g is A-valued on P it suffices to show $P' \subseteq P$. If not, let $x \bar{\in} P$ and $g(P,x) \subseteq A$. Then $j(x) \bar{\in} K$, but $\varphi_h(K, j(x)) \equiv 0$. This contradicts the nonsingularity of $((H + H^{\wedge})/K) \times K \to B/A$ induced by φ_h (1.13b). Hence P = P'.

(7.4) <u>Corollary</u>: We have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\quad k \quad} & A^n + \bar{A}^n \\ \Big\downarrow g^d & & \Big\downarrow D \\ P & \xrightarrow{\quad \bar{i} \quad} & \overline{A^n + \bar{A}^n} \end{array}$$

where k and i are in (7.2), and $D = {}_{(g_h)}d$, $\mathcal{H}(A^n) = (A^n + \bar{A}^n, g_h, q_h)$.

Proof: Left to the reader.

We have constructed P and g in a proposed λ-form (P,g,q). To construct q, let $i_1: P \to A^n$ and $i_2: P \to \bar{A}^n$ be inclusion $P \subset L' = A^n + \bar{A}^n$ followed by the coordinate projections. Let $x \in P$, $r: B \to B/A$, $r': B \to B/S_{-\lambda}(A)$ and set

$$q(x) = r'h_{L'}(i_1x, i_2x) - \varkappa(j(x)) \qquad \text{(compare (6.13))}. \tag{7.5}$$

q takes values in $A/S_{-\lambda}(A)$ since $rh_{L'}(i_1x,i_2x) = [jx,jx]$ (see (6.11) and the first sentence of the proof of (7.3)). We compute $r'g(x,x) = r'g(i_1x + i_2x, i_1x + i_2x) = r'h_{L'}(i_1x,i_2x) + r'\lambda\overline{h_{L'}(i_1x,i_2x)} = (r'h_{L'}(i_1x,i_2x) - \varkappa(jx)) + \lambda\overline{(r'h_L(i_1x,i_2x) - \varkappa(j(x))}$ ((1.16)) $= q(x) + \lambda q(x)$. The other properties (1.12a) are verified similarly so (P,g,q) is a λ-form. Set $I_0^\lambda(\theta;R) = (P,g,q) \in F_0^\lambda(A)$ and let $\wp_1^\lambda(\theta;R)$ denote its class in $W_0^\lambda(A)$.

(7.6) Proposition: For $\theta \in F_1^\lambda(B/A)$, $\wp_1^\lambda(\theta;R) \in W_0^\lambda(A)$ depends only on the equivalence class of θ in $W_1^\lambda(B/A)$.

Proof: (a) Choice of R, $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow H)$. If $(R') = (A^m \overset{\beta}{\rightarrowtail} A^m \twoheadrightarrow H)$ is another resolution of H then by (1.7) there are integers k and ℓ with $n + k = m + \ell$ and automorphisms $k_1: A^{n+k} \to A^{n+k}$, $k_2: A^{m+\ell} \to A^{m+\ell}$ with

$$k_1(\alpha + I_k)k_2 = \beta + I_\ell. \tag{7.7}$$

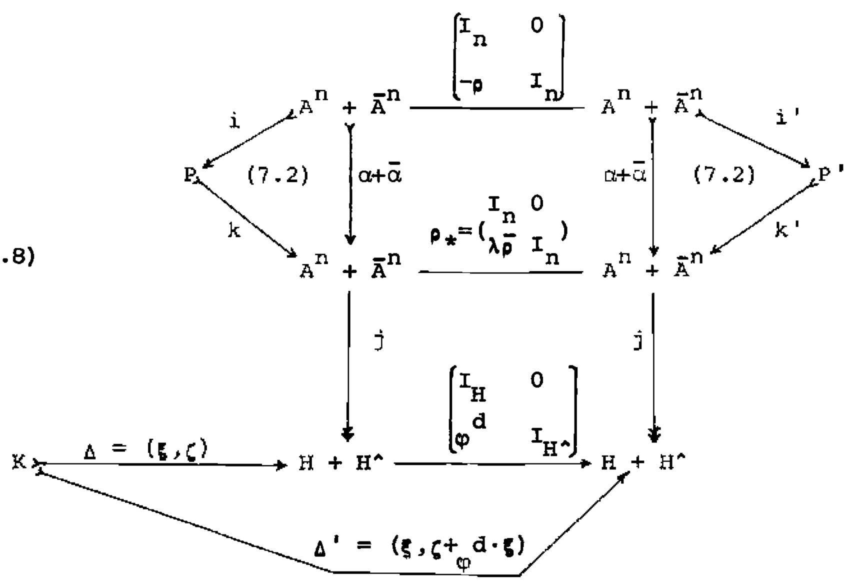
Then if $(R + A^k)$ denotes $(A^{n+k} \xrightarrow{\alpha+I_k} A^{n+k} \twoheadrightarrow H)$, $I_1^\lambda(\theta;R + A^k) = I_1^\lambda(\theta;R) \perp \mathcal{H}(A^k)$ while (7.7) implies $I_1^\lambda(\theta;R + A^k) = I_1^\lambda(\theta;R' + A^\ell)$

in $F_0^\lambda(A)$.

(b) <u>Choice of θ</u> within its equivalence class in $W_1^\lambda(B/A)$. Let $\theta = (K,H,\Delta,\varkappa)$.

(i) Let (H,φ,ψ) be a $(-\lambda)$-form and let $\chi_-(\varphi,\psi)\theta = \theta'$ $= (K,H,\Delta',\varkappa')$. We claim $I_1^\lambda(\theta';R) = I_1^\lambda(\theta;R)$ in $F_0^\lambda(A)$. Using (1.17) find τ: $A^n \times A^n \to B$ covering (H,φ,ψ) with respect to $(R) =$ $(A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow H)$. Then if $\lambda\bar{\rho} = \bar{\alpha}\tau$ and $(P',g',q') = I_1^\lambda(\theta';R)$ we have the diagram (compare (6.19)) in which the top two horizontal maps are isometries ([B,II.1.2])

(7.8)



In (7.8) $P = j^{-1}(\Delta K)$, $P' = j^{-1}(\Delta' K)$. By commutativity $\rho_*(P) = (P')$; but ρ_* is an isometry, so $I_1^\lambda(\theta';R) = I^\lambda(\theta;R)$. A similar argument shows that $I_1^\lambda(\chi_+(\varphi,\psi)\theta;R) = I_1^\lambda(\theta;R)$, for any $(-\lambda)$-form $(H^\wedge,\varphi,\psi)$.

(ii) <u>$\mathscr{s}_1^\lambda(\sigma_E\theta;R) = \mathscr{s}_1^\lambda(\theta;R)$</u>. Let $\theta' = \sigma_E\theta = (K,H,\Delta,\varkappa)$ where $(E) = (J \rightarrowtail H' \twoheadrightarrow H)$. Since $\mathscr{s}_1^\lambda(\theta';R)$ is independent of the resolution R (part (a)), we assume by the argument of (6.20) that if $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow H)$, there is β: $A^n \rightarrowtail A^n$, $\operatorname{cok}(\beta) \cong J$, such that the extension $\operatorname{cok}(\beta) \rightarrowtail \operatorname{cok}(\alpha\beta) \twoheadrightarrow \operatorname{cok}(\alpha)$ is isomorphic to E. Let $(R') = (A^n \overset{\alpha\beta}{\rightarrowtail} A^n \twoheadrightarrow H')$; then $I_1^\lambda(\theta;R) = I_1^\lambda(\sigma_E\theta;R')$. Indeed, let $L \subseteq P \subseteq L'$ be the lattices of (7.2)' associated to θ and R. Let $M = \beta(A^n) + \bar{A}^n \subseteq A^n + \bar{A}^n \equiv L$, and let M' be the dual lattice. Then $M \subseteq P \subseteq M'$ is the chain of lattices associated to $\sigma_E\theta$ and R', $M \subseteq L \subseteq P \subseteq L' \subseteq M$, and so $I_1^\lambda(\sigma_E\theta;R') = I_1^\lambda(\theta;R)$. Invariance of $\mathscr{D}_1^\lambda$ under other stabilizations is proved similarly. This completes the proof of (7.6).

§8 Exactness of the first five terms of (2.1)

(8.1): The following sequence of homomorphisms is exact:

$$W_1^\lambda(A) \xrightarrow{\mathcal{K}_1^\lambda} W_1^\lambda(B) \xrightarrow{\mathscr{L}_1^\lambda} W_1^\lambda(B/A) \xrightarrow{\mathscr{D}_1^\lambda} W_0^\lambda(A) \xrightarrow{\mathcal{K}_0^\lambda} W_0^\lambda(B).$$

Consequently, $W_1^\lambda(B/A)$ is a group.

<u>Proof</u>: (a) <u>Exactness at $W_1^\lambda(B)$</u> $\mathscr{L}_1^\lambda\mathcal{K}_1^\lambda = 0$ since if $\sigma \in U_{2n}^\lambda(A)$ we take $\nu = \eta = I_n$ in the construction of $\mathscr{L}_1^\lambda(\sigma)$, thus obtaining the zero formation. If $\sigma \in U_{2n}^\lambda(B)$ and $\mathscr{L}_1^\lambda(\sigma;\nu,\eta) = 0$ for some (hence any) choice of $\nu,\eta \in M_n(A)$, $L_1^\lambda(\sigma;\nu,\eta)$ may be converted to the zero formation by the operations of (1.34). By (6.20) each operation on $L_1^\lambda(\sigma;\nu,\eta)$ is realized by changes in ν, η and σ within its class in $W_1^\lambda(B/A)$. Hence we may find ν', η' and σ' with $[\sigma'] = [\sigma]$ in $W_1^\lambda(B)$ such that $L_1^\lambda(\sigma';\nu',\eta')$ = the zero formation. This easily implies $\nu',\eta' \in GL(A)$ so $\sigma' \in U_{2m}^\lambda(A)$ for some m.

(b) <u>Exactness at $W_1^\lambda(B/A)$</u> The lattice I in (6.4) becomes P in (7.2). I has the hyperbolic structure by construction so $\mathscr{D}_1^\lambda\mathscr{L}_1^\lambda = 0$. Suppose on the other hand that $I_1^\lambda(\theta;R) = \mathcal{H}(A^n) =$

$(A^n + \bar{A}^n, g_h, q_h)$, where $\theta = (K,H,\Delta,\kappa) \in F_1^\lambda(B/A)$ and $(R) = (A^n \overset{\mu}{\rightarrowtail} A^n \twoheadrightarrow H)$ In the notation of (7.2) there are form-preserving injections $(L,h_L) \overset{i}{\rightarrowtail} (A^n + \bar{A}^n, g_h, q_h) \overset{k}{\rightarrowtail} (L', h_{L'})$ so that under the identification $L \equiv A^n + \bar{A}^n$, i is given by a matrix

$$\begin{bmatrix} \alpha' & \beta \\ \gamma' & \delta \end{bmatrix} \quad , \ \alpha', \ \beta, \ \gamma', \ \delta \in M_n(A).$$

By (7.4), in which ${}_g d$ now equals D, k has matrix $\begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \lambda\bar{\gamma}' & \bar{\alpha}' \end{pmatrix}$ and so (7.2) $\begin{pmatrix} \bar{\delta} & \lambda\bar{\beta} \\ \lambda\bar{\gamma}' & \bar{\alpha}' \end{pmatrix} \begin{pmatrix} \alpha' & \beta \\ \gamma' & \delta \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix}$, $(R) = (A^n \overset{\mu}{\rightarrowtail} A^n \twoheadrightarrow H)$. Hence $\bar{\delta}(\alpha'\mu^{-1}) + \lambda\bar{\beta}(\gamma'\mu^{-1}) = I_n$; and $\overline{(\alpha'\mu^{-1})}(\gamma'\mu^{-1})$, $\bar{\beta}\delta \in S_{-\lambda}(B^n)$. Setting $\alpha = \alpha'\mu^{-1}$, $\gamma = \gamma'\mu^{-1}$ (1.2) shows that

$$\sigma = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in U_{2n}^\lambda(B),$$

By definition $L_1^\lambda(\sigma;\mu,I_n) = \theta$.

(c) <u>Exactness at $W_0^\lambda(A)$</u> In the construction of ∂_1^λ (7.1), the lattice L supports a form h_L which becomes hyperbolic after tensoring up to B. Since for $\theta = (K,H,\Delta,\kappa) \in F_1^\lambda(B/A)$ and R a resolution of H, $(L,h_L) \otimes B \cong L_1^\lambda(\theta;R) \otimes B$ is an isometry, $\mathcal{K}_1^\lambda \partial_1^\lambda(\theta) = 0$. Now let (P,g,q) be a λ-form, P a free A-module such that $(P,g,q) \otimes B$ is isometric to $\mathcal{H}(B^n)$. This implies there is an inclusion of $L \cong A^n + \bar{A}^n$ in P, preserving forms, where L supports a form h_L with matrix $\begin{pmatrix} 0 & \lambda\bar{\alpha} \\ \alpha & 0 \end{pmatrix}$, for some $\alpha \in M_n(A)$. Referring to the construction I_1^λ, we find $\theta = (K,H,\Delta,\kappa) \in F_1^\lambda(B/A)$ such that $I_1^\lambda(\theta;R) = (P,g,q)$ where $(R) = (A^n \overset{\alpha}{\rightarrowtail} A^n \twoheadrightarrow H)$. Details are left to the reader.

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