NAME	R. A. (1900) h	m	s	DEC. °	′	GALACTIC LATITUDE °	OBSERVED MED. MAG.	r' sin β KPC
HV 6426	0	08	20	−29	05.4	−84	16.15	15.1
RZ Scl	1	37	21	−26	43.3	−77	15.5	10.9
RY Phe	0	45	36	−56	06.4	−62	15.7	10.9
HV 6242	21	03	08	−15	38.0	−38	16.45	10.7
HV 6395	0	32	56	−54	19.0	−63	15.6	10.5
HV 6373	0	38	50	− 0	30.8	−62	15.6	10.4
AW Aqr	22	19	22	−24	08.9	−58	15.6	10.0
SX Scl	0	01	52	−30	10.2	−82	15.25	9.9
HV 6372	0	36	39	− 3	33.6	−66	15.35	9.6
AR Aqr	22	07	41	−25	13.0	−56	15.55	9.5

In northern latitudes, several variables with r' sin $\beta > 12$ kiloparsecs are known, mostly those found by Baade[3] near the globular cluster N. G. C. 4147. The very distant cluster variables in Milky Way Fields 233 and 269 stand well away from the galactic plane,[4] but because the galactic latitudes are low, the greatest values of r' sin β are, for them, between four and five kiloparsecs.

[1] These PROCEEDINGS, **19**, 29–34 (1933) and **22**, 8–14 (1936); Harvard Reprints 81 and 118.

[2] *Pub. Am. Ast. Soc.*, **9**, 239 (1939).

[3] *Ast. Nach.*, **244**, 153 (1931).

[4] *Harv. Ann.*, **90**, No. 9 (1939) and **105**, No. 13 (1936).

CRITICAL POINTS OF A MAP TO A CIRCLE

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1. *Introduction.*—We shall define the critical points of a map on a manifold to an oriented circle and describe how to classify them as to type. We shall show that inequalities analogous to the inequalities of Morse[1] are satisfied by the numbers of critical points of various types and the ranks of suitably chosen groups of homology classes. These ranks are invariants of the homotopy class of the map and are the connectivities of the given manifold for the class of maps homotopic to a point. We shall make several assumptions at the beginning of section 2 in order to simplify the exposition of those portions of the theory which are similar to known developments and to highlight the novel methods and results of this paper. The reader will bear in mind that the theory can be developed with much lighter assumptions; the principal results would be group theoretic rather than numerical in form. See the developments cited in notes 2 and 3.

We shall use the obvious definition of differential critical points in terms of the angular coördinate on the circle and shall use Morse's definition of type numbers.[4] Our method consists first of defining an induced covering space of the given manifold which can be mapped on the given manifold by a map which is locally isometric. The covering manifold is such that we can define a periodic function whose critical sets cover those of the given function and have the same type numbers. Methods due to Morse are applied to a convenient fundamental domain to yield the fundamental inequalities (Theorem I) between the numbers of critical points of various types and the excesses of new cycles over newly bounding cycles of various dimensions. Further theorems show that the excesses are positive or zero (Theorem II), are independent of the choice of fundamental domain (Theorem III), and are invariants of the homotopy class of the map (Theorem IV).

2. *The Map and the Critical Sets.*—To fix ideas, we suppose L is an n-dimensional manifold of class C^4, defined in terms of overlapping local coördinate systems and carrying a fundamental differential form of class C^3, which defines the metric. On the circle S^1 we use the angular coördinate θ, with θ increasing in the direction of positive orientation. We assume that the map on L to S^1 is defined by a point function F which is of class C^2 in terms of local coördinates. A point of L will be termed a *critical point* of F if it is a differential critical point of the function θ in the usual sense. It is assumed that the images of critical points on S^1 are isolated.

With the above assumptions it will be sufficient for our purposes to use singular chain topology with integers mod 2 as coefficients, though more refined developments demand the use of Vietoris topology.

By a *critical set* we shall mean a component of the set of critical points. The image of a critical set is a point. To a critical set is assigned a set of *type numbers* $[m_0, m_1, \ldots, m_n]$ namely, the set of type numbers assigned by Morse[4] to the set as a critical set of the function θ. We recall that the set of type numbers of a critical set depends only on the function on a neighborhood of the critical set and is independent of alterations in the function and the space elsewhere. The *count of critical points of type k* is the sum M_k of the numbers m_k over all critical sets.

3. *The Induced Covering Space.*—We shall develop the idea of the induced covering space. Let p_0 be a fixed point of L and let θ_0 be a value of θ at $F(p_0)$. Along any curve λ joining p_0 to a point p, θ is uniquely determined as a continuous function of the parameter on λ, and consequently the path λ determines θ at p uniquely. The determination of θ by two paths, λ and μ, is the same if and only if the two maps $F(\lambda)$ and $F(\mu)$ are homotopic on S^1 when their end-points are held fast. The different values of θ at p differ by integral multiples of 2π. Either (case I) there is one possible value θ_p of θ at p, or (case II) there is a positive integer α such that the possible values

of θ at p are $\theta_p + 2m\alpha\pi$, $m = 0, \pm 1, \pm 2, \ldots$, with θ_p any particular determination at p. Whichever situation holds at one point of L holds at all points, and in the second case the value of α is independent of the point p.

The points of the induced covering space K are pairs (p, θ_p) where p is a point of L and θ_p is a determination of θ at p. Let φ denote the map on K to L in which (p, θ_p) is carried into p. We readily define the metric on K in such a way that the map φ is locally one to one and isometric. Except for notation this definition is independent of p_0 and θ_0.

4. *The Function on the Covering Space.*—On K we define a point function F^* by the relation

$$F^*(p, \theta_p) = \theta_p.$$

Let $\psi(\theta)$ denote the point of S^1 with angular coördinate θ. Then ψ is a map on the line to the circle. The two maps $F\varphi$ and ψF^* on K to S^1 are identical. The critical sets of F as defined in section 2 and of F^* in the usual sense correspond under φ and have the same sets of type numbers.

The covering space admits a group of translations under which its map on S^1 remains fixed. In case I, this group is the identity, while in case II it is generated by the translation T which sends (p, θ_p) into $(p, \theta_p + 2\pi\alpha)$. In case I, K is the *fundamental domain* of the translation and is uniquely determined. In case II, the set of points where $a \leqq F^* < a + 2\pi\alpha$ is a *fundamental domain* for any value of a and we shall use only domains of this form. We let A and B denote numbers such that $A < F^* < B$ in case I and a and $a + 2\pi\alpha$ in case II.

We term a cycle on K a *new cycle* relative to (A, B) if it lies on $F < B$ and is not homologous on $F < B$ to a cycle on $F < A$. We term a cycle on K a *newly bounding cycle* relative to (A, B) if it lies on $F < A$ and is homologous to zero on $F < B$ but not on $F < A$. We count new cycles or newly bounding cycles independent if every proper linear combination is respectively a new cycle or a newly bounding cycle. We term a cycle on K a *new cycle which bounds* provided it is a new cycle and bounds on K.

The fundamental domain was so chosen that the count of critical points on it of type k is M_k. Methods similar to those of Morse[5] serve to prove the following theorem.

THEOREM I. *If* $\mathrm{M_k}$ *denotes the count of critical points on* L *of type* k *and* $\mathrm{Q_k}$ *denotes the count of new* k*-cycles less the count of newly bounding* k*-cycles relative to* (A, B) *then the following inequalities hold.*

$$\mathrm{M}_k - \mathrm{M}_{k-1} + \ldots + (-1)^k\mathrm{M}_0 \geqq \mathrm{Q}_k - \mathrm{Q}_{k-1} + \ldots + (-1)^k\mathrm{Q}_0.$$

Here $\mathrm{k} = 0, 1, \ldots, \mathrm{n}$ *and the equality holds when* $\mathrm{k} = \mathrm{n}$.

We have the following theorems.

THEOREM II. *The numbers* $\mathrm{Q_k}$ *are independent of the choice of the fundamental domain.*

THEOREM III. *The count of newly bounding* k-*cycles is equal to the count of* k-*cycles which bound. Thus* $Q_k \geqq 0$.

THEOREM IV. *The numbers* Q_k *are invariants of the homotopy class of the map* F.

Theorem II is a byproduct in the proof of Theorem I. The proof of Theorem III depends on the use of the cycle limits[6] developed by Morse and on the existence of the translation T. To prove Theorem IV we show that any deformation of a map of L can be replaced by a sequence of deformations each of which leaves a fundamental domain on K topologically invariant.

[1] Morse, M., "Calculus of Variations in the Large," *Am. Math. Soc.*, Coll. Publ., New York, 1934, Ch. VI. The basic theorem is Theorem 7.4.

[2] Morse, M., *Ann. Math.*, **38**, 386–449 (1937).

[3] Morse, M., "Functional Topology and Abstract Variational Theory," *Memoriale des Science Math.*, **92**; Paris (1939).

[4] See reference 1, sections 2 and 7.

[5] See reference 1, Theorem 7.3 and relation 7.10.

[6] See reference 2, section 4.

GROUPS CONTAINING A PRIME NUMBER OF NON-INVARIANT SUBGROUPS

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1. *General Considerations.*—For the sake of clearness we shall use the term subgroups in the present article in the sense of a proper subgroup, excluding both the identity and the entire group from the subgroups of a given group G. The non-invariant subgroups of G are transformed under G according to a substitution group to which G is α, 1 isomorphic. The subgroup of order α in G which corresponds to the identity of this substitution group includes the central of G and is invariant under G. A necessary and sufficient condition that this substitution group is transitive is that all the non-invariant subgroups of G constitute a single set of conjugate subgroups under G. If this substitution group is of degree n and involves a substitution of degree n then to the cyclic subgroup generated by this substitution there corresponds at least one cyclic invariant subgroup of G, because a group is transformed into itself by each of its own operators.

If G contains two and only two non-invariant subgroups these subgroups are transformed transitively under G. This is also true when G contains three and only three non-invariant subgroups since every substitution