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(Eds.)

Algebraic Geometry III

Complex Algebraic Varieties
Algebraic Curves and Their Jacobians

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I. Complex Algebraic Varieties: Periods of Integrals and Hodge Structures

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Translated from the Russian
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Introduction

Starting with the end of the seventeenth century, one of the most interesting directions in mathematics (attracting the attention as J. Bernoulli, Euler, Jacobi, Legendre, Abel, among others) has been the study of integrals of the form

$$A_w(\tau) = \int_{\tau_0}^{\tau} \frac{dz}{w},$$

where w is an algebraic function of z. Such integrals are now called *abelian.*

Let us examine the simplest instance of an abelian integral, one where w is defined by the polynomial equation

$$w^2 = z^3 + pz + q, \tag{1}$$

where the polynomial on the right hand side has no multiple roots. In this case the function A_w is called an *elliptic integral.* The value of A_w is determined up to $m\nu_1 + n\nu_2$, where ν_1 and ν_2 are complex numbers, and m and n are integers. The set of linear combinations $m\nu_1 + n\nu_2$ forms a lattice $H \subset \mathbb{C}$, and so to each elliptic integral A_w we can associate the torus $\mathbb{C}/H$.

On the other hand, equation (1) defines a curve in the affine plane $\mathbb{C}^2 = \{(z, w)\}$. Let us complete $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ by the addition of the "line at infinity", and let us also complete the curve defined by equation (1). The result will be a nonsingular closed curve $E \subset \mathbb{P}^2$ (which can also be viewed as a Riemann surface). Such a curve is called an *elliptic curve.*

It is a remarkable fact that the curve E and the torus $\mathbb{C}/H$ are isomorphic Riemann surfaces. The isomorphism can be given explicitly as follows.

Let $\wp(z)$ be the Weierstrass function associated to the lattice $H \subset \mathbb{C}$.

$$\wp = \frac{1}{z^2} + \sum_{h \in H, h \neq 0} \left[\frac{1}{(z-2h)^2} - \frac{1}{(2h)^2} \right].$$

It is known that $\wp(z)$ is a doubly periodic meromorphic function with the period lattice H. Further, the function $\wp(z)$ and its derivative $\wp'(z)$ are related as follows:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{2}$$

for certain constants g_2 and g_3 which depend on the lattice H. Therefore, the mapping $z \to (\wp(z), \wp'(z))$ is a meromorphic function of $\mathbb{C}/H$ onto the compactification $E' \subset \mathbb{P}^2$ of the curve defined by equation (2) in the affine plane. It turns out that this mapping is an isomorphism, and furthermore, the projective curves E and E' are isomorphic!

Let us explain this phenomenon in a more invariant fashion. The projection $(z, w) \to z$ of the affine curve defined by the equation (1) gives a double covering $\pi : E \to \mathbb{P}^1$, branched over the three roots z_1, z_2, z_3 of the polynomial $z^3 + pz + q$ and the point ∞.

The differential $\omega = dz/2w$, restricted to E is a holomorphic 1-form (and there is only one such form on an elliptic curve, up to multiplication by constants). Viewed as a C^∞ manifold, the elliptic curve E is homeomorphic to the product of two circles $S^1 \times S^1$, and hence the first homology group $H_1(E, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let the generators of $H_1(E, \mathbb{Z})$ be γ_1 and γ_2. The lattice H is the same as the lattice $\left\{m \int_{\gamma_1} \omega + n \int_{\gamma_2} \omega\right\}$. Indeed, the elliptic integral A_w is determined up to numbers of the form $\int_l \frac{1}{\sqrt{z^3+pz+q}}$, where l is a closed path in $\mathbb{C}\setminus\{z_1, z_2, z_3\}$. On the other hand

$$\int_l \frac{1}{\sqrt{z^3 + pz + q}} = \int_\gamma \omega,$$

where γ is the closed path in E covering l twice.

The integrals $\int_{\gamma_i} \omega$ are called periods of the curve E. The lattice H is called the *period lattice.* The discussion above indicates that the curve E is uniquely determined by its period lattice.

This theory can be extended from elliptic curves (curves of genus 1) to curves of higher genus, and even to higher dimensional varieties.

Let X be a compact Riemann surface of genus g (which is the same as a nonsingular complex projective curve of genus g). It is well known that all Riemann surfaces of genus g are topologically the same, being homeomorphic to the sphere with g handles. They may differ, however, when viewed as complex analytic manifolds. In his treatise on abelian functions (see de Rham [1955]), Riemann constructed surfaces (complex curves) of genus g by cutting and pasting in the complex plane. When doing this he was concerned about the periods of abelian integrals over various closed paths. Riemann called those periods (there are $3g - 3$) *moduli.* These are continuous complex parameters which determine the complex structure on a curve of genus g.

One of the main goals of the present survey is to introduce the reader to the ideas involved in obtaining these kinds of parametrizations for algebraic varieties. Let us explain this in greater detail.

On a Riemann surface X of genus g there are exactly g holomorphic 1-forms linearly independent over $\mathbb{C}$. Denote the space of holomorphic 1-forms on X by $H^{1,0}$, and choose a basis $\omega = (\omega_1, \dots, \omega_g)$ for $H^{1,0}$. Also choose a basis $\gamma = (\gamma_1, \dots, \gamma_{2g})$ for the first homology group $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then the numbers

$$\Omega_{ij} = \int_{\gamma_j} \omega_i$$

are called the *periods* of X. They form the *period matrix* $\Omega = (\Omega_{ij})$. This matrix obviously depends on the choice of bases for $H^{1,0}$ and $H_1(X, \mathbb{Z})$. It turns out (see Chapter 3, Section 1), that the periods uniquely determine the curve X. More precisely, let X and X' be two curves of genus g. Suppose

ω and ω' are bases for the spaces of holomorphic differentials on X and X', respectively, and γ and γ' be are bases for $H_1(X, \mathbb{Z})$ and $H_1(X', \mathbb{Z})$ such that there are equalities

$$(\gamma_i \cdot \gamma_j)_X = (\gamma_i' \cdot \gamma_j')_{X'}$$

between the intersection numbers of γ and γ'. Then, if the period matrices of X and X' with respect to the chosen bases are the same, then the curves themselves are isomorphic. This is the classical theorem of Torelli.

Now, let X be a non-singular complex manifold of dimension $d > 1$. The complex structure on X allows us to decompose any complex-valued C^∞ differential n-form ω into a sum

$$\omega = \sum_{p+q=n} \omega^{p,q}$$

of components of type (p,q). A form of type (p,q) can be written as

$$\omega^{p,q} = \sum_{(I,J)=(i_1,\ldots,i_p,j_1,\ldots,j_q)} h_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

If X is a projective variety (and hence a Kähler manifold; see Chapter 1, Section 7) , then this decomposition transfers to cohomology:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = \bar{H}^{q,p}. \tag{3}$$

This is the famous Hodge decomposition (Hodge structure of weight n on $H^n(X)$, see Chapter 2, Section 1). It allows us to define the periods of a variety X analogously to those for a curve. Namely, let X_0 be some fixed non-singular projective variety, and $H = H^n(X_0, \mathbb{Z})$. Let X be some other projective variety, diffeomorphic to X_0, and having the same Hodge numbers $h^{p,q} = \dim H^{p,q}(X_0)$. Fix a $\mathbb{Z}$-module isomorphism

$$\phi : H^n(X, \mathbb{Z}) \simeq H.$$

This isomorphism transfers the Hodge structure (3) from $H^n(X, \mathbb{C})$ onto $H_\mathbb{C} = H \otimes_\mathbb{Z} \mathbb{C}$. We obtain the Hodge filtration

$$\{0\} = F^{n+1} \subseteq F^n \subseteq \ldots \subseteq F^0 = H_\mathbb{C}$$

of the space $H_\mathbb{C}$, where

$$F^p = H^{n,0} \oplus \ldots \oplus H^{p,n-p}, F^{n+1} = \{0\}.$$

This filtration is determined by the variety X up to a $GL(H, \mathbb{Z})$ action, due to the freedom in the choice of the map ϕ. The set of filtrations of a linear space $H_\mathbb{C}$ by subspaces F^p of a fixed dimension f^p is classified by the points of the complex projective variety (the *flag manifold*) $F = F(f^n, \ldots, f^1; H_\mathbb{C})$. The simplest flag manifold is the Grassmanian $G(k,n)$ of k-dimensional linear subspaces in $\mathbb{C}^n$. The conditions which must be satisfied by the subspaces $H^{p,q}$ forming a Hodge structure (see Chapter 2, Section 1) define a complex submanifold D of F, which is known as the *classifying space* or the *space of period matrices.*

This terminology is easily explained. Let $h^{p,q} = \dim H^{p,q}$. Further, let the basis of $H^{p,q}$ be $\{\omega_j^{p,q}\}$, for $j = 1, \ldots, h^{p,q}$, and let the basis modulo torsion of $H_n(X, \mathbb{Z})$ be $\gamma_1, \ldots, \gamma_b$. Consider the matrix whose rows are

$$I_j^{p,q} = \left(\int_{\gamma_1} \omega_j^{p,q}, \ldots, \int_{\gamma_b} \omega_j p, q \right).$$

This is the period matrix of X. There is some freedom in the choice of the basis elements $\omega_j^{p,q}$, but, in any event, the Hodge structure is determined uniquely if the basis of H is fixed, and in general the Hodge structure is determined up to the action of the group Γ of automorphisms of the $\mathbb{Z}$-module H. Thus, if $\{X_i\}$, $i \in A$ is a family of complex manifolds diffeomorphic to X_0 and whose Hodge numbers are the same, we can define the *period mapping*

$$\Phi : A \to \Gamma \backslash D.$$

We see that we can associate to each manifold X a point of the classifying space D, defined up to the action of a certain discrete group. One of the fundamental issues considered in the present survey is the inverse problem — to what extent can we reconstruct a complex manifold X from the point in classifying space. This issue is addressed by a number of theorems of Torelli type (see Chapter 2, Section 5 for further details).

A positive result of Torelli type allows us, generally speaking, to construct a complete set of continuous invariants, uniquely specifying a manifold with the given set of discrete invariants. Let us look at the simplest example – that of an elliptic curve E. The two-dimensional vector space $H_{\mathbb{C}} = H^1(E, \mathbb{C})$ is equipped with the non-degenerate pairing

$$(\mu, \eta) = \int_E \mu \wedge \eta.$$

Restricting this pairing to $H = H^1(E, \mathbb{Z})$ gives a bilinear form

$$Q_H : H \times H \to \mathbb{Z},$$

dual to the intersection form of 1-cycles on E. We can, furthermore, pick a basis in H, so that

$$Q_H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$H_{\mathbb{C}}$ is also equipped with the Hodge decomposition

$$H_{\mathbb{C}} = \mathbb{C}\omega + \mathbb{C}\bar{\omega},$$

where ω is a non-zero holomorphic differential on E. It is easy to see that

$$\sqrt{-1}(\omega, \bar{\omega}) > 0,$$

and so in the chosen basis $\omega = (\alpha, \beta)$, where

$$\sqrt{-1}(\beta\bar{\alpha} - \alpha\bar{\beta}) > 0. \tag{4}$$

The form ω is determined up to constant multiple. If we pick $\omega = (\lambda, 1)$, then condition (4) means that $\operatorname{Im}\lambda > 0$, and so the space of period matrices D is simply the complex upper half-plane:

$$D = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Now let us consider the family of elliptic curves

$$E_\lambda = \mathbb{C}/\{\mathbb{Z}\lambda + \mathbb{Z}\}, \quad \lambda \in D.$$

This family contains all the isomorphism classes of elliptic curves, and two curves E_λ and $E_{\lambda'}$ are isomorphic if and only if

$$\lambda' = \frac{a\lambda + b}{c\lambda + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Thus, the set of isomorphism classes of elliptic curves is in one-to-one correspondence with the points of the the set $A = \Gamma\backslash D$. The period mapping

$$\Phi : A \to \Gamma\backslash D$$

is then the identity mapping. Indeed, the differential dz defines a holomorphic 1-form in each E_λ.

If γ_1, γ_2 is the basis of $H_1(E_\lambda, \mathbb{Z})$ generated by the elements $\lambda, 1$ generating the lattice $\{\mathbb{Z}\lambda + \mathbb{Z}\}$ then the periods are simply

$$\left(\int_{\gamma_1} \omega, \int_{\gamma_2} \omega\right) = (\lambda, 1).$$

The existence of Hodge structures on the cohomology of non-singular projective varieties gives a lot of topological information (see Chapter 1, Section 7). However, it is often necessary to study singular and non-compact varieties, which lack a classical Hodge structure. Nonetheless, Hodge structures can be generalized to those situations also. These are the so-called *mixed* Hodge structures, invented by Deligne in 1971. We will define mixed Hodge structures precisely in Chapter 4, Section 1, but now we shall give the simplest example leading to the concept of a mixed Hodge structure.

Let X be a complete algebraic curve with singularities. Let S be the set of singularities on X and for simplicity let us assume that all points of S are simple singularities, with distinct tangents. The singularities of X can be resolved by a normalization $\pi : \bar{X} \to X$. Then, for each point $s \in S$ the pre-image $\pi^{-1}(s)$ consists of two points x_1 and x_2, and outside the singular set the morphism

$$\pi : \bar{X} \setminus \pi^{-1}(S) \to X \setminus S$$

is an isomorphism.

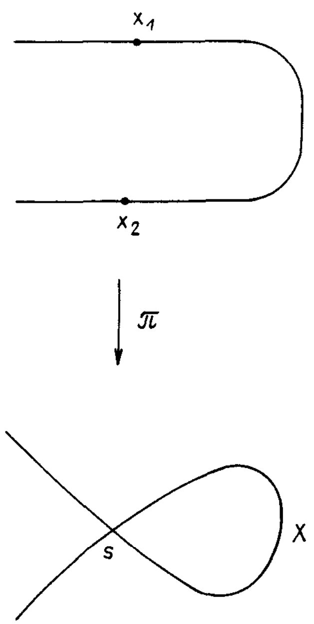


Fig. 1

For a locally constant sheaf $\mathbf{C}_X$ on X we have the exact sequence

$$0 \to \mathbf{C}_X \to \pi_* \mathbf{C}_{\bar{X}} \to \mathbf{C}_S \to 0,$$

which induces a cohomology exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(X, \mathbf{C}_S) & \to & H^1(X, \mathbf{C}_X) & \to & H^1(X, \pi_* \mathbf{C}_{\bar{X}}) & \to & 0 \\ & & \Vert\wr & & & & \Vert\wr & & \\ & & H^0(S, \mathbf{C}_S) & & & & H^1(\bar{X}, \mathbf{C}_{\bar{X}}) & & \end{array}$$

This sequence makes it clear that $H^1(\bar{X}, \mathbf{C}_X)$ is equipped with the filtration $0 \subset H^0(S, \mathbf{C}_S) = W_0 \subset H^1(X, \mathbf{C}_X) = W_1$. The factors of this filtration are equipped with Hodge structures in a canonical way – W_0 with a Hodge structure of weight 0, and W_1/W_0 with a Hodge structure of weight 1, induced by the inclusion of W_1/W_0 into $H^1(\bar{X}, \mathbf{C}_X)$.

Even though mixed Hodge structures have been introduced quite recently, they helped solve a number of difficult problems in algebraic geometry – the problem of invariant cycles (see Chapter 4, Section 3) and the description of degenerate fibers of families of of algebraic varieties being but two of the examples. More beautiful and interesting results will surely come.

Here is a brief summary of the rest of this survey.

In the first Chapter we attempt to give a brief survey of classical results and ideas of algebraic geometry and the theory of complex manifolds, necessary for the understanding of the main body of the survey. In particular, the first three sections give the definitions of classical algebraic and complex analytic geometry and give the results *GAGA (Géometrie algébrique et géométrie analytique)* on the comparison of algebraic and complex analytic manifolds.

In Sections 4, 5, and 6 we recall some complex analytic analogues of some standard differential-geometric constructions (bundles, metrics, connections).

Section 7 is devoted to classical Hodge theory.

Sections 8, 9, and 10 contain further standard material of classical algebraic geometry (divisors and line bundles, characteristic classes, extension formulas, Kodaira's vanishing theorem, Lefschetz' theorem on hyperplane section, monodromy, Lefschetz families).

Chapter 2 covers fundamental concepts and basic facts to do with the period mapping, to wit:

Section 1 introduces the classifying space D of polarized Hodge structures and explains the correspondence between this classifying space and a polarized algebraic variety. We study in some depth examples of classifying spaces associated to algebraic curves, abelian varieties and Kähler surfaces. We also define certain naturally arising sheaves on D.

In Section 2 we introduce the complex tori of Griffiths and Weil associated to a polarized Hodge structure. We also define the Abel-Jacobi mapping, and study in detail the special case of the Albanese mapping.

In Section 3 we define the period mapping for projective families of complex manifolds. We show that this mapping is holomorphic and horizontal.

In Section 4 we introduce the concept of variation of Hodge structure, which is a generalization of the period mapping.

In Section 5 we study four kinds of Torelli problems for algebraic varieties. We study the infinitesimal Torelli problem in detail, and give Griffiths' criterion for its solvability.

In Section 6 we study infinitesimal variation of Hodge structure and explain its connection with the global Torelli problem.

In Chapter 3 we study some especially interesting concrete results having to do with the period mapping and Torelli-type results.

In Section 1 we construct the classifying space of Hodge structures for smooth projective curves. We prove the infinitesimal Torelli theorem for non-hyperelliptic curves and we sketch the proof of the global Torelli theorem for curves.

In Section 2 we sketch the proof of the global Torelli theorem for a cubic threefold.

In Section 3 we study the period mapping for K3 surfaces. We prove the infinitesimal Torelli theorem. We construct the modular space of marked K3 surfaces. We also sketch the proof of the global Torelli theorem for K3 surfaces. We study elliptic pencil, and we sketch the proof of the global Torelli theorem for them.

In Section 4 we study hypersurfaces in $\mathbb{P}^n$. We prove the local Torelli theorem, and sketch the proof of the global Torelli theorem for a large class of hypersurfaces.

Chapter 4 is devoted to mixed Hodge structures and their applications.

Section 1 gives the basic definitions and survey the fundamental properties of mixed Hodge structures.

Sections 2 and 3 are devoted to the proof of Deligne's theorem on the existence of mixed Hodge structures on the cohomology of an arbitrary complex algebraic variety in the two special cases: for varieties with normal crossings and for non-singular incomplete varieties.

Section 4 gives a sketch of the proof of the invariant cycle theorem.

Section 5 computes Hodge structure on the cohomology of smooth hypersurfaces in $\mathbb{P}^n$.

Finally, in Section 5 we give a quick survey of some further developments of the theory of mixed Hodge structures, to wit, the period mapping for mixed Hodge structures, and mixed Hodge structures on the homotopy groups of algebraic varieties.

In Chapter 5 we study the theory of degenerations of families of algebraic varieties.

Section 1 contains the basic concepts of the theory of degenerations.

Section 2 gives the definition of the limiting mixed Hodge structure on the cohomology of the degenerate fiber (introduced by Schmid).

In Section 3 we construct the exact sequence of Clemens-Schmid, relating the cohomology of degenerate and non-degenerate fibers of a one-parameter family of Kähler manifolds.

Sections 4 and 5 are devoted to the applications of the Clemens-Schmid exact sequence to the degenerations of curves and surfaces.

In Section 6 we study the degeneration of K3 surfaces. We conclude that the period mapping is an epimorphism for K3 surfaces.

In conclusion, a few words about the prerequisites necessary to understand this survey. Aside from the standard university courses in algebra and differential geometry it helps to be familiar with the basic concepts of algebraic topology (Poincaré duality, intersection theory), homological algebra, sheaf theory (sheaf cohomology and hypercohomology, spectral sequences – see references Cartan–Eilenberg [1956], Godement [1958], Grothendieck [1957], Griffiths–Harris [1978]), theory of Lie groups and Lie algebras (see Serre [1965]), and Riemannian geometry (Postnikov [1971]).

We have tried to either define or give a reference for all the terms and results used in this survey, in an attempt to keep it as self-contained as possible.

Chapter 1
Classical Hodge Theory

§1. Algebraic Varieties

Let us recall some definitions of algebraic geometry.

1.1. Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) | z_i \in \mathbb{C}\}$ be the n-dimensional affine space over the complex numbers. *An algebraic set* in $\mathbb{C}^n$ is a set of the form

$$V(f_1, \ldots, f_m) = \{z \in \mathbb{C}^n | f_1(z) = \ldots = f_m(z) = 0\}.$$

where $f_i(z)$ lie in the ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of polynomials in n variables over $\mathbb{C}$. An algebraic set of the form $V(f_1)$ is a *hypersurface* in $\mathbb{C}^n$, assuming that $f_1(z)$ is not a constant.

It is clear that if $f(z)$ lies in the ideal $I = (f_1, \ldots, f_m)$ of $\mathbb{C}[z]$ generated by $f_1(z), \ldots, f_m(z)$ then $f(a) = 0$ for all $a \in V(f_1, \ldots, f_m)$. Thus, to each algebraic set $V = V(f_1, \ldots, f_m)$ we can associate an ideal $I(V) \subset \mathbb{C}[z]$, defined by

$$I(V) = \{f \in \mathbb{C}[z] | f(a) = 0, a \in V\}.$$

The ideal $I(V)$ is a finitely generated ideal, and so by Hilbert's *Nullstellensatz* (Van der Waerden [1971]) $I(V) = \sqrt{(f_1, \ldots, f_m)}$,, where $\sqrt{J} = \{f \in \mathbb{C}[z] | f^k \in J \text{ for some } k \in \mathbb{N}\}$ is the radical of J.

The ring $\mathbb{C}[V] = \mathbb{C}[z]/I(V)$ is the *ring of regular functions* over the algebraic set V. This ring coincides with the ring of functions on V which are restrictions of polynomials over $\mathbb{C}^n$.

1.2. It is easy to see that the union of any finite number of algebraic sets and the intersection of any number of algebraic sets is again an algebraic set, and so the collection of algebraic sets in $\mathbb{C}^n$ satisfies the axioms of the collection of closed sets of some topology. This is the so-called *Zariski topology.* The Zariski topology in $\mathbb{C}^n$ induces a topology on algebraic sets $V \subset \mathbb{C}^n$, and this is also called the Zariski topology. The neighborhood basis of the Zariski topology on V is the set of open sets of the form $U_{f_1,\ldots,f_k} = \{a \in V | f_1(a) \neq 0, \ldots, f_k(a) \neq 0, f_1, \ldots, f_k \in \mathbb{C}[V]\}$.

Let $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ be two algebraic sets. A map $f : V_1 \to V_2$ is called a *regular mapping* or a *morphism* if there exists a set of m regular functions $f_1, \ldots, f_m \in \mathbb{C}[V_1]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$ for all $a \in V_1$. Obviously a regular mapping is continuous with respect to the Zariski topology. It is also easy to check that defining a regular mapping $f : V_1 \to V_2$ is equivalent to defining a homomorphism of rings $f^* : \mathbb{C}[V_1] \to \mathbb{C}[V_2]$, which transforms the coordinate functions $z_i \in \mathbb{C}[V_2]$ into $f_i \in \mathbb{C}[V_1]$.

Two algebraic sets V_1 and V_2 are called *isomorphic* if there exists a regular mapping $f : V_1 \to V_2$ which possesses a regular inverse $f^{-1} : V_2 \to V_1$.

Alternatively, V_1 and V_2 are isomorphic whenever the rings $\mathbb{C}[V_1]$ and $\mathbb{C}[V_2]$ are isomorphic.

Evidently, for any algebraic set V, the ring of regular functions $\mathbb{C}[V]$ is a finitely generated (over $\mathbb{C}$) algebra. Conversely, if a commutative ring K is a finitely generated algebra over $\mathbb{C}$ without nilpotent elements, then K is isomorphic to $\mathbb{C}[V]$ for some algebraic set V. Indeed, if $z_1, \ldots, z_n$ are generators of K, then $K \simeq \mathbb{C}[z_1, \ldots, z_n]/I$, where I is the ideal of relations. Thus, $K \simeq \mathbb{C}[V]$, where $V = \{z \in \mathbb{C}^n | f(z) = 0, f \in I\}$. In other words, the category of algebraic sets is equivalent to that of finitely generated algebras over $\mathbb{C}$ without nilpotent elements.

1.3. A product of algebraic sets $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$ is the set

$$V \times W = \{(z_1, \ldots, z_{n+m}) \in \mathbb{C}^{n+m} | (z_1, \ldots, z_n) \in V, (z_{n+1}, \ldots, z_{n+m}) \in W\}.$$

It is easy to check that $V \times W$ is an algebraic set, and if $f_i(z_1, \ldots, z_n)$, $1 \leq i \leq k$ are generators of $I(V)$ and $g_j(z_1, \ldots, z_m)$, $1 \leq j \leq s$ are generators of $I(W)$, then $V \times W$ is defined by the equations $f_i(z_1, \ldots, z_n) = 0, g_j(z_{n+1}, \ldots, z_{n+m}) = 0$.

1.4. An algebraic set V is called *irreducible* if $I(V)$ is a prime ideal. An algebraic set V is irreducible if V cannot be represented as a union of closed subsets $V_1 \cup V_2$ such that $V \neq V_1, V \neq V_2, V_1 \neq V_2$. It can be shown (Shafarevich [1972]) that every algebraic set is a union of a finite number of irreducible algebraic sets.

If V is an irreducible algebraic set, then $\mathbb{C}[V]$ is an integral domain. Denote the field of quotients of $\mathbb{C}[V]$ by $\mathbb{C}(V)$. This field is called the *field of rational functions* over V, and the transcendence degree of $\mathbb{C}(V)$ over $\mathbb{C}$ is the *dimension* of V, and is denoted by $\dim V$. Elements of $\mathbb{C}(V)$ can be represented as fractions $f(z)/g(z)$ where $f(z), g(z) \in \mathbb{C}(z)$ and $g(z)$ doesn't vanish on all of V. Thus the elements of $\mathbb{C}(V)$ can be viewed as functions defined on a Zariski-open subset of V.

For each point $a \in V$ of an irreducible algebraic set V we define the *local ring* $\mathcal{O}_{V,a} \subset \mathbb{C}(V)$:

$$\mathcal{O}_{V,a} = \left\{ \frac{f}{g} \in \mathbb{C}(V) | f, g \in \mathbb{C}[V], g(a) \neq 0 \right\}.$$

The maximal ideal $m_{V,a} \subset \mathcal{O}_{V,a}$ is

$$m_{V,a} = \left\{ \frac{f}{g} \in \mathbb{C}(V) | f, g \in \mathbb{C}[V], f(a) = 0, g(a) \neq 0 \right\}.$$

In general, for any point a of an arbitrary (not necessarily irreducible) algebraic set V we can also define the local ring as a ring of formal fractions:

$$\mathcal{O}_{V,a} = \left\{ \frac{f}{g} | f, g \in \mathbb{C}[V], g(a) \neq 0 \right\}.$$

with the usual arithmetic operations. Two fractions f_1/g_1 and f_2/g_2 are considered equal if there exists a function $h \in \mathbb{C}[V], h(a) \neq 0$ such that $h(f_1g_2 - f_2g_1) = 0$.

The local rings $\mathcal{O}_{V,a}$ are the stalks of a sheaf of rings $\mathcal{O}_V$ over V, defined as follows. The sections of the sheaf $\mathcal{O}_V$ over an open set $U \subset V$ are fractions $f/g, f, g \in \mathbb{C}[V]$, such that for every $a \in U$ there exists a fraction $f_a/g_a, g_a(a) \neq 0$, which is equal to f/g at a. That is, there exists a function $h_a \in \mathbb{C}[V], h_a(a) \neq 0$, such that

$$h_a(fg_a - f_ag) = 0.$$

This sheaf of rings $\mathcal{O}_V$ is called the *structure sheaf*, and its sections over an open set U are called functions regular over U. Hilbert's *Nullstellensatz* implies that the ring of global sections of $\mathcal{O}_V$ coincides with $\mathbb{C}[V]$.

1.5. To each point $a = (a_1, \dots, a_n) \in V \subset \mathbb{C}^n$ we associate a linear space called the *tangent space* $T_{V,a}$. The tangent space $T_{V,a}$ is defined to be the subspace of $\mathbb{C}^n$, defined by the system of equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a)(z_i - a_i) = 0$$

for all $f \in I(V)$. It can be shown that $\dim T_{V,a} \geq \dim V$ for an irreducible V, and furthermore there is a non-empty Zariski-open subset $U \subset V$, such that $\dim T_{V,a} = \dim V$ for all $a \in U$. This set U is defined to be the set of $a \in V$ where the rank of the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)$ is maximal (where $I(V) = (f_1, \dots, f_m)$).

Let V_i be an irreducible component of an algebraic set V. The points $a \in V_i$ for which $\dim T_{V,a} = \dim V_i$ are called *non-singular* (or *smooth*) points of V.

The tangent space $T_{V,a}$ can be defined an yet another way, as the dual space of the $\mathbb{C}$-linear space $m_{V,a}/m_{V,a}^2$. Indeed, for every function $h = f(z)/h(z) \in \mathcal{O}_{V,a}$ define the differential

$$d_ah = \sum_{i=1}^{n} \frac{\partial h}{\partial z_i}(z_i - a_i).$$

This differential satisfies the conditions

$$d_a(h_1 + h_2) = d_ah_1 + d_ah_2 \tag{1}$$

and

$$d_a(h_1h_2) = h_1(a)d_ah_2 + h_2(a)d_ah_1. \tag{2}$$

Since $d_a(c) = 0$ for c a constant function, the differential d_a is actually determined by its values on $m_{V,a}$. For every $h \in m_{V,a}$ d_ah determines a linear function $d_ah : T_{V,a} \to \mathbb{C}$. From equation (2) it follows that $d_ah = 0$ for any $h \in m_{V,a}^2$. Thus d_a defines a mapping $d_a : m_{V,a}/m_{V,a}^2 \to T_{V,a}^*$. This map is easily checked to be an isomorphism.

Let $V \in \mathbb{C}^n$. Consider an algebraic set $T_V \in \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ defined by the equations

$$f(z_1, \dots z_n) = 0,$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z_1, \dots, z_n)(z_{i+n} - z_i) = 0,$$

for $f \in I(V)$. Let π be the projection map $\pi : T_V \to V$, where $\pi(z_1, \dots, z_{2n}) = (z_1, \dots, z_n)$. Evidently $\pi(T_V) = V$ and $\pi^{-1}(a) = T_{V,a}$ for any $a \in V$. Thus T_V fibers over V, with fibers being just the tangent spaces at the points $a \in V$. The algebraic set T_V is the *tangent bundle* to V.

1.6. Algebraic Varieties The concept of *algebraic variety* is central to algebraic geometry, and there are several ways to define this. The most general approach is that of Grothendieck (see Shafarevich [1972], Hartshorne [1977]), where an algebraic variety is defined to be a reduced separable scheme of finite type over a field k. Since we will not need such generality, we will follow A. Weil, and define an algebraic variety to be a ringed space, glued together from algebraic sets. Recall that a *ringed space* is an ordered pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ together with a family of ring homomorphisms $f_U^* : \mathcal{O}_Y|U \to \mathcal{O}_X|f^{-1}(U)$ for all open sets $U \subset Y$, which agree on intersections of open sets.

An *affine variety* is a ringed space $(V, \mathcal{O}_V)$ where V is an algebraic set and $\mathcal{O}_V$ is its structure sheaf. Note that for an affine variety V, the open sets (which are a neighborhood basis in the Zariski topology) of the form

$$U_f = \{z \in V | f(z) \neq 0\},$$

where f is a function regular on V are affine varieties. Indeed, if $V \subset \mathbb{C}^n$, then U_f is isomorphic to the algebraic set in $\mathbb{C}^{n+1}$ defined by the equations $z_{n+1} f(z_1, \dots, z_n) = 1$ and $f_i(z_1, \dots, z_n) = 0$, where $f_i(z) \in I(V) \subset \mathbb{C}[z_1, \dots, z_n]$.

Definition. A ringed space $(X, \mathcal{O}_X)$ is an *algebraic variety* if X can be covered by a finite number of open everywhere-dense sets V_i, so that $(V_i, \mathcal{O}_X|V_i)$ are isomorphic to affine varieties and X is separable: the image of X under the diagonal embedding $\Delta = (\mathrm{id}, \mathrm{id}) : X \to X \times X$ is closed in $X \times X$. (The definition of a product of affine algebraic sets can be naturally extended to ringed spaces).

Example Projective space $\mathbb{P}^n$. Let $\mathbb{P}^n$ be the set of all the lines through the origin in $\mathbb{C}^{n+1}$. Let us give $\mathbb{P}^n$ the structure of an algebraic variety. To do this, note that a line $l \subset \mathbb{C}^{n+1}$ is uniquely determined by a point $u = (u_0, \dots, u_n) \in l, u \neq 0$. The points u and $\lambda u = (\lambda u, \dots, \lambda u_n)$ define the same line. Thus

$$\mathbb{P}^n = \left\{u \in \mathbb{C}^{n+1}\right\} \setminus \{0\} / (u \sim \lambda u, \lambda \neq 0).$$

The coordinates $(u_0, \ldots, u_n)$ are the *homogeneous coordinates* for $\mathbb{P}^n$. The set U_i of $\mathbb{P}^n$ for which $u_i \neq 0$ can be naturally identified with $\mathbb{C}^n$ by means of the mapping $\phi_i : U_i \to \mathbb{C}^n$:

$$\phi_i(u_0, \ldots, u_n) = \left(\frac{u_0}{u_i}, \ldots, \frac{\widehat{u}_i}{u_i}, \ldots, \frac{u_n}{u_i} \right) = (z_1, \ldots, z_n) \in \mathbb{C}^n .$$

The transition function between U_i and U_j is given by

$$\phi_j \circ \phi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{\widehat{z}_j}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j} \right),$$

and all of the functions z_k/z_j and $1/z_j$ are rational functions on $\mathbb{C}^n = U_i$, regular on $U_i \cap U_j$. This allows us to view $\mathbb{P}^n$ as an algebraic variety.

Closed subsets of $\mathbb{P}^n$ are sets of the type

$$V_{f_1,\ldots,f_k} = \{u = (u_0, \ldots, u_n) \in \mathbb{P}^n | f_i(u_0, \ldots, u_n) = 0, 1 \leq i \leq k\},$$

where $f_i(u_0, \ldots, u_n)$ are homogeneous polynomials. The intersection $V_{f_1,\ldots,f_k} \cap U_j$ is given in $U_j = \mathbb{C}^n$ by the equations

$$f_i \left(\frac{u_0}{u_j}, \ldots, \frac{\widehat{u}_j}{u_j}, \ldots, \frac{u_n}{u_j} \right) = f_i(z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) = 0,$$

hence is an affine variety. Thus, closed sets in $\mathbb{P}^n$ are algebraic varieties. An algebraic variety isomorphic to a closed sub-variety of $\mathbb{P}^n$ is called a *projective variety.*

1.7. Let us extend the definition of a field of rational functions from affine algebraic sets to general algebraic varieties. First, note that if an affine variety V is irreducible and $U \subset V$ is an open affine sub-variety of V, then U is also irreducible, and furthermore, the restriction to U of rational functions defined on V is an isomorphism of fields $\mathbb{C}(U)$ and $\mathbb{C}(V)$. Thus, if U_1 and U_2 are non-empty affine open subsets of an irreducible algebraic variety X, then there are natural isomorphisms $\mathbb{C}(U_1) \simeq \mathbb{C}(U_1 \cap U_2) \simeq \mathbb{C}(U_2)$. Similarly we can define the field of rational functions $\mathbb{C}(X)$ on an irreducible algebraic variety X. The elements of $\mathbb{C}(X)$ are rational functions f_U defined on non-empty affine open sub-varieties $U \subset X$, where $f_{U_1} = f_{U_2}$ if the restrictions of f_{U_1} and of f_{U_2} to $U_1 \cap U_2$ agree.

The concept of rational function can be generalized to that of a *rational mapping* between algebraic varieties. A rational mapping $\phi : X \to Y$ of algebraic varieties is an equivalence class of pairs (U, ϕ_U), where U is a non-empty open subset of X while ϕ_U is a morphism from U to Y. Two pairs (U, ϕ_U) and (V, ϕ_V) are considered equivalent, if ϕ_U and ϕ_V agree on $U \cap V$. For any rational mapping we can choose a representative $(\tilde{U}, \phi_{\tilde{U}})$, such that $U \subset \tilde{U}$ for any equivalent pair (U, ϕ_U). The open set $\tilde{U}$ is called the *domain of definition of the rational mapping.* If $\phi_{\tilde{U}}$ is everywhere dense in Y, then the rational mapping ϕ defines an inclusion of fields $\phi^* : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ (if X and Y are irreducible). If ϕ^* is an isomorphism, then X and Y are said to be *bi-rationally isomorphic*. In other words, X and Y are birationally isomorphic, if there is an open dense subsets U_X and U_Y, which are isomorphic to one another.

One of the most important examples of bi-rational isomorphism is the *monoidal transformation centered on a smooth sub-variety*, which can be defined as follows. Let X be a non-singular algebraic variety, $\dim X = n$,and $C \subset X$ is a non-singular algebraic sub-variety, $\dim C = n - m$. The X can be covered by affine neighborhoods $U_k \subset X$, where C is defined by the equations $u_{k,1} = \ldots = u_{k,m} = 0$, where $u_{i,k}$ are regular in U_k and $u_{k,1}, \ldots, u_{k,m}$ generate the ideal $I(C \cap U_k)$ in $\mathbb{C}[U_k]$ (see Shafarevich [1972]). Consider a sub-variety U_k' of $U_k \times \mathbb{P}^{m-1}$ defined by the equations

$$u_{k,i} \cdot t_j = u_{k,j} \cdot t_i, \quad 1 \leq i, j \leq m,$$

where $(t_1, \ldots, t_m)$ are homogeneous coordinates in in $\mathbb{P}^{m-1}$ and let σ_k be the restriction of the projection map $p_1 : U_k \times \mathbb{P}^{m-1} \to U_k$ to C. It is easy to see that $\sigma^{-1}(x)$ is isomorphic to $\mathbb{P}^{m-1}$ for every $x \in C$ and for $x \notin C$ $\sigma_k^{-1}(x)$ is a single point, so δ defines an isomorphism between $U_k' \setminus \sigma^{-1}(C)$ and $U_k \setminus C$.It is also easy to check that the variety $U_k' \subset U_k \times \mathbb{P}^{m-1}$ doesn't depend on the choice of the equations defining the subvariety C in U_k. Therefore, the varieties U_k' can be glued together into a single variety X', and thus to obtain a morphism $\sigma : X' \to X$, such that $\sigma^{-1}(x) = \mathbb{P}^{m-1}$ for every $x \in C$ and $\sigma : X' \setminus \sigma^{-1}(C) \to X \setminus C$ is an isomorphism. The resulting map σ is called the monoidal transformation of the variety X centered on C.

Let $\phi : X \to Y$ be a rational mapping of non-singular algebraic varieties. Then, according to a theorem of Hironaka [1964], we can resolve the points where ϕ is undefined by a sequence of monoidal transformations with non-singular centers. That is, there is a commutative diagram in which σ is a composition of monoidal transformations with non-singular centers, while ϕ' is a morphism.

$$\begin{tikzcd}
X' \arrow[d, "\sigma"'] \arrow[dr, "\phi'"] & \\
\phi : X \arrow[r] & Y
\end{tikzcd}$$

§2. Complex Manifolds

2.1. Let us equip $\mathbb{C}^n$ with a topology whose neighborhood basis consists of polydisks $\Delta^n_{a,\epsilon}$, of radius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, centered at $a \in \mathbb{C}^n$:

$$\Delta^n_{a,\epsilon} = \{z \in \mathbb{C}^n \,||z_i - a_i| < \epsilon_i\}.$$

We will refer to the topology defined above as the *complex topology*.

Recall that a complex-valued function $f(z)$, defined in some neighborhood U_a of $a \in \mathbb{C}^n$, is called *analytic* (or *holomorphic*) at a if there exists a polydisk $\Delta^n_{a,\epsilon} \subset U_a$, in which f can be represented as a convergent power series:

$$f(z) = \sum_{\alpha \geq 0} c_\alpha (z-a)^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $(z-a)^\alpha = (z_1 - a_1)^{\alpha_1} \ldots (z_n - a_n)^{\alpha_n}$.

Denote by $\mathcal{O}_{n,a}$ the subring of the ring of formal power series $\mathbb{C}[[z-a]]$ at a, consisting of those $f \in \mathbb{C}[[z-a]]$ which converge in some neighborhood $U_{(f)}$ of $a \in \mathbb{C}^n$. It can be checked that $\mathcal{O}_{n,a}$ is a Noetherian local ring with unique factorization. The unique maximal ideal of $\mathcal{O}_{n,a}$ consists of the analytic functions vanishing at a. The ring $\mathcal{O}_{n,a}$ is called the ring of *germs of analytic functions* at a.

2.2. A subset $V \subset \mathbb{C}^n$ is called *analytic,* if for any $a \in V$ there exists a neighborhood U_a such that $V \cap U_a$ coincides with a zero set of a finite set of functions analytic at a. In particular, every algebraic set $V \subset \mathbb{C}_n$ is analytic.

Let f be a function defined on an analytic set V. We say that f is *analytic* at $a \in V$, if there exists a neighborhood $U_a \in V$, where f is a restriction to V of a function $F \in \mathcal{O}_{n,a}$. Just as we did for algebraic sets, we can define a local ring $\mathcal{O}_{V,a}$ of germs of functions on V analytic at a. That is, $\mathcal{O}_{V,a} = \mathcal{O}_{n,a}/I_a(V)$, where $I_a(V)$ is the ideal of functions in $\mathcal{O}_{n,a}$ which vanish on V on some neighborhood of a. The rings $\mathcal{O}_{V,a}$ can be glued into a sheaf $\mathcal{O}_V$ of functions holomorphic on V. The sections of $\mathcal{O}_V$ over an open set $U \subset V$ are functions analytic at every point $a \in U$.

A continuous mapping $\phi : V_1 \to V_2$ of analytic sets is called a *holomorphic mapping* if for every point $a \in V_1$ and every function f analytic at $\phi(a)$, the function $\phi \circ f$ is analytic at a. The holomorphic map $\phi : V_1 \to V_2$ is an *isomorphism* if there exists a holomorphic inverse ϕ^{-1}.

The tangent space $T_{V,a}$ to $V \subset \mathbb{C}^n$ at a is defined by the equations

$$\sum_{i=1}^n \frac{\partial f}{\partial z_i}(a)(z_i - a_i) = 0,$$

$$f \in I_a(V),$$

analogously to the algebraic situation. Also analogously, $T_{V,a} \simeq (m_{V,a}/m^2_{V,a})^*$, where $m_{V,a}$ is the maximal ideal of the ring $\mathcal{O}_{V,a}$. Just as in the algebraic situation, the tangent spaces $T_{V,a}$ can be glued together to mke the tangent bundle $T_V \subset \mathbb{C}^{2n}$, and there exists a projection map $\pi : T_V \to V$, such that $\pi^{-1}(a) = T_{V,a}$.

A holomorphic mapping $\phi : V_1 \to V_2,$, $\phi(a) = b \in V_2$, induces a map $\phi_* : T_{V,a} \to T_{V,b}$ as follows. By definition, ϕ induces $\phi^* : \mathcal{O}_{V_2,b} \to \mathcal{O}_{V_1,a}$, such that $\phi^*(f) = \phi \circ f$. It is easy to see that $\phi^*(m_{V_2,b}) \subset m_{V_1,a}$ and $\phi^*(m^2_{V_2,b}) \subset m^2_{V_1,a}$. Therefore, we can define a map

$$\phi^* : m_{V_2,b}/m^2_{V2,b} \to m_{V1,a}/m^2_{V1,a},$$

the dual to which is the sought after $\phi_* : T_{V_1,a} \to T_{V_2,b}$. In particular, if V_1 is an analytic subset of an analytic set V, then for any $a \in V_1$ there is a natural inclusion $T_{V_1,a} \subset T_{V,a}$.

2.3. Just as in the algebraic situation, an analytic set V is called *irreducible*, if V cannot be represented as a union of two non-empty closed subsets V_1 and V_2, such that $V_i \neq V$ and $V_1 \neq V_2$.

An analytic set V is called *irreducible at* $a \in V$ if $I_a(V)$ is a prime ideal, or equivalently $\mathcal{O}_{V,a}$ has no zero divisors. Irreducibility of V at a means that for a sufficiently small neighborhood U_a of a, the analytic set $V \cap U_a$ is irreducible. Unlike the algebraic case, irreducible analytic sets may have points where they are reducible. For example, the set $V \in \mathbb{C}^2$ defined by the equation $y^2 = x^2 + x^3$ is irreducible, and yet, at the point $(0,0)$ V is reducible, since

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-(n-1)\right)}{n!} x^n$$

is an analytic function at the origin, and hence in a small neighborhood of the origin, V has two irreducible branches, given by the equations $y - x\sqrt{x+1} = 0$, and $y + x\sqrt{x+1} = 0$ respectively.

Let V be an irreducible analytic set. A point $a \in V$ is called a *regular point* if $\dim T_{V,a} = \min_{z\in V} \dim T_{V,z}$. Regular points form a dense open subset of V. By definition, the dimension of an irreducible analytic set V is $\dim V = \dim T_{V,a}$, where a is a regular point of V.

2.4. If an analytic set V is irreducible at a, then the elements of the fraction field of the ring $\mathcal{O}_{V,a}$ are called *meromorphic fractions*. For each meromorphic fraction h there exists a neighborhood $U_a \subset V$ and functions f and g, holomorphic in U_a, such that $h = \frac{f}{g}$. In general, the fraction $\frac{f}{g}$ is a *meromorphic fraction* at $a \in V$ if g is not a zero divisor in $\mathcal{O}_{V,a}$.

A *meromorphic function* on an analytic set V is a collection $\left\{\left(U_i, \frac{f_i}{g_i}\right)\right\}$, where U_i is an open covering of V, f_i and g_i are functions holomorphic in U_i, g_i is not a zero divisor in $\mathcal{O}_{V,a}$ for any point $a \in U_i$ and in $U_i \cap U_j$ $f_i g_j = f_j g_i$. The set of meromorphic functions on V possesses natural operations of addition and multiplication. If V is irreducible, then the set $M(V)$ of meromorphic functions on V is a field. Note that a rational function on an algebraic set V is meromorphic, if V is viewed as an analytic set.

2.5. A *complex space* is a ringed Hausdorff space $(X, \mathcal{O}_X)$, for each point a of which there is a neighborhood $U_a \in X$, isomorphic to an analytic set $(V_a, \mathcal{O}_{V_a})$.

The definitions of holomorphic functions, tangent space, etc, can be used unchanged for complex spaces (since these definitions are all local).

A connected complex space all of whose points are regular is called a *complex manifold.* By the implicit function theorem (see, eg, Gunning-Rossi [1965]) it is easy to show that each point in a complex manifold X has a neighborhood isomorphic to a neighborhood of the origin in $\mathbb{C}^n$; that is, there exist neighborhoods $U_a \subset X$ and $U_0 \subset \mathbb{C}^n$ and a bi-holomorphic mapping $\phi_a : U_a \xrightarrow{\sim} U_0$. The preimages $\phi_a \circ z_i$ of coordinate functions $z_1, \ldots, z_n$ in $\mathbb{C}^n$ will be called *local coordinates* at $a \in X$, and the neighborhood U_a will be called a *a coordinate neighborhood,* or a *chart.*

One of the most important examples of complex manifolds is the complex projective space $\mathbb{P}^n_{an}$, which is defined exactly as in the algebraic case, to wit:

$$\mathbb{P}^n_{an} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \backslash \{0\}\}/\{(z_1, \ldots, z_{n+1}) \\ \sim (\lambda z_1, \ldots, \lambda z_{n+1}), \lambda \neq 0\}.$$

Note that $\mathbb{P}^n_{an}$ is a compact manifold, since P^n_{an} is the image of a compact manifold $S^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \,|\, \sum_{i=1}^{n+1} z_i \overline{z_i} = 1\}$ under a continuous map (which is the restriction of the projection map $\mathbb{C}^{n+1} \backslash \{0\} \to \mathbf{P}^n_{an}$ defined above).

§3. A Comparison Between Algebraic Varieties and Analytic Spaces

3.1. To every algebraic variety X over the complex field $\mathbb{C}$ we can associate a complex space X_{an}. To wit, since polynomials are analytic functions, each algebraic set V can be viewed as a subset of $\mathbb{C}^n$, with Zariski topology replaced by the complex topology. This induces an inclusion of the ring of regular functions on the algebraic set V into the ring of analytic functions on V_{an}, while rational functions on V can be viewed as meromorphic functions on V_{an}. By definition, an algebraic variety X is glued together from affine varieties U_i, with rational transition functions, regular on the intersections $U_i \cap U_j$. Therefore, those same U_i, regarded as analytic sets, can be glued together into a complex space X_{an} using the same transition functions (the space X_{an} is Hausdorff because X is). In other words, X_{an} is obtained from the algebraic variety X by replacing the Zariski topology on X by the complex topology, and by enlarging the rings $\mathcal{O}_{X,x}$ to $\mathcal{O}_{X_{an},x}$. Note that the identity map $\mathrm{id}_{an} : X_{an} \to X$ is a ringed space morphism.

The correspondence between the algebraic variety X and the complex space X_{an} can be extended to regular mappings – it is not hard to show that the map $f_{an} : X_{an} \to Y_{an}$, obtained from a regular map $f : X \to Y$ as $f_{an} = \mathrm{id}_{an}^{-1} \circ f \circ \mathrm{id}_{an}$ is a holomorphic map.

It can be shown (see Serre [1956]) that the variety X is connected in the Zariski topology if and only if X_{an} is connected; X is irreducible if and only if X_{an} is irreducible; $\dim X = \dim X_{an}$; X is nonsingular if and only if X_{an} is a complex manifold; X_{an} is compact if and only if X is a *complete variety,* that is, for every variety Y, the projection map $p_2 : X \times Y \to X$ is sends closed sets to closed sets.

Comparing the definitions of a rational function (see §1) on an algebraic variety and of a meromorphic function (see §2) on a complex space Y, it is seen that if $Y = X_{an}$, then every rational function f on X can be viewed as a meromorphic function on X_{an}, and if X is irreducible, then there is an inclusion of fields $\mathbb{C}(X) \subset M(X_{an})$. In general, of course, $\mathbb{C}(X) \neq M(X_{an})$. However, if X is a complete variety, then $\mathbb{C}(C) = M(X_{an})$. This claim easily follows from the following theorem:

Theorem (Siegel [1955]). *Let Y be a compact complex manifold. Then the field of meromorphic functions $M(Y)$ is finitely generated over $\mathbb{C}$ and the transcendence degree of $M(Y)$ over $\mathbb{C}$ is no greater than* $\dim Y$.

3.2. The correspondence between an algebraic variety X and the complex space X_{an} can be extended to coherent sheaves. Recall (see, eg, Hartshorne [1977]), that a coherent sheaf F is locally defined as the cokernel of a morphism of free sheaves: $\mathcal{O}_U^m \xrightarrow{\alpha} \mathcal{O}_U^n \to F \to 0$. Therefore, to each coherent sheaf F on an algebraic variety X we can associate a coherent sheaf F_{an}. Indeed, over an affine subset $U \subset X$, the morphism α is defined by a matrix of sections of the sheaf $\mathcal{O}_U$. The entries of this matrix can be viewed as sections of the sheaf $\mathcal{O}_{U_{an}}$, since α induces a map $\alpha_{an} : \mathcal{O}_{an}^m \to \mathcal{O}_{an}^n$. Therefore, F_{an} can be locally defined over U_{an} as the cokernel of α_{an}. Put another way, the sheaf F_{an} is isomorphic to the sheaf $(\mathrm{id}_{an})^{-1} F \otimes_{\mathrm{id}_{an}^{-1}\mathcal{O}_X} \mathcal{O}_{X_{an}} = \mathrm{id}_{an}^* F$, where $\mathrm{id}_{an} : X_{an} \to X$ is the pointwise identity mapping.

This way of associating to a sheaf F on X the sheaf F_{an} allows us to define natural homomorphisms of cohomology groups (see Serre [1956]).

$$i_k : H^k(X, F) \to H^k(X_{an}, F_{an}).$$

In general i_k is not an isomorphism. However, as was shown by Serre [1956], for projective varieties (later generalized by Grothendieck [1971] to complete varieties), the homomorphisms i_k are indeed isomorphisms.

One consequence of Serre's result is the following

Theorem (Chow). *Every closed complex space in $\mathbb{P}_{an}^n$ corresponds to an algebraic variety in $\mathbb{P}^n$.*

3.3. The correspondence between an algebraic variety X and an analytic space X_{an} and between coherent sheaves F on X and F_{an} on X_{an} leads to the following questions:

1) If X_{an} and Y_{an} are isomorphic, are the varieties X and Y likewise isomorphic?
2) Is every coherent sheaf on X_{an} isomorphic to F_{an} for some coherent sheaf F on the algebraic variety X?

3) If F' and F'' are coherent sheaves on X, such that $F'_{an} \simeq F''_{an}$, are F' and F'' isomorphic?

The first of these questions is a special case of a more general question:

1') Let $g : X_{an} \to Y_{an}$ be a holomorphic mapping. Does there exists a regular mapping $f : X \to Y$, such that $g = f_{an}$.

As one might expect, the answers to these questions are in general negative (for counterexamples see Hartshorne [1977] and Shafarevich [1972]). However, already in the case of complete varieties the answers are affirmative ([Serre 1956], [Grothendieck 1971]), and this allows us to use techniques of complex analysis to study algebraic varieties.

3.4. The following question arises naturally: when does a complex space come from an algebraic variety? In general, it would appear that it is not possible to give non-trivial necessary and sufficient conditions, and therefore we will only consider the question when the complex space Y is a compact complex manifold, and in this case, if Y does indeed come from an algebraic variety, we will simply say that Y is an algebraic variety.

When $\dim_{\mathbb{C}} Y = 1$ (that is, Y is a compact Riemann surface), the above question is answered by

Theorem (Riemann). *Every compact complex manifold Y with $\dim_{\mathbb{C}} Y = 1$, is is a projective algebraic variety.*

One of the many ways to prove this theorem can be found in Chapter 1, §7.

According to Siegel's theorem, in order for a compact complex manifold to be algebraic, it is necessary for the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions $M(Y)$ to be equal to $\dim Y$. It turns out that if $\dim Y = 2$, then this condition is also sufficient, by the following

Theorem (Kodaira [1954]). *Every compact complex manifold of dimension 2 with two algebraically independent meromorphic functions is a projective algebraic variety.*

It should be noted that Kodaira's theorem is false for singular surfaces. Examples of non-algebraic compact complex manifolds of dimension 2 will be given in Chapter 1, §7.

In dimension ≥ 3 the coincidence of the dimension of the manifold with the transcendence degree of the field of meromorphic functions is already insufficient to guarantee that the manifold is algebraic. However the following holds:

Theorem ("Chow's Lemma", Moishezon [1966]). *Let Y be a compact complex manifold, such that the transcendence degree of the field of meromorphic functions $M(Y)$ is equal to $\dim Y$. Then there exists a projective algebraic variety $\tilde{Y}$ and a bi-meromorphic holomorphic mapping $\pi : \tilde{Y} \to Y$, which is a composition of monoidal transformations with nonsingular centers.*

The above theorem shows that if the transcendence degree of the field of meromorphic functions of a compact complex manifold is maximal, then such a manifold is not too different from a projective variety.

Using monoidal transformations with nonsingular centers, examples can be constructed of compact manifolds which, while possessing a maximal number of meromorphic functions, are either not algebraic, or algebraic but not projective. These examples are constructed by Hironaka [1960].

Example. An algebraic variety which is not projective. Let X be an arbitrary complete algebraic variety with $\dim X = 3$ (*e. g.* $X = \mathbb{P}^3$). Choose two non-singular curves (one-dimensional sub-varieties) C_1 and C_2 on X, intersecting transversely in two points P and Q. The variety X can be covered by two open sets $X \backslash P$ and $X \backslash Q$. In each of the sets $X \backslash P$ and $X \backslash Q$ perform two monoidal transformations; in $X \backslash P$ – first a monoidal transformations centered on C_1, and then one centered in the proper preimage of the curve C_2. Perform the corresponding transformations in $X \backslash Q$ with the roles of C_1 and C_2 reversed. Since the curves C_1 and C_2 do not intersect on $X \backslash (P \cup Q)$, the monoidal transformations centered on C_1 and C_2 can be performed in any order, and will result in the same variety. Therefore, the varieties obtained in the course of monoidal transformations from $X \backslash P$ and $X \backslash Q$, respectively, can be glued together into a single algebraic variety $\tilde{X}$, together with a morphism $\pi : \tilde{X} \to X$.

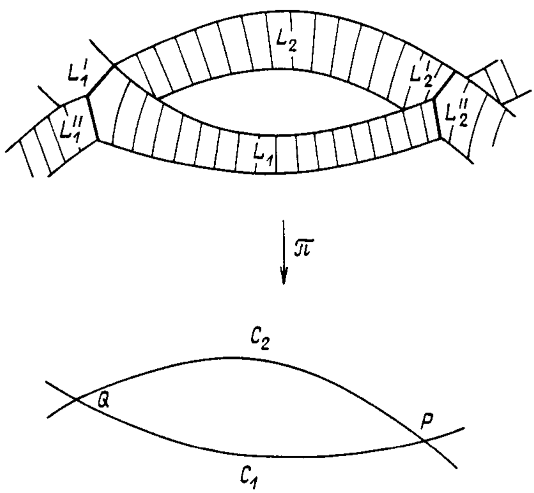


Fig. 2

The variety $\tilde{X}$ is not projective. To explain why that is true, denote by $L_1 = \pi^{-1}(x)$ the preimage over a general point $x \in C_1$, and by L_2 the preimage over a general point of the curve C_2 ($L_1 \simeq \mathbb{P}^1$, $L_2 \simeq \mathbb{P}^1$). Let us see what happens above P and Q. The preimage over Q consists of two curves L_1' and L_2'' (the curve L_1' appears after the first monoidal transformion on $X\backslash P$,and L_2'' after the second). It is easy to see that the curves L_1'' and L_2 are homologous, and also that $L_1' + L_1''$ and L_1 are homologous. Analogously, there are two curves L_2' and L_2'' sitting over P in $\tilde{X}$, and L_2'' is homologous to L_1 and $L_2' + L_2''$ is homologous to L_2. From the conditions

$$\begin{aligned} L_1'' &\sim L_2, \\ L_1'' + L_1' &\sim L_1, \\ L_2'' &\sim L_1, \\ L_2'' + L_2' &\sim L_2 \end{aligned}$$

it follows that $L_1' + L_2' \sim 0$. This, however, is impossible if $\tilde{X}$ is a projective variety. Indeed, $H_2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}L$, where L is a homology class of a projective line in $\mathbb{P}^n$, and every curve $C \subset \mathbb{P}^n$ is homologous to dL, $d > 0$, where d is the degree of C, equal to the number of intersections of C with a generic hyperplane in $\mathbb{P}^n$. In particular, a curve in $\mathbb{P}^n$ is not homologous to zero if its degree is greater than zero, hence $L_1' + L_2'$ cannot be null homologous in $\tilde{X}$ if $\tilde{X}$ is a projective variety.

Example. Non-algebraic manifold with a maximal number of meromorphic functions. This is constructed analogously to the previous example. In a complete algebraic variety take a curve C with only one singular point P. Assume that this singular point is of the simplest possible kind, that is in a sufficiently small analytic neighborhood $U \subset X_{an}$ of P, the curve C becomes reducible and falls apart into two nonsingular branches C_1 and C_2, which intersect transversely. Perform a monoidal transformation of $X\backslash P$ centered at C, and on U perform two monoidal transformations, the first centered at C_1 and the second centered at the proper preimage of C_2. It is easy to see that these transformations over $U\backslash P$ give rise to the same variety, along which the varieties obtained via monoidal transformations of $X_{an}\backslash P$ and of U may be glued. We will thus obtain a compact manifold $\tilde{X}$ and a holomorphic mapping $\pi : \tilde{X} \to X_{an}$. It is easy to see that π^* is an isomorphism between the fields $M(X)$ and $M(\tilde{X})$.

Let us show that $\tilde{X}$ can not be an algebraic variety. Denote by $L = \pi^{-1}(x)$ the preimage over a general point $x \in C \cap X\backslash P$, L_1 – the preimage over a general point $x \in C_1 \subset U$, L_2 – the preimage over a general point $x \in C_2 \subset U$, and by $(L' \cup L'')$ the preimage over the point P (L' is obtained after the first monoidal transformation in U, etc.) Then, evidently, $L_1 \sim L_2 \sim L$ and $L_2 \sim L''$, $L_1 \sim L' + L''$. Consequently, $L' \sim 0$. This, however, is impossible if $\tilde{X}$ is an algebraic variety. Indeed, if $\tilde{X}$ is algebraic, then every point $Q \in \tilde{X}$ lies in some affine neighborhood $W \subset \tilde{X}$. On an affine variety, there exists an algebraic surface S, passing through any given point, and not containing a given curve. Choose such a surface S passing through some point $Q \subset L' \subset \tilde{X}$, and denote by $\tilde{S}$ the closure of the surface S in $\tilde{X}$. Then the homology classes

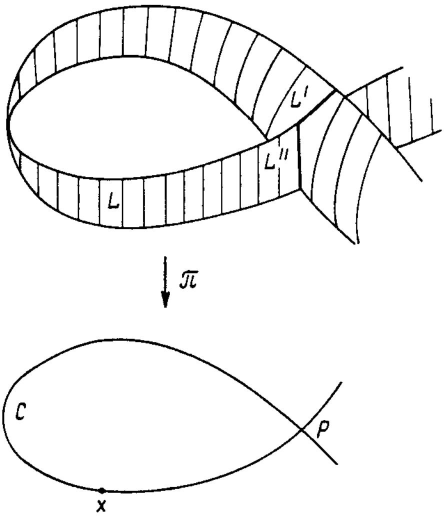


Fig. 3

$(\tilde{S})$ and (L') have a non-zero intersection number, which means that L' was not null-homologous.

3.5. At this point we should mention a sufficient condition, under which any compact complex manifold with a maximal number of algebraically independent meromorphic functions is an algebraic (and even projective) variety. Such a condition (see Moishezon [1966]) is the existence on X of a Kähler metric, which will be defined in Chapter 1, §7.

§4. Complex Manifolds as C^∞ Manifolds

If we ignore the complex structure, an n-dimensional complex manifold X can be viewed as a $2n$-dimensional differentiable manifold. This viewpoint allows us to transfer all of the differential-geometric constructions onto complex manifolds, as described in the following two sections. All of the necessary background from differential geometry can be found in Postnikov [1971] and de Rham [1955].

4.1. Tangent and Cotangent Bundles. Let $z_1, \dots, z_n$ be local coordinates in a neighborhood of a point x in a complex manifold X, where $z_j = x_j + iy_j$, $i = \sqrt{-1}$. Forgetting about the complex structure, we see that the functions $x_1, \dots, x_n, y_1, \dots, y_n$ are real local coordinates on the C^∞ manifold X. Let $TX(\mathbb{R})$ and $T^*X(\mathbb{R})$ be the tangent and cotangent bundles of X (as a C^∞

manifold) respectively, and let $TX = TX(\mathbb{R}) \otimes \mathbb{C}$ and $T^*X = T^*X(\mathbb{R}) \otimes \mathbb{C}$ be their complexifications. The bundle $TX(\mathbb{R})$ can be identified with the image of $TX(\mathbb{R})$ under the inclusion $TX(\mathbb{R}) \hookrightarrow TX$, defined by the natural inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. Analogously, $T^*X(\mathbb{R})$ can be included into T^*X.

The differentials $dx_1, \dots, dx_n, dy_1, \dots, dy_n$ form a basis of $T_x^*X(\mathbb{R})$, and thus the forms $dz_p = dx_p + idy_p$ and $d\overline{z}_p = dx_p - idy_p$ form a basis of T_x^*X. Let $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_n}$ be the basis of T_xX dual to the basis $dz_1, \dots, dz_n, d\overline{z}_1, \dots, d\overline{z}_n$:

$$\frac{\partial}{\partial z_p} = \frac{1}{2}\left(\frac{\partial}{\partial x_p} - i\frac{\partial}{\partial y_p}\right),$$

$$\frac{\partial}{\partial \overline{z}_p} = \frac{1}{2}\left(\frac{\partial}{\partial x_p} + i\frac{\partial}{\partial y_p}\right).$$

The space T_xX decomposes into a direct sum

$$T_xX = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x},$$

where $T^{1,0}_{X,x}$ is generated over $\mathbb{C}$ by the vectors $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$, while $T^{0,1}_{X,x}$ is generated by $\frac{\partial}{\partial \overline{z}_1}, \dots, \frac{\partial}{\partial \overline{z}_n}$. It is easy to see that the space $T^{1,0}_{X,x}$ is isomorphic to $T_{X,x} = (m_{X,x}/m^2_{X,x})^*$, since differentiation $\frac{\partial}{\partial z_p}$ preserves holomorphic functions. The space $T^{1,0}_{X,x}$ is called the *holomorphic tangent space* (see §2). Since $T_xX = T_xX(\mathbb{R}) \otimes \mathbb{C}$, complex conjugation $z \to \overline{z}$ can be extended to complex conjugation on T_xX, and it is easy to see that $\overline{\frac{\partial}{\partial z_p}} = \frac{\partial}{\partial \overline{z}_p}$, hence $\overline{T^{1,0}_{X,x}} = T^{0,1}_{X,x}$. Furthermore,

$$\eta = \overline{\eta},$$

for a vector $\eta \in T_xX$, if and only if $\eta \in T_xX(\mathbb{R})$. Therefore, any vector $\eta \in T_xX(\mathbb{R})$ can be represented as

$$\eta = \sum_{p=1}^{n} \left(\eta_p \frac{\partial}{\partial z_p} + \overline{\eta}_p \frac{\partial}{\partial \overline{z}_p}\right),$$

and the natural projection $T_xX \to T^{1,0}_{X,x}$ defines an $\mathbb{R}$-linear isomorphism $T_xX(\mathbb{R}) \simeq T^{1,0}_{X,x}$:

$$\eta = \sum \left(\eta_p \frac{\partial}{\partial z_p} + \overline{\eta}_p \frac{\partial}{\partial \overline{z}_p}\right) \to \sum \eta_p \frac{\partial}{\partial z_p}.$$

4.2. Orientability of a Complex Manifold. Recall that an n-dimensional real vector space V is called *oriented* when an orientation has been picked on the one-dimensional vector space $\Lambda^n V$. A locally trivial vector bundle $f : E \to X$ is called *orientable* if orientations ω_x can be chosen on all the fibers E_x in such a way that for the trivializations $f^{-1}(U) \simeq U \times V$ over sufficiently small open sets $U \subset X$ all the orientations ω_x define the same orientation on V. Finally, an *orientation of a differentiable manifold* M is an orientation of its tangent bundle.

It is easy to see that in order for an n-dimensional differentiable manifold to be orientable, it is sufficient for there to be a cover of M by open sets U_α and local coordinates $x_{\alpha,1}, \dots, x_{\alpha,n}$ on each U_α, such that the n-forms $dx_{\alpha,1} \wedge \dots \wedge dx_{\alpha,n}$ differ at each point of $U_\alpha \cap U_\beta$ by a positive multiple.

Theorem. *A complex manifold* X *is an oriented* C^∞ *manifold.*

Indeed, let $z_1, \dots, z_n$ and $w_1, \dots, w_n$ be two choices of complex local coordinates in a neighborhood of a point $x \in X$. Then

$$dw_j = \sum_{k=1}^{n} \frac{\partial w_j}{\partial z_k} dz_k.$$

Thus, $dw_1 \wedge \dots \wedge dw_n = J dz_1 \wedge \dots \wedge dz_n$, where $J = \det\left(\frac{\partial w_j}{\partial z_k}\right)$ is the Jacobian of the transition map from the z to the w coordinates. Let $z_j = x_j + iy_j$ and $w_k = u_k + iv_k$. Then $dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n = (i/2)^n dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$. Therefore $du_1 \wedge \dots \wedge du_n \wedge dv_1 \wedge \dots \wedge dv_n = J\bar{J} dx_1 \wedge \dots \wedge x_n \wedge dy_1 \wedge \dots \wedge dy_n$. But $J\bar{J} = |J|^2 > 0$ and hence X is an oriented manifold.

4.3. Denote by $\mathcal{E}_X^k$ the sheaf of complex differential k-forms on the manifold X. The local sections of the sheaf $\mathcal{E}_X^k$ are given by the forms

$$\phi = \sum_{I,J} \phi_{I,J} dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dy_{j_1} \wedge \dots \wedge dy_{j_{k-p}},$$

where $\phi_{I,J}$ are complex-valued C^∞ functions, while $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_{k-p}\}$, $0 \le p \le k$.

The exterior differentiation operator d, which acts separately on real and imaginary parts, can be extended to a differentiation operator $d : \mathcal{E}_X^k \to \mathcal{E}_X^{k+1}$. By Poincaré's Lemma, the sequence of sheaves

$$0 \to \mathbb{C}_X \to \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}_X^k \xrightarrow{d} \dots$$

is exact, that is, locally, every closed form ω ($d\omega = 0$) is exact ($\omega = d\phi$).

The sheaves $\mathcal{E}_X^k$ are fine sheaves (Godement [1958]), since the manifold X has a smooth partition of unity. The cohomology groups $H^p(X, \mathcal{E}_X^k) = 0$ for $p > 0$, and so (*de Rham's theorem*) the following relationship between cohomology groups holds:

$$H^k(X, \mathbb{C}_X) \simeq \frac{\mathrm{Ker}[d : H^0(X, \mathcal{E}_X^k) \to H^0(X, \mathcal{E}_X^{k+1})]}{\mathrm{Im}[d : H^0(X, \mathcal{E}_X^{k-1}) \to H^0(X, \mathcal{E}_X^k)]},$$

that is, the k-th cohomology group of X with complex coefficients is isomorphic to the quotient of the space of closed k forms on X by the space of exact k forms.

The decompositions $TX = T_X^{1,0} \oplus T_X^{0,1}$ and $TX^* = T_X^{1,0*} \oplus T_X^{0,1*}$ induce decompositions of k-forms into forms of type (p,q)

$$\mathcal{E}_X^k = \bigoplus_{p+q=k} \mathcal{E}_X^{p,q}.$$

If we use the notation $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, where $I = \{i_1, \ldots, i_p\}$, $|I| = p$, then the sheaves $\mathcal{E}_X^{p,q}$ over X are locally generated by the forms $dz_I \wedge d\overline{z}_J$, $|I| = p$, $|J| = q$. Note that the sheaves $\mathcal{E}_X^{p,q}$ are also fine.

It is easy to see that the differential of the form $\phi = \sum_{|I|=p,|J|=q} \phi_{I,J} dz_I \wedge d\overline{z}_J \in \mathcal{E}_X^{p,q}$ is given by

$$d\phi = \sum_{k=1}^{n} \sum \frac{\partial \phi_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J + \sum_{k=1}^{n} \sum \frac{\partial \phi_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J;$$

also $d\phi \in \mathcal{E}_X^{p+1,q} \oplus \mathcal{E}_X^{p,q+1}$. Let $\Pi_{p,q} : \mathcal{E}_X^{p+1,q} \to \mathcal{E}_X^{p,q}$ be the natural projection operator. Define

$$\partial : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q},$$
$$\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$$

by setting $\partial = \Pi_{p+1,q} \circ d$ and $\overline{\partial} = \Pi_{p,q+1} \circ d$. Then $d = \partial + \overline{\partial}$. Furthermore, $\partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = d^2 = 0$. Comparing types of the various forms, we get

$$\partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2 = 0.$$

Denote $\Omega_X^p = \mathrm{Ker}[\overline{\partial} : \mathcal{E}_X^{p,0} \to \mathcal{E}_X^{p,1}]$. The sheaves Ω_X^p are called *the sheaves of holomorphic differential p-forms on X*. It is easy to see that the sequence

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{\partial} \Omega_X^1 \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega_X^p \xrightarrow{\partial} \ldots$$

is the resolution of the constant sheaf $\mathbb{C}_X$, while the sequences

$$0 \to \Omega_X^p \to \mathcal{E}_X^{p,0} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{E}_X^{p,q} \xrightarrow{\overline{\partial}} \ldots$$

are fine resolvents of the sheaf Ω_X^p. Hence (*Dolbeault's Theorem*)

$$H^q(X, \Omega_X^p) = \frac{\mathrm{Ker}[\overline{\partial} : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q+1})]}{\mathrm{Im}[\overline{\partial} : H^0(X, \mathcal{E}_X^{p,q-1}) \to H^0(X, \mathcal{E}_X^{p,q})]},$$

and there exists a spectral sequence (the *hypercohomology spectral sequence*) with the term

$$E_1^{p,q} = H^q(X, \Omega_X^p) = H^{p,q}(X),$$

which converges to $H^{p,q}(X, \mathbb{C})$ (Godement [1958]).

§5. Connections on Holomorphic Vector Bundles

One of the central concepts of differential geometry is that of an affine connection, which makes it possible to define the concept of parallel translation on vector bundles. In this section we extend the concept of an affine connection into the complex setting.

5.1. The generalization of the concept of a vector bundle to the complex setting is the concept of a holomorphic vector bundle.

Definition. A holomorphic mapping $\pi : E \to X$ is called a *holomorphic vector bundle of rank* n if

1) There exists an open cover $\{U_\alpha\}$ of the manifold X and biholomorphic mappings $\phi_\alpha : \mathbb{C}^n \times U_\alpha \to \pi^{-1}(U_\alpha)$, such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}^n \times U_\alpha \arrow[rr, "\phi_\alpha"] \arrow[dr, "\mathrm{pr}_2"'] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

The mappings ϕ_α are called *trivializations.*

2) For every fiber $E_x = \pi^{-1}(x) \simeq \mathbb{C}^n$ over a point $x \in U_\alpha \cap U_\beta$ the mapping $h_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta^{-1}(x) : \mathbb{C}^n \to \mathbb{C}^n$, defined by the trivializations ϕ_α and ϕ_β, is a $\mathbb{C}$-linear map.

Note that if a basis for $\mathbb{C}^n$ is defined, then the trivializations ϕ_α and ϕ_β define a non-singular matrix $h_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ of order n, whose entries are functions holomorphic on $U_\alpha \cap U_\beta$ (*transition functions*). Evidently,

$$\begin{aligned} \forall x \in U_\alpha \cap U_\beta, \quad & h_{\alpha\beta} \circ h_{\beta\alpha} = \mathrm{id}, \\ \forall x \in U_\alpha \cap U_\beta \cap U_\gamma, \quad & h_{\alpha\beta} \circ h_{\beta\gamma} \circ h_{\gamma\alpha} = \mathrm{id}. \end{aligned} \tag{3}$$

It is easy to see that if we have an open cover $\{U_\alpha\}$ and matrices of holomorphic functions $h_{\alpha\beta}$ defined at every point of $U_\alpha \cap U_\beta$, then there exists a holomorphic vector bundle $E \to X$ with transition functions $\{h_{\alpha\beta}\}$. The operations of direct sum, tensor product, exterior product, and so on, can be extended without change to holomorphic vector bundles. For example, if $E \to X$ is a holomorphic vector bundle, then the vector bundle $E^* \to X$ with fibers $E_x^* = (E_x)^* = \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$ is called the dual vector bundle. Furthermore, if the transition functions for E are given by the matrices $h_{\alpha\beta}$, those for E^* are given by $g_{\alpha\beta} = h_{\alpha\beta}^{-1t}$.

A *holomorphic section* of a holomorphic vector bundle $\pi : E \to X$ over an open set $U \subset X$ is a holomorphic mapping $f : U \to E$, such that $\pi(f(x)) \equiv x$.

If f and g are two sections of the bundle U over U, then their sum $h(x) = f(x) + g(x)$ can be defined in the obvious way, and therefore holomorphic sections form a sheaf $\mathcal{O}_X(E)$. The sheaf $\mathcal{O}_X(E)$ is locally trivial (that is, it is locally isomorphic to $\mathcal{O}_X^n$). Conversely, every locally trivial sheaf of rank n is isomorphic to the sheaf of sections of a holomorphic vector bundle of rank n. In the sequel we will occasionally not distinguish between holomorphic vector bundles and their sheaves of sections.

Denote by $\mathcal{E}_X^{pq}(E)$ the sheaf of E-valued (p,q)-forms on X. If for a sufficiently small open set $U \subset X$ we choose a holomorphic basis $\{l_1, \dots, l_n\}$ of the bundle (that is, we are given a trivialization $\phi_U : \mathbb{C}^n \times U \to \pi^{-1}(U)$), then the sections of the sheaf $\mathcal{E}_x^{p,q}$ are the forms

$$\eta = \sum \eta_j \otimes l_j,$$

where $\eta_k \in \mathcal{E}_X^{p,q}|_U$.

There is an important distinction between holomorphic vector bundles and C^∞ vector bundles. While there is no natural differentiation operator d defined on the sections of a C^∞ vector bundle, for holomorphic vector bundles the differentiation operator $\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$ induces a well-defined operator

$$\overline{\partial} : \mathcal{E}_X^{p,q}(E) \to \mathcal{E}_X^{p,q+1}(E).$$

It is clear that the kernel of the operator

$$\overline{\partial} : \mathcal{E}_X^{0}(E) \to \mathcal{E}_X^{0,1}(E)$$

coincides with the sheaf $\mathcal{O}_X(E)$ of holomorphic sections of E.

Let $\pi : E \to X$ be a holomorphic vector bundle, trivial over the open set U. It is clear that there are several choices of a trivialization $\phi : \mathbb{C}^n \times U \to \pi^{-1}(U)$ over U. It can be seen that if $\psi : \mathbb{C}^n \times U \to \pi^{-1}(U)$ is another trivialization, then $\phi \circ \psi^{-1} = A(x)$, where $A(x)$ is a nonsingular over U matrix of holomorphic functions, that is, $A(x) \in H^0(U, \mathrm{GL}(n, \mathcal{O}_X|_U))$. Therefore, if $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ are two sets of transition functions defining the bundle $E \to X$, then

$$h'_{\alpha\beta} = A_\alpha(x) h_{\alpha\beta}(x) A_\beta^{-1}(x). \tag{4}$$

In particular if $E \to X$ is a *line bundle* (a bundle of rank one), then E is given by an open covering $\{U_\alpha\}$ of X, and a set of non-vanishing in $U_\alpha \cap U_\beta$ holomorphic (on $U_\alpha \cap U_\beta$) functions, which satisfy relations (3)

$$h_{\alpha\beta} h_{\beta\alpha} \equiv 1,$$
$$h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} \equiv 1.$$

The relations (3) define a Čech cocycle (see Godement [1958]) on X, with coefficients in the sheaf of invertible holomorphic functions $\mathcal{O}_X^*$. Condition (4) for the line bundle $E \to X$ shows that for two collections of transition functions $\{h_{\alpha\beta}\}$ and $\{h'_{\alpha\beta}\}$ of two trivializations of this bundle, we can find a collection of functions $f_\alpha \in H^0(U_\alpha, \mathcal{O}_X^*|_{U_\alpha})$, such that

$$h'_{\alpha\beta}(x) = (f_\alpha/f_\beta)h_{\alpha\beta}(x).$$

In other words, cocycles $\{h_{\alpha\beta}\}$ and $h'_{\alpha\beta}$ differ by a coboundary. Therefore, line bundles over X are in one-to-one correspondence with elements of $H^1(X, \mathcal{O}_X^*)$. It is easy to check that the group operation in $H^1(X, \mathcal{O}_X^*)$ corresponds to tensor product of line bundles. The group $H^1(X, \mathcal{O}_X^*)$ is called *the Picard group* of the manifold X, and denoted by $\operatorname{Pic} X$.

5.2. The analogue to the concept of a Euclidean vector bundle is that of a Hermitian vector bundle.

Definition A holomorphic vector bundle $\pi : E \to X$ is called *a Hermitian vector bundle,* if each fiber E_x is equipped with a Hermitian scalar product, which depends smoothly on $x \in X$.

Smoothness of the scalar product means that if we choose a basis $\{e_i(x)\}$, over an open set $U \subset X$, smoothly depending on $x \in U$ (in other words we choose a C^∞ trivialization $\phi_U : \mathbb{C}^n \times U \to \pi^{-1}(U)$), then the functions $h_{ij}(x) = (e_i(x), e_j(x))$ are of class C^∞.

A basis $\{e_i\}$, smoothly dependent on x, in a Hermitian bundle E over $U \subset X$ is called *unitary* if

$$(e_i(x), e_j(x)) = \delta_{ij},$$

where δ_{ij} is Kronecker's symbol.

Using the Gram-Schmidt orthogonalization process, we can always pick a unitary basis in an open set U, such that E is trivial over U.

Let $E \to X$ be a Hermitian vector bundle. Then an Hermitian scalar product on $E \to X$ induces a hermitian product on the dual vector bundle $E^* \to X$. Indeed, let $\{e_1, \ldots, e_n\}$ be a unitary basis of E over $U \subset X$, and let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis in E^*, that is, $\langle e_i, e_j^* \rangle = \delta_{ij}$. Then an Hermitian scalar product can be defined in E^* by setting $(e_i^*, e_j^*) = \delta_{ij}$.

5.3.

Definition. A connection D on a holomorphic vector bundle $E \to X$ is a mapping

$$D : \mathcal{E}_X^0(E) \to \mathcal{E}_X^1(E) = \mathcal{E}_X^1 \otimes \mathcal{E}_X^0(E)$$

satisfying the Leibnitz product rule

$$D(f \circ \alpha) = df \otimes \alpha + f D\alpha$$

for all smooth sections $\alpha \in H^0(U, \mathcal{E}_X^0(E)|_U)$ of the bundle E over an open set $U \subset X$ and for all smooth functions f.

If there is a trivialization $\phi : \mathbb{C}^n \times U \to \pi^{-1}(U)$ of the bundle $\pi : E \to X$ (which is to say, a basis $\{e_i(x)\}$ is chosen for E over U) then the connection D is determined by a matrix $\theta = (\theta_{ij})$ of one-forms:

$$De_i = \sum \theta_{ij} e_j.$$

The matrix θ is called *the connection matrix.* It can be seen that θ together with the choice of a basis $\{e_i\}$ determines the connection D. Indeed, if $\alpha = \sum \alpha_i e_i$, then

$$D\alpha = \sum d\alpha_i \otimes e_i + \sum \alpha_i De_i = \sum_j (d\alpha_j + \sum_i \alpha_i \theta_{ij}) \otimes e_j.$$

Thus, if $\{e'_i\}$ is another basis such that $e'(x) = g(x)e(x)$, then

$$\theta_{e'} = dg \cdot g^{-1} + g\theta_e g^{-1}.$$

The decomposition of 1-forms into those of types $(1,0)$ and $(0,1)$ defines a decomposition $\mathcal{E}^1_X(E) = \mathcal{E}^{1,0}_X(E) \oplus \mathcal{E}^{0,1}_X(E)$, and thus the connection D decomposes into a sum $D = D' + D''$, corresponding to the forms of different types.

Using the Leibnitz product rule, the connection D can be extended to mappings

$$D : \mathcal{E}^k_X \to \mathcal{E}^{k+1}_X(E),$$

by setting

$$D(\phi \otimes \psi) = d\phi \otimes \psi + (-1)^k \phi \wedge D\psi,$$

where $\phi \in H^0(U, \mathcal{E}^k_X|_U)$, $\psi \in H^0(U, \mathcal{E}^0_X(E)|U)$ and $\mathcal{E}^k_X(E)$ is the sheaf of E-valued k forms on X.

A simple computation shows that

$$D^2 : \mathcal{E}^0_X(E) \to \mathcal{E}^2_X(E)$$

is an $\mathcal{E}^0_X$-linear operator, that is, $D^2(f\alpha) = fD^2(\alpha)$, for C^∞ functions f. Put another way, $D^2 : \mathcal{E}^0_X(E) \to \mathcal{E}^2_X(E)$ is induced by the bundle mapping $E \to \Lambda^2 T^* E \otimes E$.

If $\{e_i\}$ is a basis for E over U, then

$$D^2 e_i = \sum \Theta_{ij} \otimes e_j,$$

where $\Theta_e = (\Theta_{ij})$ is a matrix of 2-forms. This matrix is called *the curvature matrix of D* with respect to the basis $\{e_i\}$.

If $e' = g(e)$ is another basis, then

$$\Theta_{e'} = g\Theta g^{-1}.$$

Computing:

$$D^2 e_i = D(\sum \theta_{ij} \otimes e_j) = \sum_j d\theta_{ij} \otimes e_j - \sum j, k(\theta_{ik} \wedge \theta_{kj}) \otimes e_j$$

leads to the matrix equation

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e.$$

The above is known as *Cartan's structure equation.*

5.4. Just as in differential geometry a Riemannian metric determines a unique Riemannian connection, so on an Hermitian bundle there is a unique connection which agrees with the Hermitian scalar product and with the complex structure. More precisely, the following lemma holds (see, eg, Griffiths–Harris [1978]).

Lemma. *Let $E \to X$ be an Hermitian bundle. Then there exists a unique connection D on E (the so-called metric connection) satisfying the following conditions:*

1) $D'' = \overline{\partial}$;
2) $d(\alpha, \beta) = (D\alpha, \beta) + (\alpha, d\beta)$, *where* $(\ , \ )$ *is the Hermitian product on* E.

Condition 2) of the lemma is equivalent to saying that the Hermitian scalar product is invariant under parallel translation.

It should be noted that the metric connection with respect to a holomorphic basis $\{e_i\}$ is given by a matrix θ of $(1,0)$ forms, by condition 1) of the lemma. If the basis $\{e_i\}$ is unitary (that is, e_i depends smoothly on x and $(e_i, e_j) = \delta_{ij}$), then

$$0 = d(e_i, e_j) = \theta_{ij} + \overline{\theta_{ij}},$$

so the connection matrix θ is skew-Hermitian with respect to a unitary basis.

The curvature matrix Θ of an Hermitian holomorphic vector bundle is an Hermitian matrix of $(1,1)$ forms. Indeed, since $D'' = \overline{\partial}$, then $D''^2 = 0$, and hence $\Theta^{0,2} = 0$. But with respect to a unitary basis $\{e\}$, the matrix θ_e is skew-Hermitian, hence $\Theta = d\theta - \theta \wedge \theta$ is also skew-Hermitian. Therefore,

$$\Theta^{2,0} = -{}^t\Theta^{0,2} = 0.$$

Let D be the metric connection on an Hermitian bundle $E \to X$. It defines a metric connection D^* on the dual vector bundle E^*, which can be defined by requiring that

$$d\langle \sigma, \tau \rangle = \langle D\sigma, \tau \rangle + \langle \sigma, D^* \tau \rangle,$$

for sections $\sigma \in H^0(U, \mathcal{E}^0_X(E))$ and $\tau \in H^0(U, \mathcal{E}^0_X(E^*))$, over an open set U in X. In particular, if $\{e_i\}$ is a basis of the bundle E over U and $\{e_i^*\}$ is the dual basis of E^*, and θ and θ^* are the corresponding connection matrices, then

$$0 = d\langle e_i, e_j^* \rangle = \theta_{ij} + \theta^*_{ji},$$

hence

$$\theta = -{}^t\theta^*. \tag{5}$$

§6. Hermitian Manifolds

The complex analogue of Riemannian manifolds are Hermitian manifolds.

6.1. A complex manifold X is called *a Hermitian manifold*, if its holomorphic tangent bundle $T_X^{1,0}$ has the structure of a Hermitian bundle.

It should be noted that, just like in the real setting, every complex manifold X can be made Hermitian. Indeed, locally there always exists a Hermitian structure, since locally X is isomorphic to a neighborhood of 0 in $\mathbb{C}^n$. Using a smooth partition of unity, these local Hermitian structure can be always assembled into a Hermitian structure on all of X.

A hermitian scalar product on $T_{X,x}^{1,0}$ is induced by the pairing

$$(\ , \ )_X : T_{X,x}^{1,0} \otimes \overline{T_{X,x}^{1,0}} \to \mathbb{C},$$

which depends smoothly on X. In local coordinates a Hermitian scalar product is given as

$$ds^2 = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z}_j,$$

and ds^2, as above, is hermitian, when $h_{ij}(z) = \overline{h_{ji}(z)}$.

The real and imaginary parts of the Hermitian scalar product $(\cdot, \bar{\cdot})$ determine, respectively, a Euclidean scalar product and a skew-symmetric 2-form on the vector space $T_{X,x}^{1,0}$. Therefore, under the natural isomorphism $TX(\mathbb{R}) \xrightarrow{\sim} T_X^{1,0}$, the Hermitian metric ds^2 induces the Riemannian metric

$$\operatorname{Re} ds^2 : T_x X(\mathbb{R}) \otimes T_x X(\mathbb{R}) \to \mathbb{R}$$

on X. The skew-symmetric form

$$\operatorname{Im} ds^2 : T_x X(\mathbb{R}) \otimes T_x X(\mathbb{R}) \to \mathbb{R}$$

defines a differential 2-form $\Omega = -\frac{1}{2} \operatorname{Im} ds^2$, which we will call *the associated form of a Hermitian metric.*

By the Gram-Schmidt orthogonalization process, we can find forms $\phi_1, \dots, \phi_n \in T_{x,X}^{1,0}$, such that the Hermitian metric can be locally written in the form

$$ds^2 = \sum_j \phi_j \otimes \overline{\phi_j}.$$

Let $\phi_j = \alpha_j + i\beta_j$. Then

$$\begin{aligned} ds^2 &= \sum (\alpha_j + i\beta_j) \otimes (\alpha_j - i\beta_j) \\ &= \sum (\alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j) + i \sum (-\alpha_j \otimes \beta_j + \beta_j \otimes \alpha_j). \end{aligned}$$

Hence, the Riemannian metric can be written as

$$\operatorname{Re} ds^2 = \sum_{j=1}^{n} (\alpha_j \otimes \alpha_j + \beta_j \otimes \beta_j),$$

while the associated form can be written as

$$\Omega = -\frac{1}{2} \operatorname{Im} ds^2 = \sum_{j=1}^{n} \alpha_j \wedge \beta_j = \frac{i}{2} \sum_{j=1}^{n} \phi_j \wedge \overline{\phi_j}.$$

Thus, the metric $ds^2 = \sum \phi_j \otimes \overline{\phi_j}$ can be recovered from the associated form $\Omega = \phi_j \wedge \overline{\phi_j}$. Specifically, a given real $(1,1)$ form

$$\Omega = \frac{i}{2} \sum h_{pq}(z) dz_p \wedge d\overline{z}_q,$$

this defines a *Hermitian metric*

$$ds^2 = \sum h_{pq}(z) dz_p \otimes d\overline{z}_q$$

whenever the Hermitian matrix $H(z) = (h_{pq}(z))$ is positive definite. The real $(1,1)$ forms $\Omega = \frac{i}{2} \sum h_{pq}(z) dz_p \wedge d\overline{z}_q$ for which the matrix $H(z) = (h_{pq}(z))$ is positive definite are called *positive forms*.

Finally, note that if Ω is the form associated to a Hermitian metric ds^2, then

$$\frac{1}{n!} \Omega^n = (\alpha_1 \wedge \beta_1) \wedge \ldots \wedge (\alpha_n \wedge \beta_n)$$

is the volume form dV on the Riemannian manifold X with the metric $\operatorname{Re} ds^2$.

6.2. Let X be a Hermitian manifold. Then the Hermitian metric $ds^2 = \sum h_{ij} dz_i \otimes d\overline{z}_j$ defines a metric connection D on the holomorphic tangent bundle $T_X^{1,0}$ and also, by duality, a metric connection D^* on $T_X^{1,0*}$.

For a coordinate neighborhood $U \subset X$ choose a unitary basis $\phi_1, \ldots, \phi_n \in T_X^{1,0*}$ such that $ds_2 = \sum \phi_j \otimes \phi_j$. A simple computation shows the following

Lemma. *There exists a unique matrix of* 1*-forms* (ϕ_{ij})*, such that*

1) $\phi + {}^t\overline{\phi} = 0$,
2) $d\phi_j = \sum_i \phi_{ij} \wedge \phi_j + \tau_i$, *where* τ_i *are* $(2,0)$ *forms.*

The above lemma gives an effective means to compute the connection matrix. Namely, let $v = (v_1, \ldots, v_n)$ be a basis of $T_X^{1,0}$ dual to $\phi = (\phi_1, \ldots, \phi_n)$ and let θ be the connection matrix of D with respect to the basis v, while θ^* be the matrix of the connection D^* with respect to the basis ϕ. Setting $\phi = \phi' + \phi''$, where ϕ' is the $(1,0)$ component of the 1-form ϕ, the condition $D^{*\prime\prime} = \overline{\partial}$ implies that $\theta^{*\prime\prime} = \phi''$. But then

$$\theta^* = \phi,$$

since $\phi + {}^t\overline{\phi} = 0$ and $\theta + {}^t\theta^* = 0$, since the matrix of a metric connection is skew-Hermitian with respect to a unitary basis. By equation (5) from the previous section, it follows that $\theta = -\theta^{t*}$. Consequently, the connection matrix of a holomorphic tangent bundle satisfies

$$\theta = -{}^t\phi.$$

Thus, computing the external differentials $d\phi_i$ of a unitary basis of $T_X^{1,0*}$ allows us to find the matrices θ and θ^* of the connections in holomorphic tangent and cotangent bundles, respectively.

The vector $\tau = (\tau_1, \dots, \tau_n)$, defined in the lemma above is called *the torsion.*

6.3. The Hermitian metric $ds^2 = \sum \phi_i \otimes \overline{\phi}_i$ defines a Hermitian scalar product $(\phi(z), \eta(z))$ in the fiber over the point z of the sheaf $\mathcal{E}_X^* = \bigoplus_{p,q} \mathcal{E}_X^{p,q}$, such that the basis vectors $\phi_I \wedge \overline{\phi}_J = \phi_{i_1} \wedge \dots \wedge \phi_{i_p} \wedge \overline{\phi}_{j_1} \wedge \dots \wedge \overline{\phi}_{j_q}$ are mutually orthogonal, and their lengths $\|\phi_I \wedge \overline{\phi}_J\| = 2^{p+q}$. If X is compact, then this Hermitian product gives rise to a global Hermitian product (,) on the set of sections $H^0(X, \mathcal{E}_X^*)$ of the sheaf $\mathcal{E}_X^*$:

$$(\phi, \eta) = \int_X (\phi(z), \eta(z)) dV,$$

and thus turns $H^0(X, \mathcal{E}_X^*)$ into a pre-Hilbert space.

Let $T : \mathcal{E}_X^* \to \mathcal{E}_X^*$ be a $\mathbb{C}$-linear operator. The operator T' is called *adjoint* to T if $(T\phi, \psi) = (\phi, T'\psi)$. The operator T is called *real*, if it sends real-valued forms onto real-valued forms. We will say that T has *type* (r, s) if $T(\mathcal{E}_X^{p,q}) \subset \mathcal{E}_X^{p+r,q+s}$.

In order to compute adjoints we will make extensive use of the operator (the *Hodge* $*$ *operator*)

$$* : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{n-p,n-q},$$

defined by the requirement

$$(\phi(z), \eta(z)) dV = \phi(z) \wedge *\eta(z),$$

for all $\phi, \eta \in \mathcal{E}_{X,z}^{p,q}$. If $ds^2 = \phi_i \otimes \overline{\phi}_i$, then for the form $\eta = \sum \eta_{I,J} \phi_I \wedge \overline{\phi}_J$, we have

$$*\eta = \pm 2^{p+q-n} \sum_{|I|=p, |J|=q} \eta_{IJ} \phi_{\overline{I}} \wedge \overline{\phi}_{\overline{J}},$$

where $\overline{I} = \{1, \dots, n\} \backslash I$ and the sign $+$ is used when $\phi_I \wedge \overline{\phi}_J \wedge \phi_{\overline{I}} \wedge \overline{\phi}_{\overline{J}} = \Omega^n$. It can be checked that

$$**\eta = (-1)^k \eta$$

for a k-form η.

It should be noted that the operator $*$ comes from linear algebra. Namely, let V be an oriented n-dimensional Euclidean space. Then we can define an operator $* : \bigwedge^p V \to \bigwedge^{n-p} V$, with the following properties. If $\omega = v_1 \wedge \dots \wedge v_p$

is a monomial multivector, then $*\omega$ is a monomial multivector $*\omega = v_{p+1} \wedge \ldots \wedge v_n$, such that

1) The vector spaces spanned by the sets $\{v_1, \ldots, v_p\}$ and $\{v_{p+1}, \ldots, v_n\}$ are orthogonal complements to each other;
2) The $(n-p)$-dimensional volume of the parallelopiped $\Pi * \omega$, spanned by $\{v_{p+1}, \ldots, v_n\}$ is equal to the p dimensional volume of the parallelopiped $\Pi\omega$.
3) $\omega \wedge *\omega > 0$, with the orientation chosen for V.

Let δ be adjoint to the exterior differentiation operator d. For a k form ω we have

$$\begin{aligned}(d\omega, \eta) &= \int_X d\omega \wedge *\eta \\ &= \int_X [d(\omega \wedge *\eta) + (-1)^k \omega \wedge * *^{-1} d * \eta] \\ &= (-1)^k (\omega, *^{-1} d * \eta) = -(\omega, *d * \eta).\end{aligned}$$

Thus, $\delta = - * d *$. Since $d^2 = 0$, $\delta^2 = 0$, also.

Analogously, it can be shown that the operators δ' and δ'', adjoint to ∂ and $\bar\partial$ are $\delta' = - * \partial *$, $\delta'' = - * \bar\partial *$, and hence are operators of types $(-1, 0)$ and $(0, -1)$, respectively.

6.4. The self-adjoint operator $\Delta = (d + \delta)^2 = d\delta + \delta d$ is called *the Laplace operator* (or *the Laplacian*). A form ω is called *harmonic*, if it is in the kernel of the Laplacian, that is $\Delta\omega = 0$.

It should be noted that under the usual Hermitian metric $ds^2 = \sum_{j=1}^n dz_j \otimes d\bar z_j$ on $\mathbb{C}^n$, $\Delta = \sum_{j=1}^n \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right)$, so Δ coincides with the standard Laplace operator. The operator Δ is an elliptic operator (see de Rham [1955]).

Using the commutation relations $\Delta d = d\Delta$, $\Delta\delta = \delta\Delta$, $\Delta * = * \Delta$, it can be shown that $\omega \in H^0(X, \mathcal{E}_X^*)$ is harmonic if and only if $d\omega = \delta\omega = 0$. In particular, every harmonic form is closed.

Denote by $\mathcal{H}_1$ the space of harmonic forms on X; by $\mathcal{H}_2 = dH^0(X, \mathcal{E}_X^*)$, the space of exact forms, and by $\mathcal{H}_3 = \delta H^0(X, \mathcal{E}_X^*)$. The following lemma can be easily checked:

Lemma. *$\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are mutually orthogonal, and if $\omega \perp \mathcal{H}_i$, $i = 1, 2, 3$, then $\omega = 0$.*

The lemma, in particularly, implies that a harmonic form is not exact, since $\mathcal{H}_1 \perp \mathcal{H}_2$. Therefore, $\mathcal{H}_1$ is contained in the space $H^*(X, \mathbb{C})$, isomorphic by de Rham's theorem to the quotient of the space of closed forms by the space of exact forms.

Let X be compact. It is known (de Rham [1955]) that in that case the spaces $H^k(X, \mathbb{C})$ are finite-dimensional. Therefore, the space $\mathcal{H}_1$ is also finite-dimensional. The finite dimensionality of $\mathcal{H}_1$ allows us to define the harmonic projection operator

$$H : H^0(X, \mathcal{E}_X^*) \to \mathcal{H}_1,$$

such that $(\phi, \psi) = (H\phi, \psi)$ for all $\psi \in \mathcal{H}_1$. This uniquely defines H.

By the theory of compact self-adjoint operators in Hilbert space (de Rham [1955]), it can be shown that on a compact Hermitian manifold the equation

$$\Delta\omega = \phi$$

for a prescribed form ϕ has a solution ω if and only if $\phi \perp \mathcal{H}_1$ (de Rham [1955]). This implies that there exists a unique operator G (*the Green-de Rham operator*) satisfying the following conditions:

i) For every $\omega \in H^0(X, \mathcal{E}_X^*)$

$$H\omega + \Delta G\omega = \omega.$$

ii) $(G\omega, \phi) = 0$ for every $\phi \in \mathcal{H}_1$.

It can be easily checked that H and G commute with any operator which commutes with Δ. In particular, H and G commute with Δ, d, δ, and $*$. Also, and the lemma above imply that

$$H^0(X, \mathcal{E}_X^*) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3.$$

Let ω be a closed form. Then

$$\omega = H\omega + d\delta G\omega + \delta dG\omega = H\omega + d\delta G\omega + \delta G d\omega = H\omega + d\delta G\omega.$$

This implies the following:

Theorem. *On a compact Hermitian manifold X every closed form ω is cohomologous to the harmonic form $H\omega$, and so*

$$H^*(X, \mathbb{C}) \simeq \mathcal{H}_1.$$

6.5. The theory of harmonic forms for the operator $\Delta_{\overline{\partial}} = \overline{\partial}\delta'' + \delta''\overline{\partial}$ is constructed analogously. With the notation

$$\mathcal{H}^{p,q} = \mathrm{Ker}[\Delta_{\overline{\partial}} : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q})],$$

it can be shown (see, *eg.*, [Chern 1955]) that for a compact manifold, the spaces $\mathcal{H}^{p,q}$ are finite-dimensional, and the following operators exist: the harmonic projection operator $H_{\Delta_{\overline{\partial}}} : H^0(X, \mathcal{E}_X^{p,q}) \to \mathcal{H}^{p,q}$ and the Green's operator $G : H^0(X, \mathcal{E}_X^{p,q}) \to H^0(X, \mathcal{E}_X^{p,q})$. These operators satisfy the following conditions:

i) For every form $\phi \in H^0(X, \mathcal{E}_X^{p,q})$ and $\psi \in \mathcal{H}^{p,q}$,

$$(\phi, \psi) = (H_{\Delta_{\overline{\partial}}}\phi, \psi);$$

ii) For every $\omega \in H^0(X, \mathcal{E}_X^{p,q})$

$$\omega = H_{\Delta_{\overline{\partial}}}\omega + \Delta_{\overline{\partial}} G\omega;$$

iii) For every $\psi \in \mathcal{H}^{p,q}$ and $\omega \in H^0(X, \mathcal{E}_X^{p,q})$

$$(G\omega, \psi) = 0.$$

These conditions imply that

$$H^0(X, \mathcal{E}_X^{p,q}) = \mathcal{H}^{p,q} \oplus \overline{\partial} H^0(X, \mathcal{E}_X^{p,q-1}) \oplus \delta'' H^0(X, \mathcal{E}_X^{p,q+1}), \tag{6}$$

and all of the summands are mutually orthogonal. It thus follows that every $\overline{\partial}$-closed (p,q) form is $\overline{\partial}$-cohomologous to a form in $\mathcal{H}^{p,q}$. Applying the decomposition (6) to Dolbeault's isomorphism (see §4.3) we obtain the following expression for the space of *$\overline{\partial}$-harmonic* (p,q)-forms:

$$\mathcal{H}^{p,q} = H^q(X, \Omega_X^p).$$

A simple check reveals that $\Delta_{\overline{\partial}}$ and $*$ commute. Thus, the operator $*$ induces *the Kodaira-Serre isomorphism, or Kodaira-Serre duality*

$$* : \mathcal{H}^{p,q} \to \mathcal{H}^{n-p,n-q}.$$

In particular, $\mathcal{H}^{n,n} \simeq \mathbb{C}dV$, where $dV = *1$ is the volume form of the Hermitian metric.

6.6. Every compact complex manifold has many different Hermitian metrics. For a general Hermitian manifold the operators Δ and $\Delta_{\overline{\partial}}$ are completely unrelated. However, if the Hermitian metric in question is also a *Kähler metric*, which means that it satisfies

$$d\Omega = 0,$$

where Ω is the $(1,1)$-form associated to the metric, then $\Delta = 2\Delta_{\overline{\partial}}$. The coincidence of harmonic and $\overline{\partial}$-harmonic forms has many very interesting and non-trivial cohomological consequences, which are studied in the next section.

§7. Kähler Manifolds

7.1.

Definition. A complex manifold X is called *a Kähler manifold*, if it possesses a Kähler metric, which is a Hermitian metric ds^2, such that the associated $(1,1)$-form Ω is closed: $d\Omega = 0$.

Let us give a few more equivalent definitions of a Kähler manifold. We will say that a metric ds^2 has k-th order contact with the Hermitian metric $\sum dz_j \otimes d\overline{z}_j$ on $\mathbb{C}^n$, if in a neighborhood of every point $x_0 \in X$, there exist holomorphic local coordinates $z_1, \dots, z_n$, such that

$$ds^2 = \sum (\delta_{ij} + g_{ij}(x)) dz_i \wedge d\bar{z}_j,$$

where g_{ij} have a zero of order k at x_0. It turns out (see, *eg.*, Griffiths–Harris [1978]) that a Hermitian metric on X is Kähler if and only if it has second order contact with the Hermitian metric on $\mathbb{C}^n$.

Here is a second equivalent definition. In Section 6.2 we defined the torsion vector of $(2,0)$-forms $\tau = (\tau_1, \dots, \tau_n)$ of a metric connection on the tangent bundle of a Hermitian manifold. It turns out (Griffiths–Harris [1978]) that a Hermitian metric is Kähler if and only if $\tau = 0$.

It should be pointed out immediately that the $(1,1)$-form Ω associated to a Kähler metric on a compact complex manifold X cannot be exact, since $\Omega^n = n! dV$ is a nonzero class in $H^{2n}(X, \mathbb{C})$. Thus Ω^k defines a nonzero class in all the even-dimensional cohomologies $H^{2k}(X, \mathbb{C})$, $k \leq n$.

7.2. One of the most important examples of Kähler manifolds is the projective space $\mathbb{P}^n$. Let $(u_0 : \dots : u_n)$ be homogeneous coordinates in $\mathbb{P}^n$. Consider the differential $(1,1)$ form

$$\Omega_j = i\partial\bar{\partial} \log \sum_{k=0}^{n} \left| \frac{u_k}{u_j} \right|^2$$

in the neighborhood $\{u_j \neq 0\}$. Since in the open set $\{u_j \neq 0, u_l \neq 0\}$

$$\partial\bar{\partial} \log \left| \frac{u_j}{u_l} \right|^2 = 0,$$

the forms Ω_j and Ω_l coincide in that neighborhood and thus they define a form Ω globally on $\mathbb{P}^n$. This form is closed since

$$d\partial\bar{\partial} = \partial^2\bar{\partial} - \partial\bar{\partial}^2 = 0.$$

Let $z_j = \frac{u_j}{u_0}$, $j = 1, \dots, n$ be nonhomogeneous coordinates in $U_0 = \{u_0 \neq 0\}$. Then

$$\Omega = i\partial\bar{\partial} \log \left(1 + \sum_{j=1}^{n} |z_j|^2 \right) = \sum_{r,s} \omega_{r,s} dz_r \wedge d\bar{z}_s,$$

where

$$w_{r,s} = \frac{\partial^2 H}{\partial z_i \bar{\partial} z_s},$$

$$H = \log(1 + \sum |z_j|^2).$$

Evidently, $(\omega_{r,s})$ is a Hermitian matrix. A fairly simple calculation shows that $(\omega_{r,s})$ is a positive-definite matrix. Therefore, Ω defines a Kähler metric on $\mathbb{P}^n$. This is the so-called *Fubini-Study metric.*

It is easy to see that a non-singular projective variety $X \subset \mathbb{P}^n$ with the induced metric is also a Kähler manifold.

7.3. Let X be a compact Kähler manifold and Ω the $(1,1)$-form associated to the Kähler metric. Denote by $L : \mathcal{E}_X^k \to \mathcal{E}_X^{k+2}$ the operator induced by Ω :

$$L(\phi) = \Omega \wedge \phi.$$

The operator L is a real operator of type $(1,1)$, since Ω is a real $(1,1)$-form. Since $d\Omega = 0$, $dL = Ld$. Let Λ be the adjoint operator of L. Local computations show that

$$[\Lambda, L] = \Lambda L - L\Lambda = \sum_{r=0}^{2n} (n-r)\Pi_r, \tag{7}$$

where $\Pi_r : \mathcal{E}_X^* \to \mathcal{E}_X^r$ is the projection. Let

$$\Pi = \sum_{r=0}^{2n} (n-r)\Pi_r.$$

It can be checked that

$$[\Pi, L] = -2L, \tag{8}$$
$$[\Pi, \Lambda] = 2\Lambda. \tag{9}$$

Let C be the linear operator acting on (p,q)-forms by $C\omega = i^{p-q}\omega$. It can easily be checked that C commutes with the operators $*$, L, and Λ. In addition, local computation lead to the following relations:

$$\begin{aligned} [\Lambda, d] &= -C^{-1}\delta C, \\ [L, \delta] &= C^{-1} dC, \\ C^{-1}\delta C &= i(\delta' - \delta''), \\ -C^{-1} dC &= (\partial - \overline{\partial}). \end{aligned}$$

It follows that Δ is an operator of type $(0,0)$ commuting with C, L, Λ, and $\Pi_{p,q}$, and so

$$\Delta = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}.$$

As a corollary we get a decomposition of the space of harmonic forms

$$\mathcal{H}_1^r = \bigoplus_{p+q=r} \mathcal{H}^{p,q}, \tag{10}$$
$$\mathcal{H}^{p,q} = \overline{\mathcal{H}^{q,p}}. \tag{11}$$

In particular, the spaces of holomorphic forms $\mathcal{H}^{p,0} = H^0(X, \Omega^p)$ consist of harmonic forms for any Kähler metric. Hence, holomorphic forms on a Kähler manifold are closed.

We should point out the fundamental importance of the decomposition given by equations (10) and (11). In general, of course, every form $\phi \in \mathcal{E}_X^k$

decomposes into a sum $\phi = \sum \phi^{p,q}$ of its (p,q) components. But on an arbitrary complex manifold this decomposition doesn't work on the cohomological level. For a Kähler manifold, on the other hand, it makes sense to talk of the decomposition of cohomology into (p,q) types. Indeed, let $H^{p,q}(X)$ be the quotient of the space of closed (p,q) forms on X over the space of exact (p,q) forms. Then, since Δ commutes with $\Pi_{p,q}$, the harmonic projection operator H also commutes with $\Pi_{p,q}$ (see Section 6.4). Therefore, if ω is a closed (p,q) form, then $\omega = H\omega + d\delta G\omega$, where $H\omega$ is also of type (p,q). In other words, every closed (p,q)-form is cohomologous to a harmonic (p,q)-form. Therefore,

$$H^{p,q} \simeq \mathcal{H}^{p,q}.$$

Finally, combining the decompositions (10) and (11) with the isomorphism $H^r(X,\mathbb{C}) \simeq \mathcal{H}_1^r$, we obtain the famous

Hodge Decomposition. On a compact Kähler manifold X there is the following decomposition of cohomology with complex coefficients

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

satisfying the additional relation

$$H^{p,q}(X) = \overline{H^{p,q}(X)}.$$

The dimensions $h^{p,q} = \dim H^{p,q}(X)$ are called the *Hodge numbers.*

It is noteworthy that in the proof of Hodge decomposition for cohomology of a Kähler manifold X the Kähler metric was used in a central way. On the other hand, the spaces $H^{p,q}(X)$ do not depend on the metric – different Kähler metrics lead to the same Hodge decomposition.

Hodge decomposition connects k-th Betti numbers $b_k = \dim H^k(X,\mathbb{R})$ with Hodge numbers:

$$b_k = \sum_{p+q=k} h^{p,q}.$$

As a consequence of the equality $h^{p,q} = h^{q,p}$, it follows that *odd Betti numbers of a Kähler manifold are even*

$$b_{2r+1} = 2\sum_{p=0}^{r} h^{p,2r+1-p}.$$

In addition, recall that *the even Betti numbers of a Kähler manifold are positive*

$$b_{2r} > 0$$

for $0 \le r \le n$, since the form Ω^r defines a non-zero class in $H^{2r}(X,\mathbb{R})$, and hence $h^{r,r} > 0$.

Yet another consequence of the existence of a Kähler metric on X is the assertion that if $Y \subset X$ is a closed complex submanifold of X, then Y is not null-homologous in X. Indeed, the restriction of the Kähler metric ds^2 to Y turns Y into a Kähler manifold, and the associated $(1,1)$ form of the induced metric coincides with $\Omega|_Y$. Thus

$$\int_Y \Omega^{\dim Y} \neq 0.$$

7.4. It should be noted that not every compact complex manifold can be equipped with a Kähler metric. An example is *Hopf's manifold*, constructed as follows:

Fix $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and let Γ be the group acting on $\mathbb{C}^2 \backslash \{0\}$, generated by the transformation

$$(z_1, z_2) \to (\lambda z_1, \lambda z_2).$$

It is easy to see that the quotient space $M = \{\mathbb{C}^2 \backslash \{0\}\}/\Gamma$ has a complex structure induced by the complex structure on $\mathbb{C}^2 \backslash \{0\}$. Furthermore, M is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, since $\mathbb{C}^2 \backslash \{0\}$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ under the map

$$(z_1, z_2) \to r(u_1, u_2),$$

where $r \in \mathbb{R}$, $|u_1|^2 + |u_2|^2 = 1$. It follows that $b_1(M) = 1$ and so M is not a Kähler manifold.

7.5. The Lefschetz decomposition of the cohomology of a complex Kähler manifold. Let $\mathfrak{U} = \mathbb{C}(L, \Lambda, \Pi)$ be the Lie algebra of linear operators on $\mathcal{H}_1$, generated over $\mathbb{C}$ by operators L, Λ, and Π, with multiplication given by $[A, B] = AB - BA$. The commutation relations (7), (8), and (9) show that $\mathfrak{U}$ is isomorphic to the algebra $\mathfrak{sl}_2$ of complex 2×2 matrices with trace 0 and the multiplication $[A, B] = AB - BA$. Indeed, $\mathfrak{sl}_2$ is generated over $\mathbb{C}$ by the matrices

$$\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad l = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is not hard to check that $[\lambda, l] = \pi$, $[\pi, l] = -2l$, $[\pi, \lambda] = 2\lambda$. Therefore, the identification of λ, l, and π with Λ, L, and Π defines an isomorphism between $\mathfrak{sl}_2$ and $\mathfrak{U}$. This isomorphism makes $\mathcal{H}_1$ an $\mathfrak{sl}_2$-module.

As is well known (Serre [1965]) every finite dimensional $\mathfrak{sl}_2$ module is totally reducible – it can be represented as direct sum of irreducible submodules. (Recall that a module is called irreducible if it contains no proper non-trivial submodules.)

Let V be an irreducible $\mathfrak{sl}_2$ module. Call a vector $v \in V$ *primitive*, if v is an eigenvector of π and $\lambda v = 0$. The commutation relations imply that if v is an eigenvector of π, then λv and lv are likewise eigenvectors. Thus, since $\lambda^{n+1} = 0$, (where $n = \dim V$) primitive vectors always exist.

Let v be a primitive vector and pick k so that $l^k = v \neq 0$, while $l^{k+1}v = 0$. It can be seen that the space V_k generated by vectors $v, lv, \dots, l^k v$ is an invariant $\mathfrak{sl}_2$-module, $\dim_C V_k = k+1$ and $\pi v = kv$.

Now let us apply the above construction to the $\mathfrak{sl}_2$-module $\mathcal{H}_1$. Let $\omega \in \mathcal{H}_1$ be a primitive form, that is $\Lambda\omega = 0$ and

$$\Pi\omega = \sum_{p=0}^{2n}(n-p)\Pi_p\omega = k\omega.$$

Then ω is a homogeneous form of degree $p = n-k \geq 0$. Furthermore $k+1$ is the dimension of the space spanned by $\{\omega, L\omega, \dots, L^k\omega\}$, so $k \geq 0$, $L^k\omega \neq 0$, and $L^{k+1}\omega = 0$. The converse is also true – if a harmonic $n-k$ form ω satisfies the condition $L^k\omega \neq 0$, and $L^{k+1}\omega = 0$, then ω is primitive. Therefore, if we define *primitive cohomology* as

$$\begin{aligned} P^{n-k}(X) &= \mathrm{Ker}[L^{k+1} : H^{n-k}(X) \to H^{n+k+2}(X)] \\ &= \mathrm{Ker}\,\Lambda \cap H^{n-k}(X), \end{aligned}$$

we get the the *Lefschetz decomposition*

$$H^m(X, \mathbb{C}) = \bigoplus_k L^k P^{m-2k}(X).$$

The discussion above implies the *hard Lefschetz theorem* for compact Kähler manifolds, stating that the mappings $L^k : H^{n-k}(X) \to H^{n+k}(X)$ are isomorphisms.

The hard Lefschetz theorem has the following geometric interpretation. Let $X \subset \mathbb{P}^n$ be a non-singular projective variety. Let ds^2 be the Fubini-Study metric on $\mathbb{P}^n$ (see sec. 7.2). Let Ω be the $(1,1)$ form associated to this metric. It defines a non-trivial class $[\Omega] \in H^2(\mathbb{P}^n)$. It can be shown (see, eg, Griffiths–Harris [1978]) that $[\Omega]$ is Poincaré dual to the homology class (H) of a hypersurfaces $H \subset \mathbb{P}^n$, $(H) \in H_{2n-2}(\mathbb{P}^n)$. Therefore, on $X \subset \mathbb{P}^n$ the associated $(1,1)$-form Ω_X of the induced Kähler metric is Poincaré dual to the homology class (E) of the hyperplane section $E = X \cap H$. Therefore, by duality, the strong Lefschetz theorem can be given the following dual formulation. The operation of intersection with an $n-k$-plane $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ defines an isomorphism

$$H_{n+k}(X, \mathbb{C}) \xrightarrow{\cap_{\mathbb{P}^{n-k}}} H_{n-k}(X, \mathbb{C}).$$

It should be noted that Poincaré duality identifies the primitive cohomologies $P^{n-k}(X)$ with the subgroup of $(n-k)$ cycles not intersecting the hyperplane section E, or, in other words, with the image of the map

$$H_{n-k}(X \backslash E) \to H_{n-k}(X).$$

The cycles in this subgroup are called *finite cycles*, since H can be identified with the hyperplane at infinity of $\mathbb{P}^n$.

The Lefschetz decomposition leads to the following inequalities, which must be satisfied by the Betti numbers of a Kähler manifold X :

$$b_r(X) \geq b_{r-2}(X)$$

for $r \leq \dim X$. Together with Kodaira-Serre duality this means that the even (odd) Betti numbers are "hill shaped" (see diagram).

Note that the Lefschetz decomposition agrees with the Hodge decomposition. That is, if we set

$$P^{p,q}(X) = P^{n+q}(X) \cap H^{p,q}(X),$$

then

$$P^k(X) = \bigoplus_{p+q=k} P^{p,q}(X),$$
$$P^{p,q}(X) = \overline{P^{p,q}(X)}.$$

On the other hand, since L is a real operator, then there is an analogous Lefschetz decomposition on $H^*(X,\mathbb{R}) \hookrightarrow H^*(X,\mathbb{C})$:

$$H^m(X,\mathbb{R}) = \bigoplus_k L^k P^{m-2k}(X,\mathbb{R}),$$

where $P^r(X,\mathbb{R})$ is the space of real primitive harmonic r-forms.

7.6. Hodge–Riemann bilinear relations. Consider the bilinear pairing

$$Q : H^{n-k}(X) \otimes H^{n-k}(X) \to \mathbb{C},$$

given by

$$Q(\phi,\psi) = \int_X \phi \wedge \psi \wedge \Omega^k.$$

Since the form Ω is real, Q is a positive bilinear pairing. If $n-k \equiv 0 \bmod 2$, then Q is a symmetric bilinear pairing, while if $n-k$ is odd, Q is a skew-symmetric bilinear pairing. The value $Q(\phi,\psi)$ on homogeneous forms ϕ and ψ is not zero only if $\phi \wedge \psi \wedge \Omega^k$ is a form of type (n,n), hence

$$Q(H^{p,q}, H^{r,s}) = 0, \tag{12}$$

if either $p \neq s$ or $q \neq r$.

It can be shown that for $\omega \in P^{p,q}(X)$ the following inequalities hold (see Griffiths–Harris [1978])

$$i^{p-q}(-1)^{(n-p-q)(n-p-q-1)/2} Q(\omega,\overline{\omega}) > 0,$$

which, together with (12) are known as *Hodge-Riemann bilinear relations.*

Hodge-Riemann bilinear relations together with Lefschetz decomposition imply that $Q : H^k(X) \otimes H^k(X) \to \mathbb{C}$ is a non-degenerate bilinear form, since

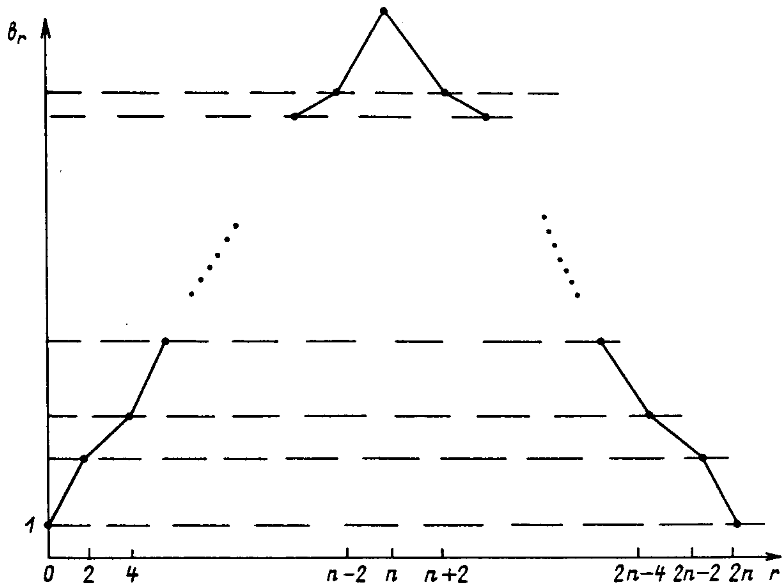


Fig. 4

$$Q(L^r\phi, L^r\psi) = Q(\phi, \psi).$$

7.7. Let X be a compact Kähler manifold of even (complex) dimension, $\dim_{\mathbb{C}} X = 2m$. Hodge-Riemann bilinear relations allow us to compute the *index* $I(X)$ of the manifold X, equal to the signature of the non-degenerate quadratic form on $H^{2m}(X, \mathbb{R})$, determined as

$$(\phi, \psi) = \int_X \phi \wedge \psi$$

for $\phi, \psi \in H^{2m}(X, \mathbb{R})$. Indeed,

$$H^{2m}(X, \mathbb{C}) = \bigoplus L^k P^{2(m-k)}(X) = \oplus_{p+q \leq 2m, p+q \equiv 0 \bmod 2} L^{m-(p+q)/2} P^{p+q}(X)$$

and Hodge-Riemann bilinear relations assert that for $p+q$ even, the symmetric form

$$i^{p-q}(-1)^{(2m-p-q)(2m-p-q-1)/2} Q$$

is positive definite on the real space

$$(P^{p,q}(X) \oplus P^{q,p}(X)) \cap H^{p+q}(X, \mathbb{R}).$$

Therefore,

$$
\begin{aligned}
I(X) &= \sum_{p+q\equiv 0 \bmod 2,\, p+q\leq 2m} i^{p-q}(-1)^{(p+q)(p+q-1)/2} \dim P^{p,q} \\
&= \sum (-1)^p \dim P^{p,q}.
\end{aligned}
$$

Lefschetz decomposition and Kodaira-Serre duality together imply the following theorem.

Index Theorem.
$$
I(X) = \sum_{p+q\equiv 0 \bmod 2} (-1)^p h^{p,q}.
$$

In the case of a compact Kähler surface X ($\dim_{\mathbb{C}} X = 2$), we have $I(X) = 2(h^{2,0}+1) - h^{1,1}$. The number $h^{2,0} = \dim H^0(X, \Omega_X^2)$ is called *the geometric genus* of the surface X.

Let n_+ be the number of positive squares of and n_- the number of negative squares of the bilinear form $(\cdot,\cdot)$ on $H^2(X,\mathbb{R})$ for the surface X. Then

$$
\begin{aligned}
n_+ - n_- &= 2(h^{2,0}+1) - h^{1,1}, \\
n_+ + n_- &= 2h^{2,0} + h^{1,1},
\end{aligned}
$$

hence $n_+ = 2h^{2,0} + 1$.

7.8. A compact Kähler manifold X is called a *Hodge manifold* if the $(1,1)$-form Ω associated to the metric is integral, that is, Ω lies in the image of the homomorphism

$$
j_* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R}),
$$

induced by the inclusion $j : \mathbb{Z} \hookrightarrow \mathbb{R}$.

Note that $\mathbb{P}^n$ is a Hodge manifold, since $\dim H^2(\mathbb{P}^n,\mathbb{C}) = 1$, and so the form Ω associated to the Fubini-Study metric is proportional to an integral form.

For an inclusion of manifolds $i : Y \hookrightarrow X$ we have a commutative diagram,

$$
\begin{tikzcd}
H^2(X,\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "j_*"] & H^2(Y,\mathbb{Z}) \arrow[d, "j_*"] \\
H^2(X,\mathbb{R}) \arrow[r, "i^*"] & H^2(Y,\mathbb{R})
\end{tikzcd}
$$

where i^* is induced by restriction of forms defined on X to Y. Therefore, a closed non-singular submanifold of a Hodge manifold is a Hodge manifold (with the induced metric). In particular, every non-singular projective variety is a Hodge manifold. Kodaira proved (Kodaira [1954]) that the converse is also true, that is, any Hodge manifold can be embedded into a projective space.

The above implies, in particular, that every compact non-singular curve ($\dim_{\mathbb{C}} X = 1$) is an algebraic variety. Indeed, every Hermitian metric on X is Kähler , since there are no differential 3-forms on X, and so the $(1,1)$ form Ω associated with the metric is closed. Furthermore, $\dim H^2(X,\mathbb{R}) = 1$ and so the class of the form Ω in $H^2(X,\mathbb{R})$ is a multiple of an integral class.

It should be further remarked that for a Hodge manifold, both the Lefschetz decomposition and the quadratic form Q are defined over $\mathbb{Z}$.

7.9. To conclude this section we will give an example of a Kähler manifold which is not a Hodge manifold. In order to do this, consider a discrete subgroup (lattice) Γ of $\mathbb{C}^n$ generated by $2n$ vectors linearly independent over $\mathbb{R}$. Let $T = \mathbb{C}^n/\Gamma$ be the quotient complex torus. The complex structure on $\mathbb{C}^n$ induces a complex structure on T and gives it the structure of a Kähler manifold, whose Kähler metric is induced by an arbitrary hermitian metric $h_{ij}dz_i \otimes d\bar{z}_j$ with constant coefficients on $\mathbb{C}^n$.

Conversely, given a Kähler metric $ds^2 = \sum h_{ij}(z)dz_i \otimes d\bar{z}_j$ on the torus T, we can integrate the coefficients of this metric over T to obtain a Kähler metric with constant coefficients $\sum h_{ij}dz_i \otimes d\bar{z}_j$, where

$$h_{ij} = \int_T h_{ij}(z)dV,$$

where dV is a translation-invariant volume form, rescaled so that the volume of T is 1. Since the integration is over the torus T and over a 2-cycle, it can be shown that if the metric we started with was a Hodge metric, then so is the metric obtained by integration.

Fix a metric ds^2 with constant coefficients on the complex torus $T = \mathbb{C}^n/\Gamma$. This metric induces a positive definitive Hermitian form $H(x,y) = \operatorname{Re} H(x,y) + i\operatorname{Im}(x,y)$ on $\mathbb{C}^n$. Linear algebra tells us that the Hermitian form $H(x,y)$ is uniquely determined by a skew-symmetric $\mathbb{R}$-bilinear form $\Omega = \operatorname{Im} H(x,y)$ on $\mathbb{C}^n$, and $\operatorname{Re} H(x,y) = \Omega(ix,y)$. From here, it is a short leap to obtaining necessary and sufficient conditions on the bilinear form $\Omega(x,y)$ for the metric ds^2 to be Hodge.

Riemann–Frobenius conditions. *The torus $T = \mathbb{C}^n/\Gamma$ is a Hodge manifold if and only if there exists a real-valued $\mathbb{R}$-bilinear form Ω on $\mathbb{C}^n$, such that*

1) *The form $\Omega(ix,y)$ is symmetric and positive-definite.*
2) *$\Omega(\alpha,\beta)$ is a rational number for any $\alpha,\beta \in \Gamma$.*

Note that the second of the Riemann-Frobenius conditions is equivalent to saying that the $(1,1)$-form associated with the metric ds^2 is rational, and hence some multiple of it is integral.

The Riemann-Frobenius conditions can be easily used to give an example of a Kähler non-Hodge manifold. Consider the lattice Γ in $\mathbb{C}^2$ generated by the vectors $e_1 = (1,0)$, $e_2 = (i,0)$, $e_3 = (\pi\sqrt{2},\pi)$, $e_4 = (\sqrt{2},i)$. We will show that the torus $\mathbb{C}^2/\Gamma$ is not a Hodge manifold. In order to do this, let $I : \mathbb{C}^2 \to \mathbb{C}^2$ be the $\mathbb{R}$-linear transformation, given by multiplication by $i = \sqrt{-1}$,

$$I(z_1, z_2) = (iz_1, iz_2).$$

It can be easily seen that if $\mathbb{C}^2$ is regarded as a vector space over $\mathbb{R}$ with the basis (e_1, e_2, e_3, e_4), then I is given by the matrix

$$I = \begin{pmatrix} 0 & -1 & \pi\sqrt{2} & \sqrt{2} \\ 1 & 0 & \pi\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & -\frac{1}{\pi} \\ 0 & 0 & \pi & 0 \end{pmatrix}.$$

By the Riemann-Frobenius conditions, in order for $\mathbb{C}^2/\Gamma$ to be a Hodge manifold, it is necessary and sufficient that there exist a skew-symmetric $\mathbb{R}$-bilinear form $\Omega(x, y)$ on $\mathbb{C}^2$, such that $\Omega(Ix, y)$ is symmetric and positive definite. Suppose that Ω is defined over the basis (e_1, e_2, e_3, e_4) by the skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

The second of the Riemann-Frobenius conditions implies that a, b, c, d, e, f are rational numbers, while the first condition implies that ${}^tI\Omega$ is a symmetric and positive matrix.

A direct computation shows that

$$ {}^tI\Omega = \begin{pmatrix} -a & 0 & d & e \\ 0 & -a & -b & -c \\ -\pi(a\sqrt{2} + c) & -\pi(a\sqrt{2} + e) & * & * \\ * & * & * & * \end{pmatrix}. \tag{13}$$

Since ${}^tI\Omega$ is supposed to be symmetric, it follows that

$$d = -\pi(a\sqrt{2} + c),$$
$$b = \pi(a\sqrt{2} + e).$$

Since $a, b, \dots, f$ are rational, (13) implies that $a = b = c = d = e = 0$, hence ${}^tI\Omega$ cannot be a positive-definite matrix, since its first two rows are zero.

§8. Line Bundles and Divisors

8.1. Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$.A complex subspace $V \subset X$ with $\dim_{\mathbb{C}} V = n-1$ will be called a *hypersurface* of X.

Definition. *A divisor D on X is* formal linear combination

$$D = \sum r_i V_i$$

of irreducible hypersurfaces of X, $r_i \in \mathbb{Z}$.

A divisor $D = \sum r_i V_i$ is called *effective* $(D \geq 0)$ if all the $r_i \geq 0$.

Recall that the local rings $\mathcal{O}_{X,x}$ are unique factorization domains (see Gunning-Rossi [1965]). Therefore, for any irreducible hypersurface V in X, the ideal $I_x(V)$ of functions holomorphic at x and vanishing on V is principal, that is, generated by a single element over $\mathcal{O}_{X,x}$. Let f be the generator of $I_x(V)$. It can be then be shown (see Gunning-Rossi [1965]) that f is a generator of $I'_x(V)$ for all points x' in a certain neighborhood U of x. This function f (more precisely $f = 0$) is usually called *the local equation* of V in a neighborhood of x.

Let g be a holomorphic function in some neighborhood of x and let V be a hypersurface. Choose a local equation f for V at x. Then

$$g = f^k h,$$

where the function h (holomorphic at x) doesn't vanish along V. Evidently, the exponent k does not depend on the choice of the local equation f for V, and it can be shown that it does not change as we move from x to another point on V. Thus, *the order* $\mathrm{ord}_V(g)$ *of the function g along V* is well defined: $\mathrm{ord}_V(g) = k$. It is easy to see that

$$\mathrm{ord}_V(g_1 g_2) = \mathrm{ord}_v(g_1) + \mathrm{ord}_V(g_2).$$

Let f be a meromorphic function on V. If f is locally represented as $f = \frac{g}{h}$, where g and h are holomorphic, we can define

$$\mathrm{ord}_V(f) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h)$$

as the *order of f* along the irreducible hypersurface V. We say that *f has a zero of order k on V*, if $\mathrm{ord}_V(f) = k > 0$ and that *f has a pole of order k on V* if $\mathrm{ord}_V(f) = -k < 0$.

Definition. *A divisor (f) of a meromorphic function f* is a divisor

$$(f) = \sum_V \mathrm{ord}_V(f) V.$$

The divisors of meromorphic functions are called *principal.*

8.2. Denote by $\mathcal{M}_X^*$ the multiplicative sheaf of meromorphic functions on X which are not identically 0, and by $\mathcal{O}_X^*$, the subsheaf of $\mathcal{M}_X^*$ of nowhere vanishing holomorphic functions. It is easy to see that a divisor D on X corresponds to a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. Indeed, a section $\{f_\alpha\}$ of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$ is a collection of meromorphic functions f_α, defined on open sets U_α, $\cup U_\alpha = X$, where

$$f_\alpha/f_\beta \in H^0(U_\alpha \cap U_\beta, \mathcal{O}_X^*|_{U_\alpha \cap U_\beta}).$$

Thus,

$$\mathrm{ord}_V(f_\alpha) = \mathrm{ord}_V(f_\beta)$$

for every hypersurface V, and hence $\{f_\alpha\}$ defines a divisor

$$D = \sum_V \mathrm{ord}_V(f_\alpha)V$$

where for each V, we choose the α in such a way that $U_\alpha \cap V \neq \emptyset$. Conversely, given a divisor $D = \sum r_i V_i$ we can choose a covering $\{U_\alpha\}$ in such a way that in each U_α the hypersurface V_i has local equation f_i. We can then set $f_\alpha = \prod_i f_i^{r_i} \in H^0(U_\alpha, \mathcal{M}_X^*|_{U_\alpha})$, which defines a global section of the sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. The functions f_α are called the *local equations* of the divisor D. It is therefore seen that

$$\mathrm{Div}\, X \simeq H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*),$$

where $\mathrm{Div}\, X$ is the group of divisors on X. The quotient group $\mathrm{Cl}\, X = \mathrm{Div}\, X/P(X)$, where $P(X)$ is the subgroup of principal divisors on X is called the ***divisor class group*** and two divisors are usually said to be *linearly equivalent* (written $D_1 \sim D_2$) if $D_1 - D_2 = (f)$ is a divisor of a meromorphic function.

The set $|D| \subset \mathrm{Div}\, X$ of all effective divisors, linearly equivalent to the divisor D is called the *complete linear system of the divisor D.*

8.3. Let us establish the relationship between the divisor class group $\mathrm{Cl}\, X$ and the group $\mathrm{Pic}\, X$ of line bundles on X (see §5).

Let D be a divisor on X. Choose local equations $\{f_\alpha\}$ for D, where $\{U_\alpha\}$ is an open covering of X. Then we define $g_{\alpha\beta} = f_\alpha/f_\beta \in H^0(U_\alpha \cap U_\beta, \mathcal{O}_X^*|_{U_\alpha \cap U_\beta})$. It can be easily checked that

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1.$$

Thus, the functions $g_{\alpha\beta}$ are the transition functions of a certain line bundle, called *the line bundle associated to the divisor* D and denoted by $[D]$. It can be seen that the line bundle $[D]$ is independent of the choice of local equations of the divisor $[D]$. It can be further seen that

$$[D_1 + D_2] = [D_1] \otimes [D_2],$$

and furthermore, the line bundle $[D]$ is trivial if and only if D is a principal divisor. Thus, there is a well-defined morphism

$$[\,]: \mathrm{Cl}\, X \to \mathrm{Pic}\, X.$$

This monomorphism has the following cohomological interpretation. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X^* \xrightarrow{\ i\ } \mathcal{M}_X^* \xrightarrow{\ j\ } \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 0.$$

This sequence induces a cohomology exact sequence

$$H^0(X, \mathcal{M}_X^*) \xrightarrow{j_*} H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*) \xrightarrow{\delta} H^1(X, \mathcal{O}_X^*).$$

It can be checked that in terms of the identifications

$$\mathrm{Div}\, X = H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*),$$
$$\mathrm{Pic}\, X = H^1(X, \mathcal{O}_X^*)$$

we have $j_* f = (f)$ for any meromorphic function f and $\delta D = [D]$ for every divisor D on X.

8.4. Consider the image of the homomorphism $[\,]: \mathrm{Cl}\, X \to \mathrm{Pic}\, X$. Let $L \to X$ be a holomorphic line bundle, with the associated sheaf $\mathcal{O}_X(L)$ of holomorphic sections. We can also consider the sheaf $\mathcal{O}_X(L) \otimes \mathcal{M}_X$ of meromorphic sections of L. Note that a meromorphic section $s \in H^0(X, \mathcal{O}_X(L) \otimes \mathcal{M}_X)$ of the bundle L is defined by a collection $s_\alpha \in H^0(U_\alpha, \mathcal{M}_X|_{U_\alpha})$ of meromorphic functions satisfying $s_\alpha = g_{\alpha\beta} s_\beta$, where $\{U_\alpha\}$ is a sufficiently fine covering of X and $\{g_{\alpha\beta}\}$ are the transition functions of the bundle L with respect to this covering. Hence, if s is a meromorphic section of L, we can define a divisor

$$(s) = \sum_V \mathrm{ord}_V(s_\alpha) V,$$

where for every V we choose an index α so that $U_\alpha \cap V \neq \emptyset$. Evidently, $[(s)] = L$ for a meromorphic section s of L if $s \neq 0$. In addition, if s_1 and s_2, $(s_2 \neq 0)$ are two meromorphic sections of L, their quotient s_1/s_2 defines a meromorphic function.

The above implies that a line bundle L lies in the image of the monomorphism $[\,]: \mathrm{Cl}\, X \to \mathrm{Pic}\, X$ if and only if the line bundle L has a non-vanishing meromorphic section. It can be shown (Gunning-Rossi [1965]) , that if X is a complete algebraic variety, then every line bundle has a non-zero meromorphic section, or

$$[\,]: \mathrm{Cl}\, X \to \mathrm{Pic}\, X$$

is an isomorphism. In other words, every line bundle on a complete algebraic variety is associated to a divisor class on X.

8.5. Chern Classes of Line Bundles. Consider the following exact sequence of sheaves on X :

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ j\ } \mathcal{O}_X \xrightarrow{\ \exp\ } \mathcal{O}_X^* \longrightarrow 0,$$

where $\exp(f) = e^{2\pi i f}$ for $f \in H^0(U, \mathcal{O}_X|_U)$. This exact sequence induces a cohomology exact sequence

$$\begin{array}{r} \longrightarrow H^1(X,\mathbb{Z}) \xrightarrow{j_*} H^1(X,\mathcal{O}_X) \xrightarrow{\exp_*} \\ \xrightarrow{\exp_*} H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \xrightarrow{j_*} H^2(X,\mathcal{O}_X) \longrightarrow \end{array} \tag{14}$$

Let $L \in \operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$. *The first Chern class* $c_1(L)$ of the line bundle L is the class $\delta L \in H^2(X,\mathbb{Z})$. By a change of coefficients, the Chern class $c_1(L)$ can be viewed as an element of $H^2(X,\mathbb{R})$ or of $H^2(X,\mathbb{C})$. To compute $c_1(L)$, let us make L a Hermitian bundle. This can be done as follows. Choose a sufficiently fine covering $\{U_\alpha\}$ of X by simply-connected open sets, and let $\{h_{\alpha\beta}\}$ be the transition functions of L corresponding to this covering. Choose real C^∞ functions a_α in U_α, so that $a_\alpha|h_{\alpha\beta}|^2 = a_\beta$ in $U_\alpha \cap U_\beta$. Such a collection of functions exists, since the sheaf $\mathcal{E}_X^+(\mathbb{R})$ of germs of positive C^∞ functions on X is a fine sheaf. Therefore, $H^1(X, \mathcal{E}_X^+(\mathbb{R})) = 0$ and hence the one-dimensional cocycle $|h_{\alpha\beta}|^2$ is null-cohomologous, which guarantees the existence of the desired collection of functions $\{a_\alpha\}$.

The functions a_α define a Hermitian scalar product on on L, since over $U_\alpha \cap U_\beta$ we have

$$a_\alpha u_\alpha \overline{u}_\alpha = a_\beta u_\beta \overline{u}_\beta,$$

where $u_\alpha(u_\beta)$ is the fiber coordinate of the trivial over U_α (or over U_β) line bundle L.

A direct computation shows that for the metric connection D in this Hermitian bundle L, the connection in the neighborhood U_α has the form

$$\theta = \partial \log a_\alpha,$$

while the curvature form is

$$\Theta = \partial\overline{\partial} \log a_\alpha.$$

By computing the boundary homomorphism δ in (14) on one hand, and of the explicit form of the de Rham isomorphism on the other, leads us to the following

Proposition. *For any line bundle L on a compact complex manifold X*

$$c_1(L) = \frac{\sqrt{-1}}{2\pi}\Theta,$$

where Θ is the curvature form of the metric connection on L.

In particular, the Chern class of a line bundle L can be represented by a differential form of type $(1,1)$.

If $X = \mathbb{P}^n$ and $V = \mathbb{P}^{n-1}$ is a hyperplane, then a direct computation (see, Griffiths–Harris [1978]) that the Chern class $c_1([\mathbb{P}^{n-1}])$ coincides with the cohomology class of the $(1,1)$-form associated with the Fubini-Study metric.

Let V be a hypersurface of X. The linear functional $\int_V \phi$ on $H^{2n-2}(X,\mathbb{Z})$ defines a homology class $(V) \in H_{2n-2}(X,\mathbb{Z})$. The Poincaré dual class $\pi_V \in H^2(X,\mathbb{C})$ is called *the fundamental class of the hypersurface* V. Define the *fundamental class* $\pi_D \in H^2(X,\mathbb{C})$ of a divisor $D = \sum r_i V_i$ as

$$\pi_D = \sum r_i \pi_{V_i}.$$

Using Stokes' theorem it is not too hard to obtain the following (see Griffiths–Harris [1978]).

Theorem. *If $L = [D]$ for some divisor D on a compact complex manifold X, then $c_1(L) = \pi_D$.*

In the exact sequence (14) the morphism $j : H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$ can be represented as a composition

$$H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{C}) \xrightarrow{\alpha} H^2(X,\Omega_X^0) = H^2(X,\mathcal{O}_X).$$

If X is a compact Kähler manifold, it can be shown that the morphism α coincides with the projection $\Pi_{0,2}$ onto the space of harmonic $(0,2)$-forms, and hence the kernel of α contains the cocycles of $H^2_{1,1}(\mathbb{Z}) \subset H^2(X,\mathbb{Z})$ which can be represented by closed $(1,1)$-forms. Since the Chern classes $c_1(L) \in H^2(X,\mathbb{C})$ are represented by $(1,1)$-forms, by exactness of the sequence (14) we get

Theorem. *On a compact Kähler manifold the set of Chern classes coincides with $H^2_{1,1}(\mathbb{Z})$.*

8.6. The adjunction formula. Let V be a non-singular hypersurface of a compact complex manifold X. The quotient line bundle

$$N_V = T_X|_V/T_V$$

(where $T_X = T_X^{1,0}$ and $T_V = T_V^{1,0}$ are holomorphic tangent bundles on X and on V) is called the *normal bundle*, and the bundle N_V^* dual to N_V is called the co-normal bundle. The dual bundle N_V^* is a subbundle of $T_X^*|_V$, and consists of all the cotangent vectors on X vanishing on $T_V \subset T_X|_V$.

Let $f_\alpha \in H^0(U_\alpha, \mathcal{O}_X)$ be a local equation for V. Since $f_\alpha \equiv 0$ on $V \cap U_\alpha$, df_α defines a section of the conormal bundle N_V^* over $V \cap U_\alpha$. Since V is a non-singular submanifold, df_α is never zero on $V \cap U_\alpha$. Furthermore, the line bundle $[V]$ is given by transition functions $\{h_{\alpha\beta} = f_\alpha/f_\beta\}$ and on $U_\alpha \cap U_\beta \cap U_\gamma \cap V$

$$df_\alpha = d(h_{\alpha\beta} f_\beta) = f_\beta dh_{\alpha\beta} + h_{\alpha\beta} df_\beta = h_{\alpha\beta} df_\beta.$$

Consequently, the sections $df_\alpha \in H^1(U_\alpha \cap V, \mathcal{O}_V(N_V^*))$ define a global nowhere vanishing section of the bundle $[N_V^*] \otimes [V]|_V$. Thus, $N_V^* \otimes [V]|_V$ is a trivial bundle and

$$N_V^* = [-V]|_V. \tag{15}$$

One of the most important line bundles on X, $\dim X = n$ is the *canonical bundle*

$$K_X = \wedge^n T_X^*.$$

Holomorphic sections of the canonical bundle are holomorphic forms of the highest degree, that is, $\mathcal{O}_X(K_X) = \Omega_X^n$. To compute the canonical bundle K_V of a non-singular hypersurface V of a complex manifold X, there is the following *adjunction formula*:

$$K_V = (K_X \otimes [V]|_V).$$

This formula is derived from the exact sequence

$$0 \to N_V^* \to T_X^*|_V \to T_V^* \to 0,$$

from which, by an elementary argument, follows

$$(\wedge^n T_C^*)|_V \simeq \wedge^{n-1} T_V^* \otimes N_V^*,$$

which, combined with (15) gives the adjunction formula.

§9. The Kodaira Vanishing Theorem

In the study of the geometry of complex manifolds, we frequently need to know whether certain cohomology groups are trivial. In this section we will describe certain sufficient conditions for the groups $H^q(X, \Omega^p(E))$ to be trivial, where E is some line bundle on a compact Kähler manifold. One corollary will be the Lefschetz theorem on hyperplane sections.

9.1.

Definition. A line bundle $E \to X$ is called *positive* if there exists a Hermitian metric with the curvature form Θ, such that the $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\Theta$ is positive. The line bundle E is called *negative* if the dual bundle E^* is positive.

The following lemma shows that a line bundle E is positive if and only if its Chern class $c_1(E) \in H^2(X, \mathbb{C})$ is represented by a positive 2-form.

Lemma. *For any real closed $(1,1)$-form ω of class $c_1(E) \in H^2(X, \mathbb{C})$ there exists a metric connection on the line bundle E with curvature form $\Theta = \frac{\sqrt{-1}}{2\pi}\omega$.*

Indeed, let $|s|^2$ be a metric on E. As we saw in section 8, if $\phi : U \times \mathbb{C} \to E_U$ is the trivialization of E over an open set U, then the metric $|s|^2$ is given by a positive function a_U :

$$|s|^2 = a_U s_U \bar{s}_U,$$

while the curvature form and Chern class are given by formulas

$$\Theta = \bar{\partial}\partial \log a_U,$$

$$c_1(E) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] \in H^2(X, \mathbb{C}).$$

For another metric $|s'|^2$ on E with the curvature form Θ' we have $|s'|^2/|s|^2 = e^f$, where f is a real C^∞ function. Therefore

$$\Theta' = \bar{\partial}\partial f + \Theta,$$

and the lemma follows from the $\partial\bar{\partial}$-lemma below.

Lemma "$\partial\bar{\partial}$-lemma." *Let X is a Kähler manifold and ω is a closed (p,q)-form. Then the following statements are equivalent:*

1) There exists a ϕ_1 such that $\omega = d\phi_1$,
2) There exists a ϕ_2 such that $\omega = \partial\phi_2$,
3) There exists a ϕ_3 such that $\omega = \bar{\partial}\phi_3$,
4) There exists a λ such that $\omega = \partial\bar{\partial}\lambda$.

In addition, if $p = q$ and the form ω is real, then λ can be chosen so that the form $\sqrt{-1}\lambda$ is also real.

The proof of the $\partial\bar{\partial}$-lemma can be found in Griffiths–Harris [1978].

9.2. As noted in the last section, the Chern class of a line bundle $[\mathbb{P}^{n-1}]$ on $\mathbb{P}^n$ is the class of the $(1,1)$ form Ω associated with the Fubini-Study metric. If $X \hookrightarrow \mathbb{P}^n$ is a non-singular projective variety, then the line bundle $[V]$ on X, where $V = X \cap \mathbb{P}^{n-1}$ is a hyperplane section, is also a positive line bundle. Indeed, $c_1(V) = j^* c_1(\mathbb{P}^{n-1})$, where $j : X \to \mathbb{P}^n$ is the inclusion map, while, on the other hand, the form $j^*\Omega$ is the associated $(1,1)$-form of the metric on X induced by the Fubini-Study metric on $\mathbb{P}^n$. Thus, $c_1(V)$ is the class of a positive $(1,1)$-form.

It can be shown that the converse is also true:

Theorem (Kodaira [1954]). *If E is a positive line bundle on the compact complex manifold X, then there exists an inclusion $j : X \to \mathbb{P}^N$, such that $E^{\otimes n} = [V]$ for some integer n, where V is a hyperplane section of X in $\mathbb{P}^N$.*

9.3. Let us study the cohomology groups $H^q(X, \Omega^p(E))$ for a positive line bundle $E \to X$ on a compact Kähler manifold X. This study is conducted by the same methods as used in the Hodge theory of complex manifolds (as in §6 and §7). Let us outline the major points.

First, for the sheaf of E-valued holomorphic p-forms on X $\Omega^p_X(E)$ we have the fine resolution

$$0 \longrightarrow \Omega^p_X(E) \longrightarrow \mathcal{E}^{p,0}_X(E) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}_X \xrightarrow{\overline{\partial}} \dots$$

Since the sheaves $\mathcal{E}^{p,q}_X$ are fine, it follows that

$$H^q(X, \Omega^p_X(E)) = H^{p,q}_{\overline{\partial}}(E),$$

where $H^{p,q}_{\overline{\partial}}(E)$ is the quotient of the space of $\overline{\partial}$-closed C^∞ differential (p,q)-forms with values in E by the subspace of $\overline{\partial}$-exact forms of the same type.

Furthermore, suppose that Hermitian metrics are defined on both the holomorphic line bundle E and on X. These metrics induce Hermitian scalar products in all the hermitian powers of the tangent and cotangent bundles and their tensor products with E and E^*. In particular, if $\{\phi_i\}$ is a unitary basis in T^*_X over some neighborhood $U \subset X$, while $\{e_k\}$ is a unitary basis for E, then for any sections

$$\eta(z) = \frac{1}{p!q!} \sum_{I,J,k} \eta_{I,J,k}(z)\phi_I \wedge \overline{\phi}_J \otimes e_k,$$

$$\psi(z) = \frac{1}{p!q!} \sum_{I,J,k} \psi_{I,J,k}(z)\phi_I \wedge \overline{\phi}_J \otimes e_k$$

of the sheaf $\mathcal{E}^{p,q}_X(E)|_U$ we can define

$$(\eta(z), \psi(z)) = \frac{2^{p+q-n}}{p!q!} \sum_{I,J,k} \eta_{I,J,K}\overline{\psi}_{I,J,k},$$

and we can define the scalar product

$$(\eta, \psi) = \int_X (\eta(z), \psi(z))dV,$$

where dV is the volume form on X.

We can also define the exterior product

$$\wedge : \mathcal{E}^{p,q}_X(E) \otimes \mathcal{E}^{p',q'}_X(E^*) \to \mathcal{E}^{p+p',q+q'}_X,$$

be setting

$$(\eta \otimes s) \wedge (\eta' \otimes s') = \langle s, s' \rangle \eta \wedge \eta'.$$

Just as in Hodge theory (see §6) we can define the $*$ operator, by setting

$$*_E : \mathcal{E}^{p,q}_X(E) \to \mathcal{E}^{n-p,n-q}_X(E^*),$$

satisfying

$$(\eta, \psi) = \int_X \eta \wedge *_E\psi$$

for all $\eta, \psi \in H^0(X, \mathcal{E}_X^{p,q}(E))$. Locally, the operator $*_E$ works as follows. Let $\{e_k\}$ and $\{e_k^*\}$ be the dual unitary bases for E and E^* over U. Then for the form

$$\eta = \sum_k \eta_k \otimes e_k \in H^0(U, \mathcal{E}_X^{p,q}(E)),$$

define

$$*_E\eta = \sum_k *\eta_k \otimes e_k^*,$$

where $*$ is the ordinary $*$ on $\mathcal{E}_X^{p,q}$, introduced in §6.

As before, the operator $*_E$ allows us to compute adjoint operators. In particular, $\overline{\partial}^* = -*_E\overline{\partial}*_E$ is adjoint to $\overline{\partial}$, meaning that for all $\phi \in H^0(X, \mathcal{E}_X^{p,q-1})$ and $\psi \in H^0(X, \mathcal{E}_X^{p,q}(E))$

$$(\overline{\partial}\phi, \psi) = (\phi, \overline{\partial}^*\psi).$$

Finally, we can define the $\overline{\partial}$-Laplacian

$$\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{E}_X^{p,q}(E) \to \mathcal{E}_X^{p,q}(E),$$

and we call an E-valued form ϕ *harmonic* if $\Delta\phi = 0$. Denote by $\mathcal{H}^{p,q}(E) = \operatorname{Ker}\Delta$ the space of harmonic E-valued (p,q)-forms. It can be shown (see Griffiths–Harris [1978]) that $\mathcal{H}^{p,q}(E)$ is a finite-dimensional space.

Furthermore, if H is the orthogonal projection $H^0(X, \mathcal{E}_X^{p,q}(E)) \to \mathcal{H}^{p,q}(E)$, then there exists an operator $G : H^0(X, \mathcal{E}_X^{p,q}(E)) \to H^0(X, \mathcal{E}_X^{p,q}(E))$ such that $G(\mathcal{H}^{p,q}(E)) = 0$, $\mathrm{Id} = H + \Delta G$, and $[G, \overline{\partial}] = [G, \overline{\partial}^*] = 0$. That implies that

$$H_{\overline{\partial}}^{p,q}(E) = \mathcal{H}^{p,q}(E)$$

and hence the operator $*$ induces an isomorphism

$$H^q(X, \Omega_X^p(E)) \simeq H^{n-q}(X, \Omega_X^{n-p}(E^*)).$$

In particular, when $p = 0$ we obtain the isomorphism

$$H^q(X, \mathcal{O}_X(E)) \simeq H^{n-q}(X, \mathcal{O}_X(E^* \otimes K_X)),$$

known as the *Kodaira-Serre duality.*

Now, suppose the line bundle E is positive. Then, by Kodaira's theorem, $E^{\otimes n} = [V]$, where V is the divisor of a hyperplane section of X under some inclusion $X \hookrightarrow \mathbb{P}^N$. Let Ω be the form associated to the Kähler metric on X, and let Θ be the curvature form of the metric connection. We can define an operator

$$L : \mathcal{E}_X^{p,q}(E) \to \mathcal{E}_X^{p+1,q+1}(E),$$

by setting

$$L(\eta \otimes s) = \Omega \wedge \eta \otimes s,$$

and we have the operator $\Theta = \frac{2\pi}{\sqrt{-1}} L$.

Let $D = D' + \overline{\partial}$ be the metric connection on E. Then the operator Θ can be interpreted as

$$\Theta\eta = D^2\eta.$$

Therefore, $\Theta = D^2 = \overline{\partial}D' + D'\overline{\partial}$.

Let $\Lambda = L^*$ be the adjoint operator of L. It can be checked that

$$[\Lambda, \overline{\partial}] = \frac{\sqrt{-1}}{2} D'^*$$

and that

$$[\Lambda, L]\eta = (n - p - q)\eta,$$

for all $\eta \in H^0(X, \mathcal{E}_X^{p,q}(E))$. Let $\eta \in \mathcal{H}^{p,q}(E)$. Then $\overline{\partial}\eta = 0$. Furthermore, $\Theta\eta = \overline{\partial}D'\eta$ and

$$\begin{aligned}
2\sqrt{-1}(\Lambda\Theta\eta, \eta) &= 2\sqrt{-1}(\Lambda\overline{\partial}D'\eta, \eta) \\
&= 2\sqrt{-1}\left(\left(\overline{\partial}\Lambda - \frac{\sqrt{-1}}{2}D'^*\right) D'\eta, \eta\right) \\
&= (D'^* D'\eta, \eta) \\
&= (D'\eta, D'\eta) \geq 0,
\end{aligned}$$

since $(\overline{\partial}\Lambda D'\eta, \eta) = (\Lambda D'\eta, \overline{\partial}^*\eta) = 0$.

A similar computation shows that

$$2\sqrt{-1}(\Theta\Lambda\eta, \eta) = -(D'^*\eta, D'^*\eta) \leq 0.$$

The last two inequalities together show that

$$2\sqrt{-1}([\Lambda, \Theta]\eta, \eta) \geq 0.$$

On the other hand, $\Theta - -2\pi\sqrt{-1}L$. Thus

$$2\sqrt{-1}([\Lambda, \Theta]\eta, \eta) = 4\pi([\Lambda, L]\eta, \eta) = 4\pi(n - p - q)(\eta, \eta) \geq 0.$$

This implies that $\mathcal{H}^{p,q}(E) = 0$, if $p + q > n$, and we get

Kodaira vanishing theorem. *Let E be a positive line bundle over a compact complex manifold X. Then*

$$H^q(X, \Omega_X^p(E)) = 0$$

for $p + q > n$.

By Kodaira-Serre duality, we see that

$$H^q(X, \Omega^p(E)) = 0$$

for $p + q < n$ for a negative line bundle $E \to X$.

9.4. Lefschetz theorem on Hyperplane Sections. Kodaira's vanishing theorem gives a way to prove Lefschetz' famous theorem, relating the cohomology of a non-singular projective variety with the cohomology of a non-singular hyperplane section.

Let X be a non-singular projective variety, $\dim X = n$, and let $V \subset X$ be a non-singular hyperplane section.

Lefschetz Theorem. *The mapping*

$$H^q(X, \mathbb{Q}) \to H^q(V, \mathbb{Q})$$

induced by the inclusion $j : V \hookrightarrow X$ is an isomorphism for $q \leq n-2$ and is an inclusion for $q = n-1$.

Evidently, it is enough to prove this theorem for cohomology with complex coefficients. The cohomologies $H^k(X, \mathbb{C})$ (and correspondingly $H^k(V, \mathbb{C})$) can be decomposed into a sum of Hodge (p, q)-spaces (see §7)

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$. It is, therefore, sufficient to show that the mapping

$$H^q(X, \Omega_X^p) \xrightarrow{j^*} H^q(V, \Omega_V^p)$$

is an isomorphism for $p+q \leq n-2$ and a monomorphism for $p+q = n-1$. To show this, decompose the restriction $\Omega_X^p \to \Omega_V^p$ as a composition

$$\Omega_X^p \xrightarrow{\ \alpha\ } \Omega_X^p|_V \xrightarrow{\ \beta\ } \Omega_V^p.$$

Obviously, the kernel of α is the sheaf of holomorphic p-forms vanishing on V. Therefore, the mapping α is part of the following sheaf exact sequence:

$$0 \longrightarrow \Omega_X^p(-V) \longrightarrow \Omega_X^p \xrightarrow{\ \alpha\ } \Omega_X^p|_V \longrightarrow 0. \qquad (16)$$

The map β is also a part of a sheaf exact sequence. Indeed, for every point $x \in V$, we have

$$0 \to N_{V,x}^* \to T_{X,x}^* \to T_{V,x}^* \to 0,$$

where N_V^* is the co-normal bundle. By taking exterior powers, obtain the exact sequence

$$0 \to N_{V,v}^* \otimes \Lambda^{p-1} T_{V,x}^* \to \Lambda^p T_{X,x}^* \to \Lambda^p T_{V,x}^* \to 0.$$

Therefore, there is a sheaf exact sequence

$$0 \longrightarrow \Omega_V^{p-1}(-V) \longrightarrow \Omega_X^p|_V \xrightarrow{\ \beta\ } \Omega_V^p \longrightarrow 0., \qquad (17)$$

since $N_V^* = [-V]|_V$.

The line bundle $[-V]$ is negative on X and so its restriction $[-V]|_V$ is also negative. Thus, by Kodaira's vanishing theorem

$$\begin{aligned} H^q(X, \Omega_X^p(-V)) &= 0, \quad p+q<n, \\ H^q(X, \Omega_V^p(-V)) &= 0, \quad p+q<n-1. \end{aligned}$$

Writing down the cohomology long exact sequences corresponding to the short exact sequences (16) and (17), we get the isomorphisms

$$H^q(X, \Omega_X^p) \overset{\alpha^*}{\simeq} H^q(V, \Omega_X^p|_V) \overset{\beta^*}{\simeq} H^q(V, \Omega_V^p)$$

for $p+q \leq n-2$, while for $p+q=n-1$ we see that α^* and β^* are injections. Thus the Lefschetz theorem is proved.

By duality, we obtain that the mappings

$$H_k(V, \mathbb{Q}) \to H_k(X, \mathbb{Q})$$

for a hyperplane section V of X are isomorphisms for $k<n-1$ and onto for $k=n-1$.

By the hard Lefschetz theorem (see §7) $H_{n+k}(X, \mathbb{Q}) \simeq H_{n-k}(X, \mathbb{Q})$. In addition, by Lefschetz decomposition, every non-primitive n-cycle can be obtained as an intersection of a cycle of dimension $> n$ with a hyperplane section. Thus, the Lefschetz hyperplane theorem together with various other results of Lefschetz show that the only "new" cohomology, beyond that of a hyperplane section, is primitive cohomology in the middle dimension. This allows one to study the topology of an algebraic variety X inductively, reducing cohomological questions about X to those of its hyperplane sections. This induction is usually effected by way of Lefschetz sheaves, described in the next section.

§10. Monodromy

10.1. In this section we will define the monodromy transform, and also describe certain classical constructions and results having to do with this transform. These results are important in the study of families of complex manifolds, and in the study of their degenerations (see Chapter 2 §3, Chapter 5, §1).

First, let us introduce some topological preliminaries. Let X, Y, B be topological spaces, and $f: X \to Y$– a continuous map. The triple $(X, Y; f)$ will be called a *locally trivial fibration* with fiber B, if for any point $y_0 \in Y$ there exists a neighborhood $U \subset Y$ and a homeomorphism ν, such that the diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\nu"] \arrow[dr, "f"'] & & B \times U \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

commutes. Here π_U is the natural projection of the product $B \times U$ onto the second factor. The homeomorphism ν is called the *local trivialization* of the fibration.

In the current work, locally trivial fibrations will usually arise as follows: Let $f : \mathcal{X} \to S$ be a smooth surjective holomorphic mapping of complex manifolds with compact fibers (a smooth surjective proper morphism (see Hartshorne [1977]). Since the morphism f is smooth (the differential *df* has maximal rank at each point $x \in \mathcal{X}$), all the fibers of f are non-singular compact complex analytic submanifolds of $\mathcal{X}$. Fix some fiber $B = f^{-1}(s_0)$ of f. Then, it can be shown that (see, eg, Wells [1973]) that $(\mathcal{X}, S; f)$ is a locally trivial fibration with fiber B. The trivialization ν can be chosen to be a diffeomorphism of the C^∞ manifolds $f^{-1}(U)$ and $B \times U$.

In the situation described above, the triple $(\mathcal{X}, S; f)$ is called a smooth family of complex analytic manifolds and the fiber $f^{-1}(s)$ over $s \in S$ is denoted as X_s.

10.2. Any locally trivial fibration $(X, Y; f)$ satisfies the *covering homotopy axiom* (see Rokhlin-Fuks [1977]). Namely, for any homotopy

$$\gamma_t : K \to Y, \quad t \in [0,1],$$

of a simplicial complex K and any continuous mapping $\beta_0 : K \to X$, such that $f \circ \beta_0 = \gamma_0$, there exists a homotopy

$$\beta_t : K \to X, \quad t \in [0,1],$$

extending β_0 and such that $f \circ \beta_t = \gamma_t$.

The homotopy β_t is called the covering homotopy for γ_t.

In the sequel we will only consider the situation where the fiber B of a locally trivial fibration $(X, Y; f)$ is a simplicial complex and the base Y is path-connected. Consider the arc

$$\gamma : [0,1] \to Y, \quad \gamma(0) = y_0, \quad \gamma(1) = y_1.$$

This curve defines a homotopy $\gamma_t : B \to Y$, defined by the condition $\gamma_t(b) = \gamma(t)$ for any $b \in B$. Let β_0 be a homeomorphism between B and $f^{-1}(y_0)$. Then there exists a homotopy $\beta_t : B \to X$, covering γ_t and extending β_0. The mapping

$$\mu : f^{-1}(y_0) \to f^{-1}(y_1),$$

defined by the formula

$$\mu(x) = \beta_1(\beta_0^{-1}(x))$$

is a homotopy equivalence of fibers. From the covering homotopy axiom it can be deduced that the homotopy class of the mapping μ depends only on the homotopy type of the arc γ, joining y_0 and y_1 in Y. The mapping μ, defined up to homotopy equivalence, is called the *monodromy transformation* of the fiber $f^{-1}(y_0)$ into the fiber $f^{-1}(y_1)$, defined by the curve γ.

Fix a point $y_0 \in Y$. By associating to the elements of the fundamental group $\pi_1(Y; y_0)$ the monodromies of the fiber $f^{-1}(y_0)$, we obtain a well-defined homomorphism of the group $\pi_1(Y; y_0)$ into the group of homotopy classes of homotopy equivalences of the fiber $f^{-1}(y_0)$. The image of the fundamental group under this homomorphism is called the *monodromy group* of the fiber $f^{-1}(y_0)$.

Let $\mu : B \to B$ be a continuous map. The homotopy class of μ defines endomorphisms of the homology and cohomology groups of the simplicial complex B. Thus, the monodromy transformation defines a homomorphism of $\pi_1(Y; y_0)$ into the group of isomorphisms of the $\mathbb{Z}$-module $H_*(f^{-1}(y_0), \mathbb{Z})$ and into the group of isomorphisms of the $\mathbb{Z}$-algebra $H^*(f^{-1}(y_0), \mathbb{Z})$. The image of $\pi_1(Y; y_0)$ under these homomorphisms will sometimes also be called the monodromy groups.

One of the simplest examples of the above, consider the n-sheeted covering $f : \Delta^* \to S^*$, $f(z) = s = z^n$ of complex unit disks punctured at the origin. The fiber of this locally trivial fibration is the space B consisting of n isolated points. Let $s_0 \in S^*$, $z_q, \ldots, z_n$; $z_k = z_1 \exp(\frac{2\pi i(k-1)}{n})$ be the preimages of s_0 in Δ^*, let γ be a curve winding once counterclockwise around the origin in S^*. Each of the points z_k uniquely defines a continuous branch of of the function $z = \sqrt[n]{s}$, which gets multiplied by $\frac{\exp 2\pi i}{n,}$ after each rotation around the origin. In this case, therefore, the monodromy transform is a cyclic permutation

$$z_1 \to z_2 \to \ldots \to z_n \to z_1$$

of the preimages.

10.3. The Picard-Lefschetz transformation. Consider a proper surjective morphism $f : \mathcal{X} \to S$ of a complex-analytic manifold onto the disk $S = \{z \in \mathbb{C} | |z| < 1\}$. Set

$$S^* = S \backslash 0, \quad \mathcal{X}^* = \mathcal{X} \backslash f^{-1}(0),$$

and assume that the restriction of the morphism f to $\mathcal{X}^*$ is smooth. Then, the triple $(\mathcal{X}^*, S^*; f)$ is a locally trivial fibration, and there is a representation of the fundamental group $\pi_1(S^*; s_0)$, $s_0 \in S^*$ on the space $H^*(X_{s_0}, \mathbb{Q})$.

The group $\pi_1(S^*, s_0)$ is isomorphic to $\mathbb{Z}$ and is generated by a rotation γ around 0 in the positive direction. This generator gives rise to the isomorphism

$$T : H^*(X_{s_0}, \mathbb{Q}) \to H^*(X_{s_0}, \mathbb{Q}),$$

which belongs to the monodromy group. The isomorphism T is called the *Picard-Lefschetz transformation of the family f*. For further discussion of the general properties of this transformation see Chapter 5, §1. Right now we will describe it in one important special case.

10.4. Vanishing cycles. Suppose that in the situation as in sec. 10.3, $x_0 \in f^{-1}(0)$ is an isolated singularity of the mapping f. We call this singularity *simple* (or *non-degenerate quadratic*) if in some choice of holomorphic local coordinates, $z = (z_0, \ldots, z_n)$ on $\mathcal{X}$ in a neighborhood of the point $x_0 = (0, \ldots, 0)$, the mapping f has the form

$$f(z) = z_0^2 + \ldots + z_n^2.$$

Here $\dim X_s = n$, $\dim \mathcal{X} = n + 1$.

Consider the case where the mapping f has a unique simple singularity x_0. Consider a sufficiently small ball

$$B_\epsilon = \{z \| z_0|^2 + \ldots + |z_n|^2 < \epsilon\}$$

in $\mathcal{X}$. Then, for s sufficiently close to 0, $s \in S$, the manifold $V_s = B_\epsilon \cap X_s$ has the homotopy type of a $2n$-dimensional sphere (see Milnor [1968]). Consider the cohomology group with compact supports $H_c^n(V_s, \mathbb{Z})$, that is, integral cohomology classes represented by a closed n-form vanishing outside some compact set in V_s. Then $H_c^n(V_s, \mathbb{Z}) \simeq \mathbb{Z}$. There is a natural inclusion

$$H_c^n(V_s, \mathbb{Z}) \to H^n(X_s, \mathbb{Z}).$$

The image δ of the generator of the group $H_c^n(V_s, \mathbb{Z})$ under this mapping is called a *vanishing cycle*. It is determined up to multiplication by -1.

The action of the Picard-Lefschetz transform on an element $\omega \in H^n(X_s)$ can be described as follows using vanishing cycles (see Arnold–Varchenko–Gusein-Zade, [1984]):

$$T(\omega) = \omega + \epsilon(\omega, \delta)\delta;$$

$$\epsilon = \begin{cases} 1, & \text{if } n \equiv 2, 3(\text{mod} 4); \\ -1, & \text{otherwise.} \end{cases}$$

$$T(\delta) = \begin{cases} \delta, & \text{if } n \equiv 1, 3(\text{mod} 4), \\ -\delta, & \text{otherwise.} \end{cases} .$$

Here (,) is the intersection form on $H^n(X_s, \mathbb{Z})$, extended to a bilinear form on

$$H^n(X_s) = H^n(X_s, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

The formulas above are known as *Picard-Lefschetz formulas*. It should be noted that for $k \neq n$ T acts on $H^k(X_s)$ by identity.

Consider, for example, the mapping

$$s = f(z_1, z_2) = z_1^2 + z_2^2$$

in a neighborhood of the origin $(0, 0) \in \mathbb{C}^2$. For small $s \neq 0$ the manifold V_s is homeomorphic to the hyperboloid of one sheet (fig. 5), while V_0 is a cone. The "core" cycle δ is contracted to a point as $s \to 0$ – that is the vanishing cycle in the cohomology of V_s. The meridian ω is sent by T (using the Picard-Lefschetz formulas) to the cycle

$$T\omega = \omega - (\omega, \delta)\delta = \omega + \delta.$$

In other words, ω is "twisted" once around the axis of the hyperboloid.

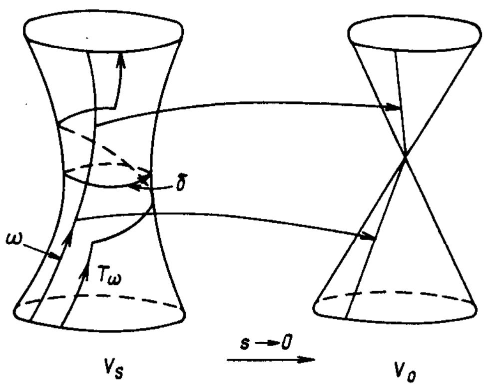


Fig. 5

10.5. Lefschetz families (Deligne [1974b]) Consider a non-singular projective variety $X \subset \mathbb{P}^N$ of dimension n. Let $L \subset \mathbb{P}^N$ be an $N-2$-plane. Then the set of hyperplanes $\{H_s\}$ in $\mathbb{P}^N$ passing through L is parametrized by a projective line $\mathbb{P}^1$ in the dual space $(\mathbb{P}^N)^* \simeq \mathbb{P}^N$.

The $N-2$-plane $L \subset \mathbb{P}^N$ can always be chosen so that the following conditions hold:

(1) L and X intersect transversally, that is $Y = X \cap L$ is a nonsingular sub-variety of X.
(2) There exists a finite subset
$$S = \{s_1, \dots, s_k\} \subset \mathbb{P}^1$$
such that for $s \notin S$ the hyperplane H_s intersects X transversally, and hence the variety $X_s = X \cap H_s$ is non-singular.
(3) For $s_j \in S$, the variety X_s has a single simple (non-degenerate quadratic) singularity $x_j \in Y \cap X_s$.

The family of manifolds $\{X_s\}$, $s \in \mathbb{P}^1$ will then be called a *Lefschetz pencil.*

We should explain why the plane L in $\mathbb{P}^N$ can be chosen so as to satisfy conditions (1)-(3). Condition (1) is, evidently, satisfied for a generic 2-plane $L \subset \mathbb{P}^N$. This follows from Bertini's theorem (see Griffiths–Harris [1978]), which says that a generic hyperplane section of a non-singular projective variety is non-singular.

Consider the set $Z \subset X \times (\mathbb{P}^N)^*$ consisting of the pairs (x, H), where $x \in X$, $H \subset \mathbb{P}^N$ is a hyperplane tangent to X at x, that is,
$$(T_X)_x \subset (T_H)_x.$$

It is easy to see that Z is a non-singular projective variety of dimension $N-1$. Consider the projection $\pi : X \times (\mathbb{P}^N)^* \to (\mathbb{P}^N)^*$. The image $\pi(Z)$ of Z under this projection is called the variety *dual* to X, and denoted by X^*. Of course, the projective variety $X^* \subset (\mathbb{P}^N)^*$ is, in general, singular. Clearly $\dim X^* \leq N-1$. If $\dim X^*$ is strictly smaller than $N-1$, then a generic line $\mathbb{P}^1 \subset (\mathbb{P}^N)^*$ does not intersect X^*, and points (2) and (3) are trivial. Suppose now that $\dim X^* = N-1$. In that case it can be shown that the the mapping

$$\pi : Z \to X^*$$

is generically one-sheeted (that is, a generic hyperplane in $\mathbb{P}^N$ tangent to X is only tangent to X at one point). Further, if (x, H) is a generic point of Z, then $H \cap X$ has a simple singularity at x.

Consider the subset $X_1^* \subset X^*$, consisting of those hyperplanes $H \subset \mathbb{P}^N$ for which the variety $X \cap H$ has either more than one simple singularity, or a non-simple singularity. Evidently, X_1^* is closed, and from the previous claims it follows that X_1^* has codimension at least 2 in $(\mathbb{P}^N)^*$. Thus, a generic line $P^1 \subset (\mathbb{P}^N)^*$ intersects X^* transversally and does not intersect X_1^*, so satisfies conditions (2) and (3).

Now, consider the Lefschetz pencil generated by a 2-plane $L \subset \mathbb{P}^N$. Associate to each element $x \in X$, $x \notin L$ an element $s \in \mathbb{P}^1$ corresponding to the unique hyperplane passing through x and L. This gives a rational mapping

$$\phi : X \to \mathbb{P}^1.$$

Let $\tilde{X}$ be the variety obtained from X by a monoidal transformation centered at Y (see Chapter 1, §1), and let π be the natural projection of $\tilde{X}$ onto X. Then there is a commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X \arrow[r, "\phi"'] & \mathbb{P}^1 \end{tikzcd},$$

where f is a morphism. The fiber $f^{-1}(s)$ of f over any point $s \in \mathbb{P}^1$ is isomorphic to the corresponding hyperplane section $X_s = X \cap H_s$ of the variety X. Identifying X_s with $f^{-1}(s)$, note that the morphism f is smooth at all points $x \in X$, except the points $x_j \in f^{-1}(s_j)$, $s_j \in S$.

Fix a point $s_0 \notin S$ and a set of disjoint disks $\mathcal{D}_i \subset \mathbb{P}^1$, centered on the points $s_i \in S$. Choose points $s_i' \in \mathcal{D}_i$, $s_i' \neq s_i$, and fix curves

$$\beta_j : [0,1] \to \mathbb{P}^1, \quad \beta_j(0) = s_0, \quad \beta_j(1) = s_j',$$

not containing any points of S. Consider an element γ_j of $\pi_1(\mathbb{P}^1 \backslash \{S\}; s_0)$, generated by the loop below: First, go from s_0 to s_j' along the curve β_j, then go around s_j in $\mathcal{D}_j$ in the positive direction, then return to s_0 along β_j.

The elements γ_j, $j = 1, \dots, k$ generate $\pi_1(\mathbb{P}^1 \backslash \{S\}; s_0)$. Restricting f to the disk $\mathcal{D}_j$, we get for each j a vanishing cycle δ_j in $H^n(X_{s'_j}, \mathbb{Z})$. Using the monodromy transformation generated by the curve β_j to identify the groups $H^n(X_{s'_j}, \mathbb{Z})$ and $H^n(X_{s_j}, \mathbb{Z})$ we get the element $\delta_j \in H^n(X_{s_0}, \mathbb{Z})$.

The space

$$E \subset H^m(X_{s_0}, \mathbb{Q}) = H^n(X_{s_0}, \mathbb{Z}) \otimes \mathbb{Q},$$

generated by the elements δ_j is called the *space of vanishing cycles.*

Let T_j be the automorphism of $H^n(X_{s_0}, \mathbb{Q})$ corresponding to the element $\gamma_j \in \pi_1(\mathbb{P}^n \backslash \{S\}; s_0)$. The following statements hold:

(1) The action of the element T_j on $\omega \in H^n(X_{s_0}, \mathbb{Q})$ is given by the formula

$$T_j(\omega) = \omega \pm (\omega, \delta_j)\delta_j.$$

The sign is determined by the Picard-Lefschetz formula.

(2) The subspace $E \subset H^n(X_{s_0}, \mathbb{Q})$ is invariant under the action of the monodromy group. In particular, this implies that E does not depend on the choice of the discs $\mathcal{D}_j$, the points s'_j, and the paths β_j.

(3) The action of $\pi_1(\mathbb{P}^1 \backslash \{S\}; s_0)$ by conjugation on the δ_j is transitive (up to sign).

(4) The subspace of of elements of $H^n(X_{s_0}, \mathbb{Q})$ fixed by the monodromy group action coincides with the orthogonal subspace to E under the intersection pairing (for more about this space see Ch 4, §4).

(5) The action of the monodromy group on $E/(E \cap E^{\perp})$ is absolutely irreducible.

This theory can be generalized to algebraic varieties over arbitrary fields of definition (see Deligne [1974b]).

Chapter 2
Periods of Integrals on Algebraic Varieties

In this Chapter we introduce the basic concepts and definitions having to do with the period mapping for algebraic varieties. The majority of the results described are due to P. Griffiths.

§1. Classifying Space

1.1. In Chapter 1 we defined the Hodge structure on the cohomology of a compact Kähler manifold. In particular, the cohomology of every non-singular projective variety is equipped with a Hodge structure. This structure allows one to get a collection of analytic invariants of the variety in question. In the sequel we will address the question of the extent to which these invariants determine the variety. Our immediate task is to formally describe the properties of Hodge structures on the cohomology of an algebraic variety, and to construct a manifold parametrizing these structures. This manifold will be called the classifying space or the space of period matrices. The points of the classifying space will be the invariants of algebraic varieties.

Let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module, $H = H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$, its complexification. Fix a natural number n. For all integers $p \geq 0$, $q \geq 0$, such that $p + q = n$, pick a complex linear subspace $H^{p,q} \subset H$.

Definition . The data $\{H_{\mathbb{Z}}, H^{p,q}\}$ is called *a Hodge structure of weight* n if the following conditions are satisfied:

$$H = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}}. \tag{1}$$

If X is a compact Kähler manifold,

$$H_{\mathbb{Z}} = H^n(X, \mathbb{Z})/(torsion), \quad H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C} \simeq H^n(X, \mathbb{C}),$$

and $H^{p,q}$ is the cohomology of type (p, q) in the Hodge decomposition, then $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a Hodge structure of weight n (see Chapter 1, §7).

For the purpose of classifying projective algebraic varieties the set of all Hodge structure of weight n on $H_{\mathbb{Z}}$ is too big. We can reduce it greatly by taking into consideration the Hodge-Riemann bilinear relations (see Chapter 1, §7).

Let $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ be a non-degenerate bilinear form. Extend Q to a bilinear form on H.

Definition. The data $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ is called a *polarized Hodge structure of weight* n if $\{H_{\mathbb{Z}}, H^{p,q}\}$ is a Hodge structure of weight n, and the following relationships are satisfied:

$$\begin{aligned} Q(\phi, \psi) &= (-1)^n Q(\psi, \phi); \\ (\psi, \phi) &= 0, \quad \text{for } \psi \in H^{p,q}, \quad \phi \in H^{p',q'}, \quad p \neq q'; \\ (\sqrt{-1})^{p-q} Q(\psi, \overline{\psi}) &> 0, \quad \text{for } \psi \in H^{p,q}, \quad \psi \neq 0. \end{aligned} \tag{2}$$

Let $\{H_{\mathbb{Z}}, H^{p,q}\}$ be a Hodge structure of weight n. By setting

$$F^p = H^{p,0} \oplus \ldots \oplus H^{p,n-p}, \quad F^{n+1} = \{0\}$$

we obtain the decreasing *Hodge filtration*

$$0 = F^{n+1} \subset F^n \subset \ldots \subset F^0 = H. \tag{3}$$

Condition (1) implies that for $p = 0, 1, \ldots, n+1$ there is a decomposition

$$H = F^p \oplus \overline{F^{n-p+1}}. \tag{4}$$

If $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ is a polarized Hodge structure, then (2) easily implies that

$$\begin{aligned} Q(F^p, F^{n-p+1}) &= 0, \\ Q(C\phi, \overline{\phi}) &> 0, \end{aligned} \tag{5}$$

where C is the Weyl operator on H, defined in Chapter 1, §4.

Conversely, suppose we are given a module $H_{\mathbb{Z}}$ and a filtration defined by (3) on $H = H_{\mathbb{C}}$. Then, if that filtration satisfies (4), by setting

$$H^{p,q} = F^p \cap \overline{F^{n-p}},$$

we can reconstruct the Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}\}$ defining (3). If, in addition, there is a bilinear form Q on $H_{\mathbb{Z}}$, which, when extended to H, satisfies conditions (5), then $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ is a polarized Hodge structure of weight n.

1.2. The primary interesting example of a polarized Hodge structure is obtained as follows. Let X be a nonsingular complex algebraic variety of dimension d, ω a closed differential form of type $(1,1)$ on X, corresponding to a polarization. This means that the cohomology class $[\omega]$ equals $rc_1(L)$ where $r > 0$ is rational and $c_1(L)$ is the Chern class of a positive line bundle on X. In particular, $[\omega]$ is a rational class.

The pair (X, ω) will be called a polarized algebraic variety. Two such varieties (X', ω') and (X'', ω'') will be considered isomorphic, if there exists an isomorphism $\phi : X' \to X''$ of algebraic varieties X' and X'', such that $\phi^*([\omega'']) = \alpha[\omega']$, for some positive rational α.

Consider a polarized algebraic variety (X, ω). The form ω defines a Kähler metric on X, and hence a Hodge decomposition on the cohomology $H^n(X, \mathbb{C})$. Let $P^n(X, \mathbb{C})$, $P^n(X, \mathbb{Q})$ be primitive cohomology corresponding to the Kähler form ω (Chapter 1, §7.5). and set

$$\begin{aligned} H_{\mathbb{Z}} &= H^n(X, \mathbb{Z}) \cap P^n(X, \mathbb{Q}); \\ H^{p,q} &= H^{p,q}(X, \mathbb{C}) \cap P^n(X, \mathbb{C}); \\ Q(\phi, \psi) &= (-1)^{n+1} \int_X \phi \wedge \psi \wedge \omega^k; \\ \phi, \psi \in P^n(X, \mathbb{C}), &\quad k = \dim_{\mathbb{C}} X - 2n. \end{aligned} \tag{6}$$

Clearly, the data $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ specifies a polarized Hodge structure of weight n. The conditions in (2) are just the Hodge-Riemann relations (see Chapter 1, §7.6).

1.3. To each polarized Hodge structure we can associate the Hodge numbers

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}, \quad f^p = \dim_{\mathbb{C}} F^p = h^{n,0} + \ldots + h^{p,n-p}.$$

Evidently

$$\begin{aligned} h^{p,q} &= h^{q,p}; \\ \sum_{p+q=n} h^{p,q} &= \operatorname{rank} H_{\mathbb{Z}}; \\ \operatorname{sgn}(Q_{\mathbb{R}}) &= \sum_{p+q\equiv 0 \bmod 2} (-1)^q h^{p,q}, \end{aligned} \tag{7}$$

where $\operatorname{sgn}(Q_{\mathbb{R}})$ is the signature of the quadratic form Q on $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$. For a polarized Hodge structure (6) the last relation is the Hodge Index theorem (Chapter 1, §7.7).

Definition. Suppose we are given

(1) A natural number n;
(2) A free $\mathbb{Z}$-module H;
(3) A nondegenerate bilinear form $Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$, satisfying the condition $Q(\phi, \psi) = (-1)^n Q(\psi, \phi)$;
(4) For all integers $p \geq 0, q \geq 0$, integers $h^{p,q}$ satisfying conditions (7).

We say that the *classifying space with data (1)-(4)* is the set D of all polarized Hodge structures $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ of weight n with given Hodge numbers $h^{p,q}$.

1.4. Consider the set $\mathcal{F} = (f^1, \ldots, f^n; H)$ of filtrations (3) of the space H by subspaces F^p of fixed dimensions $f^p = \sum_{j=0}^{n-p} h^{p,j}$. In order to introduce a complex structure on $\mathcal{F}$, first, for each pair of natural numbers $k \leq m$ we define the *Grassmann manifold (Grassmanian)* $G(k,m)$ (see Griffiths–Harris [1978]).

The points of $G(k,m)$ are in one-to-one correspondence with the set of k-dimensional linear subspaces in $\mathbb{C}^m$. The complex structure on $G(k,m)$ is introduced as follows: Let $W \subset \mathbb{C}^m$ be a k-dimensional linear subspace, then we can choose linear coordinates x_j in $\mathbb{C}^m$ such that

$$W = \{(x_1, \ldots, x_m) \in \mathbb{C}^m \,|\, x_1 = \ldots = x_{m-k} = 0\}.$$

For every matrix

$$\alpha = \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1k} \\ \ldots & \ldots & \ldots \\ \alpha_{(m-k)1} & \ldots & \alpha_{(m-k)k} \end{pmatrix}$$

define a linear subspace W_α by the system

$$x_j = \sum_{j=1}^{k} \alpha_{ij} x_{m-k+j}, \quad 1 \leq i \leq m-k.$$

Then $U = \{W_\alpha\} \subset G(k,m)$ is a neighborhood of the point $W \in G(k,m)$, and we can use the numbers α_{ij} as holomorphic coordinates on U. It can be checked that with these coordinates, $G(k, m)$ has the structure of a compact complex manifold of dimension $k(m-k)$.

It should be noted that the simplest example of a Grassmanian was studied in Chapter 1 §1 – that is the complex projective space $\mathbb{P}^m = G(1, n+1)$.

In addition to the notation $G(k, m)$ we will use $G(k, V)$ to denote the Grassmanian of k-dimensional linear subspace of an m-dimensional complex vector space V.

Consider the filtration (3). For every p, $1 \leq p \leq n$, we can associate it with the point F^p in the Grassmanian $G(f^p, H)$. This gives an inclusion

$$\mathcal{F} \hookrightarrow \prod_{p=1}^{n} G(f^p, H).$$

The image of $\mathcal{F}$ under this inclusion is a compact complex manifold in $\prod_{p=1}^{n} G(f^p, H)$. The set $\mathcal{F}$ with this complex structure is called *the flag manifold.*

1.5. Let us introduce the structure of a flag manifold on the classifying space D. The set D is obviously included in an obvious way into the flag manifold $\mathcal{F} = (f^1, \ldots, f^n; H)$ (where $f^p = \sum_{j=0}^{n-p} h^{p,j}$). Consider the subset $\check{D} \in \mathcal{F}$, consisting of the filtrations satisfying the first of the Hodge-Riemann relations (5). $\check{D}$ is an algebraic subset of $\mathcal{F}$. The space $\check{D}$ is called the dual classifying space. The group $G_{\mathbb{C}} = \mathrm{Aut}(H, Q)$ of linear automorphisms of H preserving Q acts on $\mathcal{F}$ as a group of analytic automorphisms, and leaves $\check{D}$ invariant. It can be shown that the action of G on $\check{D}$ is transitive. That implies that $\check{D}$ is non-singular. Filtrations satisfying (4) and the second of the conditions (5) form an open subset of $\mathcal{F}$. Therefore, $D \subset \check{D}$ is a non-singular open complex submanifold. Consider the subgroup $G_{\mathbb{R}} = \mathrm{Aut}(H_{\mathbb{R}}, Q) \subset G_{\mathbb{C}}$. It can be shown (see Griffiths [1968]) that $G_{\mathbb{R}}$ acts transitively on D, and $D = G_{\mathbb{R}}/K$, where K is a compact subgroup of $G_{\mathbb{R}}$, stabilizing a point of D. In turn, $\check{D} \simeq G_{\mathbb{C}}/B$, for some parabolic subgroup $B \subset G_{\mathbb{C}}$, and furthermore $K = G_{\mathbb{R}} \cap B$. The subgroup $G_{\mathbb{Z}} \subset G_{\mathbb{R}}$ of $\mathbb{Z}$-linear automorphisms of the module $H_{\mathbb{Z}}$, preserving Q acts on D by analytic automorphisms. In the future, we will be interested in the spaces $\Gamma \backslash D$, where $\Gamma \subset G_{\mathbb{Z}}$ is a subgroup.

1.6. As our first example, let us describe the classifying space of a non-singular projective curve X of genus $g > 0$. Consider the Hodge structure of weight 1 on $H^1(X, \mathbb{Z})$. In that case,

$$H^1(X, \mathbb{Z}) = P^1(X, \mathbb{Z}) = H_{\mathbb{Z}}, \quad \mathrm{rank}_{\mathbb{Z}} H_{\mathbb{Z}} = 2g,$$

$$H_{\mathbb{C}} = H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}, \quad h^{1,0} = h^{0,1} = g.$$

For any two closed differential 1-forms $\phi, \psi \in H_{\mathbb{C}}$ we have

$$Q(\phi, \psi) = \int_X \phi \wedge \psi,$$

It is known (see Griffiths–Harris [1978]) that Q is dual to the intersection form on 1-cycles on X. Thus, there exists a basis $\eta_1, \dots, \eta_g, \mu_1, \dots, \mu_g$ of $H^1(X, \mathbb{Z})$, such that the skew-symmetric bilinear form Q has a matrix of the form

$$Q = \left(\begin{array}{c|c} 0 & -E_g \\ \hline E_g & 0 \end{array}\right). \tag{8}$$

where E_g is the unit $g \times g$ matrix.

Now fix a free $\mathbb{Z}$-module H with basis $\eta_1, \dots, \eta_g, \mu_1, \dots, \mu_g$ and a bilinear form Q given by the matrix (8) in that basis. Let us construct the classifying space D of polarized Hodge structures of weight 1 with the data $H_{\mathbb{Z}}, Q$.

Let $\omega_1, \dots, \omega_g$ be a basis of $H^{1,0}$. This can be normalized by setting $\eta_i^*(\omega_j) = \delta_{ij}$, where $\eta_1^*, \dots, \eta_g^*, \mu_1^*, \dots, \mu_g^*$ is the basis of $(H_{\mathbb{Z}})^*$, dual to the basis $\eta_1, \dots, \eta_g, \mu_1, \dots, \mu_g$ of $H_{\mathbb{Z}}$. Then,

$$(\omega_1, \dots, \omega_g) = (\eta_1, \dots, \eta_g, \mu_1, \dots, \mu_g)^t\Omega,$$

where $\Omega = (E_g | Z)$, with Z a complex $g \times g$ matrix. The matrix Z is uniquely determined by the subspace $H^{1,0} \subset H_{\mathbb{C}}$. Set $Z = X + \sqrt{-1}Y$, where X and Y are real matrices. The matrices of the bilinear form $Q|_{H^{1,0}}$ and of the Hermitian form $\sqrt{-1}Q(\bullet, \bar{\bullet})|_{H^{1,0}}$, respectively, have the forms

$$\Omega = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

$${}^t\Omega = {}^tZ - Z,$$

$$\sqrt{-1}\Omega \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} {}^t\overline{\Omega} = 2Y.$$

By conditions (2), it follows that Z is symmetric and $Y = \operatorname{Im} Z$ is positive definite. Thus, D is the set of complex matrices $Z \in M(g, \mathbb{C})$, such that ${}^tZ = Z$ and $\operatorname{Im} Z > 0$; in other words, D is the *Siegel halfplane* H_g.

The group $G_{\mathbb{Z}}$ in this case is the group $\mathrm{Sp}(g, \mathbb{Z})$, or the group of matrices

$$\gamma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathrm{GL}(2g, \mathbb{Z}),$$

such that

$$\gamma \begin{pmatrix} 0 & -E_g \\ E_g & 0 \end{pmatrix} {}^t\gamma = \begin{pmatrix} 0 & -E_g \\ E_g & 0 \end{pmatrix}.$$

It is easy to see that γ acts on $Z \in D$ by the transformation rule

$$\gamma(Z) = (AZ + B)(CZ + D)^{-1}.$$

When $g = 1$, where X is an elliptic curve, the manifold H_g is the complex upper halfplane $H = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$. The group $G_{\mathbb{Z}}$ in this case is simply the group $\mathrm{SL}(2, \mathbb{Z})$, acting on H by linear-fractional transformations. It is well-known that the set of isomorphism classes of complex elliptic curves is in one-to-one correspondence with the quotient $G_{\mathbb{Z}}\backslash H$. This corresponence is given by the absolute invariant of the elliptic curve The case $g = 1$ is discussed in greater detail in Chapter 3, §1.2.

1.7. Now, let us construct the classifying space corresponding to Hodge structures on the first cohomology of a polarized abelian variety. For more information on abelian varieties, see Mumford [1968].

Let $H \subset \mathbb{C}$ be a lattice of rank $2g$. The complex torus

$$X = \mathbb{C}^g/H$$

is called an *abelian variety* if there exists a holomorphic embedding

$$\phi : X \hookrightarrow \mathbb{P}^N.$$

The embedding ϕ induces on X the structure of a polarized abelian variety (X, ω). Here, just as in Section 1.2, ω is a $(1,1)$-form on X, representing an integral cohomology class $[\omega] = \phi^*(c_1(\mathcal{O}_{\mathbb{P}^N}(1)))$.

The simplest (but extremely important) example of an abelian variety is a a one-dimensional abelian variety, or an elliptic curve (see Chapter 3, sec 1.2). In that case H is an arbitrary lattice of rank 2 in $\mathbb{C}$. The complex torus $E = \mathbb{C}/H$ is always algebraic. It can be embedded into $\mathbb{P}^2$ as a non-singular curve of degree 3. In general the lattice H must satisfy certain additional conditions (the Riemann-Frobenius conditions, see Ch 1, §7) in order for $\mathbb{C}^g/H$ to be an abelian variety.

Let $e_1, \dots, e_n$ be a basis of H, viewed as a free $\mathbb{Z}$-module, and $x_1, \dots, x_{2g}$ be the real coordinates on $\mathbb{C}^g$ with respect to this basis. Then the differential forms

$$dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

form a basis of the free $\mathbb{Z}$-module $H^k(X, \mathbb{Z})$. In particular, $dx_1, \dots, dx_{2g}$ is a basis of $H^1(X, \mathbb{Z})$. The lattice H can be identified with the module $H_1(X, \mathbb{Z})$, by associating to each element $e \in H$, the homology class of the curve $\{te\}$, $0 \le t \le 1$. Under this identification, $dx_1, \dots, dx_{2g}$ are dual basis in $H^1(X, \mathbb{Z})$ and $H_1(X, \mathbb{Z})$, respectively.

The form ω can be written as

$$\omega = \frac{1}{2} \sum_{i,j} r_{ij} dx_i \wedge dx_j,$$

where $R = (r_{ij})$ is a skew-symmetric non-singular matrix. The integrality of the form ω means that $r_{ij} \in \mathbb{Z}$.

Every skew-symmetric non-singular integral bilinear form on a $\mathbb{Z}$-module of rank $2g$ can be represented in some basis as (the Smith normal form)

$$R = \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \delta_1 & \dots & 0 \\ 0 & \ddots & \\ & \dots & \delta_g \end{pmatrix},$$

where $\delta_i \in \mathbb{Z}$, $\delta_i > 0$, $\delta_i | \delta_{i+1}$. The collection of numbers

$$\delta = (\delta_1, \ldots, \delta_{2g})$$

is an invariant of the form.

This means that we can pick a basis $e_1, \ldots, e_{2g}$ of H, such that in the corresponding real coordinates, the form ω can be written as

$$\omega = \sum_{j=1}^{g} \delta_j dx_j \wedge dx_{g+j}. \tag{9}$$

The collection of numbers δ shall be called the *polarization type* of ω. In particular, if $\delta_1 = \ldots = \delta_g = 1$, then we say that (X, ω) is a *principally polarized abelian variety.*

Pick a basis of H as above. It is then easy to show that the vectors $\delta_1^{-1}e_1, \ldots, \delta_g^{-1}e_g$ are a complex basis of $\mathbb{C}^g$. Consider complex coordinates $z_1, \ldots, z_g$ corresponding to this basis. If in those coordinates the vectors e_k, $k = 1, \ldots, 2g$, are written as

$$e_k = (\lambda_{1k}, \ldots, \lambda_{gk}),$$

then the matrix $\Omega = (\lambda_{ij})$ has the form

$$\Omega = (\Delta | Z), \tag{10}$$

where $Z = (z_{ij})$ is a complex $g \times g$ matrix.

Let us demonstrate that ω is a $(1,1)$ form representing a positive cohomology class, if and only if Z is a symmetric matrix with positive-definite imaginary part $Y = \operatorname{Im} Z$.

Observe first, that the differential forms $dz_1, \ldots, dz_g$ form a basis of the subspace

$$H^{1,0} \subset H_{\mathbb{C}} = H^1(X, \mathbb{C}).$$

Let Q be a bilinear form on $H_{\mathbb{C}}$, defined by the polarization ω using formulas (6). Then

$$Q(dz_i, dz_j) = |\delta|(-z_{ij} + z_{ji}),$$

$$Q(dz_i, d\overline{z}_j) = |\delta|(-z_{ij} + \overline{z}_{ji}),$$

where $|\delta| = \delta_1 \ldots \delta_g$. Indeed, it follows directly from (6) that

$$Q(dx_i, dx_j) = \begin{cases} 0, & \text{when } |i - j| \neq g; \\ |\delta|\delta_i^{-1}, & \text{when } j = i + g; \\ -|\delta|\delta_i^{-1}, & \text{when } i = j + g. \end{cases}$$

It is enough to observe that

$$dz_k = \delta_k dx_k + \sum_{j=1}^{g} z_{kj} dx_{g+j}. \tag{11}$$

Applying (2) get that $Z = {}^tZ, \quad Y > 0$.

Let us fix a free $\mathbb{Z}$ module $H_{\mathbb{Z}}$ of rank $2g$ with basis $\eta_1, \dots, \eta_g, \mu_1, \dots, \mu_g$. Let $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ be a form, which, in this basis looks like

$$Q = |\delta| \left(\begin{array}{c|c} 0 & \Delta^{-1} \\ \hline -\Delta & 0 \end{array} \right).$$

Let Z be a complex matrix, such that ${}^tZ = Z$, and $\operatorname{Im} \mathbb{Z} > 0$. Let $H^{1,0} \subset H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ be the subspace with basis $\omega_1, \dots, \omega_g$, where

$$\omega_k = \delta_k \eta_k + \sum_{j=1}^{g} z_{kj} \mu_g.$$

This construction uniquely defines a polarized Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ of weight 1. Different matrices Z evidently define different Hodge structures. As was shown above, any Hodge structure associated to a polarized abelian variety (X, ω) with polarization of type δ can be obtained by this means.

It has thus been shown that the classifying space D defined by the data D associated with a polarized abelian variety (X, ω) of dimension g with polarization of type δ is the Siegel upper halfspace H_g.

The role of the group $G_{\mathbb{Z}}$ is played by the group $\mathrm{Sp}(\delta, \mathbb{Z})$ of matrices $\gamma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \in \mathrm{GL}(rg, \mathbb{Z})$ satisfying the condition

$$\gamma \left(\begin{array}{c|c} 0 & \Delta^{-1} \\ \hline -\Delta & 0 \end{array} \right) {}^t\gamma = \left(\begin{array}{c|c} 0 & \Delta^{-1} \\ \hline -\Delta & 0 \end{array} \right).$$

The element γ acts on the matrix $Z \in H_g$ by the rule

$$\gamma(Z) = (AZ + B\Delta)(\Delta^{-1}CZ + \Delta^{-1}D\Delta)^{-1}.$$

Let $Z \in H_g$ be a matrix and let δ be a type of polarization. Form a matrix Ω according to the formula (10) and let us examine the column vectors $e_g \in \mathbb{C}^g$ of this matrix. If H is a lattice with basis $e_1, \dots, e_{2g}$, then $X = \mathbb{C}^g/H$ is a complex torus. Take the real coordinates $x_1, \dots, x_2g$ corresponding to the basis $\{e_j\}$ on $\mathbb{C}^g$ and define ω by the formula (9). From equation (10) and the properties of the matrix $Z \in H_g$ it follows immediately that ω is a $(1,1)$-form on X representing a positive integral cohomology class. Therefore, (X, ω) is a polarized abelian variety of dimension g with polarization of type δ.

Let (X', ω') and (X'', ω'') be polarized abelian varieties constructed over two different elements $Z', Z'' \in H$. It can be easily shown that (X', ω') and (X'', ω'') are isomorphic if and only if $Z' = \gamma(Z'')$, $\gamma \in \mathrm{Sp}(\delta, \mathbb{Z})$. Thus, the points of the complex manifold

$$\mathcal{M} = \mathrm{Sp}(\delta, \mathbb{Z}) \backslash H_g$$

are in one to one correspondence with the equivalence classes of pairs (X, ω), where X is an abelian variety of type g and ω is a polarization of type δ.

The space $\mathcal{M}$ is called the *moduli space of abelian varieties of dimension g with polarization of type δ* (see Chapter 2, §5).

Let us study the case of principal polarization in greater detail. In that case the pair (D, G_Z) coincides with the corresponding pair constructed starting with a projective curve of genus g (see 1.6). Let E be such a curve; it defines an element $Z \in H_g$ defined up to the action of $\mathrm{Sp}(g, \mathbb{Z})$. Since $\mathrm{Sp}(g, \mathbb{Z})\backslash H_g$ is the moduli space of principally polarized abelian varieties of dimension g, there is a one to one correspondence between projective curves E and principally polarized abelian varieties $(J(E), \omega)$. This abelian variety is called the *Jacobian variety,* or the *Jacobian* of the curve E. Recall, that every polarization ω on an abelian variety defines a line bundle L, such that $c_1(L) = [\omega]$. The line bundle L is defined up to a shift by an element of the torus X. If δ is the polarization type of ω, then $\dim H^0(X, L) = [\delta]$ (Mumford [1968]). In particular, if ω is a principal polarization, it corresponds to a unique divisor Θ defined up to a shift, and in the special case of a Jacobian of a curve, there is a unique, up to shifts, divisor Θ, known as the *divisor of the polarization.*

It should be emphasized that the possibility of reconstructing a polarized abelian variety from the polarized Hodge structure on its one-dimensional cohomology is the most important result established in 1.7.

1.8. Let (X, ω) be a polarized algebraic surface (that is, $\dim_{\mathbb{C}} X = 2$). As an example, let us study the classifying space D constructed from the data (6) obtained from (X, ω). In this case

$$H_{\mathbb{R}} = P^2(X, \mathbb{R}), \quad H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C} = P^2(X, \mathbb{C}).$$

If the subspace $H^{2,0} \in H_{\mathbb{C}}$ and the bilinear form Q are given, then the whole Hodge structure on $H_{\mathbb{C}}$ is uniquely defined, since $H^{0,2} = \overline{H}^{2,0}$, while $H^{1,1}$ is the orthogonal complement to $H^{0,2} \otimes H^{2,0}$ in the space $H_{\mathbb{C}}$ with respect to Q. Let $h = h^{2,0}(X)$, $k = h^{1,1}(X) - 1$. Then, there exists $\mathbb{R}$-subspaces W and S, of dimensions $2h$ and k respectively, that

$$H^{2,0} \oplus H^{0,2} = W \otimes \mathbb{C}, \quad H^{1,1} = S \otimes \mathbb{C}, \quad H_{\mathbb{R}} = W \oplus S.$$

As equations (2) show, the form Q is positive definite on $H^{1,1}$ and negative definite on $H^{2,0} \oplus H^{2,0}$. Choose a basis $\omega_1, \dots, \omega_h$ of the space $H^{2,0}$, such that $Q(\omega_i, \overline{\omega}_j) = -\delta_{ij}$. We can also choose an $\mathbb{R}$-basis $\zeta_1, \dots, \zeta_k$ for S, such that $Q(\zeta_i, \zeta_j) = \delta_{ij}$. Now, set $\eta_j = \frac{1}{2}(\omega_j + \overline{\omega}_j)$ and $\mu_j = -\frac{1}{2}\sqrt{-1}(\omega_j - \overline{\omega}_j)$. Then, expressed in terms of the basis $\eta_1, \dots, \eta_h, \mu_1, \dots, \mu_h, \zeta_1, \dots, \zeta_k$ the matrix $\tilde{Q}$ of Q has the form

$$\tilde{Q} = \begin{pmatrix} -E_{2h} & 0 \\ 0 & E_k \end{pmatrix},$$

where E_l is the $l \times l$ identity matrix. Let $G_{\mathbb{R}} = \mathbf{O}(2h, k)$ be the group of (real) transformations of $H_{\mathbb{R}}$ preserving the form Q. Then $G_{\mathbb{R}}$ acts transitively on D. Indeed, for any other Hodge structure $\{\widehat{H}^{p,q}\}$ on $H_{\mathbb{C}}$, choose elements $\widehat{\zeta}_i$, $\widehat{\eta}_i, \widehat{\mu}_i$ analogous to ζ_i, η_i, μ_i and set $T(\zeta_i) = \widehat{\zeta}_i$, $T(\eta_i) = \widehat{\eta}_i$, and $T(\mu_i) = \widehat{\mu}_i$.

Then T is in $G_{\mathbb{R}}$ and T maps the Hodge structure $\{H^{p,q}\}$ into $\{\widehat{H}^{p,q}\}$. Fix a Hodge structure $H_0^{2,0} \subset H_{\mathbb{C}}$ and set $K = \{T \in G_{\mathbb{R}} | TS_0 = S_0\}$. Then, since T is a real transformation, it follows that $T(\overline{H}_0^{2,0}) = \overline{H}_0^{2,0}$, and so T fixes $H_0^{2,0} \oplus H_0^{0,2}$. That, in turn, means that $K = U(H) \times O(k)$, where $U(h)$ is unitary and $O(k)$ is orthogonal. Finally, the classifying space D may be represented as $D = K \backslash G_{\mathbb{R}}$.

1.9. Recall that a complex structure on a variety $\mathcal{F}$ is introduced by embedding it into a product of Grassmanians $\prod G(f^p; H)$. Denote this embedding by ϕ, that is:

$$\phi = \prod \phi^p : D \hookrightarrow \prod_{p=1}^{n} G(f^p, H).$$

Let us also recall the following explicit description of the tangent bundle of a Grassmanian (see Griffiths–Harris [1978]).

Let $W \in G(f^p, F)$ be an f^p-plane. There exists a natural isomorphism between the tangent space T_W of the Grassmanian $G(f^p, F)$ at W and $\mathrm{Hom}(W, F/W)$. This isomorphism is described as follows. Let $\zeta \in T_W$. Choose a holomorphic curve $\{W_T\} \in G(f^p, F)$, such that $W_0 = W$ and ζ is the tangent vector to $\{W_t\}$ corresponding do the differentiation $\frac{d}{dt}$. For any vector $w \in W_0$ it is possible to choose a vector field $w(t) \in W_t$ which depends holomorphically on t, and such that $w(0) = w$. Then, the homomorphism $\zeta \in \mathrm{Hom}(W, F/W)$ is define by setting

$$\zeta(w) = \frac{dw(t)}{dt}|_{t=0} \bmod W.$$

Let $d \in D$ be a point and let $(F^n, \dots, F^1)$ be the corresponding flag. We obtain an induced map on the tangent spaces

$$\phi_* : T_d \hookrightarrow \bigoplus_{p=1}^{n} \mathrm{Hom}(H^p, F^0/F^p).$$

Define the *horizontal subspace* $T_{h,d} \subset T_d$ to be the subspace consisting of vectors ζ such that

$$\phi_*(\zeta) \in \bigoplus_{p=1}^{n} \mathrm{Hom}(F^p, F^{p-1}/F^p).$$

The tangent plane field $T_{h,d}$ is called the *horizontal subbundle* $T_h(D)$ of the tangent bundle $T(D)$.

1.10. On the Grassmanian $G(f^p, H)$ there is a canonically defined sheaf F^p of rank f^p – the so-called *tautological sheaf*. This is the sheaf of sections of the bundle

$$E = \{(x, h) \in G(f^p, H) \times H | h \in x\}$$

over $G(f^p, H)$ (see Griffiths–Harris [1978]), the fibers of which over the points of the Grassmanian are the f^p-subspaces of H defining them.

The sheaves $\mathcal{F}^p = (\phi^p)_* F_p$ on D thus define a filtration

$$\mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \ldots \subset \mathcal{F}^0 = H \otimes \mathcal{O}_D$$

of the constant sheaf $H \otimes \mathcal{O}_D$. Define the Hodge sheaves on D by setting

$$\mathcal{H}^{p,q} = \mathcal{F}^p / \mathcal{F}^{p+1}.$$

Thus defined, $\mathcal{H}^{p,q}$ is a locally free sheaf of $\mathcal{O}_D$ modules of rank $h^{p,q}$, and there is the C^∞ decomposition

$$\mathcal{F}^p = \mathcal{H}^{n,0} \oplus \ldots \oplus \mathcal{H}^{p,n-p}.$$

It is easy to see that the form $(\sqrt{-1})^{p-q} Q(\bullet, \bar{\bullet})$ defines a $G_{\mathbb{R}}$-invariant hermitian form on $\mathcal{H}^{p,q}$. Since there exists an embedding

$$T(D) \subset \bigoplus_{p=1}^{n} \operatorname{Hom}(\mathcal{H}^{n,0} \oplus, \ldots, \oplus \mathcal{H}^{p,n-p}, \mathcal{H}^{p_1,n_p+1} \oplus \ldots \oplus \mathcal{H}^{0,n})$$

and a $G_{\mathbb{R}}$ invariant form on $\mathcal{H}^{p,q}$ for all p, q, we thus obtain a $G_{\mathbb{R}}$ invariant metric on $T(D)$. It can be shown that this metric is generated by the Killing form on the Lie algebra $g_{\mathbb{R}}$ of $G_{\mathbb{R}}$.

Definition. The holomorphic bundles $\mathcal{F}^i$, constructed above, on the classifying space D are called *Hodge canonical bundles*

§2. Complex Tori

2.0 The primary purpose of this section is to describe the approaches to classifying polarized projective varieties (X, ω) using polarized Hodge structures on their cohomology. Let $\mathcal{E} = \{H_{\mathbb{Z}}, H^{p,q}, Q\}$ be a Hodge structure, defined by formulas (6) starting with the pair (X, ω). Consider the classifying space of polarized Hodge structures with the same data as $\mathcal{E}$. The pair (X, ω) defines a point $d \in D$, defined up to the action of the group $G_{\mathbb{Z}}$. It is interesting to understand to what extent can the pair be reconstructed from d. In the preceding section this question was solved for abelian varieties. In general, the variety $G_{\mathbb{Z}} \backslash D$ has a rather complicated structure, however when the weight of $\mathcal{E}$ is odd there is a good method to distinguish points of $G_{\mathbb{Z}} \backslash D$, as follows. To each Hodge structure $\mathcal{E} \in D$ of odd weight corresponds a pair a pair $(J(\mathcal{E}), \eta)$, where $J(\mathcal{E})$ is a certain complex torus (the so-called *Griffiths torus*) and η is a $(1,1)$-form on $J(\mathcal{E})$, representing an integral cohomology class. The form η does not, in general, correspond to a polarization, and the torus $J(\mathcal{E})$ is not necessarily an abelian variety. The pair $(J(\mathcal{E}), \eta)$ will be called the *pseudo-polarized torus* corresponding to $\mathcal{E}$. It turns out that the following theorem holds (see Griffiths [1968]):

Theorem. *Polarized Hodge structures $\mathcal{E}_1$ and $\mathcal{E}_2$ belong to the same orbit of the group $G_{\mathbb{Z}}$ if and only if the corresponding pseudo-polarized Griffiths tori are isomorphic.*

2.1. Griffiths tori. Consider a polarized Hodge structure $\mathcal{E} = \{H_{\mathbb{Z}}, H^{r,q}, Q\}$ of odd weight $2p-1$. Decompose the vector space $H_{\mathbb{C}}$ as a direct sum $H_1 \oplus H_2$, where $H_1 = \oplus_{i=0}^{p-1} H^{n-i,i}$, $H_2 = \overline{H}_1$. Let i be the projection operator mapping $H_{\mathbb{C}}$ onto H_2. Then $E = i(H_{\mathbb{Z}})$ is a discrete subgroup of rank $2\dim_{\mathbb{C}} H_2$. Consider the complex torus H_2/E.

Define a form $\mathcal{H}$ on H_2 as follows:

$$\mathcal{H}(\phi, \psi) = -2\sqrt{-1}Q(\phi, \overline{\psi}).$$

Conditions (2) and the oddness of n imply that $\mathcal{H}$ is a Hermitian form on H_2 of signature (s_1, s_2), where

$$s_1 = \sum_{j=1}^{[\frac{p+1}{2}]} h^{p-1-2j,p+2j}, \quad s_2 = \dim_{\mathbb{C}} H_2 - s_1.$$

For some choice of holomorphic coordinates $(z_1, \ldots, z_k)$ on H, $k = \dim H_2$, the form $\mathcal{H}$ can be written as

$$\mathcal{H}(z', z'') = \sum_{j=1}^{k} \varepsilon_j z'_j \overline{z''}_j,$$

where $z' = (z'_1, \ldots, z'_k)$, $z'' = (z''_1, \ldots, z''_k)$, $\epsilon_j = \pm 1$, where the number of positive ε_j is equal to s_1.

Consider the differential form η of type $(1,1)$ on H_2 written in the same coordinates as

$$\eta = \frac{\sqrt{-1}}{2} \sum_{j=1}^{k} \varepsilon_j dz_j \wedge d\overline{z}_j.$$

We will also use η to denote the induced $(1,1)$ form on the torus H_2/E. Let us show that η represents an integral 2-dimensional cohomology class.

Consider two elements z' and z'' in E. Let $\gamma_{z',z''}$ be the two-dimensional cycle generating by the following map of the square $\mathcal{D} = [0,1] \times [0,1]$ into H_2/E:

$$(t, \tau) \to tz' + \tau z''.$$

All of the elements of the second homology group of the torus H_2/E are representable by cycles of this form. However,

$$\int_{\gamma_{z',z''}} \eta = \frac{\sqrt{-1}}{2} \int \sum_{j=1}^{k} \varepsilon_j (z'_j \overline{z}''_j - \overline{z}'_j \overline{z}''_j) dt \wedge d\tau$$

$$= -\sum_{j=1}^{k} \varepsilon_j \operatorname{Im}(z'_j \overline{z}''_j) = -\operatorname{Im} \mathcal{H}(z', z'').$$

It is thus sufficient to show that $\operatorname{Im}\mathcal{H}(z', z'') \in \mathbb{Z}$ for any $z', z'' \in E$. Let $z' = i(\phi)$, $z'' = i(\psi)$, for $\phi, \psi \in H_{\mathbb{Z}}$. Then $\phi = z' + \overline{z}'$ and $\psi = z'' + \overline{z}''$. Since n is odd and since $Q(\overline{z}', \overline{z}'') = 0$, for $z', z'' \in H_2$, we get

$$\begin{aligned} -\operatorname{Im}\mathcal{H}(z', z'') &= \frac{\sqrt{-1}}{2}[\mathcal{H}(z', z'') - \mathcal{H}(z'', z')] \\ &= \frac{\sqrt{-1}}{2}[-2\sqrt{-1}Q(z', \overline{z}'') + 2\sqrt{-1}Q(z'', \overline{z}')] \\ &= Q(z', \overline{z}'') + Q(\overline{z}', z'') = Q(\phi, \psi) \in \mathbb{Z}. \end{aligned}$$

The pair $J(\mathcal{E}) = (H_2/E, \eta)$ will be called the *pseudo-polarized Griffiths torus* corresponding to the polarized Hodge structure $\mathcal{E}$.

The form η represents a positive cohomology class if and only if $s_2 = 0$. In this case the Griffiths torus is a polarized abelian variety (see 1.7). If the form Q on $H_{\mathbb{Z}}$ is *unimodular,* that is, with respect to some $\mathbb{Z}$-basis the matrix corresponding to Q has determinant ± 1, then $J(\mathcal{E})$ is a *principally polarized abelian variety.*

In general, however, the Griffiths torus $J(\mathcal{E})$ is not even necessarily algebraic.

2.2. Now, consider a polarized variety (X, ω) of dimension d, and for odd n construct a polarized Hodge structure $\mathcal{E}$ of weight n, as in Section 1.2. The corresponding pseudo-polarized torus $J^n(X) = J(\mathcal{E})$ is called the *nth intermediate Griffiths Jacobian* of the variety X.

If $d = n$, $J^n(X)$ is called the *middle* Jacobian of the variety X. The intersection form of n-dimensional cycles on the n-dimensional variety X is unimodular (see Griffiths–Harris [1978]), and so, if $s_2 = 0$, the canonically defined polarization of $J^n(X)$ is principal.

There is another way to describe the complex structure on $J^n(X)$. Consider the *Griffiths operator*

$$C_G = \sum_{p+q=n} (\sqrt{-1})^{\frac{p-q}{|p-q|}} \Pi_{p,q},$$

where $\Pi_{p,q}$ is the projection onto $H^{p,q}$. It is not hard to see that C_G is a real operator, and $C_G^2 = -1$. Thus, this operator defines a complex structure on $H^n(X, \mathbb{R})$. It can be shown that

$$J^n(X) \simeq H^n(X, \mathbb{R})/H^n(X, \mathbb{Z}),$$

as a complex torus.

Now, we can use the isomorphism

$$\theta : H^n(X, \mathbb{R}) \to H_{2d-n}(X, \mathbb{R}),$$

which associates a cycle $\gamma = \theta(\omega)$ to a form ω, which is defined by the equation

$$\int_\gamma \eta = \int_X \eta \wedge \omega.$$

It is clear that the restriction of θ to $H^n(X,\mathbb{Z})$ is an isomorphism, so

$$H^n(X,\mathbb{Z}) \simeq H_{2d-n}(X,\mathbb{Z})/(\mathrm{tor}).$$

Therefore we can also say that

$$J^n(X) = H_{2d-n}(X,\mathbb{R})/H_{2d-n}(X,\mathbb{Z}),$$

where the complex structure on $H_{2d-n}(X,\mathbb{R})$ is transferred to $H^n(X,\mathbb{R})$ by θ.

Let $\dim X = d$. In that case the Griffiths torus $J^{2d-1}(X)$ is an abelian variety, since in this case $s_1 = h^{d-1,d} = \dim H_2$. This variety is called the *Albanese variety* and denoted by $\mathrm{Alb}(X)$. Using Serre duality, obtain

$$\mathrm{Alb}(X) = H^{d-1,d}(X)/H^{2d-1}(X,\mathbb{Z}) \simeq H^0(X,\Omega^1)^*/H^1(X,\mathbb{Z}).$$

Let us give an explicit description of the Albanese variety. Let $\omega_1,\dots,\omega_h$ be a basis of the space $H^{1,0}$ of holomorphic 1-forms on X and let $\gamma_1,\dots,\gamma_{2h}$ be generators of $H_1(X,\mathbb{Z})$ modulo torsion. In the dual space $(H^{1,0})^*$ to $H^{1,0}$, pick a basis $\eta_1,\dots,\eta_h$, dual to $\omega_1,\dots,\omega_h$, and using this basis fix an isomorphism between $(H^{1,0})^*$ and $\mathbb{C}^h$. Then the map

$$\gamma \to \left(\int_\gamma \omega_1,\dots,\int_\gamma \omega_h\right)$$

is a natural inclusion of $H_1(X,\mathbb{Z})$ into the space $\mathbb{C}^h \simeq (H^{1,0})^*$. Thus, the Albanese variety is the complex torus

$$\mathrm{Alb}(X) \simeq \mathbb{C}^h/\Lambda,$$

where $\Lambda \subset \mathbb{C}^h$ is the lattice of rank $2h$, which is the image of $H_1(X,\mathbb{Z})$ under the map described above.

The complex torus $J(X)$ is called the *Picard variety* of the algebraic variety X, and is denoted by $\mathrm{Pic}^0(X)$. Using Serre duality, it can be shown that $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$ are dual abelian varieties (see Mumford [1965]). It is also not hard to see that

$$\mathrm{Pic}^0(\mathrm{Alb}(X)) = \mathrm{Pic}^0(X),$$

$$\mathrm{Alb}(\mathrm{Pic}^0(X)) = \mathrm{Alb}(X).$$

If X is a nonsingular algebraic curve, then its only intermediate Jacobian $J^1(X) = \mathrm{Alb}(X) = \mathrm{Pic}^0(X)$ is the *Jacobian variety*, or the *Jacobian* of the curve (see Section 1.7).

2.3. A choice of a basepoint $x_0 \in X$ defines the holomorphic *Albanese mapping*, as follows:

$$m : X \to \mathrm{Alb}(X), \quad x \to \left(\int_{x_0}^{x} \omega_1, \ldots, \int_{x_0}^{x} \omega_h \right) \bmod \Pi.$$

This mapping has already been used in the introduction to establish the isomorphism of two different definitions of an elliptic curve. It can be seen from the construction that the induced maps

$$\mu_* : H_1(X, \mathbb{Z})/(\mathrm{tor}) \to H_1(\mathrm{Alb}(X), \mathbb{Z})$$

and

$$\mu^* : H^0(\mathrm{Alb}(X), \Omega^1) \to H^0(X, \Omega^1)$$

are isomorphisms.

For any variety X such that $\dim X = d$, denote by $\mathrm{Ch}_0^p(X)$ the set of algebraic cycles of real codimension p on X which are algebraically equivalent to 0. In particular, $\mathrm{Ch}_0^{2d}(X)$ is the set of 0-cycles, that is to say, formal sums of the form

$$\sum n_i y_i; \quad y_i \in X, \quad \sum n_i = 0.$$

The Albanese mapping evidently depends on the choice of basepoint x_0, but it induces a basepoint-independent map

$$\emptyset : \mathrm{Ch}_0^{2d}(X) \to \mathrm{Alb}(X).$$

The map $\emptyset$ is defined as follows. Let $y = \sum n_i y_i \in \mathrm{Ch}_0^{2d}(X)$. Then, after choosing the basepoint x_0, let

$$\emptyset_{x_0}(y) = \sum n_i \mu(y_i).$$

If x_1 is another point of X, then

$$\emptyset_{x_0}(y) - \emptyset_{x_1}(y) = \left(\int_\gamma \omega_1, \ldots, \int \gamma \omega_h \right) \in \Pi,$$

where γ is a one-cycle on X. Thus, $\emptyset_{x_0}$ does not depend on x_0 and defines the map $\emptyset$.

The generalization of the Albanese map to an arbitrary intermediate Jacobian $J^n(X)$ is the *Abel-Jacobi mapping* which is a homomorphism

$$\emptyset : \mathrm{Ch}_0^{n+1}(X) \to J^n(X).$$

In order to define it, let us represent the Jacobian $J^n(X)$ as

$$J^n(X) = H_{2d-n}(X, \mathbb{R})/H_{2d-n}(X, \mathbb{Z}),$$

as described above.

Let $\delta \in \mathrm{Ch}_0^{n+1}(X)$ be an $(2d-n-1)$-cycle. Then, $\delta = \partial\gamma$, where γ is a $(2d-n)$-chain on X. The chain γ gives a linear functional

$$\omega \to \int_\gamma \omega$$

on $H^{2d-n}(X,\mathbb{R})$, and hence defines an element $\overline{\gamma}$ of the space $H_{2d-n}(X,\mathbb{R})$. Let γ_1 be any other chain, for which $\delta = \partial\gamma$. Then $\partial(\gamma - \gamma_1) = 0$, and $\overline{\gamma} - \overline{\gamma_1} \in H_{2d-n}(X,\mathbb{Z})$. Now, set

$$\emptyset(\delta) = \overline{\gamma} \bmod H_{2d-n}(X,\mathbb{Z}),$$

which is well-defined.

Suppose that a subset $S \subset \mathrm{Ch}_0^{n+1}$ is equipped with the structure of an algebraic variety. Suppose further, that there exists an algebraic cycle $\tilde{S} \in S \times X$ of real codimension $n+1$, such that any cycle $s \in S$ can be obtained as the intersection of $\tilde{S}$ with the fiber $\{s\} \times X$, where this fiber is viewed as naturally identified with X itself. In that case we say that S is an algebraic subvariety of $\mathrm{Ch}_0^{n+1}(X)$.

One example of the above situation is the set F of all lines on the cubic hypersurface X in $\mathbb{P}^4$. This set has a natural inclusion into the Grassmanian $G(2,5)$, thereby forming a nonsingular complete algebraic surface (see Chapter 3, §2.) – the so-called Fano surface.

Amazingly, the following proposition holds (see Lieberman [1968] for proof):

Proposition. *On algebraic subvarieties of* Ch_0^{n+1} *the Abel-Jacobi mapping is regular.*

2.4. Weil tori. Along with the Griffiths torus, we can associate another complex torus to a Hodge structure $\mathcal{E}$ – the Weil torus $\mathbf{I}(X)$. Consider the decomposition $H_{\mathbb{C}} = H_1 \oplus H_2$, where $H_1 = H^{n-1,1} \oplus H^{n-3,3} \oplus \ldots \oplus H^{0,n}$, $H_2 = \overline{H_1}$ (it is assumed that n is odd). Let j be the projection of $H_{\mathbb{C}}$ onto H_1, $E = j(H\mathbb{Z})$. The complex torus H_1/E with the polarization given by the Hermitian form $B = 2(\sqrt{-1})^n Q(\phi, \overline{\psi})$ is then the Weil torus $\mathbf{I}(\mathcal{E})$. It can be checked using (2) that $\mathbf{I}(\mathcal{E})$ is always an abelian variety, since the form B is always positive-definite.

Let (X,ω) be a polarized algebraic variety of dimension d, $\mathcal{E}$– the Hodge structure (6), then $\mathbf{I}^n(X) = \mathbf{I}(\mathcal{E})$ is called the n-the *intermediate Weil Jacobian* of the variety X. The torus $\mathbf{I}^n(X)$ can also be viewed as the quotient $H^n(X,\mathbb{R})/H^n(X,\mathbb{Z})$, but the complex structure is given by the *Weil operator*

$$C = \sum_{p+q=n} (\sqrt{-1})^{p-q} \Pi_{p,q}.$$

2.5. Let D be the classifying space of Hodge structures of odd weight n corresponding to some data. To each point $\mathcal{E} \in D$ we can associate the two tori $J(\mathcal{E})$ and $\mathbf{I}(\mathcal{E})$. Note that all points lying in an orbit of the $G_{\mathbb{Z}}$ action get isomorphic tori. Evidently, the Griffiths tori vary holomorphically with respect to the variation of the point $\mathcal{E} \in D$. From Griffiths' theorem (see sec 3.3) it follows that as the varieties $\{X_s\}$, $s \in S$ vary holomorphically, so do the intermediate Griffiths Jacobians $J^n(X)$. This is not so for the Weil tori – a counterexample can be constructed as follows. Let E_λ be the elliptic curve $\mathbb{C}/\{\mathbb{Z}\lambda + \mathbb{Z}\}$. If z is a holomorphic coordinate on $\mathbb{C}$, then $\omega\lambda = dz$ is a holomorphic differential on E_λ. Set

$$S = \{(\lambda_1, \lambda_2, \lambda_3)\}, \quad X_s = E_{\lambda_1} \times E_{\lambda_2} \times E_{\lambda_3}.$$

Let

$$\omega_s = \sum_{i=1}^{3} \omega_{\lambda_i} \wedge \overline{\omega_{\lambda_i}}$$

be the polarization on X_s, and $\mathcal{E}_s$ the polarized Hodge structure of weight 3 associated with the pair (X_s, ω_s). A direct computation (see Griffiths [1968]) shows that $\mathbf{I}(\mathcal{E}_s)$ does not vary holomorphically with $s \in S$.

2.6. The complex tori defined above have found interesting applications in algebraic geometry. One of the most interesting such applications is the proof of the non-rationality of the cubic threefold (see Clemens–Griffiths [1972]). The cubic threefold X is a nonsingular hypersurface of degree 3 in $\mathbb{P}^4$. It is easy to show that X is a unirational variety, that is, there exists a rational map

$$\phi : \mathbb{P}^3 \to X$$

of the projective space $\mathbb{P}^3$ onto X. For a long time the following problem of Lüroth remained open: are there unirational, but not rational algebraic varieties? In dimension 2 such varieties do not exist. The non-rationality of the cubic threefold X gave one of the currently known solutions to Lüroth's problem.

Let us briefly describe the idea of the proof. Let X be an arbitrary projective three-fold, for which there is a Hodge decomposition of type

$$H^3(X) = H^{2,1} \oplus H^{1,2}.$$

Hypersurfaces of $\mathbb{P}^4$ of degree not exceeding 4 are known to have this property. In this case the tori of Griffiths and of Weil in dimension 3 coincide, and are a principally polarized abelian variety $(J^3(X), \eta)$. It can be shown that $(J^3(X), \eta)$ has a unique decomposition as a product

$$(J^3(X), \eta) = (J_1, \eta) \times \ldots \times (J_k, \eta_k)$$

of simple principally polarized tori. Let (J_c, η_c) be the product of all of the tori which are Jacobians of curves, (J_G, η_G) the product of all of the other tori.

The polarized torus (J_G, η_G) is called the *Griffiths component* of the variety X.

If the Griffiths component of a variety X is non-trivial, then X is not rational (see Clemens–Griffiths [1972]).

This is easy to explain. For $\mathbb{P}^3$ the middle Jacobian is trivial. Let $\tilde{X} \to X$ be the morphism inverse to a monoidal transformation centered at some nonsingular curve $E \subset X$. Then it is easy to show that

$$(J^3(\tilde{X}), \tilde{\eta}) = (J^3(X), \eta) \times (J(E), \eta_E),$$

where $(J(E), \eta_E)$ is the polarized Jacobian of some curve E. From this it can be deduced that the middle Jacobian of a rational variety is a product of Jacobians of curves.

Clemens–Griffiths [1972] show that for the cubic threefold the Griffiths component is non-trivial. This is an extremely deep fact, the proof of which uses the rich geometry of the cubic threefold.

§3. The Period Mapping

3.0. In Section 1.2 it was shown that the Hodge decomposition on the n-dimensional cohomology of a polarized algebraic variety (X, Ω) defines a certain polarized Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ of weight n. Let us construct the classifying space (the space of period matrices) D using the data corresponding to $\mathcal{E}$. The point $\mathcal{E} \in D$ is called the classifying point, or the period, of the polarized variety (X, ω). In this section we will investigate how the periods of an algebraic variety vary with respect to "analytic deformations." It will be shown that for families $\{(X_s, \omega_s)\}$, $s \in S$, of varieties parametrized by points of a complex manifold S, the periods vary holomorphically with respect to the parameter value $s \in S$. Now, for some more precise definitions.

3.1. Consider a smooth morphism $f : \mathcal{X} \to S$ of complex manifolds with connected compact fibers $X_s = f^{-1}(s)$, $s \in S$. We will say that $\{X_s\}$ is a complex-analytic family of complex manifolds with connected base S.

First, note that the dimensions of all the manifolds X_s are the same. In the future we will denote $\dim X_s$ by d. Furthermore, the fibration f is C^∞ locally trivial, that is for every point $s_0 \in S$ there exists a neighborhood U such that the diagram (14) commutes,

$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "\nu"] \arrow[dr, "f"'] & & X \times U \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd} \quad , \tag{14}$$

where $X = X_{s_0}$, π_U is the canonical projection and ν is a diffeomorphism. The diffeomorphism ν is called the trivialization of the fibration Ξ over U.

Fix a trivialization ν. Then for every $s \in U$ we have a diffeomorphism

$$\gamma_s = (\pi_X \circ \nu | X_s)^{-1} : X \to X_s.$$

Here π_X is the natural projection of $X \times U$ onto X. The isomorphism

$$\gamma_s^* : H^*(X_s, \mathbb{Z}) \simeq H^*(X, \mathbb{Z})$$

already does not depend on the choice of the trivialization (see Chapter 1, Section 10).

Note that the isomorphism γ_s^* can be considered fixed for all $s \in S$, but its choice is only determined up to the action of the monodromy group Γ_s on $H^*(X, \mathbb{Z})$ (see Chapter 1, Section 10).

Consider the function

$$h^{p,q}(s) = \dim_{\mathbb{C}} H^q(X_s, \Omega^p)$$

on the base S. It is known (see Steenbrink [1974]) that $h^{p,q}$ is *upper semi-continuous* on S, that is, for any $s_0 \in S$ there is a neighborhood V, such that $h^{p,q}(s) \leq h^{p,q}(s_0)$, for all $s \in V$. Since all of the fibers are diffeomorphic, it follows that $\dim H^n(X_s, \mathbb{C})$ is constant on S. Suppose that the manifolds X_s are Kähler for all $s \in S$. then

$$\sum_{p+q=n} h^{p,q}(s) = \dim H^n(X_s, \mathbb{C}),$$

and so the functions $h^{p,q}(s) = h^{p,q}$ are all constants. We will call these constants the *Hodge numbers* of the family of Kähler manifolds $\{X_s\}$.

3.2. Suppose that the morphism f is part of a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \mathbb{P}^N \times S \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd} , \tag{15}$$

where i is an embedding and π_s is the natural projection. In this case it is said that $\{X_S\}$ is a *smooth projective family of Kähler manifolds* over the base S. This actually means that we have a family of manifolds embedded into $\mathbb{P}^N$ and parametrized by the manifold S.

If π is the natural projection of $\mathbb{P}^N \times S$ onto $\mathbb{P}^N$, then the restriction $\pi \circ i$ to the fiber X_s defines an embedding into $\mathbb{P}^N$. In particular, all of the manifolds X_s are actually algebraic varieties. The restriction to X_s of the canonical sheaf $\mathcal{O}_{\mathbb{P}^N}(1)$ defines a positive line bundle L_s on X_s. its characteristic class ω_s allows us to use formulas (6) to define a polarized Hodge structure

$$\{(H_{\mathbb{Z}})_s, (H^{p,q}), Q_s\},$$

associated with (X_s, ω_s).

Fixing $s_0 \in S$, set

$$H_{\mathbb{Z}} = (H_{\mathbb{Z}})_{s_0}, \quad (H^{p,q})_{s_0}, \quad Q = Q_{s_0}, \quad h^{p,q} = \dim H^{p,q}.$$

Let us construct the classifying space D corresponding do the data $H_{\mathbb{Z}}$, Q, $\{h^{p,q}\}$. Since the class ω_s is a restriction of a class from $H^2(\mathbb{P}^N, \mathbb{Z})$ to

$H^2(X_s, \mathbb{Z})$, it follows that $\gamma_s^*(\omega_s) = \omega_{s_0}$. Note also that γ_s^* maps the forms Q_s and Q_{s_0} into one another, since those forms are defined topologically, once the class ω_s is chosen. Therefore, γ_s^* restricts to an isomorphism between $(H_{\mathbb{Z}})_s$ and $H_{\mathbb{Z}}$, such that $(\gamma_s^*)^*(Q) = Q_s$. In particular, the monodromy group is a subgroup Γ_s of the group $G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q)$. Fix a subgroup $\Gamma \subset G_{\mathbb{Z}}$ containing Γ_s. Extending γ_s^* linearly to $(H)_s = H^n(X_s, \mathbb{C})$, get a polarized Hodge structure

$$\{H_{\mathbb{Z}}, \gamma_s^*((H^{p,q})_s, Q\}.$$

This structure is determined by the Γ_s orbit of the point $s \in S$. Thus, we get a mapping

$$\Phi_\Gamma : S \to \Gamma \backslash D. \tag{16}$$

Definition. The mapping Φ_Γ is called the *period mapping* of the smooth projective family $\{X_s\}$.

3.3.

Theorem 1. (Griffiths)

1. *The mapping Φ_Γ is holomorphic.*
2. *The mapping Φ_Γ has the local lifting property, that is for each point $s_0 \in S$ there exists a neighborhood $U \subset S$ and a holomorphic map Φ_U, such that the diagram*

$$\begin{tikzcd} U \arrow[rr, "\tilde{\Phi}_U"] \arrow[dr, "\Phi_\Gamma"'] & & D \arrow[dl, "\lambda"] \\ & \Gamma / D & \end{tikzcd}$$

commutes. Here λ is the canonical projection map.

3. *The mapping $\tilde{\Phi}_U$ in the above diagram is horizontal, that is, its differential $(\tilde{\Phi}_U)_*$ maps the tangent space T_s at each point $s \in U$ into a horizontal subspace $T_{h,\tilde{\Phi}(s))} \subset T_{\tilde{\Phi}(s)}$ (see sec 1.9).*

The proof of Theorem 1, based on results of Kuranishi, was given by Griffiths [1968]. Despite the fact that the statement of the theorem sounds very plausible, the theorem is far from trivial. A second proof, due to Grothendieck, is based on theorems on direct images of sheaves (see the survey Griffiths [1970]). We will make a few remarks *a propos* the proof of the theorem.

First, note that the statement of the theorem is local, so it is enough to prove it for a polydisk S. From the triviality of the monodromy group in that case, and from the construction of the period mapping it follows that Φ can be lifted to a map

$$\tilde{\Phi} : S \to D.$$

Consider the map

$$\phi^p : D \to G(f^p, H),$$

defined in Section 1.9. The complex structure on D is introduced using the embedding $\phi = \Pi \phi^p$.

Consider the composition $\lambda^p = \phi^p \circ \tilde{\Phi}$. This can be described explicitly as follows. Let $\{F^p\}$ be the Hodge filtration on the n-dimensional primitive cohomology of the space $X = X_{s_0}$. Consider the basis F^p consisting of primitive harmonic differential forms $\omega_1, \dots, \omega_{f^p}$ on X. Suppose that ω_k is of type (r_k, l_k). Set

$$\eta_k(s) = \gamma_s^* \circ \Pi_s^{r_k, l_k} \circ (\gamma_s^{-1})^*(\omega_k),$$

where $\Pi_s^{r,s}$ is the projection map onto the subspace $H^{r,l}(X_s)$ of the space $H^n(X_s)$. The forms $\eta_k(s)$ are C^∞ (with respect to s) primitive harmonic forms on X, such that $\eta_k(s_0) = \omega_k$. Let $F^p(s)$ be the subspace generated by the set $\{\eta_k(s)\}$ in $H = P^n(X)$. Then, the mapping λ^p associates to each $s \in S$ a subspace $F^p(s) \in G(f^p, H)$.

To prove that λ^p is holomorphic, it is sufficient to consider the case where S is a one-dimensional disk, and to show that

$$\frac{\partial}{\partial s} \eta_k(s)|_{s=s_0} \in F^p.$$

Indeed, from the definition of the tangent space to the Grassmanian (Section 1.9) it follows that in that case

$$\lambda_*^p(\frac{\partial}{\partial \bar{s}}) = 0.$$

To show claim (3) of Theorem 1, it is enough to show that

$$\frac{\partial}{\partial s} \eta_k(s)|_{s=s_0} \in F^{p-1}.$$

Indeed, in that case

$$\lambda_*^p(\frac{\partial}{\partial \bar{s}}) \in \mathrm{Hom}(F^p, F^{p-1}/F^p),$$

and so $\tilde{\Phi}_*(T_{s_0}) \subset T_{h, \tilde{\Phi}(S_0)}$.

To conclude this section, let us note that the holomorphicity theorem (Theorem 1) is valid also in the following more general setting. Let $\{X_s\}$, $s \in U$ be a complex analytic family of compact Kähler manifolds over the polydisk U, and ω_s is the Kähler form on X_s.

Suppose that the family $\{X_s\}$ is C^∞-trivial, and the trivialization is fixed. Also fix the center s_0 of the polydisk U.

Suppose now that ω_s depends smoothly on s, or, more precisely, the form $\gamma_s^*(\omega_s)$ depends smoothly on s as a form on $X = X_{s_0}$. Consider the Hodge filtration $\{F_s^p\}$ on $H_s = H^n(\tilde{X}_s)$, defined by the Kähler form ω_s. If we associate to each point $s \in S$ the subspace $F^p(s) = \gamma_s^*(F_s^p)$ of the space $H = H^n(X)$, we obtain a mapping

$$\lambda^p : U \to G(f^p, H),$$

defined in some neighborhood of s_0.

Proposition. *The mapping γ^p is holomorphic.*

The proof can be found in Griffiths [1968].

§4. Variation of Hodge Structures

4.1. Consider a smooth complex manifold S and a subgroup $\Gamma \subset G_{\mathbb{Z}}$. Suppose that we are given a map $\Phi : S \to \Gamma\backslash D$, satisfying the conditions:

1. Φ is holomorphic,
2. Φ is locally liftable,
3. Φ is horizontal. Then the map Φ is called a *variation of Hodge structures* (vHs). An important example of vHs is given by Theorem 1.

Consider the universal cover $\tilde{S}$ of the manifold S. Conditions (1) and (2) are equivalent to the existence of a holomorphic mapping $\tilde{\Phi}$ which makes the diagram

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\mu"] & D \arrow[d, "\lambda"] \\ S \arrow[r, "\Phi"] & \Gamma\backslash D \end{tikzcd} \tag{17}$$

commute. Here μ and λ are canonical projections. Define a representation $\rho : \pi_1(S, s_0) \to \Gamma$, having the property that $\tilde{\Phi}(\gamma\tilde{s}) = \rho(\gamma)\tilde{\Phi}(\tilde{s})$, for all $\gamma \in \pi_1(S, s_0)$, $\tilde{(s)} \in \tilde{S}$. By factoring $\tilde{S} \times H_{\mathbb{Z}}$ by the action of $\pi_1(S, s_0)$ obtain a locally constant sheaf $\mathcal{H}_{\mathbb{Z}}$ of free abelian groups on S. The sheaves $\tilde{\Phi}^*(\mathcal{F}^p)$ are invariant with respect to the $\pi_1(S, s_0)$ action, and hence define locally free sheaves $\mathcal{F}^p_s$ on S. This also defines a filtration

$$\mathcal{F}^n_s \subset \ldots \subset \mathcal{F}^1_s \subset \mathcal{F}^0_s = \mathcal{H}_S$$

of the locally constant sheaf $\mathcal{H}_S = \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S$. Setting $\mathcal{H}^{p,q}_S = \mathcal{F}^p_S / \mathcal{F}^{p+1}_S$ makes $\mathcal{H}^{p,q}_S$ into a locally free sheaf of rank $h^{p,q}$. There is also a C^∞ decomposition

$$\mathcal{H}_S = \mathcal{H}^{n,0}_S \oplus \ldots \oplus \mathcal{H}^{0,n}_S,$$

with $\mathcal{H}^{p,q}_S = \overline{\mathcal{H}}^{q,p}_S$, and there is a C^∞ decomposition

$$\mathcal{F}^r_S = \mathcal{H}^{n,0}_S \oplus \ldots \oplus \mathcal{H}^{n-r,r}_S.$$

The bilinear form Q on $H_{\mathbb{Z}}$, invariant with respect to $G_{\mathbb{Z}}$ defines a bilinear form $Q_S : \mathcal{H}_S \times \mathcal{H}_S \to \mathcal{O}_S$, satisfying conditions (2), where $H^{p,q}$ is the fiber of $\mathcal{H}^{p,q}_S$ at any fixed point $s_0 \in S$.

Exterior differentiation in $\mathcal{O}_S$ and the equality $\mathcal{H}_S = \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_S$ defines a locally integrable connection (see Chapter 1, Section 5), the so-called *Gauss – Manin connection,*

$$\nabla : \mathcal{H}_S \to \mathcal{H}_S \otimes \Omega^1_S.$$

Conversely, the choice of data $\{S, \mathcal{H}_{\mathbb{Z}}, \{\mathcal{F}^p_s, Q_S, \nabla\}\}$, satisfying the conditions above allows one to reconstruct the vHs which produced that data.

4.2. Call the map $\Phi : S \to \Gamma \backslash D$ an *extended variation of Hodge structure,* if there exists an open everywhere dense set $S' \subset S$, such that $\Phi|_{S'} : S' \to \Gamma\backslash D$ is vHs. It can be shown that in this situation the image $\rho(\pi_1(S, s_0)) \in \Gamma$ does not depend on the choice of S'. This image is called the monodromy group of the extended vHs.

§5. Torelli Theorems

5.1. R. Torelli [1914] showed that the jacobian variety of an algebraic curve determines the curve. That is, the algebraic curve X is uniquely determined by its polarized Hodge structure on $H^1(X, \mathbb{C})$, which is to say its image under the period mapping. The questions regarding the degree to which the periods of a variety determine the variety are usually called *Torelli problems.* There are four types of Torelli problems:

1. infinitesimal,
2. global,
3. local,
4. global generalized, (or weak global).

Our immediate goal is to explain the formulation of these problems. First, some definitions.

5.2. The Kodaira-Spencer Mapping. Let $f : \mathcal{X} \to S$ be a smooth family of complex manifolds over a complex manifold base S, let s_0 be some point of S, $X = X_{s_0} = f^{-1}(s_0)$. Such a family is called a *deformation* of the complex manifold X.

Let us denote by T_X the tangent bundle of the manifold X, and by T_{s_0} the tangent space to S at the point s_0. Consider the exact sequence of fiber bundles on X :

$$0 \to T_X \to T_{\mathcal{X}}|_X \to N|_{X|_{\mathcal{X}}} \to 0.$$

Here $N|_{X|_{\mathcal{X}}}$ is the normal bundle of the submanifold $X \subset \mathcal{X}$.

The map df defines an isomorphism of the fibers of the bundle $N_{X|_{\mathcal{X}}}$ at all points $x \in X$ and the space T_{s_0}. Therefore, any element $\zeta \in T_{s_0}$ defines a global section $\tilde{\zeta} \in H^0(X, N_{X|_{\mathcal{X}}})$. Consider the coboundary homomorphism

$$\delta : H^0(X, N_{X|_{\mathcal{X}}}) \to H^1(X, T_X).$$

Setting $\rho(\zeta) = \delta(\tilde{\zeta})$, we obtain the linear *Kodaira-Spencer mapping*

$$\rho : T_{s_0} \to H^1(X, T_X).$$

Let us give an explicit description of this mapping. Without loss of generality we can consider just the case $\dim S = 1$. Denote the base coordinate by s, in that case $\frac{\partial}{\partial s}|_{s=s_0}$ is the basis vector of the space T_{s_0}.

Consider the atlas $\{U_\alpha\}$, which covers a neighborhood of the fiber $X = X_{s_0} \in \mathcal{X}$. On each of the sets U_α pick complex coordinates of the form

$$(x_\alpha; s) = (x_\alpha^1, \dots, x_\alpha^d; s), \quad d = \dim X.$$

Then

$$(X_\alpha; s) = F_{\alpha\beta}(F_{\alpha\beta}^1, \dots, F_{\alpha\beta}^d; S),$$

for some choice of transition functions $F_{\alpha\beta}^j$. The differentiation $\left(\frac{\partial}{\partial s}\right)_\alpha$ defines on $X \cap U_\alpha$ a section of the sheaf $T_{\mathcal{X}|X}$. Since on $U_\alpha \cap U_\beta \cap X$ the differentials $\left(\frac{\partial}{\partial s}\right)_\alpha$ and $\left(\frac{\partial}{\partial s}\right)_\beta$ define the same element in $N_{X|\mathcal{X}}$, then the difference

$$t_{\alpha\beta} = \left(\frac{\partial}{\partial s}\right)_\beta - \left(\frac{\partial}{\partial s}\right)_\alpha = \sum_j \left(\frac{\partial F_{\alpha\beta}^j}{\partial s}\right)\Bigg|_{s=s_0} \otimes \frac{\partial}{\partial x_\alpha^j}$$

is a correct definition of the section of the sheaf T_X over $U_\alpha \cap U_\beta \cap X$. Evidently

$$t_{\alpha\beta} = -t_{\beta\alpha}$$

on $U_\alpha \cap U_\beta \cap X$, and

$$t_{\alpha\beta} + t_{\beta\gamma} + t_{\gamma\alpha} = 0$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap X$. Thus the collection of sections $\{t_\alpha\beta\}$ forms a 1-cocycle. The element of $H^1(X, T_X)$ defined by this cocycle is then the image of the differential $\frac{\partial}{\partial s} \in T_s$ under the map ρ.

Call a deformation $\mathcal{X} \to S$ *trivial*, if there exists a holomorphic diffeomorphism ν which makes the following diagram commute.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\nu"] \arrow[dr, "f"'] & & X \times S \arrow[dl, "\pi_S"] \\ & S & \end{tikzcd}$$

Here π_s is the natural projection of the product $X \times S$ onto the second multiplicand. The deformation is called *locally trivial* if this diagram commutes for some neighborhood of a point s_0 in S.

It is clear that the construction of ρ gives a trivial map for a trivial deformation. The converse is not, in general, true. Indeed, let $\phi : P \to S$ be a base change, $\phi(p_0) = s_0$. Consider the induced family $\mathcal{X} \times_s P \to P$. Then the

Kodaira-Spencer mapping $\rho' : T_{p_0} \to H^1(X, T_x)$ of this family is related with ρ by the relation

$$\rho' = \rho \circ d\phi.$$

Now, take any family over the unit disk S with non-isomorphic fibers, and after making the base change $s = p^2$, obtain a nontrivial deformation of the manifold X over P with a trivial Kodaira-Spencer mapping.

On the other hand, the following result holds (Frölicher–Nijenhuis [1957]). Let X be a compact complex manifold for which $H^1(X, T_X) = 0$, $f : \mathcal{X} \to S$ a deformation of X. Then there exists a neighborhood U of the point $s_0 \in S$, such that all of the fibers X_s, $s \in U$ are isomorphic as complex manifolds.

It should be noted that there are locally non-trivial deformations all of whose fibers are isomorphic (for example with fiber $\mathbb{P}^1$.)

The elements of the group $H^1(T, T_X)$ are called *infinitesimal deformations* of a manifold X. If $H^1(X, T_X) = 0$, we say that the manifold X is *rigid.*

5.3. Universal families. Consider the deformation

$$f : \mathcal{X} \to S$$

of a complex manifold $X = f^{-1}(s_0)$.

Call the deformation f *complete* if any other deformation of X is obtained from f by a local change of base. More precisely, let

$$\phi : \mathcal{F} \to P$$

be an arbitrary deformation of the same manifold $X = \phi^{-1}(p_0)$, then there exists a neighborhood $U \subset P$ of the point p_0 and a holomorphic map $\pi : P \to S$, $\pi(p_0) = s_0$, that the family ϕ defined over U is isomorphic to the induced family $\mathcal{X} \times_S U \to U$. If the differential $(d\pi)_{p_0}$ is uniquely defined, then the family f is called *versal.* If the germ of the map f is defined uniquely as well, the f is called *universal.*

A deformation is called *effective* if the corresponding Kodaira-Spencer mapping is a monomorphism. Clearly, an effective complete deformation is versal.

If there is a complete versal deformation for a manifold X, then such a deformation is called the *Kuranishi family* of this manifold. The questions of existence of such families were studied by Kuranishi [1962]. The simplest and most concrete example of this subject is the following (Kodaira–Nirenberg–Spencer [1958]).

Let X be a compact complex manifold, for which $H^2(X, T_X) = 0$. Then there exists a complete deformation $f : \mathcal{X} \to S$ of X over some polydisk S, for which the Kodaira-Spencer mapping is an isomorphism.

The role of the cohomology group $H^2(X, T_X)$ in this theorem can be explained as follows. Consider a non-trivial infinitesimal deformation $h \in H^1(X, T_X)$ of the manifold X and let us attempt to construct a deformation $f : \mathcal{X} \to S$ of X over the disk S, for which $\rho(\frac{\partial}{\partial s}) = h$.

In the notation of Section 5.2, let

$$F_{\alpha\beta}(s) = f^0_{\alpha\beta} + s f^1_{\alpha\beta} + s^2 f^2_{\alpha\beta} + \dots$$

The tensor $f^1_{\alpha\beta}$ defines a cocycle generating h. In order for $F_{\alpha\beta}$ to be transition functions, it must be true that

$$F_{\alpha\beta} \circ F_{\beta\gamma} - F_{\alpha\gamma} = 0.$$

Let us expand the left-hand side into a power series with respect to s; the coefficient of s^2 then defines a 2-cocycle on X with coefficients in T_X, which gives rise to an element $h_1 \in H^2(X, T_X)$, called the first obstruction. It can be shown that h_1 depends on h only. The coefficient of s^2 can be nullified by a choice of $f^2_{\alpha\beta}$ if and only if $h_1 = 0$.

Trying to make the coefficients of higher powers of s equal to 0, we will obtain second, third, etc, obstructions in $H^2(X, T_X)$.

If $H^2(X, T_X) \neq 0$ it is useful consider the deformations of X over arbitrary (not necessarily nonsingular) bases. Let S be an analytic space (see Chapter 1, Section 2). Suppose that the topological space $\mathcal{X} = X \times S$ has the structure of an analytic space, so that the natural projection $f : \mathcal{X} \to S$ is a morphism of analytic spaces, and the analytic structure on $\mathcal{X}$ induces an analytic structure on each fiber $X_s = f^{-1}(s)$, so that $X \simeq X_{s_0}$. Such a morphism f will be called a deformation of the manifold X over the base S. Obviously this restricts to the old definition when S is nonsingular.

Kuranishi [1962, 1965] constructed for each complex manifold X such a deformation

$$f : \mathcal{X} \to S$$

over an analytic set X, $X \simeq f^{-1}(s_0)$, that any other deformation $\phi : F \to T$, $\phi^{-1}(t_0) \simeq X$ can be locally obtain from f by a base change $\pi : T \to S$, $\pi(t_0) = s_0$. The differential $(d\pi)_{t_0}$ is then uniquely determined.

The deformation f is called the *versal Kuranishi deformation.* The analytic set S is constructed as follows. First, an analytic map of affine complex varieties

$$\gamma : H^1(X, T_X) \to H^2(X, T_X)$$

is defined, so that $S = \gamma^{-1}(0)$, $s_0 = 0$.

This map is such that $\gamma(0) = 0$, $(d\gamma)_0 = 0$. The second differential of γ coincides with the map $h \to h_1$, which associates the first obstruction to an infinitesimal deformation $h \in H^1(X, T_x)$. It is clear that if $H^2(X, T_X) = 0$, then $S = H^1(X, T_X)$. It can be shown that if the function $\dim H^0(X_s, T_{X\,s})$ is constant in some neighborhood of $0 \in S$, then Kuranishi's versal deformation is actually universal.

5.4. Moduli Space. The problem of classification of spaces (complex, algebraic, etc) is usually approached as follows. First the set of all the spaces under consideration is divided into a countable set of classes, by considering a certain number of discrete invariant, such as dimension, rank of various homology and cohomology groups, and so on. These discrete invariants are chosen in such a way, that the spaces of the same class are described up to isomorphism by a collection of continuous complex parameters – "moduli." The term "moduli" was introduced by Riemann, who showed that a nonsingular complex projective curve of genus $g \geq 2$ can be specified by $3g - 3$ complex parameters.

Let X be a complex manifold. The manifolds that can result from X as the moduli change have to correspond to different complex structures on the C^∞ manifold X. The complex structures must change continuously with respect to the moduli.

Consider the space $\mathfrak{M}$ of all complex structures on the manifold X. The construction of the *moduli space* for X corresponds to equipping M with a complex structure. It is, of course, preferable that the resulting complex space (or manifold) $\mathfrak{M}$ be well-behaved with respect to the deformations of X.

Suppose that there exists a smooth family

$$f : \mathcal{X} \to \mathfrak{M}$$

of complex manifolds with base $\mathfrak{M}$, that for every $m \in \mathfrak{M}$ the preimage $f^{-1}(m)$ is isomorphic to the manifold parametrized by m. Furthermore, if $\phi : \mathcal{F} \to S$ is any deformation of X, then there exists a unique morphism $\pi : S \to \mathfrak{M}$, such that the family ϕ is isomorphic to the induced family.

If the manifold $\mathfrak{M}$ has the properties described above, then it is called the *fine moduli space* of the manifold X.

Unfortunately, such a space exists rather infrequently. For example, it is impossible to construct it for curves of a fixed genus g. Indeed, there exist examples of locally nontrivial families of curves, all fibers of which are isomorphic. If $\phi : \mathcal{F} \to S$ is such a family, then the corresponding morphism π must send S into a single point $m \in \mathfrak{M}$, which contradicts the nontriviality of the family ϕ.

For this reason, the conditions on $\mathfrak{M}$ are usually weakened. Namely, the usual requirement is that for any deformation $\phi : \mathcal{F} \to S$ of X there exists a morphism $\pi : S \to \mathfrak{M}$ that for every point $s \in S$ the manifold $\phi^{-1}(s)$ is isomorphic to the manifold corresponding to the point $\pi(s) \in \mathfrak{M}$. Such a manifold $\mathfrak{M}$ is called the *coarse moduli space* of X.

The definitions of coarse and fine moduli spaces for an algebraic variety X are analogous. In that case $\mathfrak{M}$ is the space of all algebraic structures on a C^∞ manifold X, and all of the morphisms involved are algebraic.

As an example, consider a nonsingular projective curve X of genus g. The set $\mathfrak{M}_g$ of complex structures on the smooth manifold X is then the same as the space of algebraic structures on this manifold. Mumford [1965] obtained the following result:

For $g \geq 2$ the set $\mathfrak{M}_g$ can be endowed with the structure of a quasiprojective algebraic variety of dimension $3g-3$. This structure makes $\mathfrak{M}_g$ into the coarse moduli space parametrizing the nonsingular projective curves of genus g.

It should be noted that Mumford's result is valid not only over $\mathbb{C}$, but over any algebraically closed field of definition.

If $g = 0$, then the space $\mathfrak{M}_g$ is a single point, corresponding to $\mathbb{P}^1$. If $g = 1$, then $\mathfrak{M}_g$ is isomorphic to the affine line $\mathbf{A}^1$ (see Section 1.2 of Chapter 3 for more details).

Sometimes it is convenient to study not all complex (or algebraic) structures on X, but only those corresponding to some special type. This gives rise to different moduli spaces, for example the space of hyperelliptic curves of genus g.

Another common moduli space is the moduli space of polarized algebraic varieties. Let (X, ω) be a polarized algebraic variety. Then for all n the variety (X, ω) defines a polarized Hodge structure of weight n, according to the formulas (6). The sets of data corresponding to these structures for different choices of n are usually chosen to be the collection of discrete invariants, as discussed above. As the set $\mathfrak{M}$ consider the set of such structures $(\tilde{X}, \tilde{\omega})$ of a polarized algebraic variety on a smooth manifold X, which give the polarized Hodge structure with the same data as (X, ω). Suppose that we have succeeded in endowing $\mathfrak{M}$ with the structure of an algebraic variety. Let

$$\phi : \mathcal{F} \to S$$

be any algebraic deformation of (X, ω). Then ϕ is an algebraic deformation of $X = \phi^{-1}(s_0)$. In addition, in each fiber $\phi^{-1}(s)$ we have a positive integral cohomology class of type $(1,1)$, invariant with respect to the monodromy action (see Chapter 1, sec. 10, and Chapter 4, Section 4), and furthermore, $(\phi^{-1}(s_0), \omega_{s_0}) \simeq (X, \omega)$. Suppose that fora every such deformation there exists a morphism $\pi : S \to \mathfrak{M}$, such that for every point $s \in S$ the pair $(\phi^{-1}(s), \omega_s)$ corresponds to the point $\pi(s) \in \mathfrak{M}$. In this case we say that $\mathfrak{M}$ is the (coarse) moduli space of polarized algebraic varieties corresponding to (X, ω).

In Chapter 3, sec. 3 we discuss the example of constructing a moduli space for K3 surfaces.

5.5. Infinitesimal Torelli theorem. Suppose that a compact Kähler manifold X admits a universal Kuranishi family

$$f : \mathcal{X} \to S$$

with a nonsingular base $S \simeq H^1(X, T_X)$, $X = f^{-1}(s_0)$. Suppose further that on each fiber X_s, $s \in S$, we have a Kähler form ω_s, smoothly dependent on s (see the end of Section 3.3). Then, by virtue of the simply-connectedness of S there is a well-defined holomorphic map

$$\lambda = \Pi \lambda^p : S \to \Pi G(f^p, H).$$

Here λ^p is the mapping defined in Section 3.3. We will call λ the period mapping of an unpolarized Kuranishi family.

It is said that an *infinitesimal Torelli theorem* holds for a Kähler manifold X, if λ is an isomorphic embedding of some neighborhood of a point $s_0 \in S$

into $\Pi G(f^p, H)$. This is equivalent to saying that the differential $(d\lambda)_{s_0}$ is injective.

Suppose that the Kähler form ω on the fiber $X = f^{-1}(s_0)$ defines a positive integral class cohomology class. Consider a submanifold $S_\omega \subset S$ on which ω has type $(1,1)$. The restriction of the family f to S_ω is the *universal family* of the polarized algebraic variety (X, ω). This means that any other deformation $\phi : \mathcal{F} \to T$ of a polarized algebraic variety (X, ω), $(X, \omega) = \phi^{-1}(t_0)$ (see Section 5.4) is locally obtained from f by a unique change of base $\pi : T \to S_\omega$, $\pi(t_0) = s_0$.

Suppose Φ is the period mapping (16) (here $\Gamma = \{1\}$) for the family f over S_ω. If Φ_T is the period mapping for T, then we have a (locally) commutative diagram

$$\begin{tikzcd} T \arrow[rr, "\pi"] \arrow[dr, "\Phi_T"'] & & S_\omega \arrow[dl, "\Phi"] \\ & D & \end{tikzcd} .$$

We say that for the infinitesimal Torelli theorem holds for a polarized algebraic variety (X, ω), if Φ is a local embedding. This is the same as saying that the differentials $d\Phi$ and $d\lambda$ are injective on $(T_{S_{0\omega}})_{s_0}$.

Let us describe a criterion that ensures that the infinitesimal Torelli theorem holds for a polarized abelian variety (X, ω). This criterium was obtained by Griffiths [1968].

Recall that the tangent space $T_{\lambda(s_0)}$ to $\Pi G(f^p, H)$ at the point $\lambda(s_0)$ is isomorphic to $\bigoplus \mathrm{Hom}(F^p, H/F^p)$. By the results of item 3 of Theorem 1, it follows that $\lambda_*(T_{s_0})$ lies in the subspace $\bigoplus \mathrm{Hom}(H^{n-p,p}, H^{n-p-1,p+1})$ of the space $\bigoplus \mathrm{Hom}(F^p, H/F^p)$. The bilinear mapping

$$T_X \times \Omega_X^{p-1}$$

defines a pairing

$$H^1(X, T_X) \times H^{n-p,p} \to H^{n-p-1,p+1}.$$

This pairing defines a homomorphism

$$\varepsilon : H^1(X, T_X) \to \bigoplus \mathrm{Hom}(H^{n-p,p}, H^{n-p-1,p+1}).$$

Suppose $H^1(X, T_X)_\omega$ is the subspace of those elements $\xi \in H^1(X, T_X)$ for which $\xi \wedge \omega = 0$. It can be observed that if $\mu \in H^1(X, T_X)_\omega$, $\lambda \in P^{n-p,p}$, then $\mu \times \lambda \in P^{n-p-1,p+1}$. Thus, there is a homomorphism

$$\varepsilon_0 : H^1(X, T_X)_\omega \to \bigoplus \mathrm{Hom}(P^{n-p,p}, P^{n-p-1,p+1}).$$

Griffiths [1968] proved that

$$(d\lambda)_{s0} = \varepsilon_0 \circ \rho,$$

where ρ is the Kodaira-Spencer mapping for the family f over S_ω.

Since for effective families the map ρ is a monomorphism, the infinitesimal Torelli theorem follows from the injectivity of ε_0. Occasionally it is more convenient to check that the infinitesimal Torelli theorem holds by using the surjectivity of the map

$$\mu : \bigoplus_{p=0}^{n} P^{n-p,p} \otimes P^{d-n+p+1,d-p-1} \to H^{d-1}(X, \Omega^1 \otimes \Omega^d)_\omega, \tag{19}$$

dual to ε_0.

5.6. Local, global, and generalized Torelli theorems. Let $\mathfrak{M}$ be the moduli space of the polarized algebraic variety (X, ω), while D is the classifying space of polarized Hodge structures of weight n associated to (X, ω). Since, by definition, the points of $\mathfrak{M}$ parametrize the polarized varieties which give rise to the Hodge structures parametrized by the points of D, there is a natural mapping

$$\psi : \mathfrak{M} \to G_{\mathbb{Z}}\backslash D. \tag{20}$$

Suppose that ψ is an extended variation of Hodge structures (see Section 4.2.)

Definition. The *global Torelli theorem* holds for the moduli space $\mathfrak{M}$ if ψ is an embedding of closed points of $\mathfrak{M}$ into the set of closed points of $G_{\mathbb{Z}}\backslash D$.

Definition. Let (X, ω) be a polarized algebraic variety, while $[X]$ is the corresponding point of $\mathfrak{M}$. We say that the *local Torelli theorem* holds for (X, ω), if the differential $d\psi$ of ψ defines an inclusion of the tangent space $T_{[X]}$ into $T_{\psi([X])}$.

Definition. The *generalized global (or weak global)* Torelli theorem holds for the moduli space $\mathfrak{M}$ if there exists an open Zariski dense subset $\mathfrak{M}' \subset \mathfrak{M}$, such that $\psi|\mathfrak{M}'$ is an embedding into the set of closed points.

In other words, a global Torelli theorem must hold for a generic point of $\mathfrak{M}$.

Together with problems addressed by Torelli-type theorems, it is also interesting to consider the questions of surjectivity of the period mapping ψ. Sometimes a simple dimension count allows one to obtain a negative answer (for example, for algebraic curves). In general, however, this is a rather subtle problem – a solution for K3 surfaces, obtained in Kulikov [1977b] is described in Chapter 3, Section 3.

5.7. Let us note the difference between the infinitesimal and the local Torelli theorems. Let $\phi : \mathcal{X} \to S$ be the Kuranishi family of a variety X, while $S_\omega \subset S$ is a smooth submanifold described in Section 5.5. Then the neighborhood U of a point $s_0 \in S$ is not always locally isomorphic to a neighborhood $V \subset \mathfrak{M}$

of a point $[X]$ corresponding to a polarized variety (X, ω) in the moduli space. It can happen that V is locally isomorphic to U/Γ, where Γ is some group of locally analytic automorphisms of U. Examples of this sort are discussed in Sections 1.3 and 3.7 of Chapter 3.

§6. Infinitesimal Variation of Hodge Structures

Definition 6.1. An infinitesimal variation of Hodge structures (ivHs) of weight n is a collection of data

$$V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\},$$

where $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ is a polarized Hodge structure of weight n, T is a finite-dimensional complex vector space, and δ is a linear map

$$\delta : \bigoplus_{p=1}^{n} \delta_p : T \to \bigoplus_{p=1}^{n} \mathrm{Hom}(H^{p,q}, H^{p-1,q+1},$$

satisfying the conditions

1 $\delta_{p-1}(\xi_1)\delta_p(\xi_2) = \delta_{p-1}(\xi_2)\delta_p(\xi_1)$,
2 $Q(\delta(\xi)\psi, \eta) + Q(\psi, \delta(\xi)\eta) = 0$, $\psi \in H^{p,q}$, $\eta \in H^{q+1,p-1}$.

The concept of ivHs was introduced in Carlson–Griffiths [1980] and was studied in detail in a series of papers by Griffiths and others (Carlson–Green–Griffiths–Harris [1983], Cattani–Kaplan [1985], Griffiths [1983a]), and has already been found useful in questions having to do with global Torelli problems. In Section 4 of Chapter 3 we give a sketch of a weak global Torelli theorem, which was obtained by Donagi [1983] for a large class of hypersurfaces, and where the concept of ivHs was used to great advantage.

6.2. The main example of ivHs of concern to us here is obtained as follows. Let $\mathcal{X} \to S$ is a family of polarized algebraic varieties (see Section 3.2). Then, to each point $s_0 \in S$ we can associate a polarized Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ of weight n on $P^n(X_{s_0}, \mathbb{C})$, as in Section 1.2. We can also introduce a complex vector space $T = (T_S)_{s_0}$ of dimension $k = \dim S$. Let

$$\Phi_\Gamma : S \to \Gamma \backslash D$$

be the period mapping associated to our family, as in Section 3.2. By Theorem 1 of Section 3, in a certain neighborhood $U \subset S$ of the point s_0 the mapping Φ_Γ can be lifted to a holomorphic map

$$\Phi_U : U \to D.$$

Then the differential $(\Phi_U)_*$ sends T to a horizontal subspace T_{h,d_0} (see setion 1.7) of the tangent space $T_{d_0}(G)$ to D at the point $d_0 = \Phi_U(s_0)$. There is a canonical isomorphism

$$T_{h,d_0} \simeq \bigoplus_{p=1}^{n} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1})$$

(see Section 1.7), and thus we obtain a linear mapping

$$\delta = (\Phi_U)_* : T \to \bigoplus_{p=1}^{n} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1}).$$

To obtain properties (1) and (2) from the definition of ivHs, let $\tilde{\xi}$, $\tilde{\xi}_1$, $\tilde{\xi}_2$ be the images of ξ, ξ_1, ξ_2 in $H^1(X_{s_0}, T_{X_{s_0}})$ under the Kodaira-Spencer mapping (see Section 5.2). Then $\delta(\xi_i)\psi$, for $\psi \in H^{p,q}$ is the image of the pair $(\tilde{\xi}_i, \psi)$ under the natural pairing

$$H^1(X_{s_0}, T) \times H^q(X_{s_0}, \Omega^p) \to H^{q+1}(X_{s_0}, \Omega^{p-1}),$$

generated by the convolution

$$T \times \Omega^p \to \Omega^{p-1}.$$

Property (1) is then apparent.

To show (2), note that $\delta(\xi)$ can be viewed as differentiation in the cohomology algebra $H^*(X_{s_0}, \mathbb{C})$ with respect to the wedge product. Then, since $\tilde{\xi} \in H^1(X_{s_0}, T_{X_{s_0}})_\omega$ (see Section 5.2), it follows that $\delta(\xi)\omega = 0$. Therefore the one-parameter group of automorphisms $\{\exp(\delta(\xi)t)\}$ of the algebra $H^*(X_{s_0}, \mathbb{C})$ preserves ω, and hence leaves the form Q on $H^n(X_{s_0}, \mathbb{Z})$ invariant. Condition (2) is just a differential way to express this invariance.

6.3. Consider an arbitrary complex analytic manifold Y, and a locally free sheaf $\mathcal{F}$ of rank m on Y. Fix a natural number $k < m$. Define a manifold $G_Y(k, \mathcal{F})$, called the *grassmanization of the sheaf* $\mathcal{F}$ on Y. The manifold $G_Y(k, \mathcal{F})$ is a fiber bundle

$$\pi : G_Y(k, \mathcal{F}) \to Y,$$

each fiber $\pi^{-1}(y)$ of which is isomorphic to the Grassmanian $G(k, \mathcal{F}_y)$. In order to introduce a complex structure on $G_Y(k, \mathcal{F})$, consider a neighborhood $U_y \subset Y$ of an arbitrary point $y \in Y$, such that the sheaf $\mathcal{F}$ is trivial in that neighborhood. We can treat $\mathcal{F}|U_Y$ as the sheaf of sections of a trivial bundle $U_y \times \mathbb{C}^m$. In an obvious way we can fix a bijection

$$f_y : U_y \times G(k, m) \simeq \pi^{-1}(U_y).$$

The bijections f_y are compatible on the intersections of the neighborhoods U_y, and define a complex structure on $G_Y(k, \mathcal{F})$.

6.4. Let us return again to the setting of Section 6.2. Let $G_D(k) = G_D(k, T_h(D))$, where D is the classifying space associated to the fiber X_{s_0}, where $k = \dim S$, $T_h(D)$ is the horizontal sheaf (see Section 1.7) over D. Suppose that for a generic point s_0 of the manifold S the tangent map $(\Phi_U)_*$ is injective. Then, by mapping generic points $s_0 \in S$ to the subspace $(\Phi_U)_* T \subset T_{h,d_0}(D)$ of dimension k we obtain an almost everywhere defined holomorphic map

$$\tilde{\Phi} : S \to G_D(k).$$

An obvious but useful remark is that if the map $\tilde{\Phi}$ is a bijection in a generic point, then so is Φ_Γ. Indeed, any map of complex manifolds which is locally an injection, and for which a generic point of the image has only one preimage is an embedding on an open everywhere dense set.

Let $\mathfrak{M}$ be the moduli space of a class of algebraic varieties (see Section 5.4), ψ the map given by equation (20). Then the above can be rephrased into the following general principle:

Suppose that for a generic point $m \in \mathfrak{M}$ an ivHs associated with X_m determines X_m. Then the *generic global Torelli theorem* holds for $\mathfrak{M}$.

6.5. Consider the example of a nonsingular projective curve X of genus g. Let $T = H^1(X, T_X)$ be the tangent space to $\mathfrak{M}_g$ at the point $[X]$. Consider the ivHs associated to the point $[X]$ of the moduli space $\mathfrak{M}_g$. Here, the map

$$\delta : T \to \operatorname{Hom}(H^{1,0}, H^{0,1})$$

has a dual

$$\delta^* : \operatorname{Sym}^2 H^{1,0} \to T^* = H^0(X, K_X^{\oplus^2}),$$

where Sym^k is the k-th symmetric power of the vector space, and K_X is the canonical bundle over X.

Note that $H^{1,0} = H^0(X, K_X)$. Therefore, the definition of δ^* above is equivalent to the natural map

$$\delta^* : \operatorname{Sym}^2 H^0(X, K_X) \to H^0(X, K_X^{\oplus^2}).$$

Thus, there is a naturally defined subspace

$$\operatorname{Ker} \delta^* \subset \operatorname{Sym}^2 H^0(X, K_X).$$

Let $\mathbb{P}^{g-1}$ be the projectivization $[H^0(X, K_X)]^*$, and $X' \subset \mathbb{P}^{g-1}$ is the canonical curve, that is, the image of X under the mapping given by the complete linear system $|K_X|$. Then $\operatorname{Ker} \delta^*$ is identified with the set of quadrics on $\mathbb{P}^{g-1}$ passing through the canonical curve X'. However, it is known that a generic canonical curve of genus $g \geq 5$ is the intersection of quadrics passing through it (Griffiths–Harris [1978]). Thus, we have established a generic Torelli theorem for curves of genus $g \geq 5$.

Chapter 3.
Torelli Theorems

In this Chapter we describe the main concrete results currently known concerning theorems of Torelli type (see Chapter 2, Section 5). Our goal is to demonstrate the main ideas of the proofs of this kind of results, and so we will often omit the technical details, referring the reader to the original papers.

§1. Algebraic Curves

1.1. In this section we will briefly outline the main results having to do with the period mappings for algebraic curves. A more complete exposition can be found in Griffiths–Harris [1978].

We will be looking at nonsingular irreducible projective curves over the field $\mathbb{C}$. Let X be such a curve. Then the Hodge structure on $H^1(X)$ has the form

$$H^1(X) = H^{1,0} \oplus H^{0,1},$$

where $H^{1,0}$ is the space of one-dimensional holomorphic differential forms on X, while $H^{0,1} = \overline{H^{1,0}}$. The sheaf ω_X^1 of holomorphic 1-forms on X is the canonical sheaf. The corresponding divisor class – the canonical class – will be denoted by K_X. The genus $g = g(X)$ of the curve X is the number

$$g = \dim_{\mathbb{C}} H^{1,0} = \dim H^0(X, K_X).$$

Let $D = \sum_{i=1}^k n_i(x_i)$, $n_i \in \mathbb{Z}, x_i \in X$ a divisor on X. The degree of the divisor is the number

$$\deg D = n_1 + \ldots + n_k.$$

The Riemann-Roch theorem states that

$$\dim H^0(X, D) - \dim H^0(X, K_X - D) = \deg D + 1 - g.$$

This implies, in particular, that

$$\deg K_X = 2g - 2.$$

The complete linear system $|K_X|$ for $g \geq 2$ is defined by the map

$$i : X \to \mathbb{P}^{g-1},$$

called the canonical map. If $\omega_1, \ldots, \omega_g$ form a basis of holomorphic 1-forms on X, x a local coordinate in some neighborhood of $x_0 \in X$, then the canonical mapping is given in that neighborhood explicitly as

$$x \to \left(\frac{\omega_1(x)}{dx} : \ldots : \frac{\omega_g(x)}{dx}\right).$$

Since the map i is defined by a complete linear system, then its image $i(X)$, the so-called canonical curve, does not lie in any proper linear subspace $\mathbb{P}^{g-1}$. This means that for $k \leq g-1$ a generic set of k points of the canonical curve is linearly independent in $\mathbb{P}^{g-1}$, that is, it does not lie in a linear subspace of dimension $k-2$.

Call the curve X hyperelliptic if there exists a double cover $X \to \mathbb{P}^1$. If the curve is hyperelliptic, than its canonical map i is a double cover of $i(X)$. If X isnot hyperelliptic, than for $g > 2$, the canonical map is an embedding (it should be noted thatall curves of genus 2 are hyperelliptic).

1.2. Let $\mathfrak{M}_g$ be a coarse moduli space of curves of genus $g \geq 2$ (see Chapter 2, Section 5).

Recall that $\mathfrak{M}_g$ is a quasiprojective (possibly singular) algebraic variety of dimension $3g-3$. The closed points of $\mathfrak{M}_g$ parametrize the isomorphism classes of nonsingular projective curves of genus g. Furthermore, for any flat algebraic family $\{X_s\}$, $s \in S$ of curves of genus g there exists a morphism $\pi : S \to \mathfrak{M}_g$, such that for any closed point $s \in S$ the curve X_s belongs to the class parametrized by the point $\pi_s \in \mathfrak{M}_g$.

The existence of such a variety $\mathfrak{M}_g$ was proved by Mumford [1965].

Deligne and Mumford [1969] proved the irreducibility of the moduli space $\mathfrak{M}_g$. For $g = 1$ the coarse moduli space can be described as follows. Any curve of genus 1 (an *elliptic curve*) can be embedded into $\mathbb{P}^2$. After an appropriate choice of homogeneous coordinates $(x : y : z)$ in $\mathbb{P}^2$ its equation can be written in the form

$$zy^2 = 4x^3 - g^2xz^2 - g_3z^3.$$

The discriminant of the right-hand side

$$\Delta = g_2^3 - 27g_3^2$$

is then different from O. The number

$$j = j(X) = 2^6 3^2 g_2^3/\Delta$$

is called the j *invariant* of the elliptic curve X. The curves X_1 and X_2 are isomorphic if and only if $j(X_1) = j(X_2)$.

Conversely, for any $j_0 \in \mathbb{C}$ there exists an elliptic curve X, such that $j(X) = j_0$.

Thus, the coarse moduli space $\mathfrak{M}$ for elliptic curves is nothing but the affine line $\mathbf{A}^1 = \mathbb{C}$.

If $\{X_s\}$, $s \in S$ is a flat family of elliptic curves, than the mapping π is given by the formula

$$\pi(s) = j(X_s).$$

We already know (see Introduction and Chapter 2, Section 1) that an elliptic curve is a quotient of $\mathbb{C}$ modulo the lattice $\Pi = \{\mathbb{Z}e_1 + \mathbb{Z}e_2\}$. We can assume that the basis of Π consists of the numbers 1 and λ, $\operatorname{Im}\lambda > 0$. Curves corresponding to the lattices $\mathbb{Z}+\mathbb{Z}\lambda_1$ and $\mathbb{Z}+\mathbb{Z}\lambda_2$ are isomorphic if and only if $\lambda_1 = a\lambda_2 + b/c\lambda_2 + d$; where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Thus, $\mathfrak{M}_g$ can be viewed as the manifold $SL(2,\mathbb{Z})\backslash H$, where $H = H^1$ is the complex upper halfplane, and the group $SL(2,\mathbb{Z})$ acts on H by fractional linear transformations. It can be seen that $SL(2,\mathbb{Z})\backslash H \simeq \mathbf{A}^1$. The equivalence of these two descriptions of moduli space are established as follows.

Define the following functions of $z \in H$.

$$g_2(z) = 60 \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0,0\}} (mz+n)^{-4},$$

$$g_3(z) = 140 \sum_{(m,n)\in\mathbb{Z}^2\backslash\{0,0\}} (mz+n)^{-6},$$

$$j(z) = 2^6 3^2 (g_2(z))^3/((g_2(z))^3 - 27(g_3(z))^2).$$

The function $j(z)$ is called the *modular invariant.* For $z_1, z_2 \in H$ the equality $j(z_1) = j(z_2)$ holds if and only if z_1 and z_2 are equivalent with respect to the $SL(2,\mathbb{Z})$ action on H. If $\lambda \in H$, then the elliptic curve $X = \mathbb{C}/\{\mathbb{Z}+\mathbb{Z}\lambda\}$ has j invariant $j(X)$ which is equal to $j(\lambda)$.

1.3. Recall (see Chapter 2, Section 1.6) that the classifying space of Hodge structures of weight 1 for a nonsingular projective curve g is the Siegel half-plane of genus g, denoted by H_g. This is the set of complex square matrices $Z = X + iY$, which are symmetric (${}^tZ = Z$, and with positive-definite imaginary part: $Y > 0$. The complex structure on H_g is induced by the natural complex structures on the space of complex $g \times g$ matrices. The group $G_\mathbb{Z}$ (see Chapter 2, §1.) is in this case the group $\mathrm{Sp}(g,\mathbb{Z})$ of integral $2g \times 2g$ matrices

$$\gamma = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

such that

$$\gamma \Delta {}^t\gamma = \Delta$$

for

$$\Delta = \left(\begin{array}{c|c} 0 & -E_g \\ \hline E_g & 0 \end{array}\right).$$

The element of the group $G_\mathbb{Z}$ acts on the matrix $Z \in H_g$ by the formula

$$\gamma(Z) = (AZ+B)(CZ+D)^{-1}.$$

In Chapter 2, §5 we have defined a natural period mapping on the moduli space $\mathfrak{M}_g$ denoted by

$$\psi_g : \mathfrak{M}_g \to G_\mathbb{Z}\backslash H_g.$$

On the variety $\operatorname{reg}\mathfrak{M}_g$ of nonsingular points of $\mathfrak{M}_g$ the mapping ψ_g is an extended variation of Hodge structures (see Chapter 2, Section 4).

Note that for $g = 1$, as we just showed, the map ψ_g is an isomorphism. In general, it will be shown that ψ_g restricted to the set of closed points is an embedding (Torelli theorem, Section 1.4).

As in Chapter 2, Section 2, we can associated a Griffiths complex torus $J(Z)$ with principal polarization ω to each element $Z \in H_g$. The two polarized tori corresponding to two points $Z_1, Z_2 \in H_g$ are isomorphic if and only if $Z_1 \in G_{\mathbb{Z}}(Z_2)$. Thus, the polarized tori are in correspondence to the points of the quotient space $G_{\mathbb{Z}}\backslash H_g$.

Let X be a curve of genus g, and let $[X] \in \mathfrak{M}_g$ be the corresponding point in the moduli space. Let $Z = \psi_g([X])$. Then the polarized torus corresponding to the point $Z \in H_g$ is called the *Jacobian of the curve* X and denoted by $J(X)$ (see Chapter 2, Sections 1.7 and 2.2).

Let A be an abelian variety of dimension g with polarization ω of the type

$$\delta = \begin{pmatrix} \delta_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \delta_g \end{pmatrix}$$

(see Chapter 2, Section 1.7), and let L be the line bundle on A corresponding to this polarization. Then, from the general theory of abelian varieties (see Mumford [1968]), it follows that L is defined uniquely, up to translation by $a \in A$. Also,

$$\dim H^0(X, L) = \delta_1 \dots \delta_g.$$

In particular, for a principal polarization, this dimension equals 1. Hence, the polarization of the Jacobian defines a unique, up to translation, divisor θ on $J(X)$, called the theta-divisor. The pair $(J(X), \theta)$ will also be called the polarized Jacobian of the curve X.

1.4. Let $\gamma_1, \dots, \gamma_{2g}$ be a basis of the integral homology of the curve X, while $\omega_g, \dots, \omega_g$ is the basis of the holomorphic 1-forms. We have a lattice $\Pi \subset \mathbb{C}^g$ with basis $e_1, \dots, e_{2g}$, where

$${}^t e_k = \left(\int_{\gamma_k} \omega_1, \dots, \int_{\gamma_k} \omega_g \right).$$

The complex torus $J(X)$ is then identified with $\mathbb{C}^g/\Pi$. Fix a point $x_0 \in X$, and consider the holomorphic Albanese mapping (see Chapter 2, Section 2) $\mu : X \to J(X)$ given by the formula

$${}^t\mu(X) = \left(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_g \right).$$

Let $S^{(k)}X$ is the k-th symmetric power of the curve X, that is, the quotient of the product of k copies of X by the standard action of the permutation group on k letters. If $(x_1, \dots, x_k) \in S^{(k)}X$, $x_i \in X$, let

$$\mu_k(x_1, \ldots, x_k) = \mu(x_1) + \ldots + \mu(x_k).$$

Then the following theorem holds (see Griffiths–Harris [1978]):

Theorem (Abel) *Let $X_1, \ldots, X_k$, $p_1, \ldots, p_k$ be points on X. Then the divisor $\sum_{i=1}^k (x_i) - (p_i)$ is linearly equivalent to 0 on X if and only if*

$$\mu_k(x_1, \ldots, x_k) = \mu_k(p_1, \ldots, p_k).$$

If $W_k = \mu_k(S^{(k)}X)$, then Abel's theorem implies that the points of W_k parametrize the equivalence classes of divisors of degree k on X. Thus the fibers of μ_k are projective spaces.

Let z_j be the local parameter at the point $x_j \in X$, and let $\Omega_{ij} = \frac{\omega_i(x_j)}{dz_j}$. Then for a generic point $(x_1, \ldots, x_k) \in S^{(k)}X$, the differential $d\mu_k$ has, in coordinates $z_1, \ldots, z_k$ the form

$$d\mu_k = (\Omega_{ij}).$$

Here, $T_{J(X)}$ is naturally identified with $\mathbb{C}^g$. If $i : X \to \mathbb{P}^{g-1}$ is the canonical mapping (see Section 1.1) , then it follows from the explicit description of i that $d\mu_k$ is degenerate if and only if the points $i(x_1), \ldots, i(x_k)$ of the canonical curve $i(x) \subset \mathbb{P}^{g-1}$ lie in a $(k-2)$-dimensional linear subspace. This is not so for a generic set of of $k \leq g$ points in X. Thus, for $k \leq g$ the map μ_g is bijective in a neighborhood of a generic point. In particular, setting $k = g - 1$ get a divisor $W_{g-1} \subset J(X)$.

Theorem (Riemann) *The divisors θ and W_{g-1} on $J(X)$ coincide up to a translation by an element of $J(X)$.*

Note that the divisor of the polarization θ is determined by the Hodge structure on X, while the divisor W_{g-1} is closely related to the geometry of X. The remarkable result above gives us the main approach to the proof of the Torelli theorem for curves.

It will be necessary to describe the set of nonsingular points of the divisor W_{g-1} (see Griffiths–Harris [1978]). Let $D = \sum_{i=1}^{q-1}(x_i)$ be a divisor on X, $\omega = \mu_{g-1}(x_1, \ldots, x_{g-1})$ the corresponding point of M_{g-1}. Call the divisor D *regular* if $\dim D = 0$, or, in other words, $\dim H^0(X, D) = 1$. Then, the point $\omega \in W_{g-1}$ is nonsingular if and only if the divisor D is regular.

The geometric formulation of the criterion above is as follows. Let $i : X \to \mathbb{P}^{g-1}$ be the canonical mapping. Let X be a non-hyperelliptic curve, $g > 2$. Let us identify the curve X with its image $i(X) \subset \mathbb{P}^{g-1}$.

The point $\omega \in W_{g-1}$ is nonsingular if and only if there exists a unique hyperplane H in $\mathbb{P}^{g-1}$, such that $D \subseteq H \cap i(X)$.

Indeed, by the Riemann-Roch theorem, in that case

$$\dim |D| = \dim |K_X - D|,$$

while the elements of the linear system $|K_X - D|$ admit a natural interpretation as hyperplanes in $\mathbb{P}^{g-1}$ passing through D.

1.5. We will need another general construction having to do with abelian varieties. Let A be an abelian variety of dimension g, $V \subset A$ a subvariety of dimension $k < g$, and $\operatorname{reg}(V)$ the set of regular points of V. By parallel translation, let us identify all of the tangent spaces T_a at points $a \in A$ with the tangent space T_0 at the identity element of A. The correspondence between a point $v \in \operatorname{reg}(V)$ and the tangent space $(T_V) \subset T_0$ to V at v defines a holomorphic mapping

$$i_V : \operatorname{reg}(V) \to G(k, g),$$

called the *Gauss mapping* of the subvariety V. Note that i_V is invariant under the translations of V by the elements $a \in A$.

Suppose now that $A = J(X)$, $\mathbb{P}^{g-1} = \mathbb{P}(T_0) \simeq G(1, g)$. Consider the map $i = i_{\alpha(X)} \circ \alpha$, where α is the Albanese mapping. Then i is the canonical mapping, which follows from an explicit description of the canonical mapping and the mapping α.

1.6. Torelli theorem. In Chapter 2, Section 5 it was shown that the infinitesimal Torelli theorem fails for curves. More precisely, let $f : \mathcal{X} \to S$ be the universal Kuranishi family of the curve $X \simeq f^{-1}(s_0)$ over a polydisk S and let $\lambda : S \to H_g$ be the corresponding holomorphic period mapping (see Chapter 2, Section 5.5). Then the differential d is degenerate at the point s_0 in case of hyperelliptic curves of genus $g > 2$. This is related to the observation that the neighborhood of the point $[X]$ in the moduli space $\mathfrak{M}_g$ is analytically isomorphic to the quotient of S by an action of an involution (Chapter 2, Section 5). This makes it even more remarkable that the global Torelli theorem holds for the moduli space $\mathfrak{M}_g$ (see Chapter 2, Section 5.6). This theorem was first obtained by Torelli [1914]. A modern proof was given by Andreotti [1958]. A complete proof of the theorem with a detailed exposition of the necessary theory of algebraic curves and their jacobians can be found in Griffiths–Harris [1978].

Theorem 1.1. (Global Torelli Theorem). *Let X_1 and X_2 be two nonsingular projective curves of genus g. Then if their Jacobians are isomorphic as polarized abelian varieties, the curves are isomorphic.*

It is clear that Theorem 1.1 implies injectivity on the set of closed points of the period mapping

$$\psi_g : \mathfrak{M}_g \to G_{\mathbb{Z}} \backslash H_g,$$

(see Section 1.2). Indeed, the elements of H_g which belong to the same orbit of the group $G_{\mathbb{Z}}$ correspond to isomorphic Griffiths tori.

1.7. We will give a proof of Torelli's theorem while omitting some technical details.

First, suppose that X is a non-hyperelliptic curve. Let us show that X may be reconstructed from the polarized jacobian $(J(X), \Theta)$.

By Riemann's theorem, $\Phi = W_{g-1}$ up to a translation by $x \in J(X)$. Consider the translation-invariant Gauss map

$$i_W : \operatorname{reg}(W_{g-1}) \to G(g-1, g) \simeq (\mathbb{P}(T_0))^*.$$

Here, T_0 is the tangent space to $J(X)$ at the identity element, $(\mathbb{P}(T_0)^*$ is the dual projective space to the projectivization $\mathbb{P}(T_0) \simeq \mathbb{P}^{g-1}$. In other words, $(\mathbb{P}(T_0))^*$ is the space of hyperplanes in $\mathbb{P}(T_0)$.

From the explicit form of the differential $d\mu_k$ it follows that the point $w = \mu_{g-1}(x_1, \dots, x_{g-1})$ is nonsingular on W_{g-1} if and only if the points $i(X_1), \dots, i(x_{g-1})$ of the canonical curve $C = i(X) \subset \mathbb{P}^{g-1}$ generate a unique hyperplane (see Section 1.4). This hyperplane is then the image of w under the Gauss map i_W. Since each hyperplane intersects the canonical curve in a finite set of points (generally in $\deg K_X = 2g-2$ points), the mapping i_W is finite-to-one, and its degree at a generic point is equal to $\binom{2g-2}{g-1.}$

Consider the set B of branch points of the mapping i_W, which is the set of images of points $w \in \operatorname{reg}(W_{g-1})$ where di_W is degenerate. Let $\overline{B}$ be the closure of the set B in $(\mathbb{P}(T_0))^*$.

Lemma. *The set $\overline{B}$ is dual to the canonical curve $C \in \mathbb{P}(T_0)$:*

$$\overline{B} = C^*.$$

Recall that if $Y \subset \mathbb{P}^N$ is a closed algebraic subvariety, then the dual subvariety $Y^* \subset (\mathbb{P}^N)^*$ is the set of hyperplanes $H \in \mathbb{P}^N$ tangent to Y in at least one point. By the main theorem of projective duality

$$(Y^*)^* = Y.$$

Hence, the lemma implies that

$$C = (\overline{B})^*.$$

Since B is determined by the pair $(J(X), \Theta)$, Torelli's theorem follows.

Let us give a sketch of the proof of the lemma. Let $H \in (\mathbb{P}^{g-1})^*$ be a hyperplane. It intersects C in $2g-2$ points, counting multiplicity. Consider the various divisors of degree $g-1$ corresponding to sums of points in this set. If all of these divisors determine points in $\operatorname{reg}(W_{g-1})$ then we say that the hyperplane H is in general position. Let $V \subset (\mathbb{P}^{g-1})^*$ be the set of hyperplanes which are not in general position.

Denote by B_1 the set of hyperplanes $H \subset (\mathbb{P}^{g-1})^* \backslash V$, which intersect the canonical curve transversely everywhere, except at one point, where the intersection has multiplicity 2. Then

$$C^* \cap ((\mathbb{P}^{g-1})^* \backslash V) \subset \overline{B}_1,$$

where $\overline{B}_1$ is the closure of B_1 in $(\mathbb{P}^{g-1})^*$.

Let $H \in B_1$, and let $i(x_1)$ be the double point of the intersection of H with $C = i(X)$. Let $x_2, \dots, x_{g-1}$ be points of X, such that $i(x_j) \in H$, and in the set $x_1, \dots, x_{g-1}$ all of the points are distinct. Let $z_1, \dots, z_{g-1}$ be local coordinates on X in the neighborhood of the points $x_1, \dots, x_{g-1}$. Then $z_1, \dots, z_{g-1}$ can be chosen as local coordinates in a neighborhood of the point $x = (x_1, \dots, x_{g-1}) \in S^{(g-1)}X$ and also as local coordinates on W_{g-1} in the neighborhood of the point $w = \mu_{g-1}(X)$. Note that the point w is nonsingular, since $H \notin V$. Then,

$$i_W(w) = H,$$

and since H is tangent to $i(X)$ at the point $i(x_1)$, evidently

$$\frac{\partial}{\partial z_1} i_W(w) = 0.$$

Thus, di_W is degenerate at the point w and $H \in B$.

To summarize, so far we have shown that $B_1 \subset B$, and so

$$C^* \cap ((\mathbb{P}^{g-1})^* \backslash V) \subset \overline{B}.$$

On the other hand, let H be any hyperplane in $\mathbb{P}^{g-1}$ intersecting C in $2g-2$ distinct points. Let $x_1, \dots, x_{g-1} \in X$ be some set of distinct points, for which $i(x_j) \in H$ and let $z_1, \dots, z_{g-1}$ be the local coordinates in the neighborhoods of the points $X_1, \dots, x_{g-1}$ respectively. Since the intersection of H with C is transverse at $i(x_1), \dots, i(x_{g-1})$, it follows that $z_1, \dots, z_{g-1}$ are holomorphic functions on $(\mathbb{P}^{g-1})^*$ in some neighborhood of H. Henceforth, if $w = \mu_{g-1}(x_1, \dots, x_{g-1})$ is nonsingular, then the map i_w has an inverse in a neighborhood of w and so the differential di_w is nondegenerate. We have shown that

$$B \subset C^*.$$

The irreducibility of C^* now finishes the proof of the lemma, and of Torelli's theorem for non-hyperelliptic curves. If the curve X is hyperelliptic, almost the same argument goes through. In this case, the canonical map $i : X \to \mathbb{P}^{g-1}$ is a double cover of a nonsingular rational canonical curve $C = i(X)$ branched in $2g+2$ points. The set B in this case is precisely the set of hyperplanes $H \subset \mathbb{P}^{g-1}$ which are either tangent to the canonical curve C or pass through one of the branch points of the canonical map i. Thus, in this case, $B \subset (\mathbb{P}^{g-1})^*$ is the union of the set C^* with the set of hyperplanes of the form

$$p^* = \{H \in (\mathbb{P}^{g-1})^* | p \in H\},$$

where p is a branch point of the canonical mapping.

It follows that in this case we can reconstruct the curve C and $2g+2$ branch points on it. But there is only one hyperelliptic curve which covers a fixed rational curve with fixed branch points. This finishes the proof of Theorem 1.1.

§2. The Cubic Threefold

2.1. In this section we will be studying the nonsingular complex hypersurface X of degree 3 in four-dimensional projective space $\mathbb{P}^4$ – the *cubic threefold.* The study of cubic hypersurface of projective space has always been very interesting for a large circle of mathematicians, and has yet to be completed. One of the key questions posed in this research is the question of the rationality of these hypersurfaces. Let us recall that a variety is called rational, if it is birationally isomorphic to the projective space of the same dimension. The cubic curve in $\mathbb{P}^2$ (an elliptic curve) is the first example of a non-rational algebraic curve. The cubic hypersurface in $\mathbb{P}^3$ – the cubic surface (see Section 2.3) turns out to be a rational variety. It has long been known that the cubic threefold is unirational, that is, there is a rational covering $\mathbb{P}^3 \dashrightarrow X$. The question of rationality had long remained open, before being resolved negatively by Clemens and Griffiths (see Clemens–Griffiths [1972], see also Chater 2, Section 2.6). In addition to the previously discovered unirationality result this gave one of the presently known negative solutions to the Lüroth problem in dimension 3. The question of the rationality of a generic cubic fourfold (see Section 4.9) remains open at the time of this writing (1988).

In the current section we will summarize the basic ideas of the proof of the global Torelli theorem for cubics. This theorem was obtained by Clemens and Griffiths [1972] and by Tyurin [1971].

2.2. Let X be a cubic threefold. The Hodge structure of weight 3 on $H^3(X)$ has the form (see Chapter 4, Section 5):

$$H^3(X) = H^{2,1} + H^{1,2},$$

where $\dim H^{2,1} = h^{2,1} = 5$. Thus, for the cubic threefold, the polarized Griffiths torus coincides with the Weil torus (see Chapter 2, §2) and is a five-dimensional principally polarized abelian variety $(J^3(X), \omega)$ (see Chapter 2, Section 1.7). Since ω is a principal polarization, it defines a unique (up to translation) divisor Θ on $J^3(X)$ (see Section 1.3) – the *divisor of the polarization.* The pair $(J^3(X), \Theta)$ will also be called the polarized *middle Jacobian* of X. The main result described in this section is the following.

Theorem 2.1. (The global Torelli theorem). *The nonsingular cubic threefold X is uniquely determined by its polarized middle Jacobian $(J(X), \Theta)$.*

A complete proof of theorem 2.1 can be found in Tyurin [1971] or in Clemens–Griffiths [1972].

The proof of theorem 2.1 is in large part the same as the proof of the Torelli theorem for curves (theorem 1.1). First, consider the Gauss mapping (see Section 1.5):

$$i_\Theta : \mathrm{reg}(\Theta) \to (\mathbb{P}(T_0))^* = (\mathbb{P}^4)^*$$

on the abelian variety $J = J^3(X)$. Let B be the set of branch points of i_Θ on $(\mathbb{P}^4)^*$, that is, the set consisting of the images of the points $x \in \mathrm{reg}(\Theta)$ where the differential di_Θ is degenerate. Denote by $\overline{B}$ the closure of B in $(\mathbb{P}^4)^*$. The projective space $(\mathbb{P}^4)^*$ is the set of hyperplanes in $\mathbb{P}^4$. It turns out that it can be assumed that the cubic X is embedded into $\mathbb{P}^4$ in such a way that the following proposition holds.

Proposition 2.2. *The variety $X^* \subset (\mathbb{P}^4)^*$, dual to X, coincides with B (see Section 1.7).*

Since the pair $(J(X), \Theta)$ determines B up to projective isomorphism, Theorem 2.1 follows from Proposition 2.2, since

$$X \simeq (\overline{B})^*.$$

The remainder of this section is devoted to the description of the main concepts and results leading to Proposition 2.2. These results use the rich geometry of the cubic threefold.

2.3. First, let us recall some basic properties of cubic hypersurfaces in $\mathbb{P}^3$ – cubic surfaces. A detailed exposition of the theory of these surfaces can be found in Griffiths–Harris [1978].

A nonsingular cubic surface is a rational algebraic surface. It can be obtained from the projective plane $\mathbb{P}^2$ by blowing up (see Chapter 1, Section 1) along six points not all contained in a conic. Such a surface contains exactly 27 projective lines. These consist of, firstly, the six curves obtained by the blowing up along the six points, secondly the 15 proper transforms of the lines joining pairs of the blown-up points, and finally the 6 proper transforms of the six conic sections passing through quintuples of the blown-up points.

The description above shows that any line l on a nonsingular cubic surface intersects exactly 10 others, and its self-intersection index is

$$(l, l) = -1.$$

2.4. The lines in $\mathbb{P}^4$ are in one-to-one correspondence with the points of the Grassmanian $G(2,5)$. Consider the set $F(X)$ of lines l lying on the cubic three-fold $X \subset \mathbb{P}^4$. We can write 4 local equations defining $F(X) \in G(2,5)$. Hence, $F(X)$ is a subvariety of the Grassmanian, and $\dim F(X) \geq 2$. Consider a nonsingular hyperplane section $S \subset X$, containing the line l. Then S is a cubic surface, and $(l, l)_S = -1$.

Let $X_1 \subset X_2$ be a pair of complex manifolds, and let T_{X_1} and T_{X_2} be the corresponding tangent spaces. Then T_{X_1} is naturally included into the restriction $T_{X_2}|X_1$ of T_{X_2}. The quotient bundle $T_{X_2}|X_1/T_{X_1}$ is called the normal bundle of X_1 in X_2 and is denoted by $N_{X_2}^{X_1}$, or N_{X_1/X_2}. The sheaf of sections of the normal bundles is called the normal sheaf.

Recall that if X_1 is a divisor in an algebraic variety X_2 then the sheaf of sections of $N_{X_2}^{X_1}$ is isomorphic to the restriction to X_1 of the sheaf $\mathcal{O}_{X_2}(X_1)$ (see Griffiths–Harris [1978]).

Now, if N_S^l and N_X^l are the normal sheaves of the line l in S and in X respectively, while N_X^S is the normal sheaf of S in X, then

$$N_S^l = \mathcal{O}_l(-1), \quad N_X^S = \mathcal{O}_S(S) = \mathcal{O}_{\mathbb{P}^4}(1)|S.$$

Hence, there is an exact sequence

$$0 \to \mathcal{O}_l(-1) \to N_X^l \to \mathcal{O}_l(1) \to 0$$

of sheaves on $l \simeq \mathbb{P}^1$. Consequently,

$$H^1(l, N_X^l) = 0, \quad \dim H^0(l, N_X^l) = 2.$$

Deformation theory now implies that $F = F(X)$ is a non-singular surface, called the *Fano surface of the cubic* X. The geometry of this surface plays an important role in the study of the cubic threefold.

We will need some properties of the Fano surface of a cubic threefold. Most of these properties were discovered by Fano [1904], while a modern exposition can be found in Altman–Kleiman [1977].

2.5. Let $V \simeq \mathbb{C}^5$ be the constant fiber bundle over the Grassmanian $G = G(2,5)$. Define the *tautological subbundle* $\tau \subset V$ as follows: the fiber of τ above a point in the Grassmanian is exactly the two-dimensional subspace parametrized by that point. Looking at the dual bundles, we get an exact sequence

$$V^* \to \tau^* \to 0$$

of bundles over G. The bundle τ^* is called the *antitautological* bundle. The images of five linearly independent sections of the constant bundle V^* are a basis of the space $H^0(G, \tau^*)$. The natural map

$$H^0(G, \tau^*) \wedge H^0(G, \tau^*) \to H^0(G, \wedge^2 \tau^*)$$

is an isomorphism. The vector bundle $\wedge^2\tau^*$ defines a Plücker embedding i of the Grassmanian into $\mathbb{P}^9$. Let us identify G with its image in $\mathbb{P}^9$.

Recall that a morphism of a variety Y into the Grassmanian $G(m,n)$ is equivalent to an m-dimensional vector bundle E over Y together with n global sections of E generating E over each point. The subspace $L \subset H^0(Y, E)$ generated by these global sections defines a morphism of Y into $G(m,n)$ uniquely (up to automorphisms of the Grassmanian).

The standard embedding

$$\lambda : F \hookrightarrow G(2,5)$$

is defined by the bundle $\tau^*|_F$ and by the images of five linearly independent global sections of τ^* under the restriction map

$$r : H^0(G, \tau^*) \to H^0(F, \tau^*|_F).$$

It turns out that r is an isomorphism.

Let Ω be the bundle associated with the sheaf of holomorphic 1-forms on F. The following proposition (see Altman–Kleiman [1977]) is quite important:

Proposition 2.3. *The sheaves Ω and $\tau^*|_F$ are isomorphic.*

The above proposition implies that

$$\dim H^0(F,\Omega) = 5;$$
$$K_F = \wedge^2\tau^*|_F;$$
$$\dim H^0(F,K_F) = 10.$$

Furthermore, the standard embedding λ can be defined by the bundle Ω and fiveof its linearly independent global sections, to wit a basis $\omega_1,\dots,\omega_5$ of the space of holomorphic 1-forms on F.

2.6. Fix a point x_0 in F and consider the Albanese mapping $a : F \to \mathrm{Alb}(F)$ (see Section 2 of Chapter 3). If $\omega_1,\dots,\omega_5$ form a basis of the space of global holomorphic 1-forms on F while $\gamma_1,\dots,\gamma_{10}$ are a basis of $H^1(F,\mathbb{Z})$, then

$$a(x) = \left(\int_{x_0}^{x}\omega_1,\dots,\int_{x_0}^{x}\omega_5\right) \bmod \Pi,$$

where Π is the lattice in $\mathbb{C}^5$ generated by

$$\delta_i = \left(\int_{\gamma_i}\omega_1,\dots,\int_{\gamma_i}\omega_5\right), \quad i=1,\dots,10.$$

Here $\mathrm{Alb}(F)$ is identified with $\mathbb{C}^5/\Pi$, where the tangent space T_0 to $\mathrm{Alb}(F)$ at the point $(0,\dots,0) \bmod \Pi$ is naturally identified with $\mathbb{C}^5$. Let x_1, x_2 be local coordinates in a neighborhood of a point $x \in F$, $\omega_j = \alpha_i dx_1 + \beta_i dx_2$. Consider the Gauss mapping (see Section 1.9) i_A on the subvariety $a(F) \subset \mathrm{Alb}(F)$. Then $i_A \circ a(x)$ is a two-dimensional subspace $L \subset \mathbb{C}^5$ generated by the vectors $(\alpha_1,\dots,\alpha_5)$ and $(\beta_1,\dots,\beta_5)$. Thus the map

$$i_A \circ a : F(X) \to G(2,5)$$

coincides with the map induced by the two-dimensional vector bundle Ω and its global sections $\omega_1,\dots,\omega_5$. By virtue of Proposition 2.3, $i_A \circ a$ coincides with the standard inclusion λ up to an automorphism of $G(2,5)$.

2.7. Let $\emptyset$ be the Abel-Jacobi mapping (see Section 2, Chapter 2) for one-dimensional algebraic cycles on X algebraically equivalent to 0. Fixing a line $l_0 \subset X$ and setting $\emptyset_0(l) = \emptyset(l - l_0)$ get the holomorphic map

$$\emptyset_0 : F(X) \to J^3(X),$$

which we will call the *Abel-Jacobi mapping for F.*

From the universality of the Albanese mapping it follows that there exists a unique morphism b of abelian varieties, such that the diagram below commutes.

$$\begin{tikzcd} F(X) \arrow[rr, "\emptyset_0"] \arrow[dr, "a"'] & & J^3(X) \arrow[dl, "f"] \\ & \mathrm{Alb}(F) & \end{tikzcd}$$

Consider the Gauss mapping

$$i_F : \mathrm{reg}(\emptyset_0(F)) \to G(2,5).$$

Here $G(2,5)$ is the set of two-dimensional subspaces in the tangent space $(T_J)_0$ to J at 0. If we can show that b is an isogeny, that is to say, the tangent mapping db is an isomorphism, it will follow that $i_A \circ a$ and $i_F \circ \emptyset_0$ give the same embedding of F into G up to automorphism of G. Using previous results, we obtain

Proposition 2.4. *The map*

$$i_F \circ \emptyset_0 : F \to G$$

coincides with the standard embedding of the Fano surface F into G.

2.8. Let us sketch the proof of the fact that b is an isogeny. Consider the variety $P(X) \subset F(X) \times \mathbb{P}^4$ consisting of pairs (x, l) for which $x \in l$. Then $P(X) = \mathbb{P}_F(\tau^*)$ is the projectivization of the antitautological bundle on F. The projections onto $F(X)$ and onto $\mathbb{P}^4$ give maps

$$\pi_1 : P(X) \to F(X), \quad \pi_2 : P(X) \to X.$$

One can check that π_2 is a finite morphism of degree 6.

To show that b is an isogeny, it is enough to show that the map

$$b_* : H_1(\mathrm{Alb}(F), \mathbb{Z}) \to H_1(J, \mathbb{Z})$$

is injective. The module $H_1(\mathrm{Alb}(F), \mathbb{Z})$ can be naturally identified with the lattice $\Pi \subset \mathbb{C}^5$, and hence with $H_1(F(X), \mathbb{Z})$. Fix this identification. Analogously, identify $H_1(J, \mathbb{Z})$ with $H_3(X, \mathbb{Z})$. We will thus think of b_* as a map from $H_1(F(X), \mathbb{Z})$ to $H_3(X, \mathbb{Z})$. The map b_* is not hard to describe. Let $\gamma \in H_1(F, \mathbb{Z})$, $\tilde{\gamma}$ the cycle representing γ, $\pi_1^{-1}(\tilde{\gamma})$ the complete preimage of the cycle $\tilde{\gamma}$ in $P(X)$. Then $b_*(\gamma)$ is the homology class corresponding to the cycle $\pi_2(\pi_1^{-1}(\tilde{\gamma}))$. The map b_* is called the *cylindrical* mapping.

Let $l \subset X$ be a line. Consider the set $F_l \subset F(X)$, consisting of the lines $l_1 \subset X$ for which $l \cap l_1 \neq \emptyset$, $l \neq l_1$. It can be shown that for a generic line $l \subset X$, the set F_l is a smooth curve on the surface F. Let X be the blow up of

X along the curve l. Then $\pi : \tilde{X} \to \mathbb{P}^2$ is a *conic bundle*, that is, the preimage of each point is a curve of degree 2. Consider the curve $C \subset \mathbb{P}^2$ consisting of the points whose preimages under π are degenerate. This curve is called the *discriminant* curve of the bundle π. Let $x \in C$. Then, $\pi^{-1}(x) = l_1 \cup l_2$, where l_1, l_2 are lines on X such that $l_1 \cap l_2 \neq \emptyset$ and $l_j \cap l \neq \emptyset$, $j = 1, 2$. This defines a map

$$\alpha : F_l \to C,$$

which for a generic line l is an unbranched covering of degree 2. The natural involution $i : F_l \to F_l$ corresponding to this covering interchanges the lines l_1 and l_2 in each fiber. The degree of $C \subset \mathbb{P}^2$ is 5. Indeed, consider the generic line $m \in \mathbb{P}^2$. Let $H \subset \mathbb{P}^4$ be the hyperplane generated by the lines m and l, $S = H\dot{X}$ be the cubic surface. The lines on X intersecting l are precisely the preimages of the intersections of m and C. Since on a non-singular cubic surface each line intersects precisely 10 others, while α is a two-fold covering, it follows that $\deg C = 5$. It is easy to compute that $g(C) = 6$, $g(F_i) = 11$.

The divisor $F_i \subset F$ is ample. Indeed, if $l_1, l_2, l_3 \in X$ are three lines on the same 2-plane in $\mathbb{P}^4$ then $F_{l_1} + F_{l_2} + F_{l_3}$ is a very ample divisor on F by virtue of being the restriction to F of the very ample divisor on $G(2,5)$ generated by the lines in $\mathbb{P}^4$ having non-empty intersection with a fixed 2-plane. Therefore, by the Lefschetz theorem, the natural map

$$j_* : H_1(F_l, \mathbb{Z}) \to H_1(F, \mathbb{Z})$$

induced by the inclusion $j : F_l \hookrightarrow F$ is surjective (see Chapter 1, Section 9). Denote the map $b_* \circ j_*$ by Φ. Let $\gamma_1, \gamma_2 \in H_1(F_l, \mathbb{Z})$. A straightforward computation produces the following equation for the intersection indices of cycles.

Lemma 2.5.

$$\Phi(\gamma_1) \cdot \Phi(\gamma_2))_{\tilde{X}} = -(\gamma_1 \cdot \gamma_2)_{F_l} + (i_*\gamma_1 \cdot \gamma_2)_{F_l}.$$

Here $(\bullet \cdot \bullet)_{\tilde{X}}$, $(\bullet \cdot \bullet)_{F_l}$ are intersection indices of three-dimensional cycles on $\tilde{X}$ and one-dimensional cycles on F_l respectively, while i_* is the involution on $H_1(F_l, \mathbb{Z})$ generated by i.

Let $H_1(F_l, \mathbb{Z}) = H' \oplus H''$ be the decomposition of the $\mathbb{Z}$-module $H_1(F_l, \mathbb{Z})$ into the direct sum of invariant (H') and anti-invariant (H'') homologies, with respect to the action of i_*. Then $H' \simeq H_1(C, \mathbb{Z})$, $\operatorname{rank}_{\mathbb{Z}} H' = 2g(C) = 12$, $\operatorname{rank} H'' = 22 - 12 = 10$. Therefore, $\operatorname{rank}_{\mathbb{Z}}(\operatorname{Im} \Phi) = 10$. Since the dimensions of $\operatorname{Alb}(F)$ and $J^3(X)$ coincide, we get the injectivity of the cylindrical map b_*, which shows that b is an isogeny.

2.9. Let l_1 and l_2 be lines on X. Setting $\emptyset_{1,-1}(l_1, l_2) = \emptyset(l_1 - l_2)$ we obtain a holomorphic map

$$\emptyset_{l_1, l_2} : F(X) \times F(X) \to J^3(X).$$

The main fact used in the proof of theorem 2.1 is the following analogue of Riemann's theorem (see Section 1).

Theorem 2.6. *The image of the map $\emptyset_{1,-1}$ coincides, up to shift, with the polarization divisor θ.*

The proof of this theorem will be omitted, and can be found in Clemens–Griffiths [1972] or Tyurin [1987].

Let the point $\zeta\Theta$ be $\emptyset_{(1,-1)}(l_1, l_2)$. Suppose that $(l_1, l_2) \in F(X) \times F(X)$, while $l_1 \cap l_2 = \emptyset$. Let us determine $i_\Theta(\zeta)$.

The tangent space T to $F \times F$ at the point (l_1, l_2) is a direct sum $T_1 \oplus T_2$, where T_i is the tangent space to the surface F at the point l_i. Evidently, i_Θ is a hyperplane of $\mathbb{P}^4 = \mathbb{P}((T_I)_0)$ generated by the images of T_1 and T_2, which are the lines $i_F(\emptyset_0(l_1))$ and $i_F(\emptyset_0(l_2))$. Proposition 2.4 implies that $i_\Theta(\zeta)$ is a hyperplane $\langle l_1, l_2\rangle$ in $\mathbb{P}^4$ generated by l_1 and l_2. Assume that $l_1 \cap l_2 = p \in X$, and neither l_1 nor l_2 is a double line on X. Then, it follows from local considerations (see Tyurin [1971], Artin–Mumford [1972]) that the map i_Θ can be extended to the point $\emptyset_{(1,-1)}(l_1, l_2)$ by continuity. The image is the hyperplane tangent to the cubic X at the point p.

2.10. Now let's prove (omitting some details) Proposition 2.2, which implies the Torelli theorem for the cubic threefold.

Proposition 2.7. *The branching divisor B of the map i_Θ coincides with the subvariety $X^* \in (\mathbb{P}^4)^*$ of tangent hyperplanes to the cubic X.*

The variety X^* is irreducible, and thus, by Theorem 2.6, it is enough to show that X^* is the branching divisor of the map $\lambda = i_\Theta \circ \emptyset_{(1,-1)}$. Let l_1 and l_2 be skew lines on the cubic, and let $H = \langle l_1, l_2\rangle$ be the hyperplane spanned by them. Let $Y = XH$ be the section of X by H. Then Y is a cubic surface, and if Y is non-singular, then by Section 2.2 there are 27 lines on Y, forming a standard configuration. Thus, the point $\lambda((l_1, l_2))$ has 27İ6 preimages under λ. These preimages are pairs (l_1, l_2) disjoint from lines on Y. If Y is a singular surface, then Y can be obtained by blowing up six points on $\mathbb{P}^2$ all lying on one quadric, with the subsequent contraction of the quadric (see Griffiths–Harris [1978]). It is easy to see that such a surface has 21×20 ordered pairs of lines, which is fewer than on the nonsingular surface. Therefore, $H \in (\mathbb{P}^4)^*$ belongs to the branch divisor of λ if and only if Y is a singular surface, hence H is tangent to X.

§3. K3 Surfaces and Elliptic Pencils

3.1. Most of this section is devoted to the exposition of the main ideas of the classical work of Piatetsky-Shapiro–Shafarevich [1971], where they obtain a global Torelli theorem of K3 surfaces. In the last part of this section we give a sketch of the proof of the generic Torelli theorem for elliptic pencils, as obtained by K. Chakiris.

A detailed exposition of the theory of K3 surfaces can be found in Shafarevich *et al* [1965]. The same paper contains the proof of the holomorphicity of the period mapping for K3 surfaces and an infinitesimal Torelli theorem for K3 surfaces. These results are due to G. I. Tyurina. They were substantially used by Shafarevich *et al* [1965] to obtain a global Torelli theorem for marked K3 surfaces. In actuality, much more is obtained by Piatetsky-Shapiro–Shafarevich [1971] – they give a complete description of the automorphisms of K3 surfaces. This description can be expressed in terms of linear algebra. The work of Piatetsky-Shapiro–Shafarevich [1971] gave rise to a substantial body of work on K3 surfaces and Enriques surfaces. The result described in Section 3.7 also owes its existence to the synthesis of the ideas of Piatetsky-Shapiro–Shafarevich [1971] with the results (Kulikov [1977a]) of one of the authors of the present survey concerning the modification of degenerations. Note that the paper Kulikov [1977b] finished the study of the period mapping for K3 surfaces, by showing it to be surjective (see Chapter 5, Section 6). Let us briefly describe the main properties of K3 surfaces, while referring to Shafarevich *et al* [1965] for details.

A *K3 surface* is a compact complex manifold, of complex dimension two, for which $H^1(X, \mathbb{Z}) = 0$, and for which there is a unique nonvanishing holomorphic 2-form.

In this section, we shall treat projective (hence algebraic) K3 surfaces. The canonical class K_X of such a surface X is trivial, and the geometric genus

$$p_g = \dim H^0(X, \Omega_X^2) = h^{2,0}$$

equals 1. From the simple connectivity it follows that

$$h^{1,0} = \dim H^0(X, \Omega_X^1) = 0.$$

Noether's formula (see Griffiths–Harris [1978]) says that

$$K_X^2 + \chi = 12(1 + h^{1,0} + h^{2,0}).$$

It follows that the Euler characteristic χ of such a surface equals 24. Consequently, the Hodge structure of weight 2 on $H^2(X)$ has the form

$$H^2(X) = H^{2,0} + H^{1,1} + H^{0,2},$$

where $\dim H^{2,0} = \dim H^{0,2} = 1$, $\dim H^{1,1} = 20$.

The group $H^2(X,\mathbb{Z})$ is torsion-free, and hence is isomorphic to $\mathbb{Z}^{22}$. From the Hodge-Riemann bilinear relations (see Chapter 2, Section 1) it follows that the intersection form $\langle , \rangle$ extended to a bilinear form on

$$H^2(X,\mathbb{R}) = H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

has signature $(3,19)$.

The Riemann-Roch formula for a divisor D on X has the form

$$\dim H^0(X,D) + \dim H^2(X,-D) = D^2/2 + 2 + \dim H^1(X,D).$$

For an effective divisor D this formula simplifies to

$$\dim H^0(X,D) = D^2/2 + 2.$$

For the arithmetic genus $p_a(C)$ of a curve $C \subset X$ we have

$$p_a(C) = C^2/2 + 1.$$

This implies that $C^2 \geq -2$, with equality exactly for nonsingular algebraic curves $C \subset X$.

The first example of a K3 surface is a nonsingular hypersurface of degree 4 in $\mathbb{P}^3$ (a quartic surface). Let us give a crude count of the number of moduli (see Chapter 2, Section 5.4) of such hypersurfaces. The dimension of the space of homogeneous forms on four symbols of degree 4 is 35. The dimension of the group $GL(2,\mathbb{C})$ acting on that space is 16. The orbits of the action correspond to isomorphic quartic surfaces. Thus, the dimension of the moduli space ought to be $35 - 16 = 19$. We will see that this is true in general (see Section 3.5).

To understand the results of this section it is important to keep the following example in mind. Let A be a two-dimensional abelian variety, and let τ be the involution on A, defined in terms of the group structure on A as

$$\tau(x) = -x.$$

The involution τ has exactly 16 fixed points – these are points of order 2 and the identity. Let g be the group $\{1,\tau\}$. The complex analytic space A/g is a normal algebraic variety with 16 simple singularities (see Chapter 1, Section 6.3). Resolving each of these singularities by a σ-process, we get a nonsingular algebraic surface $\tilde{A}$. It can be shown that $\tilde{A}$ is a K3 surface. This is the so-called *Kummer surface*. There are 16 nonsingular rational curves on the Kummer surface, attached instead of the singularities of A/g.

If the initial abelian variety A contains an elliptic curve (or, equivalently, admits a morphism onto an elliptic curve), then the corresponding surface $\tilde{A}$ is called a *special Kummer surface*.

3.2. In this section we will verify the infinitesimal Torelli theorem for K3 surfaces. This result, together with the holomorphicity of the period mapping for K3 surfaces was first obtained by G. N. Tyurina in Shafarevich *et al* [1965].

Let X be an arbitrary compact complex manifold of dimension n, such that the canonical bundle $K_X = \Omega_X^n$ is trivial. Consider a holomorphic form η_0 of type $(n, 0)$ on X. We can choose local coordinates $x_1, \ldots, x_n$ in a neighborhood U of any point of X, so that

$$\eta_0 = \Phi dx_1 \wedge \ldots \wedge dx_n,$$

where Φ is a non-vanishing holomorphic function on U. Then the mapping

$$\sum_{i=1}^{n} A_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \to \frac{1}{\Phi} \sum_{i=1}^{n} (-1)^{i+n} A_i \frac{\partial}{\partial x_i}$$

defines an isomorphism between the bundles Ω_X^{n-1} and T_X on X. The inverse isomorphism $\phi_{\eta_0} : T_X \to \Omega_X^{n-1}$ is induced by the convolution of vector fields with the form η_0. The bundle isomorphism ϕ_{η_0} induces an isomorphism

$$\Phi_k : H^k(X, T_X) \to H^k(X, \Omega_X^{n-1})$$

on cohomology.

Now, let X be a K3 surface. Then $H^{2,0}(X) = \{\mathbb{C}\eta_0\}$, where η_0 is the unique holomorphic $(2, 0)$ form on X. From the above discussion it follows that the convolution $T_X \times \eta_0 \to \Omega_X^1$ defines a non-degenerate pairing

$$H^1(X, T_X) \times H^{2,0} \to H^{1,1}.$$

Let X be an algebraic surface and let L be a positive line bundle on X with $\omega = c_1(L)$. Since ω is the class corresponding to an algebraic cycle on X, and is thus a class of type $(1, 1)$, it follows that $\eta_0 \wedge \omega = 0$ and $H^{2,0} = P^{2,0}$. Hence, it follows that the pairing

$$H^1(X, T_x)_\omega \times P^{2,0} \to P^{1,1}$$

is non-degenerate.

We will use Griffiths' criterion (Chapter 2, Section 5.4) to see that the infinitesimal Torelli theorem holds for K3 surfaces.

In particular, let $\mathcal{X} \xrightarrow{\phi} S$ be a family of polarized K3 surfaces over the base S. This means that for each point $s \in S$ the variety $X_s = \phi^{-1}(s)$ is a K3 surface with a polarization ω_s, so that $\gamma_s^*(\omega_s) = \omega$, where ω is a fixed polarization on $X = X_{s_0}$ (see the notation of Chapter 2, Section 3). Suppose that the family is effectively parametrized, that is, for each $s \in S$ the Kodaira-Spencer mapping

$$\rho : (T_S)_s \to H^1(X_s, T_{X_s})$$

is injective (see Chapter 2, Section 5). Let D be the classifying space of Hodge structures of weight 2, associated with the pair (X, ω), as in Chapter 2, Section 1. This defines a period mapping

$$\Phi : S \to G_{\mathbb{Z}} \backslash D.$$

The choice of a representative $d_0 \in D$ of the image d_0 of the point s_0 under the period mapping defines a local holomorphic lifting

$$\Phi_{\tilde{d}_0} : S \to D$$

of the map Φ The infinitesimal Torelli theorem guarantees that $\Phi_{\tilde{d}_0}$ is a local embedding in the Zariski topology.

3.3. In this section we will describe the classifying space of Hodge structures of weight 2 associated with the polarized K3 surface (X, ω) (see Chapter 2, Section 1).

First, let us give some useful definitions. An *Euclidean lattice* is a free $\mathbb{Z}$ module of finite rank, with a non-degenerate pairing $\langle , \rangle$ with values in $\mathbb{Z}$. The Euclidean lattice E is called *unimodular* if the Gram determinant is equal to ± 1. The lattice is called *even* if $\langle x, x\rangle = 0 \bmod 2$ for all $x \in E$.

Consider the $\mathbb{Z}$-module

$$H_X = H_2(X, \mathbb{Z}) \simeq \mathbb{Z}^{22}.$$

The non-degenerate symmetric pairing $\langle , \rangle$ defined by the intersection of cycles on X defines a Euclidean lattice structure on H_X. This lattice is even and unimodular. In addition, the signature of the form $\langle , \rangle$ on $H_X \otimes \mathbb{R}$ is $(3, 19)$. As shown in Serre [1970], there exists a unique (up to isomorphism) Euclidean lattice with these properties. Denote this lattice by L.

Let $\phi : H_X \to L$ be an isomorphism of Euclidean lattices, and let l be the image of the homology class $c \in H_X$ Poincaré-dual to the polarization class ω. The classifying space which we are attempting to construct is uniquely determined by the vector $l \in L$. Hence we will denote it (the classifying space) by $D(l)$.

Having fixed the isomorphism ϕ, we will not distinguish the lattices H_X and L. Consider the dual lattice $L^* = \mathrm{Hom}(L, \mathbb{Z})$. Since the pairing $\langle , \rangle$ on L is unimodular, there is an isomorphism $\nu : L \to L^*$, such that $\nu(r)(s) = \langle r, s\rangle$ for any $r, s \in L$. This isomorphism allows us to transfer the bilinear form from L to L^*. Let $H^* \subset L^*$ be the free submodule generated by those elements $f \in L^*$ for which $f(l) = 0$.

The module L^* is uniquely identified with $H^2(X, \mathbb{Z})$, and the primitive cohomology classes correspond to the elements of the module H^*. Let Q be the restriction of the inner product on L^* to H^*. The pair (H^*, Q) and the numbers $h^{2,0} = h^{0,2} = 1$, and $h^{1,1} = 19$ define a classifying space of Hodge structures of weight 2.

Note that the Hodge structures in question are uniquely reconstructible from the subspace $H^{2,0} = \{\mathbb{C}\omega_0\} \subset H^* \otimes \mathbb{C}$. The conditions (2) of Chapter 2 are equivalent to saying that

$$Q(\omega_0, \omega_0) = 0, \quad Q(\omega_0, \overline{\omega}_0) > 0. \tag{1}$$

Thus, $D(l)$ is an open subset of a quadric hypersurface in $\mathbb{P}^{20}$. It can be shown that $D(l)$ is a disjoint union of two isomorphic complex manifolds.

Each of these manifolds is a symmetric domain in $\mathbb{C}^{19}$. The group G_l of $\mathbb{R}$-linear transformations $L^* \otimes \mathbb{R}$ preserving the inner product and the hyperplane $H^* \otimes \mathbb{R}$ acts transitively on $D(l)$.

3.4. We will say that a *marked K3 surface* is an ordered triple (X, ϕ, ζ), where X is a K3 surface, $\phi : H_X \to L$ is an isomorphism of Euclidean lattices, and $\zeta \in H_X$ is a class representing a very ample divisor on X with $\phi(\zeta) = l$.

An isomorphism of marked surfaces (X, ϕ, ζ) and (X', ϕ', ζ') is an isomorphism $f : X \to X'$ of the underlying surfaces which sends ϕ to ϕ' and ζ to ζ'.

To each marked surface $\tilde{X} = (X, \phi, \zeta)$ we can associate a well-defined point $\Phi(\tilde{X}) \in D(l)$ as follows. Let η_0 be a holomorphic differential of type $(2,0)$ on X. The map

$$\gamma \to \int_\gamma \eta_0$$

is an element of $\omega_0 \in L^* \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{Z}}(H_X, \mathbb{C})$ satisfying the conditions (1) and such that $\omega_0(l) = 0$. Then $\Phi(\tilde{X}) \in D(l)$ is a subspace $\{\mathbb{C}\omega_0\} \subset H^* \otimes \mathbb{C}$. The point $\Phi(\tilde{X})$ will be called the period of the marked surface X.

The aim of this section is to sketch the proof of the following proposition:

Global Torelli theorem for K3 surfaces. *A marked K3 surface is determined by its periods.*

3.5 An important role in the proof of the global Torelli theorem is played by the existence of a "good" moduli space for marked K3 surfaces. We will not give the details of the construction of such a moduli space (see Piatetsky-Shapiro–Shafarevich [1971]), but we will give the main steps and some comments.

A family of marked K3 surfaces is a smooth family of projective varieties

$$f : \mathcal{X} \to S$$

over a complex-analytic base S, such that each fiber $X_s = f^{-1}(s)$ has the structure

$$X_s = (X_s, \phi_s, \zeta_s)$$

of a marked K3 surface, and such that ϕ_s and ζ_s vary smoothly with s as follows. For any s_1 and s_2 in the same connected component of S, and any path γ joining them, let μ be the monodromy transformation from X_{s_1} to X_{s_2} induced by γ. Then

$$\mu_* : H_{X_{s_1}} \to H_{X_{s_2}}$$

is the induced isomorphism on homology, and the condition that ϕ_s and ζ_s vary smoothly is simply the condition that $(X_{s_1}, \phi_{s_1}, \zeta_{s_1})$ be isomorphic to $(X_{s_1} = \phi \circ \mu_*, \mu_*^{-1}(\zeta))$ as marked K3 surfaces.

In particular, by definition, he action of each element of the monodromy group on the homology of the fiber X_s is generated by some automorphism of the K3 surface X_s.

Fix an integer $n \geq 3$. The main result consists of the existence of a family of marked K3 surfaces

$$f : \mathcal{X} \to \mathfrak{M}_n,$$

satisfying the following conditions:

(1) The family f is effectively parametrized, that is, the Kodaira-Spencer mapping (see Chapter 2, Section 5) is injective at each point $m \in \mathfrak{M}_n$.

(2) Let $\tilde{X} = (X, \phi, \zeta)$ be a marked K3 surface, and let $C \subset X$ be a very ample divisor defining ζ, and such that the complete linear system $|C|$ defines an embedding $X \hookrightarrow \mathbb{P}^n$. Then there exists a unique element $m \in \mathfrak{M}_n$, such that the fiber $\tilde{X}_m = \tilde{f}^{-1}(m)$ is isomorphic to $\tilde{X}$ as a marked K3 surface.

(3) The dimension of $\mathfrak{M}_n$ is 19.

The variety $\mathfrak{M}_n$ will be called the *moduli space* of marked K3 surfaces in $\mathbb{P}^n$.

In Section 3.4 we have defined the period $\Phi(\tilde{X}) \in D(l)$ of a marked K3 surface $\tilde{X}$. By associating the periods to the fibers of f, we obtain a holomorphic (see Chapter 2, Section 3) period mapping

$$\Phi : \mathfrak{M}_n \to D(l).$$

The infinitesimal Torelli theorem, together with property (1) imply that Φ is a local embedding.

Comparing the dimensions of the moduli space $\mathfrak{M}_n$ and $D(l)$, we get that

Proposition 3.1. *The period mapping Φ of the moduli space $\mathfrak{M}_n$ of marked K3 surfaces into the classifying space of polarized Hodge structures $D(l)$ is a local isomorphism.*

Let us make some comments regarding the construction of the space $\mathfrak{M}_n$. The main technical problem is to prove that the set of K3 surfaces in $\mathbb{P}^n$ embedded into $\mathbb{P}^n$ by a complete linear system of some very ample divisor is parametrized by some nonsingular algebraic variety M. Given that, let $S \subset M$ be a subset parametrizing surfaces X_s admitting the structure of a marked K3 surfaces $\tilde{X}_s = (X_s, \phi, \zeta)$ where ζ is the class of a hyperplane section in $\mathbb{P}^n$. It can be shown that S is a connected component of M. Hence, we get some projective family (see Chapter 2, Section 3)

$$\phi : \mathcal{F} \to S$$

of K3 surfaces. The fiber of this family over a point $s \in S$ will be denoted by X_s.

Let $\tilde{X}_1 = (X_1, \phi_1, \zeta_1)$, $\tilde{X}_2 = (X_2, \phi_2, \zeta_2)$ be two marked K3 surfaces, let $C_i \subset X_i$ be very ample divisors corresponding to ζ_i and realizing the embedding of X_i into $\mathbb{P}^n$. Note that if $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic, then the isomorphism should be realizable by a projective transformation of $\mathbb{P}^n$. This is so, because the mapping $\nu : X_1 \to X_2$ inducing the isomorphism between $\tilde{X}_1$ and $\tilde{X}_2$ sends ζ_1 to ζ_2.

Now, let us construct an unramified covering $\nu : S' \to S$, such that each fiber $\nu^{-1}(s)$ corresponds to the set of isomorphisms of the lattices H_{X_s} and L sending ζ_i to l. Let Γ be the group of automorphisms of L fixing l. The monodromy action of $\pi_1(S, s)$ preserves the hyperplane class ζ_s, and by choosing an identification $(H_{X_s}, \zeta_s) \equiv (L, 1)$ we get a homomorphism $\pi_s(S, s) \to \Gamma$. Now, consider the family $\pi' : \mathcal{F}' \to S'$, obtained from π by the base change ν. This means that $\mathcal{F}'$ is a fiber product $\mathcal{F} \times_S S'$ relative to the morphisms π and ν.

The projective group $G = \mathrm{PGL}(n)$ acts on $\mathcal{F}$ and on S via its action on $\mathbb{P}^n$. This action commutes with π. Let us define the action of G on S' which commutes with ν. Let $\gamma \in G$, $s' \in S'$, $\nu(s') = s$. Then the element s' is defined by the choice of the isomorphism $\phi_s : H_{X_s} \to L$. The element γ defines by the surface isomorphism $a_\gamma : X_s \to X_{\gamma(s)}$. Consider an isomorphism $\phi : H_{X_{\gamma(s)}} \to L$, for which $\phi_s = \phi \circ (a_\gamma)_s$. Then the pair $(\gamma(s), \phi)$ uniquely determines the point $\gamma(s') \in S'$. Hence, the action of the group G on $\mathcal{F}$, S, S' agrees with the morphisms π and ν, and thus the action on $\mathcal{F}' = \mathcal{F} \times_S S'$ agrees with the morphism π'.

Let us now define the varieties $\mathcal{X}$ and $\mathfrak{M}_n$ to be, respectively, the quotients $\mathcal{F}'/G$ and S'/G. It can be shown that these quotients exist in the category of complex analytic spaces. In addition, it is shown by Piatetsky-Shapiro–Shafarevich [1971] that every automorphism of a K3 surface X acting trivially on H_X is trivial. It follows that the group G acts on S' and $\mathcal{F}'$ without fixed points, and so $\mathcal{X}$ and $\mathfrak{M}_n$ are complex manifolds.

3.6. In this section we give a sketch of the proof of the global Torelli theorem for marked K3 surfaces (Section 3.4).

Let $Y \subset \mathfrak{M}_n$ be the subset parametrizing the special Kummer surfaces (see Section 3.1). Let us consider the period mapping

$$\Phi : \mathfrak{M}_n \to D(l),$$

and let $Z = \Phi(Y)$. The set Z has the following properties:

(1) Z is everywhere dense in $D(l)$.

(2) $\Phi^{-1}(z)$ is a single point for any $z \in Z$.

The main part of the proof of the global Torelli theorem consists of demonstrating properties (1) and (2). Let us first deduce the theorem from these properties.

Let $d \in D(l)$ be a point such that there are at least two points m_1 and m_2 in the preimage $\Phi^{-1}(d)$. By proposition 3.1, the map Φ is a local isomorphism, and, consequently, there are non-intersecting neighborhoods U_1 and U_2 of m_1 and m_2 respectively, which are isomorphically mapped onto the same neighborhood U of the point $d \in D(l)$. But in that case, each $z \in Z \cap U$ would have at least two preimages, and thus $Z \cap U = \emptyset$. This contradicts the density of Z in $D(l)$.

In the remaining part of this section we will try to explain the derivation of properties (1) and (2) of the set Z. We will need some definitions.

For any nonsingular algebraic surface X, let us define S_X as the group of equivalence classes of divisors with respect to algebraic equivalence. It is known that if X is a K3 surface, then $S_X \simeq \operatorname{Pic} X$ is a finitely generated group without torsion. The intersection index makes S_X into a Euclidean lattice. We will also use S_X to denote the sublattice generated by S_X in the homology group $H_X = H_2(X, \mathbb{Z})$.

Let $a \in S_X$. Let $S_{X'} \in S_X$ be a sublattice generated by elements b such that $(b, b) = -2$, $(b, a) = 0$. We will denote the quotient lattice $S_{X'}/\{\mathbb{Z}a\}$ by $S_X(a)$. The Euclidean lattice structure on S_X induces one on $S_X(a)$.

Consider the morphism $\pi_1 : A \to E$ of the two-dimensional abelian variety A onto an elliptic curve E. Let τ be an involution of A (of the form $x \to (-x)$ and let τ_0 be the induced involution on E. Let $C = E/\{1, \tau\} \simeq \mathbb{P}^2$, and let $X = \tilde{A}$ be the special Kummer surface obtained by resolving the singularities of the complex analytic space $A/\{1, \tau\}$. The morphism π_1 induces a morphism $\pi_2 : X \to C$, all of whose non-singular fibers are curves of genus 1. There are four singular fibers of π_2, corresponding to double points on E. Each such fiber is reducible. It consists of a double curve $2l_0$ generated by the fiber of π_1 and also of the four curves l_i, $i = 1, \ldots, 4$ which are the preimages of the singular points on $A/\{1, \tau\}$. We have

$$\begin{aligned} &(l_i, l_i) = -2, \quad i = 0, \ldots, 4; \\ &(l_i, l_j) = 0, \quad 1 \le i, j \le 4, \quad i \ne j; \\ &(l_0, l_i) = 1, \quad i = 1, \ldots, 4. \end{aligned} \tag{2}$$

Let $a \in S_X$ be the element generated by the fiber of the morphism π_2, then $a = 2l_0 + l_1 + l_2 + l_3 + l_4$. Evidently, $(a, l_i) = 0$, $i = 0, \ldots, 4$. Let G_4 be the Euclidean lattice which is isomorphic as a $\mathbb{Z}$-module to the quotient $(\oplus_{i=0}^4 \mathbb{Z} l_i)/\mathbb{Z}a$, with the pairing defined by (2). It can be checked that G_4 is well-defined.

Let $b \in S_X$ be such that $(b, b) = -2$. Then, either b or $-b$ is effective. Indeed, let b be generated by a divisor D. Then, by the Riemann-Roch theorem (Section 3.1)

$$\dim H^0(X, \mathcal{O}_X(D)) + H^0(X, \mathcal{O}_X(-D) \ge \frac{D^2}{2} + 2 = 1.$$

If, in addition, $(b, a) = 0$, then b is a sum of components of the fibers of π_2, since an effective divisor must have a non-empty intersection with some fiber of π_2. It follows that if a is an element in S_X generated by a fiber of π_2, then, for a special Kummer surface

$$S_X(a) \simeq (G_4)^4.$$

Theorem 3.1. *A K3 surface X is a special Kummer surface if and only if S_X contains an element a such that $(a, a) = 0$, the class of a contains an irreducible divisor and $S_X(a) \simeq (G_4)^4$.*

The discussion above has almost proved the necessity part of Theorem 3.1. Let us sketch the proof of sufficiency. Let D be an irreducible divisor of class a. Then, by the Riemann-Roch theorem for X, recalling that $K_X = 0$, it follows that

$$\dim H^0(X, \mathcal{O}_X(D)) \geq 2.$$

Thus, a certain subsystem of the linear system $|D|$ defines a morphism $\pi : X \to \mathbb{P}^1$. By Bertini's theorem, the generic fiber of π is non-singular. By the adjunction theorem, the genus of the generic fiber is $\frac{1}{2}D(D + K_X) + 1 = 1$. Thus, π is a fibration with fiber an elliptic curve. It can be shown that if the divisor S is such that $S^2 = -2$, $S\dot{D} = 0$, then S is an irreducible component of a fiber of π. The structure of possible reducible fibers of π is well-known (see Kodaira [1960]). From the classification, it follows that in order of the group $S_X(a)$ to be isomorphic to $(G_4)^4$ it is necessary and sufficient for the morphism π to have exactly 4 reducible fibers. Each of these fibers contains a rational double curve $2l_0$, and also 4 rational curves l_i, $i = 1, \ldots, 4$. Their mutual intersection indices are given by formulas (2). There are no singular fibers of π, other than the reducible ones. This follows from the well-known formula

$$\chi = \sum_{i=1}^{r} \chi_i$$

for the Euler characteristic χ of an elliptic surface (see Griffiths–Harris [1978]). Here χ_i is the Euler characteristic of the degenerate fiber D_i, and the summation is over the degenerate fibers. Since $\chi = 24$, and $\chi_i = 6$ for each reducible fiber, it follows that all other fibers have $\chi_i = 0$. This means that the other fibers are either non-singular or multiple, and it can be easily shown that there are no multiple fibers.

Consider a double cover $\lambda : C \to \mathbb{P}^1$, ramified over points over which π has reducible fibers, let A be the normalization of the surface $X \times_{\mathbb{P}^1} C$, and let $\pi : A \to C$ be the natural projection map. All fibers of π are now smooth elliptic curves. It is then possible to introduce a group structure on C in such a way that the morphism λ coincides with the quotient with respect to the automorphism group generated by the reflection $x \to -x$. This involution can be extended to an involution θ of the variety A. Evidently, X can be obtained by a minimal resolution of singularities of the surface A/θ. The regular $(2,0)$-form on X defines a regular non-zero 2-form on A, and thus $p_g(A) > 0$. Now theorem 3.1 is reduced to the following statement.

Lemma 3.2. *Let C be an elliptic curve, A a smooth surface with $p_g(A) \neq 0$, and $\pi_1 : A \to C$ a morphism, all fibers of which are elliptic curves. Then A is an abelian variety.*

Furthermore, suppose that there is an involution θ on A, such that $\theta\pi_1 = -\pi_1$, and $p_g(X') \neq 0$ for the variety $X' = A/\theta$. Then it is possible to introduce a group law on A, so that θ assumes the form $\theta x = -x$.

The proof of the lemma is based on the fact that the family π_1 of elliptic curves without singular and multiple fibers always becomes trivial after an unramified base change $\mu : C' \to C$. Thus, $A = A'/\Gamma$, where $A' = C' \times_C A$, $A' = B \times C'$, where B is an elliptic curve and Γ is some finite group of automorphisms of A'. All automorphisms $\gamma \in \Gamma$ have the form

$$\gamma(b, c') = (\phi_\gamma(b), c' + e_\gamma).$$

Here, $\phi_\gamma(b) = a_\gamma(b) + f_\gamma$, where a_γ is an automorphism of B as an abelian variety, and f_γ is an element of B. It can be seen that if a_γ is non-trivial, then γ acts non-trivially on a holomorphic 2-form on $B \times C'$. Since $H^0(A, \Omega^2) = H^0(A', \Omega^2)^\Gamma$, and $p_g(A) \neq 0$, that is impossible. Hence, γ is generated by translations, so A is also an abelian variety. The second part of the lemma can be also established in a straightforward fashion, after transferring the automorphism θ to $C' \times_C A$ and using the fact that θ leaves invariant a holomorphic 2-form on A'.

Note that the statement of theorem 3.1 can be strengthened, by omitting the requirement that class a contain an irreducible curve. Thus, the period $\Phi(\tilde{X})$ of a marked surface $\tilde{X} = (X, \phi, \zeta)$ allows us to determine whether or not it is a special Kummer surface.

Indeed, let the period be a subspace $\mathbb{C}\omega_0 \subset H^* \otimes \mathbb{C}$, using the notation of Section 3.4. Then, using the isomorphisms $\phi : H_X \to L$ and $\nu : L \simeq L^*$ we can assume that the group $S_X \subset H_X$ is the set of elements $\lambda \in L$ for which $(\omega_0, \lambda) = 0$. Indeed, in that case λ is an integral $(1,1)$-cycle, which is algebraic by Hodge's theorem.

Thus, it is a matter of linear algebra to show that the set $Z \subset D(l)$ of Hodge structures is everywhere dense. In fact, it can even be shown that that the set of Hodge structures of special Kummer surfaces with $\operatorname{rank}(S_X) = 20$ is everywhere dense.

Now, let $\tilde{X} = (X, \phi, \zeta)$ and $\tilde{X}' = (X', \phi', \zeta')$ be two marked K3 surfaces, where X' is a special Kummer surface. Suppose that these two surfaces have same period $\mathbb{C}\omega_0 \subset H^* \otimes \mathbb{C}$. If we could show that under these assumptions $\tilde{X} \simeq \tilde{X}'$ then it will be shown that $\Phi^{-1}(z)$ contains a single point for any $z \in Z$ and the proof of the Torelli theorem will be finished.

Consider an isomorphism $\psi : H_X \to H_{X'}$ such that $\phi' \circ \psi = \phi$. Evidently, $\psi(S_X) = S_{X'}$. Indeed, in the notation of Section 3.3, the class $a \in H_X$ is an element of S_X if and only if $\nu(\phi(a))$ is orthogonal to $\mathbb{C}\omega_0$. Furthermore, ψ maps effective cycles to effective cycles. To show this, consider a class $a \in H_X$ generated by an irreducible curve $D \subset X$. Then, by the Riemann-Roch theorem on X'

$$\dim H^0(X', \psi(a)) + \dim H^0(X', \psi(-a)) \geq \frac{\psi^2(a)}{2} + 2.$$

But by the adjunction formula, $\psi^2(a) = a^2 = D^2 \geq -2$. Hence, either $\psi(a)$ or $\psi(-a)$ is effective. On the other hand,

$$(\psi(a).\ \zeta') = (\psi(a).\psi(\zeta)) = (a.\zeta) > 0,$$

and so the cycle $\psi(a)$ is effective.

Let $a \in S_X$ be the cycle generated by an irreducible divisor, such that $(a.a) = 0$ and $S_X(a) = (G_4)^4$. Then, from the above discussion it follows that $\psi(a)$ has analogous properties, and hence by Theorem 3.1, the surface X' is a special Kummer surface.

Let A be a two-dimensional abelian variety, let X be the special Kummer surface generated by A and let $\pi : A \to X$ be the corresponding rational map. Then

$$H_2(X, \mathbb{Z}) = \pi_*(H_2(A, \mathbb{Z})) \oplus \Pi_X,$$

where the module Π_X is generated by the classes l_i, $i = 1, \dots, 16$, corresponding to the curves obtained by resolving the singularities of A/θ. Furthermore, $\pi_*(H_2(A, \mathbb{Z}))$ is the orthogonal complement to Π_X in H_X. The map π_* is injective. The following statement holds.

Lemma 3.3. *Any automorphism of the lattice H_X which restricts to the identity on the orthogonal complement to Π_X and which transposes vectors l_i, $i = 1, \dots, 16$ and fixing at least one of these vectors, is in fact the identity on H_X.*

Let us return to the problem at hand. It is easy to see that $\psi(\Pi_X) = \Pi_{X'}$, since the classes l_i are exactly those components of the fiber that have multiplicity 1.

Suppose that the special Kummer surfaces X and X' are obtained from the abelian varieties A and A' respectively. Let $\pi_1 : A \to X$, $\pi_2 : A' \to X'$ be the corresponding rational maps. Then $\psi[(\pi_1)_*(H_2(A, \mathbb{Z}))] = (\pi_2)_*(H_2(A', \mathbb{Z}))$, thus we can assume that we are given an isomorphism between $H_2(A, \mathbb{Z})$ and $H_2(A', \mathbb{Z})$ which, evidently, preserves periods. This isomorphism is always induced by an isomorphism $\lambda : A \to A'$ of complex tori. That is so, because a 2×4 matrix of rank 2 is determined by its 2×2 minors up to multiplication by a non-degenerate 2×2 matrix. Since a translation by an element $x \in A$ induces the identity automorphism of the group $H_2(A, \mathbb{Z})$, the map λ can be picked in such a way that $0 \in A$ gets mapped to a prescribed element $a' \in A'$. Let l be the homology class on X obtained by resolving the singularity at some 2-torsion point $a' \in A$. Pick λ in such a way that $\lambda(0) = a'$. It can be seen that there exists an isomorphism $\eta : X \to X'$, such that the diagram below commutes.

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\pi_1"] & A' \arrow[d, "\pi_2"'] \\ X \arrow[r, "\eta"] & X' \end{tikzcd}$$

The isomorphism $\lambda_* : H_X \to H_{X'}$ agrees with ψ on $(\pi_1)_*(H_2(A, \mathbb{Z}))$ and maps Π_X to $\Pi_{X'}$ while preserving effective cycles, and also coincides with ψ on the class l. Thus, by Lemma 3.3, λ_* coincides with ϕ. This concludes the proof of the isomorphism of the marked surfaces $\tilde{X}$ and $\tilde{X}'$ and thus of the Torelli theorem.

3.7. The generic global Torelli theorem (see Chapter 3, Section 5) for elliptic pencils was proved by K. Chakiris [1984], using the ideas of Piatetsky-Shapiro–Shafarevich [1971]. In this section we shall formulate this result, and give the basic ideas of the proof.

Let $\psi : V \to \mathbb{P}^1$ be a morphism of a non-singular surface V satisfying the following conditions.

(1) For a generic point $x \in \mathbb{P}^1$ the fiber $\psi^{-1}(x)$ is an elliptic curve.
(2) There exists a section $\tau : \mathbb{P}^1 \to V$ of the morphism ψ (in particular ψ has no multiple fibers). This section defines the divisor $\mathfrak{U} = \tau(\mathbb{P}^1) \subset V$.
(3) V is simply connected.

item(4) $p_g(V) = n > 0$.

Then the triple $(V, \mathfrak{U}, \psi)$ is called an *elliptic pencil of genus* n. The number n will stay fixed in this section. Isomorphism of elliptic pencils is defined in the obvious way. Elliptic pencils occupy an important position in the classification of algebraic surfaces (see Shafarevich *et al* [1965], Griffiths–Harris [1978]). Together with two-dimensional abelian varieties and K3 surfaces they form almost the whole set of algebraic surfaces with Kodaira dimension equal to 0.

Chakiris [1984] constructs the following objects:

(1) The quasiprojective variety $\mathfrak{M}$, $\dim \mathfrak{M} = 10n + 8$, the closed points of which are in one-to-one correspondence with the isomorphism classes of elliptic families of genus n.

(2) A classifying space D of polarized Hodge structures of weight 2, and a discrete group Γ of its automorphisms.

There is a regular map $\Phi : \mathfrak{M} \to \Gamma \backslash D$, which is an extended variation of Hodge structures (see Chapter 2, Section 4).

Theorem 3.2. *There exists an open everywhere dense set* $\mathfrak{M}' \subset \mathfrak{M}$ *on which the map* Φ *is injective.*

3.8 Note that in order to prove Theorem 3.2 it is sufficient to find at least one point $x \in \mathfrak{M}$ having the following properties.

(A) The map Φ is a local Zariski embedding at x (that is, $d\Phi$ is injective on $(T\mathfrak{M})_x$).
(B) $\Phi^{-1}(\Phi(X)) = \{x\}$.
(C) There exists a neighborhood U of the point $\Phi(x)$ such that $\Phi|_{\Phi^{-1}(U)}$ is a proper morphism.

First, let us describe the classifying space D. Let $\psi : V \to \mathbb{P}^1$ be an elliptic pencil, $\mathfrak{U}$ a section, C_u, a fiber. Consider the sublattice $H_V \subset H^2(V, \mathbb{Z})$ consisting of the classes γ such that $\gamma.C_u = \gamma.\mathfrak{U} = 0$. Note that H_V is a unimodular lattice, and it can be computed that the intersection form continued to $H_V \otimes \mathbb{R}$ has $2n$ positive and $10n + 8$ negative eigenvalues. There is exactly one isomorphism class (see Serre [1970]) of Euclidean lattices with this property. Denote this lattice by H, and the inner product by Q. Then D is the classifying space of polarized Hodge structures of weight 2 with data $(H, Q, h^{p,q})$, where $h^{2,0} = n$, $h^{1,1} = 10n + 8$.

3.9. The main role in the proof of Theorem 3.2 is played by *special elliptic pencils* – the analogue of the special Kummer surfaces. Let C_1 be an elliptic curve, let $p \in C_1$ be a point, and let $\gamma : C_n \to \mathbb{P}^1$ be a double cover ramified at $2(n+1)$ points. Then C_n is a hyperelliptic curve of genus n. Let τ_1 be an involution of C_1 such that $\tau_1(p) = p$, and let τ_n be the involution of C_n transposing the points in the fibers of γ. Let G be the group of transformations of $C_1 \times C_n$ generated by $\tau = (\tau_1, \tau_n)$. Let V be the smooth surface obtained as a result of minimal resolution of singularities of the complex space $C_1 \times C_n/G$. The projection of $C_1 \times C_n$ onto C_1 defines a morphism $\psi : V \to \mathbb{P}^1$, with a section generated by the divisor $\{p\} \times C_n \subset C_1 \times C_n$. It can be observed that V is an elliptic pencil of genus n. The triple (V, U, ψ) obtained in this way is called a *special elliptic pencil.*

Evidently, if $\psi : V \to \mathbb{P}^1$ is a special elliptic pencil, then the morphism ψ has $2(n+1)$ reducible fibers, each of which has the same structure as in the case of special Kummer surfaces. For an arbitrary elliptic pencil V, denote by L_V the submodule of H_V generated by effective algebraic cycles γ such that $\gamma^2 = -2$, $\gamma \cdot C_u = \gamma \cdot \mathfrak{U} = 0$. For the special elliptic pencil, it is easy to see that $L_V \simeq (G_4)^{2(n+1)}$ (the lattice G_4 is defined in Section 3.6).

The element $x \in \mathfrak{M}$, satisfying conditions (A), (B), and (C) can be chosen in such a way that the corresponding surface V_x is a special elliptic pencil. The following result necessary for establishing condition (B) is the following.

Lemma 3.4. *The elliptic pencil V of genus n is special if and only if*

$$L_V \simeq (G_4)^{2(n+1)}.$$

The proof of Lemma 3.4 is almost word-for-word the same as the proof of Theorem 3.1 of Section 3.6.

The gist of Lemma 3.4 is that the image $\Phi(x)$ of an element $x \in \mathfrak{M}$ contains enough information to determine whether or not the corresponding V_x is a special elliptic surface. Indeed, L_{V_X} is isomorphic to the submodule of those elements in L which are orthogonal to $H^{2,0} \subset H \otimes \mathbb{C}$.

It is *not* shown in Chakiris [1984] that the special elliptic pencil V_x is uniquely determined by its image $\Phi(x)$ in D/Γ. This is proved, however, for a sufficiently generic elliptic pencil.

3.10. Property (A) uses the work of K. I. Kii [1978], who develops some conditions that guarantee that the infinitesimal Torelli theorem holds.

Let $\phi : V \to \mathbb{P}^1$ be an elliptic pencil, and let $\rho : \mathcal{O} \to N$ be a Kuranishi family generated by the surface V. This means that ρ is a smooth morphism of complex manifolds such that $\rho^{-1}(n_0) = V$,

$$\dim N = \dim_{\mathbb{C}} H^1(V, T_V) = 11n + 8,$$

and the family is effectively parametrized. Consider the submanifold $N_0 \subset N$, corresponding to those elliptic pencils for which the divisor of the section $\mathfrak{U}$ remains effective. It can be shown that $\dim N_0 = 10n + 8$, and N_0 is a nonsingular submanifold. Let $\rho_0 : \mathcal{O}_0 \to N_0$ is the restriction of the morphism ρ to $\mathcal{O}_0 = \rho^{-1}(N_0)$. Assuming that N_0 is simply-connected we can lift the period mapping to a holomorphic mapping

$$\theta_0 : N_0 \to D.$$

From the results of Kii [1978] it follows that θ_0 is a Zariski local embedding. However, this is not quite what we need. The problem is that if $x \in \mathfrak{M}$ is a special elliptic pencil, then the neighborhood $U \subset \mathfrak{M}$ of x is not locally isomorphic to N_0, but rather there exists a discrete group Γ_x, such that U is locally isomorphic to N_0/Γ_x. Let us describe some automorphisms in Γ_x.

Let $l \subset V_x$ be the curve obtained by resolving one of the singularities of $C_1 \times C_n/g$, and let $N_1 \subset N_0$ be the submanifold of codimension 1, where the class of the divisor l remains effective. Consider this divisor over each point $x_1 \in N_1$. The union of all these is a submanifold S of $\mathcal{O}$ of codimension 2. This is a $\mathbb{P}^1$ bundle over N_1. Construct a σ-process over $\mathcal{O}_0$ with center in S. The preimage $\tilde{S}$ of the manifold S with respect to this σ-process is a conic bundle over N_0. Contracting all of these conics to lines in another direction, we obtain a new family $\rho_0' : \mathcal{O}' \to N_0$, which is a bi-rational modification of $\mathcal{O}_0$. Note, however, that for every point $n \in N_0$ we have $(\rho_0')^{-1}(n) \simeq \rho_0^{-1}(n)$. By the universality of the family $\mathcal{O}_0 \to N_0$ it follows that the new family is obtained by means of a base change $\tau_l : N_0 \to N_0$, $\tau_l(n_0) = n_0$. Clearly, $V_x \simeq V_{\tau_l(x)}$. Thus, all of the τ_l are contained in Γ_x. Furthermore, N_0 is acted upon by the standard involution of the pencil $\psi : V \to \mathbb{P}^1$, generated by the section. This is an involution, which acts on the generic fiber by an involution of the elliptic curve $\psi^{-1}(u)$ fixing the point $\mathfrak{U} \cap \psi^{-1}(u)$.

For a generic special elliptical family one can get around these difficulties.

3.11. Finally, the most subtle point in the proof of Theorem 3.2 is the establishment of property (C). It is not hard to see that the Hodge structure on a generic special elliptic pencil does not split; that is, there is not a pair of subspaces

$$F_1, F_2 \subset H \otimes \mathbb{Q}, \quad F_1 \neq \{0\}, \quad F_2 \neq \{0\},$$

such that

$$H^{2,0} = (H^{2,0} \cap (F_1 \otimes \mathbb{C})) \oplus (H^{2,0} \cap (F_2 \otimes \mathbb{C})).$$

Let $x \in \mathfrak{M}$, with V_x a special elliptic pencil for which the Hodge structure does not split. We check property (C) for such an element x. The main idea is roughly as follows. Consider a semistable completion $\overline{\mathfrak{M}}$ of $\mathfrak{M}$ – this means that $\overline{\mathfrak{M}} \backslash \mathfrak{M}$ is a divisor with normal intersections and such that the monodromy (see Chapter 5, Section 1) action on the cohomology $H^2(V_y, \mathbb{Z})$ of a generic surface V_y, $y \in \mathfrak{M}$ has finite order. Let $\Phi(x) = \Phi(x')$ for $x' \in \overline{\mathfrak{M}} \backslash \mathfrak{M}$. We want to show that this is impossible. Consider the complex disk $S \simeq \{z \in \mathbb{C} \mid |z| < 1\}$ centered at s_0, and such that $S \subset \overline{\mathfrak{M}}$ and $S \cap (\overline{\mathfrak{M}} \backslash \mathfrak{M}) = s_0$. We obtain a degeneration $\pi : D \to S$ of elliptic pencils over S (see Chapter 5, Section 1). After a base change, we can assume that the monodromy action is trivial on the two-dimensional cohomology of the generic fiber, and that the degeneration is semistable. Instead of the degeneration of elliptic pencils, it will be more convenient to study the degeneration $\tilde{\pi} : \tilde{D} \to S$ of Weierstrass pencils obtained by contracting all of the components of the fibers having trivial intersection with the divisor $\mathfrak{U}_s$ of the section. All of the fibers $\tilde{\pi}^{-1}(s)$, $s \neq s_0$ are normal surfaces with, at worst, simple singularities. The fiber $\pi^{-1}(s)$ can be the union of a collection of irreducible components $V_1, \ldots, V_p$. Call those V_i for which $p_g(V_i) > 0$ marked. From the triviality of the monodromy action, it follows that

$$\sum_{i=1}^{p} p_g(V_i) = n,$$

(see Chapter 5, Section 5). However, if there were several marked components, the limiting Hodge structure would split. But since the limiting structure is the same as that on V_x, this cannot be. Now, all we need is

Theorem 3.3. *Let $\rho : Y \to D$ be a semistable (see Chapter 5, Section 1) degeneration of Weierstrass families with trivial monodromy. Let the central fiber have a single marked component. Then there exists a base change $D' \to D$, such that the degeneration $D' \times_D Y \to D'$ is D'-birationally equivalent to a degeneration, whose central fiber is a surface having at worst double point singularities.*

The proof of the above statement is the most difficult place in Chakiris [1984].

§4. Hypersurfaces

4.1 In this section we describe two results on smooth projective hypersurfaces. These are the infinitesimal Torelli theorem (see Chapter 2, Section 5) obtained by P. Griffiths [1969] and the generic global Torelli theorem obtained by Donagi [1983]. In Section 4.9 we briefly describe the global Torelli theorem for four-dimensional cubics, proved by Voisin [1986].

In Chapter 4, Section 5 of this survey we compute the Hodge structure on the cohomology of a nonsingular projective hypersurface. We shall describe the necessary facts here; more details can be found in Griffiths [1969].

Let $X \subset \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d, defined by the equation

$$f(x_0, \ldots, x_{n+1}) = 0.$$

Set $V = H^0(\mathbb{P}^{n+1}, \mathcal{O}(1))$ and let $S = \oplus_{k=0}^{\infty} S^{(k)} V$ be a graded symmetric algebra of the vector space V. The homogeneous component $S(k)(V)$ of this algebra will be also denoted by S^k. Consider the homogeneous ideal J_f in S, generated by homogeneous polynomials

$$\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_{n+1}}.$$

This is the so-called *Jacobian ideal of the polynomial f*.

The ring $R_f = S/J_f$ is a graded algebra. Denote its homogeneous component of degree a by R_f^a. Then $R_f^a = S^a/J_f^a$, where J_f^a is the homogeneous component of the ideal J_f of degree a.

We will be interested in the Hodge structure on the primitive cohomology of dimension n of the variety X

$$P^n(X) = F^0 \supset F^1 \supset \ldots \supset F^n \supset F^{n+1} = \{0\}.$$

Theorem 4.1. *There exists a natural isomorphism, which depends holomorphically on f*

$$\lambda_a : R_f^{t_a} \mapsto F^a/F^{a+1},$$

where $t_a = (n - a + 1)d - (n + 2)$.

Let us describe the structure of this isomorphism. The Poincaré residue operator Res, defines an isomorphism (see Chapter 4, Section 3)

$$\mathrm{Res} : H^{n+1}(\mathbb{P}^{n+1} \backslash X) \to H^n(X).$$

Any class in $H^{n+1}(\mathbb{P}^{n+1} \backslash X)$ can be represented by a holomorphic differential

$$\alpha = \frac{A\Omega}{f^{n+1}},$$

where

$$\Omega = \sum_{i=0}^{n+1} (-1)^i x_i dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{n+1}.$$

Above, A is a polynomial such that $\deg \alpha = 0$, that is, $\deg A = d(n+1) - (n+2)$. Furthermore, $\mathrm{Res}\, \alpha \in F^a$ if and only if α has a pole along X of order no greater than $n-a+1$ (thus f^a divides A). This defines a map $\mathrm{Res}_a : S^t \mapsto F^a$. The proof of Theorem 4.1 is then reduced to establishing that

$$\mathrm{Res}^{-1}(F^{a+1}) = J_f^{t_a}.$$

4.2. We will need to understand the ring R_f better. Let $f_0, \ldots, f_{n+1}$ be homogeneous polynomials without common zeros in $\mathbb{P}^{n+1}$, $I = (f_0, \ldots, f_{n+1})$ is the ideal generated by these polynomials in S, and let $R = S/I$.

The elements f_i form a regular sequence (see Griffiths–Harris [1978]), that is, for every i, the element f_{i+1} is not a zero-divisor in the ring $S/(f_0, f_1, \ldots, f_{n+1})$.

Consider a basis $e_0, e_1, \ldots, e_{n+1}$ of the vector space V^*. For every $k = 1, 2, \ldots, n+1$ we obtain an S-linear map

$$\partial_k : S \otimes \wedge^k V^* \to S \otimes \bigwedge^{k-1} V^*,$$

by setting

$$\partial_k(e_{j_1} \wedge \ldots \wedge e_{j_k}) = \sum_{\nu=1}^{k} (-1)^{\nu-1} f_{j_\nu} e_{j_1} \wedge \widehat{e_{j_\nu}} \wedge \ldots \wedge e_{j_k}.$$

It is easily checked that $\partial_{k-1}\partial_k = 0$, and so we obtain a complex

$$0 \to S \otimes \wedge^{n+1} V^* \xrightarrow{\partial_{n+1}} \ldots \quad S \otimes V^* \xrightarrow{\partial_1} S \longrightarrow R \to 0,$$

called the *Koszul complex* of the collection $(f_0, \ldots, f_{n+1})$. Since the sequence $\{f_i\}$ is regular, the Koszul complex is exact (see Griffiths–Harris [1978]).

The grading on the ring S induces a grading

$$R = R^0 \oplus R^1 \oplus \ldots$$

of the ring R. From the exactness of the Koszul complex it follows that $\dim R^a$ depends only on a and the degrees $d_i = \deg f_i$. Indeed, $\dim R^a$ can be computed as the alternating sum of the dimensions of the graded components of the free modules $S \times \wedge^k V^*$. Set

$$\sigma = \sum_{i=1}^{n+1} (d_i - 1);$$

then the following holds

Lemma 4.1.

(1) The homogeneous components R^a of the ring R vanish for $a > \sigma$.
(2) $\dim R^a = 1$.
(3) The pairing $R^a \times R^{\sigma - a} \to R^\sigma$, induced by the multiplication in the ring R is nondegenerate for $a = 0, 1, \ldots, \sigma$.

The preceding lemma is an easy consequence of the following result, due to Macaulay [1916]. Let $\zeta_0, \ldots, \zeta_{n+1}$ be a basis of V, and let $\widehat{m}$ be the maximal ideal $(\zeta_0, \ldots, \zeta_{n+1})$ of the ring S. Then, with notation as above we have

Lemma 4.2. (Macaulay's theorem). *Denote by A_l the subring of polynomials in S such that*

$$A_i \widehat{m}^l \subset (f_0, \ldots, f_{n+1}).$$

Then,

$$A_l = (f_0, \ldots, f_{n+1}) + \widehat{m}^{\sigma+1-l}.$$

In particular, for $a + b \leq \sigma$ *the pairing* $R^a \times R^b \to R^{a+b}$ *is nondegenerate on each component.*

4.3. Every element $g \in S^d$ defines a hypersurface $X_g = \{g = 0\}$ of degree d in $\mathbb{P}^{n+1}$. Let $U \subset S^d$ be an open dense set of points g for which X_g is non-singular. The group $G = \mathrm{PGL}(n)$ acts naturally on S^d and on U.

Let $f \in U$, then the linear subspace S^d can be identified with the tangent space $(T_U)_f$ to U at the point f. Let $(T_G)_f$ be the tangent space to the orbit Gf at the point f. Then

$$(T_G)_f \subset (T_U)_f.$$

Lemma 4.3. $(T_G)_f = J_f^d$.

Proof. The group G is locally generated by one parameter subgroups $g_{ij}(t)$, where

$$g_{ij}(t)x_k = x_k, \quad k \neq i, \quad g_{ij}(t)x_i = x_i + tx_j.$$

It can be seen that

$$\frac{d}{dt}[g_{ij}(t)f]_{t=0} = x_i \frac{\partial f}{\partial x_i},$$

that is, $(T_G)_f$ coincides with the component J_f^d of the ideal J_f. Note that by Section 4.3 $\dim J_f^d$ does not depend on $f \in U$.

It is possible to introduce the structure of a quasi-projective algebraic variety on the quotient space $\mathfrak{M} = U/G$ (see Mumford [1965]). The variety $\mathfrak{M}$ is the *moduli space of nonsingular hypersurfaces.* Its closed points correspond precisely to the isomorphism classes of hypersurfaces of degree d.

For $\deg f \geq 3, n \geq 2$ the automorphism group of a nonsingular hypersurface X of degree d in $\mathbb{P}^{n+1}$ is finite. This follows from the fact that there are no global non-vanishing vector fields on X, that is, $H^0(X, T_X) = 0$. Moreover, the automorphism group of a generic hypersurface is trivial. In this case the point $[f] \in \mathfrak{M}$ is nonsingular, and the tangent space to $\mathfrak{M}$ at this point is naturally isomorphic to the quotient $R_f^d = S^d/I_f^d$.

Consider the projective family

$$\pi : \tilde{\mathcal{X}} \to U$$

of hypersurfaces of degree d in $\mathbb{P}^{n+1}$. Here the set $\tilde{\mathcal{X}} \subset U \times \mathbb{P}^{n+1}$ is the set of pairs (g, x), $g \in U$, $x = (x_0, \ldots, x_{n+1}) \in \mathbb{P}^{n+1}$, such that $g(x_0, \ldots, x_{n+1}) = 0$. The G actions on $\tilde{\mathcal{X}}$ and U are compatible. Pick a neighborhood G_0 of the identity in G, small enough to contain no non-trivial automorphisms of X. Let U_0 be a sufficiently small neighborhood of the point f, $\tilde{\mathcal{X}}_0 = \pi^{-1}(U_0)$. The subset $G_0 \subset G$ defines an equivalence relation on $\tilde{\mathcal{X}}_0$ and on U_0, $f_1 \simeq f_2$, if $f_1 = \gamma(f_2)$, $\gamma \in G_0$. The quotient spaces $\mathcal{X}$ and S (of $\tilde{\mathcal{X}}_0$ and U_0, respectively) with respect to this equivalence relation have natural complex manifold structures.

We obtain a smooth family

$$\beta : \mathcal{X} \to S$$

of complex manifolds. The sheaf $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ defines a polarization on each of the fibers of β. The resulting family β is the universal family for the polarized variety $(X, c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1)|_x))$ (see Chapter 2, Section 5.5).

Pick a simply-connected neighborhood of the point $[f] \in S$ corresponding to $f \in U$. Let

$$\Phi_V : V \to D$$

be some lifting of the period mapping of the polarized family β (see Chapter 2, Sections 3 and 5).

Theorem (Infinitesimal Torelli theorem). *The mapping Φ_V is a local embedding if*

(1) $d > 2$, $n \neq 2$;

(2) $d > 3, n = 2$.

There is another formulation of this theorem (Griffiths [1969]). Consider the lifting

$$\Phi_U : U_0 \to D$$

of the period mapping of the projective family $\pi : \tilde{\mathcal{X}} \to U$. Here, U_0 is a simply-connected neighborhood of $f \in U$. Then the alternative formulation has the form:

If the differential $d\Phi_U$ of Φ_U vanishes on the tangent vector $\zeta \in (T_U)_f$, then ζ is contained in the tangent space of the G-orbit of f.

Indeed, the tangent space $(T_S)_{[f]}$ is naturally identified with $(T_U)_f/(T_G)_f \simeq R_f^d$.

4.4. Let us prove the theorem formulated in the previous section. Let $d_0 = \Phi_V([f])$, and let

$$v : T_{[f]} \to T_{d_0}$$

be the differential of the map Φ_V at the point $[f]$. Let us show that this differential is nondegenerate under the hypotheses of the theorem. Let us recall (Chapter 2, Section 1) that there exists an embedding

$$T_{d_0} \hookrightarrow \bigoplus_{p=1}^{n} \operatorname{Hom}(F^p, F^0/F^p).$$

By Theorem 1 of Chapter 2

$$\operatorname{Im} v \subset \bigoplus_{p=1}^{n} \operatorname{Hom}(F^p/F^{p+1}, F^{p-1}/F^p).$$

Thus, we have a mapping

$$v = \oplus v_p : T_{[f]} \mapsto \bigoplus_{p=1}^{n} \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p).$$

The spaces F^p/F^{p+1} and F^{p-1}/F^p can be identified, thanks to theorem 4.1, with the homogeneous components $R_f^{t_p}$ and $R_f^{t_{p-1}} = R_f^{t_p+d}$ of the ring R_f. Furthermore, as previously observed, $T_{[f]}$ can be identified with R_f^d.

Lemma 4.4. *The map*

$$v_p : T_{[f]} \to \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p),$$

with the identifications indicated above, is induced by the multiplication

$$R_f^d \times R_f^{t_p} \to R_f^{t_p+d}$$

in the ring R_f.

Proof. Let $g \in S^d$, $A \in S^{t_p}$, and $\overline{g}$, $\overline{A}$ are images of g and A in $R^d \simeq T_{[f]}$ and $R^{t_p} \simeq F^p/F^{p+1}$ respectively. To compute $v_p(\overline{g})(\overline{A})$ it is sufficient to differentiate the form

$$\omega_t = \frac{A\Omega}{(f+tg)^{n+1-p}}$$

with respect to t at $t = 0$. Up to multiplication by a scalar we obtain the form $gA\Omega/f^{n+2-p}$.

Now it is a simple matter to finish the proof of the main theorem. Using the notation of Lemma 4.1 we have $\sigma = (n+2)(d-2)$. Let $t_a = \min_{t_p \geq 0} t_p$, then for n and d satisfying the hypotheses of the theorem it is not hard to check that

$$d + t_a \leq \sigma.$$

But this implies that the pairing

$$R^d \times R^{t_a} \to R^{t_{a-1}} = R^{t_a+d}$$

is nondegenerate, hence the map

$$v_a : T_{[f]} \to \mathrm{Hom}(F^a/F^{a+1}, F^{a-1}/F^a)$$

is injective.

4.5. Now let us consider the period mapping

$$\Phi : \mathfrak{M} \to G_{\mathbb{Z}}\backslash D,$$

defined in Chapter 2, Section 5. The following is proved by Donagi [1983]:

Theorem 4.2. (The generic Torelli theorem for hypersurfaces). *The map Φ is injective at a generic point, with the possible exception of the following cases*

(0) $n = 2$, $d = 3$;
(1) d *divides* $n + 2$;
(2) $d = 4$, $n = 4m$ *or* $d = 6$, $n = 6m + 1$.

The result is known to be false in the case (0) and open in cases (1) and (2). We will not give a complete proof, but rather will give the basic idea and examine some special cases.

4.6. Consider the infinitesimal variation of Hodge structures associated to the period mapping Φ (see Chapter 2, Section 6). It is enough to show that ivHs at a generic point $[f] \in \mathfrak{M}$ allows us to reconstruct the hypersurface X_f (see Chapter 2, Sections 6 and 4).

The period mapping brings out the interaction of two kinds of structures on the cohomology $P^n(X, \mathbb{C})$:

(a) algebraic – the filtration on the space $P^n(X)$ and the inner product on that space satisfying certain conditions.

(b) transcendental – the lattice $P^n(X, \mathbb{Z}) \subset P^n(X)$.

It turns out that in order to prove Theorem 4.2 is it sufficient to use the algebraic structure alone. More precisely, the ivHs allows us to reconstruct a filtration and a bilinear form on $P^n(X)$, which, in turn, allow us to determine the hypersurface uniquely.

In order to formulate the main idea, we will need one more statement about the bilinear form induced by the $\wedge$-product on $P^n(X)$ obtained by Carlson–Griffiths [1980].

Lemma 4.5. *Let us identify the space F^a/F^{a+1} with the homogeneous components $R_f^{t_a}$ of the ring R_f, just as in Theorem 4.1. Then the bilinear form*

$$F^a/F^{a+1} \times F^{n-a}/F^{n-a+1} \to H^{2n}(X),$$

generated by the $\wedge$-product, is naturally identified with the ring multiplication

$$R_f^{t_a} \times R_f^{t_{n-a}} \to R_f^{(d-2)(n+2)} = R_f^{a+t_{n-a}}.$$

Now, suppose that we are given an ivHs at some point $[X] \in \mathfrak{M}$, associated with the period mapping Φ, just as in Chapter 2, Section 6.2. In view of the Lemmas 4.4 and 4.5, it can be assumed that for integer values $t_i = (i+1)d - (n+2)$, $0 \leq i \leq n$, and also for the number d we have vector spaces W^{t_i} and W^d and bilinear maps

$$\begin{aligned} &v_i : W^d \times W^{t_i - d} \to W^{t_i}, \\ &Q_i : W^{t_i} \times W^{\sigma - t_i} \to \mathbb{C}, \quad \sigma = (d-2)(n+2). \end{aligned} \tag{3}$$

It is known that there exists a homogeneous polynomial $f \in S^{(d)}V$, such that the spaces W^{t_i} and W^d are isomorphic to the components $R_f^{t_i}$ and R_f^d of the graded ring R_f. This isomorphism sends the bilinear forms (3) into the ring products in R_f.

It turns out that in some cases these conditions uniquely determine a homogeneous polynomial f up to a projective coordinate transformation in $\mathbb{P}^{n+1}$. This approach was first used by Carlson–Griffiths [1980] and further developed by R. Donagi [1983] to prove Theorem 4.2.

4.7. To reconstruct the hypersurface X from the ivHs at the point $[X] \in \mathfrak{M}$, an important role is played by the following¿

Lemma 4.6. *If $f, g \in S^{(d)}V$ have the same Jacobian ideal J, then the polynomials f and g are the same up to a projective transformation of $\mathbb{P}^{n+1}$.*

The proof of Lemma 4.6 is based on the following statement (see Donagi [1983]). Let G be a Lie group, acting on a variety T, $U \subset Y$ be a connected, locally closed subset, such that

(a) for each $x \in U$, $(T_U)_x \subset (T_{G_x})_x$, that is, U is tangent at each point to the G-orbit of the point.

(b) $\dim_{\mathbb{C}}(T_{G_x})_x$ does not depend on $x \in U$.

Then U is contained in an orbit of G.

Let us apply this statement to $Y = S^{(d)}V$, $G = \mathrm{GL}(V^*)$, $\overline{U} = \{tf+(1-t)g\}$, $t \in \mathbb{C}$. Identify the tangent space to Y with $S^{(d)}V$, then by Lemma 4.3

$$(T_{Gf})_f = J_f^d = J_g^d, \quad (T_{\overline{U}})_f = f - g \in J^d,$$

thus (a) holds for any Zariski-open subset $U \subset \overline{U}$. Furthermore, elt $x = tf + (1-t)g$. Then

$$(T_{Gx})_x = J_x^d \subset J_f^d + J_g^d = J^d,$$

thus the function $\dim_{\mathbb{C}}(T_{Gx})_x$ is semicontinuous on $\overline{U}$ and achieves its maximum at points f and g. Thus, there exists a Zariski-open subset $U_1 \in U$, such that $f, g \in U_1$ and

$$(T_{Gx})_x = (T_{Gf})_x,$$

for any $x \in U$. Thus, (b) also holds for U.

4.8. We will give the proof of Theorem 4.2 in some very simple special cases. The main ideas of the required constructions come from Carlson–Griffiths [1980].

Let t be the smallest non-negative number among $t_i = (i+1)d - (n+2)$. From the hypothesis of the theorem it can be claimed that $1 \leq t \leq d-1$. However, we will study only the case

$$\begin{aligned} t &< d-1; \\ 2t &\geq d-1. \end{aligned} \tag{4}$$

Lemma 4.7. *Suppose the conditions (4) hold and the isomorphism*

$$\lambda_t : R^t \to W^t$$

can be uniquely reconstructed from the conditions (3). Then the Jacobian ideal J_f can also be uniquely reconstructed.

Before starting the proof of this lemma, note that the condition $t < d-1$ implies that $R^t = R^t_f = S^{(t)}V$ does not depend on f. Fixing the isomorphism λ_t is essentially equivalent to equipping W^t with a polynomial structure, i.e., an isomorphism $S^{(t)}W \simeq W^t$.

Let us prove Lemma 4.7. By repeatedly applying the maps (3) we get a map

$$W^t \times W^d \times \ldots \times W^d \to W^{\sigma-t}.$$

Combining this map with polarization, obtain a multilinear map

$$W^t \times W^t \times W^d \times \ldots \times W^d \to W^d \times W^{\sigma-t} \to \mathbb{C},$$

which generates a bilinear map

$$S^2W \times S^p(W^d) \to \mathbb{C}.$$

The pairing above defines a linear map

$$\phi : S^2W^t \to [S^p(W^d)]^*.$$

We know that for all $a = t_i, d$ there are isomorphisms

$$\lambda_a : R^a_f \simeq W^a,$$

compatible with the bilinear maps (3). Composing ϕ on the left with the known isomorphism λ_t and the unknown isomorphism λ_d get a map

$$\psi : S^2(S^{(t)}V) = S^2R^t \to [S^p(R^d_f)]^*.$$

Since ψ must be induced by the ring product in R_f, this map can be composed with the multiplication map

$$\mu : S^2(S^{(t)}V) \to S^{(2t)}V$$

and by lemma 4.2

$$\psi^{-1}(0) = \mu^{-1}(J^{2t}).$$

Thus, $J^{2t} = \mu(\operatorname{Ker}\psi)$ is uniquely reconstructible, and by Macaulay's theorem $J^{d-1} \subset S^{d-1}V$ can also be reconstructed uniquely (we used the hypothesis $2t \geq d-1$). Since J^{d-1} generates J the lemma is proved.

Corollary. *Theorem 4.2 is true for cubics of dimension $n = 3m$.*

Indeed, in this case $d = 3$, $t = 1$, and any two isomorphisms $S^1V = V$ and W^1 are equivalent up to a projective transformation of $\mathbb{P}^{n+1}$. We will describe some other simple cases where the isomorphism λ_t is uniquely reconstructed ip to a projective transformation V. Recall that $V = H^0(\mathbb{P}^{n+1}, \mathcal{O}(1))$, thus $\mathbb{P}(V) = \mathbb{P}^{n+1}$. Consider the Veronese embedding

$$s : \mathbb{P}(V) \simeq S \subset \mathbb{P}(S^t V),$$

defined by the complete linear system $|\mathcal{O}_{\mathbb{P}(V)}(t)|$. The following statement is self-evident.

Lemma 4.8.

(1) $\mathrm{Ker}(\mu)$ *is a quadric system in* $\mathbb{P}(S^t V)$ *containing the Veronese variety* S*;*
(2) S *is a basis set of the system* $\mathrm{Ker}(\mu)$*;*
(3) $\mathrm{Ker}(\mu)$ *is generated by quadrics of rank* 4.

Recall that $\mu : S^2(S^t V) \to S^{2t} V$ is the multiplication map.

Let W be a vector space, $\lambda : S^t V \simeq W$ an isomorphism which we would like to reconstruct up to an isomorphism of V. This isomorphism will be called the polynomial structure on W.

Lemma 4.9. *A polynomial structure on* W *is defined by a linear subspace*

$$\lambda_*(\mathrm{Ker}(\mu)) \subset S^2 W$$

or by the image $\lambda_*(S) \subset \mathbb{P}(W)$ *of the Veronese mapping.*

To prove the lemma, note that we can assume that $\lambda_*(S) \subset \mathbb{P}(W)$ is known. Indeed, if we know $\lambda_*(\mathrm{Ker}(\mu))$, then $\lambda_*(S)$ can be reconstructed as the basis set of the quadric system. Fix some isomorphism

$$p : \mathbb{P}(V) \to T = \lambda_*(S).$$

Consider the isomorphisms

$$W = H^0(\mathbb{P}(W), \mathcal{O}(1)) \overset{q}{\simeq} H^0(T, \mathcal{O}_T(1))^{p^*} \to H^0(\mathbb{P}(V), \mathcal{O}(t)) = S^t V.$$

The isomorphism q is given, since the embedding $T \subset \mathbb{P}(W)$ is defined. The composition map is the sought-after isomorphism λ, determined up to automorphism.

Corollary. *Theorem 4.2 is holds when* $d = 2n + 3$, $n \geq 2$.

Proof. In order to reconstruct the subspace

$$(\lambda_t)_*(\mathrm{Ker}(\mu)) \subset S^2 W^t,$$

we shall use the map $\phi : S^2 W^t \to [S^p(W^d)]^*$ as in the proof of Lemma 4.7.

Let $K_4 \subset S^2 W^t$ be the set of quadrics of rank 4. Then, if $X \subset \mathbb{P}(V)$ is a sufficiently general hypersurface, it can be shown that

$$(\lambda_t)_*(\mathrm{Ker}(\mu)) = \mathcal{L}[\mathrm{Ker}(\phi) \cap K_4],$$

where $\mathcal{L}(U)$ is the vector space spanned by the set U. Indeed, the last assertion is equivalent to saying that

$$\mathrm{Ker}(\mu) = \mathcal{L}[\mathrm{Ker}(\psi) \cap K_4'],$$

where ψ is the map used in the proof of Lemma 4.7, and $K_4' \subset S^2(R^t)$ is the set of quadrics of rank 4. In this case the inclusion in one of the direction is proved in Lemma 4.8. Since for $d = 2n+3$

$$\mu(\mathrm{Ker}(\psi)) = J^{2t} = J^{d-1},$$

the reverse inclusion is equivalent to saying that

$$\mu(K_4') \cap J^{d-1} = \{0\}.$$

For a generic homogeneous polynomial $f \in S^{(d)}V$ this follows from a simple dimension count.

4.9. In this section X is a nonsingular complex hypersurface of degree 3 in $\mathbb{P}^5$ – a four-dimensional cubic. This case is not covered by the theorem of Donagi considered above. Nonetheless, Voisin [1986] succeeded in obtaining an even stronger result than Donagi's in that case – a global Torelli theorem.

Consider the free $\mathbb{Z}$-module $H = H^4(X, \mathbb{Z})$. Let $Q : H \times H \to \mathbb{Z}$ be the intersection form. Then, H is a unimodular Euclidean lattice (see Section 3.3). Let h be the cohomology class in $H^2(X, \mathbb{Z})$ generated by a hyperplane section of the cubic in $\mathbb{P}^5$. Then the primitive cohomology classes in $H^4(X)$ are those elements η for which $\eta \cdot h^2 = 0$.

The Hodge structure on $H^4(X) = H \otimes \mathbb{C}$ has the form

$$H_{\mathbb{C}} = H^{3,1} + H^{2,2} + H^{1,3},$$

where $\dim H^{3,1} = \dim H^{1,3} = 1$, $\dim H^{2,2} = 21$.

It follows that specifying a polarized Hodge structure is equivalent to specifying a one-dimensional subspace $\mathbb{C}\eta \subset H_{\mathbb{C}}$, defined by an η such that

$$\eta \cdot \eta = 0, \quad \eta \cdot h^2 = 0, \quad \eta \cdot \overline{\eta} > 0.$$

Therefore, D is a 20-dimensional complex manifold. We saw in Section 4.3 that the tangent space to the moduli space $\mathfrak{M}$ of nonsingular complex hyperplanes of degree d at a generic point $[f]$ is naturally identified with R_f^d. A simple computation shows that for the four-dimensional cubic, the dimension of $\mathfrak{M}$ is 20. The coincidence between the dimensions of D and of $\mathfrak{M}$ is one of the reasons why Donagi's method fails in this case – the ivHs gives us no new information.

The global Torelli theorem is obtained by Voisin [1986] in the following form.

Theorem. *Let X, X' be two non-singular cubic fourfolds and let*

$$i : H^4(X, \mathbb{Z}) \to H^4(X', \mathbb{Z})$$

be an isometry of Euclidean lattices preserving the class h^2 and inducing an isomorphism on Hodge structures

$$i \otimes \mathbb{C} : H^4(X, \mathbb{C}) \to H^4(X', \mathbb{C}).$$

Then there exists an isomorphism $I : X' \to X$ of the hypersurfaces, such that $i = I^*$.

In conclusion, let us note that the extremely interesting question of the rationality of the generic cubic fourfold remains open. There are examples of nonsingular rational cubics in $\mathbb{P}^5$, for example, the cubics containing two non-intersecting planes L_1 and L_2 (an 18-dimensional family in moduli space). Indeed, consider the points x_1 and x_2 on the planes L_1 and L_2. Construct a line between them. This line will, in general, contain another intersection with X. The map $(x_1, x_2) \mapsto x_3$ defines a birational isomorphism between $\mathbb{P}^2 \times \mathbb{P}^2$ and X. There are no known examples of irrational cubics.

§5. Counterexamples to Torelli Theorems

5.1. In the preceding sections of this Chapter we saw some of the theorems of Torelli type proved to date. The proof of each one of these theorems, as we have observed, is based on a deep understanding of the geometry of the objects in question. This is not surprising, since if a point in the classifying space uniquely determines an algebraic variety, then all of the properties of the variety must somehow be contained in that point. This is most clearly seen in the classical case of the Torelli theorem for algebraic curves (Section 1), which gives an explicit method for the reconstruction of the curve by its image in the classifying space (more precisely, by the Griffiths torus, the Jacobian of the curve). It should be noted that a large number of the properties of algebraic curves can be easily obtained from the properties of the embedding of the curve into its Jacobian – an object intimately connected with the appropriate point in classifying space. The remarkable work of Piatetsky-Shapiro–Shafarevich [1971], which contains the proof of the global Torelli theorem for K3 surfaces (Section 3) has helped us to get a whole new understanding of these objects. It gave rise to a whole field of study of automorphisms of K3 surfaces and Enriques surfaces. The proofs of future Torelli theorems are surely expected to give us new insights.

On the other hand, at present it is not at all clear how wide the class of varieties is for which we have any right to expect Torelli theorems to hold. The one thing that is clear, is that they certainly do not always hold. In this section we will examine some known counterexamples.

5.2. There is a large collection of counterexamples to the infinitesimal Torelli theorem. The simplest and most interesting one of these is in the case of projective curves. This example is even more interesting since in this case the global Torelli theorem holds (see Section 1)!

Let X be a non-singular projective curve of genus $g > 1$ (see Section 1). Since $\dim X = 1$, $H^2(X, T_X) = 0$, and so there exists a universal Kuranishi family $f : \mathcal{X} \to S$ of the curve $X \simeq f^{-1}(s_0)$ (see Chapter 2, Section 5.3). This family has a nonsingular base S, $\dim S = \dim_{\mathbb{C}} H^1(X, T_X) = \dim H^2(X, 2K_X) = 3g - 3$. By Griffiths' criterion (Chapter 2, Section 5.5) in order to check whether the infinitesimal Torelli theorem holds, we can check the surjectivity of the map

$$\mu : H^0(X, \Omega^1_X) \otimes H^0(X, \Omega^1_X) \to H^0(X, S^2\Omega^1_X).$$

Consider the canonical map

$$i : X \to \mathbb{P}^{g-1}$$

(see Section 1.1). If μ is an epimorphism, the preimages of the quadrics in $\mathbb{P}^{g-1}$ form a complete linear system $2K_X$ on X. If X is a hyperelliptic curve of genus $g \geq 3$, then this is not so. For example, if $g = 3$, the dimension of the space of quadrics $H^0(\mathbb{P}^2, \mathcal{O}(2))$ in $\mathbb{P}^2$ is equal to 3. The dimension of $H^2(X, 2K_x)$ is also equal to 3. However, if $i(X) \subset \mathbb{P}^2$ is a quadric, then an the element of $H^0)\mathbb{P}^2, \mathcal{O}(2))$ defining $i(X)$ evidently vanishes on X. If, on the other hand, X is not hyperelliptic, then μ is an epimorphism. This is the classical theorem of Nöether-Castelnuovo (Griffiths–Harris [1978]).

Thus, the infinitesimal Torelli theorem holds for non-hyperelliptic curves, and does not hold for hyperelliptic curves of genus $g > 2$. This can be explained as follows. The base S of the Kuranishi family, isomorphic to the $3g - 3$-dimensional polydisk, is not isomorphic, in general, to a neighborhood U of the point $[X]$ of the moduli space $\mathfrak{M}_g$ (see Chapter 2, Section 5.4; Chapter 3, Section 1) corresponding to the curve X.

For a general hyperelliptic curve of genus $g > 2$, the neighborhood U is locally isomorphic to the quotient of S by the involution corresponding to the canonical involution of the curve X. More precisely, U has the local structure of $\mathbb{C}^{2g-1} \times (\mathbb{C}^{g-2}/(\mathbb{Z}/2\mathbb{Z}))$, where the group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}^{g-2}$ by multiplying all of the coordinates by -1. The subvariety $\mathbb{C}^{2g-1}$ of S corresponds to the hyperelliptic curves. The differential of the period mapping is injective on the subspace $V \subset (T_S)_{s_0}$ generated by this subvariety, and vanishes on a subspace of dimension $g - 2$.

5.3. There are also counterexamples to the infinitesimal Torelli theorem for certain surfaces of general type (Kynev [1977], Catanese [1979], Todorov [1980]). For these surfaces the global Torelli theorem also fails (Catanese [1980], Chakiris [1980]). Let us describe some of these results in greater detail.

We will be studying a non-singular projective variety X of dimension 2 (a surface), satisfying the following conditions:

(1) X is simply connected, that is $H^1(X, \mathbb{Z}) = 0$.

(2) The geometric genus $p_g = \dim_{\mathbb{C}} H^0(X, \Omega^2_X)$ of X equals 1.

(3) The canonical class K_X is ample, and $K^2_X = 1$. From Noether's formula (Griffiths–Harris [1978]) for surfaces

$$\frac{1}{12}(K_X^2 + \chi) = 1 - h^{1,0} + h^{2,0}$$

it follows that the Euler characteristic χ of X equals 23. The Hodge structure of weight 2 on $H^2(X)$ thus has the form

$$H^2(X) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2},$$

where $\dim H^{2,0} = \dim H^{0,2} = 1$, $\dim H^{1,1} = 19$.

Let $\omega = c_1(K_X)$; then the classifying space D of Hodge structures of weight 2, corresponding to the polarized variety (X, ω) is constructed as follows. Let $H = H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^{21}$, and let $k \in H$ be the class corresponding to the polarization ω. Let Q be the bilinear form on H (see Chapter 2, Section 1) corresponding to the intersection form. The Hodge structure on $H_{\mathbb{C}} = H \otimes \mathbb{C}$ is defined by choosing a non-zero element $\zeta \in H^{2,0}$. Since the cycle k is algebraic, and since ζ corresponds to a $(2,0)$ form, it follows that

(a) $Q(\zeta, k) = 0$. The Hodge-Riemann bilinear relations (see Chapter 2, Section 1) give us further

(b) $Q(\zeta, \zeta) = 0$;

(c) $Q(\zeta, \bar{\zeta}) > 0$.

These constraints define an open set D on a quadric in $\mathbb{P}^{19}$. Thus, D is an 18-dimensional complex manifold.

Furthermore, the Kuranishi family of any such surface has a nonsingular base of dimension $\dim H^1(X, T_X) = 18$ (this is so since $H^2(X, T_X) = 0$, Todorov [1980]).

Any such surface can be realized as the complete intersection of two smooth hypersurfaces in the weighted projective space $\mathbb{P}^4(1,2,2,3,3)$ (see Todorov [1980]). This proves the existence of a coarse moduli space $\mathfrak{M}$ of such surfaces (see Chapter 2, Section 5.4). The space $\mathfrak{M}$ is a connected quasi-projective variety of dimension 18.

Consider the period mapping

$$f : \mathfrak{M} \to \Gamma \backslash D$$

(see Chapter 2, Section 5). Chakiris [1980] shows that for certain points $d \in \Gamma \backslash D$, the preimage $f^{-1}(d)$ has positive dimension. This gives a counterexample to the global Torelli theorem.

The idea of the proof is as follows. Construct a smooth family $\phi : \mathcal{X} \to S$ of surfaces over the polydisk S satisfying the following conditions:

(1) For any $s \in S$ the surface X_s satisfies conditions (a)-(c).

(2) $\dim S = 15$.

(3) For $s_1, s_2 \in S$, $s_1 \neq s_2$, the surfaces X_{s_1} and X_{s_2} are not birationally isomorphic.

(4) There exist five linearly independent elements $k_1, \ldots, k_5 \in H^2(X_s, \mathbb{Z})$ corresponding to algebraic cycles on X_s for all $s \in S$.

Here we use the identification of the groups $H^2(X_s, \mathbb{Z})$ for all fibers X_s by monodromy, and the simple-connectivity of S.

Condition (3) guarantees that the natural embedding of S into the moduli space $\mathfrak{M}$ is an embedding on the set of closed points. The composed embedding of S into $\Gamma\backslash D$, which coincides with the image of the image under the period map of the family of polarized varieties $(X_s, c_1(K_{X_s}))$ has image of dimension at most 14. Indeed, since the cycles h_j are algebraic, it follows that $h_j \in H^{1,1}$ (see Chapter 1, §8), and so by the Hodge–Riemann relations we see that

$$Q(\xi, h_j) = 0, \quad j = 1, \ldots, 5.$$

These conditions limit the possible dimension of the image to 14.

Chapter 4. Mixed Hodge Structures

§1. Definition of Mixed Hodge Structures

In Chapter 1 it was shown that the cohomology of Kähler manifolds (and thus also projective varieties) comes equipped with Hodge structure. With the help of Chow's lemma (see Chapter 1, §3) the existence of Hodge structure on the cohomology of projective varieties can be extended to the case of nonsingular complete algebraic varieties. However, if X is an incomplete, or non-smooth variety, then one cannot say anything about the existence of Hodge structure of weight n on the cohomology group $H^n(X, \mathbb{C})$. For example, if $X = \mathbb{C}^1 \setminus \{0\}$ is the punctured line, then $\dim H^1(X) = 1$, and thus there is no Hodge structure on $H^1(X)$.

In order to understand this phenomenon better, let us study the following example.

1.1. Example. Let the curve X consist of two components X_1 and X_2, intersecting transversally in two points Q_1 and Q_2. In addition, assume that the curves X_i are incomplete, and are obtained from complete curves $\overline{X_i}$ by removing a collection of points $P_1, \ldots, P_N$. Since the cohomology group $H^1(X)$ is dual to the homology group $H_1(X)$, let us study $H_1(X)$ for simplicity. The elements of the group $H_1(X)$ are of three kinds. The first kind are the loops α_i around the punctures P_i. The second kind are the elements β_j, which came from $H_1(\overline{X_i})$. These elements are determined only modulo elements of the first kind. Finally, the remaining elements in $H_1(X)$ can be represented modulo elements of the first two kinds by a combinatorial cycle γ : first we travel from Q_1 to Q_2 along some path in X_1, and then return from Q_2 to Q_1 along a path in X_2.

This reasoning suggests that there may be a filtration on $H^1(X)$ of the form

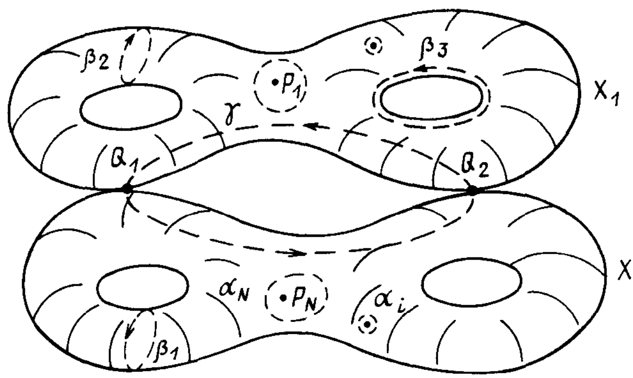


Fig. 6

$$0 \subset W_0 \subset W_1 \subset W_2 = H^1(X)$$

such that the factors W_i/W_{i-1} come from the cohomology of smooth complete varieties, and thus it might be possible to introduce Hodge structure on the quotients W_i/W_{i-1}.

1.2. The Hodge structure defined in Chapter 2, Section 1 shall be called the *pure Hodge structure.*

A *morphism of pure Hodge structures of type* (p, q) is a $\mathbb{Q}$-linear mapping $\phi : H_1 \to H_2$ such that

$$\phi(H_1^{r,s}) \subseteq H_2^{r+p,s+q},$$

where $H_i = \oplus_{r+s=n_i} H_i^{r,s}$ are the Hodge decompositions.

A morphism of pure Hodge structures of type (p, q) is a *strict morphism*, that is

$$\phi(H_1^{r,s}) = \phi(H_1) \cap H_2^{r+p,s+q}.$$

A morphism of pure Hodge structures of type (l, l) will be called a *morphism of weight* $2l$. On the level of Hodge filtrations a morphism of pure Hodge structures of weight $2l$ satisfies the condition

$$\phi(F_1^p) \subseteq F_2^{p+l}$$

and is also strict, that is

$$\phi(F_1^p) = \phi(H_1) \cap F_2^{p+l}.$$

It is not too hard to check the following lemma.

Lemma. *The kernel, the cokernel and the image of a morphism of pure Hodge structures are pure Hodge structures.*

1.3. A mixed Hodge structure on a finite-dimensional vector space $H_{\mathbb{Q}}$ consists, by definition, of the following:

(i) An increasing (*weight*) filtration W on $H_{\mathbb{Q}}$,

(ii) A decreasing filtration (also called the *Hodge filtration*) F on $H = H_{\mathbb{Q}} \otimes \mathbb{C}$, satisfying the following condition: the filtration F induces a pure Hodge structure of weight n on $\mathrm{Gr}_n^W H = W_n \otimes \mathbb{C}/W_{n-1} \otimes \mathbb{C}$. More precisely, the elements of the filtration $F^p \mathrm{Gr}_n^W H$ on $\mathrm{Gr}_n^W H$, induced by the filtration F are

$$F^p \mathrm{Gr}_n^W H = (F^p \cap W_n \otimes \mathbb{C} + W_{n-1} \otimes \mathbb{C})/W_{n-1} \otimes \mathbb{C}.$$

It should be noted that the concept of a pure Hodge structure is a special case of the concept of a mixed Hodge structure. Indeed, if there exists a pure Hodge structure of weight n, defined by the filtration F on a space H, then the filtrations W and F define a mixed Hodge structure on H by setting

$$0 = W_{n-1} \subset W_n = H.$$

In the sequel, whenever we study several spaces with mixed Hodge structures, then the elements of the filtrations W and F on each of these spaces H will be denoted, correspondingly by $W_i H$ and $F^p H$. In addition, whenever it is clear from context, the spaces $W_i \otimes \mathbb{C}$ will also be denoted by W.

1.4.

Definition. *A morphism of mixed Hodge structures of weight* $2l$ is a $\mathbb{Q}$-linear map $\phi : H_1 \to H_2$ of spaces equipped with Hodge structures, which is compatible with the filtrations, that is

$$\phi(W_p H_1) \subseteq W_{p+2l} H_2,$$

$$\phi(F^p H_1) \subseteq F^{p+l} H_2.$$

Note that if $\phi : H_1 \to H_2$ is a morphism of mixed Hodge structures of weight $2l$, then the induced map

$$\phi_n : \mathrm{Gr}_m^W H_1 \to \mathrm{Gr}_{m+2l}^W H_2$$

is a morphism of pure Hodge structures of weight $2l$.

1.5. Let (H, W, F) be a mixed Hodge structure. Then $W_n H$ and $H/W_n H$, together with the filtrations induced by the filtrations W and F are mixed Hodge structures, and the natural maps $i : W_n \hookrightarrow H$ and $j : H \to H/W_n H$ are morphisms of mixed Hodge structures of weight 0.

1.6. If H_1 and H_2 are mixed Hodge structures, then, by equipping $H_1 \otimes H_2$ with filtrations

$$W_n(H_1 \otimes H_2) = \sum_{i+j \leq n} W_i H_1 \otimes W_j H_2,$$

and

$$F^n(H_1 \otimes H_2) = \sum_{i+j \geq n} F^j H_1 \otimes F^j H_2,$$

we turn $H_1 \otimes H_2$ into a space with a mixed Hodge structure. Note that

$$\mathrm{Gr}_n^W(H_1 \otimes H_2) = \bigoplus_{i+j=n} (\mathrm{Gr}_i^W H_1 \otimes \mathrm{Gr}_j^W H_2).$$

1.7. The field of rational numbers $\mathbb{Q}$, viewed as a $\mathbb{Q}$-vector-space, can be equipped with a mixed Hodge structure, denoted by $\mathbb{Q}(n)$. The weight filtration on $\mathbb{Q}(n)$ consists of $W_{-2n-1} = 0$ and $W_{-2n} = \mathbb{Q}$, while the Hodge filtration is $F^{-n} = \mathbb{C}$, and $F^{-n+1} = 0$. This is a pure Hodge structure, and $H = \mathbb{Q}(n) \otimes \mathbb{C} = H^{-n,-n}$.

1.8. Let (H, W, F) be a mixed Hodge structure. Let $H(n) = H \otimes \mathbb{Q}(n)$. On $H(n)$ the elements of the weight filtration are $W_p H(n) = W_{p+2n} H$, while the elements of the Hodge filtration are $F^p H(n) = F^{p+n} H$. The identity map $\mathrm{id} : H(n) \to H$ is a morphism of mixed Hodge structures of weight $2n$.

1.9. The dual space $H_{\mathbb{Q}}^{\vee} = \mathrm{Hom}(H_{\mathbb{Q}}, \mathbb{Q})$ can be equipped with the mixed Hodge structure

$$W^p H^{\vee} = \{x \in H^{\vee} | x(y) = 0, \forall y \in W_{-p-1} H\},$$

and

$$F^p H^{\vee} = \{x \in H^{\vee} | x(y) = 0, \forall y \in F^{1-p} H\}.$$

1.10. If there are mixed Hodge structures on H_1 and H_2 then we can also introduce a mixed Hodge structure on $\mathrm{Hom}(H_1, H_2) = \mathrm{Hom}(H_1, \mathbb{Q}) \otimes H_2$. It is easily checked that

$$F^p(\mathrm{Hom}(H_1, H_2)) = \{\phi \in \mathrm{Hom}(H_1, H_2) | \phi(F^i H_1) \subseteq F^{i+p} H_2, \forall i\},$$

and

$$W_q(\mathrm{Hom}(H_1, H_2)) = \{\phi \in \mathrm{Hom}(H_1, H_2) | \phi(W_j H_1) \subseteq W_{j+2q} H_2, \forall j\}.$$

1.11. An extremely important and remarkable aspect of the theory of mixed Hodge structures is the fact that the compatibility of a morphism of mixed Hodge structures with the filtrations implies its *strict compatibility,* that is the following holds:

Proposition (Deligne [1972]). *Let $\phi : H_1 \to H_2$ be a morphism of mixed Hodge structures of weight $2l$. Then ϕ is a strict morphism, that is*

$$\phi(W_p H_1) = W_{p+2l} H_2 \cap \phi(H_1),$$

and

$$\phi(F^p H_1) = F^{p+l} H_2 \cap \phi(H_1).$$

The proof of this proposition is based on the observation that although a space with a mixed Hodge structure does not admit a decomposition into a direct sum of (p,q) components, as is the case for pure Hodge structures, if we set

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F}^q \cap W_{p+q} + \sum_{i \geq 1} \overline{F^{q-i}} \cap W_{p+q-i-1}),$$

the following holds:

Lemma (Griffiths–Schmid [1975])

1. $I^{p,q} \equiv \overline{I^{q,p}} \bmod W_{p+q-2}$,
2. $W_m = \bigoplus_{p+q \leq m} I^{p,q}$,
3. $F^p = \bigoplus_{i \geq p} \bigoplus_q I^{i,q}$,
4. *The projection $W_m \to \mathrm{Gr}_m^W H$ isomorphically maps $I^{p,q}$ with $m = p+q$ onto the Hodge subspace $(\mathrm{Gr}_m^W H)^{p,q}$.*
5. *If $\phi : H_1 \to H_2$ is a morphism of mixed Hodge structures of weight $2l$, then $\phi(I^{p,q} H_1) \subseteq I^{p+l,q+l} H_2$.*

1.12. Using the proposition and the lemma above, it is easy to obtain the following proposition:

Proposition. *Let $\phi : H_1 \to H_2$ be a morphism of mixed Hodge structures of weight $2l$. Then, the induced weight filtrations and Hodge filtrations on* $\operatorname{Ker}\phi$ *and* $\operatorname{Coker}\phi$ *define mixed Hodge structures.*

Corollary. *Let*

$$0 \to H_1 \xrightarrow{\alpha} H \xrightarrow{\beta} H_2 \to 0$$

be an exact sequence of morphisms of mixed Hodge structures, with α of weight $2l$ and β of weight $2r$. Then the induced sequence

$$0 \to \mathrm{Gr}_{n-2i}^W H_1 \to \mathrm{Gr}_n^W H \to \mathrm{Gr}_{n+2r}^W H_2 \to 0$$

is exact.

Proof. To prove the corollary, it is enough to check the exactness of the sequence

$$0 \mapsto W_{n-2l} H_1 \xrightarrow{\bar{\alpha}} W_n H \xrightarrow{\bar{\beta}} W_{n+2r} H_2 \to 0.$$

Obviously, $\tilde{\alpha} : W_{n-2l}H_1 \to W_nH$ is an inclusion. The epimorphicity of $\tilde{\beta}$ follows from the strictness of the morphism β :

$$\tilde{\beta}(W_nH) = W_{n+2r}H_2 \cap \beta(H) = W_{n+2r}H_2,$$

since β is an epimorphism. It remains to check the exactness at the middle term:

$$\tilde{\alpha}(W_{n-2l}H_1) = \alpha(H_1) \cap W_nH = \operatorname{Ker}\beta \cap W_nH = \operatorname{Ker}\tilde{\beta}.$$

1.13. The importance of the concept of mixed Hodge structures introduced above is explained by the following theorem of Deligne:

Theorem (Deligne). *Let X be an algebraic variety defined over $\mathbb{C}$. Then the cohomology groups $H^n(X, \mathbb{C})$ can be equipped with a natural mixed Hodge structure. If $f : X \to Y$ is a morphism of algebraic varieties, then*

$$f^* : H^n(Y, \mathbb{Q}) \to H^n(X, \mathbb{Q})$$

is a morphism of mixed Hodge structures of weight 0.

Furthermore, Deligne's mixed Hodge structure possesses a number of nice properties even aside from functoriality. To wit, the mixed Hodge structure coincides with the classical Hodge structure for complete smooth varieties, and it is compatible with algebraic constructions: duality, Künneth formulas, and so on. The essence of of the theory of mixed Hodge structures lies in the observation that while the cohomology classes of an arbitrary algebraic variety have different weights, nontheless cohomology classes with different weights do not interact, and can always be separated.

The proof of Deligne's theorem uses heavily the machinery of spectral sequences, used to compute the cohomology (see Cartan–Eilenberg [1956], Godement [1958], Grothendieck [1957]). Roughly speaking, the construction of the mixed Hodge structure on the cohomology of an arbitrary algebraic variety X can be described as follows. The variety X is represented as the sum or difference of certain nonsingular complete varieties, and the corresponding operations are performed at the chain-complex level. The weight filtration W and the Hodge filtration F arise at the chain level already.

In the next two sections we will sketch the proof of Deligne's theorem in two cases important for applications – if X has normal crossings and is and complete and if X is smooth but not complete. This will also give some idea of the methods used in the proof of Deligne's theorem.

§2. Mixed Hodge Structure on the Cohomology of a Complete Variety with Normal Crossings

2.1.

Definition. A complex manifold V is called a *manifold with normal crossings of dimension n* if for each $x \in V$ there exists a neighborhood U, which can be realized as the union of coordinate hyperplanes in $\mathbb{C}^n$:

$$U \simeq \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} | z_1 \cdot \ldots \cdot z_k = 0, |z_i| < \epsilon\}.$$

Let us try to understand the structure of the cohomology of manifolds with normal crossings by looking at two very simple examples.

In the first example V consists of two smooth components, each a projective variety: $V = V_1 \cup V_2$. In this case we have a resolution for the locally constant sheaf $\mathbb{C}_V$:

$$0 \mapsto \mathbb{C}_V \mapsto \mathbb{C}_{V_1} \oplus \mathbb{C}_{V_2} \mapsto \mathbb{C}_{V_1 \cap V_2} \mapsto 0,$$

which induces the *Mayer-Vietoris long exact sequence* on cohomology:

$$\longrightarrow H^{k-1}(V_1 \cap V_2) \xrightarrow{\gamma_{k-1}} H^k(V) \xrightarrow{\alpha_k} H^k(V_1) \oplus H^k(V_2) \xrightarrow{\beta_k}$$

$$\xrightarrow{\beta_k} H^k(V_1 \cap V_2) \xrightarrow{\gamma_k}$$

The morphism β_k in this sequence looks as follows:

$$\beta_k(\omega_1 \oplus \omega_2) = i_1^*(\omega_1) - i_2^*(\omega_2),$$

where the morphisms $i_j^* : H^k(V_j) \to H^k(V_1 \cap V_2)$ are induced by the inclusions $i_j : V_1 \cap V_2 \to V$.

From the Mayer-Vietoris sequence it is clear that there is a filtration

$$0 \subset W_{k-1} \subset W_k = H^k(V)$$

on $H^k(V)$, where $W_{k-1} = \operatorname{Im} \gamma_{k+1} \simeq \operatorname{Coker} \beta_{k-1}$. Since the varieties V_1, V_2, and $V_1 \cap V_2$ are projective, there are pure Hodge structures of weight k on $H^k(V_1) \oplus H^k(V_2)$ and on $H^k(V_1 \cap V_2)$. Furthermore, it is easy to see that the morphism β_k, thanks to its geometric origins, is a morphism of pure Hodge structures. Therefore, there is a pure Hodge structure of weight $k - 1$ on $W_{k-1} = \operatorname{Coker} \beta_{k-1}$. Further, $W_k / W_{k-1} \simeq \operatorname{Im} \alpha_k \simeq \operatorname{Ker} \beta_k$. Again, since β_k is a morphism of pure Hodge structures, there is a pure Hodge structure of weight k on $\operatorname{Ker} \beta_k$. Thus we see that there is a natural mixed Hodge structure on $H^k(V)$.

Another, analogous, example is one where V is a one-dimensional variety with normal crossings. Let us decompose V into irreducible components: $V = V_1 \cup \ldots \cup V_N$. Let all of the V_i be irreducible complete curves. Denote the intersection of components V_i and V_j by $P_{ij} = V_i \cap V_j$.

Just as in the previous example, the exact sequence

$$0 \to \mathbb{C}_V \to \oplus \mathbb{C}_{V_i} \to \oplus \mathbb{C}_{P_{ij}} \to 0$$

induces a Mayer-Vietoris sequence

$$0 \to H^0(V) \xrightarrow{\alpha} \oplus H^0(V_i) \xrightarrow{\beta} \oplus H^0(P_{ij}) \xrightarrow{\gamma} H^1(V) \xrightarrow{\delta} \oplus H^1(V_i) \to 0,$$

and we see that there is a filtration

$$0 \subset W_0 \subset W_1 = H^1(V)$$

on $H^1(V)$, with $W_0 = \operatorname{Im}\gamma$. The quotient $W_1/W_0 \simeq \oplus H^1(V_i)$ is equipped with a pure Hodge structure of weight 1, induced by pure Hodge structures on $H^1(V_i) = H^{1,0}(V_i) \oplus H^{0,1}(V_i)$ (see Chapter 1, Section 7). Let us note, for future reference, that

$$\dim W_1/W_0 = 2\sum_{i=1}^{N} p_g(V_i),$$

where $p_g(V_i)$ is the genus of the curve V_i.

The space W_0 also has a natural geometric interpretation. Indeed, let $\Gamma(V)$ be the *dual graph* of V, defined as follows. To each curve V_i we associate a vertex $v_i \in \Gamma(V)$, and for each point P_{ij} an edge p_{ij} of the graph $\Gamma(V)$, joining the vertices v_i and v_j. Then, the beginning segment of the Mayer-Vietoris sequence

$$0 \to H^0(V) \to \oplus_i H^0(V_i) \to \oplus_{i<j} H^0(P_{ij})$$

can be interpreted as follows. The space $H^0(V) = \oplus \mathbb{C}$, where the sum is over all of the connected components of the graph $\Gamma(V)$,

$$\oplus H^0(V_i) \simeq \bigoplus_{v_i \in \Gamma(V)} \mathbb{C}, \quad \oplus H^0(P_{ij}) = \bigoplus_{p_{ij} \in \Gamma(V)} \mathbb{C}$$

and, moreover, the morphism β becomes a combinatorial differential

$$\beta(c_1, \dots, c_N) = (d_{ij})$$

where $d_{ij} = c_i - c_j$, where $i < j$. Therefore, the space $W_0 = \operatorname{Im}\gamma = \operatorname{Coker}\beta$ is naturally identified with the cohomology group $H^1(\Gamma(V))$ of the graph $\Gamma(V)$, and, in particular,

$$\dim W_0 = \dim H^1(\Gamma(V)).$$

2.2. Consider the general case. Let $V = V_1 \cup \ldots \cup V_N$, where V_i are irreducible components of a complete algebraic variety with normal crossings V. In the future we will assume, in order to simplify the exposition, that each component V_i is a smooth variety. It should only be noted that if the V_i are not smooth, then they can be replaced by their normalizations, since V_i, in turn, have normal crossings, and the procedure can be iterated.

Let $V_{i_1,\dots,i_k} = V_{i_1} \cap \dots \cap V_{i_k}$, and let $V^{(k)} = \bigsqcup_{i_1<\dots<i_k} V_{i_1,\dots,i_k}$, $k \geq 1$, be the disjoint union taken over all the nondecreasing sequences $i_1, \dots, i_k$, $1 \leq i_1 \leq \dots \leq N$. Let $a_k : V^{(k)} \to V$ be the natural mapping, and let $\delta_j : V^{(k)} \to V^{(k-1)}$ be the mapping which has the structure of an inclusion

$$V_{i_1,\dots,i_k} \hookrightarrow V_{i_1} \cap \dots \cup V_{i_{j-1}} \cap V_{i_{j+1}} \cap \dots \cap V_{i_k}$$

on $V_{i_1,\dots,i_k}$.

Consider the sheaves of C^∞ differential m-forms $\mathcal{E}^m_{V^{(k)}}$ on $V^{(k)}$. Each of the sheaves $\mathcal{E}^m_{V^{(k)}}$ can be decomposed into a direct sum of sheaves $\mathcal{E}^m_{V^{(k)}} = \oplus_{p+q=m}\mathcal{E}^{p,q}_{V^{(k)}}$ of forms of type (p,q). The differentiation of forms $d : \mathcal{E}^m_{V^{(k)}} \to \mathcal{E}^{m+1}_{V^{(k)}}$ can also be decomposed as a sum $d = \partial + \overline{\partial}$, where

$$\partial : \mathcal{E}^{p,q}_{V^{(k)}} \to \mathcal{E}^{p+1,q}_{V^{(k)}},$$

and

$$\overline{\partial} : \mathcal{E}^{p,q}_{V^{(k)}} \to \mathcal{E}^{p,q+1}_{V^{(k)}}.$$

The sheaves $\mathcal{E}^m_{V^{(k)}}$ define a bi-complex

$$\mathfrak{U}^{p,q} = (a_{q+1})_*\mathcal{E}^p_{V^{(k)}}, \quad p,q \geq 0,$$

with differentials $D_1 : \mathfrak{U}^{p,q} \to \mathfrak{U}^{p+1,q}$ (D_1 is the differentiation of forms) and $D_2 : \mathfrak{U}^{p,q} \to \mathfrak{U}^{p,q+1}$, where D_2 is defined as

$$D_2 = \sum_{j=1}^{q+1}(-1)^{p+j}\delta_j^*,$$

thus D_2 is a combinatorial differential.

Let $(\mathfrak{U}^\cdot, D_1 + D_2)$ be the total complex of the bicomplex $(\mathfrak{U}^{\cdot\cdot}, D_1, D_2)$:

$$\mathfrak{U}^m = \bigoplus_{p+q=m} \mathfrak{U}^{p,q}.$$

2.3. The constant sheaf $\mathbb{C}_V$ is contained in $\mathfrak{U}^0 = (a_1)_*\mathcal{E}^0_{V^{(1)}}$. It should be noted that the sheaves $\mathcal{E}^p_{V^{(q+1)}}$ are fine. From this and from the Poincaré Lemma we have:

Lemma (de Rham theorem for varieties with normal crossings). *The sequence*

$$0 \to \mathbb{C}_V \to \mathfrak{U}^*$$

is an acyclic resolution, and thus

$$H^m(V, \mathbb{C}) = H^m(A^\cdot, D),$$

where $H^m(A^*, D)$ is the cohomology of the complex of sections

$$(A^{\cdot} = H^0(V, \mathfrak{U}^{\cdot}), \quad D = D_1 + D_2).$$

2.4. Our goal is to define a mixed Hodge structure on $H^m(V, \mathbb{C})$ by defining two filtrations W and F on $H^m(V, \mathbb{C}) = H^m(A, D)$. It turns out that these filtrations can be introduced already at the level of the bicomplex $(A^{p,q}, D_1, D_2)$. Set

$$\widetilde{W}^p = \bigoplus_r \bigoplus_{s \geq p} A^{r,s}, \tag{3}$$

$$F^p = \bigoplus_{r,s} F^p(A^{r,s}), \tag{4}$$

where $F^n(A^{r,s})$ is the usual Hodge filtration on the differential r-forms of the $n - s$-dimensional manifold, defined by the decomposition of the forms into (p, q) types.

The filtration $\widetilde{W}$ induces a decreasing filtration $\widetilde{W}$ on $H^m(V, \mathbb{C})$. Define an increasing filtration W on $H^m(V, \mathbb{C})$, by setting $W_p = \widetilde{W}^{m-p}$.

Theorem 1. *The filtrations W and F introduced above induce a mixed Hodge structure on $H^m(V, \mathbb{C})$.*

The remainder of this section will be devoted to proving this theorem and some of its consequences.

2.5. Consider the spectral sequence associated with the decreasing filtration $\widetilde{W}$ (see Godement [1958])

$$\widetilde{W}^p = \bigoplus_r \bigoplus_{s \geq p} A^{r,s}$$

of the bi-complex $A^{\cdot\cdot}$. The graded module $\bigoplus \operatorname{Gr}_p^W H^m(V, \mathbb{C})$ associated with the weight filtration W is isomorphic to the E_∞^p term of this spectral sequence. We need to show that the filtration F induces a pure Hodge structure of weight p on E_∞^p.

2.6. The differential $D_1 : A^{r,s} \to A^{r+1,s}$ coincides with the differentiation of r-forms. Therefore,

$$E_1^{s,r} = H^r(V^{(s+1)}, \mathbb{C}), \tag{5}$$

since $0 \mapsto (a_{s+1})_* \mathbb{C}_{V^{(s+1)}} \mapsto \mathfrak{U}^{\cdot,s}$ is a fine resolution of the sheaf $(q_{s+1})_* \mathbb{C}_{V^{(s+1)}}$. Note that on each term $E_1^{s,r} = H^r(V^{(s+1)}, \mathbb{C})$ there is a pure Hodge structure, since $V^{(s+1)}$ is a nonsingular projective variety, and, furthermore, this Hodge structure is defined by the filtration F.

The differential $d_1 : E_1^{s,r} \to E_1^{1,r+1}$ is induced by the combinatorial differential $D_2 = \sum_{j=1}^{q+1} (-1)^{p+j} \delta_j^*$, that is, d_1 coincides up to sign with the morphism $H^r(V^{(s+1)}, \mathbb{C}) \to H^r(V^{(s+2)}, \mathbb{C})$ induced by the embedding $V^{(s+2)} \hookrightarrow V^{(s+1)}$.

By Lemma 1.2, the filtration F induces a pure Hodge structure of weight r on $E_2^{s,r}$.

In order to prove Theorem 1, it is enough to show that the spectral sequence collapses at the E_2 term. Then

$$E_2^{s,r} = E_\infty^{s,r} = \mathrm{Gr}_r^W H^{s+r}(V, \mathbb{C}),$$

that is, the filtrations W and F induce a mixed Hodge structure on each $H^m(V, \mathbb{C})$.

2.7. In order to show that the spectral sequence collapses at the E_2 term it is enough to show that the map

$$\widetilde{W}^r H^m(V, \mathbb{C}) \to E_2^{r,m-r}$$

is an epimorphism. The element $x \in E_2^{r,m-r}$ is represented by a harmonic $(m-r)$-form $\omega_r \in H^{m-r}(V^{(r+1)}, \mathbb{C})$, such that $D_2\omega_r = 0$ in $H^{m-r}(V^{(r+2)}, \mathbb{C})$, that is, $D_2\omega_r$ is an exact form on $V^{(r+2)}$. To show the required epimorphicity, it is enough to show that the form ω_r can be extended to a point $\omega = \omega_r + \omega_{r+1} + \ldots + \omega_m$, such that $D\omega = 0$, where $\omega_j \in W^j A^{\cdot}$.

By Hodge's theorem, the harmonic form $\omega_r \in H^{m-r}(V^{(r+1)}, \mathbb{C})$ can be uniquely decomposed into the (p,q)-harmonic components

$$\omega_r = \sum_{p+q=m-r} \omega_r^{p,q}.$$

Since d_1 is a morphism of Hodge structures, and $D_2\omega_r = 0$, it follows that $D_2(\omega_r^{p,q}) = 0$. Thus, it is enough to extend each of the (p,q)-components of ω_r to a closed form. Let us apply the $\partial\bar{\partial}$-lemma (Chapter 1, Section 9) to the exact form $D_2\omega_r^{p,q}$. According to the $\partial\bar{\partial}$ lemma, $D_2(\omega_r^{p,q}) = \partial\bar{\partial}\gamma_{r+1}$, where γ_{r+1} is a $(p-1, q-1)$ form on $V^{(r+2)}$. Set $\omega_{r+1}^{p,q-1} = \partial\gamma_{r+1}$. A direct computation shows that $D(\omega_r^{p,q} + \omega_{r+1}^{p,q-1}) = D_2(\omega_{r+1}^{p,q-1})$. Moreover, $D_2(\omega_{r+1}^{p,q-1})$ is a d-closed form on $V^{(r+2)}$ and, in addition, $D_2(\omega_{r+1}^{p,q-1} = D_2(\partial\gamma_{r+1}) = \partial(D_2\gamma_{r+1})$, thus $D_2(\omega_{r+1}^{p,q-1})$ is a ∂-coboundary. Thus, we can again apply the $\partial\bar{\partial}$-lemma to the form $D_2(\omega_{r+1}^{p,q-1})$ to find a form $\omega_{r+2}^{p,q-2}$ on $V^{(r+3)}$ such that

$$D(\omega_r^{p,q} + \omega_{r+1}^{p,q-1} + \omega_{r+2}^{p,q-2}) = D_2(\omega_{r+2}^{p,q-2})$$

and such that $D_2(\omega_{r+2}^{p,q-2})$is again d-closed and is a ∂-coboundary. Repeating this reasoning q times, we finally get a D-closed form

$$\omega = \sum_{i=0}^{q} \omega_{r+i}^{p,q-i},$$

the existence of which implies the collapsing of the spectral sequence (5) at the E_2 term. This completes the proof of Theorem 1.

2.8. Let us mention a few consequences, which follow immediately from Theorem 1 and its proof.

Corollary 1. *Mixed Hodge structures on the cohomology of varieties with normal crossings are functorial, that is, if $f : X \to Y$ is a morphism of varieties with normal crossings, then*

$$f^* : H^m(Y, \mathbb{Q}) \to H^m(X, \mathbb{Q})$$

is a morphism of mixed Hodge structures of weight 0.

Corollary 2. *Let V be a variety with normal crossings. Then the weight filtration on $H^n(V, \mathbb{Q})$ has the form*

$$W_0 \subset \ldots \subset W_n = H^n(V, \mathbb{Q}).$$

Corollary 3. *Let $f : V \to X$ be a morphism of a variety with normal crossings into a nonsingular complete algebraic variety. Then*

$$W_{n-1}H^n(V, \mathbb{Q}) \cap \operatorname{Im} f^* = 0.$$

Corollary 4. *Let $V = \bigcup_{i=1}^N V_i$ be a variety with normal crossings, such that $V_{i_1,\ldots,i_k} = \emptyset$ when $s \geq d$. Then*

$$W_{n-d}H^n(V) = 0.$$

Corollary 5. *Let X be a smooth projective variety of dimension $d+1$, and let V be an ample divisor on X with normal crossings (hence a variety with normal crossings) Then $W_{n-1}H^n(V, \mathbb{Q}) = 0$, for $n < d$.*

Indeed, by Lefschetz' theorem (Chapter 1, §9) the maps $H^n(X, \mathbb{Q}) \to H^n(V, \mathbb{Q})$ induced by the inclusions $V \hookrightarrow X$ are isomorphisms for $n < d$. By corollary 1, these isomorphisms are morphisms of the mixed Hodge structures of weight 0. Thus, for $n < d$ the mixed Hodge structure on $H^n(V, \mathbb{Q})$ the mixed Hodge structure is pure.

Definition. *The polyhedron $\Pi(V)$ of a variety with normal crossings* $V = \bigcup_{i=1}^N V_i$, $\dim V_i = d$ is the polyhedron whose vertices correspond to the irreducible components V_i of the variety V. The vertices $V_{i_1}, \ldots, V_{i_k}$ form a $(k-1)$-simplex if $V_{i_1,\ldots,i_k} \neq \emptyset$.

Corollary 6. *Let V be a variety with normal crossings. then*

$$W_0H^m(V, \mathbb{Q}) = H^m(\Pi(V), \mathbb{Q}).$$

Indeed, the spectral sequence defined in Section 2.5 collapses at the E_2 term, and by definition of the mixed Hodge structure on $H^m(V, \mathbb{Q})$

$$W_0H^m(V, \mathbb{Q}) = E_2^{m,0}.$$

The terms $E_2^{m,0}$ are, by definition, the m-dimensional cohomology of the complex $\{E_1^{m,0}, d_1\}$, with $E_1^{m,0} \simeq H^0(V^{(m+1)}, \mathbb{Q})$, and where d_1 coincides with the combinatorial differential D_2. But $H^0(V^{(m+1)}, \mathbb{Q}) \simeq \oplus H^0(V_{i_1,\ldots,i_{m+1}}, \mathbb{Q})$, where the sum is over all of the m-dimensional simplices of the polyhedron $\Pi(V)$. Thus, $E_2^{m,0} \simeq H^m(\Pi(V), \mathbb{Q})$.

2.9. To conclude this section, we will study the structure of the weight spaces on the cohomology of a variety with normal crossings $V = \bigcup_{i=1}^N V_i$ in the case $\dim_{\mathbb{C}} V_i = 2$, that is, for surfaces. Let $C_{ij} = V_i \cap V_j$ be the *double curves* of the variety V. Then:

(a) $W_0 H^m(V) = H^m(\Pi(V))$,
(b) $H^1(V) = W_1 \supset W_0 \supset 0$ and

$$\mathrm{Gr}_1^W H^1(V) = \mathrm{Ker}\left(\bigoplus_{i=1}^N H^1(V_i) \xrightarrow{D_2} \bigoplus_{i<j} H^1(C_{ij})\right),$$

(c) $H^2(V) = W_2 \supset W_1 \supset W_0 \supset 0$, and

$$\mathrm{Gr}_1^W H^2(V) = \mathrm{Coker}\left[\bigoplus_{i=1}^N H^1(V_i) \xrightarrow{D_2} \bigoplus_{i<j} H_1(C_{ij})\right],$$

$$\mathrm{Gr}_2^W H^2(V) = \mathrm{Ker}\left[\bigoplus_{i=1}^N H^2(V_i) \xrightarrow{D_2} \bigoplus_{i<j} H^2(C_{ij})\right],$$

$$\mathrm{Gr}_2^W H^2(V) = H_2^{2,0} \oplus H_2^{1,1} \oplus H_2^{0,2}$$

where $H_2^{0,2} = \overline{H}_2^{2,0}$ and $H_2^{2,0} = \bigoplus_{i=1}^N H^{2,0}(V_i)$.
Indeed, the spectral sequence (5) has the form

$$\begin{array}{l|lllll}
 & \bigoplus H^4(V_i) & & & & \\
q & \bigoplus H^3(V_i) & & & & \\
 & \bigoplus H^2(V_i) & \xrightarrow{D_2} \bigoplus\limits_{i<j} H^2(C_{ij}) & \longrightarrow 0 & & \\
 & \bigoplus H^1(V_i) & \xrightarrow{D_2} \bigoplus\limits_{i<j} H^1(C_{ij}) & \longrightarrow 0 & & \\
 & \bigoplus H^0(V_i) & \xrightarrow{D_2} \bigoplus\limits_{i<j} H^0(C_{ij}) & \longrightarrow \bigoplus\limits_{i<j<k} H^0(P_{ijk}) & \longrightarrow 0 & \\
\hline
\end{array}$$

where $P_{ijk} = V_i \cap V_j \cap V_k$ are the *triple points of the divisor* V. The claims (a), (b), and (c) follow immediately from the computation $E_2^{p,q} = E_\infty^{p,q} = \mathrm{Gr}_p^W H^{p+q}(V)$ and it should be noted that D_2 is a morphism of pure Hodge structures of weight 0, and $H^2(C_{ij} = H^{1,1}(C_{ij})$, since C_{ij} are curves and thus carry no holomorphic 2-form. Thus

$$\bigoplus_{i=1}^{N} H^{2,0}(V_i) \subset \mathrm{Ker}[\oplus H^2(V_i) \xrightarrow{D_2} \oplus H^2(C_{ij})] = \mathrm{Gr}_2^W H^2(V).$$

§3. Cohomology of Smooth Varieties

Let X be a smooth complex algebraic variety of dimension d. In this section we will sketch the proof of the following theorem.

Theorem 2. *The cohomology $H^m(X,\mathbb{Q})$ can be naturally equipped with a functorial mixed Hodge structure.*

The proof of this theorem implies

Corollary 1. *The weight filtration on the cohomology $H^m(X,\mathbb{Q})$ of a smooth variety X has the form*

$$0 = W_{m-1} \subset W_m \subset \ldots \subset W_{2m} = H^m(X,\mathbb{Q}).$$

Corollary 2. *Let $j : X \hookrightarrow \overline{X}$ be the smooth compactification of a variety X. Then*

$$j^*(H^m(\overline{X},\mathbb{Q})) = W_m H^m(X,\mathbb{Q}).$$

3.1. A *proper compactification* of a variety X is an open embedding $j : X \hookrightarrow \overline{X}$ into a complete smooth algebraic variety $\overline{X}$, such that $\overline{X}\backslash X = V = \bigcup_{i=1}^N V_i$ is a divisor with normal crossings (that is, V is a variety with normal crossings). According to Hironaka's theorem (Hironaka [1964]), a proper compactification always exists. Just as in the previous section, let $V^{(k)} = \bigsqcup_{i_1<\ldots i_k} V_{i_1,\ldots,i_k}$ for $k \geq 1$, $V^{(0)} = V$, and let $a_k : V^{(k)} \to \overline{X}$ be the natural inclusions.

3.2. For the constant sheaf $\mathbb{C}_X$ we have the resolution

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \ldots \to \Omega_X^n \to \ldots.$$

Thus,

$$H^{\cdot}(X,\mathbb{C}) = \mathbf{H}^{\cdot}(X,\Omega_X^{\cdot}),$$

where $\mathbf{H}^{\cdot}(X,K^{\cdot})$ denotes the hypercohomology of the complex of sheaves $\{K^{\cdot},d\}$ (Grothendieck [1957], Godement [1958]).

The sheaves Ω_X^p are j^*-acyclic, that is, $R^q J_*(\Omega_X^p) = 0$ for $q > 0$. Indeed, any point $x \in \overline{X}$ has a neighborhood U isomorphic to the polydisk Δ^d where the divisor V is defined by the equation $z_1 \cdot \ldots \cdot z_k = 0$. Then $U \cap X \simeq \Delta^{*k} \times \Delta^{d-k}$, where $\Delta^* = \Delta \backslash \{0\}$. The manifold $\Delta^{*k} \times \Delta^{d-k}$ is Stein, and so by Cartan's Theorem B (see Gunning–Rossi [1965]), it follows that $H^q(X \cap U, \Omega_X^p) = 0$.

From the j_*-acyclicity of the sheaves Ω_X^p it follows (Godement [1958]) that

$$H^{\cdot}(X, \mathbb{C}) \simeq \mathbf{H}^{\cdot}(\overline{X}, j_* \Omega_X^{\cdot}).$$

The sheaves $j_* \Omega_X^p$ are too big for the computation of $H^{\cdot}(X, \mathbb{C})$, and as we shall see below, it will be enough to deal with sheaves of holomorphic forms on X with poles of first order along V.

Let the divisor V be defined by equations $z_1 \cdot \ldots \cdot z_k = 0$ in the neighborhood $U \subset \overline{X}$, where the z_i are local coordinates in U.

3.3.

Definition. The sheaf

$$\Omega_{\overline{X}}^n(\log V) = \wedge^n(\Omega_{\overline{X}}^1(\log V))$$

is called the *sheaf of holomorphic n-forms on $\overline{X}$ with logarithmic poles along* V, where $\Omega_{\overline{X}}^1$ is the locally free $\mathcal{O}_{\overline{X}}$-module, generated over U by the differentials

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_k.$$

In other words, the sections of the sheaf $\Omega_{\overline{X}}^n(\log V)$ in the neighborhood U are n-forms

$$\omega \wedge \frac{dz_I}{z_I},$$

where ω is a holomorphic form on U, $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, k\}$ and

$$\frac{dz_I}{z_I} = \frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_p}}{z_{i_p}}.$$

It is easily checked that the definition of the sheaf $\Omega_{\overline{X}}^n(\log V)$ does not depend on the choice of the equation defining the divisor V.

Let $\mathcal{E}_{\overline{X}}^{p,q}(\log V) = \Omega_{\overline{X}}^p(\log V) \otimes_{\mathcal{O}_{\overline{X}}} \mathcal{E}_{\overline{X}}^{0,q}$.

We have

$$\Omega_{\overline{X}}^n \subset \Omega_{\overline{X}}^n(\log V) \subset j_* \Omega_X^n,$$

and

$$\mathcal{E}_{\overline{X}}^{p,q} \subset \mathcal{E}_{\overline{X}}^{p,q}(\log V) \subset j_* \mathcal{E}_X^{p,q}.$$

Clearly, the differentials d, ∂ and $\overline{\partial}$ send forms with logarithmic poles into forms with logarithmic poles. Thus, by Dolbeault's theorem, the bicomplex $\{\mathcal{E}_{\overline{X}}^{p,q}(\log X), \partial, \overline{\partial}\}$ is a fine resolution of the complex $\{\Omega_{\overline{X}}^{\cdot}(\log V), d\}$.

Let us introduce increasing filtrations W on the sheaves $\Omega^{\cdot}_{\overline{X}}(\log V)$, by setting

$$W_k\Omega^p_{\overline{X}}(\log V) = \Omega^k_{\overline{X}}(\log V) \wedge \Omega^{p-k}_{\overline{X}},$$

an similarly on $\mathcal{E}^{p,q}_{\overline{X}}(\log V)$. Clearly, $d(W_k) =\subset W_k$, and $\overline{\partial}(W_k) \subset W_k$. Thus, the following complex $\{W_k/W_{k-1}, d\}$ is well defined:

$$\{0 \mapsto W_k\Omega^k_{\overline{X}}(\log V) \overset{d}{\mapsto} W_k\Omega^{k+1}_{\overline{X}}(\log V)/W_{k-1}\Omega^{k+1}_{\overline{X}}(\log V) \overset{d}{\mapsto} \dots\}$$

and, analogously, the complexes

$$\{W_k\mathcal{E}^p_{\overline{X}}(\log V)/W_{k-1}\mathcal{E}^l_{\overline{X}}(\log V), d\}$$

and

$$\{W_k\mathcal{E}^{p,q}_{\overline{X}}(\log V)/W_{k-1}\mathcal{E}^{p,q}_{\overline{X}}(\log V), \overline{\partial}\}.$$

3.4. Poincaré Residue. In the neighborhood U of every point $x \in V$ there is a map

$$R^l : W_l(H^0(U, \Omega^n_{\overline{X}}(\log V))) \mapsto H^0(U \cap V^{(l)}, \Omega^{n-l}_{V^{(l)}}),$$

which maps the form $\omega \wedge \frac{dz_I}{z_I}$ to

$$R^l(\omega \wedge \frac{dz_I}{z_I}) = (2\pi\sqrt{-1})^{(I)}\omega|_{V_{i_1,\dots,i_l}},$$

where $|I| = l$ for $I = \{i_1, \dots, i_l\}$.

It can be checked that this map does not depend on the choice of the local defining equation of the divisor V and is well-defined on the intersection of neighborhoods. Thus, there is a well-defined sheaf map

$$R^l : W_l\Omega^n_{\overline{X}}(\log X) \to \Omega^{n-l}_{V^{(l)}},$$

which we will call the *Poincaré residue map.*

We can analogously define

$$R^l : W_l\mathcal{E}^{p,q}_{\overline{X}}(\log V) \to \mathcal{E}^{p-l,q}_{V^{(l)}},$$

and

$$R^l : W_l\mathcal{E}^p_{\overline{X}}(\log V) \to \mathcal{E}^{p-l}_{V^{(l)}}$$

Lemma 1.

(1) $R^l(W_{l-1}) = 0$;
(2) R^l *commutes with the differentials* d, ∂, *and* $\overline{\partial}$.

Let us recall that two complexes $\{K^{\cdot}, d\}$ and $\{L^{\cdot}, d\}$ are called quasi-isomorphic, if there is exists a morphism of complexes $\phi : K^{\cdot} \to L^{\cdot}$ inducing isomorphisms $\widetilde{\phi}_n : H^n_d(K^{\cdot}) \to H^n_d(L^{\cdot})$, where

$$H^n_d(K^{\cdot}) = \operatorname{Ker}[d : K^n \to K^{n+1}]/\operatorname{Im}[d : K^{n-1} \to K^n].$$

The following lemma is an easy consequence of Poincaré's lemma (see Nickerson [1958]).

Lemma 2. *The maps*

$$R^l : \{W_l\Omega^{\cdot}/W_{l-1}\Omega^{\cdot}, d\} \to \{\Omega^{\cdot-l}_{V^{(l)}}, d\},$$

$$R^l : \{W_l\mathcal{E}^{\cdot\cdot}_{\overline{X}}(\log V)/W_{l-1}\mathcal{E}^{\cdot\cdot}_{\overline{X}}(\log V), \overline{\partial}\} \to \{\mathcal{E}^{\cdot-l,\cdot}_{V^{(l)}}, \overline{\partial}\},$$

$$R^l : \{W_l\mathcal{E}^{\cdot}_{\overline{X}}(\log V)/W_{l-1}\mathcal{E}^{\cdot}_{\overline{X}}(\log V), d\} \to \{\mathcal{E}^{\cdot-l}_{V^{(l)}}, d\}$$

are quasi-isomorphisms.

The complexes $j_*\Omega^{\cdot}_X$ and $\Omega^{\cdot}_{\overline{X}}(\log V)$ are evidently isomorphic in sufficiently small neighborhoods of points $x \in j(X)$. Let us look near points $x \in V \subset \overline{X}$. Choose a neighborhood U of x, which is isomorphic to the polydisk Δ^d, where $U \cap X = \Delta^{*k} \times \Delta^{d-k}$. Then $U \cap X$ is homotopy equivalent to the torus $T^k(S^1)^k$, therefore $H^n(X \cap U, \mathbb{C}) = H^n(T^k, \mathbb{C}) = \wedge^n H^1(T^k, \mathbb{C}) = \wedge^n(\mathbb{C}^k)$. Furthermore, $H^1(X \cap U, \mathbb{C})$ has a canonical basis over $\mathbb{Z}$, consisting of the forms $\xi_i = \frac{1}{2\pi\sqrt{-1}}\frac{dz_i}{z_i}$, $1 \le i \le k$. Thus, in the neighborhood U, the complex $j_*\Omega^{\cdot}_X$ is quasi-isomorphic to the complex $\wedge^{\cdot}\mathbb{C}(\xi_1, \ldots, \xi_k)$ with the trivial differential. Let

$$\alpha : \wedge^{\cdot}\mathbb{C}(\xi_1, \ldots, \xi_k) \hookrightarrow H^0(U, \Omega^{\cdot}_{\overline{X}}(\log V))$$

be an embedding of the complexes in the neighborhood U. Let us show that α_* is an isomorphism on cohomology. For this purpose, let us introduce a filtration $\widetilde{W}_p$ on $\wedge^{\cdot}\mathbb{C}(\xi_1, \ldots, \xi_k)$, analogous to the filtration on $\Omega^{\cdot}_{\overline{X}}(\log V)$. It is clear that

$$\alpha(\widetilde{W}_p) \subset W_p(U, \Omega^{\cdot}_{\overline{X}}(\log V)) = W_p.$$

Consider the commutative diagram

$$\begin{tikzcd}
W_p/W_{p-1} \arrow[rr, "R^p"] & & H^0(V^{(p)} \cap U, \Omega^{\cdot-p}_{V^{(p)}}) \\
& \widetilde{W}_p/\widetilde{W}_{p-1} \arrow[ul, "\alpha_p"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where β is also the Poincaré residue ($\beta(c\xi_I) = c$). It is easy to see that β is an isomorphism on cohomology by Poincaré lemma, and R^p is also an isomorphism by Lemma 2. Thus α_p is also an isomorphism. By induction on p, we get quasi-isomorphisms

$$\alpha_p : \widetilde{W}_p \to W_p.$$

For $p = k$ we obtain a quasi-isomorphism of complexes $\wedge^{\cdot}\mathbb{C}(\xi_1, \ldots, \xi_k)$ and $H^0(U, \Omega^{\cdot}_{\overline{X}}(\log V))$. Thus, we have proved

Lemma 3. *The complexes* $\{j_* \Omega^{\cdot}_X, d\}$ *and* $\Omega^{\cdot}_{\overline{X}}(\log V), d\}$ *are quasi-isomorphic.*

Corollary. $H^{\cdot}(X, \mathbb{C}) = \mathbf{H}^{\cdot}(\overline{X}, \Omega^{\cdot}_{\overline{X}}(\log V))$.

As was noted above, the bicomplex $\mathcal{E}^{\cdot\cdot}_{\overline{X}}(\log V)$ is a fine resolution of the complex $\Omega^{\cdot}_{\overline{X}}(\log V)$. The complex $\mathcal{E}^{\cdot}_{\overline{X}}(\log V)$ is the full complex associated with $\mathcal{E}^{\cdot\cdot}_{\overline{X}}(\log V)$, and so the fineness of the sheaves $\mathcal{E}^{n}_{\overline{X}}(\log V)$ we get the following de Rham theorem:

Theorem. $H^n(X, \mathbb{C}) = H^n_d(H^0(\overline{X}, \mathcal{E}^{\cdot}_{\overline{X}}(\log V)))$. *In other words, the cohomology classes of a smooth algebraic variety* X *are represented by closed forms on* $\overline{X}$ *with logarithmic singularities along* V *modulo exact forms of the same kind.*

3.5. Now we are set to impose a mixed Hodge structure on $H^n(X, \mathbb{Q})$. The weight filtration W on $H^n(X, \mathbb{C})$ will be induced by the filtration $W_l\mathcal{E}^{n}_{\overline{X}}(\log V)$ (where the indices have to be changed). Set

$$W_{k+n}H^n(X, \mathbb{C}) = H^n_d(H^0(\overline{X}, W_k\mathcal{E}^{\cdot}_{\overline{X}}(\log V))). \tag{6}$$

Using the Poincaré residue, it can be shown that the weight filtration is defined over $\mathbb{Q}$.

The Hodge filtration is induced by the filtration

$$F^p\mathcal{E}^{n}_{\overline{X}}(\log V) = \bigoplus_{i \geq p} \mathcal{E}^{i,n-i}_{\overline{X}}(\log V). \tag{7}$$

The remaining part of this section is devoted to showing that the filtrations defined in (6) and (7) induced a mixed Hodge structure on $H^n(X, \mathbb{C})$.

To compute $H^n_d(H^0(\overline{X}, \mathcal{E}^{\cdot}_{\overline{X}}(\log V)))$ let us use the spectral sequence defined by the decreasing filtration

$$\mathcal{W}^{-l} = W_l\mathcal{E}^{\cdot}_{\overline{X}}(\log V).$$

The E_1 term of this spectral sequence is equal to

$$E_1^{-p,q} = H_d^{-p+q}(W^{-p}/W^{-p+1}), \tag{8}$$

where $W^{-p} = H^0(\overline{X}, \mathcal{W}^{-p})$.

Lemma 4. *The Hodge filtration (7) induces a pure Hodge structure of weight* $p+q$ *on* $H_d^{-p+q}(W^{-p}/W^{-p+1})$.

Proof. Applying Lemma 2 from Section 3.4 we obtain

$$H^{\cdot}_d(H^0(\overline{X}, W_l\mathcal{E}^{\cdot}_{\overline{X}}/W_{l-1}\mathcal{E}^{\cdot}_{\overline{X}})) \simeq H_d^{\cdot-l}(V^{(l)}, \mathcal{E}_{V^{(l)}}). \tag{9}$$

The sheaves $W_l\mathcal{E}^{p}_{\overline{X}}(\log V)$ are fine. Therefore, from the sheaf exact sequence

$$0 \mapsto W_{l-1}\mathcal{E}^{p}_{\overline{X}}(\log V) \mapsto W_l\mathcal{E}^{p}_{\overline{X}}(\log V) \mapsto W_l\mathcal{E}^{p}_{\overline{X}}(\log V)/W_{l-1}\mathcal{E}^{p}_{\overline{X}}(\log V) \mapsto 0$$

we get that

$$H^0(\overline{X}, W_l\mathcal{E}^p_{\overline{X}}(\log V)/W_{l-1}\mathcal{E}^p_{\overline{X}}(\log V)) \simeq \frac{H^0(\overline{X}, W_l\mathcal{E}^p_{\overline{X}}(\log V))}{H^0(\overline{X}, W_{l-1}\mathcal{E}^p_{\overline{X}}(\log V))}.$$

Applying (9) we get that

$$E_1^{-p,q} = H_d^{-p+q}(W^{-p}/W^{-p+1}) \simeq H^{q-2p}(V^{(p)}, \mathbb{C}). \tag{10}$$

Let $W_l^{p,q} = H^0(\overline{X}, W_l\mathcal{E}^{p,q}_{\overline{X}}(\log V))$. Then,

$$F^r(W^{-p}/W^{-p+1}) = \left(\bigoplus_{i>r} W_p^{i,\cdot}\right)/W_{p-1}.$$

The Poincaré residue agrees with the Hodge filtration:

$$R^p : F^r(W^{-p}/W_{p+1}) \to F^{r-p}H^0(V^{(p)}, \mathcal{E}^{\cdot-p}_{V^{(p)}}),$$

and the filtration $F^qH^0(V^{(p)}, \mathcal{E}^{\cdot-p}_{V^{(p)}})$ defines a pure Hodge structure on $H^\cdot(V^{(p)}, \mathbb{C})$. It follows that the isomorphism (10) is a morphism of pure Hodge structures of weight -2, and so the lemma is proved.

3.6. The Gysin mapping. The Poincaré residue identifies the E_1 terms of the spectral sequence with the cohomology of the varieties $V^{(p)}$:

$$R^p : E_1^{-p,q} = H^{-p+q}(W_p/W_{p-1}) \xrightarrow{\sim} H^{q-2p}(V^{(p)}, \mathbb{C}).$$

It turns out that under this identification that differential $d_1 : E_1 \to E_1$ coincides with the Gysin map. Let us recall the definition.

Let Y be a Kähler manifold, $\dim_{\mathbb{C}} Y = d+1$ and let $j : D \hookrightarrow Y$ be a smooth submanifold of codimension 1. The inclusion j induces a map on homology:

$$j_* : H_{2d-n}(D) \to H_{2d-n}(Y). \tag{11}$$

On the other hand, on Y (and similarly on D) there is a non-degenerate pairing

$$(\cdot,\cdot) : H^n(Y) \times H^{2d+2-n}(Y) \to \mathbb{C},$$

such that for $\phi \in H^n(Y)$ and $\psi \in H^{2d+2-n}(Y)$

$$(\phi, \psi) = \int_Y \phi \wedge \psi.$$

This pairing defines the Poincaré duality isomorphism

$$\pi_Y : H_{2d+2-n}(Y) \to H^n(Y).$$

The map

$$\gamma : H^n(D) \to H^{n+2}(Y),$$

dual to the map (11) (by virtue of Poincaré duality) is called the *Gysin map* $\gamma = \pi_Y \cdot j_* \cdot \pi_D^{-1}$.

Note that the Gysin map $\gamma : H^n(D) \to H^{n+1}(Y)$ is a morphism of Hodge structures of weight $(1,1)$. Indeed, $H_m(X) = H^m(X)^\vee$ and so for a Kähler manifold X $H_m(X)$ comes equipped with a pure Hodge structure of weight $-m$ (see Section 1.7). The morphism (11) is a morphism of Hodge structures of type $(0,0)$ with respect to this Hodge structure. It is easy to check that the Poincaré duality isomorphism $\pi_X : H_{2k-n}(X) \to H^n(X)$ is type (k,k) morphism of Hodge structures for a k-dimensional X. It follows that $\gamma = \pi_Y \cdot j_* \cdot \pi_D^{-1}$ is a morphism of Hodge structures of type $(1,1)$, as a composition of morphisms of Hodge structures of types $(d+1, d+1)$, $(0,0)$, and $(-d,-d)$, respectively.

Let us study the Gysin map on the level of forms. Let $[D]$ be a line bundle corresponding with the divisor D (see Chapter 1, Section 5) and pick an Hermitian metric $|\cdot|$ on $[D]$. Let L be a section of the sheaf $\mathcal{O}_Y(D)$, whose zero-set coincides with D. Set

$$\eta = \frac{1}{2\pi\sqrt{-1}} \partial \log |s|^2$$

and

$$\alpha = \frac{1}{2\pi\sqrt{-1}} \bar{\partial}\partial \log |s|^2.$$

The form α is a $(1,1)$-form on Y, representing the Chern class $c_1[D]$ of the line bundle $[D]$ (see Chapter 1, Section 5).

In a neighborhood U of a point $x \in Y$ the divisor D can be given by the equation $z_1 = 0$. In this neighborhood

$$\eta = \frac{1}{2\pi\sqrt{-1}} \frac{dz_1}{z_1} + \beta,$$

where β is some C^∞ form of type $(1,0)$.

Let ω be a closed form on D, $\deg \omega = p$. Let us extend ω to a C^∞ form $\tilde{\omega}$ on Y, so that $\tilde{\omega}|D = \omega$, and set

$$\tilde{\gamma}(\omega) = d(\tilde{\omega} \wedge \eta) = d\tilde{\omega} \wedge \eta \pm \tilde{\omega} \wedge \alpha.$$

Since ω is a closed form on D, the Poincaré residue $R^1(\tilde{\gamma}(\omega)) = 0$. Therefore,

$$\tilde{\gamma}(\omega) \in W_*^{p+2} = \operatorname{Ker}[R^1 : \mathcal{E}_Y^{p+2}(\log D) \to \mathcal{E}_Y^{p+1}].$$

The sheaves W_*^p are fine. By applying a version of Poincaré lemma (see Nickerson [1958]), it can be shown that the sequence of sheaves

$$0 \to \mathbb{C}_Y \to W_*^0 \xrightarrow{d} W_*^1 \xrightarrow{d} \dots$$

is exact. Thus

Lemma 5. $H_d^n(H^0(Y, W_*^\cdot)) = H^n(Y, \mathbb{C})$.

Using this lemma, we can show that $\tilde{\gamma}(\omega) = H_d^*(H^0(Y, W_*^{\cdot}))$ equals, up to sign, the image of the form $\omega \in H_d^{*-2}(D)$ under the Gysin map. To do this, it is sufficient to show that for any closed form ω on Y

$$\int_Y \tilde{\gamma}(\omega) \wedge \phi = \pm \int_D \omega \wedge \phi.$$

To prove the above equality, consider a tube T_ϵ of radius ϵ around the divisor D. By Stokes' theorem

$$\int_Y \tilde{\gamma}(\omega) \wedge \phi = -\lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \tilde{\omega} \wedge \eta \wedge \phi = \pm \int_D \omega \wedge \phi,$$

since $\lim_{\epsilon \to 0} \int_0^{2\pi} f(\epsilon e^{i\theta}) d\theta = f(0)$ for any C^∞ function f.

Lemma 6. *$d_1 : E_1 \to E_1$ coincides with the Gysin map.*

Proof. The class $\tilde{\phi} \in E_1^{-p,q} = H^{-p+q}(W_p/W_{p-1})$ can be represented by form $\phi \in W_p \mathcal{E}_{\overline{X}}^{-p+q}(\log V)$, such that $d\phi = 0$ in W_p/W_{p-1}, that is,

$$R^p d\phi = dR^p \phi = 0.$$

Thus, $d\phi \in W_{p-1}$.

By definition of the differential d_1 in the spectral sequence, the class $d_1 \tilde{\phi}$ equals the class $d\phi$ in W_{p-1}/W_{p-2}, that is $d_1 \tilde{\phi} = R^{p-1} d\phi$, in view of the identification $E_1^{p,q} = H^{q-2p}(V^{(p)}, \mathbb{C})$.

Let $\eta_I = \eta_{i_1} \wedge \ldots \wedge \eta_{i_p}$, where $\eta_{i_e} = \frac{1}{2\pi\sqrt{-1}} \partial \log |s_{i_e}|^2$ is the form defined above for the divisor $D = V_{i_e}$ on X. It is not hard to check that for the form $\psi = \phi_I \wedge \eta_I$

$$R^{|I|}(\phi)|_{V_I} = \phi_I|_{V_I}.$$

Simple computations show that if

$$\phi = \sum_{|I|=p} \phi_I \wedge \eta_I,$$

represents the class $\tilde{\phi} \in E_1^{-p,q}$ (where $E_1^{p,q}$ is identified with $H^{q-2p}(V^{(p)}, \mathbb{C})$), then the differential d_1 gives

$$R^{p-1} d\phi|_{V_{i_1,\ldots,i_{p-1}}} = \pm \sum_j (d\phi_{i_1,\ldots,i_{p-1}} \wedge \eta_j \pm \phi_{i_1,\ldots,i_{p-1},j} \wedge \alpha_j).$$

Thus, d_1 coincides up to sign with the Gysin map. The lemma is proved.

Since the Gysin map is a morphism of Hodge structures, so $d_1 : E_1 \to E_1$ is a morphism of pure Hodge structures of type $(1, 1)$. Thus, the terms $E_2^{-p,q}$ of the spectral sequence are equipped with a pure Hodge structure of weight $p + q$.

Lemma 7. *The spectral sequence (8) collapses in the E_2 term, that is, $d_2 = d_3 = \ldots = 0$.*

The proof of this lemma uses the same arguments as were used in Section 2.7 to show an analogous result for a variety with normal crossings.

3.7. To complete the proof of Theorem 2, we need to check the independence of the definition of the mixed Hodge structure on $H^{\cdot}(X, \mathbb{C})$ from the choice of the proper compactification and to check the functoriality of the definition.

First, let us note that if a morphism $f : X \to Y$ of smooth varieties is extended to a morphism $\overline{f} : \overline{X} \to \overline{Y}$ of the proper compactifications, then the induced map

$$f^* : \mathcal{E}^{p,q}_{\overline{Y}}(\log(\overline{Y} \backslash Y)) \to \mathcal{E}^{p,q}_{\overline{X}}(\log(\overline{X} \backslash X))$$

agrees with the weight and Hodge filtrations, and so $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ is a morphism of mixed Hodge structures. Thus, the definition of the mixed Hodge structure on $H^{\cdot}(X, \mathbb{C})$ is functorial. In particular, if $X = Y$ and $f : X \to Y$ is the identity morphism, then the isomorphism $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ is an isomorphism of Hodge structures.

To show the independence of the definition of the Hodge structure on $H^{\cdot}(X, \mathbb{C})$ on the choice of compactification, it is enough to note that if $i : X \hookrightarrow \overline{\overline{X}}$ is another proper compactification, then by Hironaka's theorem, the varieties $\overline{X}$ and $\overline{\overline{X}}$ are dominated by a third proper compactification $k : X \hookrightarrow \tilde{X}$, so that the diagram below commutes (Hironaka [1964]):

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl] \arrow[dr] & \\
\overline{X} & X \arrow[l, hook', "j"'] \arrow[r, hook, "i"'] \arrow[u, hook, "k"] & \overline{\overline{X}}
\end{tikzcd}$$

3.8. As was previously noted (see Section 3.5) the Hodge filtration on $H^{\cdot}(X, \mathbb{C})$ is induced by the Hodge filtration

$$F^p \mathcal{E}^n_{\overline{X}}(\log V) = \bigoplus_{i \geq p} \mathcal{E}^{i, n-i}_{\overline{X}}(\log V).$$

Furthermore, the bicomplex $\mathcal{E}^{\cdot\cdot}(\log V)$ with the Hodge filtration is a fine resolution of the complex $\Omega^{\cdot}_{\overline{X}}(\log V)$ with the "stupid" filtration

$$F^p \Omega^{\cdot}_{\overline{X}}(\log V) = \{\ldots \to 0 \to \Omega^p_{\overline{X}}(\log V) \to \Omega^{p+1}_{\overline{X}}(\log V) \to \ldots\}.$$

Therefore, the "stupid" filtration on $\Omega^{\cdot}_{\overline{X}}(\log V)$ also induces a Hodge filtration on

$$H^{\cdot}(X, \mathbb{C}) = \mathbf{H}^{\cdot}(\overline{X}, \Omega^{\cdot}_{\overline{X}}(\log V)).$$

The term ${}_F W_1^{p,q}$ of the spectral sequence associated with this filtration has the form

$$ {}_FE_1^{p,q} = H^q(\overline{X}, \Omega^p_{\overline{X}}(\log V)) \Rightarrow H^{p+q}(X, \mathbb{C}). \tag{12} $$

The following theorem is shown by Deligne [1972].

Theorem 3. *The spectral sequence (12) collapses at the E_1 term.*

§4. The Invariant Subspace Theorem

4.1 Consider the following situation: let $f: X \to S$ be a smooth projective morphism of smooth complex algebraic varieties. In addition, suppose that S is connected. Fix an $s \in S$ and a fiber $f^{-1}(s) = X_s = V$.

As a mapping of C^∞ manifolds, the morphism f is a locally trivial fibration with fibers diffeomorphic to V. Therefore, the fundamental group $\pi_1(S, s)$ acts on cohomology $H^n(X_s, \mathbb{Q})$. In particular, for each fiber there is a π_1-invariant subspace

$$ H^n(X_s, \mathbb{Q})^{\pi_1} \hookrightarrow H^n(X_s, \mathbb{Q}). $$

Since f is a C^∞ locally trivial fibration, the choice of trivialization of the map f over sufficiently small open set $U \subset S$ identifies the spaces $H^n(X_s, \mathbb{Q})^{\pi_1}$ for various s in U. Therefore, these invariant subspaces can be glued together into a constant sheaf I^n. This sheaf can be described as follows. Let $R^n f_* \mathbb{Q}$ be the sheaf associated to the presheaf which associates to a an open set U in S the group $H^n(f^{-1}(U), \mathbb{Q})$. Then I^n coincides with the constant sheaf on S with the fiber $H^0(S, R^n f_* \mathbb{Q})$. Indeed, the global section h of the sheaf $R^n f_* \mathbb{Q}$ defines in each fiber a local section $h_s \in (R^n f_* \mathbb{Q})_s$, invariant under the action of π_1, and conversely.

For each point $s \in S$ we have an isomorphism

$$ \phi_s : H^0(S, R^n f_* \mathbb{Q}) \xrightarrow{\sim} H^n(X_s, \mathbb{Q})^{\pi_1}, $$

and the space $H^n(X_s, \mathbb{Q})^{\pi_1}$ is a subspace of $H^n(X_s, \mathbb{Q})$ equipped with a pure Hodge structure of weight n, since X_s is a smooth projective variety.

Theorem 4.

(1) The invariant subspace $H^n(X_1, \mathbb{Q})^{\pi_1}$ in $H^n(X_s, \mathbb{Q})$ has a Hodge substructure.

(2) The Hodge structure on $H^0(S, R^n f, \mathbb{Q})$ obtained by means of an isomorphism ϕ_s does not depend on s.

(3) If $\overline{X}$ is a smooth compactification of a variety X, then the composition of morphisms

$$ H^n(\overline{X}, \mathbb{Q}) \to H^n(X, \mathbb{Q}) \to H^0(S, R^n f_* \mathbb{Q}) = I^n $$

is surjective.

Proof. For a smooth compactification $\overline{X}$ we have the commutative diagram

$$\begin{tikzcd} H^n(X,\mathbb{Q}) \arrow[r, "i_s^*"] & H^n(X_s,\mathbb{Q}) \\ H^n(\overline{X},\mathbb{Q}) \arrow[u] \arrow[ur, "\overline{i}_s^*"'] & \end{tikzcd}$$

where $i_s; X_s \to X$ and $\bar{i}_s : \overline{X}_s \to \overline{X}$ are natural inclusions.

By Theorem 2, the cohomology $H^n(X,\mathbb{Q})$ have a functorial mixed Hodge structure $(H^n(X,\mathbb{Q}), W, F)$ and

$$W_n H^n(X,\mathbb{Q}) = \operatorname{Im}(H^n(\overline{X},\mathbb{Q}) \to H^n(X,\mathbb{Q})).$$

In addition, $H^n(\overline{X},\mathbb{Q})$ and $H^n(X_s,\mathbb{Q})$ also have a pure Hodge structure of weight n and all of the arrows in the diagram are morphisms of the mixed Hodge structures. From the strictness of the morphisms of the mixed Hodge structures, it follows that

$$\operatorname{Im} \bar{i}_s^* = \operatorname{Im} i_s^*. \tag{13}$$

Consider the *Leray spectral sequence* (Godement [1958]) with the term

$$E_2^{p,q} = H^q(S, R^p f_* \mathbb{Q}) \Rightarrow H^{p+q}(X,\mathbb{Q}).$$

It turns out that the following theorem is true (Deligne [1972]).

Theorem. *The Leray spectral sequence for a smooth projective morphism $f : X \to S$ of smooth algebraic varieties collapses in the term*

$$E_2^{p,q} = H^q(S, R^p f_* \mathbb{Q}).$$

The proof of this theorem is based on the hard Lefschetz theorem (Chapter 1, Section 7).

From the collapsing of the Leray spectral sequence at the E_2 term it follows that the canonical mapping

$$j : H^n(X,\mathbb{Q}) \to H^0(S, R^n f_* \mathbb{Q})$$

is surjective. In addition, the map

$$i_s^* : H^n(X,\mathbb{Q}) \to H^n(X_s,\mathbb{Q})$$

can be decomposed as follows:

$$i_s^* : H^n(X,\mathbb{Q}) \to H^0(S, R^n f_* \mathbb{Q}) \xrightarrow{\phi} H^n(X_s,\mathbb{Q})^{\pi_1} \hookrightarrow H^n(X_s,\mathbb{Q}).$$

On the other hand, i_s^* is a morphism of mixed Hodge structures. Therefore, $H^n(X_s,\mathbb{Q})^{\pi_1}$, as an image of a Hodge structure, is a Hodge substructure in $H^n(X_s,\mathbb{Q})$, which proves (1).

We can introduce a Hodge structure on $H^0(S, R^n f_* \mathbb{Q})$, as a quotient structure on $H^n(X, \mathbb{Q})$. This quotient structure is independent of s, which proves (2). Part (3) now follows from equation (13).

In homological terms, Theorem 4 can formulated as follows. We say that a cycle c on X is *vanishing* (globally), if it is null-homologous in $\overline{X}$. For example, a cycle of the form $c - g^*c$, where $g \in \pi_1(S, s)$, is vanishing. Theorem 4 claims that the converse is true; that is, the space of vanishing cycles is generated by cycles of the form $c - g^*c$.

4.2. Using Theorem 4, Deligne [1972] reached the following conclusion about the semi-simplicity of the monodromy action.

Complete Reducibility. The monodromy representation on $H^k(X_s)$ is completely reducible.

In other words, the theorem claims that for any π_1-invariant subspace $V \subset H^k(X_s)$ there exists a π_1-invariant $W \subset H^k(X_s)$, complementary to V. The idea of the proof is to check this first for $V = H^k(X_s)^{\pi_1}$. On $H^k(X_s)$ we have the Hodge bilinear form (see Chapter 1, Section 7), and we can set $W = V^\perp$. From theorem 4 it follows that V is equipped with a Hodge structure, and this Hodge structure is a Hodge substructure of that on $H^k(X_s)$. It follows that the Hodge bilinear form on V is non-degenerate and $V \cap W = 0$. The general case is reduced to this case by formal manipulation.

4.3. As a corollary of Theorem 4 we see that if a section ω of the sheaf I^n is of type (p, q) (as an element of $H^n(X_s, \mathbb{C})$) at some point s, then it has type (p, q) everywhere.

In particular, let D_1 be a divisor in the fiber X_s, and let $c_1(D_s) = \omega_1 \in H^2(X_s, \mathbb{Z})$ is the Chern class of this divisor. Then ω_s has type $(1,1)$ (see Chapter 1, Section 8). If ω_s is invariant under the monodromy action $\pi_1(S, s) \to \mathrm{Aut}(H^2(X_s))$, then it is produced by some class $\omega \in H^2(X, \mathbb{Q})$, also of type $(1,1)$, according to Theorem 4. Therefore, (Chapter 1, Section 8), a rational multiple of ω is the Chern class of a divisor D on $\overline{X}$, such that $D \cap X_s$ is homologous to nD_s for some n. In particular, if the base S is simply connected, then every divisor D_s on X_s is produced (up to rational homology) by a global divisor on $\overline{X}$.

4.4. Let us formulate another consequence of Theorem 4, having to do with the period mapping. Let the algebraic variety S be simply connected, then the Hodge structure on $H^n(X_s, \mathbb{Q})$ is constant (not dependent on s.) If in this case the local Torelli theorem holds for the fibers X_s (see Chapter 2, Section 5), then the morphism $f : X \to S$ is a locally trivial family in the complex topology. In particular, if $f : X \to \mathbb{C}^1$ is a smooth family of varieties for which the local Torelli theorem holds, then this family is locally constant in the complex topology. A similar statement holds when $S = \mathbb{C}^1 \setminus \{0\}$. Indeed, in this case the monodromy is local in character (it comes from going around

0), and hence is quasi-unipotent, according to Landman's theorem (Chapter 5, Section 1.3). Replacing $\mathbb{C}^1\backslash\{0\}$ by an unramified covering, we can assume that monodromy is unipotent. On the other hand, the global monodromy is semi-simple, and hence trivial. This implies that a non-trivial fibration over $\mathbb{P}^1$ of varieties for which the local Torelli theorem holds have at least three degenerate fibers.

§5. Hodge Structure on the Cohomology of Smooth Hypersurfaces

5.1. Let X be smooth hypersurface of degree N in complex projective space $\mathbb{P}^n = \mathbb{P}$. In this section, using the existence of a mixed Hodge structure on cohomology $H^i(U, \mathbb{C})$ of the affine variety $U = \mathbb{P}\backslash X$, we will compute the Hodge numbers $h^{p,q} = \dim H^{p,q}(X)$.

The hypercohomology exact sequence induced by the exact sequence

$$0 \to \Omega^{\cdot}_{\mathbb{P}}(\log X) \xrightarrow{\text{Res}} \Omega^{\cdot-1}_X \to 0,$$

where Res is the Poincaré residue map, is an exact sequence of morphisms of mixed Hodge structures:

$$\ldots \to H^{m-2} \xrightarrow{\gamma} H^m(\mathbb{P}) \xrightarrow{i^*} H^m(U) \xrightarrow{\text{Res}} H^{m-1}(X) \xrightarrow{\gamma} H^{m+1}(\mathbb{P}) \to \ldots, \quad (14)$$

where i^* is induced by the inclusion $i : U \to \mathbb{P}$, γ is the Gysin homomorphism (see Section 3.6) and Res is a morphism of mixed Hodge structures of weight -2 (see Section 3.4).

According to the weak Lefschetz theorem, for $m < n - 1$ the maps

$$j^* : H^m(\mathbb{P}) \to H^m(X)$$

are isomorphisms, where $j; X \hookrightarrow \mathbb{P}$ is the natural inclusion. By duality,

$$\gamma : H^m(X) \to H^{m+2}(\mathbb{P})$$

are also isomorphisms of Hodge structures for $m > n - 1$ and epimorphisms for $m = n - 1$, and, as is well known, for $k \leq n$, $H^{2k}(\mathbb{P}, \mathbb{C}) = H^{k,k}(\mathbb{P})$, and $H^{2k-1}(\mathbb{P}, \mathbb{C}) = 0$. Thus, by Poincaré duality, it is sufficient to compute the Hodge structure on H^{p-1}.

5.2. First of all, let us note that in the exact sequence (14),

$$i^* : H^n(\mathbb{P}) \to H^n(U)$$

is the zero map. Indeed, if n is odd, $H^n(\mathbb{P}) = 0$, while if n is even, then by duality it is sufficient to show that $i^* : H_n(U) \to H_n(\mathbb{P})$ is the zero map. The space $H_n(\mathbb{P})$ is generated by the homology class of $\mathbb{P}^{n/2} \hookrightarrow \mathbb{P}^n$. Suppose that for $\alpha \in H_n(U)$ we have $i_*(\alpha) = k[\mathbb{P}^{n/2}]$. Consider the intersection of $i_*(\alpha)$ with a plane section of X of dimension $n/2$. We have $0 = k \cdot \deg X$, so $k = 0$.

Thus, we have the exact sequence

$$0 \to H^n(U) \xrightarrow{\text{Res}} H^{n-1}(X) \xrightarrow{\gamma} H^{n+1}(\mathbb{P}) \to \dots. \tag{15}$$

Using the Gysin map, it is easy to show that $\operatorname{Ker}[H^{n-1}(X) \to H^{n+1}(\mathbb{P})]$ coincides with the primitive cohomology $P^{n-1}(X, \mathbb{C})$ of the variety X Thus,

$$P^{n-1}(X) = H^n(U)(1), \tag{16}$$

since the Gysin map is a morphism of mixed Hodge structures of weight 2.

5.3. Let us compute the mixed Hodge structure on $H^n(U)$. From (15) it follows that the Hodge structure on $H^n(U)$ is a pure structure of weight $n+1$. As we saw in Section 3.8, the Hodge filtration F on $H^{\cdot}(U, \mathbb{C})$ is induced by the "stupid" filtration on the logarithmic de Rham complex $\Omega_{\mathbb{P}}(\log X)$ and is the limit filtration in the collapsing spectral sequence

$$E_1^{p,q} = H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(\log X)) \Rightarrow H^{p+q}(U, \mathbb{C}).$$

Therefore,

$$F^p H^n(U)/F^{p+1}H^n(U) \simeq H^{n-p}(\mathbb{P}, \Omega_{\mathbb{P}}^p(\log X)).$$

It remains to compute the dimension of the cohomology $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(\log X))$, which is equal to $h^{p,n-p+1}(U)$.

5.4. In order to compute $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(\log X))$ consider the sequence of sheaves

$$0 \to \Omega_{\mathbb{P}}^p(\log X) \to \Omega_{\mathbb{P}}^p(X) \xrightarrow{d} \frac{\Omega_{\mathbb{P}}^{p+1}(2X)}{\Omega_{\mathbb{P}}^{p+1}(X)} \xrightarrow{d} \dots \frac{\Omega_p^{p+k}((k+1)X)}{\Omega_{\mathbb{P}}^{p+k}(kX)} \to \dots, \tag{17}$$

where $\Omega_{\mathbb{P}}^i(mX)$ denotes the sheaf of holomorphic i-forms on U which have poles of at most m-th order along X. It turns out that the sequence (17) is exact. Indeed, let $z_1, \dots, z_n, x$ be local coordinates on $\mathbb{P}$, such that $x = 0$ is the local equation of the divisor X. For $k > 0$ the local section of the sheaf $\Omega_{\mathbb{P}}^{p+k}((k+1)X)/\Omega_{\mathbb{P}}^{p+k}(kX)$ can be written as

$$\omega = \frac{\phi_{p+k-1}(z) \wedge dx}{x^{k+1}} + \frac{\psi_{p+k}(z)}{x^{k+2}}.$$

where ϕ_{p+k-1} and ψ_{p+k} are holomorphic forms, depending on $z_1, \dots, z_{n-1}$. Thus, the image $d\omega \in \Omega_{\mathbb{P}}^{p+k+1}((k+2)X)/\Omega_{\mathbb{P}}^{p+k+1}((k+1)X)$ is

$$d\omega = (-1)^{p+k+1} \frac{(k+1)\psi_{p+k} \wedge d}{x^{k+2}}.$$

Thus, the exact forms are those that can be written as $\frac{\phi \wedge dx}{x^{k+1}}$, but these are also the closed forms. It remains to check the exactness of the sequence (17) in the $\Omega_{\mathbb{P}}^p(X)$ term. We have

$$d\left(\frac{\phi}{x}\right) = \frac{d\phi}{x} + (-1)^{p+1}\frac{\phi}{x^2}\wedge dx \equiv (-1)^{p+1}\frac{\phi}{x^2}\wedge dx \in \Omega_{\mathbb{P}}^{p+1}(2X)/\Omega_{\mathbb{P}}^{p+1}(X).$$

Thus, $\frac{\phi}{x} \in \operatorname{Ker} d$ if and only if $\phi = \psi \wedge dx$, hence $\frac{\phi}{x} = \omega \wedge \frac{dx}{x} \in \Omega_{\mathbb{P}}^p(\log X)$.

5.5. Let us use Bott's theorem (Hartshorne [1977]) to compute $H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k\mathbb{P}^{n-1})) = H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k))$.

Theorem (Bott). *$H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p(k)) = 0$ for $q > 0$ and $k > 0$.*

By examining the cohomology exact sequence associated with the exact sequence of sheaves

$$0 \to \Omega_{\mathbb{P}}^q(k) \to \Omega_{\mathbb{P}}^q(k+1) \to \Omega_{\mathbb{P}}^q(k+1)/\Omega_{\mathbb{P}}^q(k) \to 0,$$

we obtain the following corollary from Bott's theorem:

Corollary.

(1) $H^i(\mathbb{P}, \Omega_{\mathbb{P}}^{p+k}((k+1)X)/\Omega_{\mathbb{P}}^{p+k}(kX)) = 0$ for $i > 0$,
(2)

$$H^0(\mathbb{P}, \frac{\Omega_{\mathbb{P}}^{p+k}((k+1)X)}{\Omega_{\mathbb{P}}^{p+k}(kX)}) = \frac{H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{p+k}((k+1)X))}{H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{p+k}(kX))}$$

for $k > 0$,
(3) $H^q(\mathbb{P}, \Omega^p(\log X))$ coincides with the q-th cohomology of the complex $\{H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{p+k}((k+1)X))/H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{p+k}(kX)), d\}$.

Thus in the case of interest, where $p + q = n$, we have

$$H^{n-p}(\mathbb{P}, \Omega_X^p(\log X)) = \frac{H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n((n-p+1)X))}{H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n((n-p)X)) + dH^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}((n-p)X))}.$$

We now have to compute the bases of $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$ and of $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX))$ and to compute the differential d on $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX))$.

5.6. The basis of $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$. Let $z_0, \ldots, z_n$ be homogeneous coordinates in $\mathbb{P}$, and let X be defined by the homogeneous equation $f(z) = 0$ of degree N.

The projective space $\mathbb{P}^n$ is covered by affine charts $U_i = \{z \in \mathbb{P} | z_i \neq 0\} \simeq \mathbb{C}^n$, $i = 0, 1, \ldots, n$, with the coordinates $\left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)$. Denote the coordinates in U_0 by $x_1 = \frac{z_1}{z_0}, \ldots, x_n = \frac{z_n}{z_0}$, and the coordinates in U_1 by $y_1 = \frac{z_0}{z_1}, \ldots, y_n = \frac{z_n}{z_1}$. Then the section of the sheaf $\Omega_{\mathbb{P}}^n(kX)$ over U_0 can be written in the form

$$\omega = \sum_{\alpha} \frac{c_\alpha x^\alpha}{g_0^k} dx, \tag{18}$$

where

$$g_0(x_1, \ldots, x_n) = \frac{f(z_0, \ldots, z_n)}{z_0^N} = f(1, x_1, \ldots, x_n) = 0$$

is the equation of X in U_0, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $x^\alpha = x^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ and $dx = dx_1 \wedge \ldots \wedge dx_n$. This form ω is in $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$ if ω has no poles over the points of the hyperplane section $z_0 = 0$ (that is, on $\mathbb{P} \backslash U_0$). Thus, we need to move into the chart U_1 and see: under what conditions on α is y_1 not in the denominator of the expression of ω in the local coordinates $y_1, \ldots, y_n$.

We have $x_1 = y_1^{-1}$ and $x_i = y_i y_1^{-1}$ for $i > 1$, thus $dx_1 = -y_1^{-2} dy_1$ and $dx_1 = y_1^{-1} dy_i - y_i y_1^{-2} dy_1$. In addition,

$$g_0(X) = \frac{f(z)}{z_0^N} = \frac{f(z)}{z_1^N} \left(\frac{z_1}{z_0}\right)^N = g_1(y) y_1^{-N},$$

where $g_1(y) = 0$ is the equation of X in the chart U_1. Substituting all of this into (18), we get

$$\omega = \sum_\alpha c_\alpha y_1^{-\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} y_1^{-\alpha_2 - \ldots - \alpha_n} \frac{y_1^{kN}}{g_1^k(y)} y_1^{-n-1} dy_1 \wedge \ldots \wedge dy_n,$$

which simplifies to:

$$\omega = \sum_\alpha c_\alpha y_1^{kN - \alpha_1 - \ldots - \alpha_n - n - 1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} \frac{dy}{g_1^k(y)}.$$

It follows that ω has no poles over the hyperplane section $z_0 = 0$ if and only if, for all non-zero c_α

$$kN - |\alpha| - n - 1 \geq 0,$$

where $|\alpha| = \sum_{i=1}^n \alpha_j$. Setting $\alpha + 1 = (\alpha_1 + 1, \ldots, \alpha_n + 1)$, we finally see that the forms

$$\omega_\alpha = \frac{x^\alpha}{g_0^k(x)} dx, \quad \alpha \in \mathbb{Z}_+^n, \quad |\alpha + 1| < kN$$

are a basis of the space $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$.

5.7. Basis for $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX))$. Let $d\widehat{x}_i = dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n$. Then one can write a section of the sheaf $\Omega_{\mathbb{P}}^{n-1}(kX)$ in the form

$$\sum_{\alpha, i} c_{\alpha, i} \frac{x^\alpha}{g_0^k} d\widehat{x}_i.$$

A computation, quite similar to the one above, to find the conditions for the regularity of this form at infinity (that is, over $\mathbb{P} \backslash U_0$) lead to the following basis for the space $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX))$:

$$\eta_{\alpha, i} = \frac{x^\alpha}{g_0^k} d\widehat{x}_i, \quad |\alpha + 1| \leq kN, \quad i = 1, \ldots, n.$$

$$\tau_\alpha = \frac{x^\alpha}{g_0^k} \left(\sum_{j=1}^n (-1)^j x_j d\widehat{x}_j \right), \quad |\alpha + 1| = kN.$$

5.8. Computation of the differential. We need to compute

$$d : H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX)) \to H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n((k+1)X)).$$

Omitting elements of $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$, we have

$$\begin{aligned} d(\eta_{\alpha,i}) &= d\left(\frac{x^\alpha}{g_0^k} d\widehat{x}_i\right) \\ &\equiv -k\frac{x^\alpha}{g_0^{k+1}}\frac{\partial g_0}{\partial x_i} dx_i \wedge d\widehat{x}_i \\ &= (-1)^i k \frac{\partial g_0}{\partial x_i}\frac{x^\alpha}{g_0^{k+1}} dx \\ &= (-1)^i k \frac{\partial g_0}{\partial x_i}\omega_\alpha, \end{aligned} \tag{19}$$

and

$$\begin{aligned} d\tau_\alpha = d\left(\frac{x_\alpha}{g_0^k}\sum(-1)^j x_j d\widehat{x}_j\right) &\equiv -k\frac{x^\alpha}{g_0^{k+1}} dg_0 \wedge \left(\sum(-1)^j x_j \wedge d\widehat{x}_j\right) \\ = \frac{kx^\alpha}{g_0^{k+1}}\left(\sum x_j \frac{\partial g_0}{\partial x_j}\right) dx. \end{aligned} \tag{20}$$

5.9. Let us recall that the computations of the bases for $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX))$ and $H^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kx))$ and of the differential d was needed for the computation of the Hodge numbers $h^{p,q}(X)$. On the other hand, it is known (see Chapter 2, Section 3), that the Hodge numbers are invariant under smooth deformation. We can thus take X in $\mathbb{P}$ is given by the equation

$$z_0^N + \ldots + z_n^N = 0,$$

and compute the Hodge numbers for this particular variety X.

We have $g_0 = 1 + \sum_{i=1}^n x_i^N$ and $\sum x_j \frac{\partial g_0}{\partial x_j} = N g_0 - N$. From equations (19) and (20) we get that

$$d(\eta_{\alpha i}) = (-1)^i kN x_i^{N-1}\omega_\alpha,$$

$$d(\tau_\alpha) \equiv -kN\omega_\alpha.$$

We see that the basis of the space

$$H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n((k+1)X))/H^0(\mathbb{P}, \Omega_{\mathbb{P}}^n(kX)) + dH^0(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(kX))$$

consists of the forms ω_α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfy the following conditions:

5.9.1 $0 \le \alpha_i \le N-2$, $i = 1, \ldots, n$.

5.9.2. $kN < |\alpha + 1| < (k+1)N$.

5.10. Collecting the results of Sections 5.2-5.9, we finally see that for $p+q=n-1$ the Hodge numbers of the primitive part P^{n-1} of the cohomology $H^{n-1}(X,\mathbb{C})$ of a smooth hypersurface X in $\mathbb{P}^n$ of degree N are equal to

$$h_0^{p,q} = \operatorname{card}\{\beta \in \mathbb{Z}^n | qN < |\beta| < (q+1)N, 0 < \beta_i < N\}$$

– the number of integer points in the hypercube $[1, N-1]^n$ lying strictly between the hyperplanes $\sum_{i=1}^n \beta_i = qN$ and $\sum_{i=1}^n \beta_i = (q+1)N$.

Let $b_0^{n-1} = \dim P^{n-1}(X)$ for a smooth hypersurface X of degree N in $\mathbb{P}^n$. Then

$$b_0^{n-1} = \sum_{p+q=n-1} h_0^{p,q}(X),$$

and is equal to the number of the integer points of the hypercube $[1, N-1]^n$ not contained in the hyperplanes $|\beta| = kN$. Note that projection onto one of the coordinate hyperplanes establishes a one-to-one correspondence between the integer points of the hypercube $[1, N-1]^n$ lying on the hyperplanes $|\beta| = kN$ and the integer points of the hypercube $[1, N-1]^{n-1}$ not lying on the hyperplanes of this same type. Thus, we have a recurrence of the form

$$b_0^{n-1} + b_0^{n-2} = [N-1]^n.$$

For small n we have

$$b_0^0 = N-1, \quad b_0^1 = (N-1)^2 - (N-1).$$

In general, an inductive argument shows that

$$b_0^{n-1} = (N-1)^n - (N-1)^{n-1} + \ldots + (-1)^{n-1}(N-1) = \frac{N-1}{N}[(N-1)^n - (-1)^n].$$

5.11. Let us compute the geometric genus $p_g(X) = \dim H^0(X, \Omega_X^{n-1})$ of a smooth hypersurface of degree N in $\mathbb{P}^n$. From Section 5.10 we know that

$$p_g(X) = h_0^{n-1,0}(X) = \operatorname{card}\{\beta \in \mathbb{Z}^n | \beta_i > 0, 0 < |\beta| < N\}.$$

Thus,

$$p_g(X) = \sum_{k=1}^{N-1} \operatorname{card}\{\beta \in \mathbb{N}^n || \beta| = k\} = \sum_{k=1}^{N-1} \operatorname{card}\{\alpha \in \mathbb{Z}^n | \alpha_i \geq 0, |\alpha| = k-n\}.$$

In particular, $p_g(X) = 0$ for $N \leq n$.

The summands $\operatorname{card}\{\in \mathbb{Z}^n | \alpha_i \leq 0, |\alpha| = l\}$ are equal to the number of monomials of degree l in n variables, and it is easy to check that

$$\operatorname{card}\{\in \mathbb{Z}^n | \alpha_i \geq 0, |\alpha| = l\} = \binom{n+1-1}{l}.$$

Thus, for $N > n$

Table

N	2	3	4	5	6	7
$h^{3,0}(V_N)$	0	0	0	1	5	15
$h^{2,1}(V_N)$	0	5	30	101	255	379

$$p_g(X) = \sum_{k=n}^{N-1} \binom{k-1}{k-n} = \binom{n-1}{N-n-1},$$

or, finally, for $N > n$,

$$p_g(X) = \binom{N-1}{n}.$$

5.12. Let us compute the Hodge numbers of smooth hypersurfaces of degree N in $\mathbb{P}^n$ for some specific values of n and N.

5.12.1. The genus of a plane curve of degree N.

$$g = h^{1,0} = \binom{N-1}{2} = \frac{(N-1)(N-2)}{2}.$$

5.12..2. Surfaces S_N of degree N in $\mathbb{P}^3$.

$$b_0^2 = (N-1)(N^2-3+3),$$

$$h^{2,0} = h^{0,2} = \frac{(N-1)(N-2)(N-3)}{6},$$

$$h_0^{1,1} = b_0^2 - 2h^{2,0} = \frac{(N-1)(2N^2-4N+3)}{3}.$$

Table

N	2	3	4	5	6
$h^{2,0}(S_N)$	0	0	1	4	10
$h^{1,1}(S_N)$	1	6	19	44	85

5.12.3. Threefold of degree N in $\mathbb{P}^4$.

$$b_0^3 = b_3 = (N-1)(N^3-4N^2+6N-4),$$

$$h^{3,0} = h^{0,3} = \frac{1}{24}(N-1)(N-2)(N-3)(N-4),$$

$$h^{2,1} = h^{1,2} = \frac{1}{2}b^3 - h^{3,0} = \frac{1}{24}(N-1)(11N^3-39N^2+46N-24).$$

5.12.4. Quadrics in $\mathbb{P}^b$. Here $N = 2$, and so the cube $[1, N-1]^n$ consists of the single point $(1, \ldots, 1)$, and if n is even then $b_0^{n-1} = 0$, while if $n = 2k+1$, then $b_0^{2k} = h_0^{k,k} = 1$.

5.13. Hodge numbers of complete intersections. Consider k non-singular hypersurfaces $X^{(a_1)}, \ldots, X^{(a_k)}$ of degrees $a_1, \ldots, a_k$ in $\mathbb{P}^{N+k}$. If these hypersurfaces are in general position, then the complete intersection $X(a_1, \ldots, a_k) = X^{(a_1)} \cap \ldots \cap X^{(a_k)}$ is a non-singular projective variety. Hodge numbers of a complete intersection, just as in the case of a single hypersurface, depend only on the numbers N and $a_1, \ldots, a_k$, since Hodge numbers do not change under holomorphic deformation.

There is also a beautiful formula of Hirzebruch (Hirzebruch [1966]), relating the Hodge numbers of complete intersections. In order to state it here, we will need the following notation. For an arbitrary coherent sheaf F on X let

$$\chi^p(X, F) = \chi(X, F \otimes \Omega_X^p),$$

where

$$\chi(X, F) = \sum_{q=0}^{n} (-1)^q \dim H^q(X, F)$$

is the Euler characteristic of the sheaf F. Introduce the formal polynomials

$$\chi_y(X, F) = \sum_{p \geq 0} \chi(X, F) y^p.$$

In the special case where $F = \mathcal{O}_X$ is the structure sheaf on X,

$$\begin{aligned} \chi_y(X, \mathcal{O}_X) &= \sum_{p \geq 0} \chi(X, \Omega_X^p) y^p \\ &= \sum_{p,q \geq 0} (-1)^q y^p \dim H^q(X, \Omega_X^p) \\ &= \sum_{p,q \geq 0} (-1)^q y^p h^{p,q}. \end{aligned}$$

Hirzebruch Signature Formula. Let $X_N = X(a_1, \ldots, a_k)$ be a complete intersection in $\mathbb{P}^{n+k}$ and let $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes^m}$, where $\mathcal{O}_X(1)$ is the sheaf associated with the hyperplane section on X_N. Then

$$\sum_{N=0}^{\infty} \chi_y(X_N, \mathcal{O}_X(m)) z^{N+k} = \frac{(1+zy)^{m-1}}{(1-z)^{m+1}} \prod_{i=1}^{k} \frac{(1+zy)^{a_i}(1-z)^{a_i}}{(1+zy)^{a_i} + y(1-z)^{a_i}}.$$

Hirzebruch's formula follows from the proof of the Riemann-Roch formula (see Hirzebruch [1966]).

We are interested in the numbers $h^{p,q} = \dim H^q(X, \Omega_X^p)$, and so we should set $m = 0$. Then

$$\sum_{N=0}^{\infty} \chi_y(X_N, \mathcal{O}_X) z^{N+k} = \frac{1}{(1+zy)(1-z)} \prod_{i=1}^{k} \frac{(1+zy)^{a_i} - (1-z)^{a_i}}{(1+zy)^{a_i} + y(1-z)^{a_i}}.$$

This formula, together with the Lefschetz theorem on hyperplane section (see Chapter 1, Section 9) allows us to compute Hodge numbers of complete intersections.

§6. Further Development of the Theory of Mixed Hodge Structures

In recent years, the theory of mixed Hodge structures continues developing explosively. In this section we will briefly indicate some of the directions of this development.

6.1. Variation of mixed Hodge structures with a graded polarization.

Definition. *A variation of mixed Hodge structures* is an ordered quadruple $(S, H_{\mathbb{Z}}, W, F)$, consisting of a local system $H_{\mathbb{Z}}$ of free $\mathbb{Z}$-modules of finite rank over a connected complex manifold S, and two filtrations: a decreasing filtration W on $H_{\mathbb{Z}}$ by primitive local subsystems and an increasing filtration F by holomorphic subbundles of the holomorphic bundle $H_{\mathcal{O}} = H_{\mathbb{Z}} \otimes \mathcal{O}_S$ satisfying the following the conditions:

(1) For each point $s \in S$, the fiber $(H_{\mathbb{Z}}, W, F)(s)$ is a mixed Hodge structure (see Chapter 4, Section 1).
(2) The Gauss-Manin connection ∇ on $H_{\mathcal{O}}$ corresponding to the local system $H_{\mathbb{Z}}$ (see Chapter 2, Section 4) satisfies the condition

$$\nabla F^p H_{\mathcal{O}} \subset \Omega_S^1 \otimes F^{p-1} H_{\mathcal{O}}$$

for all p.

Suppose that we have a collection $Q = \{\mathcal{O}_k\}$ of locally constant $(-1)^k$-symmetric $\mathbb{Q}$-valued bilinear forms on

$$\operatorname{Gr}_k^W H_{\mathbb{Q}} = W_k \otimes \mathbb{Q} / W_{k-1} \otimes \mathbb{Q},$$

such that for all k and for all $s \in S$, the form Q_k defines a polarization in the fiber $(\operatorname{Gr}_k^W H_{\mathbb{Z}})(s)$, that is,

$$Q_k((F^p \operatorname{Gr}_k^W H_{\mathcal{O}})(s), (F^{k-p+1} \operatorname{Gr}_k^W H_{\mathcal{O}})(s)) = 0$$

for all p, and

$$Q_k(Cu, \overline{u}) > 0$$

for all the non-zero $u \in (\operatorname{Gr}_k^W H_{\mathcal{O}})(s)$, where C is the Weil operator on $\operatorname{Gr}_k^W H_{\mathcal{O}}$ defined by the filtration F (see Chapter 1, Section 6). Under these assumption, the variation $(S, H_{\mathbb{Z}}, W, F)$ is called *variation of mixed Hodge structures with a graded polarization.* In Section 1 of Chapter 2 we defined a classifying space for polarized Hodge structures. This definition can be extended to the case of mixed Hodge structures with a graded polarization. To wit, let $(H_{\mathbb{Z}}(0), W, F, Q)$ be the fiber of the variation of mixed Hodge structures with graded polarization over the point $0 \in S$. Let $s^p = \dim F H_{\mathbb{C}}(0)$ and let $f_k^p = \dim F^p \operatorname{Gr}_k^W H_{\mathbb{C}}(0)$. Just as in Chapter 2, let

$$\mathcal{F} = \{F \in \operatorname{Flag}(H_{\mathbb{C}}; \ldots, f^p, \ldots) | \dim F^p \operatorname{Gr}_k^W H_{\mathbb{C}}(0) = f_k^p \, \forall p, k\}$$

be the flag space, and let

$$\pi_k : \mathcal{F} \to \mathcal{F}_k = \operatorname{Flag}(\operatorname{Gr}_k^W H_{\mathbb{C}}(0); \ldots, f_k^p, \ldots)$$

be the map sending $F H_{\mathbb{C}}(0)$ to $F \operatorname{Gr}_k^W H_{\mathbb{C}}(0)$.

The polarization Q allows us to define spaces

$$\check{D}_k = \{F \in \mathcal{F}_k | Q_k(F^p, F^{k-p+1}) = 0, \forall p\},$$

$$D_k = \{F \in \check{D}_k | Q_k(Cu, \overline{u}) > 0, \forall u \in \operatorname{Gr}_k^W H_{\mathbb{C}}(0), u \neq 0\}.$$

Set $\check{D} = \bigcap_k \pi_k^{-1}(\check{D}_k) \in \mathcal{F}$, and $D = \bigcap_k \pi_k^{-1}(D_k) \subset \check{D}$.

Let us introduce maps $\check{\pi} : \check{D} \to \prod_k \check{D}_k$, which sends F to $(\ldots, \pi_k(F), \ldots)$ and $\pi : D \to \prod_k D_k$, which is the restriction of $\check{\pi}$ to D.

On the space $\mathcal{F}$ there is an action by the group

$$\operatorname{GL}_W H_{\mathbb{C}}(0) = \{g \in \operatorname{GL} H_{\mathbb{C}}(0) | g W_k = W_k \quad \forall k\}.$$

Let

$$\begin{aligned} G_{\mathbb{C}} &= \{g \in \operatorname{GL}_W H_{\mathbb{C}}(0) | \operatorname{Gr}_k^W g \text{ preserves } Q_k \text{ for all } k\}, \\ G_{\mathbb{R}} &= \{g \in G_{\mathbb{C}} | g H_{\mathbb{R}}(0) = H_{\mathbb{R}}(0)\}, \\ G_{\mathbb{Z}} &= \{h \in G_{\mathbb{C}} | g H_{\mathbb{Z}}(0) = H_{\mathbb{Z}}(0)\}. \end{aligned}$$

Let G be the group

$$G = G'_{\mathbb{C}} \cdot (G_{\mathbb{R}} \cap G''_{\mathbb{C}}),$$

where $G_{\mathbb{C}} = G'_{\mathbb{C}} \cdot G''_{\mathbb{C}}$ is the decomposition of $G_{\mathbb{C}}$ into the unipotent radical $G'_{\mathbb{C}}$ and the semistable part $G''_{\mathbb{C}}$.

Just as in Chapter 2, it can be shown (see Saito–Shimizu–Usui [1987], Siegel [1955]) that G acts transitively on D, $G_{\mathbb{Z}}$ acts properly discontinuously on D, and $\check{\pi} : \check{D} \to \prod_k \check{D}_k$ is a homogeneous algebraic vector bundle relative to the action of the group $G_{\mathbb{C}}$. In the tangent space $T_{\check{D}}$ we can define a holomorphic subbundle of $T_{\check{D}}^{\text{eh}}$ of horizontal vectors, and (Siegel [1955]) $T_{\check{D}}^{\text{eh}}$ agrees under the $\check{\pi}$ action with the horizontal subbundle $\oplus T_{\check{D}_k}^k$ on $\prod_k \check{D}_k$.

For a variation of mixed Hodge structures with graded polarization $(S, H_{\mathbb{Z}}, W, F, Q)$ we can define a mixed period mapping

$$\Phi : S \to \Gamma \backslash D_k,$$

where $\Gamma = \mathrm{Im}(\pi_1(S,0) \to G_{\mathbb{Z}})$. This map Φ is compatible with respect to π with the period mappings

$$\Phi_k : S \to \Gamma_k \backslash D_k,$$

defined by the variations of polarized Hodge structures $(S, \mathrm{Gr}_k^W H_{\mathbb{Z}}, F\,\mathrm{Gr}_k^W H_{\mathcal{O}}, Q_k)$ for all k, where $\Gamma_k = \mathrm{Gr}_k^W \Gamma$.

It can be shown (Saito–Shimizu–Usui [1987], Siegel [1955]) that the map Φ has a locally horizontal lift, since this is so for the period maps Φ_k (see Chapter 2, Section 3).

6.2. Deformation of a smooth pair. *A smooth family of a pair* is a quadruple $(\mathcal{H}, \mathcal{Y}, f, S)$, consisting of a connected complex manifold S, a proper smooth morphism $f : \mathcal{H} \to S$ of a complex manifold $\mathcal{H}$ and a divisor with normal crossings $\mathcal{Y} = \cup \mathcal{Y}_i$ in $\mathcal{H}$, such that the intersections $\mathcal{Y}_{i_1} \cap \ldots \cap \mathcal{Y}_{i_k}$ are smooth manifolds on S for all of the choices $i_1, \ldots, i_k$.

Let X be a compact complex manifold and let Y be a divisor with normal crossings in X. Let

$$T_X(-\log Y) = \{\Theta \in T_X | \Theta I_Y \subset I_Y\},$$

where I_Y is a sheaf of ideals of the divisor Y in X. It can be shown (Kashiwara [1985]), that there exists a semi-universal family of deformations of the pair (X, Y) (analogous to the Kuranishi family, see Chapter 2, Section 5); furthermore, $T_X(-\log Y)$ coincides with the sheaf of infinitesimal automorphisms of the pair (X, Y). The cohomology $H^1(T_X(-\log Y))$ coincides with the set of infinitesimal deformations of the pair (X, Y), and $H^2(T_X(-\log Y))$ coincides with the space of obstructions.

Just as in the classical case, for the smooth family of the pair $(\mathcal{H}, \mathcal{Y}, f, S)$ we can define the Kodaira-Spencer mapping

$$\rho_s : T_s(s) \to H^1(T_{\mathcal{H}_s}(-\log \mathcal{Y}_s)),$$

where $\mathcal{H}_s = f^{-1}(s)$ is the fiber over the point $s \in S$ (analogous to $\mathcal{Y}_s$).

6.3. The period mapping for the smooth family of a pair. Let $(\mathcal{H}, \mathcal{Y}, f, S)$ be a smooth family of a pair, and let f be a projective morphism. Let us define the period mapping for this family. It can be shown (Saito–Shimizu–Usui [1987], Siegel [1955], see also Chapter 4, Section 2) that the cohomology spectral sequences of the relative logarithmic de Rham complex $\Omega^{\cdot}_f(\log \mathcal{Y})$, defined by the weight filtration and the Hodge filtration F, collapse, respectively in the ${}_WE_2 = {}_WE_\infty$ and ${}_FE_1 = {}_FE_\infty$. Thus, we have a variation of mixed Hodge structures with graded polarization $(S, R^n_{\mathbb{Z}}(\breve{f}), W[n], F, Q)$, where $R^n_{\mathbb{Z}}(\breve{f})$ denotes the the sheaf $R^n f_* \mathbb{Z}_{\mathcal{H} \setminus \mathcal{Y}}$ modulo torsion, and $(W[n]_k)_{\mathbb{Z}}$ denotes the primitive part of the sheaf $(W[n]_k)_{\mathbb{Q}} \cap R^n_{\mathbb{Z}}(\breve{f})$. For the variation $(S, R^n_{\mathbb{Z}}(\breve{f}), W[n], F, Q)$ we also have the period mapping of mixed Hodge structures

$$\Phi : S \to \Gamma\backslash D.$$

Let $\tilde{\Phi}$ be the local lifting of Φ at s. Then (Saito–Shimizu–Usui [1987]) the diagram (compare Chapter 2, §5)

$$\begin{tikzcd}
T_S(s) \arrow[r, "d\tilde{\Phi}_s"] \arrow[d, "\rho_S"'] & T_D^{eh}(\tilde{\Phi}(S)) \subset T_D(\tilde{\Phi}(S)) \\
H^1(T_{\mathcal{X}_s}(-\log \mathcal{Y}_s)) \arrow[r, "\epsilon"] & \bigoplus_p S_p \arrow[u, "\wr"']
\end{tikzcd}$$

is commutative up to sign (where
$S_p = \mathrm{Hom}_{(W,Q)}(H^{n-p}(\Omega^p_{\mathcal{X}_s}(\log \mathcal{Y}_s)), H^{n-p+1}(\Omega^{p-1}_{\mathcal{X}_s}(\log \mathcal{Y}_s))))$.

Using this diagram, we can obtain an infinitesimal mixed Torelli theorem (see Chapter 2) for various classes of smooth pairs. Let us mention one such result, due to Griffiths [1983b] and Green [1984]. Let (X, Y) be a smooth pair, $\dim X \geq 2$, and Y be a very ample divisor on X. Let $\mathcal{O}_X(1) = \mathcal{O}_X(Y)$ and let $D_1(\mathcal{O}_X(1), \mathcal{O}_X(1))$ be the sheaf of first-order differential operators on the sections of the sheaf $\mathcal{O}_X(1)$. Let $\Delta \subset X \times X$ be the diagonal and let $p_i : \Delta \to X$ be the projection maps. We have the following:

Theorem (Green [1984], Griffiths [1983b]). *Let Y be a smooth subvariety of X such that*

$$H^q(\wedge^{q-1} D_1(\mathcal{O}_X(1), \mathcal{O}_X(1))(-q)) = 0$$

for $1 \leq q \leq n-1$ and, furthermore,

$$H^1(I_\Delta \otimes p_1^* K_X(1) \otimes p_2^* K_x(n-1)) = 0,$$

where K_X is the canonical sheaf on X and $K_X(m) = K_X \otimes \mathcal{O}_X(-1)^{\otimes^m}$. Then the map

$$\epsilon_n : H^1(T_X(-\log Y)) \to \mathrm{Hom}(H^0(\Omega^n_X(\log Y)), H^1(\Omega^{n-2}_Y))$$

is an inclusion.

There are also several results (Donagi [1983], Saito [1986], Green [1984]) having to do with the generic Torelli theorem. Here is one:

Theorem (Green [1984]). *Let X be a smooth projective variety of dimension $n \geq 2$, the canonical class of which is very ample, and let L be a sufficiently ample sheaf on X. Then the period mapping*

$$\Phi_{n+1} : |L|_{\mathrm{reg}}/\operatorname{Aut}(X, L) \to G_{n+1,\mathbb{Z}}\backslash D_{n+1}$$

has degree one over its image, where $|L|_{\mathrm{reg}}$ is the set of smooth elements of the linear system $|L|$.

6.4. Mixed Hodge structures on homotopy groups. The methods used by Deligne to introduce mixed Hodge structures on the cohomology of algebraic varieties can in certain cases be used successfully to introduce such structures on the homotopy groups of algebraic varieties. This has to do with the fact that the real homotopy groups

$$\pi_n(X,\mathbb{R}) = \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R} \quad (n > 0)$$

of a C^∞ manifold X can be obtained by certain formal algebraic construction from the de Rham complex of these manifolds. Let us briefly describe some aspects of this theory. For a more detailed introduction the reader is referred to the survey Deligne- Griffiths-Morgan-Sullivan [1975].

Let $\mathcal{A}$ be a *graded differential algebra* over a field K. This means that

$$\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}^k$$

and multiplication satisfies

$$x \cdot y = (-1)^{kl} y \cdot x, x \in \mathcal{A}^k, y \in \mathcal{A}^l.$$

In addition, the algebra $\mathcal{A}$ has a differential, that is, a map $d : \mathcal{A} \to \mathcal{A}$ satisfying the conditions

(1) $d^2 = 0$;
(2) $d(\mathcal{A}^k) \subset \mathcal{A}^{k+1}$;
(3) $d(x \cdot y) = dx \cdot y + (-1)^k x \cdot dy, x \in \mathcal{A}^k$.

To any graded differential algebra we can associated the graded differential algebra of cohomology groups

$$H(\mathcal{A}) = \bigoplus_{k\geq 0} H^k(\mathcal{A}),$$

equipped with the zero differential. We will call a graded differential algebra *connected* if $\mathcal{A}^0 \simeq K$, and *simply-connected*, if it is connected and $H^1(\mathcal{A}) = \{0\}$.

For a simply-connected graded differential algebra $\mathcal{A}$ we can define the *augmentation ideal*

$$A(\mathcal{A}) = \oplus_{k>0}\mathcal{A}^k$$

and the *space of indecomposable elements*

$$I(\mathcal{A}) = A(\mathcal{A})/A(\mathcal{A}) \cdot A(\mathcal{A}).$$

The differential d is called *decomposable*, if for any element $x \in \mathcal{A}$

$$dx \in A(\mathcal{A})A(\mathcal{A}).$$

Let V be a vector space, and let n be a natural number. The free algebra $\Lambda_n(V)$ over V is defined to be the polynomial algebra generated by V for even

n, and the exterior algebra over n if n is odd. The grading will be chosen in such a way that the elements of the generating space are elements of weight n.

An *elementary extension* of a graded differential algebra $\mathcal{A}$ is a graded differential algebra of the form $\mathcal{B} = \mathcal{A} \otimes \Lambda_n(V)$, if the differentials $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$ of the algebras satisfy the conditions

$$d_{\mathcal{B}}|_{\mathcal{A}} = d_{\mathcal{A}}; \quad d_{\mathcal{B}}(V) \subset \mathcal{A}.$$

It is clear that $d_{\mathcal{B}}$ is decomposable if and only if $d_{\mathcal{A}}$ is decomposable and $d_{\mathcal{B}}(V) \subset A(\mathcal{A})A(\mathcal{A})$.

A graded differential algebra $\mathcal{M}$ is called *minimal,* if it can be represented as an increasing union of graded differential subalgebras:

$$\mathcal{M}_0 = K \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \bigcup_{i \geq 0} \mathcal{M}_i = \mathcal{M},$$

where each extension $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ is elementary and $d_{\mathcal{M}}$ is decomposable.

A triple $(\mathcal{A}, \mathcal{M}, \rho)$, where $\mathcal{A}$ and $\mathcal{M}$ are graded differential algebras and $\rho : \mathcal{M} \to \mathcal{A}$ is a morphism of graded differential algebras is called a *minimal model* of the graded differential algebra $\mathcal{A}$ if

(1) $\mathcal{M}$ is minimal;
(2) ρ induces an isomorphism of cohomology algebras.

It turns out that any simply connected graded differential algebra $\mathcal{A}$ has a minimal model, defined up to isomorphism.

Let us now look at the topological applications of these constructions. Let X be a C^∞-manifold, and

$$\mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \ldots$$

its de Rham complex $\mathcal{E}(X)$. The de Rham complex is obviously a graded differential algebra. Let X be simply connected, then $\mathcal{E}(X)$ is a simply-connected graded differential algebra. Let $(\mathcal{E}, \mathcal{M}, \rho)$ be its minimal model.

Consider, for $i \geq 2$, the space

$$\mathcal{L}_i = (I(\mathcal{M})^i)^*,$$

conjugate to the space of the indecomposable elements of weight i. On the graded space

$$\mathcal{L} = \bigoplus_{i=2} \mathcal{L}_i$$

we can naturally define a graded Lie algebra structure:

$$[\mathcal{L}, \mathcal{L}_l] \subset \mathcal{L}_{h+l-1}.$$

This structure is defined by the map dual to the map

$$d : I(\mathcal{A}) \to I(\mathcal{A}) \otimes I(\mathcal{A}).$$

It is a remarkable fact that for $n \geq 2$ there are isomorphisms

$$\mathcal{L}_n \simeq \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R} = \pi_n(X, \mathbb{R}).$$

These isomorphisms can be chosen in such a way that the bracket [,] on $\mathcal{L}$ coincides with the Whitehead product

$$\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X)$$

on homotopy groups.

These results, due, to the most part, to D. Sullivan, were applied by John Morgan [1978] to compute mixed Hodge structures on homotopy groups.

First, let us note (Deligne- Griffiths-Morgan-Sullivan [1975]) that if X is a Kähler manifold, then the minimal model of the de Rham complex $\mathcal{E}(X)$ of X is the minimal model of its cohomology complex $H^*(X)$. Indeed, for a Kähler manifold there is an inclusion of graded differential algebras

$$H^*(X) \xrightarrow{\phi} \mathcal{E}^*(X),$$

since $H^n(X)$ is just the set of harmonic forms on X, and exterior products of harmonic forms on a Kähler manifolds are harmonic forms. The inclusion ϕ defines an isomorphism on cohomology. Thus, if we take a minimal model of the algebra $H^*(X)$, we, by definition, obtain a minimal model for the de Rham complex. The filtrations defining a Hodge structure on the algebra $H^*(X)$ can be formally transferred to the minimal model, thereby defining a mixed Hodge structure on $\mathcal{M}$ and hence on the groups $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

In general, if X is a non-singular algebraic variety over $\mathbb{C}$, X can be embedded into a compact Kähler manifold as the complement to a divisor with normal crossings. The weight filtration and the Hodge filtration on differential forms with logarithmic singularities also induce certain filtrations on the minimal model of the complex $\mathcal{E}(X)$, which leads to the appearance of mixed Hodge structures on the homotopy groups of the variety X.

An analogous approach can be applied to the fundamental group of an algebraic variety X, see the details in the references cited above. We will merely formulate some of the results.

Theorem (Morgan [1978]). *Let X be a non-singular complex algebraic variety with $\pi_1(X) = 0$. Then there exists a natural finite mixed Hodge structure on $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Whitehead product*

$$\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X)$$

is a morphism of mixed Hodge structures.

Recall that if X is simply connected, then for any two points $x, y \in X$ the groups $\pi_n(X, x)$ and $\pi_n(X, y)$ are naturally isomorphic. It turns out that this isomorphism is an isomorphism of mixed Hodge structures.

Now consider a non-simply-connected algebraic variety X with a basepoint $x \in X$. Let $\mathbb{Z}\pi_1(X,x)$ be the group ring of the group $\pi_1(X,x)$ and let $I \subset \mathbb{Z}\pi_1(X,x)$ be the augmentation ideal, that is, the kernel of the natural map

$$\mathbb{Z}\pi_1(X,x) \to \mathbb{Z}.$$

Theorem (Morgan [1978], Hain [1987]). *Let X be a non-singular algebraic variety, and let $x \in X$. Then for any $s \geq 0$, there is a natural mixed Hodge structure on the $\mathbb{Z}$-module $\mathbb{Z}\pi_1(X,x)/I^{s+1}$. The structures thus defined are functorial with respect to morphisms of varieties with basepoints.*

Chapter 5
Degenerations of Algebraic Varieties

§1. Degenerations of Manifolds

1.1. Let $\pi : X \to \Delta$ be a proper map of a Kähler manifold X onto the unit disk $\Delta = \{t \in \mathbb{C} | |t| < 1\}$, such that the fibers $X_t = \pi^{-1}(t)$ are nonsingular compact complex manifolds for every $t \neq 0$. We will call such a map π a *degeneration*, and the fiber $X_0 = \pi^{-1}(0)$ will be called the *degenerate fiber.*

Let us call a map $\psi : Y \to \Delta$ a *modification* of a degeneration π if there exists a bimeromorphic map $f : X \to Y$, biholomorphic outside the degenerate fibers, and such that the diagram below commutes.

$$\begin{tikzcd} Y \arrow[rr, dashed, "f"] \arrow[dr, "\psi"'] & & X \arrow[dl, "\pi"] \\ & \Delta & \end{tikzcd}$$

According to Hironaka's theorem (Hartshorne [1977]), every map can be modified into a degeneration such that the degenerate fiber X_0 is a divisor with normal crossings, that is, the map π in a neighborhood of each point $x \in X_0$ is defined by equations

$$x_1^{a_1} \cdot x_2^{a_2} \cdot \ldots x_{n+1}^{a_{n+1}} = t, \quad a_i \in \mathbb{Z}, a_i \geq 0, \tag{1}$$

where $x_1, \ldots, x_{n+1}$ is a local coordinate system in a neighborhood of the point x. The degeneration is called *semistable* if $a_i \leq 1$ in equation (1) above. In other words, the degeneration is semistable, if the degenerate fiber is a reduced divisor with normal crossings.

When studying many aspects of degenerations of manifolds, it is often enough to restrict ones attention to semistable degenerations. However, not every degeneration can be modified into a semistable one. Nonetheless, there is the following theorem (Mumford, Kempf *et al* [1973]), which makes it possible to reduce any degeneration to a semistable one after a change of base.

Let $\alpha : \Delta \to \Delta$ be a holomorphic self-map of the disk Δ, such that $\alpha(0) = 0$. Starting with a degeneration $\pi : X \to \Delta$, we can construct a new degeneration $\pi_\alpha = \mathrm{pr}_\Delta : X_\alpha = X \times_\Delta \Delta \to \Delta$, obtained from the degeneration π by way of the change of base $\alpha : \Delta \to \Delta$:

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d, "\pi_\alpha"] & X \arrow[d, "\pi"] \\ \Delta \arrow[r] & \Delta \end{tikzcd}$$

The semistable reduction theorem. Let $\pi : X \to \Delta$ be a degeneration. Then there exists a base change $\alpha : \Delta \to \Delta$ (defined as $t \mapsto t^k$, for some $k \in \mathbb{N}$) and a semistable degeneration$\psi : Y \to \Delta$, which is a modification of the degeneration $\pi_\alpha : X_\alpha \to \Delta$. Furthermore, the modification $f : Y \to X_\alpha$ is a composition of blowings-up and blowings-down of nonsingular submanifolds of the degenerate fibers.

1.2. Topology of semistable degenerations. Let us start the study of semistable degenerations with the study of degenerations of curves. For each $p \in X$ of a semistable degeneration of curves $\pi : X \to \Delta$ we can choose a neighborhood U and local coordinates x and y so that $U \simeq \{|x| < 2, |y| < 2\}$, and the map π can be written (after a linear change of coordinates) as either

$$\pi(x, y) = x, \tag{1}$$

or

$$\pi(x, y) = xy, \tag{2}$$

where in the second case the point p is a singularity of the degenerate fiber X_0. Set $U_t = X_t \cap U$.

In the first case the fiber U_t and the degenerate fiber U_0 have the same structure. We actually have an isomorphism $c_t : U_t \to U_0$, defined by $(t, y) \to (0, y)$.

In the second case let $U_x = \{y = 0\}$ and $U_y = \{x = 0\}$. Then $U_0 = \{xy = 0\} = U_x \cup U_y$ and $U_x \cap U_y = p$. The fiber $U_1 = \{xy = 1\}$ can also be covered by two charts $U_{1,y} = \{|x| \leq 1\}$ and $U_{1,x} = \{|y| \leq 1\}$. Define $\rho_y : U_{1,uy} \to U_y$ and $\rho_x : U_{1,x} \to U_x$ by setting $\rho_y(x, y) = (0, y)$ and $\rho_x(x, y) = (x, 0)$. Let $\tilde{U}_x = \rho_x(U_{1,x})$ and $\tilde{U}_y = \rho_y(U_{1,y})$ and let $\partial\tilde{U}_x = \rho_x(U_{1,x} \cap U_{1,y})$, and $\partial\tilde{U}_y = \rho_y(U_{1,x} \cap U_{1,y})$. It is easy to see that $\tilde{U}_x$ is the annulus $\{1 \leq |x| < 2\}$ and $\tilde{U}_y = \{1 \leq |y| < 2\}$ while $\partial\tilde{U}_x = \{|y| = 1\}$ is the boundary of $\tilde{U}_x$ and similarly $\partial\tilde{U}_y$ is the boundary of $\tilde{U}_y$). Furthermore, the fiber U_1 can be obtained from $\tilde{U}_x$ and $\tilde{U}_y$ by gluing them along their boundaries $\partial\tilde{U}_x$ and $\partial\tilde{U}_y$.

Let $f : \{1 \leq |t| < 2\} \to \{|z| < 2\}$ be the map defined by the formula $f(t) = (|t| - 1)t$. The map f contracts the boundary $\{|t| = 1\}$ to 0, and outside the boundary f is a bijective map from the annulus $\{1 < |t| < 2\}$ onto the punctured disk $\{0 < |z| < 2\}$. Thus, the maps $f \circ \rho_x$ and $f \circ \rho_y$ define a map $c_1 : U_1 \to U_0$ (where U_1 is viewed as $\tilde{U}_x$ glued with $\tilde{U}_y$). The map c_1 is bijective over all points other than p, and $c_1^{-1}(p) \simeq S^1 = \{|y| = 1\} = \partial\tilde{U}_x$.

Using standard partition of unity arguments, we can combine the local maps $c_1 : U_1 \to U_0$ into a global map $c_1 : X_1 \to X_0$, which is bijective outside the singularities of the fiber X_9, and if p is a singularity of X_0, then $c_1^{-1}(p) \simeq S^1$.

The construction above can be generalized to arbitrary dimension. Specifically, for a fiber X_t of a semistable degeneration $\pi : X \to \Delta$ there exists a map $c_t : X_t \to X_0$, such that c_t is bijective outside the fiber X_0, and for singularities p

$$c_t^{-1} = (S^1)^k,$$

if p lies in the intersection of precisely $k + 1$ different components of the degenerate fiber X_0. Furthermore, the map $c_t : X_t \to X_0$ can be obtained as the restriction of the map $c : X \to X_0$ onto x_t, where c is a deformation retract of X onto X_0, compatible with the radial retraction $\Delta \to \{0\}$ (see Clemens [1977]). This is the so-called *Clemens mapping.*

1.3. Let $\pi^* : X^* \to \Delta^*$ be the restriction of the map π onto the punctured disk $\Delta^* = \Delta\backslash 0$. The restriction π^* is a smooth proper morphism. Therefore, we are in the situation described in Chapter 1, Section 10, and thus we have the monodromy map $T : H^k(X_t) \to H^k(X_t)$ on the cohomology of a fixed fiber X_t. This map is generated by the $\mathbb{R}$-action on X, which lifts the rotation of Δ by the angle $2\pi\phi$.

We should note that the Clemens mapping $c : X \to X_0$ can be constructed so as to commute with the above-mentioned $\mathbb{R}$ action of X. Thus the monodromy T acts on the sheaf $R^q c_{t_*}\mathbb{Z}$, and on the Leray spectral sequence corresponding to the map $c_t : X_t \to X_0$ (Deligne [1972])

$$E_2^{p,q} = H^q(X_0, R^q c_{t_*}\mathbb{Z}\} \Rightarrow H^{p+q}(X_t, \mathbb{Z}).$$

This allows us to deduce the theorem on the quasi-unipotence of the monodromy action:

Theorem (Landman, see Griffiths [1970]). *Let $\pi : X \to \Delta$ be a degeneration, and let $T : H^m(X_t) \to H^m(X_t)$ be the Picard-Lefschetz transformation. Then*

(1) T is quasi-unipotent with nilpotence index m, that is, there exists a $k > 0$, such that

$$(T^k - \mathrm{id})^{m+1} = 0.$$

(2) If π is a semistable degeneration, then T is unipotent, that is, $k = 1$.

The idea of the proof of this theorem is as follows. Consider the fiber of the sheaf $R^q c_{t_*}\mathbb{Q}$ at the point $p \in X_0$. Suppose that in the neighborhood of this point the fiber X_0 is defined by the equation

$$z_1^{a_1} \cdot z_2^{a_2} \cdot \ldots \cdot z_r^{a_r} = 0,$$

where we are assuming that the fiber X_0 is a divisor with normal crossings. Let $a = \gcd(a_1, \ldots, a_r)$ and let $b_1 a_1 + \ldots + b_r a_r = a$. In the neighborhood of the point p the rotation of the disk Δ by angle ϕ can be locally lifted to the transformation of the neighborhood by the map

$$(z_1, \ldots, z_r, z_{r+1}, \ldots, z_{n+1}) \to (\exp(\frac{ib_1\phi}{a})z_1, \ldots, \exp(\frac{ib_r\phi}{a}), z_{r+1}, \ldots, z_{n+1})).$$

The monodromy T is induced by the rotation by 2π, and since this is the trivial action, so is the action of T^a on $(R^* c_{t_*}\mathbb{Q})_p$. It follows that T^k, where $k =$ gcd of the multiplicities of the components of the divisor X_0 acts trivially on the E_2 term of the Leray spectral sequence for c_t. Therefore, T^k acts trivially on E_∞, and thus T^k acts unipotently on $H^k(X_t)$.

1.4. Let consider degenerations of curves in greater detail. Let $\pi : X \to \Delta$ be a degeneration, whose general fiber X_t is a complete curve of genus g. For different t, the corresponding fibers X_t are homologous to each other, and do not intersect. Thus, the self-intersection index vanishes:

$$(X_0^2)_X = 0.$$

Let us write the degenerate fiber as $X_0 = \sum r_i C_i$, where C_i are the irreducible components. We can assume that none of the components C_i are exceptional curves of the first kind, that is, such that $C_i \simeq \mathbb{P}^1$, and $(C_i^2)_X = -1$. By Castelnuovo's blowing-down criterion (Moishezon [1966]), these curves can we blown down to points by monoidal transformations (see Chapter 1, Section 1).

There are certain topological conditions which must be satisfied by the components C_i. Firstly, the intersection matrix $((C_i, C_j)_X)$ is negative semi-definite, and

$$(\sum s_i C_i, \sum s_i C_i))_X = 0$$

if and only if the divisor $\sum s_i C_i$ is a multiple of the degenerate fiber X_0. This follows from the Hodge index theorem (see Chapter 1, Section 7).

Furthermore, it is known that the arithmetic genus $p_a(C)$ of an irreducible curve C on the surface X is

$$p_a(C) = \frac{(C, K_X)_X + (C^2)_X}{2} + 1 \geq 0,$$

where K_X is the canonical class of the nonsingular surface X. In particular, since X_t and X_0 are homologous and $(X_t^2)_X = 0$,

$$\sum r_i(C_i, K_X) = 2g - 2.$$

If we denote the arithmetic genus of the curve C_i by p_i, , we get the following relation between the self-intersection indices $(C_i^2)_X$ and the genera p_i.

$$\sum r_i[2p_i - (C_i^2)_X - 2] = 2g - 2.$$

It turns out (see Moishezon [1965]) that the converse is also true, that is, if we are given a collection of data $(A < r_i, p_i, g)$, where A is a negative-semidefinite matrix, $A = (a_{ij})$, of order n, r are positive integers, and p_i and g are non-negative integers, $1 \leq i \leq n$, satisfying the relations

$$r^t Ar = 0,$$

$$\sum r_i[2p - a_{ii} - 2] = 2g - 2,$$

then there exists a degeneration of curves of genus g with degenerate fiber $X_0 = \sum r_i C_i$, such that $(C_i, C_j)_X = a_{ij}$ and $p_a(C_i) = p_i$.

The conditions above allow us to describe the types of degenerations of curves. In particular, it is not hard to show that if π is a degeneration of rational curves ($g = 0$), then it's trivial: X_0 is a non-singular rational curve (recall that we have assumed the absence of exceptional curves of the first kind in the degenerate fiber).

The degenerate fibers of the degenerations of elliptic curves were first described by Kodaira [1960]. These fall into the following categories:

type ${}_mI_0$ is a non-singular elliptic curve; types ${}_mI_1$ and II are rational curves with one singularity of order 2, locally defined by the equation $x^2 + y^2 = 0$ (type ${}_mI_1$) or $x^2 + y^3 = 0$, (type II); the degenerate fiber of type III consists of two non-singular rational curves, tangent in one poin; type IV consists of three nonsingular rational curves, intersecting in one point; all of the remaining types are divisors with normal crossings, and consists of nonsingular rational curves C_i with $(C_i^2)_X = -2$. Figure 7 shows the degenerate fibers of these types (each line represents an irreducible component C_i of the divisor X_0, while the integers are the multiplicities r_i).

All of the possible types of degenerate fibers have been also described for degenerations of curves of genus 2 (see Namikawa–Ueno [1973], Ogg [1966]). In this case there are already around a hundred possible types.

In Section 4 we shall consider the connections between the type of the degenerate fiber and monodromy.

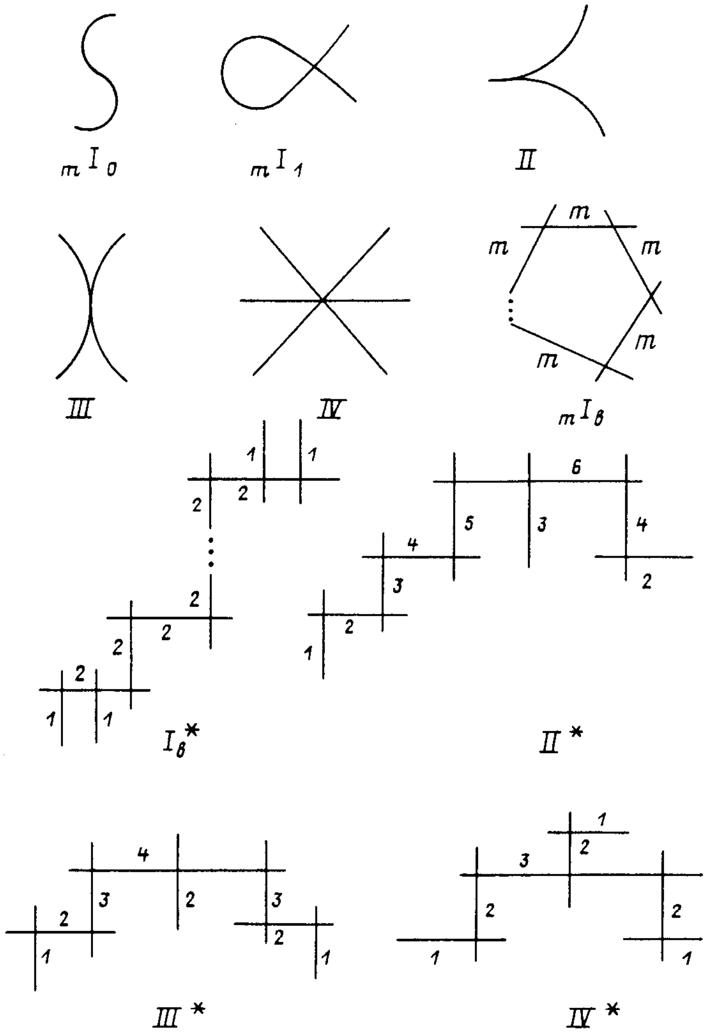


Fig. 7

§2. The Limit Hodge Structure

Schmid [1973] in the Kähler situation and Steenbrink [1974] in the algebraic situation introduced mixed Hodge structure on the cohomology $H^*(X_t)$ of the fiber X_t of a degeneration. Here, we shall briefly describe Schmid's approach to the construction of the limit mixed Hodge structures.

2.1. The limit Hodge filtration F_∞. Let $\pi : X \to \Delta$ be a degeneration of Kähler manifolds. Looking at pure Hodge structures on the cohomology $H^m(X_t)$ of the fibers X_t, $t \neq 0$, we have the period mapping

$$\tau : \Delta^* \to D/\Gamma,$$

where D is the corresponding space of Hodge structures, while Γ is a certain discrete group. Let

$$\exp(2\pi i\cdot) : U = \{z \in \mathbb{C} |\, \mathrm{Im}\, z > 0\} \to \Delta^*$$

is the universal covering of the punctured disk Δ^*, $t = \exp(2\pi iz)$. Let $\tilde{\pi} : \tilde{X} \to U$ be the lifting of the family $\pi^* : X^{\to}\Delta^*$ to U, where $X^* = \pi^{-1}(\Delta^*)$. The period mapping τ can be lifted to the map.

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d] & D \arrow[d] \\ \Delta^* \arrow[r, "\tau"] & D/\Gamma \end{tikzcd}$$

It is clear that

$$\Phi(z+1) = T\Phi(z),$$

where T is the Picard-Lefschetz transformation. Using the theorems of Mumford and Landman, we can assume that T is unipotent. Let

$$N = \log T = T - \mathrm{id} - \frac{(T-\mathrm{id})^2}{2} + \ldots + (-1)^{m+1}\frac{(T-\mathrm{id})^m}{m},$$

and let

$$\tilde{\Phi} : U \to \check{D},$$

where $\tilde{\Phi} = \exp(-zN)\Phi(z)$. It can be checked that $\tilde{\Phi}(z+1) = \tilde{\Phi}(z)$. Thus, the map $\tilde{\Phi}$ induces the map $\psi : \Delta^* \to \check{D}$, where $\psi(t) = \tilde{\Phi}\left(\frac{1}{2\pi i}\log t\right)$.

Schmid [1973] showed that the map ψ can be continued to a map $\psi : \Delta \to \check{D}$. The filtration corresponding to $\psi(0) \in \check{D}$ is the limit Hodge filtration F_∞.

2.2. The limit weight filtration W^∞. The limit filtration W^∞ is defined using the monodromy T, acting on $H^m_\infty = H^m(X_t)$. By Landman's theorem, $N = \log T$ is a nilpotent map: $N^{m+1} = 0$.

Proposition. *Let H be a linear space, and let $N : N \to H$ be a nilpotent map ($N^{m+1} = 0$). Then there exists a natural filtration*

$$0 \subset W_0 \subset \ldots \subset W_{2m} = H,$$

satisfying the conditions
(1) $N(W_k) = W_{k-2}$,

(2) $N(W_k) = \operatorname{Im} N \cap W_{k-2}$,
(3) $N : \operatorname{Gr}^W_{m+k} H \to \operatorname{Gr}^W_{m-k} H$ *is an isomorphism,*
(4) $N^k : H \to H$ *is the zero map if and only if* $W_{m-k} = 0$.

The filtration above is constructed as follows. Let $W_0 = \operatorname{Im} N^m$, and let $W_{2m-1} = \operatorname{Ker} B^m$. Then, if for some $k < m$ we already have

$$0 \subset W_0 \subset \ldots \subset W_{k-1} \subset W_{2m-k} \subset \ldots \subset W_{2m} = H,$$

satisfying

$$N^{m-k+1}(W_{2m-k}) \subset W_{k-1},$$

then we can set

$$W_k / W_{k-1} = \operatorname{Im}(N^{m-k}|_{W_{2m-k}/W_{l-1}})$$

and

$$W_{2m-k-1}/W_{k-1} = \operatorname{Ker}(N^{m-k}|_{W_{2m-k}/W_{k-1}})$$

and define W_k and W_{2m-k-1} as the corresponding preimages of the spaces W_k/W_{k-1} and W_{2m-k-1}/W_{k-1} under the map $H \to H/W_{k-1}$. It can be checked that $W_k \subset W_{2m-k-1}$ and that $N^{m-k}(W_{2m-k-1}) \subset W_k$. Therefore, the inductive hypothesis holds, and we can continue the construction of the filtration W.

Let us apply this proposition to $N : H^m(X_t) \to H^m(X_t)$, and let us denote the resulting filtration by

$$\subset W_0^\infty \subset \ldots \subset W_{2m}^\infty = H^m_\infty \simeq H^m(X_t).$$

It turns out that the following theorem holds:

Theorem (Schmid [1973]).
(1) $(H^m_\infty, F_\infty, W^\infty)$ *is a mixed Hodge structure.*
(2) $N : H^m_\infty \to H^m_\infty$ *is a morphism of Hodge structures of weight* -2.

The complete proof of this theorem turns out to be quite technical and cumbersome, and so we will omit it.

In the next section we will show the relationship between this limit Hodge structure and the Hodge structure of the degenerate fiber.

§3. The Clemens–Schmid Exact Sequence

In this section we describe the construction of the Clemens-Schmid exact sequence, which connects the cohomology with complex coefficients of the degenerate and non-degenerate fibers of a Kähler degeneration of manifolds.

3.1. Let $\pi : X \to \Delta$ be a semistable degeneration of n-dimensional complex manifolds. By shrinking $\Delta = \{t \in \mathbb{C} | |t| \leq 1\}$ we can assume that π is defined over a neighborhood of the closed disk $\overline{\Delta}$. Let $\overline{X} = \pi^{-1}(\overline{\Delta})$, $\partial X = \pi^{-1}(\partial \Delta)$, where $\partial\Delta = S^1 = \{|t| = 1\}$. Fix a fiber $i : X_t \hookrightarrow \partial X$ over the point $t = 1 = \exp(2\pi i 0)$. We thereby obtain a triple of spaces $(\overline{X}, \partial X, X_t)$. Topologically, the Clemens-Schmid exact sequence is obtained from the exact sequences of the pairs $(\partial X, X_t)$ and $(\overline{X}, \partial X)$.

3.2. The Wang exact sequence. The circle S^1 can be viewed as the segment $I = [0, 1]$ with ends identified:

$$\exp(2\pi i) : I \to S^1,$$

and the pair $(\partial X, X_t)$ can be viewed as the quotient of the pair $(X_t \times I, X_t \times \{0\} \cup X_t \times \{1\})$, where $X_t \times \{0\}$ and $X_t \times \{1\}$ are identified by the monodromy

$$T : X_t \times \{0\} \to X_t \times \{1\}.$$

We have an isomorphism of relative cohomology groups

$$H^m(\partial X, X_t) \xrightarrow{\sim} H^m(X_t \times I, X_t \times \{0\} \cup X_t \times \{1\}).$$

From the exact cohomology sequence of the pair $(X_t \times I, X_t \times \{0\} \cup X_t\{1\})$ we see can compute $H^m(X_t \times I, X_t \times \{0\} \cup X_t \times \{1\})$ by means of the sequence:

$$\begin{tikzcd}
{} \arrow[r] & H^m(X_t \times I) \arrow[r, "i^*"] \arrow[d, equal] & H^m(X_t \times \{0\} \cup X_t \times \{1\}) \arrow[r, "\partial"] \arrow[d, equal] & H^{m+1}(X_t \times I, \ldots) \arrow[r] & {} \\
& H^m(X_t) \arrow[r, "(,)"] & H^m(X_t) \oplus H^m(X_t) \arrow[r] & H^{m+1}(X_t \times I) \arrow[r] \arrow[d, equal] & {} \\
& & & H^{m+1}(X_t) &
\end{tikzcd}$$

The map i^* coincides with the diagonal inclusion $a \to (a, a)$, which implies that ∂ is an epimorphism and, consequently

$$H^{m+1}(X_t \times I, X_t \times \{0\} \cup X_t \times \{1\}) \simeq H^m(X_t).$$

Under this identification, the morphism ∂ is the same as the subtraction morphism $''-'' : H^m(X_t) \oplus H^m(X_t) \to H^m(X_t)$, where $" - " : (a, b) \to a - b$.

The map of exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^m(\partial X) \arrow[r] \arrow[d] & H^m(X_t) \arrow[r, "\partial"] \arrow[d, "(\mathrm{Id}, T)"] & H^{m+1}(\partial X, X_t) \arrow[r] \arrow[d, "\simeq"'] & H^{m+1}(\partial X) \arrow[r] & {} \\
0 \arrow[r] & H^m(X_t) \arrow[r] & H^m(X_t) \oplus H^m(X_t) \arrow[r, "\text{``}-\text{''}"] & H^m(X_t) \arrow[r] & 0
\end{tikzcd}$$

associated with the map of pairs

$$(X_t \times I, X_t \times \{0\} \cup X_t \cup \{1\}) \to (\partial X, X_t),$$

identifies the map $\partial : H^m(X_t) \to H^{m+1}(\partial X, X_t) \simeq H^m(X_t)$ with the map $\mathrm{id} - T$. Replacing $\mathrm{id} - T$ with $T - \mathrm{id}$ and $T - \mathrm{id}$ by

$$N = \log T = (T - \mathrm{id}) - \frac{(T - \mathrm{id})^2}{2} + \dots$$

does not change either the kernel or the image, and hence preserves the exactness of the sequence, since $T - \mathrm{id} = \exp(\log T) - \mathrm{id} = \log T + \frac{\log T^2}{2} + \dots$ (we use the fact that π is a semistable degeneration and $(T - \mathrm{id})^{m+1} = 0$).

After replacing $\mathrm{id} - T$ by N in the exact sequence of the pair $(\partial X, X_t)$ we get the *Wang exact sequence*

$$\xrightarrow{i^*} H^m(X_t) \xrightarrow{N} H^m(X_t) \xrightarrow{j} H^{m+1}(\partial X) \xrightarrow{i^*} H^{m+1}(X_t) \to$$

3.3. The exact sequence of the pair $(\overline{X}, \partial X)$

$$\to H^{m-1}(\partial X) \to H^m(\overline{X}, \partial X) \xrightarrow{r} H^m(\overline{X}) \xrightarrow{i^*} H^m(\partial X) \to$$

can be transformed with the help of

(a) Lefschetz duality (Dold [1972])

$$\mathrm{Lef} : H^m(\overline{X}, \partial X) \simeq H_{2n+2-m}(\overline{X}),$$

where $2n + 2 = \dim_{\mathbb{R}} \overline{X}$ and

(b) the deformation retraction isomorphisms

$$H^m(\overline{X}) \overset{c^*}{\underset{\sim}{\leftarrow}} H^m(X_0)$$

and

$$H_{2n+2-m}(\overline{X}) \overset{c_*}{\underset{\sim}{\to}} H_{2n+2-m}(X_0),$$

where $c : \overline{X} \to X_0$ is the Clemens map of the space $\overline{X}$ to the degenerate map X_0. The restriction of c to ∂X will also be denoted by $i : \partial X \to X_0$, while the restriction of c to X_t will be denoted by $c_t : X_t \to X_0$.

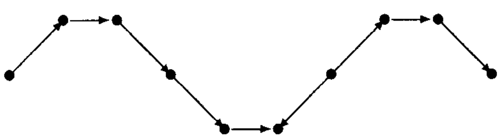
We get the exact sequence

$$\xrightarrow{c^*} H^{m-1}(\partial X) \to H_{2n+2-m}(X_0) \xrightarrow{\mu} H^m(X_0) \xrightarrow{c^*} H^m(\partial X) \to, \tag{2}$$

3.4. The Clemens-Schmid sequence. The Wang sequence and (2) share terms $H^m(\partial X)$. Let us braid these sequences together using the shared terms $H^m(\partial X)$:

$$\begin{tikzcd}
{} \arrow[rr, dashed, "\psi"] \arrow[dr] & & H_{2n+2-m}(X_0) \arrow[r, "\mu"] & H^m(X_0) \arrow[rr, dashed, "\nu"] \arrow[dr] & & H^m(X_t) \arrow[r, "N"] & H^m(X_t) \\
& H^{m-1}(\partial X) \arrow[ur] \arrow[dr] & & & H^m(\partial X) \arrow[ur] \arrow[dr] & & \\
{} \arrow[rr, dashed, "\nu"] \arrow[ur] & & H^{m-1}(X_t) \arrow[r, "N"] & H^{m-1}(X_t) \arrow[rr, dashed, "\psi"] \arrow[ur] & & H_{2n+2-(m+1)}(X_0) \arrow[r] & H^{m+1}(X_0)
\end{tikzcd}$$

so that the sequences of the form



are exact. Adding dashed arrows ν and ψ we obtained two sequences in the upper and lower lines (they are similar, though in one of them m is always even and in the other, m is always odd):

$$\xrightarrow{N} H^{m-2}(X_t) \xrightarrow{\psi} H_{2n+2-m}(X_0) \xrightarrow{\mu} H^m(X_0) \xrightarrow{\nu} H^m(X_t) \xrightarrow{N} H^m(X_t) \xrightarrow{\psi}$$

The above sequence is the aforementioned *Clemens-Schmid exact sequence*, where the maps are as follows:

(1) $N = \log T = (T - \mathrm{id}) - \frac{(T-\mathrm{id})^2}{2} + \ldots$, where T is the monodromy map,
(2) μ is the composition

$$\mu : H_{2n+2-m}(X_0) \overset{c_*}{\underset{\sim}{\leftarrow}} H_{2n+2-m}(\overline{X}) \overset{\mathrm{Lef}\sim}{\to} H^m(\overline{X}, \partial X) \overset{r}{\to} H^m(\overline{X}) \overset{c^*}{\underset{\sim}{\leftarrow}} H^m(X_0),$$

(3) $\nu = c_t^* : H^m(X_0) \to H^m(X_t)$,
(4) ψ is the composition

$$\psi : H^m(X_t) \overset{j}{\to} H^{m+1}(\partial X) \overset{\partial}{\to} H^{m+2}(\overline{X}, \partial X)$$
$$\overset{\sim}{\to} H_{2n-m}(\overline{X}) \overset{(c_t)_*}{\to} H_{2n-m}(X_0),$$

and it can be checked that

$$\psi = (c_t)_* P : H^m(X_t) \overset{P}{\to} H_{2n-m}(X_t) \overset{(c_t)_*}{\to} H_{2n-m}(X_0),$$

where P is Poincaré duality map.

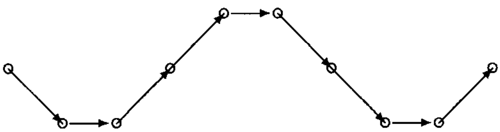
3.5. As we had already noted, the Clemens-Schmid sequence is obtained by the braiding of the two exact sequences of pairs $(\partial X, X_t)$ and $(\overline{X}, \partial X)$. It is easy to braid in two more strands, consisting of the exact sequence of the pair $(\overline{X}, X_t)$ and the exact sequence of the triple $(\overline{X}, \partial X, X_t)$:

$$\to H^m(\overline{X}, \partial X) \to H^m(\overline{X}, X_t) \to H^m(\partial X, X_t) \to H^{m+1})(\overline{X}, \partial X) \to .$$

We obtain the commutative diagram

$$\begin{tikzcd}
{} \arrow[rr, "\psi"] \arrow[dr] & & H^m(\overline{X}, \partial X) \arrow[rr, "\mu"] \arrow[dr] & & H^m(\overline{X}) \arrow[rr, "\nu"] \arrow[dr] & & H^m(X_t) \arrow[rr, "N"] \arrow[dr] & & {} \\
& H^{m-1}(\partial X) \arrow[ur] \arrow[dr] & & H^m(\overline{X}, X_t) \arrow[ur] \arrow[dr] & & H^m(\partial X) \arrow[ur] \arrow[dr] & & {} & \\
{} \arrow[rr, "\nu"'] \arrow[ur] & & H^{m-1}(X_t) \arrow[rr, "N"'] \arrow[ur] & & H^m(\partial X, X_t) \arrow[rr, "\psi"'] \arrow[ur] & & H^{m+1}(\overline{X}, \partial X) \arrow[rr, "\mu"'] \arrow[ur] & & {}
\end{tikzcd}$$

where every sequence that looks like



is exact.

The diagram (3) can be constructed starting with any triple $(\overline{X}, \partial X, X_t)$. It is clear that the upper and lower lines of (3) are complexes (that is, $d^2 = 0$. These sequences are not necessarily exact.

3.6. It is not hard to show (by chasing arrows) that if one of the lines in the sequence (3) is exact in some term, then the other line is also exact in the corresponding term (one lying directly above or below, as the case may be).

3.7. The question of the exactness of the Clemens-Schmid sequence in the term

$$\xrightarrow{\nu} H^m(X_t) \xrightarrow{N}$$

is called the *problem of the local invariance of cycles.* If $\phi \in H^m(X_t)$ is an invariant cycle (that is, $T(\phi) = \phi$) then does there exists a class $\phi \in H^m(X)$, whose restriction to X_t coincides with ϕ. In general, this is not so, as is shown by the surface family of Hopf (Clemens [1977]).

Using the theory of mixed Hodge structures Deligne [1972] and Steenbrink [1974] for algebraic varieties, and Clemens and Griffiths in the Kähler situation

Clemens–Griffiths [1972] have shown that the problem of invariant cycles has a positive solution in these cases.

3.8. Let us show that the exactness of the Clemens-Schmid sequence can be reduced to the problem of local invariance of cycles, that is, to the exactness of the sequence at the term preceding the map N.

By Section 3,6, exactness in the term preceding N implies exactness at the term preceding μ. By 3.6 the exactness in the term preceding ν follows from the exactness at the term preceding ψ, which is also the term after N. To show the exactness in the term preceding ψ, let's note that the map $\nu = c_1^*$ is dual to $\psi = (c_t)_* P$, and the segment

$$\to H^m(X_t) \xrightarrow{N} H^m(X_t) \xrightarrow{\psi} H^{m+2}(\overline{X}, \partial X) \to$$

of the Clemens-Schmid sequence is dual to the segment

$$\to H^{2n-m}(\overline{X}) \xrightarrow{\nu} H^{2n-m}(X_t) \xrightarrow{N} H^{2n-m}(X_t) \to,$$

the exactness of which follows from the positive solution to the problem of local invariance of cycles.

3.9. In the sequel, suppose that X is a Kähler manifold.. In this case all of the terms of the Clemens-Schmid sequence can be equipped with mixed Hodge structures. On the term $H^m(X_t)$ we consider the limit Hodge structure, and on the term $H^m(X_0)$ the mixed Hodge structure introduced in Chapter 4, Section 2 (recall that we are assuming throughout that $\pi : X \to \Delta$ is a semistable degeneration). On the terms $H_m(X_0)$ We can introduce mixed Hodge structures on the terms $H_m(X_0)$ by duality (Chapter 4, Section 1.7), using the fact that $H_m(X_0) = H^m(X_0)^\vee$, and the fact that we have already introduced a mixed Hodge structure on $H^m(X_0)$.

Theorem. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration. Then the Clemens-Schmid sequence is an exact sequence of morphisms of mixed Hodge structures. The maps μ, ν, N, and ψ are morphisms of weights $2n+2$, 0, $-2n$ respectively.*

A complete proof of this theorem is contained in Clemens [1977] and Steenbrink [1974] and is quite technical and complicated, so we shall omit it. We should note again that to show the exactness of the Clemens-Schmid sequence it is enough to show the local invariant cycle theorem.

In addition, let us note that Steenbrink [1974] has introduced a mixed Hodge structure on the terms of the Wang exact sequence (the structure on $H^m(X_t)$ is the limit Hodge structure). This turns the Wang exact sequence into an exact sequence of mixed Hodge structure morphisms.

§4. An Application of the Clemens-Schmid Exact Sequence to the Degeneration of Curves

4.1. Let $\pi : V \to D$ be a degeneration of curves of genus g, that is, $V_t = \pi^{-1}(t)$ is a non-singular compact complex curve for $t \neq 0$, $\dim H^0(C, \Omega^1_{V_t}) = g$. By the Mumford stable reduction theorem we can assume (after a base change $t = s^N$) that $V_0 = \pi^{-1}(0) = C_1 \cup \ldots \cup C_r$ is a divisor with normal crossings in V, C_i are non-singular curves of genus $g(C_i) = g_i$. Let $D_{V_0} = \bigcup_{i<j}(C_i \cap C_j)$ be the set of double points of the divisor V_0, and let there be d such double points.

4.2. The mixed Hodge structure on $H^1(V_0)$ is defined by the spectral sequence

$$E_1^{p,q} = H^q(V_0^{(p+1)}, \mathbb{C}) \Rightarrow H^{p+q}(V_0),$$

which, in this case, turns into the Mayer-Vietoris exact sequence

$$0 \to H^0(V_0) \to \bigoplus_{i=1}^{r} H^0(C_i) \xrightarrow{d_1} \bigoplus_{p \in D_{V_0}} H^0(P) \to H^1(V_0) \to \bigoplus_{i=1}^{r} H^1(C_i) \to 0.$$

The weight filtration ${}^0W$ on $H^1(V_0)$ has the form

$$H^1(V_0) = {}^0W_1 \supset {}^0W_0 \supset 0$$

and

$${}^0W_0 = \operatorname{Coker} d_1 = H^1(\Gamma),$$

where $\Gamma = \Pi(V_0)$ is the dual graph of the curve $V_0 = C_1 + \ldots + C_r$ (see Chapter 4, Section 2), and

$$\operatorname{Gr}_1^{{}^0W} H^1(V_0) = {}^0W_1/{}^0W_0 = \bigoplus_{i=1}^{r} H^1(C_i).$$

Since the Euler characteristic of the graph Γ is

$$e(\Gamma) = 1 - h^1(\Gamma) = r - d,$$

it follows that

$$\dim {}^0W_0 = h^1(\Gamma) = d - r + 1.$$

We get the following formula for $h^1(V_0) = \dim H^1(V_0)$:

$$h^1(V_0) = 2\sum_{i=1}^{r} g_i + h^1(\Gamma).$$

4.3. The weight filtration W' on $H_1(V_0)$ has the weights

$$H_1(V_0) = W'_0 \supset W'_{-1} \supset 0.$$

4.4. For the limit mixed Hodge structure on $H^1(X_t)$ we have (see Section 2.2)

$$H^1 = H^1(X_t) = W_2 \supset W_1 \supset W_0 \supset 0.$$

On W_0 there is a pure Hodge structure of weight 0, that is, $W_0 = H_0^{0,0}$ (recall that $H^{p,q}$ is the subspace of (p,q) forms in Hodge decomposition (see Chapter 2, Section 1), and since N is a morphism of weight -2,

$$N : W_2/W_1 \stackrel{\sim}{\to} W_0 = H_0^{0,0}$$

it follows that $W_2/W_1 = H_2^{1,1}$.

By definition of mixed Hodge structure, W_1/W_0 is the space with pure Hodge structure of weight 1. Therefore,

$$W_1/W_0 = H_1^{1,0} \oplus H_1^{0,1}.$$

Let $\omega = \dim W_0$. We have $\dim H^1(X_t) = 2g$. In addition

$$\dim H^1(X_t) = \sum \dim W_i/W_{i-1}.$$

Therefore

$$2g = 2\omega + \dim W_i/W_0. \tag{5}$$

Note that the monodromy T is trivial (or, equivalently, $N = 0$) on $H^1(X_t)$ if and only if $W_0 = 0$. Indeed, $N(W_1) = 0$ and $N : W_2/W_1 \stackrel{\sim}{\to} W_2$ is an isomorphism, therefore $N = 0$ on $H^1(X_t)$ if and only if $W_0 = 0$.

4.5. In the case of curve degeneration, the maps μ, ν, N, and ψ in the Clemens-Schmid exact sequence have weights $4, 0, -2$, and -2 respectively, and the (odd) sequence itself has the form:

$$\begin{tikzcd}
0 \arrow[r] & H^1(V_0) \arrow[r, "\nu"] \arrow[d, equal] & H^1(V_t) \arrow[r, "N"] \arrow[d, equal] & H^1(V_t) \arrow[r, "\psi"] \arrow[d, equal] & H_1(V_0) \arrow[r] \arrow[d, equal] & 0 \\
 & {}^0W_1 \arrow[d, phantom, "\cup"] \arrow[dr] & W_2 \arrow[d, phantom, "\cup"] \arrow[ddr] & W_2 \arrow[r] \arrow[d, phantom, "\cup"] & W_0' \arrow[d, phantom, "\cup"] & \\
 & {}^0W_0 \arrow[d, phantom, "\cup"] \arrow[dr] & W_1 \arrow[d, phantom, "\cup"] \arrow[ddr] & W_1 \arrow[r] \arrow[d, phantom, "\cup"] & W_{-1}' \arrow[d, phantom, "\cup"] & \\
 & 0 & W_0 \arrow[d, phantom, "\cup"] & W_0 \arrow[r] \arrow[d, phantom, "\cup"] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

From the strictness of the morphisms of mixed Hodge structures we get

$$ {}^0W_0 \simeq \nu({}^0W_0) = W_0, \quad {}^0W_1 \simeq \nu({}^0W_1) = W_1. $$

This implies that

$$ \omega = \dim W_0 = \dim WW_0 = h^1(\Gamma). $$

Applying (4) and (5) we see that

$$ \dim W_1/W_0 = 2\sum_{i=1}^{r} g_i. $$

One consequence is the following

Theorem. *The monodromy T is trivial on $H^1(V_t)$ for a Kähler degeneration of curves if and only if $H^1(\Gamma) = 0$, that is, the graph Γ is a tree. In that case $g = \sum_{i=1}^{r} g_i$.*

§5. An Application of the Clemens-Schmid Exact Sequence to Surface Degenerations. The Relationship Between the Numerical Invariants of the Fibers X_t and X_0.

Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces. Let $X_0 = V_1 + \ldots + V_r$ be the degenerate fiber, let $D_{X_0} = \{C\}$ be the set of double curves, let τ be the set (and also the number) of triple points of the fiber X_0 (see Chapter 4, Section 2.9).

5.1. The Clemens-Schmid exact sequences for surface degenerations have the form

$$0 \to H^0(X_t) \xrightarrow{\psi} H_4(X_0) \xrightarrow{\mu} H^2(X_0) \xrightarrow{\nu} H^2(X_t)$$
$$\xrightarrow{N} H^2(X_t) \xrightarrow{\psi} H^2(X_0) \xrightarrow{\mu} H^4(X_0) \xrightarrow{\nu} H^4(X_t) \to 0$$

and

$$0 \to H^1(X_0) \xrightarrow{\nu} H^1(X_t) \xrightarrow{N} H^1(X_t) \xrightarrow{\psi} H_3(X_0)$$
$$\xrightarrow{\mu} H^3(X_0) \xrightarrow{\nu} H^3(X_t) \xrightarrow{N} H^3(X_t) \xrightarrow{\psi} H^1(X_0) \to 0.$$

The morphisms μ, ν, ψ, and N are morphisms of mixed Hodge structures of weights 6, 0, -4, and -2, respectively.

In Chapter 4, Section 2 we computed the weight filtration on $H^m(X^0)$. Let us introduce the notation

$$kh^i = \dim \operatorname{Ker}\left(\bigoplus_{j=1}^{r} H^i(V_j) \to \bigoplus_{C} H^i(C)\right) = \dim \operatorname{Gr}_i^{{}^0W} H^i(X_0),$$

and

$$\mathrm{ckh}^i = \dim \operatorname{Coker}\left(\bigoplus_{j=1}^{r} H^i(V_j) \to \bigoplus_{C} H^i(C)\right) = \dim \operatorname{Gr}_i^{{}^0W} H^{i+1}(X_0),$$

and finally

$$h^i(X_t) = \dim H^i(X_t).$$

As usual, $p_g(X_t) = \dim H^0(X_i, \Omega^2_{X_i})$ is the geometric genus of the surface X_t.

Theorem (Kulikov, Vik. S. and Kulikov, V. S. [1981], Persson [1977]). *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, then*

$$h^1(X_t) = \sum_{i=1}^{r} h^1(V_i) - \sum_{C} h^1(C) + 2h^1(\pi) + \mathrm{ckh}^1, \tag{6}$$

$$h^2(X_t) = \sum_{i=1}^{r} h^2(V_i) + 3h^2(\Pi) - h^1(\Pi) - d - r + 1, \tag{7}$$

$$p_g(X_t) = \sum_{i=1}^{r} p_g(V_i) + h^2(\Pi) + \frac{1}{2}\,\mathrm{ckh}^1\,. \tag{8}$$

In addition,

$$\dim W_0 H^1(X_t) = \dim {}^0\!W H^1(X_0) = h^1(\Pi), \tag{9}$$
$$\dim W_1 H^2(X_t) = \dim {}^0\!W_1 H^2(X_0)/W W_0 H^2(X_0) = \mathrm{ckh}^1, \tag{10}$$
$$\dim W_0 H^2(X_t) = \dim {}^0\!W H^2(X_0) = h^2(\Pi), \tag{11}$$
$$\mathrm{ckh}^2 = h^1(\Pi), \tag{12}$$

where r is the number of components, d is the number of double curves of the fiber X_0, $\Pi = \Pi(X_0)$ is the polyhedron of the degeneration.

We will prove one of these formulas (formula (8)) to demonstrate how much information can be obtained from the strictness of mixed Hodge structures.

Consider the even Clemens-Schmid sequence

$$
\begin{tikzcd}
0 \arrow[r, mapsto] & H^0(X_t) \arrow[r, "\psi"] & H_4(X_0) \arrow[r, "\mu"] \arrow[d, equal] & H^2(X_0) \arrow[r, "\nu"] \arrow[d, equal] & H^2(X_t) \arrow[r, "N"] \arrow[d, equal] & H^2(X_t) \arrow[r, "\psi"] \arrow[d, equal] & H_2(X_0) \arrow[r, mapsto] \arrow[d, equal] & {} \\
& H_{-4}^{-2,-2} \arrow[r, equal] & {}^0W_{-4} \arrow[r] & {}^0W_2 \arrow[ddddr] & W_4 \arrow[ddddr] & W_4 \arrow[r] & {}^0W_0 & \\
& & \cup & \cup & \cup & \cup & \cup & \\
& & 0 \arrow[r] & {}^0W_1 \arrow[ddddr] & W_3 \arrow[ddddr] & W_3 \arrow[r] & {}^0W_{-1} & \\
& & & \cup & \cup & \cup & \cup & \\
& & & {}^0W_0 \arrow[ddddr] & W_2 \arrow[ddddr] & W_2 \arrow[r] & {}^0W_{-2} & \\
& & & \cup & \cup & \cup & \cup & \\
& & & 0 & W_1 \arrow[ddddr] & W_1 \arrow[r] & 0 & \\
& & & & \cup & \cup & & \\
& & & & W_0 & W_0 & & \\
& & & & \cup & \cup & & \\
& & & & 0 & 0 & &
\end{tikzcd}
$$

Comparing the weights and the types of morphisms, we get the sequences

$$
\begin{tikzcd}
0 \arrow[r] & H^0(X_t) \arrow[r, "\psi"] & H_4(X_0) \arrow[r, "\mu"] & {}^0W_2 \arrow[r, "\nu"] & W_2 \arrow[r, "N"] & W_0 \arrow[r] & 0 \\
& & & \cup & \cup & & \\
& & 0 \arrow[r] & {}^0W_1 \arrow[r] & W_1 \arrow[r] & 0 & \\
& & & \cup & \cup & & \\
& & 0 \arrow[r] & {}^0W_0 \arrow[r] & W_0 \arrow[r] & 0 &
\end{tikzcd}
$$

From the strictness of the morphisms of mixed Hodge structures, it follows that these sequences are exact. This implies formulas (10) and (1), since $\dim {}^0W_0 H^2(X_0) = h^2(\Pi)$ by virtue of Chapter 4, Section 2.9.

From the exactness of the sequences (13) it follows that the sequence

$$0 \to H^0(X_t) \xrightarrow{\psi} H_4(X_0) \xrightarrow{\mu} {}^0W_2/{}^0W_1 \xrightarrow{\nu} W_2/W_1 \xrightarrow{N} W_0 \to 0$$

is exact also (note that ${}^0W_2/{}^0W_1 = {}^0H_2^{2,0} \oplus {}^0H_2^{1,1} \oplus {}^0H_2^{0,2}$, $W_2/W_1 = H_2^{2,0} \oplus H_2^{1,1} \oplus H_2^{0,2}$, and $W_0 = H_0^{0,0}$).

Each term of the sequence (14) is a space with a pure Hodge structure. By definition of the dual Hodge structure, there exists a pure Hodge structure of weight -4 on $H_4(X_0) = H^4(X_0)^\vee$, and also $H_4(X_0) = H_{-4}^{-2,-2}$ since $H^4(X_0) = \oplus H^4(V_i)$. On $H^4(V_i$ there is only one non-trivial Hodge summand $H^{2,2}$ of type $(2,2)$. Therefore, $\operatorname{Ker}\nu = \operatorname{Im}\mu \subset {}^0H_0^{1,1}$ and the morphism ν is an inclusion on ${}^0H_2^{2,0} \oplus {}^0H_2^{0,2}$. SInce $\operatorname{Im}\nu = \operatorname{Ker} N \supset H_2^{2,0} \oplus H_2^{0,2}$, it follows that $\nu : {}^0H_2^{2,0} \xrightarrow{\sim} H_2^{2,0}$ and $\nu : {}^0H_2^{0,2} \xrightarrow{\sim} H_2^{0,2}$ are isomorphisms. By Chapter 4, Section 2.9

$$\dim H_2^{2,0} = \dim {}^0H_0^{2,0} = \sum_{i=1}^{r} p_g(V_i). \tag{15}$$

We want to compute $p_g(X_t) = h^{2,0}(X_t)$. Using the connection between the Hodge decomposition and the Hodge filtration, we have

$$p_g(X_t) = \dim F^2 H^2(X_t),$$

where it can be considered (see Section 2.1) that F is the limit Hodge filtration:

$$H^2(X_t) = F^0 \supset F^1 \supset F^2 \supset F^3 = 0.$$

Setting

$$F_k^i = F^i \cap W_k / F^i \cap W_{k-1},$$

and

$$f_k^i = \dim F_k^i,$$

get

$$p_g(X_t) = f_4^2 + f_3^2 + f_2^2.$$

Above we used the observation that $f_1^2 = f_0^2 = 0$, since the filtration induces a pure Hodge structure of weight k on $\operatorname{Gr}_k^W = W_k/W_{k-1}$, and so $f_k^i = 0$ for $i > k$.

On Gr_4^W the filtration F induces a pure Hodge structure of weight 4, and so $\operatorname{Gr}_4^W = F_4^2 \oplus \overline{F}_4^3$. But $\overline{F}_4^3 = 0$, since $F^3 = 0$. Therefore,

$$f_4^2 = \dim W_4/W_3 = \dim W_0 = h^2(\Pi),$$

since $N^2 : W_4/W_3 \xrightarrow{\sim} W_0$ is an isomorphism.

On Gr_3^W the filtration F induces a pure Hodge structure of weight 3. As above, since F^3, we see that $\operatorname{Gr}_3^W = \overline{F}_3^1$. Therefore, $F_3^2 = F_3^2 \cap \overline{F}_3^1 = H_3^{2,1}$. In addition, $\operatorname{Gr}_3^W = F_3^2 \oplus \overline{F}_3^2 = H_3^{2,1} \oplus \overline{H}_3^{2,1}$, and so, by formula (10),

$$f_3^2 = \frac{1}{2}\dim W_3/W_2 = \frac{1}{2}\dim W_1/W_0 = \frac{1}{2}\operatorname{ckh}^1,$$

since $N : W_3/W_2 \xrightarrow{\sim} W_1/W_0$ is an isomorphism.

Equation (15) implies that

$$f_2^2 = \sum_{i=1}^{r} p_g(V_i),$$

which finally proves formula (8).

5.2.

Theorem. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, then:*

(1) The monodromy $T = \mathrm{id}$ on $H^1(X_t)$ if and only if $h^1(\Pi) = 0$.

(2) The monodromy T has unipotency index 1 on $H^2(X_t)$ (that is, $(T-\mathrm{id})^2 = 0$ if and only if $H^2(\Pi) = 0$.

(3) The monodromy $T = \mathrm{id}$ on $H^2(X_t)$ if and only if $H^2(\Pi) = 0$ and $\mathrm{ckh}^1 = 0$.

Proof. It is clear that $(T - \mathrm{id})^k = 0$ if and only if $N^k = 0$.

(1) The morphism $N : W_2/W_1 \to W_0$ is an isomorphism on the space $H^1(X_t)$, and so $N = 0$ if and only if $W_0 = 0$, that is, when $\dim W_0 = h^1(\Pi) = 0$.

(2) On $H^2(X_t)$ we have $N^2 : W_4/W_3 \xrightarrow{\sim} W_0$ and $N^2(W_3) = 0$. Therefore, $N^2 = 0$ if and only if $W_0 = 0$, that is, $\dim W_0 = h^2(\Pi) = 0$.

(3) Since $N^2 : W_4/W_3 \xrightarrow{\sim} W_0$ and $N : W_3/W_2 \xrightarrow{\sim} W_1/W_0$ are isomorphisms, it follows that $N = 0$ on $H^2(X_t)$ if and only if $W_1/W_0 = 0$ and $W_0 = 0$. But by equation (10), $\dim W_1/W_0 = \mathrm{ckh}^1$.

5.3. The algebraic Euler characteristic $\chi(X_t)$. Let $\chi(V) = h^0(\mathcal{O}_V) - h^1(\mathcal{O}_V) + h^2(\mathcal{O}_V) = p_g - q + 1$ be the Euler characteristic of the structure sheaf $\mathcal{O}_V$ of the algebraic surface V.

Theorem. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, $X_0 = V_1 + \ldots + V_r$, then*

$$\chi(X_t) = \sum_{i=1}^{r} \chi(V_i) - \sum_{C} \chi(C) + \tau.$$

Proof. It is known (Mumford [1966]) that the Euler characteristic of a flat coherent sheaf is constant. In particular, $\chi(X_t) = \chi(X_0)$. For the structure sheaf $\mathcal{O}_{X_0}$ there is the resolution

$$0 \to \mathcal{O}_{X_0} \to (a_1)_*\mathcal{O}_{V^{(1)}} \xrightarrow{\delta} (a_2)_*\mathcal{O}_{V^{(2)}} \xrightarrow{\delta} \ldots$$

and the theorem follows from the additivity of the Euler characteristic (note that the Euler characteristic of a point is 1.)

5.4. The intersection index of the canonical class. Let K be the canonical class on X (see Chapter 1, Section 8), let K_t be the canonical class of a non-singular fiber X_t and let K_i be the canonical class of the component V_i of a degenerate fiber $X_0 = V_1 + \ldots + V_r$ of a semistable degeneration of surfaces.

If $D \in \operatorname{Pic} X$, and V is a component of a fiber, then let $D_V = i^*(D) = D \cdot V$, where $i : V \hookrightarrow X$ is the natural inclusion. For $D, D' \in \operatorname{Pic} X$, the intersection index $D \cdot D' \cdot V$ is defined, and $D \cdot D' \cdot V = D_V \cdot D'_V$ is the intersection index on V. The fibers X_t and X_0 are linearly equivalent (homologous), and so

$$D \cdot D' \cdot X_t = D \cdot D' \cdot X_0 = \sum_{i=1}^{r} D_{V_i} \cdot D'_{V_i}, \tag{16}$$

and, in addition, $V_i \sim -(V_1 + \ldots + \widehat{V}_i + \ldots V_r)$, hence

$$V_i|_{V_i} = -V_i \cdot \sum_{j=1} V_j. \tag{17}$$

By the adjunction formula (Chapter 1, Section 8), for the surface $V \subset X$, the canonical class satisfies $K(V) = (K + V) \cdot V$, and, in particular,

$$K_t = K \cdot X_t = K_{X_t}, \quad K_i = (K + V_i)_{V_i}. \tag{18}$$

By the adjunction formula for curves, we have

$$(K_i \cdot C)_{V_i} = \deg K(C) - (C^2)_{V_i} = 2(g(C) - 1) - (C^2)_{V_i}, \tag{19}$$

where $(C^2)_{V_i}$ is the intersection index of the double curve C on the surface V_i.

Lemma. *Let $C = V_i \cap V_j$ be a double curve of a semistable degeneration of surfaces, then*

$$(C^2)_{V_i} + (C^2)_{V_j} = -T_C, \tag{20}$$

where T_C is the number of triple points of the fiber X_0 incident to C.

Proof. Note that C is a union of non-singular curves, since X_0 is a divisor with normal crossings. We have $V_i \cdot V_j \cdot X_0 = 0$, since $X_0 \sim X_t$. On the other hand,

$$\begin{aligned} V_i \cdot V_j \cdot X_0 &= V_i \cdot V_j \cdot (V_i + V_j + \sum_{k \neq i,j} V_k) \\ &= V_i V_j V_i + V_i V_j V_j + T_C = (C^2)_{V_j} + (C^2)_{V_i} + T_C. \end{aligned}$$

The next theorem follows from equations (16)-(20):

Theorem (Persson [1977]). *Let $\pi : X \to \Delta$ be a semistable degeneration of surfaces, then*

$$(K_t^2)_{X_t} = \sum_{i=1}^{r} (K_i^2)_{V_i} + 8 \sum_C (g(C) - 1) + 9\tau.$$

5.5. The topological Euler characteristic $e(X_t)$. Let $e(V) = \sum_{i=0}^{4} (-1)^i b_i = 2 - 2h^1 + h^2$ be the topological Euler characteristic of the Kähler surface V, where $b_i = h^i$ is the i-th Betti number. For a curve C, the topological Euler characteristic satisfies $e(C) = 2 - h^1 = 2 - 2g(C) = 2\chi(C)$.

From Noether's formula (see Shafarevich *et al* [1965]) for a surface, we have

$$\chi(V) = \frac{(K^2) + e(V)}{12},$$

and from Theorems 5.3 and 5.4 we get the following formula for $e(X_t)$:

Theorem. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, then*

$$e(X_t) = \sum_{i=1}^{r} e(V_i) - 2 \sum_C e(C) + 3\tau.$$

§6. The Epimorphicity of the Period Mapping for K3 Surfaces

In this section we will show that the period mapping for K3 surfaces is onto. In order to do that, we will need to study semistable degenerations of K3 surfaces.

6.1. Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of K3 surfaces; the generic fiber X_t has the following properties: $p_g(X_t) = 1$, $q(X_t) = 0$, and the canonical class K_X is trivial. The canonical class K_X of the degeneration is not, in general, trivial. However, there is the following:

Theorem (Kulikov [1977a, 1980]). *Let π be a semistable Kähler degeneration of K3 surfaces. Then, there exists a reconstruction $\pi' : X' \to \Delta$ of the degeneration π, such that*

1. π' is a semistable degeneration,

2. K'_X is trivial.

The proof of this theorem is based on a thorough study of the degenerate fibers of semistable degenerations of K3, by constructing the birational automorphisms of the threefold X. In essence, this theorem shows that every semistable degeneration of K3 surfaces is obtained from a semistable degeneration with trivial canonical class by the following sequence of transformations. At every step, the space X is covered by open sets, and in each open sets we construct some blowings-up (monoidal transformations, see Chapter 1, Section

1), centered at points or (possibly singular) curves, and some blowings-down. These blowings-up and blowings-down are constructed in such a way that in the end we can reglue the open sets into a singular non-singular variety, and obtain a semistable degeneration of K3 surfaces.

In the sequel we will suppose that π is a semistable degeneration of K3 surfaces, and that K_X is trivial. In the sequel, we will describe the possible types of degenerate fibers and their relationship with monodromy.

6.2. For degenerations of K3 surfaces, formulas (8) and (9) of theorem 5.1 imply that

$$h^1(\Pi(X_0)) = 0, \tag{21}$$

and

$$\sum_{i=}^{r} p_g(V_i) + h^2(\Pi(X_0)) + \frac{1}{2}\,\mathrm{ckh}^1 = 1. \tag{22}$$

Let $C_{ij} = V_i \cap V_j$, where V_i are the components of the degenerate fiber X_0. From formulas (18) and (17) and the triviality of the canonical class K_X, it follows that

$$K_i = -\sum_{j\neq i} C_{i,j}.$$

Theorem (Kulikov [1977a]). *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of K3 surfaces, such that K_X is trivial. Then the degenerate fiber X_0 can be one of the following three types:*

1. *$X_0 = V_1$ is a non-singular K3 surface.*
2. *$X_0 = V_1 + \ldots + V_r$, $r > 1$, V_1 and V_r are rational surfaces, while V_i are ruled elliptic surfaces for $1 < i < r$. The double curves $C_{1,2}, \ldots, C_{r-1,r}$ are elliptic curves, and the polyhedron $\Pi(X_0)$ has the form*

$$\underset{V_1}{\circ}\!-\!\!-\!\!-\underset{V_2}{\circ}\!-\!\!-\!\!-\cdots-\!\!-\!\!-\underset{V_{r-1}}{\circ}\!-\!\!-\!\!-\underset{V_r}{\circ}$$

3. *$X_0 = V_1 + \ldots + V_r$, $r > 1$, with all V_i rational surfaces, and all the double curves C_{ij} rational. The polyhedron $\Pi(X_0)$ is a triangulation of S^2.*

These three types of degenerations are distinguished via the monodromy action T on $H^2(X_t, \mathbb{Z})$:

1. $T = \mathrm{id}$;
2. $(T - \mathrm{id}) \neq 0, \quad (T - \mathrm{id})^2 = 0$;
3. $(T - \mathrm{id})^2 \neq 0, \quad (T - \mathrm{id})^3 = 0$.

Proof. Case 1 is the case when X_0 has a single component.

Let $r > 1$. Then the canonical class $K_i = -\sum_{i\neq j} C_{i,j}$ is anti-effective, and thus (Shafarevich *et al* [1965]) all of the V_i are ruled surfaces. Consider a double curve C_{ij} on V_i. We have

$$2g(C_{ij}) - 2 = (K_i + C_{ij}, C_{ij})_{V_i} = -\sum_{k\neq i,j} (C_{ik}, C_{ij})_{V_i} = -T_{C_{ij}},$$

where $T_{C_{ij}}$ is the number of triple points of the fiber X_0 lying on the curve C_{ij}. Since $T_{C_{ij}} \geq 0$ and $g(C_{ij}) \geq 0$, there are two possibilities:

(R) $g(C_{ij}) = 0$ and $T_{C_{ij}} = 2$, so that C_{ij} is a rational curve, and there are exactly two triple points on C_{ij}.

(E) $g(C_{ij} = 1$, and there are no triple points on C_{ij}, that is, C_{ij} is an elliptic curve, and C_{ij} does not intersect any other double curves.

In the case (R) we see that C_{ij} intersects some other double curve, which must also be rational, and which also contains two triple points of X_0.

Thus, every V_i is a ruled surface, and the set of double curvers on V_i consists of a disjoint union of a finite union of elliptic curves, and a finite number of cycles of rational curves.

Let $V = V_{i_0}$ be one of the components, let $\phi : V \to \overline{V}$ be the morphism onto the minimal model $\overline{V}$ (Shafarevich *et al* [1965]) (ϕ is the composition of monoidal transformations centered at points), and let L be an exceptional curve of first type on V, that is, $L \simeq \mathbb{P}^1$, and $(L^2)_V = -1$, and L is blown down to a point by the morphism ϕ. Then $(L, K_V)_V = -1$, so $(L, \sum_{j \neq i_0} C_{i_0 j})_V = 1$. Thus, either L intersects just one of the connected components of the divisor $\sum C_{i_0 j}$ or L coincides with one of the $C_{i_0,j}$. It follows that the number of connected components of the divisor $\sum \phi_* C_{i_0 j}$ equals the number of connected components of the divisor $\sum C_{i_0,j}$, since $K_{\overline{V}} = \phi_* K_V$. It can be checked that the reduced divisor

$$-\sum j \neq i_0 \phi_* C_{i_0 j} \sim K_{\overline{V}},$$

on the ruled surface $\overline{V}$ either consists of one connected component, or of two, and in the last situation, $\overline{V}$ is a ruled elliptic surface, and

$$\sum \phi_* C_{i_0 j} = C_1 + C_2,$$

where C_1 and C_2 are elliptic curves. Furthermore, if the divisor $\sum \phi_* C_{i_0 j}$ is connected, then either $\phi_* C_{i_0 j}$ are rational curves, and $\overline{V}$ a rational surface, or $\sum \phi_* C_{i_0 j} = C$ is an elliptic curve while $\overline{V}$ is either a rational or ruled elliptic surface.

Therefore, the following are possible:

(a) V_i is a rational surface, and $\sum_{i \neq j} C_{ij}$ is a cycle of rational curves;

(b) V is a rational or a ruled elliptic surface, and $\sum_{i \neq j} C_{ij} = C$ is a single elliptic curve.

(c) V is a ruled elliptic surface and $\sum C_{ij} = C_1 + C_2$ consists of two disjoint elliptic curves.

There are two possibilities for the fiber X_0.

Case 1. One of the V_i is of type (a). Then the double curves on the components adjacent to V_i also contain triple points, and thus the adjacent components also are of type (a). Since X_0 is connected, it follows that all V_i are rational surfaces, and their double curves form a cycles. Therefore, the polyhedron $\Pi(X_0)$ is a triangulation of some compact surface without boundary (there are precisely two triple points of X_0 on each double curves).

Since the V_i are rational surfaces and C_{ij} are rational curves, we know that $p_g(V_i) = 0$, ckh$^1 = 0$ (since $h^1(C_{ij}) = 0$). Formulas (21) and (22) tell us that

$$h^0(\Pi(X_0)) = 1, \quad h^1(\Pi(X_0)) = 0, h^2(\Pi(X_0)) = 1.$$

There is only one surface with such properties – the sphere S^2. Thus, in this case the degenerate fiber falls into type 3 in the statement of the theorem.

Case 2. All the V_i have type (b) or (c). Then X_0 has no triple points, and thus $\Pi(X_0)$ is one-dimensional, and $h^2(\Pi(X_0)) = 0$. By formula (21), $h^1(\Pi(X_0)) = 0$, and so the graph $\Pi(X_0)$ is a tree, and since each component V_i has at most two double curves, $\Pi(X_0)$ must be a simple path

$$\underset{V_1}{\circ}\!\!-\!\!-\!\!-\!\!\underset{V_2}{\circ}\!\!-\!\!-\!\!-\!\ldots\!-\!\!-\!\!-\!\!\underset{V_{r-1}}{\circ}\!\!-\!\!-\!\!-\!\!\underset{V_r}{\circ}$$

Applying Theorem 5.3 to the case $\chi(X_t) = 2$, we see that in this case

$$\sum_{i=1}^{r} \chi(V_i) = 2,$$

since $\tau = 0$, and $\chi(C) = 0$ for an elliptic curve C. For a ruled elliptic surface V_i the Euler characteristic $\chi(V_i)$ is 0, while for a rational surface $\chi(V_i) = 1$. Therefore, there are exactly two rational V_i, and the rest are elliptic. Since there is only one double curve on a rational V_i (type (c)), it follows that the endpoints V_1 and V_r are the rational surfaces, and so X_0 falls into type 2 of the theorem.

The connection between the type of the degenerate fiber and the monodromy immediately follows from Theorem 5.2, and Theorem 6.2 is proved.

6.3. Let us give some examples of degenerations of K3 surfaces of types 2 and 3 of Theorem 6.2. As we know, a nonsingular hypersurface $X_4 \subset \mathbb{P}^3$ defined by a homogeneous equation $F_4(x_0 : \ldots : x_4) = 0$ of degree 4 is a K3 surface (see Chapter 4, Section 5). Take two non-singular quadrics $Q_1, Q_2 \in \mathbb{P}^3$, intersecting each other and X_4 along smooth curves. It is easy to see that Q_1 and Q_2 intersect along an elliptic curve. Let $F_2'(x_0 : \ldots : x_4) = 0$ and $F_2''(x_0 : \ldots : x_4) = 0$ be the equations of the quadrics Q_1 and Q_2. Then, these quadrics and X_4 define a rational map

$$f : \mathbb{P}^3 \overset{(F_2'F_2'':F_4)}{\longrightarrow} \mathbb{P}^1.$$

After resolving the singularities of this map by monoidal transformations centered at $X_4 \cap Q_1$ and $X_4 \cap Q_2$ we get a regular map

$$\tilde{f} : \tilde{\mathbb{P}^3} \to \mathbb{P}^1,$$

generic fiber of which is a K3 surface. The degenerate fiber, coming from the quadrics will, evidently, consists of two rational surfaces – the proper preimages of the quadrics under the resolution of points of indefiniteness of the mapping f. Since these rational surfaces intersect along an elliptic curve, we have a degeneration of type 2 of Theorem 6.2.

To get a degeneration of the third type, instead of two quadrics, take four planes in general position.

6.4. Let us show how theorems 6.1 and 6.2 imply that the period mapping is onto for K3 surfaces. In the notation of Chapter 3, Section 3, let $D(l)$ be the space of periods of marked K3 surfaces. By the global Torelli theorem, there is an effectively parametrized family $\pi : \mathcal{F} \to S$ of marked K3 surfaces, such that $\dim S = 19$, the period mapping $\Phi : S \to D(l)$ is one-to-one, and $\Phi(S)$ is an open everywhere dense set in $D(l)$.

Theorem (Kulikov [1977b]). *For every point $x \in D(l)$ there exists a marked K3 surface $\tilde{V}_x = (V_x, \phi, \xi)$ of type l, such that $\Phi(V_x) = x$. The class $\xi \in H_2(V_x, \mathbb{Z})$ corresponds to an ample modulo "-2 curves" divisor class on V_x.*

A divisor D on V is called ample modulo "-2 curves" if for some positive integer n, the linear system $|nD|$ defines a morphism $f : V \to \mathbb{P}^{\dim |nD|}$, satisfying the following conditions:

1. $f(V)$ is a normal surface in $\mathbb{P}^{\dim |nD|}$ with the simplest rational singularities,
2. $f : V \backslash f^{-1}(x_1 \cup \ldots \cup x_k) \to f(V) \backslash \{x_1, \ldots, x_k\}$ is an isomorphism, where $x_1, \ldots, x_k$ are singularities of $f(V)$,
3. $f^{-1}(x_1 \cup \ldots \cup x_k) = \bigcup_{i=1}^{N} L_i$, where L_i are "-2 curves", that is, L_i are rational curves, and $(L_i^2)_V = -2$.

Proof. Let $F = \mathcal{O}_V(1)$ be a very ample sheaf, corresponding to the class $\xi \in H^2(V, \mathbb{Z})$ for some marked K3 surface V. Let $P(k) = a_2k^2 + a_1k + a_0 = \chi(\mathcal{O}_V(k))$ be the Hilbert polynomial, and let $\tilde{\mathcal{H}} \to \tilde{M}$ be the Hilbert scheme with Hilbert polynomial $P(k)$ (Hartshorne [1977]). The fibers of this scheme are K3 surfaces of type l over an open set in $\tilde{M}$. Thus, we have a family of nonsingular marked K3 surfaces, which shall denote by $f : \mathcal{H} \to M$, where M is some quasi-projective variety. For this family we have the period mapping

$$\Phi_M : M \to D(l)/\Gamma_l, \tag{23}$$

where Γ_l is an arithmetic group of transformations of $L \simeq H_V$ preserving the intersection form and leaving the vector l invariant (see Chapter 3, Section 3). From the global Torelli theorem we know that $\Phi_M(M)$ is everywhere dense in $D(l)/\Gamma_l$. Let

$$\begin{array}{ccc} \mathcal{H} & \subset & \overline{\mathcal{H}} \\ \downarrow & & \downarrow \\ M & \subset & \overline{M} \end{array}$$

be the compactification of the family $\mathcal{H} \to M$, such that $\overline{\mathcal{H}}$ and $\overline{M}$ are projective varieties, and $\overline{\mathcal{H}} \backslash \mathcal{H}$ and $\overline{M} \backslash M$ are divisors with normal crossings (this is always possible by Hironaka's theorem). Now we can use a theorem of Borel [1972].

Theorem. *Let D be a bounded symmetric domain and let $\overline{M} \backslash M$ be a divsor with normal crossings. Then the period map $\Phi : M \to D/\Gamma$ can be continued to a holomorphic map $\overline{\Phi} : \overline{M} \to \overline{D/\Gamma}$, where $\overline{D/\Gamma}$ is the Baily-Borel compactification of the space D/Γ (Baily-Borel [1966]).*

Note. The Baily-Borel compactification is constructed as follows. Let K_D be the canonical sheaf on D. The Γ-invariant sections $s \in H^0(D, K_D^{\otimes n})$ induce sections $\overline{s} \in H^0(D/\Gamma, K^{\otimes n})$. It can be shown that the ring

$$\bigoplus_{n \geq 0} H^0(D/\Gamma, K^{\otimes n})$$

is finitely generated, while the map

$$D/\Gamma \to \operatorname{Proj}\left(\bigoplus_{n \geq 0} H^0(D/\Gamma, K^{\otimes})\right)$$

is an inclusion. In other words, for a sufficiently large n, the sections in $H^0(D/\Gamma, K^{\otimes n})$ define an inclusion

$$D/\Gamma \to \mathbb{P}^{\dim H^0(D/\Gamma, K^{\otimes n})-1},$$

whose image is an open subset of some projective variety $\overline{D/\Gamma}$.

Let us apply Borel's theorem to the period mapping (23). Let $\overline{\Phi} : \overline{M} \to \overline{D(l)/\Gamma_l}$ be the continuation to $\overline{M}$ of the period mapping $\Phi_M : M \to D(l)/\Gamma_l$. Since $\overline{M}$ and $\overline{D(l)/\Gamma_l}$ are compact and $\Phi_M(M)$ is everywhere dense in $D(l)/\Gamma$, we know that $\overline{\Phi} : \overline{M} \to \overline{D(l)/\Gamma_l}$ is onto.

Let $\tilde{x}$ be the image of a point x under the map $D(l) \to D(l)/\Gamma_l$. Pick an arbitrary curve $i : \overline{S} \hookrightarrow \overline{M}$ passing through a point $\overline{y} \in \overline{\Phi}^{-1}(\tilde{x}) \in \overline{M}$ and such that S is not contained in $\overline{M} \backslash M$. Let $j : S \to \overline{S}$ be the resolution of singularities, and let $y \in j^{-1}(\overline{y})$. Let $\overline{X} \to S$ be the preimage of the family $\overline{\mathcal{H}} \to \overline{M}$ under the map $i \circ j : S \to \overline{M}$. The period map $\Phi_S : S \to \overline{D(l)/\Gamma_l}$ coincides with the composition $\overline{\Phi} \circ i \circ j$, and $\Phi_S(y) = \tilde{x}$.

Let $\Delta = \{|t| < 1\} \subset S$ be a sufficiently small neighborhood of the point y, and let $\pi : X \to \Delta$ be the restriction of the family $\overline{X} \to S$ to Δ. From the condition $\Phi_s(y) = \tilde{x} \in D(l)/\Gamma_l$ it is easy to see that the monodromy group of the family $\pi^* : X^* \to \Delta^*$, acting on $H_2(V, \mathbb{Z})$ is finite. Thus, after passing to a finite cover if necessary, we can use Mumford's semistable reduction theorem, and assume that $\pi : X \to \Delta$ is a semistable degeneration of K3 surfaces, and $T = \mathrm{id}$. According to theorem 6.1, we can modify the degeneration $\pi : X \to \Delta$ into a degeneration $\pi' : X' \to \Delta$ with trivial canonical class, without changing the monodromy. Applying Theorem 6.2 we see that the degeneration $\pi' : X' \to \Delta$ has no degenerate fibers, since $T = \mathrm{id}$. Thus, X_0' is a non-singular K3 surface. The period of the surface X_0' coincides with $x \in D(l)$. Thus, it is enough to show that the class $\xi_0 = \phi^{-1}(l)$ is the class of an ample modulo "-2 curves" divisor. To this end, note that the polarizing classes ξ_t on the fibers X_t are invariant cycles. Thus, by the invariant cycle theorem (see Chapter 4, Section 4) some multiple $n\xi_t$ is carried by a global divisor $\eta \in H^4(X, \mathbb{Z})$ for $t \neq 0$. Thus $\xi_0 = \frac{1}{n}\overline{\eta}X_0'$, where $\overline{\eta}$ is the proper image of the divisor η under the reconstruction. We then know that $(\xi_0^2)_{X_0'} = (\xi_t^2)_{X_t'} > 0$. Thus (see Griffiths–Harris [1978]) some multiple of the divisor ξ_0 defines a birational morphism $\phi_{\xi_0}; X_0' \to \phi_{\xi_0}(X_0') \subset \mathbb{P}^N$. If the morphism ϕ_{ξ_0} blows an irreducible curve $C \subset X_0'$ down to a point, then $(\xi_0, C)_{X_0'} < 0$. In addition, $(C, K_{X_0'})_{X_0'} = 0$, since $K_{X_0'} = 0$. On the other hand, the arithmetic genus $g_a(C)$ of an irreducible curve C is non-negative, and equals

$$\frac{(C^2)_{X_0'} + (C, K_{X_0'})_{X_0'}}{2} + 1.$$

Therefore, $(C^2)_{X_0'} \geq -2$, and thus $(C^2)_{X_0'} = -2$, and so $g_a(C) = 0$, thus C is a "-2 curve."

Comments on the bibliography

1. The best introduction to the concept of algebraic variety is Shafarevich [1972], which also contains a brief historical survey of algebraic geometry. For a more extensive introduction to complex algebraic geometry we suggest Griffiths–Harris [1978]. The methods of modern algebraic geometry over arbitrary ground fields are explained in Hartshorne [1977]. The theory of analytic functions of several complex variables and the theory of analytic sets can be found in Gunning–Rossi [1965]. A good basic introduction to the theory of complex manifolds is Wells [1973], The classical theory of Hermitian and Kähler manifolds is treated in Chern [1957], Griffiths–Harris [1978], and Wells [1973]. A comparison between algebraic and analytic categories can be found in Serre [1956].

2. The concept of periods of integrals as analytic parameters which determine a complex manifold goes back to Riemann's paper "Theory of abelian functions". The first proof of Torelli's theorem for curves was given in Torelli [1914]; a modern proof is given in Andreotti [1958].

The explosive developement of the theory of periods of integrals and their application to Torelli-type theorems began with Griffith [1968] and [1969]. Griffiths [1968] introduces the concepts of the space of period matrices and the period mapping and proves the that the period mapping is holomorphic and horizontal. The conditions for the infinitesimal Torelli theorem to hold obtained by Griffiths [1968] provided the impetus for an extensive literature on the subject. The remarkable paper Griffiths [1969] studies the period mapping for hypersurfaces, and proves a local Torelli theorem. A relatively complete survey of the key results on the period mapping can be found in Griffiths [1970] and Griffiths–Schmid [1975]. Griffiths tori are introduced in Griffiths [1968]. Their comparison with Weil tori is undertaken in that same paper.

The theorems on the existence of global deformations of complex manifolds are contained in Kodaira–Nirenberg–Spencer [1958], Kuranishi [1962, 1965]. A rigorous formulation of the theory of moduli spaces is contained in Mumford [1965]. Various questions having to do with infinitesimal variations of Hodge structure are discussed in Carlson–Griffiths [1980], Carlson–Green–Griffiths–Harris [1983], Griffiths–Harris [1983] and Griffiths [1983a].

3. There is a good exposition of the theory of Jacobians of algebraic curves in Griffiths–Harris [1978], which also contains a complete proof of Torelli's theorem for curves. The most spectacular papers proving global Torelli theorems are Piatetski-Shapiro–Shafarevich [1971], Andreotti [1958] and Clemens-Griffiths [1972]. These papers largely determined the later progress of the theory. Counterexamples to Torelli theorems can be found in Chakiris [1980], Kynev [1977], Todorov [1980], Griffiths [1984].

4. The theory of mixed Hodge structures owes its existence to the work of Pierre Deligne ([1971], [1972], [1974b]). This theory was applied to the theorem on invariant local cycles in Deligne [1972].

The limit mixed Hodge structure on the cohomology of a degenerate fiber was introduced and studied in Schmid [1973] and Steenbrink [1974]. Clemens [1977] constructs the Clemens-Schmid sequence, introduces the mixed Hodge structures on its members, and proves its exactness in the case of Kähler degenerations. A general survey of the theory of mixed Hodge structures can be found in Griffiths [1984] and Griffiths–Schmid [1975].

The generalizations of the concepts of the variation of Hodge structures, period mapping and Torelli theorem to the case of mixed Hodge structures can be found in Cattani–Kaplan [1985], Cattani–Kaplan–Schmid [1987a], Cattani-Kaplan–Schmid [1987b], Griffiths [1983a, 1983b], Kashiwara [1985], Kashiwara [1986], Kashiwara–Kawai [1987], Saito [1986], Shimizu [1985], Usui [1983]. A brief survey is presented in Saito–Shimizu–Usui [1987].

The mixed Hodge structure on homotopy groups is introduced in Morgan [1978]. This is based on the theory of D. Sullivan, a survey of which can be found in Deligne- Griffiths-Morgan-Sullivan [1975]. A somewhat different approach, based on the theory of iterated integrals is presented in Hain [1987].

5. Degenerations of algebraic varieties were first studied systematically in the classical work Kodaira [1960]. That paper considered the degenerations of elliptic curves in conjunction with the problem of classifying compact complex manifolds. Degenerations of curves of degree 2 were considered in Namikawa–Ueno [1973] and Ogg [1966].

A considerable advance in the study of degeneration of surfaces was provided by Kulikov [1977], where the degenerations of K3 surfaces were investigated. A survey of the results on the degeneration of surfaces is contained in Friedman–Morrison [1983] and Persson [1977].

As a first introduction to the local degeneration theory we suggest Milnor's book [1968]. The retraction of the degenerate fiber onto a non-degenerate one (the Clemens map) is constructed by Clemens [1977]. A survey of the main results on the connection of topological characteristics of the degenerate and non-degenerate fibers of algebraic surfaces can be found in Kulikov [1981] and Persson [1977].

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II. Algebraic Curves and Their Jacobians

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Contents

Introduction

The current article is a continuation of the survey "Riemann Surfaces and Algebraic Curves" of Volume 23 of the current series. The results presented here are also classical, although completely rigorous proofs have been obtained only recently. The analytical aspects of the theory of Jacobians, theta-functions, and their applications to the equations of mathematical physics can be found in the survey "Integrable systems I" of volume 4 of the present series (by Dubrovin, Krichever, and Novikov).

§1. Applications

Here we discuss some remarkable recent applications of the theory of algebraic curves. We show that the class of theta functions of complex algebraic curves (more precisely, of their period matrices) is quite sufficient to solve several important problems. Throughout this section, the ground field k is assumed to have characteristic 0.

1.1. Theory of Burnchall–Chaundy–Krichever. There is a natural bijective correspondence between the following sets of data:

Data A. A complete irreducible curve C over k, p a nonsingular k point of C, a tangent vector v at p, and a torsion-free sheaf $\mathcal{F}$ over C of rank 1 with $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$.

Data B. A commutative subring $R \subset k[[x]][d/dx]$ with $k \subset R$, and such that there exist two operators $A, B \in R$ of the form

$$A = \left(\frac{d}{dx}\right)^m + a_1(x)\left(\frac{d}{dx}\right)^{m-1} + \ldots + a_m(x),$$

$$B = \left(\frac{d}{dx}\right)^n + b_1(x)\left(\frac{d}{dx}\right)^{n-1} + \ldots + b_n(x)$$

with $(m, n) = 1$. Two such subrings $R_1, R_2 \subset k[[x]][d/dx]$ will be identified if

$$R_1 = u(x) \cdot R_2 \cdot u(x)^{-1},$$

where the formal powerseries $u(x) \in k[[x]]$, $u(0) \neq 0$, is viewed as the operator corresponding to multiplication by $u(x)$.

Let us examine the correspondence between data A and data B most important for applications. In order to do this, construct a deformation $\mathcal{F}^*$ of the sheaf $\mathcal{F}$ over $C \times_k k[[x]]$ and a differential operator

$$\nabla : \mathcal{F}^* \to \mathcal{F}^*(p),$$

such that

(a) $\nabla(as) = a\nabla s + \frac{\partial a}{\partial x}s$ for all $a \in \mathcal{O}_{\mathbb{C}} \otimes_k k[[x]]$, $\quad s \in \mathcal{F}^*$;

(b) $\nabla s = \frac{s}{z} +$(a section of $\mathcal{F}^*$), where z is a local parameter at p with $\frac{d}{dz} = v$.

For $k = \mathbb{C}$, the desired sheaf $\mathcal{F}^*$ is easily described analytically. Let U be a neighborhood of p, where z is a local coordinate. Then $\mathcal{F}^*$ is taken to be equal to $\mathcal{F} \otimes \mathcal{O}_{\mathbb{C}}$ over $U \times \mathbb{C}$ and on $(C - p) \times \mathbb{C}$. These sheaves are glued together over $(U - p) \times \mathbb{C}$ by multiplying by the transition function $\exp(x/z)$. On the sections of $\mathcal{F}^*$ over $(C - p) \times \mathbb{C}$ the operator ∇ is defined to be the partial differentiation operator $\frac{\partial}{\partial x}$. This can be extended to the required differential operator from $\mathcal{F}^*$ to $\mathcal{F}^*(p)$, since

$$\exp(-x/z)\frac{\partial}{\partial x}(\exp(x/z)f(z,x)) = \frac{1}{z}f(z,x) + \frac{\partial}{\partial x}f(z,x),$$

and on the sections of $\mathcal{F}^*$ over $U \times \mathbb{C}$ this operator will be the linear differential operator $\frac{\partial}{\partial x} + \frac{1}{z}$. In order to implement this construction in general, the function $\exp(x/z)$ must be replaced by a formal power series in x. Analogously, we define operators

$$\nabla : \mathcal{F}^*(lp) \to \mathcal{F}^*((l+1)p).$$

From the hypotheses on $\mathcal{F}$ it follows that $h^i(C \times_k k[[x]], \mathcal{F}^*) = 0$ for $i = 0, 1$. By the exact sequence of restriction it follows that $H^0(C \times_k k[[x]], \mathcal{F}^*(p))$ is a free $k[[x]]$ module of rank 1. Denote its generator by s_0, and correspondingly let

$$s_l = \nabla^l s_0 \in H^0(C \times_k k[[x]], \mathcal{F}^*((l+1)p)).$$

By construction, $s_l = s_0/z^l +$(higher order terms). Thus, the sections $s_0, \ldots, s_l$ form a $k[[x]]$-basis of $H^0(C \times_k k[[x]], \mathcal{F}^*((l+1)p))$. Set $R = \Gamma(C - p, \mathcal{O}_C)$. For every $a \in R$, such that $a = \alpha/z^l +$ (higher order terms), $as_0 \in H^0(C \times_k k[[x]], \mathcal{F}^*((l+1)p))$, and thus

$$as_0 = \alpha s_l + \sum_{i=0}^{l-1} a_i(x)s_i = \left(\alpha\nabla^l + \sum_{i=0}^{l-1} a_i(x)\nabla^i\right) s_0.$$

Mapping a to the differential operator

$$D(a) = \alpha\left(\frac{d}{dx}\right)^l + \sum_{i=0}^{l-1} a_i(x)\left(\frac{d}{dx}\right)^i,$$

we obtain an inclusion of R into $k[[x]][d/dx]$. It is not hard to check that this is a homomorphism, and its image is a commutative ring isomorphic to R. Under the transformation sending s_0 to $u(x)s_0$ (with $u(0) \neq 0$), the inclusion D is transformed into the inclusion

$$a \to u(x) \cdot D(a) \cdot u(x)^{-1}.$$

In a more explicit description of the above correspondence for a nonsingular curve C of genus g, the sheaf $\mathcal{F}$ is defined by a (general) ineffective divisor D

of degree $g-1$ (compare the beginning of Section 1.2). The section s_0 in that case is a Baker-Akhiezer function, which can be represented in terms of the Riemann theta-function of the curve C (see Shiota [1983]).

Data A can be reconstructed from Data B using the spectral properties of the operators in the subring $D(R)$ identified with R. This is done by considering the space of formal power series $f(x) = \sum a_i x^i$ with coefficients a_i in extensions $K \supseteq k$, which are eigenvectors for all $a \in R$:

$$D(a)f = \lambda(a)f.$$

Note that the homomorphisms

$$\lambda : R \to K$$
$$a \to \lambda(a)$$

are K-points of $C - p$.

Proposition. *There is a natural isomorphism of the space of eigenvectors*

$$\{f \in K[[x]] | D(a)f = \lambda(a)f \text{ for all } a \in R\}$$

and the space

$$\mathrm{Hom}_R(\mathcal{F}_q/\mathfrak{m}_q\mathcal{F}_q, K),$$

where $\mathcal{F}_q/\mathfrak{m}_q\mathcal{F}_q$ is the fiber of $\mathcal{F}$ over the K-point q corresponding to the homomorphism $\lambda : R \to K$; the field K is viewed as a λ-module.

Indeed, every homomorphism $\varphi : \mathcal{F}_q/\mathfrak{m}\mathcal{F}_q \to K$ defines a unique homomorphism

$$\begin{array}{c} \mathcal{F}_q \\ \cup \\ \varphi : \bigoplus_{l=0}^{\infty} s_l k = \Gamma(C-p,\mathcal{F}) \to K. \end{array}$$

In turn, the homomorphism φ can be uniquely extended to an $R[[x]]$-linear map

$$\varphi^* : \bigoplus_{l=0}^{\infty} s_l k[[x]] = \Gamma((C-p) \times k[[x]], \mathcal{F}^*) \to K[[x]]$$

where

$$\varphi^* \nabla a = \frac{d}{dx} \varphi^* a.$$

On the other hand, such a homomorphism φ^* is determined by a single value $f(x) = \varphi^*(s_0)$:

$$\varphi^* \left(\sum s_l a_l(x) \right) = \sum a_l(x) \left(\frac{d}{dx} \right)^n f(x).$$

By R-linearity,

$$\varphi^*(as_0) = \lambda(a)\varphi^* s_0$$

for all $a \in R$, that is, if $as_0 = \left(\sum a_i(x)\nabla^i\right) s_0$, then

$$\sum a_i(x) \left(\frac{d}{dx}\right)^i f(x) = \lambda(a) f(x),$$

– that is, f is a λ-eigenvector. The correspondence $\varphi \mapsto f$ establishes the required isomorphism.

Finally, $C - p = \operatorname{Spec} R$, where the point p corresponds to the valuation $a \to \operatorname{ord}_p a = \deg D(a)$. The eigenspaces associated to each point of $C - p$ form a bundle. The sheaf $\mathcal{F}$ over $C - p$ is a sheaf of regular, fiberwise linear functions on this bundle. An example is the sheaf of functions $f \to f^{(l)}(0)$. Furthermore, $f \to f^{(l)}(0) \in \Gamma(C, \mathcal{F}((l+1)p))$, and for $m \gg 0$ such functions (with $l \leq m$) generate $\mathcal{F}((m+1)p)$. Moreover, as found by Burnchall and Chaundy and rediscovered by Krichever, the Data B always correspond to some choice of Data A. This dictionary is extended to the case of sheaves of rank d and commutative rings of differential operators of degree divisible by d in by Mumford [1978]. A somewhat different version of Data A and B is given in Shiota [1983].

1.2. Deformation of Commuting Differential Operators. In this section we assume $k = \mathbb{C}$, unless explicitely stated otherwise. In Data A, fix a curve C, a smooth point $p \in C$, a tangent vector v at p, and also a local parameter z, as in the last section. Then, varying the sheaf $\mathcal{F}$ does not change the ring $R = \Gamma(C - p, \mathcal{O}_C)$, but does change the embedding D and its image – the subring $D(R) \subset \mathbb{C}[[x]][d/dx]$. By the Riemann-Roch formula, the hypotheses on $\mathcal{F}$ (when the curve C is irreducible) imply that $\mathcal{F}$ is an invertible sheaf of degree $g - 1$, where g is the genus of C. Let us recall that every torsion-free sheaf of rank 1 is invertible in the neighborhood of the regular points of C. Therefore, in this case, the sheaf $\mathcal{F}$ can be identified (up to isomorphism) with a point of the Jacobian variety $\operatorname{Pic}^{g-1} C$, and furthermore $\mathcal{F} \notin \Theta$ (where Θ is the canonical polarization divisor) by virtue of the hypothesis $h^0(\mathcal{F}) = 0$. For a singular curve C, the 1-parameter deformations of $\mathcal{F}$ are given by tensor products $\mathcal{F} \otimes \mathcal{F}_t$, where $\mathcal{F}_t \in J(C) = \operatorname{Pic}^0 C$ is a deformation on the Jacobian with $\mathcal{F}_0 = \mathcal{O}_C = 0 \in J(C)$. Thus, the deformation of the sheaf on the Jacobian corresponds to the deformation of the embedded subring $D_t(R)$ in $\mathbb{C}[[x]][d/dx]$, called the *Jacobian flow*. Considering the ring R fixed, we indicate the evolution law of the Jacobian flow by

$$D_t : R \hookrightarrow \mathbb{C}[[x]] \left[\frac{d}{dx}\right].$$

This evolution law is described fairly simply as follows:

Theorem (Mumford [1978]). *The deformation of any pair of operators $a, b \in R$ satisfies a Lax-type equation*

$$\frac{d}{dt}D_t(a) = [(D_t(b)^{l/n})_+, D_t(a)],$$

where $[\ ,\ ]$ *is the commutator,* n *is the degree of* b, $l \geq 1$ *is an arbitrary integer, and* $(D_t(b)^{l/n})_+ \in \mathbb{C}[[x]][d/dx]$ *is the sum of terms of non-negative degree of the pseudo-differential operator* $D_t(b)^{l/n}$, *which is the operator* $D_t(b)$ *raised to the* l/n*-th power.*

Example. Consider the operator $A = \left(\frac{d}{dx}\right)^2 + a(x)$ of degree two. It can be checked directly that

$$A^{1/2} = \left(\frac{d}{dx}\right) + \frac{a(x)}{2}\left(\frac{d}{dx}\right)^{-1} - \frac{a'(x)}{4}\left(\frac{d}{dx}\right)^{-2} + \\ + \frac{a''(x) - a(x)^2}{8}\left(\frac{d}{dx}\right)^{-3} + \frac{6a(x)a'(x) - a'''(x)}{16}\left(\frac{d}{dx}\right)^{-4} + \dots,$$

and so

$$A^{3/2} = \left(\frac{d}{dx}\right)^3 + \frac{3a(x)}{2}\left(\frac{d}{dx}\right) + \frac{3a'(x)}{4} + \frac{a''(x) + 3a(x)^2}{8}\left(\frac{d}{dx}\right)^{-1} + \dots$$

and

$$\left[A, (A^{3/2})_+\right] = -\frac{1}{4}(a'''(x) + 6a(x)a'(x)).$$

Therefore, if $D(a) = D(b) = A$, then the Jacobian flow

$$A_t = \left(\frac{d}{dx}\right)^2 + a(t,x) \tag{1}$$

satisfies the Korteweg–de Vries (KdV) equation

$$\frac{\partial a}{\partial t} = \frac{1}{4}\left(\frac{\partial^3 a}{\partial x^3} + 6a\frac{\partial a}{\partial x}\right)$$

after rescaling of the coefficients. Thus, Jacobian flows give explicit solutions to this equation. On the other hand, in order to construct the flow (1), one must have a function $a \in R = \Gamma(C - p, \mathcal{O}_C)$ with a pole of second order at p. Thus, we should take a hyperelliptic curve C and a Weierstrass point p on it. Then there exists a function $a \in R$ with a pole of second order at p, and by choosing the section s_0 (see Section 1.1) appropriately,

$$D(a) = \left(\frac{d}{dx}\right)^2 + a(x).$$

Thus the flows on hyperelliptic Jacobians define 1-parameter families of operators

$$D_t(a) = \left(\frac{d}{dx}\right)^2 + a(t,x),$$

where $a(t,x)$ satisfy the KdV equation. For a smooth curve C such solutions were found by McKean and Van Moerbeke [1975], and Dubrovin, Matve'ev and Novikov [1976]. For singular hyperelliptic curves

$$y^2 = xf(x)^2$$

these solutions are known as Kay-Moses [1956] solitons, while for unicursal curves

$$y^2 = x^{2n+1}$$

they are known as the rational solitons of @Airault–McKean–Moser [1977].

1.3. Kadomtsev–Petviashvili Equations. In the proof of the theorem of the last section it is natural to extend the inclusions D_t to an inclusion of the field of rational functions on the curve C into the ring of pseudo-differential operators:

$$D_t : \mathbb{C}(C) \hookrightarrow P_sD\{x\}.$$

With an appropriate choice of the section s_0, the element $1/z$ corresponds to the pseudo-differential operator $L_t = D_t(1/z) \in \frac{d}{dx} + \Psi^-$, where Ψ^- is the space of pseudo-differential operators of degree ≤ -1. Such an operator L_t depends solely on $\mathcal{F}_t$ – the commutative ring $R_t = D_t(R)$ can be reconstructed uniquely.

Proposition (Shiota [1983].)

$$R_t = \left\{ A \in \mathbb{C}[[x]]\left[\frac{d}{dx}\right] \mid [A, L_t] = 0 \right\}.$$

This means that the Jacobian flows R_t of this type are completely determined by the deformations of the operators L_t. To determine the evolution of L completely as a function of $\mathcal{F}$, choose a set of complex variables $t_1, \dots, t_N$ and, instead of the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{C}}$ on $C \otimes_{\mathbb{C}} \mathbb{C}[[z]]$, let us consider the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{C}^N}$ on $U \times \mathbb{C}^N$ and $(C - p) \times \mathbb{C}^N$ glued on $(U - p) \times \mathbb{C}^N$ by multiplication by the function $\exp\left(\sum_{j=1}^N t_j z^{-j}\right)$. The deformation $\mathcal{F}^*$ of this sheaf on $C \times_{\mathbb{C}} \mathbb{C}[[x, t_1, \dots, t_N]]$ is obtained by formally replacing t_1 by $t_1 + x$. The dependence of the inclusion $D_{t_1,\dots,t_N}$ and its image $D_{t_1,\dots,t_N}(R)$ is again described by a Lax equation for the operator $L = L(t_1, \dots, t_N) \in \frac{d}{dx} + \Psi^- \otimes \mathbb{C}[[t_1, \dots, t_N]]$:

$$\frac{\partial}{\partial t_n} L = [(L^n)_+, L], \quad n = 1, \dots, N.$$

Evidently, the form of this system does not depend on N. For this reason, people usually consider an infinite chain of variables $t_1, t_2, \dots$, and the system of equations

$$\frac{\partial}{\partial t_n} L = [(L^n)_+, L]$$

controlling the evolution of the operator $L = L(t_1, t_2, \ldots) \in \frac{d}{dx} + \Psi^- \otimes \mathbb{C}[[t_1, t_2, \ldots]]$ is then called the *Kadomtsev–Petviashvili hierarchy*, or, shorter, the *KP hierarchy*.

Remark-Example. *To study differential equations on functions, associate to the operator $L(t_1, t_2, \ldots)$ a complex-valued function $\tau(t_1, t_2, \ldots)$. The corresponding equations for τ, written using Hirota's bilinear differential operator can be found in Shiota [1983]. In particular, for the function*

$$\tau(t_1, t_2, t_3, 0, \ldots) = \exp(Q(t_1, t_2, t_3))\vartheta(t_1 a_1 + t_2 a_2 + t_3 a_3 + \zeta),$$

where $\vartheta(u) = \vartheta(u, Z)$ is Riemann's theta function, $a_i \in \mathbb{C}^g$, $Q(t_1, t_2, t_3)$ is a quadratic form, and ζ is a parameter in $\mathbb{C}^g$, the first of the equations of the KP hierarchy is equivalent to the system

$$\left(\frac{\partial^4}{\partial t_1^4} + 3\frac{\partial^2}{\partial t_2^2} - 4\frac{\partial^2}{\partial t_1 \partial t_3} + c_1 \frac{\partial^2}{\partial t_1^2} + c_2\right) \theta \begin{bmatrix} \delta \\ 0 \end{bmatrix} \left(2\sum_{i=1}^{3} t_i a_i, 2Z\right) = 0$$

for all $\delta \in \{0,1\}^g$. The constants c_1 and c_2 depend on Q as follows: if

$$Q(t_1, t_2, t_3) = \sum_{i,j=1}^{3} Q_{ij} t_i t_j, \quad Q_{ij} = Q_{ji},$$

then

$$c_1 = 24Q_{11}, \quad c_2 = 48Q_{11}^2 + 12Q_{22} - 16Q_{13},$$

where Q may be chosen so that $Q_{1j} = Q_{j1} = 0$ for all j, and, in particular, $c_1 = 0$. As will become clear later, this is almost enough to determine the period matrices (I, Z) of Jacobians of curves (see Section 4.5).

1.4. Finite Dimensional Solutions of the KP Hierarchy. For sufficiently large N ($\geq 2g+1$), the dimension of the image $L(t_1, \ldots, t_N)$ becomes equal to g – the genus of C, (arithmetic genus for a singular curve C). This property of finite-dimensionality characterizes Jacobian flows. In general, an operator $L \in \frac{d}{dx} + \Psi^-$ is called *finite-dimensional* if the linear map

$$dL : \mathbb{C}^\infty \to \psi^-$$

$$\sum c_n \frac{\partial}{\partial t_n} \to \left[\sum c_n (L^n)_+, L\right]$$

has finite rank, called the *dimension* of L. If L is a function of $t = (t_1, t_2, \ldots)$ and satisfies the KP hierarchy, $dL(t)$ coincides with the tangent map to the map $L : t \to L(t)$ at the point t, where we make the identifications $T_t\mathbb{C}^\infty = T_0\mathbb{C}^\infty = \mathbb{C}^\infty$, and $T_{L(t)}\frac{d}{dx} + \Psi^- = \Psi^-$. It turns out that the finite-dimensionality of the operator and its dimension are independent of t. Therefore a *solution* $L(t)$ is called *finite-dimensional* if it is finite-dimensional for some t. It is known that every finite-dimensional solution is meromorphic on

$\mathbb{C}^\infty$, and the associated function τ is entire. If the solution L is g-dimensional, then the map dL (meromorphic in t) factors through $\mathbb{C}^\infty/K_L \simeq \mathbb{C}^g$:

$$\begin{tikzcd} \mathbb{C}^\infty \arrow[rr] \arrow[dr, "dL"'] & & \mathbb{C}^\infty/K_L, \; K_L = \operatorname{Ker} dL \arrow[dl, "\overline{dL}"] \\ & \Psi^- & \end{tikzcd}$$

Therefore, the space of effective parameters

$$T_L = (\mathbb{C}^\infty/K_L)/\Gamma$$

is a complex abelian Lie group, where

$$\Gamma = \{\gamma \in \mathbb{C}^\infty/K_L \,|\overline{dL}(\gamma) = \overline{dL}(0)\} = \{\overline{dL}(t+\gamma) = \overline{dL}(t), \quad \text{for all } t \in \mathbb{C}^\infty\}$$

is a discrete subgroup of $\mathbb{C}^\infty/K_L$. If T_L is compact, then the operator L is is called *compact* and the corresponding solution is called *quasi-periodic.*

Theorem (Mulase, Shiota [1983]). *Every g-dimensional solution $L(t_1, t_2, \ldots)$ of the KP hierarchy corresponds to a Jacobian flow of some curve C of genus g and there is an isomorphism*

$$T_L \cong J(C)$$

of complex Lie groups. In particular, L is quasi-periodic if the curve C is non-singular.

Note. If C is non-singular, $T_L \simeq J(C)$ as principally polarized abelian varietis, where the polarization on T_L is given by the zero divisor of the function τ (see Shiota [1983]).

1.5. Solutions of the Toda Lattice. There are other operator variations on the theory of Burnchall–Chaundy–Krichever. Let us briefly examine the case of finite difference operators discovered by Mumford and Van Moerbeke (see Mumford [1978]). For an arbitrary field k let $M_\infty^\Delta(k)$ be the ring of finite difference operators over k, that is, maps $A : \Pi_{-\infty}^{+\infty} k \to \Pi_{-\infty}^{+\infty} k$, ($\Pi_{-\infty}^{+\infty}$ is the set of doubly infinite sequences) defined by the rule

$$A(x)_n == \sum_{m=n-N_1}^{n+N_2} A_{nm} x_m \quad \text{for all } n \in \mathbb{Z}.$$

The minimal interval $[N_1, N_2]$ such that $A_{mn} = 0$ for $m - n \notin [N_1, N_2]$ is called the *carrier* of A. Furthermore, the carrier is called *exact* if $A_{n,n+N_1} \neq 0$ and $A_{n,n+N_2} \neq 0$ for all $n \in \mathbb{Z}$. There is a natural bijective correspondence between the following sets of data.

Data A.
(a) C is a complete irreducible curve over k.
(b) p, q are non-singular k points of C.
(c) $\mathcal{F}$ is a torsion-free sheaf of rank 1 on C, such that $\chi(\mathcal{F}) = 0$, and $h^1(\mathcal{F}(np - nq)) = 0$ for all $n \in \mathbb{Z}$.

Data B. A commutative subring $R \subset M_\infty^\Delta(k)$, with $k \subset R$, and such that there are $A, B \in R$, with exact carriers $[a_1, a_2]$, $[b_1, b_2]$ with $(a_1, b_1) = 1$ and $(a_2, b_2) = 1$, $a_2 b_1 < a_1 b_2$. Two subrings $R_1, R_2 \subset M_\infty^\Delta(k)$ are identified if there exists an invertible element $\Lambda = (\lambda_n, \delta_{nm}), \lambda_n \in k \backslash \{0\}$ with

$$R_1 = \Lambda \circ R_2 \circ \Lambda^{-1}.$$

This Jacobian flow in this case satisfies the equation

$$\frac{d}{dt} D_t(a) = \frac{1}{2}[D_t(b)_+ - D_t(b)_-, D_t(a)],$$

where $(\ )_+$ is the operation of taking the "upper triangular part" of an operator, and $(\ )_-$ is the "lower triangular part":

$$(A_+)_{ij} = \begin{cases} A_{ij}, & i < j \\ 0, & i \geq j \end{cases}$$

$$(A_-)_{ij} = \begin{cases} 0, & i \leq j \\ A_{ij}, & i > j. \end{cases}$$

Example. Consider the evolution of n-periodic operators A, that is, operators such that $A_{n+m,n+l} = A_{ml}$ for all $m, l \in \mathbb{Z}$, with carrier $[-1, 1]$. If

$$A = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e^{\alpha_n - \alpha_1} & \beta_1 & e^{\alpha_1 - \alpha_2} & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & e^{\alpha_1 - \alpha_2} & \beta_2 & e^{\alpha_2 - \alpha_3} & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & e^{\alpha_2 - \alpha_3} & \beta_3 & e^{\alpha_3 - \alpha_4} & 0 \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\alpha_{n-1} - \alpha_n} & \beta_n & e^{\alpha_n - \alpha_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $\sum_{i=1}^n \alpha_j = 0$, and $\sum_{i=1}^n \beta_i = 0$, then α_i and β_i satisfy the evolution equations

$$\begin{aligned} \dot{\alpha}_i &= \beta_i, \\ \dot{\beta}_i &= e^{\alpha_{i-1} - \alpha_i} - e^{\alpha_i - \alpha_{i+1}}, \end{aligned}$$

known as the *Toda lattice equations*, describing the dynamics of n particles on a circle, each interacting with the two neighboring particles by an exponential force law. The hypotheses on the matrix $A = D(a)$, with the exception of the last normalizations express the following hypotheses on the element $a \in \Gamma(C - p - q, \mathcal{O}_{\mathbb{C}})$:

n-periodicity: $np \sim nq$, where $\sim$ is linear equivalence, and
exact carrier $[-1, 1]$: the function a has divisor of poles $p + q$.

The solutions corresponding to a rational curve C with m ordinary second-order singularities are called m-solitons of the system.

1.6. Solution of Algebraic Equations Using Theta-Constants. The Babylonians, the Hindus, and the Chinese knew how to solve quadratic equations by the second millenium B.C. In the sixteenth century, formulas for the solution of cubics and quartics were found in Italy. These are now known as Cardano's and Ferrari's formulas. As Abel discovered in 1826, the general equation of degree greater than four cannot be solved in radicals. This result played an important part in the development of algebra. However, neither Abel's work, nor the more precise results given by Galois theory stopped work on finding explicit formulas for the solution of higher degree algebraic equations, using special functions other than radicals. For example, in 1858 Hermite and Kronecker proved that the equation of degree five could be solved using an elliptic modular function of level five. Kronecker's formula was generalized by Klein, and in 1870 Jordan showed that an algebraic equation of arbitrary degree could be solved using modular functions. Tomae's formula (see Mumford [1983]) shed further light on Jordan's proof. However, much more convenient is the more recent formula of Umemura, which can be easily deduced from Tomae's formula (see Mumford [1983]). Let $f(x)$ be a complex polynomial of odd degree $2g + 1$ with simple roots $x_1, \ldots, x_{2g+1}$. Then the equation $y^2 = f(x)$ gives a hyperelliptic curve C of degree g. Let (I, Z) be its normalized period matrix. It is uniquely determined by the choice of a standard basis for $H^1(C, \mathbb{Z})$, which, in turn, is completely determined by the ordering of the roots if $f(x)$. Thus, the theta constants $\vartheta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (Z) \stackrel{\text{def}}{=} \vartheta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (0, Z)$ are completely determined.

Umemura's formula.

$$\frac{x_1-x_3}{x_1-x_2}=\left(\vartheta\begin{bmatrix}1&0&\dots&0\\0&0&\dots&0\end{bmatrix}(Z)^4\vartheta\begin{bmatrix}1&1&0&\dots&0\\0&0&0&\dots&0\end{bmatrix}(Z)^4\right.$$

$$+\vartheta\begin{bmatrix}0&\dots&0\\0&\dots&0\end{bmatrix}(Z)^4\vartheta\begin{bmatrix}0&1&0&\dots&0\\0&0&0&\dots&0\end{bmatrix}(Z)^4$$

$$\left.-\vartheta\begin{bmatrix}0&0&\dots&0\\1&0&\dots&0\end{bmatrix}(Z)^4\vartheta\begin{bmatrix}0&1&\dots&0\\1&0&\dots&0\end{bmatrix}(Z)^4\right)$$

$$\times\left(2\vartheta\begin{bmatrix}1&0&\dots&0\\0&0&\dots&0\end{bmatrix}(Z)^4\vartheta\begin{bmatrix}1&1&\dots&0\\0&0&\dots&0\end{bmatrix}(Z)^4\right)^{-1}.$$

This formula can be used to find the roots of an algebraic equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0,\quad a_0\neq 0,\quad a_i\in\mathbb{C},\quad 1\leq i\leq n. \tag{2}$$

Evidently, we can restrict to the case of simple roots $\alpha_1,\ldots,\alpha_n\neq 0,1$ and $\neq 2$ for even n. Then the right hand side of Umemura's formula gives the value of the root α_1 if Z is the Siegel matrix of the hyperelliptic curve $y^2=f(x)$ with

$$f(x)=\begin{cases}x(x-1)(a_0x^h+\ldots+a_n) & \text{for odd } n,\\ x(x-1)(a_0x^h+\ldots+a_n)(x-2) & \text{for even } n.\end{cases}$$

The roots of $f(x)$ are ordered as follows:

$$x_1=0,\quad x_2=1,\quad x_{i+2}=\alpha_i\quad\text{for odd } n$$

$$x_1=0,\quad x_2=1,\quad x_{i+2}=\alpha_i,\quad x_{n+3}=2\quad\text{for even } n.$$

Indeed, in that case

$$\frac{x_1-x_3}{x_1-x_2}=x_3=\alpha_1.$$

So, to solve equation (2) we need to find the matrix Z, and for that we must order the roots. If the exact values of the roots are unknown, to choose an ordering it is enough to separate them, that is, to find the regions of $\mathbb{C}$ containing exactly one root each. The complex version of Sturm's theorem provides an algorithm to do that. The right hand side of Umemura's formula can thus be effectively used when solving algebraic equations by means of theta constants. Note that these constants are none other than Siegel modular forms. In comparison with the radicals

$$\sqrt[n]{a}=\exp\left(\frac{1}{n}\log a\right)=\exp\left(\frac{1}{n}\int_1^a\frac{1}{x}dx\right),$$

in the formulas above, the exponential function is replaced by Siegel modular functions, and the integrals $\int_0^a\frac{1}{x}dx$ are replaced by hyperelliptic integrals $\int(x^i/\sqrt{f(x)})dx$, (for $0\leq i\leq g-1$) defining the matrix Z.

§2. Special Divisors

As was already explained in the first part of the survey, the study of projective embeddings of curves is equivalent to the study of linear systems on them. To get more precise results, it is not enough to study some individual linear systems or divisors which reveal the general appearance of the curve, but rather we must study the configurations of all linear systems or divisors of a prescribed type.

In this section, C will denote a complete nonsingular curve of genus g over an arbitrary algebraically closed field k.

2.1. Varieties of Special Divisors and Linear Systems. We will be interested in the following configurations of divisors and linear systems on the curve C :

$$C_d^r \stackrel{\text{def}}{=} \{D \in C_d \mid \dim |D| \geq r\}$$

is the subset of the d-fold symmetric power C_d, consisting of divisors of degree d on C, lying in linear systems of dimension no less than r;

$$W_d^r = W_d^r C \stackrel{\text{def}}{=} \{|D| \mid \deg D = d, \dim |D| \geq r\}$$

is the subset of the Picard variety $\operatorname{Pic}^d C$ consisting of complete linear systems of degree d and dimension at least r;

$$G_d^r = G_d^r C \stackrel{\text{def}}{=} \{g_d^r \text{ on } C\}$$

is the set of linear systems on C of degree d and dimension *exactly* r. The connection between C_d^r and W_d^r is given by Abel's mapping

$$\mu_d : C_d \to \operatorname{Pic}^d C;$$

to wit

$$\mu_d(C_d^r) = W_d^r.$$

The first important fact is that all these configurations are algebraic varieties in a natural way. More precisely, C_d^r and W_d^r are subvarieties of C_d and $\operatorname{Pic}^d C$, respectively, while C_d^r with the natural projection onto W_d^r (for r and d corresponding to special linear systems), is a canonical resolution of singularities of the variety W_d^r.

Example 1. $C_d^0 = G_d^0 = C_d$.

Example 2. $W_{g-1}^0 = \Theta \subset \operatorname{Pic}^{g-1} C$ is the canonical polarization divisor.

The first interesting properties of these varieties are their dimensions, the number of their connected components, their singularities, and their connectivity. These questions are at least partly answered by the Brill-Noether theory described below. It should be noted right away that the varieties C_d^r, W_d^r and

G^r_d consisting of nonspecial divisors or linear systems are uninteresting, since then $C^r_d = C_d$ and $W^r_d = \operatorname{Pic}^d C$, while $G^r_d \to W^r_d = \operatorname{Pic}^d C$ is a bundle of Grassmannians.

2.2. The Brill-Noether matrix. The Brill-Noether Numbers. The fact that C^r_d and W^r_d are subvarieties of C_d and $\operatorname{Pic}^d C$ easily follows from the semicontinuity of the dimension of the cohomology groups on a family of sheaves or divisors. However, in the current situation this can be explained and proved by more elementary considerations, which also help explain some of the later developments.

Let $D = \sum p_i$ be an effective divisor of degree d on C. By the geometric interpretation of the Riemann–Roch formula, $\dim |D| \geq r$, if and only if the dimension of the linear hull of D does not exceed $\deg D - r - 1$. When the points p_i are distinct, this is equivalent to the inequality

$$\operatorname{rank}\begin{pmatrix} \omega_1(p_1) & \dots & w_1(p_d) \\ \dots\dots & \dots\dots & \dots\dots \\ \omega_g(p_1) & \dots & \omega_g(p_d) \end{pmatrix} \leq d - r,$$

where $\omega_1, \dots, \omega_g$ is the basis of the space of regular differentials on C. The matrix above is the *Brill-Noether matrix*. Its definition for a general divisor is somewhat more complicated (see Arbarello–Cornalba–Griffiths–Harris[1984]). The fact that $D \in C^r_d$, if and only if the rank of the Brill-Noether matrix does not exceed $d - r$, shows that in a neighborhood of the divisor D the subset C^r_d is the zero set of $(d - r + 1)$ minors. Indeed,

$$\operatorname{rank}(\omega_i(p_j)) = \operatorname{rank}(f_i(z_j)),$$

where z_j are local parameters at the points p_j and $\omega_i = f_i(z_j)dz_j$. Therefore, the minors of the above matrix are regular functions in a neighborhood of D on C_d, which shows that C^r_d is a subvariety of C_d.

The aforementioned determinantal description of the variety C^r_d allows us to get a lower bound on the dimension of its components. Since the components C^r_d are locally defined by the simultaneous vanishing of all of the $(d - r + 1)$ minors of a $g \times d$ matrix, the codimension does not exceed

$$[d - (d - r)][g - (d - r)] = r(g - d + r).$$

Therefore:

Proposition 1.

1. *The dimension of the components of C^r_d is at least $r + \rho$.*
2. *The dimension of the components of W^r_d is at least ρ.*

Here

$$\rho = \rho(g, r, d) \overset{\text{def}}{=} g - (r + 1)(g - d + r)$$

is the well-known *Brill-Noether number.* By the above proposition, its role is explained by the fact that $r+\rho$ and ρ are lower bounds for the dimensions of the varieties C_d^r and W_d^r, respectively (assuming that these varieties are non-empty).

2.3. Existence of Special Divisors. There arises a natural question: assuming that ρ is non-negative, does it follow that W_d^r and thus C_d^r and G_d^r are non-empty?

An affirmative answer is given by the following:

Existence theorem. (Kleiman, Laksov, Kempf). *For $d \geq 1$, $r \geq 0$ and*

$$\rho = g - (r+1)(g-d+r) \geq 0,$$

W_d^r, and thus C_d^r and G_d^r, is non-empty. Furthermore, for $r \geq d-g$, which is equivalent to the inequality $\rho \leq g$, each component of W_d^r, C_d^r, and G_d^r has dimension at least ρ, $\rho + r$, and ρ, respectively.

Fig. 1 illustrates this result: $d = 2r$ is the Clifford straight line and $\rho = 0$ is the curve of Brill-Noether, the vertically dashed region is the region of existence of linear systems g_d^r, the non-existence region is undashed.

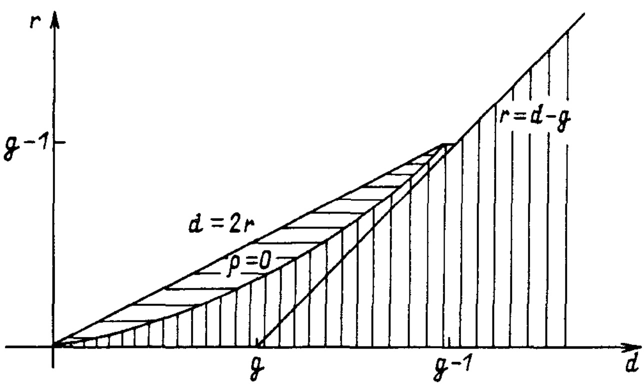

Fig. 1

The proof of the proposition follows directly from the explicit formula for the fundamental class of the subvarieties W_d^r and C_d^r, computed with the aid of Porteous' formula.

Proposition. *If W_d^r has expected dimension ρ, then the fundamental class has the form*

$$w_d^r = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} \Theta^{(r+1)(g-d+r)}.$$

Example 1. $\dim C^1_{g-1} \geq g - 3$ for $g \geq 4$, which can be checked directly. The case of a hyperelliptic curve C is self-evident: $C^1_{g-1} \supseteq g^1_2 + C_{g-3}$ and $\dim C^1_{g-1} \geq g - 2$ for $g \geq 3$. A non-hyperelliptic curve can be identified with its canonical model $C \subset \mathbb{P}^{g-1}$. It is then enough to show that a general effective divisor $p_1 + \ldots + p_{g-3}$ of degree $g - 3$ on C can be completed, by adding two points, to an effective divisor D of degree $g - 1$ with $\dim |D| = 1$, or, equivalently, with $\dim \overline{D} = g - 3$. If the $(g-4)$-plane $\overline{p_1 + \ldots + p_{g-3}}$ intersects C in an additional point or is tangent to C, then there exists a point $q \in C$, such that $\overline{p_1 + \ldots + p_{g-3} + q} = \overline{p_1 + \ldots + p_{g-3}}$, and the second point can be picked arbitrarily. Otherwise, the projection from this plane $\pi : C \to \mathbb{P}^2$ is birational, but is not an inclusion, since $\deg \pi(C) = 2g - 2 - (g-3) = g + 1$, and $g(\pi(C)) = \frac{g(g-1)}{2} > g$, for $g \geq 4$. Since π is not an inclusion, then there exist points p and q on C with $\pi(p) = \pi(q)$, and for them $\dim \overline{p_1 + \ldots + p_{g-3} + p + q} = g - 3$. Combining these observations with the theorem on general position, it is easy to see that $\dim W^1_{g-1} = g - 3$ for a hyperelliptic curve of genus $g \geq 2$, and $\dim W^1_{g-1} = g - 4$ for a non-hyperelliptic curve of genus $g \geq 3$.

Example 2. In particular, for a non-hyperelliptic curve $C \subset \mathbb{P}^3$ of genus 4, there are two g_3s which is consistent with the proposition. These linear systems are cut out by linear generators of the unique quadric passing through C.

2.4. Connectedness. As was observed by Fulton and Lazarsfeld, the existence theorem essentially follows from the ampleness properties of the complex of sheaves which gives W^r_d as the degeneracy locus. From similar considerations and using general results such as the theorems of Lefschetz and Bertini, it is possible to obtain the following

Connectedness theorem (Fulton, Lazarsfeld). *When $d \geq 1$, $r \geq 0$ and*

$$\rho = g - (r+1)(g - d + r) \geq 1$$

the variety W^r_d, and hence the varieties C^r_d and G^r_d, are connected.

2.5. Special Curves. The General Case. The existence theorem gives a lower bound on the dimension of W^r_d. A natural question is how sharp is this bound, andin particular whether W^r_d are empty for $\rho < 0$. For a generic curve C the answer is affirmative. However, there exist curves for which this does not hold, the so-called *special* curves. The possible values of r and d for special linear systems g^r_d on a curve of genus g for $\rho < 0$ are to be found in the horizontally dashed region ("lune") in Fig. 1.

Question. The author does not know whether every one of these can be realized for complete linear systems, that is, whether for any r and d in the "lune" there exists a curve C of genus g with a complete linear system g^r_d.

Example 1. By definition, there exists a g^1_2 on a hyperelliptic curve C, but $\rho(g, 1, 2) = 2 - g < 0$ for $g \geq 3$. Thus, the existence of a g^1_2 on a curve of genus $g \geq 3$ is not typical, which is easy to verify by counting parameters.

Example 2. Analogously, on a trigonal curve there is a g_3^1, but $\rho(g,1,3) = 4 - g < 0$, for $g \geq 5$.

Example 3. In order to better understand the meaning of the condition $\rho < 0$, consider the general question of the existence of a system of type g_d^1. The corresponding Brill-Noether number is

$$\rho = 2d - g - 2.$$

On the other hand, the free linear system g_d^1 defines a d-sheeted covering $C \to \mathbb{P}^1$ with branching divisor of degree $2d + 2g - 2$ by the Hurwitz formula. Since the automorphism group of $\mathbb{P}^1$ is three-dimensional, a d-sheeted cover of $\mathbb{P}^1$ of genus g depends on $d + 2g - 5$ parameters. This means that a generic curve of genus g has no g_d^1 when

$$2d + 2g - 5 < 3g - 3,$$

which is equivalent to the inequality $\rho = 2d - g - 2 < 0$.

Example 4. A generic curve of genus ≥ 4 is not a plane curve, that is, it does not admit an embedding into $\mathbb{P}^2$. In particular, a generic curve of genus 6 is not a plane quintic.

Example 5. Another special kind of curves is important for the discussion below. These are the *bi-elliptic* curves C, characterized by the existence of a two-sheeted covering $\epsilon : C \to E$ onto some elliptic curve . For such curves $\dim W_4^1 \geq 1$. Indeed,

$$W_4^1 \supseteq \{g_4^1 = \varepsilon^* g_2^1 \mid g_2^1 \in G_2^1 E\} = \varepsilon^* G_2^1 E$$

and

$$G_2^1 E = W_2^1 E = \operatorname{Pic}^1 E$$

by the Riemann-Roch formula. The corresponding Brill-Noether number is $\rho = 6 - g \leq 0$ for $g \geq 6$ and, as is easily checked by counting dimensions, a general curve of genus ≥ 6 is not bi-elliptic. Furthermore, no bi-elliptic curve of genus ≥ 6 is hyperelliptic, and every bi-elliptic curve admits a unique covering $\varepsilon : C \to E$ of the type prescribed above. This covering is realized on the canonical curve $C \subset \mathbb{P}^{g-1}$ as a projection with center at $O \in \mathbb{P}^{g-1}$, lying outside C (Shokurov [1983], [1981]).

Let us now study general curves. The statement below was formulated by Brill and Noether, although it was proved only in 1979 by Griffiths and Harris.

Dimension theorem. *Let C be a general curve of genus g, let $d \geq 1$, $r \geq 0$, and $\rho = g - (r+1)(g - d + r)$. Then*

(a) W_d^r is an irreducible (reduced) variety of pure dimension $\min(\rho, g)$ *for* $\rho > 0$;

(b) (Castelnuovo) $G_d^r = W_d^r$ is a set of cardinality

$$g! \prod_{\alpha=0}^{r} \frac{i!}{(g-d+r+\alpha)!}$$

for $\rho = 0$ (compare with the proposition of Section 2.3).
(c) W_d^r, C_d^r, and G_d^r are empty when $\rho < 0$.

The proof is based on a perturbation method: if the theorem is true for some curve of genus g, then it is true for the general curve of genus g. This follows essentially from the irreducibility of the moduli space of curves of a fixed genus. The difficulties in the proof stem from the fact that all of the known examples of curves: hyperelliptic, trigonal, plane, and so on, are special for $g \gg 0$. Thus, the original approach was to try to find a curve lying on the boundary of moduli space – a general rational curve with double points of simplest type, which, regrettably, takes one outside the class of non-singular curves. Recently, Lazarsfeld (see also Tyurin [1987]), by using the theory of vector bundles on algebraic surfaces, showed that one could take the generic curve to be a generic curve in a polarized linear system of a K3 surface, with a polarized complete linear system of degree $2g-2$ without multiple curves. A generic K3 surface of degree $2g-2$ satisfies this condition. Amazingly, such curves do not fill up the moduli space of curves of genus $g \gg 0$, and so are not general curves in the sense of Grothendieck.

2.6. Singularities. In roughly the same fashion one can establish the following:

Smoothness theorem (Gieseker). *Let C be a generic curve of genus g, $d \geq 1$ and $r \geq 0$. Then $G_d^r C$ is a smooth variety of dimension ρ.*

Theorem on singularities. *Let C be a general curve of genus g, $d \geq 1$, $r \geq 0$, and $r > d - g$. Then*

$$\operatorname{Sing} W_d^r C = W_d^{r+1} C.$$

These results will be explained in the next section.

2.7. Infinitesimal Theory of Special Linear Systems. Let us view a linear system L of degree d as a point of the Picard variety $\operatorname{Pic}^d C$. First, note that there are canonical isomorphisms of tangent spaces

$$T_L(\operatorname{Pic}^d C) = \Omega^\vee = H^0(K)^\vee,$$

where Ω is the space of regular differentials on C, while K is the canonical divisor on C. The first equality (in the case $k = \mathbb{C}$) follows from the observation that $\operatorname{Pic}^d C$ is a principal homogeneous space of the Jacobian

$$J(C) = \operatorname{Pic}^0 C = \Omega^\vee / \Lambda,$$

where
$$\Lambda = \left\{ \int_c : \Omega \to \mathbb{C} \,\middle|\, c \in H_1(C, \mathbb{Z}) \right\}$$
is a lattice in the complex space $\Omega^\vee$. In particular, the elements of the cotangent space $T_L^\vee(\operatorname{Pic}^d C)$ to $\operatorname{Pic}^d C$ at L can be naturally identified with regular differentials. Let us fix a divisor $D \in L$. Then, there is a canonical pairing
$$\mu : H^0(D) \otimes H^0(K - D) \to H^0(K)$$
$$f \otimes g \to f \cdot g.$$
There is the following description of the Zariski tangent space $T_L(W_d^r)$ to W_d^r at L; it is assumed that W_d^r is locally defined by the minors of the Brill-Noether matrix (see Section 2.2).

Proposition.

(a) If $L \in W_d^r$ but $L \notin W_d^{r+1}$, and thus $r \geq d - g$, then
$$T_L(W_d^r) = (\operatorname{Im} \mu)^\perp.$$
(b) If $L \in W_d^{r+1}$, then
$$T_L(W_d^r) = T_L(\operatorname{Pic}^d C).$$
In particular, if W_d^r has the expected dimension ρ and $r > d - g$ (so that $\rho < g$), then L is a singular point of W_d^r.

To illustrate point (a) of the above proposition, let us use the following geometric interpretation, valid for $r = 0$.

Example. Let $L \in W_d^0 \backslash W_d^1$, so that $L = |D|$ is a linear system consisting of one effective divisor D of degree d, and $g \geq d$. If the curve C is not hyperelliptic, then it is canonically embedded into the projectivization
$$\mathbb{P}(T_L(\operatorname{Pic}^d C)) = \mathbb{P}(H^0(K)^\vee) = \mathbb{P}^{g-1}.$$
The claim is that part (a) of the proposition is equivalent to the equality
$$\mathbb{P}(T_L(W_d^0)) = \overline{D}.$$
On the other hand, this equality can be easily reduced to the special case: $d = 1$, $D = p \in \mathbb{C}$
$$\mathbb{P}(T_{|p|}(W_1^0)) = p$$
by the relation $W_d^0 = W_1^0 + \ldots + W_1^0$, where $+$ denotes addition in the Picard group $\operatorname{Pic} C$. For $k = \mathbb{C}$, this can be obtained immediately from the analytic description of the Abel mapping
$$a_1 : C \to J(C) = \Omega^\vee / \Lambda = \mathbb{C}^g / \Lambda,$$
$$a_1(p) = \left(\int_{p_0}^p \omega_1, \ldots, \int_{p^0}^p \omega_g \right) \bmod \Lambda,$$

(where $\Omega^\vee = \mathbb{C}^g$) is the isomorphism defined by the choice of basis $\omega_1, \dots, \omega_g$ in Ω. Indeed, $W_1^0 = a_1(C) + p_0$, and so differentiating a_1 with respect to the local coordinate at the point p we obtain the tangent vector proportional to $(\omega_1(p), \dots, \omega_g(p))$.

Corollaries.

1. $\operatorname{Sing} W_1^0 = W_d^1$ *for* $d < g$.

2. $\operatorname{Sing} W_{g-1}^0 = W_{g-1}^1$. *This is a special case of Riemann's theorem on singularities (see below in Section 2.10), since* $W_{g-1}^0 = \Theta$. *Therefore, by Example 1 of Section 2.3 it follows that*

3. $\dim \operatorname{Sing} \Theta = g - 3$ *for a hyperelliptic curve of genus* $g \geq 2$ *and* $\dim \operatorname{Sing} \Theta = g - 4$ *for a non-hyperelliptic curve of genus* $g \geq 3$.

Part (b) of the proposition is discussed below in Section 2.10 in connection with Kempf's theorem. It implies the following weaker version of the theorem on singularities:

4. *If* $\dim W_d^r < g$, *(and in particular* $d < g$*), then*

$$\operatorname{Sing} W_d^r \supseteq W_d^{r+1}.$$

The full theorem on singularities is equivalent to the following

Petri–Gieseker theorem. *Let C be a general curve of genus g, and let D be an effective divisor. Then the pairing*

$$\mu : H^0(D) \otimes H^0(K - D) \to H^0(K)$$
$$f \otimes g \to fg$$

is injective.

Indeed, if $L \in W_d^r \backslash W_d^{r+1}$ and if μ is injective, then by the Riemann-Roch formula $\dim \operatorname{Im} \mu = (r+1)(g-d+1)$, and so

$$\dim T_L(W_d^r) = \dim \operatorname{Im} \mu^\perp = \rho.$$

Thus, $\rho \leq g$, $\dim_L W_d^r = \rho$ and L is nonsingular on W_d^r. The dimension theorem falls out of this also. The Petri-Gieseker theorem itself is proved by a perturbation method. The versions of the theorems on the tangent spaces of C_d^r and G_d^r are similar and can be found in Arbarello–Cornalba–Girffiths–Harris [1984]. Now we can apply some of the results discussed above.

2.8. Gauss Mappings. In view of the homogeneity of the Picard variety $\operatorname{Pic}^d C$ we have the rational Gauss mapping

$$\gamma; W_d \stackrel{\text{def}}{=} W_d^0 - - \to G(d-1, \mathbb{P}^{g-1}),$$
$$L \to \mathbb{P}(T_L(W_d)) \subseteq \mathbb{P}(T_L(\operatorname{Pic} C)) = \mathbb{P}^{g-1}$$

for $d \leq g$, associating to the system $L \in W_d$ the projectivization of the tangent d-plane to W_d at L, viewed as an element of the Grassmannian $G(d-1, \mathbb{P}^{g-1})$. Evidently this map is defined on the set $W_d \backslash \operatorname{Sing} W_d$, which is the same as $W_d \backslash W_d^1$ for $d \leq g-1$, by Corollary 1 of the preceding section. Furthermore, by the example of Section 2.7, we associate the $(d-1)$-plane $\overline{D}$ to the 0-dimensional linear system $L = |D|$.

Example 1. For $d = 1$ and $g \geq 1$, the Gauss map $\gamma : W_1 \to \mathbb{P}^{g-1}$ is, in essence, just the canonical mapping. More precisely, the composition

$$\begin{array}{ccccc} C & \xrightarrow{a_1} & W_1 & \xrightarrow{\gamma} & \mathbb{P}^{g-1} \\ & & \cap\vert & & \\ & & \operatorname{Pic}^1 C & & \end{array}$$

is canonical.

Example 2. The best-known example of the Gauss map is one where

$$\gamma : \Theta = W_{g-1} \dashrightarrow (\mathbb{P}^{g-1})^{\vee}.$$

It is an essential ingredient of several proofs of the Torelli theorem, due to the remarkable geometric properties of this mapping. We give two such for the case of a non-hyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$. The degree of the Gauss map $\gamma : \Theta \to (\mathbb{P}^{g-1})^{\vee}$ is equal to $\binom{2g-2}{g-1}$ that being the number of collections of divisors of degree $g-1$ in a general hyperplane section of C. To explain the next property of the Gauss map, define a subvariety

$$\Gamma_p = \{L \in \Theta \,|\, p \in \gamma(L)\}$$

in Θ. If $p \in C$, then Γ_p is reduced and consists of two irreducible components Γ_1' and Γ_1'' : the general point of $\Gamma_1' = p + W_{g-2}^0$ is a zero-dimensional system $L = |D|$ of an effective divisor D of degree $g-1$ containing p, and the general point of $\Gamma_1'' = |K| - \Gamma_1'$ is the complementary linear system $|K - D|$. It is also easy to show that for $g \geq 5$

$$\{p \in \mathbb{P}^{g-1} \,|\, \Gamma_p \text{ reduced}\} = C \cup \{\text{a finite set of points}\}.$$

More precisely, for $g \geq 6$ the finite point set (referred to in the formula above) for a bi-elliptic curve C contains a single point 0, which is the center of the projection onto an elliptic curve (see Example 5 of Section 2.5) and is otherwise empty. Thus, a non-hyperelliptic curve of genus ≥ 5 can be uniquely reconstructed from its principally polarized Jacobian , which is a significant part of the following fundamental result:

Torelli theorem. *If the Jacobians of the curves C and C' are isomorphic as principally polarized abelian varieties, then the curves C and C' are isomorphic.*

The details of the above-mentioned approach to the proof of this theorem can be found in Arbarello–Cornalba–Griffiths–Harris [1984]. That book also has the more traditional approach due to Andreotti, which uses the duality of the branching divisor of the Gauss mapping $\gamma : \Theta \to (\mathbb{P}^{g-1})^{\vee}$ to the canonical curve $C \subset \mathbb{P}^{g-1}$ (see Griffiths–Harris [1978]).

2.9. Sharper Bounds on Dimensions. The existence theorem tells us only a lower bound on the dimension of the components W_d^r. There are rare cases (such as when $r = 0$) when we have a complete answer:

$$\dim W_d^g = \begin{cases} d & \text{for } d \leq g, \\ g & \text{for } d \geq g, \end{cases}$$

which follows from the relationship $W_d^0 = \mu_d(C_d)$ by the fact that μ_d is birationalwhen $d \leq g$ and surjective when $d \geq g$. It is often necessary to have an upper bound for the dimension in some less trivial situations. The first result in this direction is

Martens theorem. *Let C be a curve of genus $g \geq 3$, $2 \leq d \leq g - 1$, and $0 < 2r \leq d$. Then*

(a) If C is not hyperelliptic, then each component of W_d^r has dimension no greater than $d - 2r - 1$.
(b) If C is hyperelliptic, then

$$W_d^r = rg_2^1 + W_{d-2r}^0$$

is an irreducible variety of dimension $d - 2r$.

Indeed, by the proposition of Section 2.7, the dimension of any component $Z \subseteq W_d^r$ in a generic point $L \notin W_d^{r+1}$ does not exceed the dimension of the Zariski tangent space $T_L(W_d^r)$, which is equal to $g - \dim \operatorname{Im} \mu^{\perp} \leq g - 1 - h^0(D) - h^0(K - D)$. The last inequality follows from the inequality $\dim \operatorname{Im} \mu \geq h^0(D) + h^0(K - D) - 1$, which holds whenever $h^0(D), h^0(K - D) \geq 1$ (compare with the lemma of Chapter 2, Section 3.5 of the first part of the survey). Then, by the Riemann–Roch formula

$$\dim Z \leq \deg D - 2h^0(D) + 2 = d - 2r,$$

since $h^0(D) = r + 1$ by the choice $L = |D|$. Just as in the proof of Clifford's inequality, equality implies hyperellipticity of C, and we can determine the form of $L = |D|$. Martens' theorem can be viewed as a considerable generalization of Corollary 3 of Section 2.7.

Note. Clifford's theorem implies that on a curve of genus $g \geq 3$ there are no linear systems g^r_d with $r \leq d \leq g-1$, and therefore W^r_d is empty.

The following sharper version is proved in roughly the same way.

Mumford's theorem. *Let C be a non-hyperelliptic curve of genus $g \geq 4$, for which there are integral r and d such that $r \leq d \leq g-2$ and $d \geq 2r > 0$, and there exists a component $Z \subseteq W^r_d$ with*

$$\dim Z = d - 2r - 1.$$

Then the curve C is trigonal, bi-elliptic or is a plane quintic (for $g = 6$).

In principle, these results can be improved further, but this leads to a growing number of exceptions in lower genera (compare with Keem's theorem in Arbarello–Cornalba–Griffiths–Harris [1984]).

2.10. Tangent Cones. As we have already learned, the point $|D| \in \Theta$ is singular if and only if $\dim |D| \geq 1$. Furthermore, the following result holds.

Riemann's theorem on singularities. *For any effective divisor D of degree $g-1$ on a curve C of genus g,*

$$\operatorname{mult}_{|D|} \Theta = h^0(D) = r + 1,$$

and the tangent cone $Q_{|D|}(\Theta)$ to Θ at $|D|$ is given by an $(r+1)$-form

$$\det(f_i g_j) = 0,$$

where $f_1, \dots, f_{r+1}$ is a basis of $H^0(D)$, and $g_1, \dots, g_{r+1}$ is a basis of $H^0(K - D)$. For a canonical curve this is equivalent to the geometric statement

$$\mathbb{P}(Q_{|D|}(\Theta)) = \bigcup_{D' \in |D|} \overline{D'}.$$

In this geometric reformulation, the theorem can be deduced from the example in Section 2.7 for the case $r = 0$, and by a limiting argument in general. For $k = \mathbb{C}$ there is the analytic proof due to Riemann, using the heat equation for theta-functions (see Arbarello–Cornalba–Griffiths–Harris [1984]). For a purely algebraic proof see Shokurov [1983].

Example. Let $C \in \mathbb{P}^{g-1}$ be a non-hyperelliptic canonical curve of genus $g \geq 5$. Then a generic singularity $|D| \in \operatorname{Sing}\Theta$ is quadratic, and the projectivization of the tangent cone $\mathbb{P}(Q_{|D|}(\Theta))$ is a quadric of rank 4, swept out by $(g-3)$-planes $\overline{D'}$, $D' \in |D|$. Evidently this quadric passes through C. It can be checked by a parameter count that such quadrics form a $(g-4)$-dimensional component of the space of rank 4 quadrics through C.

In view of this, the following fact is quite remarkable:

M. Green's theorem. *The tangent quadratic forms to double points on the theta-divisor Θ generate the quadratic ideal $I_2(C)$ of a non-hyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$.*

Using this and the theorems of Enriques, Babbidge and Petri one can deduce the Torelli theorem for non-hyperelliptic, non-trigonal curves, and curves not isomorphic to the plane quintic. A natural generalization of Riemann's theorem on singularities of W_d is

Kempf's theorem. *Let $g_d^r = |D|$ be a point in W_d and $d \leq g-1$. Then the tangent cone $Q_{|D|}(W_d)$ is a variety of degree $\binom{r+g-d}{r}$, whose ideal is generated by maximal minors of the matrix $(f_i g_j)$ where $f_1, \dots, f_{r+1}$ are a basis of $H^0(D)$, while $g_1, \dots, g_{g-d+r}$ are a basis of $H^0(K-D)$, or, geometrically, for a canonical curve C*

$$\mathbb{P}(Q_{|D|}(W_d)) = \bigcup_{D' \in |D|} \overline{D'}.$$

The last statement implies that $\mathbb{P}(Q_{|D|}(W_d))$ contains the canonical curve C for $r \geq 1$. Since C is linearly normal, we have the equality $\mathbb{P}(T_{|D|}(W_d)) = \mathbb{P}^{g-1}$ (compare part (b) of the proposition in Section 2.7). A description of the tangent cones to W_d^r can be found in Arbarello–Cornalba–Griffiths–Harris [1984].

§3. Prymians

In this section we discuss Prymians, associated with double unbranched covers of curves. These are principally polarized abelian varieties which play an important role in the geometry of higher-dimensional varieties. Unless specified otherwise, C is a non-singular curve of genus g over an algebraically closed field of characteristic not equal to 2.

3.1. Unbranched Double Covers. Let us recall that an involution is an automorphism of order two. Specifying an unbranched double covering

$$\pi : \tilde{C} \to C$$

is equivalent to defining an involution without fixed points

$$I : \tilde{C} \to \tilde{C}$$

permuting its sheets.

Proposition 1.

$$\operatorname{Ker}[\pi^* : J(C) \to J(\tilde{C})] = \{0, \sigma\} \qquad \sigma \in J_2(C)$$
$$\textit{class of } D \to \textit{class } \pi^* D.$$

where $J_2(C) \stackrel{\text{def}}{=} \{p \in J(C) \,|\, 2p = 0\}$.

Proposition 2. *The correspondence* $\pi \to \sigma$ *defines a bijection between the set of unbranched double covers* $\pi : \tilde{C} \to C$ *and the set* $J_2(C) - 0$ *of non-trivial points of second order on the Jacobian* $J(C)$.

The covering $\pi : \tilde{C} \to C$ corresponding to an element $\sigma \neq 0 \in J_2(C)$ is given as a map corresponding to the field extension $k(C) \subset k(C)(\sqrt{f})$, where $(f) = 2D$ and the linear equivalence class of D is σ.

Corollary. *A curve* C *of genus* g *has* $2^{2g} - 1$ *unbranched double covers.*

Note. The reader can learn more about constructing normal coverings of curves with abelian automorphism groups, or, equivalently, abelian extensions of algebraic function fields of transcendence degree one arising from separable isogenies of their generalized Jacobians from Serre [1959].

3.2. Prymians and Prym Varieties. Fix a non-singular curve $\tilde{C}$ together with an involution $I : \tilde{C} \to \tilde{C}$ without fixed points. This involution induces an involution of Jacobians

$$\begin{gathered} I^* : J(\tilde{C}) \to J(\tilde{C}) \\ \text{class of } D \to \text{class of } I^*D. \end{gathered}$$

Lemma-Definition. *The Prymian of the pair* $(\tilde{C}, I)$ *is the abelian variety*

$$\begin{aligned} \mathrm{Pr}(\tilde{C}, I) &\stackrel{\text{def}}{=} \{p - I^*p \,|\, p \in J(\tilde{C})\} \\ &= \text{connected component of } 0 \text{ in } \operatorname{Ker}\mathrm{Nm} \end{aligned}$$

where

$$\begin{gathered} \mathrm{Nm} : \operatorname{Pic}\tilde{C} \to \operatorname{Pic} C, \\ \mathrm{Nm}(\text{class of } \textstyle\sum a_i p_i) = \text{class of } \textstyle\sum a_i \pi(p_i), \end{gathered}$$

is the norm induced by the factorization $\pi : \tilde{C} \to C = \tilde{C}/I$.

The inclusion $\{p - I^*p \,|\, p \in J(\tilde{C})\} \subseteq \operatorname{Ker}\mathrm{Nm}$ can be checked directly. The desired equality follows by counting parameters, using the relations

$$\pi^* \circ \mathrm{Nm} = \mathrm{id} + I^* \quad \text{and} \quad \mathrm{Nm} \circ \pi^* = 2,$$

where the 2 is the isogeny of multiplication by 2. Indeed, from these relations it follows that Nm is an epimorphism while $\pi^* \circ \mathrm{Nm}$ is the isogeny of multiplication by 2 on the elements of the kernel of $\mathrm{id} - I^*$. It follows that

$$\dim P_r(\tilde{C}, I) = \dim J(\tilde{C}) - \dim J(C) = g(\tilde{C}) - g(C).$$

By the Hurwitz formula $g(\tilde{C}) = 2g(C) - 1$, which proves

Proposition 1. $\dim \mathrm{Pr}(\tilde{C}, I) = g - 1$.

Evidently, $\mathrm{Pr}(\tilde{C}, I)$ is the "odd" part of $J(\tilde{C})$ relative to I^* because $I^*(p - I^*p) = -p + I^*p$, and, by counting parameters, we see that it is the connected component of identity. On the other hand, the involution $I^* : \Omega_{\tilde{C}} \to \Omega_{\tilde{C}}$ splits the cotangent space

$$T_L^\vee(J(\tilde{C})) = \Omega_{\tilde{C}} = \Omega_{\tilde{C}}^+ \oplus \Omega_{\tilde{C}}^-$$

into the invariant

$$\Omega_{\tilde{C}}^+ = \pi^* \Omega_C = H^0(K_C),$$

and antiinvariant

$$\Omega_{\tilde{C}}^- = \{\omega \text{ a regular differential on } \tilde{C} \mid I^*\omega = -\omega\}$$

components.Therefore, the last space is canonically isomorphic to the cotangent space

$$T_L^\vee(\mathrm{Pr}(\tilde{C}, I)) = \Omega_{\tilde{C}}^-, \quad L \in \mathrm{Pr}(\tilde{C}, I).$$

In particular, the elements of the cotangent space $T_L^\vee(\mathrm{Pr}(\tilde{C}, I))$ to $\mathrm{Pr}(\tilde{C}, I)$ at L are naturally identified with the regular differentials on $\tilde{C}$, antiinvariant with respect to I^*, – the *Prym differentials*. Likewise, for $k = \mathbb{C}$ we get the analytic representation of the Prymian:

$$\mathrm{Pr}(\tilde{C}, I) = (\Omega_{\tilde{C}}^-)^\vee / \Lambda,$$

where Λ is the lattice given by integration with respect to antiinvariant 1-cycles (see Arbarello–Cornalba–Griffiths–Harris [1984]).

Note. There is a natural isomorphism

$$\Omega_{\tilde{C}}^- = H^0(K_C + D),$$

where the class $D \in J_2(C)$ is a double point corresponding to the covering $\pi : \tilde{C} \to C$.

The next result shows that $\mathrm{Nm}^{-1}(0)$ has two connected components, and gives yet another description of the Prymian.

Proposition 2. $\mathrm{Nm}^{-1}(K_C) \subseteq \mathrm{Pic}^{2g-2}\tilde{C}$ *consists of two non-intersecting components* P^+ *and* P^-, *which are translates of* $\mathrm{Pr}(\tilde{C}, I)$. *More precisely,*

$$P^+ = \{L \in \mathrm{Pic}^{2g-2}\tilde{C} \mid \mathrm{Nm}\, L \subseteq K_C \text{ and } h^0(L) \text{ even}\}$$

This follows from the conservation of parity of $h^0(L)$ on each connected component of $\mathrm{Nm}^{-1}(K_C)$. The variety $P(\tilde{C}, I) \stackrel{\text{def}}{=} P^+$ is called the *Prym variety* of the pair $(\tilde{C}, I)$. It is a principal homogeneous space with respect to the natural action of $\mathrm{Pr}(\tilde{C}, I)$.

The concept of the Prymian has been extended in two directions. On the one hand, one can allow ordinary double points on $\tilde{C}$ and fixed points of the involution $I : \tilde{C} \to \tilde{C}$. Despite the fact that the Jacobian $J(\tilde{C})$ of a singular, and perhaps reducible, connected curve C is not, as a rule, complete, and hence not an abelian variety, its odd part – the Prymian – will be an abelian variety whenever the involution preserves the singularities and does not transpose the branches. On the other hand, one can put more general quadratic constraints on I (of the form $I^2 + aI + b = 0$) which leads to Prym–Tyurin varieties (see Tyurin [1972]).

3.3. Polarization Divisor. Returning to the previous situation, where $\tilde{C}$ is a nonsingular curve with a fixed-point-free involution I, note that the polarization $J(\widetilde{C})$ induces by restriction a polarization on the Prymian $\Pr(\widetilde{C}, I)$. Furthermore, this polarization is twice a principal polarization on $\Pr(\widetilde{C}, I)$. When $\Pr(\widetilde{C}, I)$ is described analytically over $k = \mathbb{C}$, this comes out by explicit computation of the intersection form on antiinvariant 1-cycles on $\widetilde{C}$. In general, it can be shown that $\Theta|_{\Pr(\widetilde{C},I)} \sim 2\Xi$ for the effective polarization divisor $\Theta \subset J(\widetilde{C})$, where Ξ is the principal polarization divisor. The reader can find out how to compute the degree of polarization of the divisor Ξ and how to show that it is indeed principal in Shokurov [1983] and Mumford [1974]. Here we will only note that the divisibility by two follows from Riemann's theorem on singularities. Indeed, we have the following

Proposition. *The canonical polarization divisor $\Theta \subset \operatorname{Pic}^{2g-2} \widetilde{C}$ defines the canonical polarization divisor*

$$\Xi = P(\widetilde{C}, I) \cap \Theta = \{L \in P(\widetilde{C}, I) \mid h^0(L) > 0\}$$

on $P(\widetilde{C}, I)$.

Since $h^0(L)$ is even, the divisor Θ cuts out a divisor on $P(\widetilde{C}, I)$ whose components all have multiplicity ≥ 2. That these components have multiplicity exactly 2 follows from the computation of the degree of polarization (see Shokurov [4]).

For a singular curve $\widetilde{C}$ with involution I one can also define a principally polarized abelian variety, if the involution preserves ordinary quadratic double points, preserves their branches, and has no fixed smooth points. Such pairs $(\widetilde{C}, I)$ are called *Beauville pairs* (Beauville [1977]). Polarizations can also be divided by two to get a principal polarization if there are exactly two smooth fixed points of the involution. For a greater number of fixed smooth points or smooth branch points of the projection $\pi : \widetilde{C} \to C$ the Prymian does not have a natural principal polarization.

Example 1. (Mumford, Dalalian). Let C be a hyperelliptic curve of genus g, and let $\gamma : C \to \mathbb{P}^1$ be its hyperelliptic projection with branch points $p_1, \ldots, p_{2g+2}$. All of the unbranched double covers $\pi : \widetilde{C} \to C$ are constructed as follows. The points p_i are divided into two non-empty sets with even numbers of elements: $\{p_1, \ldots, p_{2g+2}\} = I' \cup I''$, $\operatorname{card} I' = 2h + 2$, $\operatorname{card} I'' = 2l + 2$, and $I' \cap I'' = \emptyset$, so that $h + l + 1 = g$. These point sets define hyperelliptic curves

C' and C'' with projections $\gamma' : C' \to \mathbb{P}^1$ and $\gamma'' : C'' \to \mathbb{P}^1$, branched over I' and I'' respectively. The curve $\widetilde{C}$ is then defined as the desingularization of the fiber product $C' \times_{\mathbb{P}^1} C$. The curve $\widetilde{C}$ is acted upon by the automorphism group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which defines a commutative diagram

$$\begin{tikzcd} & \widetilde{C} \arrow[dl, "\pi"'] \arrow[d, "\pi'"'] \arrow[dr, "\pi''"] & \\ C \arrow[dr, "\gamma"'] & C' \arrow[d, "\gamma'"'] & C'' \arrow[dl, "\gamma''"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

of factorization with respect to the three subgroups of order two. The curve $\widetilde{C}$ will also be the desingularization of $C \times_{\mathbb{P}^1} C'' = C' \times_{\mathbb{P}^1} C''$. The claim is that

$$\begin{aligned} \mathrm{Pr}(\widetilde{C}, I) &= J' \times J'', \\ \Xi &= J' \times \Theta'' + \Theta' \times J'', \end{aligned}$$

where J' and J'' are Jacobians of the curves C' and C'' respectively (see Mumford [1974]). If $h = 0$ or $l = 0$, then one of these multiplicands disappears, and the Prymian becomes the hyperelliptic Jacobian.

Example 2. (Clemens, Tyurin, Masiewicky, Donagi, and Smith). Now, suppose that C is a non-hyperelliptic, non-trigonal (canonical) curve of genus 5, and let Γ be the curve of quadrics of rank 4 through C. This last is a possibly singular quintic in the plane of all quadrics through C. There is a double cover $\pi : \operatorname{Sing}\Theta \to \Gamma$, where Θ is a canonical polarization divisor on $\mathrm{Pic}^4 C$. The corresponding involution has the form $|D| \to |K_C - D|$. It turns out that $(\operatorname{Sing}\Theta, I)$ is a Beauville pair and that $\mathrm{Pr}(\operatorname{Sing}\Theta, I) \approx J(C)$ as principally polarized abelian varieties. Note that in this case $h^0(\operatorname{Sing}\Theta, \pi^* M)$ is odd (where M is the divisor of the hyperplane section Γ). Such coverings are called *even*, since in general, when Γ is a non-singular plane quintic, this is equivalent to the evenness of $h^0(\Gamma, M + D)$ of the theta-characteristic of the class $M + D$, where the σ-class D is a double point corresponding to the unbranched cover $\pi : \operatorname{Sing}\Theta \to \Gamma$.

The interest in Prymians is largely due to the fact that they arise as the intermediate Jacobians of threefolds of special types.

Example 3. (Tyurin, Beauville). A *bundle of conics* usually means a flat, relatively minimal morphism $f : V \to S$ of a non-singular three-dimensional variety V onto a nonsingular algebraic surface S, the general fiber of which is a conic – an anticanonical model of the projective line $\mathbb{P}^1$. There is a curve of degenerations $C \subset S$, with ordinary double-point singularities, such that over its smooth points, the fiber $f^{-1}(c)$ separates into a pair of disjoint lines, and over the singular points the fiber is one double line (see fig. 2). The lines of these fibers are, in general, parametrized by a non-connected curve $\tilde{C}$, together with an involution I which permutes the lines of the degenerate fibers. It turns out that $(\tilde{C}, I)$ is a Beauville pair, and that the intermediate Jacobian $J(V)$ of the variety V is isomorphic (as a principally polarized abelian variety) to $\Pr(\tilde{C}, I)$, whenever S is a rational surface.

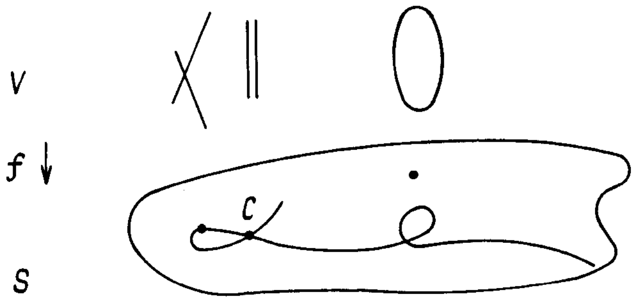

Fig. 2

Example 4 (Griffiths, Clemens). Consider a nonsingular cubic threefold $V \subset \mathbb{P}^4$. Its intermediate Jacobian can also be represented as a Prymian. Indeed, the projection of V from a line $l \subset V$, after blowing up at l, becomes a bundle of conic sections over $\mathbb{P}^2$, without changing the intermediate Jacobian V. The curve of degenerations will be a smooth plane quintic $\Gamma \subset \mathbb{P}^2$, with an *odd* covering $\tilde{\Gamma} \to \Gamma$, where $\tilde{\Gamma}$ is the curve parametrizing the lines on V intersecting l.

Prymians also arise as the intermediate Jacobians of the intersection of three quadrics in an even-dimensional projective space $\mathbb{P}^{2n+4}$ (see Tyurin [1972]). In this case the quotient curve C is a discriminant curve, which is a plane curve of degree $2n+5$ (compare with Example 2, for $n = 0$). More details on the geometric applications of Prymians can be found in the survey of Iskovskikh on higher-dimensional algebraic geometry. Prymians for involutions with two fixed points play an important role in mathematical physics – they can be used to construct explicit solutions of Schroedinger's equation, analogous to Jacobian flows in §1.

3.4. Singularities of the Polarization Divisor. The points of the polarization divisor Ξ of the Prym variety can, according to the discussion above, be identified with odd-dimensional linear systems $|D|$, where D is an effective divisor of degree $2g-2$ on $\widetilde{C}$ with $\operatorname{Nm} D \sim K_C$. By the Hurwitz formula for the canonical divisor, the last condition is the same as the linear equivalence $D + I^*D \sim K_{\widetilde{C}}$. Therefore, the pairing μ of Section 2.7 for $|D| \in \Xi$ can be written as

$$\mu : H^0(D) \otimes H^0(I^*D) \to H^0(K_{\widetilde{C}}) = \Omega_{\widetilde{C}}.$$

The splitting of $\Omega_{\widetilde{C}}$ into the even and odd parts (see Section 3.2) allows us to define a skew-symmetric pairing

$$[\,] : \wedge^2 H^0(D) \to \Omega^-_{\widetilde{C}}$$
$$s \wedge t \to [s \wedge t] \stackrel{\text{def}}{=} sI^*t - tI^*s.$$

In particular, for every basis (f_i) of $H^0(D)$ we can define a skew-symmetric matrix (ω^-_{ij}) with $\omega^-_{ij} = [f_i \wedge f_j]$. We denote its Pfaffian by $\operatorname{Pf}(\omega^-_{ij})$. Riemann's theorem on singularities immediately gives

Theorem.

$$\operatorname{mult}_{|D|} \Xi \geq h^0(D)/2.$$

More precisely,

(a) *If* $\operatorname{Pf}(\omega^-_{ij}) \neq 0$, *then the form* $\operatorname{Pf}(\omega^-_{ij})$ *gives the tangent cone to* Ξ *at* $|D|$ *of degree*

$$\operatorname{mult}_{|D|} \Xi = h^0(D)/2.$$

(b) *If* $\operatorname{Pf}(\omega^-_{ij}) = 0$, *then*

$$\operatorname{mult}_{|D|} \Xi > h^0(D)/2.$$

Singularities of type (b) are called *Mumford singularities*, and their appearance is connected with a tangency of the canonical polarization divisor Θ as it cuts out Ξ from $P(\widetilde{C}, I)$ (see the beginning of Section 3.3). According to Welters [1985], these do not exist on a general Prymian. However, as we shall see below, they give a substantial contribution to $\operatorname{Sing} \Xi$ on special Prymians.

As a direct consequence of part (a) of the proposition in Section 2.7 we get

Lemma. *Let* $|D|$ *be a point of the subvariety*

$$P^r = \{|D| \in P(\widetilde{C}, I) \mid h^0(D) \geq r+1\} \subset P(\widetilde{C}, I)$$

with $h^0(D) = r+1$. *Then the Zariski tangent space*

$$T_{|D|}(P^r) \subseteq T_{|D|}(P(\widetilde{C}, I)) = \left(\Omega^-_{\widetilde{C}}\right)^\vee$$

is contained in the zero set of forms in $\operatorname{Im}\left[[\,] : \wedge^2 H^0(D) \to \Omega^-_{\widetilde{C}}\right]$. *In particular,*

$$\dim_{|D|} P^r \leq g - 1 - \dim \operatorname{Im}[\ ].$$

Proposition. $\dim \operatorname{Sing} \Xi \leq g-5$, *that is, all of the components of* $\operatorname{Sing} \Xi$ *have dimension* $\leq g-5$, *if and only if the curve* C *is not hyperelliptic.*

The proof of necessity is based on the following claim: If there is an irreducible component $Z \subseteq \operatorname{Sing} \Xi$ of dimension $\geq g-5$, then for a general point $|D| \in Z$, the following property (P) holds: there exist linearly independent $s, t \in H^0(D)$ with

$$sI^*t = tI^*s.$$

This is equivalent to the relation $[s \wedge t] = 0$, which holds whenever $h^0(D) = 2$, by part (b) of the theorem. If, on the other hand, $h^0(D) \geq 4$, then, by the lemma, $\dim \operatorname{Im}[\ ] \leq 4$, while the subvariety of decomposable forms $s \wedge t$ in $\wedge^2 H^0(D)$ has dimension ≥ 5. Thus, there is a decomposable non-zero form $s \wedge t \in \operatorname{Ker}[\ ]$. Property (P) can also be written as $I^*\varphi = \varphi$, where $\varphi = s/t$. This means that $\varphi = \pi^*\psi$ for some rational function ψ on C, and so

$$|D| = |\pi^* M + \sum p_i|,$$

where $\dim |M| \geq 1$. By Martens' theorem, the number of parameters of the system $|M|$ does not exceed $d-2$, where $d = \deg M$. On the other hand, $\operatorname{Nm}(\sum p_i) \in |K_C - 2M|$ so by Clifford's theorem, $\dim |K_C - 2M| \leq g - d - 1$. Finally,

$$\dim Z \leq (d-2) + (g-d-1) = g-3,$$

and is $\geq g-4$ only for a hyperelliptic curve C. We get sufficiency from Example 1 of Section 3.3 and Corollary 3 of Section 2.7. Moreover, when $\dim \operatorname{Sing} \Xi \geq g-3$ the Prymian is decomposable, and equals the sum of two hyperelliptic Jacobians.

Question. Let (A, Θ) be a principally polarized abelian variety with $\dim \operatorname{Sing} \Theta = \dim A - 2$. Is it then true that it is decomposable, that is, representable as a sum of principally polarized varieties of smaller dimension?

By the same method as above one can prove

Mumford's theorem (Mumford [1974]). *If* C *is a non-hyperelliptic curve of genus* $g \geq 5$, *then* $\dim \operatorname{Sing} \Xi = g-5$ *precisely when*

(a) the curve C *is trigonal, or*
(b) the curve C *is bi-elliptic, or*
(c) $g = 5$ *and* C *has an even theta-characteristic* L *with* $h^0(L) > 0$ *and* $L + \sigma$ *even, or*
(d) $g = 6$ *and* C *has an odd theta-characteristic* L *with* $H^0(L) \geq 3$ *and* $L + \sigma$ *even.*

where $\sigma \in J_2(C)$ *is a double point corresponding to the covering* $\pi : \widetilde{C} \to C$.

A complete list for curves $\widetilde{C}$ with ordinary double points, found by Beauville [1977], contains more than ten cases. Here is an example:

(e) $C = C_1 \cup C_2$ where C_1 and C_2 are connected curves of genus ≥ 1 intersecting in four points.

Note. The lower bound

$$\dim \operatorname{Sing} \Xi \geq g - 7$$

is due to Welters [1985].

3.5. Differences Between Prymians and Jacobians. As one consequence of Mumford's theorem we see that if the curve C has genus $g \geq 5$, is not hyperelliptic, is not trigonal, is not bi-elliptic, and is not covered by the special cases (c) and (d) of the theorem, then the Prymian is indecomposable, and is not the Jacobian of a curve. Indeed, by Corollary 3 of Section 2.7 the Prymian can be a Jacobian only if $\dim \operatorname{Sing} \Xi \geq g - 5$. In particular, it follows that the intermediate Jacobian of a non-singular cubic threefold (see Example 4 of Section 3.3) is not the Jacobian of a curve. On the other hand, according to Griffiths, the intermediate Jacobian of a three-dimensional rational algebraic variety is either itself a Jacobian of a curve, or the sum of such Jacobians (see Tyurin [1972]). We thus establish the irrationality of the cubic threefold, first established by Clemens and Griffiths in 1972. The cubic threefold is one of the first and simplest counterexamples to Lüroth's problem. A careful analysis of the polarization divisor Ξ shows that the Prymian will not be a Jacobian in some cases of Mumford's theorem.

Theorem. $\Pr(\widetilde{C}, I)$ *is a Jacobian or the sum of Jacobians of curves, if and only if the quotient curve* $\widetilde{C}/I$ *is hyperelliptic, trigonal, or is a plane quintic with an even covering* $\pi : \widetilde{C} \to C$.

Sufficiency in the hyperelliptic case and for the plane quintic follows from Examples 1 and 2 of Section 3.3. Sufficiency in the trigonal case is given by

Theorem (Recillas). *The Prymian* $\Pr(\widetilde{C}, I)$ *of a pair* $(\widetilde{C}, I)$ *with trigonal quotient* $C = \widetilde{C}/I$ *is the Jacobian of a curve with* g^1_4.

Curves possessing a g^1_4 are called *tetragonal.* Let us study a canonical tetragonal curve $S \subset \mathbb{P}^{g-2}$ of genus $g - 1$. The planes $\overline{D}, D \in g^1_4$ sweep out a three-dimensional $\mathbb{P}^2$ bundle $V \to \mathbb{P}^1$. Let $\widetilde{V}$ be the blowing-up of V in S. Then, as is well known, the intermediate Jacobian $J(\widetilde{V})$ will be isomorphic to the Jacobian $J(S)$ of the curve S being blown-up. On the other hand, $\widetilde{V}$ is equipped with the natural structure of a conic bundle over a rational ruled surface $\mathbf{F}_n$ (see Fig. 3). The conics over $\overline{D}$ passing through the points p_1, p_2, p_3, p_4 of the divisor $D = p_1 + p_2 + p_4 + p_4 \in g^1_4$ upon blowing-up turn into the conics of the bundle $\widetilde{V}$ over $\mathbf{F}_n$. By construction, the curve of degenerations C is trigonal, and according to Example 3 of Section 3.3, $\Pr(\widetilde{C}, I) = J(V) = J(S)$.

Moreover, Recillas checked that the Prymian of any pair $(\widetilde{C}, I)$ with a trigonal quotient curve C arises in this fashion. This is also borne out by counting parameters. For special Beauville pairs $(\widetilde{C}, I)$ we essentially get one new case, where $\mathrm{Pr}(\widetilde{C}, I)$ is a Jacobian (see Shokurov [1983], [1981]). From the last result we get

Rationality Criterion. *A three-dimensional, relatively minimal conic bundle $V \to S$ over a minimal rational surface S, that is, over $S = \mathbb{P}^2$ or $\mathbf{F}_n$, is rational if and only if its intermediate Jacobian is a Jacobian of a curve, or the sum of Jacobians of curves.*

In the process of proof it is found that for such conic bundles, rationality implies the relationship $|2K_S + C| = \emptyset$, where C is the curve of degenerations (Shokurov [1983]). This is generalized by the

Conjecture. *Let $V \to S$ be a relatively minimal conic section bundle with the curve of degenerations $C \subset S$. If V is rational, then*

$$|2K_S + C| = \emptyset.$$

Other formulations of this conjecture and some approaches to the proof are discussed by Iskovskikh [1987].

3.6. The Prym Map. Associating to a pair $(\widetilde{C}, I)$ the Prymian $\mathrm{Pr}(\widetilde{C}, I)$ defines the regular *Prym map*

$$\mathrm{Pr} : \mathcal{R}_g \to \mathcal{A}_{g-1},$$

where $\mathcal{R}_g$ is the moduli space of pairs $(\widetilde{C}, I)$ or, equivalently, of the unramified covers $\pi : \widetilde{C} \to C$ onto a curve of genus g, while $\mathcal{A}_{g-1}$ is the moduli space of principally polarized abelian varieties of dimension $g - 1$.

Theorem. *The Prym map is*
(a) dominant, with general fiber of dimension ≥ 1 for $g \leq 5$;
(b) (Donagi, Smith) dominant, of finite degree 27 for $g = 6$;
(c) (Kanev, Friedman, Smith) birational on a proper subvariety of $\mathcal{A}_{g-1}$ for $g \geq 7$.

This was initially proved by studying infinitesimal properties of a special fiber of the boundary (Donagi–Smith [1981], Kanev [1982], Friedman–Smith [1982]). Part (c) is usually called the generic Torelli theorem for Prym varieties in a generic point. Welters [1987] suggests a technique for reconstructing the pair $(\widetilde{C}, I)$ from a general Prymian $\mathrm{Pr}(\widetilde{C}, I)$. It has also been recently established that the Prym map is birational (Friedman–Smith [1986]) and even bijective (Debarre [1989]) for the pairs $(\widetilde{C}, I)$ corresponding to the intersection of three quadrics of odd dimension ≥ 5. The last fact establishes the Torelli theorem for such complete non-singular intersections. Of course, all of these

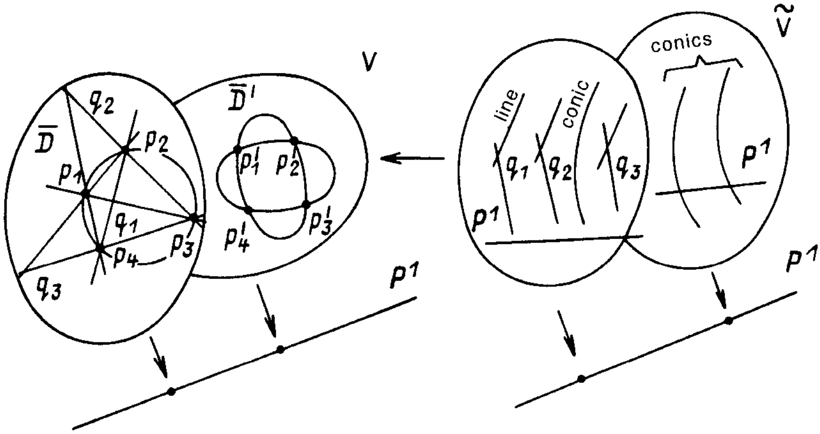

Fig. 3

facts follow easily from the following conjectural picture of the domain where the Prym map is one-to-one.

Conjecture (Donagi [1981]) *The Prym map is bijective on the open subset*

$$\{(\widetilde{C}, I) \mid C = \widetilde{C}/I \text{ has no } g_4^1\}.$$

More precisely, for two Prymians to coincide, it is necessary and sufficient for one of the pairs defining them to be obtained from the other by using the tetragonal construction, which is possible if the quotient curves have a g_4^1.

Note. A. Verra found a counterexample, for $g(C) = 10$.

§4. Characterizing Jacobians

Here we discuss some geometric and analytic means of distinguishing Jacobians of curves among all principally polarized abelian varieties of the same dimension. The reader should be able to glean more information from Mumford's lectures (Mumford [1975]).

4.1. The Variety of Jacobians. Like special divisors, special abelian varieties – Jacobians – are best studied from a moduli standpoint. Jacobians comprise an irreducible quasi-projective variety

$$J_g = \{J(C) \mid C \text{ a nonsingular curve of genus } g\}$$

in $\mathcal{A}_g$ – the moduli space of principally polarized abelian varieties of dimension g. Its closure $\overline{J_g}$ in $\mathcal{A}_g$ is also called the *variety of Jacobians.*

Theorem (Hoyt [1963]) *The space $\overline{J_g}$ consists of Jacobians and direct sums of Jacobians (of total dimension g.).*

Corollary 1. *J_g is a closed subvariety of $\mathcal{A}_g^0$ – the subvariety of indecomposable principally polarized abelian varieties.*

Corollary 2. *$J_g = \mathcal{A}_g^0$ for $g \leq 3$, that is, each indecomposable principally polarized abelian variety of dimension ≤ 3 is a Jacobian of a curve.*

The last Corollary easily follows from the Torelli theorem and the irreducibility of $\mathcal{A}_g$ (recently established for every g in positive characteristic) by counting dimensions. From the same considerations we see that for $g \geq 4$, $\overline{J_g} \subset \mathcal{A}_g$ and $J_g \subset \mathcal{A}_g^0$ are proper closed subvarieties. Thus, for $g \geq 4$ Jacobians must have special properties which distinguish them among all principally polarized abelian varieties.

Note. The subvariety of Prymians $\Pr(\mathcal{R}_g) \subseteq \mathcal{A}_{g-1}$ is not closed even in $\mathcal{A}_{g-1}^0$. It can be closed by adding Prymians of Beauville pairs and Wirtinger pairs (see Beauville [1977], Donagi–Smith [1981]).

4.2. The Andreotti-Meyer Subvariety. As we already know, the theta-divisor of the Jacobian of a curve of genus g has a singular subvariety of dimension $\geq g-4$. Andreotti and Mayer, when studying the characterizations of Jacobians by this condition introduced the subvariety

$$\mathcal{N}_{g-4} = \{(A, \Theta) \mid \dim \operatorname{Sing} \Theta \geq g-4\}$$

in $\mathcal{A}_g$.

Theorem (Andreotti-Meyer). *$\overline{J_g}$ is the only irreducible component $\mathcal{N}_{g-4}$ containing J_g.*

It should be noted that both $\overline{J_g}$ and J_g are irreducible and have dimension $3g-3$ for $g \geq 2$. Therefore it must be shown that the dimension of any component $\mathcal{N}_{g-4}$ containing $\overline{J_g}$ does not exceed $3g-3$. For $k = \mathbb{C}$, the space $\mathcal{A}_g$ can be replaced by the Siegel halfplane $\mathbf{H}_g$, while $\mathcal{N}_{g-4}$ can be replaced by the subspace distinguished by the singularities of the Riemann theta-function.

$$\mathcal{N}_{g-4} = \{Z \in \mathbf{H}_g \mid \dim \operatorname{Sing} \Theta(Z) \geq g-4\},$$

where

$$\operatorname{Sing} \Theta(Z) = \{u \in \mathbb{C}^g \mid \vartheta(u, Z) = 0, \text{ and } \frac{\partial \vartheta}{\partial u_i}(u, Z) = 0, \quad \text{for all } 1 \leq i \leq g\}.$$

For an arbitrary point $Z_0 \in \mathcal{N}_{g-4}$ with $\dim \operatorname{Sing} \Theta(Z_0) = g-4$ it is not hard to check that the tangent vectors $\sum q_{ij}(\partial/\partial Z_{ij}) \in T_{Z_0}(\mathcal{N}_{g-4})$ satisfy

$$\sum \frac{\partial \vartheta}{\partial Z_{ij}}(u_0, Z) q_{ij} = 0,$$

with $u_0 \in \operatorname{Sing} \Theta(Z_0)$. It is enough to check that if Z_0 is the Siegel matrix of a sufficiently general (non-hyperelliptic) Jacobian and $u, \ldots, u^{\frac{(g-2)(g-3)}{2}}$ (recall that $\dim \mathcal{A}_g = \frac{g(g+1)}{2}$) are $\dim \mathcal{A}_g - (3g-3)$ sufficiently generic points on $\operatorname{Sing} \Theta(Z_0)$, then the $(g-2)(g-3)/2$ vectors

$$\left(\frac{\partial \vartheta}{\partial Z_{ij}}(u^l, Z_0) \right), \quad 1 \leq l \leq (g-2)(g-3)/2,$$

are linearly independent. By the heat equation

$$\frac{\partial^2 \vartheta}{\partial u_i \partial u_j}(u, Z) = 2\pi\sqrt{-1}(1 + \delta_{ij}) \frac{\partial \vartheta}{\partial Z_{ij}}(u, Z)$$

this linear independence is equivalent to the linear independence of $(g-2)(g-3)/2$ quadrics

$$Q_l = \left\{ \sum_{i,j} \frac{\partial^2 \vartheta}{\partial u_i \partial u_j}(u^l, Z_0) u_i u_j = 0 \right\}.$$

On the other hand, these quadrics Q_l are the tangent cones to the Riemann theta-divisor Θ of the corresponding Jacobian at the points $u^l \bmod \Lambda$ (see example of Section 2.10). Thus the result follows from M. Green's theorem. The analogous fact for a general curve is fairly straightforward and used by Andreotti and Meyer. For $g \geq 4$ the varieties $\mathcal{N}_{g-4}$ also have non-Jacobian components. Examples of (even indistinguishable) abelian varieties in $\mathcal{N}_{g-4} - \overline{J_g}$ can be constructed in the same way as we constructed Prymians (in the discrepancy between Mumford's theorem and the theorem of Section 3.5).

Example. $\mathcal{N}_0 \subset \mathcal{A}_4$ consists of two irreducible subvarieties of codimension 1: $\overline{J_4}$ and the closure of

$$J_4' = \{\Pr(\widetilde{C}, I) \mid \text{for } (\widetilde{C}, I) \text{ of type (c) in Mumford's theorem of Section 3.4}\}$$

(see Beauville [1977]). The theta-divisor of a generic abelian variety in J_4' has exactly one singular point, while the theta divisor of a generic Jacobian of a curve of genus 4 has two (compare with Example 2 of Section 2.3).

One can also obtain a description of the components of $\mathcal{N} \subset \mathcal{A}_5$ by using the theory of Prym varieties (see Beauville [1977]). Much less is known about the components of $\mathcal{N}_{g-4}$ with $g \geq 6$ (see Shokurov [1983], Beauville [1986]).

4.3. Kummer Varieties. As in Section 3, we will assume in the sequel that char $k \neq 2$. Every abelian variety A has the antipodal involution

$$\begin{aligned} - : A &\to A \\ p &\to -p. \end{aligned}$$

The quotient variety $A/-$ is called the *Kummer variety* of the abelian variety A. The generic Kummer variety of dimension g has a natural embedding into $\mathbb{P}^{2g-1}$.

Theorem. *Let A be an indecomposable principally polarized abelian variety of dimension g, Θ its Riemann theta-divisor. Then the map*

$$\varphi_A = \varphi_{|2\Theta|} : A \to \mathbb{P}^{2^g-1},$$

associated with the complete linear system $|2\Theta|$, corresponds to factoring out by the involution $-$, and so its image is the Kummer variety $A/-$.

By the linear equivalence $2(\Theta+\eta) \sim 2\Theta$ for second order points $\eta \in A_2$, the map $\varphi_{|2\Theta|}$ does not depend on the choice of Riemann theta-divisor. Since the polarization is principal, $\dim H^0(2\Theta) = 2^g$. In the complex situation, for an abelian variety A corresponding to a Siegel matrix $Z \in \mathbf{H}_g$, the space $H^0(2\Theta)$ is identified with the space L_2 of automorphic functions with multipliers $\mu_i = 1$ and $\mu_{g+i} = \exp(-2\pi\sqrt{-1}(2ui + Z_{ii}))$, $2 \le i \le g$. This space has a standard basis of theta functions with characteristics

$$\vartheta_2[\sigma](u, Z) = \sum_{m\in\mathbb{Z}^g} \exp(2\pi\sqrt{-1}\left(\langle m + \frac{\sigma}{2}, (m + \frac{\sigma}{2})\mathbb{Z}\rangle + 2\langle m + \frac{\sigma}{2}, u\rangle\right)),$$

where σ are $0, 1$ vectors of length g. Therefore, the map $\varphi_{|2\Theta|}$ can be analytically represented as

$$u \bmod \Lambda \to (\ldots : \vartheta_2[\sigma](u, Z) : \ldots).$$

These theta functions are even, and thus $\phi_{|2\Theta|}$ factors through the Kummer variety $A/-$. The complete proof requires further analysis of the linear system $|2\Theta|$.

4.4. Reducedness of $\Theta \cap (\Theta + p)$ and Trisecants. This approach to the characterization of Jacobians stems from the following observation of A. Weil [1957]. For every $p \neq q \in C$ the intersection $\Theta \cap (\Theta +$ class of $(p-q))$ is reduced, to wit

$$\Theta \cap (\Theta + \text{ class of } (p-q)) \subset (\Theta + \text{ class of } (p-r)) \cup (\Theta + \text{ class of } (s-q)),$$

for any choice of distinct $p, q, r, s \in C$. Indeed, examine the canonical polarization divisor $\Theta = W_{g-1}$, to get

$$\begin{aligned} &W_{g-1} \cap (W_{g-1} + \text{ class of } (p-q)) \\ &= (W_{g-2} + \text{ class of } (p)) \cup (W_g^1 - \text{ class of } (q)) \\ &\subset (W_{g-1} + \text{ class of } (p-r)) \cup (W_{g-1} + \text{ class of } (s-q)) \end{aligned}$$

for any $r, s \in C$. For any principally polarized abelian variety (A, Θ) this leads to conditions

(i) There exists a $p \neq 0 \in A$ such that $\Theta \cap (\Theta + p)$ is reduced.
(ii) There are nonzero distinct $p, q, r \in A$ such that (scheme-theoretically)

$$\Theta \cap (\Theta + p) \subset (\Theta + q) \cup (\Theta + r).$$

(iii) The Kummer subvariety $A/- \subset \mathbb{P}^{2g-1}$ (in the indecomposable case) has a triple secant – -called a *trisecant.*

Condition (i) is an obvious weakening of (ii), while conditions (ii) and (iii) are equivalent, which can be checked by reducing them to Fay's trisecant identities (see Mumford [1983]). More precisely, for every $s \in A$ with $2s = q+r$, the points $\phi_A(s)$, $\phi_A(s-p)$, and $\phi_A(s-q) = \phi_A(s-r)$ lie on the same line in $\mathbb{P}^{2g-1}$. Thus, Jacobians satisfy all of the above conditions.

Theorem (Beauville–Debarre [1986])
(a) A principally polarized abelian variety (A, Θ) satisfying one of the equivalent conditions (ii) or (iii) satisfies the Andreotti–Meyer condition, that is, for such an abelian variety $\dim \operatorname{Sing} \Theta \geq g-4$.
(b) Condition (i) implies membership in $\mathcal{N}_{g-4}$ modulo a certain irreducible component not containing $\overline{J_g}$.

Together with the Andreotti-Meyer theorem this implies

Corollary. *$\overline{J_g}$ is the only irreducible component of the subvariety of principally polarized abelian varieties, defined by one of the conditions (i), (ii), or (iii).*

So, the existence of a trisecant of $A/-$ implies that $A \in \mathcal{N}_{g-4}$. It is also known that abelian varieties in some of the components of $\mathcal{N}_{g-4}$ satisfy (i) but not (ii). In conjunction with these observations, there is

Trisecant conjecture (Beauville [1987]) *An irreducible abelian variety satisfying (ii) or, equivalently, (iii), is the Jacobian of a curve.*

Before discussing the results leading towards the resolution of this conjecture, let us make one general observation:

Note. All of the known characterizations of Jacobians are connected with various methods of proof of the Torelli theorem (see Mumford [1975]). For example, Section 2.8 uses the property of theta-divisor derived in Example 2, which is just an infinitesimal version of condition (ii).

Jacobians have non-trivial families of trisecants. This gives the first complete characterization of Jacobians, obtained by Gunning. Complete means precise, that is, it does not allow any "parasitic" components, as in theorems of Andreotti–Meyer type.

Theorem (Gunning). *An indecomposable principally polarized abelian variety A is a Jacobian if and only if the subvariety*

$$C = \{2p \in A \mid \varphi_A(p+p_1), \varphi_A(p+p_2), \varphi_A(p+p_3) \text{ lie on the trisecant } A/-\}$$

has dimension ≥ 1 *for some* $p_1, p_2, p_3 \in A$. *Moreover, in this case* C *is a smooth irreducible curve of genus* $g = \dim A$, *and* A *is its Jacobian.*

Gunning then also generalized this result to the case of m-planes intersecting $A/-$ in at least $m+2$ points (see Van der Geer [1985]). However, more interesting is the infinitesimal version of Welters, obtain by coalescing the points p_i.

Theorem (Gunning–Welters). *In the statement of Gunning's theorem, instead of the submanifold* C, *defined by three-point subset* $\{p_1, p_2, p_3\} \subset A$, *consider the submanifolds*

$$C_Y = \{2p \in A \,|\, p + Y \subseteq \phi_A^{-1}(l) \text{ for some line } l \subset \mathbb{P}^{2g-1}\},$$

where $Y \subset A$ *is an artinian (zero-dimensional) subscheme of length* 3.

Following these results, there followed a whole flood of characterizations of the Jacobian (see Arbarello [1986], Beauville [1987]), and including:

4.5. The Characterization of Novikov–Krichever. Already Mumford (see Mumford–Fogarty [1982]) noticed that when the three points $p, q, r \in A$ coalesce to 0 in condition (ii) of Section 4.4, the corresponding Fay trisecant identity leads to the first of the equations of the KP hierarchy. To understand this, let's restrict to the complex case $k = \mathbb{C}$ and $Y = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^3)$ – an artinian subscheme containing 0. It is easy to check the following coincidence for the second order germ $(C_Y)_2$ of the curve C_Y at 0.

$$(C_Y)_2 = Y.$$

Moreover, according to Welters, the existence of a third-order germ $(C_Y)_3$ is equivalent to the existence of constant vector fields $D_1 \neq 0$, D_2, D_3 on A, and of a constant $d \in \mathbb{C}$, such that all of the theta-functions $\vartheta_2[\sigma](u, Z)$ satisfy the equation

$$\left(D_1^4 - D_1 D_3 + \frac{3}{4} D_2^2 + d\right) \vartheta[\sigma](0, Z) = 0, \tag{2}$$

where Z is the Siegel matrix of the abelian variety A (compare with the system of equations in the Note-Example of Section 1.3). By the Gunning–Welters theorem the theta functions $\vartheta_2[\sigma](u, Z)$ of the Jacobian matrix Z really does satisfy equation (2). On the other hand, Dubrovin showed that equations (2) are equivalent to the *Novikov–Krichever condition*:

There are three vectors $a_1 \neq 0, a_2, a_3 \in \mathbb{C}^g$, such that for every $\zeta \in \mathbb{C}^g$ the function

$$u(x, y, t; Z) = \frac{\partial^2}{\partial x^2} \log \vartheta(\zeta + x a_1 + y a_2 + t a_3, Z)$$

is the solution of the *Kadomtsev–Petviashvili equation*

$$3u_{yy} = \frac{\partial}{\partial x}(u_t - 3u u_x - 2u_{xxx}).$$

Thus, we get the result of Krichever that this condition holds for Jacobian matrices. Based on this, Novikov conjectured that this condition is only satisfied by Jacobian matrices. This is indeed so, according to the following result.

Theorem (Shiota). *The Siegel matrix $Z \in \mathbf{H}_g$ of an indecomposable principally polarized abelian variety $A = \mathbb{C}^g(\mathbb{Z}^g + Z\mathbb{Z}^g)$ is a Jacobian if and only if all of the theta functions $\theta_2[\sigma](u, Z)$ satisfy equation (2) for some constant vector fields $D_1 \neq 0, D_2, D_3$ on A and a constant $d \in \mathbb{C}$.*

This result proves not only Novikov's conjecture, but also that indecomposable abelian varieties for which $(C_Y)_3$ exists are Jacobians. There are two approaches to the proof. The first, due to Shiota [1983], uses the fact that the solution of the first KP equation can be extended to a solution of the whole KP hierarchy, and the theorem of Section 1.4. Another, more geometric, approach of Arbarello–de Concini [1987] uses the ideas of Welters, and basically establishes the existence of the curve C_Y from the existence of the formal third-order germ $(C_Y)_3$.

Note. All of the above characterizations have analogies for the Prymian (see Beauville [1986]). For example, the variety of Prymians $\overline{\Pr(\mathcal{R}_{g+1})} \subseteq \mathcal{A}_g$ is the unique irreducible component containing the Prymians in

$$\mathcal{N}_{g-6} = \{(A, \Theta) \mid \dim \operatorname{Sing} \Theta \geq g - 6\} \subseteq \mathcal{A}_g.$$

No characterization of principally polarized Prym–Tyurin varieties is currently known.

4.6. Schottky Relations. Many of the above characterizations of Jacobians are easily written as analytic relations on the period matrices. The first such relations, written as polynomials in theta-constants were found using Prym varieties by Schottky–Jung [1909]. Thereafter, any approach to distinguishing Jacobians has been known as the *Schottky problem.* Even earlier, in 1888, Schottky found a non-trivial relation, now known as the *Schottky relation*, for theta-constants of the principally polarized abelian varieties of dimension 4, vanishing for Jacobian of curves of genus 4. By counting dimensions, one of the components defined by this relation will be the variety of Jacobians $\overline{J}_4$ (compare with example of Section 4.2). The following statement was proved only quite recently:

Theorem (Igusa [1981]). *Schottky's relation defines an irreducible subvariety in $\mathcal{A}_4$, which thus coincides with $\overline{J_4}$.*

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