The Self-Linking Number of a Closed Space Curve

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Gauss gave an integral formula for the linking number of two disjoint closed space curves. By studying the limiting behavior of this formula as one curve approaches the other, G. Călugăreanu [5, 6, 7] discovered an integer invariant for a *single* simple closed space curve. We give here a new, clearer, and much simplified treatment of his results on this invariant.

1. Let us consider two closed oriented curves in ordinary (real) space $f_1 : C_1 \to E^3$ and $f_2 : C_2 \to E^3$, where C_1, C_2 denote distinct circles and f_1, f_2 are differentiable of class C^1. We assume that $f_1(C_1)$ and $f_2(C_2)$ are disjoint loci. We may regard C_1 as the boundary of a disc D, which we assume oriented compatibly with C_1. By the Thom Transversality Theorem f_1 extends to a C^1 map $f_1 : D \to E^3$, the singularities of which are of dimension 1 or less in D, and so that the image of the singularities does not meet $f_2(C_2)$. We assume that E^3 is oriented. The *linking number* of the two curves, $L(f_1, f_2)$, is defined to be the intersection number $f_1(D) \cdot f_2(C_2)$. Note that this number is independent of the extended map $f_1 : D \to E^3$; for if another such map f_1' were chosen, then since $f_1'(D) - f_1(D)$ is homologous to zero, we have

$$0 = [f_1'(D) - f_1(D)] \cdot f_2(C_2) = f_1'(D) \cdot f_2(C_2) - f_1(D) \cdot f_2(C_2).$$

In order to give an integral formula for the linking number it will be convenient to assume that $f_2(C_2)$ intersects $f_1(D)$ transversally, and hence in a finite number of points $x_1 = f_1(d_1) = f_2(c_1), \cdots, x_N = f_1(d_N) = f_2(c_N)$. (This can always be achieved by a small alteration of the map f_1 on the interior of D.) To each point $(x, y) \in D \times C_2$ other than $(d_1, c_1), \cdots, (d_N, c_N)$ we associate the unit vector $e_1(x, y)$ directed from $f_1(x)$ to $f_2(y)$. We surround each (d_i, c_i) by a sphere Σ_i in $D \times C_2$ which contains no other (d_j, c_j) and which does not meet the boundary of $D \times C_2$, that is $C_1 \times C_2$. Let dS denote the pull-back of the element of area on the unit sphere S^2 under the map e_1. Note that $d(dS) = 0$ since dS is the pull-back of a 2-form defined on a 2-dimensional manifold. We

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orient $D \times C_2$ in the canonical way. By Stokes' Theorem we have

$$\int_{C_1 \times C_2} dS = -\sum_i \int_{\Sigma_i} dS.$$

But the j-th integral on the right gives -4π times the intersection number of $f_1(D)$ and $f_2(C_2)$ at (d_j, c_j) [1]. This proves the *Formula of Gauss*:

$$L(f_1, f_2) = \frac{1}{4\pi} \int_{C_1 \times C_2} dS.$$

This integral is called the *Gauss Integral*. Let us note that if C_1 and C_2 are interchanged then the orientation of the product is reversed and e_1, and hence the integrand, is just reversed in sign. Consequently the integral is unchanged and we have $L(f_1, f_2) = L(f_2, f_1)$. We note also that $L(f_1, f_2)$ is just the degree of the map $e_1 : C_1 \times C_2 \to S^2$. And finally the Gauss Integral does not involve D, which shows again that $L(f_1, f_2)$ is independent of the extension $f_1 : D \to E^3$.

The Formula of Gauss enables us to study the behavior of the linking number under deformations. First, we observe that if f_1 and f_2 are smoothly deformed in such a way that the curves never meet, then the integral changes in a continuous fashion but the linking number remains integer-valued. Consequently the linking number must remain constant. However, if the curves are deformed in such a way that an arc of one passes through an arc of another then the intersection number $f_1(D) \cdot f_2(C_2)$ will change by ± 1 (a circumstance which might be used to define the passing-through), and hence the linking number will change by that amount. This implies that the Gauss Integral jumps by $\pm 4\pi$. We can derive some analytic information from this. Let f_{1t}, f_{2t}, $0 \leqq t \leqq 1$, denote smooth deformations of f_1 and f_2 respectively. (For fixed t we assume that f_{1t} and f_{2t} are closed curves of class C^1.) We assume that in the deformation only a single passing-through occurs. Analytically this may be expressed by saying that there are points $x_0 \in C_1$, $x_1 \in C_2$, $t_0 \in [0, 1]$ so that if $f_{1t}(x) = f_{2t}(y)$, then $t = t_0$, $x = x_0$, and $y = y_0$, and that a passing through actually occurs at $t = t_0$. We may surround $(x_0, y_0) \in C_1 \times C_2$ by an arbitrarily small square Q. We can set

$$4\pi L(f_{1t}, f_{2t}) = I_t = I_{1t} + I_{2t},$$

where

$$I_{1t} = \int_Q dS \quad \text{and} \quad I_{2t} = \int_{C_1 \times C_2 - Q} dS.$$

Now I_{2t} varies continuously in t throughout, so that as t goes through t_0 the jump of I_t must occur entirely in I_{1t}.

We shall now reinterpret the Formula of Gauss to give another way of finding the linking number. Locally on $C_1 \times C_2$ let us associate smoothly with every point unit vectors e_2, e_3, which are perpendicular to one another and to e_1, and so that $e_1 e_2 e_3$ is a right-handed frame in E^3. Let $\omega_{ij} = de_i \cdot e_j$. Then since

$e_i \cdot e_j = \delta_{ij}$, we find by differentiation that $\omega_{ij} + \omega_{ji} = 0$. By expanding $d\,de_i = 0$, we obtain the equations of structure:

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} .$$

It is elementary that

$$dS = \omega_{12} \wedge \omega_{13} = d\omega_{32} .$$

This shows that if $e_2 e_3$ could be chosen globally on $C_1 \times C_2$ in a smooth fashion to complement e_1, then dS would be exact, and hence $L(f_1, f_2) = 0$.

There is, however, a natural choice of frames in general. Let us assume that f_1 is differentiable of class C^2 and free from cusps. For each $(x, y) \in C_1 \times C_2$ at which the direction $e_1(x, y)$ and the positive tangent vector to C_1 at x, $f_1'(x)$, are in general position, we consider the plane spanned by $e_1(x, y)$ and $f_1'(x)$ with orientation $e_1 f_1'$. Let e_2 be a unit vector in this plane normal to e_1 and so that $e_1 e_2$ agrees with the given orientation. The third leg of the frame is completely determined: $e_3 = e_1 \times e_2$. Where this construction becomes indeterminate, that is, at a point $(x, y) \in C_1 \times C_2$ such that the tangent line to f_1 at x contains $f_2(y)$, we shall call (x, y) a *cross tangent of* f_1 *with respect to* f_2.

Now the totality of tangent lines to f_1 forms a developable surface, called the tangent developable of f_1. Let us now assume that the curvature of f_1 never vanishes. This will guarantee that the tangent developable is an immersed surface everywhere except at $f_1(C_1)$ (which is called the edge of regression). Let us also assume that the curve f_2 crosses the tangent developable of f_1 transversally and note that a point where such a crossing occurs is just a cross tangent of f_1 with respect to f_2. It follows that the cross tangents are finite in number, say $(x_1, y_1), \cdots, (x_M, y_M)$. We surround each cross tangent by a box of width ϵ, $B_{i\epsilon} = \{(x, y) \in C_1 \times C_2 \mid |x - x_i| \leqq \epsilon, |y - y_i| \leqq \epsilon\}$ with ϵ taken small enough that $B_{i\epsilon}$ contains no other (x_j, y_j).

Let $A_{i\epsilon}$ denote the boundary of $B_{i\epsilon}$. On $C_1 \times C_2 - \cup B_{i\epsilon}$ the frames described above are well-defined, and we may apply Stokes' Theorem:

$$\int_{C_1 \times C_2} dS = \lim_{\epsilon \to 0} \sum_i \int_{A_{i\epsilon}} \omega_{32} . \tag{1}$$

We assert that in the limit the integrals on the right are integral multiples of 2π. To prove this we apply the operation of *directed dilatation* [4] to each $(x_i, y_i) \in C_1 \times C_2$; that is, we replace each (x_i, y_i) with the circle of oriented tangent directions Σ_i to $C_1 \times C_2$ at (x_i, y_i). In this way we obtain a manifold with boundary consisting of $\Sigma_1, \cdots, \Sigma_M$. We assert that the frames defined on $C_1 \times C_2 - \{(x_1, y_1), \cdots, (x_M, y_M)\}$ extend smoothly to the boundary. To prove this we take power-series expansions of f_1 and f_2 in neighborhoods of x_i and y_i respectively. We take the origin of E^3 at $f_1(x_i)$ and we let s_1, s_2 denote arc length on f_1 and f_2 respectively, so taken that $s_1(x_i) = s_2(y_i) = 0$. Let $e_{10} = \pm e_1(x_i, y_i)$, e_{20}, e_{30} denote the tangent, principal normal and binormal, respectively, of f_1 at x_i, and let t_0 denote the positive tangent to f_2 at y_i. Then

$$f_1(s_1) = s_1 e_{10} + \tfrac{1}{2}\kappa s_1^2 e_{20} + O(s_1^3)$$
$$f_2(s_2) = f_2(0) + s_2 t_0 + O(s_2^2).$$

But $f_2(0) = \lambda e_{10}$ for some λ, and we may write $t_0 = a_1 e_{10} + a_2 e_{20} + a_3 e_{30}$. We have assumed that f_2 crosses the tangent developable of f_1 transversally, and since the tangent space to this developable along the tangent line to f_1 at x_i is spanned by e_{10} and e_{20}, we must have $a_3 \neq 0$. Now $e_1(s_1, s_2)$ is proportional to $f_2(s_2) - f_1(s_1)$ and $e_3(s_1, s_2)$ is proportional to

$$[f_2(s_2) - f_1(s_1)] \times f_1'(s_1) = a_3 s_2 e_{20} + (\kappa\lambda s_1 - s_2 a_2)e_{30} + \text{second order terms.}$$

This shows that the direction of e_3 is dominated by first-order terms so that it extends continuously to the boundary Σ_i. The leg e_1 is well-defined and constant on Σ_i; and $e_2 = e_3 \times e_1$. Clearly

$$\lim_{\epsilon \to 0} \int_{A_{i\epsilon}} \omega_{32} = \int_{\Sigma_i} \omega_{32} .$$

But since e_1 is constant on Σ_i, $\omega_{32} = \pm d\varphi$, where $\varphi = \angle(e_3, e_{30})$. Consequently the integral on the right is an integral multiple of 2π, which proves the previous assertion. We call this integer the *index* of the cross tangent (x_i, y_i). From Formula (1) we now obtain our first theorem: *$L(f_1, f_2)$ is equal to half the sum of the indices of the cross tangents of f_1 with respect to f_2.*

As we have already observed, $L(f_1, f_2) = L(f_2, f_1)$. This gives our second theorem: *the sum of the indices of the cross tangents of f_1 with respect to f_2 equals the sum of the indices of the cross tangents of f_2 with respect to f_1.*

In obtaining our first theorem we assumed that f_2 crosses the tangent developable of f_1 transversally, and in obtaining the second we assume moreover that f_1 crosses the tangent developable of f_2 transversally. However, the results are true without these assumptions, provided that the cross tangents are counted according to the rules of the art of geometrical counting. Moreover the general position can always be achieved by a small deformation of f_2. Finally, the index of a cross tangent is stable under deformations which preserve the general position. We may deform therefore to some standard form, say f_1 a circle in a neighborhood of x_i, and f_2 a line in a neighborhood of y_i. As one may now check, the index of the cross tangent is just the local intersection number of f_2 with the tangent developable of f_1, where this developable is oriented in the following fashion: in each tangent space, which is spanned by the tangent vector t and the principal normal at some point of the curve f_1, we take nt as a right handed frame, where n is the principal normal on the concave side of f_1. Since the intersection number is also invariant under the previous deformation we may assert without qualification that *the index of a cross tangent of f_1 with respect to f_2 is just the local intersection number of f_2 and the tangent developable of f_1 canonically oriented.*

Our first theorem may be approached more directly as follows. Let us observe that the tangent developable of f_1 divides into two parts, the part swept out by the forward, or positive, tangent rays, and that swept out by the backward, or negative, tangent rays. We denote these parts by T_f and T_b respectively. It is easily checked that the orientations of T_f and T_b agree with the orientation of C_1 ; let us recall that we have spanned $f_1(C_1)$ by a disc D the orientation of which also agrees with that of C_1. Consequently $T_f - D$ and $T_b - D$ are both homologous to zero. Therefore

$$0 = (T_b - D) \cdot f_2(C_2) = T_b \cdot f_2(C_2) - L(f_1, f_2)$$

and

$$0 = (T_f - D) \cdot f_2(C_2) = T_f \cdot f_2(C_2) - L(f_1, f_2).$$

We call a cross tangent of f_1 with respect to f_2 a *forward* cross tangent if f_2 meets the forward tangent ray, and a backward cross tangent otherwise. This gives our third theorem: *the sum of the indices of the forward cross tangents equals the sum of the indices of the backward cross tangents and both sums are equal to* $L(f_1, f_2)$.

Our theory may be generalized in the following way. Let us consider the ruled surface swept out by the principal normal lines, or by the binormal lines. We may define forward and backward cross normals and cross binormals of f_1 with respect to f_2 in analogy with our definition of forward and backward cross tangents. Our proofs go over almost exactly and we obtain our fourth theorem: *the sums of the indices (suitably defined) of the forward cross normals, backward cross normals, forward cross binormals, backward cross binormals, are each equal to the linking number* $L(f_1, f_2)$.

2. The definition we have given of the linking number of two space curves (which is in fact the standard one) requires that the two curves do not meet, for otherwise the intersection numbers used in the definition are indeterminate. It is therefore useless for defining a "self-linking number" of a single curve with respect to itself. Another candidate for such a definition is the Gauss Integral. In fact if $f: C \to E^3$ is a simple closed curve differentiable of class C^1, we assign to each $(x, y) \in C \times C$, $(x \neq y)$, the unit vector $e_1(x, y)$ oriented from $f(x)$ to $f(y)$. Let dS denote the pull-back of the element of area on the unit sphere under this map. We call

$$\int_{C \times C} dS$$

the *Gauss Integral for f*. However, as we shall see before long this integral may assume a continuum of values, so that it does not define an integral "self-linking number". But our first theorem offers a point of departure. We consider a closed curve $f: C \to E^3$ which is differentiable of class C^3, which is free from cusps and double points (i.e. is embedded) and which has the property that the (second-order) curvature never vanishes. We assume C oriented. We define the *self-*

linking number of f, SL, to be half the sum of the indices of the cross tangents of f with respect to itself. It is easy to see that this number is independent of the orientation of C. Note that the discussion of the local properties of cross tangents already given applies in the new context.

Let us give an integral formula for SL. Let $S(C)$ denote the space of abstract secant directions of the circle C. (This space was constructed in a canonical fashion in [4]; it has been studied by Whitney, Lashof and Smale, *inter al.*) $S(C)$ is a differentiable manifold with boundary; the interior consists of $C \times C - \Delta$ where $\Delta = \{(x, x)\} \subset C \times C$, and the boundary is the unit tangent bundle of C. It is constructed by replacing Δ in $C \times C$ with its bundle of oriented normal directions, that is to say, by cutting $C \times C$ along Δ. The result may be represented graphically in the following fashion. We regard $C \times C$ as the Cartesian plane modulo points of the form (m, n), m, n integers. If we take the parallelogram $0 \leqq x \leqq 1$, $0 \leqq y - x \leqq 1$ and identify the vertical sides by $(0, y) \equiv (1, 1 + y)$ we obtain $S(C)$ (Figure 1). To each $(x, y) \in C \times C - \Delta$ we associate the unit vector $e_1(x, y)$ in E^3 directed from $f(x)$ to $f(y)$. To each $t \in \partial S(C)$ we associate $e_1(t) = t$, regarded as a unit vector in E^3. This map $e_1 : S(C) \rightarrow S^2$ is differentiable, as is shown in [4]. To each $(x, y) \in C \times C - \Delta$ such that the tangent line to f at x does not pass through $f(y)$ we associate $e_2(x, y)$, the unit vector in the plane spanned by the tangent line to f at x and the secant line $f(x)f(y)$, perpendicular to e_1 and so oriented that $e_1 e_2$ agrees with the orientation $e_1 f'(x)$, where f' is the positive tangent vector to the curve at x. The vector function e_2 also extends smoothly to the boundary of $S(C)$

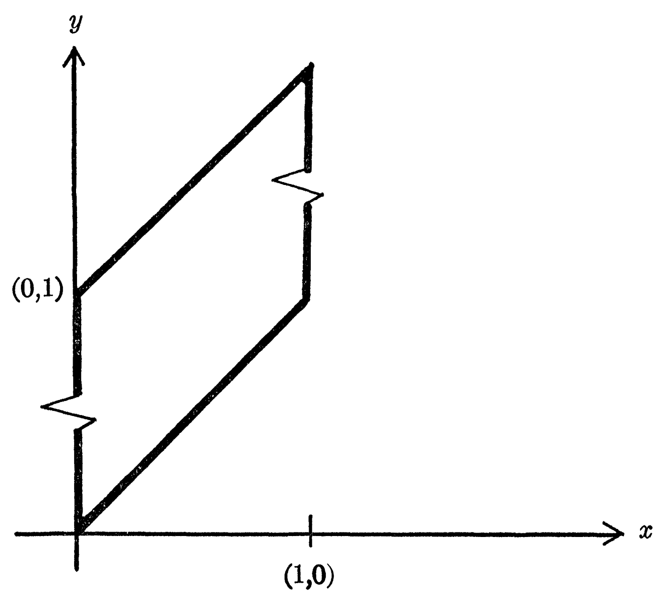


FIGURE 1.

(as is shown in [3]), and gives there the principal normal of f lying on the concave side of the boundary component consisting of the negative tangents and the principal normal lying on the convex side as a positive tangent is approached. Note that e_2 is undefined at the cross tangents. We let $e_3 = e_1 \times e_2$. Let us take the canonical orientation on $C \times C$; this induces an orientation on $S(C)$. Let us now assume that $f(C)$ crosses its tangent developable transversally at the cross tangents. This guarantees that the cross tangents are finite in number, $(x_1, y_1), \cdots, (x_M, y_M)$. We surround each by a box $B_{i\epsilon}$ of width ϵ in $C \times C - \Delta \subset S(C)$, and apply Stokes' Theorem.

$$\int_{S(C)} \omega_{12} \wedge \omega_{13} = \int_{\partial S(C)} \omega_{32} + \lim_{\epsilon \to 0} \sum_i \int_{\partial B_{i\epsilon}} \omega_{32} . \tag{2}$$

The integrals under the limit give the indices of the cross tangents times 2π; consequently the entire sum gives $4\pi SL$. Now $\partial S(C)$ consists of two parts, the forward unit tangent vectors $C(1)$ and the backward tangents $C(2)$. Let $T: C \times C - \Delta \to C \times C - \Delta$ be defined by $T(x, y) = (y, x)$; this map extends smoothly to a map $T: S(C) \to S(C)$. Now T reverses the orientation of $C \times C - \Delta$; hence it reverses the orientation of $\partial S(C)$. It is clear that $e_1 T = -e_1$, and as we have remarked, on the boundary $e_2 T = -e_2 T$. Consequently $e_3 T = e_3$, so that $T^* \omega_{32} = -\omega_{32}$ on the boundary. It follows that

$$\int_{C(1)} \omega_{32} = \int_{C(2)} \omega_{32} .$$

But on $C(1)$ the chosen frames are just the Frenet frames of the curve, so that $\omega_{32} = -\tau \, ds$ where τ is the torsion of f and ds is the positive element of arc on the curve. And the induced orientation on $C(1)$ agrees with that of the curve, as one verifies by inspecting Figure 1. Consequently

$$\int_{C(1)} \omega_{32} = -\int_C \tau \, ds.$$

Using these considerations, and the fact that $\omega_{12} \wedge \omega_{13} = dS$ is the pull-back of the element of area on the unit sphere under the map e_1 , we may rewrite (2) as

$$SL = \frac{1}{4\pi} \int_{C \times C} dS + \frac{1}{2\pi} \int_C \tau \, ds. \tag{3}$$

This remarkable formula constitutes our fifth theorem. The first integral in the formula is called the *Gauss Integral for f*, and the second integral is called *the total torsion.* Observe that both integrals are invariant under reversal of the orientation of the curve.

In establishing this formula we have assumed that the curve crosses its tangent developable transversally. But any curve can be brought into such a state by an arbitrarily small deformation, which will preserve the sum of the indices of the cross tangents, and hence SL, and will disturb the integrals by an arbitrarily small amount. Hence the formula holds for any curve satisfying our original hypotheses.

We have defined SL to be half of an integer; it remains to show that SL itself is an integer. We shall give three proofs, and the first is as follows. We deform the curve f by moving it up the binormal at each point a distance d to obtain a new curve f_d. If d is sufficiently small, this may be done without passing the new curve through the original curve or through itself, and we take d to be that small. This operation moves the curve off of its original tangent developable. The sum of indices of the cross tangents of f with respect to itself equals the sum of the indices of the cross tangents of f with respect to f_d. Consequently $SL = L(f, f_d)$ which constitutes our sixth theorem and proves that SL is an integer. Note that the forwardness of cross tangents is preserved under the operation so that by our second theorem we conclude that *the sum of the indices of the forward cross tangents of f with respect to itself equals the sum of the indices of the backward cross tangents of f with respect to itself, and both are equal to SL.* This is our seventh theorem.

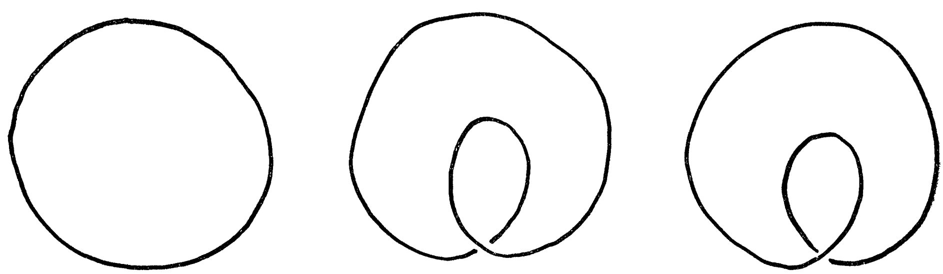
Our second proof that SL is an integer is by homology theory. We identify the special orthogonal group, $SO(3)$, with all right-handed frames $e_1e_2e_3$ in E^3. Recall that the fundamental group of $SO(3)$ is the group of order 2, Z_2, and that a generating curve consists of all $e_1e_2e_3$ with e_1 fixed. Also the map $T : SO(3) \to SO(3)$ defined by $T(e_1e_2e_3) = (-e_1)(-e_2)e_3$ induces the identity map on the fundamental group. We may identify the first homology group of $SO(3)$ with coefficients in Z_2, $H_1(SO(3); Z_2)$, with the fundamental group. We now assume that our curve f has been deformed slightly so that it crosses its tangent developable transversally and hence in a finite number of points. We assume that the cross tangents have been blown up in $S(C)$, so that $S(C)$ is transformed into a new manifold $S^*(C)$, the boundary of which consists of $C(1)$, $C(2)$ as before and circles Σ_i. The frames defined on $S(C)$ except at the cross tangents extend, as we have shown, to the Σ_i. We thus have a map $F : S^*(C) \to SO(3)$. Now $F(C(1)) + F(C(2))$ is homologous to zero in $SO(3)$ since one curve differs from the other by the map T, hence are homologous, and the homology group has order 2. But $C(1) + C(2) + \Sigma_1 + \Sigma_2 + \cdots$ is homologous to zero in $S^*(C)$ since it forms the boundary. Hence $F(\Sigma_1) + F(\Sigma_2) + \cdots$ is homologous to zero in $SO(3)$. And each $F(\Sigma_i)$ gives the generator of $H_1(SO(3); Z_2)$. Hence the Σ_i must be even in number. But the cross tangent corresponding to Σ_i has index ± 1. Hence the sum of the indices is even, which proves that SL is an integer. Our third proof that SL is an integer will be given in the next §.

In proving (3) we could have chosen our frames so that e_2 is in the plane spanned by e_1 and the principal normal at x, or in the plane spanned by e_1 and the binormal at x. The frames become undefined when the normal (or binormal) lines cross the curve again. As with the cross tangents we may assign indices to the cross normals or cross binormals. Following the proof of (3) we obtain our eighth theorem: *SL equals half the sum of the indices of the cross normals,* *and also equals half the sum of the indices of the cross binormals.* Furthermore the sum of the indices of the forward cross normals equals the sum of the indices of the backward cross normals, and similarly for cross binormals.

3. We shall next investigate the behavior of SL under deformations. Let us begin with some definitions. By a *smooth regular deformation* of closed curves we mean a thrice continuously differentiable map $H : C \times I \to E^3$, where C is an abstract circle and $I = [0, 1]$ is the closed unit interval of real numbers, such that for fixed t, $f_t(x) = H(x, t)$ is a closed immersed curve. We call such an H a *non-degenerate deformation* if each f_t is non-degenerate, *i.e.* has nowhere-vanishing curvature. We call a smooth regular deformation an *isotopy* if each f_t is embedded (*i.e.* free from double points).

It is well-known that the total torsion of a closed space curve may assume a continuum of values; in fact if a curve is non-degenerately deformed into a plane curve the total torsion goes to zero. However it is a striking consequence of (3) that the sum on the right is an integer. (It follows from these statements that the Gauss Integral of a single closed space curve may take a continuum of values.) Let us now observe that under a non-degenerate deformation the Frenet frames vary smoothly, which implies that the total torsion varies continuously. And under an isotopy the Gauss Integral varies continuously. Since SL remains integral, (3) implies that *SL is invariant under a non-degenerate isotopy.* This is our ninth theorem. (I find that this helps me understand the difficulties of untangling coiled telephone cords.)

Let us next consider some examples. Any plane convex curve has no cross tangents, so that $SL = 0$. Figure 2.b depicts a curve with $SL = 1$, and Figure 2.c depicts a curve with $SL = -1$. Now curve 2.b may be deformed non-degenerately into curve 2.c by passing it through itself at the apparent crossing point in the obvious way. Since the total torsion varies continuously in a non-degenerate deformation, the Gauss Integral must jump by 8π in this self-passage. Now if an arbitrary curve is passed through itself, the deformation may be changed so that locally the self-passage has the form of that just considered. Hence the Gauss Integral must jump by 8π in an *arbitrary* self passage of an



(a) (b) (c)

FIGURE 2.

arbitrary curve. Hence *in a self-passage of a curve during a non-degenerate homotopy SL changes by* 2. This is our tenth theorem.

Now curve 2.b may be isotopically deformed into a circle by "pulling out the kink", a procedure familiar to any one who has handled fine copper wire, but which we shall not attempt to define mathematically. In this operation the Gauss Integral varies continuously, but the total torsion jumps by 2π. The jump occurs in a neighborhood of the kink so it is local. (This might be used to define the operation precisely.) Thus we find that *under an isotopy in which a kink is pulled out SL changes by* 1. This is our eleventh theorem. I think it indicates that the theory of SL belongs not to topology, nor even to differential topology properly speaking, but to differential geometry. We next state a

Theorem of Feldman [2]. *Any non-degenerate closed space curve may be non-degenerately deformed into either of curves* 2.a *or* 2.b. *Neither of these can be non-degenerately deformed into the other.*

Since during a non-degenerate deformation SL changes by twos its parity is preserved. It follows that *a closed space curve may be non-degenerately deformed into curve* 2.a *if and only if SL is even*; *it may be non-degenerately deformed into curve* 2.b *if and only if SL is odd.* This is our twelfth theorem. It also follows that SL is an integer and this is our third proof of this fact.

We can say more: *if SL is even, a non-degenerate deformation of a curve to curve* 2.a *requires at least* $\frac{1}{2}$ $|SL|$ *self-passage; and if SL is odd, a non-degenerate deformation of a curve to curve* 2.b *requires at least* $\frac{1}{2}(|SL| - 1)$ *self-passages.* (This is our thirteenth theorem). And *an arbitrary regular deformation of a curve to a circle requires at least s self-passages and k "kink pullings", so that* $2s + k \geqq |SL|$. (This is our fourteenth theorem.)

4. Sometimes it may be tedious to locate and count the cross tangents in order to determine the self-linking number of a curve. There is another method, however, which permits it to be found very easily in certain cases. Let us introduce coordinates (x, y, z) in E^3 and consider the transformation S_t of E^3 defined by $S_t(x, y, z) = (x, y, tz)$. This is affine for $t \neq 0$, so that as t varies from 1 to $c > 0$ S_t gives a non-degenerate isotopy of any non-degenerately embedded space curve. Suppose we have such a curve C and suppose that $S_0(C)$ has nowhere vanishing curvature. As t goes from 1 to 0 the total torsion goes to zero, so that $4\pi SL$ equals the limit value of the Gauss Integral alone. Now to evaluate this it suffices to determine how many times a general point of the sphere is covered by the e_1 map in the limit, and what the signs of the Jacobian are at the inverse images of that point. But it is readily seen that this number is the number of double points of $S_0(C)$, and that these are to be counted in the following fashion. Let t_1 be the positive tangent to the top branch and t_2 the positive tangent to the bottom branch at a double point. Then if t_1t_2 appears to form a right-handed frame we count the double point positively, and if t_1t_2 appears to form a left-handed frame we count negatively. Of course it is not

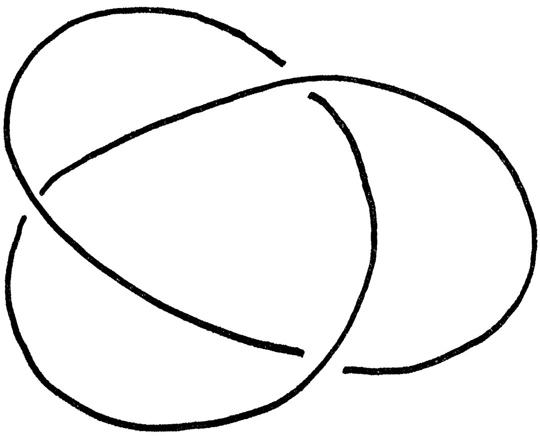


FIGURE 3.

necessary actually to deform the curve into the plane. If one constructs a model of the curve out of wire, stands at some distance from it, and looks at it, then if there are no apparent second-order inflection points, one counts the apparent double points in the fashion indicated and obtains SL. This method is applicable to the curves in Figure 2, as well as the trefoil in Figure 3. Here $SL = -3$.

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