Introduction

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Charles François Sturm was born in Geneva on 29 September 1803[1]. He received his scientific education in this city, in which science has traditionally been of such great importance. Though he was later drawn to Paris, where he settled permanently in 1825 and carried out most of his scientific work, he has left his mark also on the city of Geneva, where his name is commemorated by the Place Sturm and the Rue Charles-Sturm. On the first floor of the Museum of History of Science, in its beautiful setting with magnificent views over Lake Geneva, you can see some of the equipment with which his friend and collaborator Daniel Colladon pursued his research on the lake into the propagation of sound through water[2].

Sturm's family came to Geneva from Strasbourg a few decades before his birth. He frequently moved house, and at least two of the addresses where he spent some of his early years can still be found in Geneva's old town[3,4].

Not only did Charles Sturm leave his mark on Geneva, but his rich scientific legacy is recognized by mathematicians and scientists the world over, and continues to influence the direction of mathematical development in our own times[5]. In

[1]This corresponds to the sixth day of the month of Vendémiaire in year XII of the French revolutionary calendar then in use in the Département du Léman.

[2]Colladon was the physicist and experimentalist of this partnership, while Sturm played an important role as theoretician. Their joint work on sound propagation and compressibility of fluids was recognized in 1827 by the award of the Grand Prix of the Paris Academy of Sciences.

[3]The address 29, Place du Bourg-de-Four was home to ancestors of Charles Sturm in 1798. The present building appears on J.-M. Billon's map of Geneva, dated 1726, which is the earliest extant cadastral map of the city. The home of Charles Sturm in 1806, with his parents and first sister, was 11, Rue de l'Hôtel-de-Ville. The building now on this site was constructed in 1840. The two houses are in close proximity.

[4]For details on Sturm's life, see the biographical notice by J.-C. Pont and I. Benguigui in *The Collected Works of Charles François Sturm*, J.-C. Pont, editor (in preparation), as well as Chapter 21 of the book by P. Speziali, *Physica Genevensis, La vie et l'oeuvre de 33 physiciens genevois*, Georg, Chêne-Bourg (1997).

[5]Sturm was already judged by his contemporaries to be an outstanding theoretician. Of the numerous honours which he received during his lifetime, special mention might be made of the Grand Prix in Mathematics of the Paris Academy, in 1834, and membership of the Royal Society of London as well as the Copley Medal, in 1840. The citation for membership of the Royal Society was as follows: "Jacques Charles François Sturm, of Paris, a Gentleman eminently distinguished for his original investigations in mathematical science, is recommended by us as a proper person

bringing together leading experts in the scientific history of Sturm's work with some of the major contributors to recent and contemporary mathematical developments in related fields, the Sturm Colloquium provided a unique opportunity for the sharing of knowledge and exchange of new ideas.

Interactions of this kind between individuals from different academic backgrounds can be of great value. There is, of course, a powerful argument for mathematics to take note of its history. Mathematical results, concepts and methods do not spring from nowhere. Often new results are motivated by existing or potential applications. Some of Sturm's early work on sound propagation in fluids is a good example of this, as are his fundamental contributions to the theory of differential equations, which were partly motivated by problems of heat flow. Some of the later developments in areas that Sturm had initiated proceeded in parallel with one of the revolutions in twentieth century physics, namely quantum mechanics. New ideas in mathematics need to be considered in the light of the mathematical and cultural environment of their time.

Sturm's mathematical publications covered diverse areas of geometry, algebra, analysis, mechanics and optics. He published textbooks in analysis and mechanics, both of which were still in use as late as the twentieth century[6].

To most mathematicians today, Sturm's best-known contributions, and those which are usually considered to have had the greatest influence on mathematics since Sturm's day, have been in two main areas.

The first of Sturm's major contributions to mathematics was his remarkable solution, presented to the Paris Academy of Sciences in 1829 and later elaborated in a memoir of 1835[7], of the problem of determining the number of roots, on a given interval, of a real polynomial equation of arbitrary degree. Sturm found a complete solution of this problem, which had been open since the seventeenth century. His solution is algorithmic; a sequence of auxiliary polynomials (now called Sturm

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to be placed on the list of Foreign members of the Royal Society". The Copley Medal was in recognition of his seminal work on the roots of real polynomial equations and was the second medal awarded that year, the first having gone to the chemist J. Liebig. The citation for the Medal was: "Resolved, by ballot. – That another Copley Medal be awarded to M. C. Sturm, for his "Mémoire sur la Résolution des Equations Numériques," published in the Mémoires des Savans Etrangers for 1835". Sturm is also one of the few mathematicians commemorated in the series of plaques at the Eiffel tower in Paris.

[6] Both of these books were published posthumously, Sturm having died on 18 December 1855. The analysis text went through 15 editions, of which the last printing was as late as 1929. A reference for the first edition is: *Cours d'analyse de l'École polytechnique* (2 vols.), published by E. Prouhet (Paris, 1857–59). The text was translated into German by T. Fischer as: *Lehrbuch der Analysis* (Berlin, 1897–98). The first edition of the mechanics text was: *Cours de mécanique de l'École polytechnique* (2 vols.), published by E. Prouhet (Paris, 1861). The fifth and last edition, revised and annotated by A. de Saint-Germain, was in print at least until 1925.

[7] The full text of Sturm's resolution of this problem is to be found in: *Mémoire sur la résolution des équations numériques*, in the journal Mémoires présentés par divers savans à l'Académie Royale des Sciences de l'Institut de France, sciences mathématiques et physiques **6** (1835), 271–318 (also cited as Mémoires Savants Étrangers). See also *The Collected works of Charles François Sturm*, J.-C. Pont, editor (in preparation) for further discussion of this work.

functions), is calculated, and the number of roots on an interval is determined by the signs of the Sturm functions at the ends of the intervals. Sturms work on zeros of polynomials undoubtedly influenced his work on related problems for solutions of differential equations, which was to follow.

His second major mathematical contribution, or rather a whole series of contributions, was to the theory of second-order linear ordinary differential equations. In 1833 he read a paper to the Academy of Sciences on this subject, to be followed in 1836 by a long and detailed memoir in the Journal de Mathématiques Pures et Appliquées. This memoir was one of the first to appear in the journal, which had recently been founded by Joseph Liouville, who was to become a collaborator and one of Sturm's closest friends in Paris. It contained the first full treatment of the oscillation, comparison and separation theorems which were to bear Sturm's name, and was succeeded the following year by a remarkable short paper, in the same journal and in collaboration with Liouville, which established the basic principles of what was to become known as Sturm-Liouville theory[8]. The problems treated in this paper would be described today as Sturm-Liouville boundary value problems (second-order linear differential equations, with linear dependence on a parameter) on a finite interval, with separated boundary conditions. Sturm's earlier work had shown that such problems led to an infinity of possible values of the parameter. The collaboration between Sturm and Liouville took the theory some way forward by proving the expansion theorem, namely that a large class of functions could be represented by a Fourier-type expansion in terms of the family of solutions to the boundary value problem. In modern terminology, the solutions would later be known as eigenfunctions and the corresponding values of the parameter as eigenvalues.

The 1837 memoir, published jointly by Sturm and Liouville, was to become the foundation of a whole new branch of mathematics, namely the spectral theory of differential operators. Sturm-Liouville theory is central to a large part of modern analysis. The theory has been successively generalized in a number of directions, with applications to Mathematical Physics and other branches of modern science. This volume provides the reader with an account of the evolution of Sturm-Liouville theory since the pioneering work of its two founders, and presents some of the most recent research. The companion volume will treat aspects of the work of Sturm and his successors as a branch of the history of scientific ideas. We believe that the two volumes together will provide a perspective which will help to make clear the significant position of Sturm-Liouville theory in modern mathematics.

Sturm-Liouville theory, as originally conceived by its founders, may be regarded, from a modern standpoint, as a first, tentative step towards the development of a spectral theory for a class of second-order ordinary differential operators.

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[8] For a more extended treatment of the early development of Sturm-Liouville theory, with detailed references, see the paper on Sturm and differential equations by J. Lützen and A. Mingarelli in the companion volume, as well as the first contribution by Everitt to this volume.

Liouville had already covered in some detail the case of a finite interval with two regular endpoints and boundary conditions at each endpoint. He regarded the resulting expansion theorem in terms of orthogonal eigenfunctions[9] as an extension of corresponding results for Fourier series, and the analysis was applicable only to cases for which, in modern terminology, the spectrum could be shown to be pure point. In fact the term "spectrum" itself, in a sense close to its current meaning, only began to emerge at the end of the nineteenth and the beginning of the twentieth century, and is usually attributed to David Hilbert.

The first decade of the twentieth century was a period of rapid and highly significant development in the concepts of spectral theory. A number of mathematicians were at that time groping towards an understanding of the idea of continuous spectrum. Among these was Hilbert himself, in Göttingen. Hilbert was concerned not with differential equations (though his work was to have a profound impact on the spectral analysis of second-order differential equations) but with what today we would describe as quadratic forms in the infinite-dimensional space l^2. Within this framework, he was able to construct the equivalent of a spectral function for the quadratic form, in terms of which both discrete and continuous spectrum could be defined. Examples of both types of spectrum could be found, and from these examples emerged the branch of mathematics known as spectral analysis. For the first time, spectral theory began to make sense even in cases where the point spectrum was empty. The time was ripe for such developments, and the theory rapidly began to incorporate advances in integration and measure theory coming from the work of Lebesgue, Borel, Stieltjes and others.

As far as Sturm-Liouville theory itself is concerned, the most significant progress during this first decade of the twentieth century was undoubtedly due to the work of the young Hermann Weyl. Weyl had been a student of Hilbert in Göttingen, graduating in 1908. (He was later, in 1930, to become professor at the same university.) His 1910 paper[10] did much to revolutionise the spectral theory of second-order linear ordinary differential equations. Weyl's spectrum is close to the modern definition via resolvent operators, and his analysis of endpoints based on limit point/limit circle criteria anticipates later ideas in functional analysis in which deficiency indices play the central role. For Weyl, continuous spectrum was not only to be tolerated, but was totally absorbed into the new theory. The expansion theorem, from 1910 onwards, was to cover contributions from both discrete and continuous parts of the spectrum. Weyl's example of continuous spectrum, corresponding to the differential equation $-d^2f(x)/dx^2 - xf(x) = \lambda f(x)$ on the

[9]Liouville's proof of the expansion theorem was not quite complete in that it depended on assumptions involving some additional regularity of eigenfunctions. Later extensions of this theory, as well as a full and original proof of completeness of eigenfunctions, can be found in the article by Bennewitz and Everitt in this volume.

[10]A full discussion of Weyl's paper and its impact on Sturm-Liouville theory is to be found in the first contribution by Everitt to this volume.

half line $[0, \infty)$, could hardly have been simpler[11]. And, perhaps most importantly, with Weyl's 1910 paper complex function theory began to move to the center stage in spectral analysis.

The year 1913 saw a further advance through the publication of a research monograph by the Hungarian mathematician Frigyes Riesz[12], in which he continued the ideas of Hilbert, with the new point of view that it was the linear operator associated with a given quadratic form, rather than the form itself, which was to be the focus of analysis. In other words, Riesz shifted attention towards the spectral theory of linear operators. In doing so he was able to arrive at the definition of spectrum in terms of the resolvent operator, to define a functional calculus for linear operators, and to explore the idea of what was to become the resolution of the identity for bounded self-adjoint operators. An important consequence of these results was that it became possible to incorporate many of Weyl's results on Sturm-Liouville problems into the developing theory of functional analysis. Thus, for example, the role of boundary conditions in determining self-adjoint extensions of differential operators could then be fully appreciated.

The modern theory of Sturm-Liouville differential equations, which grew from these beginnings, was profoundly influenced by the emergence of quantum mechanics, which also had its birth in the early years of the twentieth century. At the heart of the development of a mathematical theory to meet the demands of the new physics was John von Neumann[13].

Von Neumann joined Hilbert as assistant in Göttingen in 1926, the very year that Schrödinger first published his fundamental wave equation. The Schrödinger equation is, in fact, a partial differential equation, but, in the case of spherically symmetric potentials such as the Coulomb potential, the standard technique of separation of variables reduces the equation to a sequence of ordinary differential equations, one for each pair of angular momentum quantum numbers. In this way, under the assumption of spherical symmetry, Sturm-Liouville theory can be applied to the Schrödinger equation.

Von Neumann found in functional analysis the perfect medium for understanding the foundations of quantum mechanics. Quantum theory led in a natural way to a close correspondence (one could almost say identification, though that would not quite be true) of the physical objects of the theory with mathematical objects drawn from the theory of linear operators (usually differential operators) in Hilbert space. The state of a quantum system could be described by a normalized element (or vector, or wave function) in the Hilbert space. Corresponding to each

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[11] Later it was to emerge that examples of this kind could be interpreted physically in terms of a quantum mechanical charged particle moving in a uniform electric field.

[12] F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Gauthier-Villars, Paris (1913). See also J. Dieudonné, *History of functional analysis*, North-Holland, Amsterdam (1981). With Riesz we begin to see the development of an "abstract" operator theory, in which the special example of Sturm-Liouville differential operators was to play a central role.

[13] Von Neumann established a mathematical framework for quantum theory in his book *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin (1932). An English translation appeared as *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955).

quantum observable was a self-adjoint operator, the spectrum of which represented the range of physically realizable values of the observable. Both point spectrum and continuous spectrum were important – in the case of the hydrogen atom the energy spectrum had both discrete and continuous components, the discrete points (eigenvalues of the corresponding Schrödinger operator) agreeing closely with observed energy levels of hydrogen, and the continuous spectrum corresponding to states of positive energy.

Von Neumann quickly saw the implications for quantum mechanics of the new theory, and played a major part in developing the correspondence between physical theory and the analysis of operators and operator algebras. Physics and mathematical theory were able to develop in close parallel for many years, greatly to the advantage of both. He developed to a high art the spectral theory of self-adjoint and normal operators in abstract Hilbert space. A complete spectral analysis of self-adjoint operators in Hilbert space, generalizing the earlier results of Riesz, was just one outcome of this work, and a highly significant one for quantum theory. Similar results were independently discovered by Marshall Stone, who expounded the theory in his book published in 1932. (See the first article by Everitt.)

Of central importance for the future development of applications to mathematical physics, particularly in scattering theory which existed already in embryonic form in the work of Heisenberg, was the realization that the Lebesgue decomposition of measures into its singular and absolutely continuous (with respect to Lebesgue measure) components led to an analogous decomposition of the Hilbert space into singular and absolutely continuous subspaces for a given self-adjoint operator. Moreover, these two subspaces are mutually orthogonal. The singular subspace may itself be decomposed into two orthogonal components, namely the subspace of discontinuity, spanned by eigenvectors, and the subspace of singular continuity. Physical interpretations have been found for all of these subspaces, though in most applications only the discontinuous and absolutely continuous subspaces are non-trivial. In the case of the Hamiltonian (energy operator) for a quantum particle subject to a Coulomb force, the discontinuous subspace is the subspace of negative energy states and describes bound states of the system, whereas the absolutely continuous subspace corresponds to scattering states, which have positive energy.

The influence of the work of Charles Sturm and his close friend and collaborator Joseph Liouville may be found in the numerous modern developments of the theory which bears their names. A principal aim of this volume is to follow in detail the evolution of the theory since its early days, and to present an overview of the most important aspects of the theory as it stands today at the beginning of the twenty-first century.

We are grateful indeed to Norrie Everitt for his contributions to this volume, as author of two articles and coauthor of another. Over a long mathematical career, he has played an important role in the continuing progress of Sturm-Liouville theory.

The first of Norrie's articles in this volume deals with the development of Sturm-Liouville theory up to the year 1950, and covers in particular the work of

Weyl, Stone and Titchmarsh, of whom Norrie was himself a one-time student. (He also had the good fortune, on one occasion, to have encountered Weyl, who was visiting Titchmarsh at the time.)

Don Hinton's article is concerned with a series of results which follow from Sturm's original oscillation theorems developed in 1836 for second-order equations. Criteria are obtained for the oscillatory nature of solutions of the differential equation, and implications for the point spectrum are derived. Extensions of the theory to systems of equations and to higher-order equations are described.

Joachim Weidmann's contribution considers the impact of functional analysis on the spectral theory of Sturm-Liouville operators. Starting from ideas of resolvent convergence, it is shown how spectral behavior for singular problems may in appropriate cases be derived through limiting arguments from an analysis of regular problems. Conditions are obtained for the existence (or non-existence) of absolutely continuous spectrum in an interval.

Spectral properties of Sturm-Liouville operators are often derived, directly or indirectly, as a consequence of an established link between large distance asymptotic behavior of solutions of the associated differential equation and spectral properties of the corresponding differential operator. In the case of complex spectral parameter, the existence of solutions which are square-integrable at infinity may be described by the values of an analytic function, known as the Weyl-Titchmarsh m-function or m-coefficient, and spectral properties of Sturm-Liouville operators may be correlated with the boundary behavior of the m-function close to the real axis. The article by Daphne Gilbert explores further the link between asymptotics and spectral properties, particularly through the concept of subordinacy of solutions, an area of spectral analysis to which she has made important contributions.

A useful resource for readers of this volume, particularly those with an interest in numerical approaches to spectral analysis, will be the catalogue of Sturm-Liouville equations, compiled by Norrie Everitt with the help of colleagues. More than 50 examples are described, with details of their Weyl limit point/limit circle endpoint classification, the location of eigenvalues, other spectral information, and some background on applications. This collection of examples from an extensive literature should also provide a reference to some of the sources in which the interested reader can find further details of the theory and its applications, as well as numerical data on spectral properties.

In collaboration with Christer Bennewitz, Everitt has contributed a new version of the proof of the expansion theorem for general Sturm-Liouville operators, incorporating both continuous and discontinuous spectra.

The article by Barry Simon presents some recent results related to Sturm's oscillation theory for second-order equations. The cases of both Schrödinger operators and Jacobi matrices (which may be regarded as a discrete analogue of Schrödinger operators) are considered. A focus of this work is the establishment of a connection between the dimension of spectral projections and the number of zeros of appropriate functions defined in terms of solutions of the Schrödinger