DIFFERENTIABLE STRUCTURES ON SPHERES.*

By John Milnor.[1]

According to [5] the sphere S^7 can be given several differentiable structures which are essentially distinct. A corresponding result for the 15-sphere has been proved by Shimada [10] and Tamura [12]. The object of this note is to prove corresponding theorems for other dimensions of the form $4k-1$.

In § 1 certain differentiable manifolds $M(f_1, f_2)$ are constructed and studied; where $(f_1) \in \pi_m(SO_{n+1})$, $(f_2) \in \pi_n(SO_{m+1})$. In most cases these manifolds are topologically spheres. In § 2 an invariant λ is defined for differentiable $(4k-1)$-manifolds which are both homology spheres and boundaries. In § 3 the invariant $\lambda(M(f_1, f_2))$ is computed.

For $k \leqq 8$ the calculations are carried out explicitly. It is shown that there exist non-standard differentiable structures on S^{4k-1} for $k = 2, 4, 5, 6, 7, 8$. For example S^{31} has over sixteen million distinct differentiable structures. It is conjectured that the same argument works for all $k \geqq 4$; but I have not succeeded in solving the number theoretic problem which arises.

For $k = 1, 3$ the argument does not work. This is not surprising in the case $k = 1$, since J. Munkres, S. Smale, and J. H. C. Whitehead have shown (independently) that two differentiable 3-manifolds which are homeomorphic must necessarily be diffeomorphic.

The word *manifold* will always be used for a compact, oriented manifold, with or without boundary. The symbol D^k will stand for the unit disk in the euclidean space R^k.

1. Construction of manifolds homeomorphic to spheres. Given any diffeomorphism[2] $f\colon S^m \times S^n \to S^m \times S^n$, a manifold M of dimension $m+n+1$ is obtained by matching the boundaries of $D^{m+1} \times S^n$ and $S^m \times D^{n+1}$ under the correspondence f. That is: M is the identification space obtained from the disjoint union of $D^{m+1} \times S^n$ and $S^m \times D^{n+1}$ by identifying each point (x, y) in the boundary of $D^{m+1} \times S^n$ with $f(x, y)$, considered as a point in boundary of $S^m \times D^{n+1}$.

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[2] A *diffeomorphism* is a differentiable homeomorphism with differentiable inverse.

An alternative definition of M, which makes it into a differentiable manifold, is the following. Let $f(x,y)=(x',y')$ and define $t'=1/t$. Start with disjoint spaces $R^{m+1}\times S^n$ and $S^m\times R^{n+1}$. Let M be obtained from these by matching (tx,y) with $(x',t'y')$ for every $x\in S^m$, $y\in S^n$, $0<t<\infty$.

[As an example suppose that f is the identity map of $S^m\times S^n$. Then M is diffeomorphic to the unit sphere $S^{m+n+1}\subset R^{m+1}\times R^{n+1}$. In fact the correspondence

$$(tx,y)\to(tx/(1+t^2)^{\frac{1}{2}},\,y/(1+t^2)^{\frac{1}{2}}),$$
$$(x',t'y')\to(x'/(1+t'^2)^{\frac{1}{2}},\,t'y'/(1+t'^2)^{\frac{1}{2}})$$

defines a diffeomorphism $M\to S^{m+n+1}$.]

If $y\in S^n$ has coordinates $(y_0,\cdots,y_n)$, define $h(y)=y_n$. The function $h\colon S^n\to[-1,1]$ has just two critical points.

Lemma 1. *Suppose that the diffeomorphism*

$$(x,y)\xrightarrow{\;f\;}(x',y')$$

satisfies the restriction $h(y)=h(y')$ for all (x,y). Then the manifold M constructed above is homeomorphic to S^{m+n+1}.

Proof. A differentiable map $g\colon M\to[-1,1]$ is defined by the correspondence

$$(tx,y)\to h(y)/(1+t^2)^{\frac{1}{2}} \text{ (in the first coordinate system)},$$
$$(x',t'y')\to t'h(y')/(1+t'^2)^{\frac{1}{2}} \text{ (in the second)}.$$

It is easily verified that g has just two critical points, and that these are non-degenerate. Together with [5] Theorem 2, this completes the proof.

One way to construct such a diffeomorphism $(x,y)\to(x',y')$ is the following. Start with differentiable maps of spheres into rotation groups

$$f_1\colon S^m\to SO_{n+1},\qquad f_2\colon S^n\to SO_{m+1},$$

and let

$$y'=f_1(x)\cdot y,\qquad x'=f_2(y')^{-1}\cdot x=f_2(f_1(x)\cdot y)^{-1}\cdot x$$

for all $x\in S^m$, $y\in S^n$. This defines a diffeomorphism with inverse

$$x=f_2(y')\cdot x',\qquad y=f_1(x)^{-1}\cdot y'=f_1(f_2(y')\cdot x')^{-1}\cdot y'.$$

[More generally the rotation groups SO_{n+1} and SO_{m+1} could be replaced by the groups Diff S^n, Diff S^m consisting of all diffeomorphisms.] The condition $h(y)=h(y')$ is equivalent to the requirement that $f_1(S^m)$ be contained in the subgroup $SO_n \subset SO_{n+1}$. Whether this condition is satisfied or not, *the resulting* $(m+n+1)$*-manifold will be denoted by* $M(f_1, f_2)$.

Next we will show that $M(f_1, f_2)$ is a boundary. Start with three copies of the space $D^{m+1} \times D^{n+1}$. The notation (x_i, y_i) will be used for a point of $(D^{m+1} \times D^{n+1})_i$, $i = 1, 2, 3$. Identify $(S^m \times D^{n+1})_1$ with $(S^m \times D^{n+1})_2$ by the correspondence $(x_1, y_1) \to (x_2, y_2)$ where

$$x_2 = x_1, \qquad y_2 = f_1(x_1) \cdot y_1.$$

The resulting space $(D^{m+1} \times D^{n+1})_1 \cup (D^{m+1} \times D^{n+1})_2$ can be considered as a fibre bundle over the $(m+1)$-sphere $(D^{m+1})_1 \cup (D^{m+1})_2$ with fibre D^{n+1}, group SO_{n+1}, and characteristic map f_1. (See Steenrod [11] § 18.) It follows that this union can be given a differentiable structure in a natural way.

Next identify $(D^{m+1} \times S^n)_2$ with $(D^{m+1} \times S^n)_3$ by the correspondence $(x_2, y_2) \leftarrow (x_3, y_3)$, where $y_2 = y_3$, $x_2 = f_2(y_3) \cdot x_3$. Thus

$$(D^{m+1} \times D^{n+1})_2 \cup (D^{m+1} \times D^{n+1})_3$$

becomes a fibre bundle over an $(n+1)$-sphere with fibre D^{m+1} and characteristic map f_2.

Let W_1 denote the union of all three copies of $D^{m+1} \times D^{n+1}$. Clearly W_1 is a topological manifold with boundary:

$$\partial W_1 = (D^{m+1} \times S^n)_1 \cup (S^m \times D^{n+1})_3.$$

The intersection

$$(D^{m+1} \times S^n)_1 \cap (S^m \times D^{n+1})_3$$

of the two halves of the boundary is equal to $(S^m \times S^n)_1 = (S^m \times S^n)_3$. These two copies of $S^m \times S^n$ are identified under the composite correspondence

$$(x_1, y_1) \to (x_2, y_2) \to (x_3, y_3),$$

where

$$y_3 = y_2 = f_1(x_1) \cdot y_1, \qquad x_3 = f_2(y_3)^{-1} \cdot x_2 = f_2(y_3)^{-1} \cdot x_1.$$

But this is just the correspondence which was used to define the manifold $M(f_1, f_2)$. Thus ∂W_1 is homeomorphic to $M(f_1, f_2)$.

As it stands W_1 is not a differentiable manifold since there is an "angle" along the subset $(S^m \times S^n)_1$ of ∂W_1. Let W denote a differentiable manifold obtained by "straightening" this angle. (See the appendix to [7].) Clearly ∂W is diffeomorphic to $M(f_1, f_2)$.

2. The invariant $\lambda(M)$. First recall the index theorem of Hirzebruch [4]. If M_1 is a $4k$-manifold without boundary having Pontrjagin classes

$p_1, \cdots, p_k$, then the index $I(M_1)$ is equal to $L_k(p_1, \cdots, p_k)[M_1]$; where L_k is a certain polynomial.[3] For example

$$L_1 = p_1/3, \qquad L_2 = (7p_2 - p_1{}^2)/45, \cdots.$$

The coefficient s_k of p_k in L_k is particularly important. Hirzebruch expresses s_k in terms of the Bernoulli number B_k as follows (page 14):

$$s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!.$$

For example $s_2 = 7/45$, $s_3 = 62/945$, $s_4 = 127/4725$.

Now let M be a differentiable $(4k-1)$-manifold which

1) has the same rational homology groups as the $(4k-1)$-sphere, and

2) is a boundary: $M = \partial W$ with W differentiable.[4] Then a rational number modulo 1,

$$\lambda(M) \in Q/Z,$$

is defined as follows. The natural homomorphism

$$j: H^i(W, M; Q) \to H^i(W; Q)$$

is an isomorphism for $0 < i < 4k-1$. Hence the Pontrjagin classes $p_1, \cdots, p_{k-1}$ of W can be lifted back to $H^*(W, M; Q)$. Define $\lambda(M)$ as the residue class of

$$(I(W) - L_k(j^{-1}p_1, \cdots, j^{-1}p_{k-1}, 0)[W])/s_k$$

modulo 1. (Here the symbol $[W]$ stands for the homomorphism $H^{4k}(W, M; Q) \to Q$ associated with the orientation of W; and $I(W)$ denotes the index of the quadratic form $\alpha \to (\alpha \cup \alpha)[W]$, where $\alpha \in H^{2k}(W, M; Q)$.)

LEMMA 2. *This residue class $\lambda(M)$ is an invariant of M: that is it does not depend on the choice of W.*

The proof is completely analogous to that in [5], [10] or [12]. If M is the boundary of both W_1 and W_2, then an unbounded $4k$-manifold M_1 is obtained from W_1, W_2 by

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[3] The symbol $[M_1]$ is used to denote the homomorphism of $H^{4k}(M_1; Q)$ into the rational numbers Q which is determined by an orientation for M_1. The index $I(M_1)$ is defined as the index of the quadratic form over $H^{2k}(M_1; Q)$ which is given by the formula $\alpha \to (\alpha \cup \alpha)[M_1]$.

[4] This second condition follows automatically if $H_*(M; Z)$ has no torsion. In fact every homology $(4k-1)$-sphere is a π-manifold (see [7]), and every π-manifold is a boundary (see [8]).

1) reversing the orientation of W_2;

2) matching W_1 and W_2 along the common boundary M;

3) constructing a differentiable structure in a neighborhood of $W_1 \cap W_2 = M$. (See [6] Lemma 4 or [7]). Then $I(M_1) = I(W_1) - (W_2)$; and each Pontrjagin number $p_{i_1} \cdots p_{i_r}[M_1]$ other than $p_k[M_1]$ is equal to the difference of corresponding Pontrjagin numbers for W_1, W_2. Now the index theorem for M_1 implies that the two definitions of $\lambda(M)$ differ only by the integer $p_k[M_1]$.

Example 1. For the $(4k-1)$-sphere it is clear that

$$\lambda(S^{4k-1}) \equiv 0 \pmod 1.$$

Example 2. For a 3-manifold the definition reduces to

$$\lambda(M^3) \equiv 3 \cdot I(W) \equiv 0 \pmod 1.$$

Example 3. For the 7-manifold $M_3{}^7$ of [5] the values

$$I(W) = 1, \quad (j^{-1}p_1)^2[W] = 36$$

give

$$\lambda(M_3{}^7) \equiv (45 I(W) + (j^{-1}p_1)^2[W])/7 \equiv 4/7 \pmod 1.$$

Remark. If $H^*(M;Z)$ has no torsion, then the classes $j^{-1}p_i$ can be considered as integral cohomolgy classes, hence the Pontrjagin numbers of W are integers. This sharply restricts the denominator which $\lambda(M)$ can have. (For example $7\lambda(M^7)$ must be an integer.) On the other hand, if $H^*(M;Z)$ has torsion then arbitrarily large denominators may occur. (See the examples studied by Tamura.)

Example 4. In [7] § 4 certain homotopy spheres $M_0{}^{4k-1}$ are constructed for $k > 1$. These have the property that $M_0{}^{4k-1} = \partial W$, where W is parallelizable, and $I(W) = 8$. Thus

$$\lambda(M_0{}^{4k-1}) \equiv 8/s_k \pmod 1.$$

For $k = 2$ this gives $\lambda(M_0{}^7) \equiv 3/7$ with denominator 7. For $k = 3, 4, 5, 6, 7$ the denominator of $\lambda(M_0{}^{4k-1})$ is 31, 127, 73, 1414477, and 8191 respectively. (These numbers are prime, except for $1414477 = 23 \cdot 89 \cdot 691$.) I do not know whether the inequality $8/s_k \not\equiv 0 \pmod 1$ holds for all $k > 1$.

In conclusion, the following three properties of the invariant λ are easily verified.

1) If the orientation of M is reversed, then λ changes sign.

2) For the connected sum of manifolds (see [7]), λ satisfies

$$\lambda(M_1 \# M_2) \equiv \lambda(M_1) + \lambda(M_2) \pmod 1.$$

3) λ is an invariant of the J-equivalence class of M. (See Thom [13] or [7].)

3. Computation of $\lambda(M(f_1, f_2))$. Define the *Pontrjagin homomorphism*

$$p_r\colon \pi_{4r-1}(SO_q) \to Z$$

as follows. Every map $f\colon S^{4r-1} \to SO_q$ induces a bundle ξ over S^{4r} with Pontrjagin class $p_r(\xi) \in H^{4r}(S^{4r}; Z) \approx Z$. Define $p_r(f)$ as the corresponding integer $p_r(\xi)[S^{4r}]$.

Let $f_1\colon S^m \to SO_{n+1}$, $f_2\colon S^n \to SO_{m+1}$ be arbitrary differentiable maps, with $m + n + 1 = 4k - 1$. First suppose that $m \neq n$.

LEMMA 3. *If $m \neq n$ then $M(f_1, f_2)$ is a topological sphere. The invariant $\lambda(M(f_1, f_2))$ is zero if m, n are not of the form $4r - 1$. If $m = 4r - 1$, $n = 4(k - r) - 1$, then*

$$\lambda \equiv \pm\, p_r(f_1)\, p_{k-r}(f_2)\, s_r s_{k-r}/s_k \pmod 1.$$

Proof. We may assume that $m < n$. The exact sequence

$$\pi_m(SO_n) \to \pi_m(SO_{n+1}) \to \pi_m(S^n) = 0$$

implies that f_1 is homotopic to a map f_1' which carries S^m into the subset $SO_n \subset SO_{n+1}$. According to Lemma 1 the manifold $M(f_1', f_2)$ is homeomorphic to S^{m+n+1}. But it can be verified that $M(f_1, f_2)$ is homeomorphic to $M(f_1', f_2)$, and therefore is also homeomorphic to the sphere.

Next consider the manifold W constructed in Section 1. Recall that W is the union of a fibre bundle over S^{m+1} with fibre D^{n+1} and a fibre bundle over S^{n+1} with fibre D^{m+1}. Call these sets W_2 and W_3 respectively, Thus $W_2 \cup W_3$ is W and $W_2 \cap W_3$ is a topological cell.

These bundles have canonical cross-sections corresponding to the center point of the disk. Hence S^{m+1} and S^{n+1} are imbedded in W. It follows easily that W has the same homology groups as $S^{m+1} \vee S^{n+1}$ (the union with a single point in common). That is $H_i(W; Z)$ is infinite cyclic for i equal to 0, $m + 1$, or $n + 1$, and zero otherwise.

The homology intersection ring of W (see Lefschetz [14]) is described as follows. Let a and b stand for generators in dimensions $n + 1$, $m + 1$ respectively. Clearly a and b have intersection number ± 1. The self-intersections $a \cdot a$ and $b \cdot b$ are zero. For example $a \cdot a$ is represented by a cycle of dimension

$$\dim a + \dim a - \dim W = n - m$$

which lies on the sphere $S^{n+1} \subset W_3$. Since $H_{n-m}(S^{n+1}; Z) = 0$, this cycle is homologous to zero.

Applying Poincaré duality it follows that $H^*(W, M; Z)$ *is free abelian on three generators, say* α *in dimension* $m+1$, β *in dimension* $n+1$, *and* $\alpha\beta$ *in dimension* $m+n+2$. *The cup products* $\alpha\alpha$ *and* $\beta\beta$ *are zero.* This implies that the index $I(W)$ is zero.

Computation of the Pontrjagin numbers of W. We may assume that $m = 4r - 1$, $n = 4k - 4r - 1$. (If the dimensions are not of this form, then the Pontrjagin numbers are certainly zero, hence $\lambda = 0$.) First consider the tangent bundle of W_2. This splits into a Whitney sum $\xi \oplus \eta$, where ξ is the bundle of vectors tangent to the fibre and η is the bundle of vectors normal to the fibre. Restricting η to the sphere $S^{m+1} \subset W_2$ we obtain the tangent bundle of S^{m+1} with trivial Pontrjagin classes. Restricting ξ to S^{m+1} we obtain the bundle determined by $(f_1) \in \pi_m(SO_{n+1})$. Thus $p_r(W_2) = p_r(\xi)$ is equal to the integer $p_r(f_1)$ multiplied by a generator of the infinite cyclic group $H^{4r}(W_2; Z)$. Using the isomorphisms

$$H^{4r}(W_2; Z) \leftarrow H^{4r}(W; Z) \xleftarrow{j} H^{4r}(W, M; Z)$$

it follows that

$$p_r(W) = \pm p_r(f_1) j(\alpha).$$

Similarly

$$p_{k-r}(W) = \pm p_{k-r}(f_2) j(\beta).$$

Thus *the Pontrjagin number* $(j^{-1}p_r)(j^{-1}p_{k-r})[W]$ *is equal to* $\pm p_r(f_1) p_{k-r}(f_2)$. All other Pontrjagin numbers of W are zero (except $(j^{-1}p_k)[W]$ which is not defined).

Computation of the coefficients of $p_r p_{k-r}$ in the Hirzebruch polynomial L_k. Define the symmetric function $\sum t_1^{i_1} \cdots t_a^{i_a}$ in indeterminates $t_1, \cdots, t_N$ as the sum of all monomials which can be obtained from $t_1^{i_1} \cdots t_a^{i_a}$ by permuting $t_1, \cdots, t_N$. Each possible monomial should be included only once in the sum. (For example $\sum t_1^r = t_1^r + \cdots + t_N^r$.) Hirzebruch showed[5] that the coefficient of $p_{i_1} \cdots p_{i_a}$ in L_k can be expressed in the form $\sum t_1^{i_1} \cdots t_a^{i_a}$, where $t_1, \cdots, t_N$ are certain fixed complex numbers. (Here N stands for

[5] See [4] § 1.4.1.

some fixed integer greater than or equal to k.) In particular, the coefficient s_k of p_k is equal to $\sum t_1{}^k$.

The product rule

$$(\sum t_1{}^r)(\sum t_1{}^{k-r}) = \sum t_1{}^k + \sum t_1{}^r t_2{}^{k-r} \qquad \text{for } r \neq k - r$$

is easily verified. Hence *the coefficient* $\sum t_1{}^r t_2{}^{k-r}$ *of* $p_r p_{k-r}$ *in* L_k *is equal to* $s_r s_{k-r} - s_k$.

Thus we have $I(W) = 0$ and

$$L_k(j^{-1}p_1, \cdots, j^{-1}p_{k-1}, 0)[W] = \pm\, p_r(f_1) p_{k-r}(f_2)(s_r s_{k-r} - s_k).$$

Dividing by s_k and reducing modulo one, this yields the required formula

$$\lambda(M) \equiv \pm\, p_r(f_1) p_{k-r}(f_2) s_r s_{k-r}/s_k \pmod 1$$

Now consider the case $m = n$. Again it is necessary to assume that m has the form $4r - 1$ in order to obtain a non-trivial λ.

LEMMA 4. *If the maps* f_1, f_2 *both carry* S^m *into the subgroup* $SO_m \subset SO_{m+1}$, *then the formula*

$$\lambda(M) \equiv p_r(f_1) p_r(f_2) s_r s_r / s_{2r}$$

holds, just as in Lemma 3.

Proof. Just as above, $H_*(W; Z)$ is isomorphic to $H_*(S^{m+1} \vee S^{m+1})$. If $b, a \in H_{m+1}(W; Z)$ are the generators corresponding to the two spheres, then the intersection number $a \cdot b$ is ± 1. The hypothesis $f_1(S^m) \subset SO_m$ implies that the normal bundle of the first $(m+1)$-sphere in W has a cross-section. Hence the self-intersection number $a \cdot a$ is zero. Similarly $b \cdot b = 0$. It follows that W has index zero.

The computation of Pontrjagin classes for W proceeds as before. Thus

$$p_r(W) = \pm\, p_r(f_1) j\alpha \pm p_r(f_2) j\beta.$$

However the Pontrjagin number $(j^{-1}p_r)^2[W]$ is now equal to $\pm\, 2p_r(f_1)p_r(f_2)$. On the other hand, the coefficient of $p_r p_r$ in L_{2r} is equal[5] to $\frac{1}{2}(s_r s_r - s_{2r})$. Thus the factor of $\frac{1}{2}$ cancels the 2, so that

$$\lambda(M(f_1 f_2)) \equiv \pm\, p_r(f_1) p_r(f_2) s_r s_r / s_{2r}$$

as before.

In order to make use of Lemmas 3, 4 it is necessary to know what integers $p_r(f)$ can occur.

THEOREM OF BOTT [2], [3]. *In the stable range* $q \geqq 4r$ *the Pontrjagin homomorphism*

$$p_r : \pi_{4r-1}(SO_q) \to Z$$

has image generated by

$$(2r-1)! \text{ if } r \text{ is even}$$

$$2(2r-1)! \text{ if } r \text{ is odd.}$$

For smaller values of q this result can be augmented as follows.

LEMMA 5. *If* $q \leqq 2r$ *then the homomorphism* p_r *is zero. If* $q > 2r$ *then* p_r *is non-zero. In fact there exists an element*

$$(f) \in \pi_{4r-1}(SO_q)$$

such that the prime factors of $p_r(f)$ *are all less than* $2r$.

Proof by descending induction on q. Suppose that the assertion has been proved for $q+1$, and that $q > 2r$. In the exact sequence

$$\pi_{4r-1}(SO_q) \to \pi_{4r-1}(SO_{q+1}) \to \pi_{4r-1}(S^q),$$

the third group is stable. According to Serre [9] a prime π can divide the order of this group only if $2\pi - 3$ is less than or equal to the difference $4r - q - 1$. The inequalities $2\pi - 3 \leqq 4r - q - 1$, $q > 2r$, yield $\pi \leqq r$. Thus any element of $\pi_{4r-1}(SO_{q+1})$, after being multiplied by primes less than or equal to r, can be lifted back to $\pi_{4r-1}(SO_q)$. This completes the induction.

If $q < 2r$, then the Pontrjagin class p_r of any SO_q-bundle is zero. If $q = 2r$, then $p_r(\xi^{2r})$ is the square of the Euler class of ξ^{2r}. (See Borel and Serre [1].) Since our base space is S^{4r}, this implies that $p_r(\xi^{2r}) = 0$; which completes the proof of Lemma 5.

Combining Lemmas 3, 4, 5 this proves:

THEOREM 1. *Suppose that* r *is an integer satisfying*

$$k/3 < r \leqq k/2.$$

Then there exists a differentiable manifold M *homeomorphic to* S^{4k-1} *for which* $\lambda(M)$ *is congruent* modulo 1 *to* $s_r s_{k-r}/s_k$ *times some integer with prime factors all less than* $2(k-r)$.

The proof is straightforward. (The inequality $k/3 < r$ guarantees the existence of a map $f_2 : S^{4(k-r)-1} \to SO_{4r}$ such that $p_{k-r}(f_2) \neq 0$.)

Note. Given k, the inequality $k/3 < r \leqq k/2$ has a solution r providing that $k = 2$ or $k \geqq 4$. It has no solution for $k = 1$ or 3.

THEOREM 2. *There exist at least 7 distinct differentiable structures on* S^7*; at least:*

127 on the 15-sphere,
73 on the 19-sphere,
$23 \cdot 89 \cdot 691$ on the 23-sphere,
8191 on the 27-sphere, and at least
$31 \cdot 151 \cdot 3617$ on the 31-sphere.

Proof. These results follow immediately from Theorem 1. As an example, for $k = 5$, taking $r = 2$, we have

$$s_2 s_3 / s_5 = 341/365.$$

Cancelling all prime factors less than 6 from the denominator, this leaves 73. But if M is a 19-manifold such that the denominator of $\lambda(M)$ is 73, then the first 73 manifolds

$$S^{19}, M, M \# M, M \# M \# M, \cdots$$

must be pairwise distinct. (Alternatively, if the homotopy class (f_1) is replaced by $q(f_1)$, $0 \leqq q < 73$, then we obtain 73 different values for the invariant λ.) Each one represent a possible differentiable structure for the 19-sphere.

In conclusion, here are two unsolved problems.

Problem 1. Does Theorem 1 imply the existence of non-standard differentiable structures on S^{4k-1} for all $k \geqq 4$? I have checked this only for k up to 14.

Problem 2. Is the invariant $\lambda(M^{4k-1})$ of a homotopy sphere always a multiple of the invariant

$$\lambda({M_0}^{4k-1}) \equiv 8/s_k ?$$

This question is of interest since, for any manifold M^{4k-1} which bounds a parallelizable manifold W, we have

$$\lambda(M^{4k-1}) \equiv I(W)/s_k \pmod 1,$$

and it can be shown that $I(W)$ is a multiple of 8. (Compare [7].)

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