# Polynomial invariants of knots of codimension two*

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In classical knot theory, a useful set of invariants of knot type are the *elementary ideals* in the integral group ring $\Lambda$ of the integers (see [2]). These can be expressed as invariants of the homology of the universal abelian covering of the complement of the knot. In this paper we study the natural generalization of these considerations to higher dimensional knots of codimension two.

Just as the knot polynomials (again see [2]) are a weaker but more tractable set of invariants, so we find more generally that the *rational* homology of the universal abelian covering of the complement of a knot (which, as will be apparent, is the appropriate generalization of the knot polynomials) permits a more complete analysis. We refer the reader to Kervaire [6, Ch. I, II] for a study of the first non-zero homotopy group of the complement and, thereby, an indication of the complexity of the integral homology. One of the more important reasons for this complexity is that the integral homology must be treated as a $\Lambda$-module, and $\Lambda$ is a relatively bad ring, while the rational homology is a module over $\Lambda \otimes Q$, which is a principal ideal domain.

Our aim is to give a complete classification of those modules which can appear as rational homology of the universal abelian covering of the complement of a knot. This will be *almost* accomplished; the sole obstruction being the appearance of an analogue to the Kervaire invariant [7: § 8].

An interesting by-product of this investigation will be the fact that these invariants seem to detect the differentiable structure on the knot in half the cases (i.e.; knots in $(4q-1)$-space. See § 3.2 and 3.3), but ignore it in the other half ($(4q+1)$-space).

Some of the arguments and results of this paper are similar to those of Kervaire [6, Ch. II].

## 1. Definition of the Alexander invariants

1.1. An *n-knot* $K$ will be a smooth oriented $(n-2)$-dimensional submanifold of the $n$-sphere $S^n$, diffeomorphic to $S^{n-2}$. The *complement* of $K$ is the space $X = S^n - K$. The *universal abelian covering* $\tilde{X}$ of $X$ is the covering

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associated with the commutator subgroup of $\pi_1(X)$. By Alexander duality, the group of covering translations of $\tilde{X}$ is infinite cyclic; the orientations of $S^n$ and $K$ determine a preferred generator of this group corresponding to a small oriented circle linking $K$ with linking number 1. Thus the (integral) homology groups of $\tilde{X}$ have a well-defined action by the integers $Z$. If $\Lambda$ is the integral group ring of $Z$, they become $\Lambda$-modules, finitely generated because $X$ has a finite polyhedron as a deformation retract, and $\Lambda$ is noetherian.

Similarly the rational homology of $\tilde{X}$,

$$H_q(\tilde{X}; Q) \approx H_q(\tilde{X}) \otimes_Z Q ,$$

is a finitely-generated module over the rational group ring, $\Gamma \approx \Lambda \otimes_Z Q$, of the integers.

1.2. We shall consider $\Lambda$ as a subring of $\Gamma$, and write the elements of both as Laurent polynomials in a *variable* $t$, the identity of $Z$, with integral or rational coefficients. Thus an element $\lambda$ of $\Gamma$ will also be written $\lambda(t)$. This notation allows us to, e.g. write $\lambda(t^{-1})$ for the image of $\lambda$ under the unique automorphism of $\Gamma$ which maps $t$ to $t^{-1}$, or $\lambda(a)$, $a \in Q$, for the image of $\lambda$ under the unique homomorphism $\Gamma \longrightarrow Q$ mapping $t$ to $a$.

1.3. If $R$ is any integral domain and $\lambda \in R$, we denote the $R$-module of rank 1, with a single generator of order $\lambda$, by $M_R(\lambda)$. Notice that $M_R(\lambda) = M_R(\mu)$ if and only if $\lambda$ and $\mu$ are *associate in* $R$, i.e., for some unit $\alpha \in R$, $\lambda = \alpha\mu$; we write $\lambda \sim \mu$(*in* $R$). If $\lambda$ is a multiple of $\mu$ by an element of $R$, we write $\mu \mid \lambda$ (in $R$).

1.4. An element of $\Lambda$ is *primitive* if its coefficients are relatively prime. It is clear that every associate class in $\Gamma$ contains a primitive element of $\Lambda$; that all such are associate in $\Lambda$ follows from the easily proved:

LEMMA. *Let $\lambda$ and $\mu$ be primitive elements of $\Lambda$. Then $\lambda \mid \mu$ (in $\Gamma$) if and only if $\lambda \mid \mu$ (in $\Lambda$). Consequently, $\lambda \sim \mu$ (in $\Gamma$) if and only if $\lambda \sim \mu$ (in $\Lambda$).*

Thus, given a module $M_\Gamma(\lambda)$, we can require that $\lambda$ be a primitive element of $\Lambda$, and such a $\lambda$ is determined up to associate class in $\Lambda$.

1.5. We will occasionally have to do with an associate class in $\Lambda$ in which an element $\mu$ satisfies $\mu(t) \sim \mu(t^{-1})$ in $\Lambda$, $\mu(1)$ is odd.

LEMMA. *Any such associate class in $\Lambda$ contains a unique element $\lambda$ satisfying $\lambda(t) = \lambda(t^{-1})$ and $\lambda(1) > 0$.*

This is proved by a standard argument (see [2, p. 137]). We have $\mu(t) = \varepsilon t^a \mu(t^{-1})$, where $\varepsilon = \pm 1$ and $a$ is some integer. Since $\mu(1) \neq 0$, we have $\varepsilon = 1$, while, if $a$ were odd, then $\mu(1)$ would have to be even. Now set $\lambda(t) = t^{-a/2}\mu(t)$.

We call the element $\lambda$, whose existence is asserted by the lemma, the *normal* element of the associate class.

1.6. Note that $\Gamma$ is a principal ideal domain. This follows readily from the fact that the ordinary polynomial ring $Q[t]$ is a principal ideal domain, and an argument similar to that of [2, Ch. VII, § 2]. Since $H_q(\tilde{X}; Q)$ is a finitely-generated $\Gamma$-module, there is a finite sequence $\lambda_1^q, \lambda_2^q, \cdots, \lambda_k^q$ of elements of $\Gamma$, satisfying $\lambda_{i+1}^q \mid \lambda_i^q$ in $\Gamma$, and unique up to associate class in $\Gamma$, such that (for a proof, see e.g. [3])

$$H_q(\tilde{X}; Q) \approx \textstyle\sum_{i=1}^{k} M_\Gamma(\lambda_i^q) \ .$$

As pointed out in (1.4) we can choose $\lambda_i^q$ to be a primitive element of $\Lambda$; its associate class in $\Lambda$ is then uniquely determined. With this convention we refer to the set $\{\lambda_i^q\}$, $0 < q < n - 1$, as the Alexander invariants of the knot, where $\lambda_i^q$ is, strictly speaking, an associate class in $\Lambda$.

If $n = 2q + 1$, we also consider the element

$$\Delta = \lambda_1^q \lambda_2^q \cdots \lambda_k^q$$

which is, by the Gauss lemma, a primitive element of $\Lambda$ and well-defined only up to associate class. In the case $n = 3$, $\Delta$ is the Alexander polynomial (see [11, § 2]); when $\tilde{X}$ is $(q - 1)$-connected, $\Delta$ is related to the *torsion* [12] of $\tilde{X}$ (see [6, Lem. III. 11]).

## 2. Some properties of the Alexander invariants

2.1. The aim of this section will be to prove the following theorem.

THEOREM 1. *Let* $\{\lambda_i^q\}$, $0 < q < n - 1$, $0 < i < k$, *be the family of Alexander invariants of an n-knot,* $n \geqq 3$. *Then, the following properties must be satisfied*:

(a) $\lambda_{i+1}^q \mid \lambda_i^q$ in $\Lambda$,

(b) $\lambda_i^q(1) = \pm 1$,

(c) $\lambda_i^q(t) \sim \lambda_i^{n-q-1}(t^{-1})$ *in* $\Lambda$,

(d) *if* $n = 2q + 1$, *q even, and* $\Delta$, *as defined in* (1.6), *is assumed normal, by* (b), (c), *and* (1.5), *then* $\Delta(-1)$ *is an odd square.*

REMARK. This theorem and proof are valid for knots which are *homotopy* spheres.

For $n = 3$, the theorem is well-known (see [2], [13]). Our proof will use a generalization of the technique of Seifert [13].

2.2. We begin with the notions of presentation of a module and elementary ideals of a finitely generated module. We refer the reader to [16, p. 117] for details.

Let $R$ be a ring and $A$ an $R$-module. It is well-known that there exists an exact sequence of $R$-modules,

$$F_1 \xrightarrow{d} F_0 \longrightarrow A \longrightarrow 0 ,$$

where $F_1$ and $F_0$ are free $R$-modules. If $\{\alpha_i\}$ and $\{\beta_i\}$ are bases of $F_1$ and $F_0$, respectively, we can write

$$d(\alpha_j) = \textstyle\sum_i \lambda_{ij}\beta_i .$$

The collection $(\lambda_{ij})$ of elements of $R$ forms a matrix (with perhaps an infinite number of rows or columns) which we refer to as a *presentation matrix* of $A$.

If $A$ is finitely-generated, we may insist that $F_0$ be finitely generated and, therefore, only consider presentation matrices with a finite number of rows. Suppose such is the case, and $(\lambda_{ij})$ has $m$ rows. If, in addition, $R$ is commutative, we may consider, for each integer $k \geqq 0$, the $(m-k) \times (m-k)$ sub-determinants of $(\lambda_{ij})$. We call these the *minors of order* $k$. They generate an ideal in $R$, which depends only on $A$, called the $k^{\text{th}}$ *elementary ideal* of $(\lambda_{ij})$ or $A$, denoted $E_k(A)$. If $k \geqq m$, we define $E_k(A) = R$. It is clear that $E_k(A) \subset E_{k+1}(A)$.

2.3. If $R$ is a principal ideal domain, we choose a generator $\Delta_k(A)$ of $E_k(A)$; $\Delta_k(A)$ is determined up to associate class. We have $\Delta_{k+1}(A) \mid \Delta_k(A)$.

By the structure theorem for finitely-generated $R$-modules (see [3]), $A$ has a diagonal presentation matrix with entries $\lambda_1, \cdots, \lambda_r$ satisfying $\lambda_{i+1} \mid \lambda_i$. It is readily checked that $\Delta_k = \Delta_k(A)$ may be chosen so that

$$\Delta_k = \begin{cases} \lambda_{k+1}\lambda_{k+2} \cdots \lambda_r & \text{for } k < r , \\ 1 & \text{for } k \geqq r . \end{cases}$$

We see immediately that the sequence $\{\Delta_k\}$ is a faithful invariant of the isomorphism class of $A$.

2.4. Given an $n$-knot $K$ with complement $X$, we will obtain a presentation for $H_q(\tilde{X}; Q)$ with the aid of a submanifold $V$ of $S^n$ bounded by $K$ (see [1], [4], [6], and [13], for related discussion).

According to [6, Lem. II. 10] or [10, Lem. 2] $V$ exists. Let $Y$ be the manifold (with corner) obtained from $S^n$ by *cutting* along $V$; then $\partial Y$ consists of two copies of $V$, $V_1$ and $V_2$, identified along their boundaries. Note the isomorphism $H_q(S^n - V) \approx H_q(Y)$ by inclusion. $\tilde{X}$ may be constructed from $Y$ in the following way. Let $(Y_i, V_1^i, V_2^i)$, $-\infty < i < \infty$, be an infinite number of copies of $(Y - K, V_1 - K, V_2 - K)$. Then $\tilde{X}$ is obtained as a quotient space of the disjoint union of the $Y_i$ by identifying $V_2^i$ and $V_1^{i+1}$ for every $i$. A generator of the group of covering translations maps $Y_i$ onto $Y_{i+1}$

in the obvious manner. This construction of $\tilde{X}$ permits a convenient application of the Mayer-Vietoris sequence. Define, for $\varepsilon = 0, 1$, $Y^\varepsilon = \bigcup_i Y_{2i+\varepsilon}$; then $(\tilde{X}; Y^0, Y^1)$ is a proper triad, and $Y^0 \cap Y^1 = \bigcup_i V_1^i = \bar{V}$. Note that $\bar{V}$ is invariant under the covering transformations of $\tilde{X}$. Thus

$$H_q(\bar{V}; Q) \approx \sum_i H_q(V_1^i; Q)$$

is a $\Gamma$-module and is, clearly, isomorphic to $H_q(V; Q) \otimes_Q \Gamma$ with $H_q(V_1^i; Q)$ corresponding to $H_q(V; Q) \otimes Qt^i$. Also

$$H_q(Y^0; Q) \oplus H_q(Y^1; Q) \approx \sum_i H_q(Y_i; Q)$$

can be identified with $H_q(Y; Q) \otimes_Q \Gamma$, with $H_q(Y_i; Q)$ corresponding to $H_q(Y, Q) \otimes Qt^i$ (see (1.2)). By the Mayer-Vietoris theorem we have an exact sequence of $\Gamma$-modules:

$$(1) \qquad H_q(V; Q) \otimes_Q \Gamma \xrightarrow{d} H_q(Y; Q) \otimes_Q \Gamma \xrightarrow{e} H_q(\tilde{X}; Q) .$$

From the construction of $\tilde{X}$ and the action of the covering transformations, it follows that $d$ can be described as follows. Let $i_1, i_2\colon V \to Y$ be defined by the identification of $V$ with $V_1$ or $V_2$. If $t$ is the element of $\Gamma$ corresponding to the covering transformation $Y_i \to Y_{i+1}$, then, for $\alpha \in H_q(V; Q)$

$$\begin{aligned} d(\alpha \otimes 1) &= i_{1*}(\alpha) \otimes t - i_{2*}(\alpha) \otimes 1 \\ &= t(i_{1*}(\alpha) \otimes 1) - i_{2*}(\alpha) \otimes 1 . \end{aligned}$$

We will see below that $d$ is a monomorphism for all $q$; by exactness, this implies $e$ is onto for every $q$ and, therefore, we can derive a presentation matrix from (1).

2.5. For any space $S$, let $B_q(S)$ denote the image $H_q(S) \to H_q(S; Q)$. Then $B_q(S)$ is free abelian, and $H_q(S; Q) = B_q(S) \otimes_Z Q$. The elements of $H_q(S; Q)$ which lie in $B_q(S)$ will be called *integral*.

If $V$ is as in (2.4) and $p = n - 1 - q$, there is a homomorphism:

$$L\colon B_q(V) \otimes_Z B_p(S^n - V) \longrightarrow Z ,$$

which is defined by linking numbers (see [14, p. 277]). By Alexander duality, $L$ is a *completely dual pairing*, i.e., given any basis $\alpha_1, \cdots, \alpha_r$ of $B_q(V)$, there is a *dual* basis $\beta_1, \cdots, \beta_r$ of $B_p(S^n - V)$ such that

$$L(\alpha_i \otimes \beta_j) = \delta_{ij} .$$

Note that $L$ extends to a completely dual pairing:

$$L\colon H_q(V; Q) \otimes_Q H_q(S^n - V; Q) \longrightarrow Q .$$

Given $\alpha \in H_q(V; Q)$, $\beta \in H_p(V; Q)$, we have the relation

$$(2) \qquad L(\alpha \otimes i_{1*}(\beta)) = (-1)^{pq+1} L(\beta \otimes i_{2*}(\alpha)) ,$$

where $i_1$, $i_2$ are defined in (2.4) using the isomorphism $H_q(S^n - V) \approx H_q(Y)$. This is an immediate consequence of the definitions and the commutativity ([14, p. 278]) of linking numbers.

Finally we have, for $\alpha \in H_q(V; Q)$, $\beta \in H_p(V; Q)$,

$$L(\alpha \otimes i_{2*}(\beta)) - L(\alpha \otimes i_{1*}(\beta)) = \alpha \cdot \beta \,, \tag{3}$$

where $\alpha \cdot \beta$ is the intersection number in $V$. This follows from the existence of a $(q+1)$-chain $c$ in $S^n$ satisfying

$$\partial c = i_{1*}(\beta) - i_{2*}(\beta)$$
$$c \cap V = \beta \,,$$

where we have confused cycles and homology classes. The chain $c$ is constructed from a cycle $z$ representing $\beta$ by constructing $I \times z$ transversal to $V$ along $1/2 \times z$. The formula follows from a consideration of the intersection $\alpha \cdot c$ in $S^n$.

2.6. For every $q$, choose a basis $\{\alpha_i^q\}$ of $B_q(V)$ and a dual basis $\{\beta_i^p\}$ of $B_p(S^n - V) \approx B_p(Y)$. Then $\{\alpha_i^q \otimes 1\}$ and $\{\beta_i^p \otimes 1\}$ form bases of the free $\Gamma$-modules $H_q(V; Q) \otimes_Q \Gamma$ and $H_p(Y; Q) \otimes_Q \Gamma$. Set

$$i_{2*}(\alpha_j^q) = \sum_i \lambda_{ij}^q \beta_i^q \,,$$
$$i_{1*}(\alpha_j^q) = \sum_i \mu_{ij}^q \beta_i^q \,,$$

Note that $\lambda_{ij}^q$ and $\mu_{ij}^q$ are integers. Thus

$$d(\alpha_j^q \otimes 1) = \sum_i (t\mu_{ij}^q - \lambda_{ij}^q)(\beta_i^q \otimes 1) \,,$$

and we can define $P_q(t)$, a presentation matrix for $H_q(\tilde{X}; Q)$, with entries *in* $\Lambda$ by

$$P_q(t) = (t\mu_{ij}^q - \lambda_{ij}^q) \,.$$

Note that $P_q(t)$ is a square matrix since, by Alexander duality,

$$H_q(S^n - V; Q) \approx H_p(V; Q) \,;$$

and by Poincaré duality, $H_p(V; Q) \approx H_q(V; Q)$, $0 < q < n-1$.

2.7. We now translate (2) and (3) into information about $P_q(t)$. First, we have

$$L(\alpha_j^q \otimes i_{1*}(\alpha_k^p)) = \sum_i \mu_{ik}^p L(\alpha_j^q \otimes \beta_i^p) = \mu_{jk}^p \tag{4}$$

$$L(\alpha_k^p \otimes i_{2*}(\alpha_j^q)) = \sum_i \lambda_{ij}^q L(\alpha_k^p \otimes \beta_i^q) = \lambda_{kj}^q \,. \tag{5}$$

By (2), we have $\mu_{jk}^p = (-1)^{pq+1}\lambda_{kj}^q$. This implies

$$(-1)^{pq} P_q(t) = t P_p(t^{-1})' \,, \tag{6}$$

where $'$ indicates *transpose*.

As a consequence of (3), (4), and (5), we have

$$(7) \qquad \lambda_{ij}^q - \mu_{ij}^q = \alpha_i^p \cdot \alpha_j^q .$$

Thus $P_q(1)$ is, up to sign, the matrix of the intersection pairing

$$B_q(V) \otimes_Z B_p(V) \longrightarrow Z ;$$

since $\partial V$ is a sphere, $P_q(1)$ is unimodular, i.e.,

$$(8) \qquad \det P_q(1) = \pm 1 .$$

As a consequence of (8), $P_q(t)$ is non-singular and, therefore, the homomorphism $d$ in (1) is injective. This implies $e$ is onto, and so $P_q(t)$ is, indeed, a presentation matrix for $H_q(\tilde{X}; Q)$.

2.8. Now $\Delta_i(H_q(\tilde{X}; Q))$ can be chosen as a greatest common divisor $\Delta_i^q$ of the $i^{\text{th}}$ order minors of $P_q(t)$; $\Delta_i^q$ is an element of $\Lambda$. We see that $\Delta_{i+1}^q \mid \Delta_i^q$ in $\Lambda$. By (8), $\Delta_0^q(1) = \det P_q(1) = \pm 1$ and, therefore,

$$\Delta_i^q(1) = \pm 1 .$$

Furthermore, by (6),

$$\Delta_i^q(t) \sim \Delta_i^p(t^{-1}) \qquad \text{in } \Lambda .$$

Now, define $\mu_i^q = \Delta_i^q / \Delta_{i+1}^q$, an element of $\Lambda$. From the above properties of $\Delta_i^q$ we have

$$\mu_i^q(1) = \pm 1$$

and

$$\mu_i^q(t) \sim \mu_i^p(t^{-1}) \qquad \text{in } \Lambda .$$

This implies that $\mu_i^q$ is primitive. Finally, by (2.3) and (1.4), $\mu_i^q \sim \lambda_i^q$ in $\Lambda$. This proves (b) and (c) of Theorem 1.

2.9. Suppose $n = 2q + 1$, $q$ even. Then (6) implies that $P_q(-1)$ is skew-symmetric. Note that

$$\Delta = \Delta_0\big(H_q(\tilde{X}; Q)\big) \sim \det P_q(t) .$$

Thus $\Delta(-1) = \pm \det P_q(-1)$. But the determinant of a skew-symmetric matrix is square (see [5, 9.3]) and, therefore, $|\Delta(-1)|$ is square.

Since $\Delta$ is normal, we have $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$. These properties imply $\Delta(-1) - \Delta(1) = \Delta(-1) - 1$ is a multiple of 4. But this is impossible if $\Delta(-1)$ were the negative of a square; or, if it were an even square. This proves (d) of Theorem 1.

## 3. Further restriction on $\Delta(-1)$

3.1. When $n = 2q + 1$, $q$ odd, Theorem 1 contains no restriction on $\Delta(-1)$. In this section we will see that, when $q \neq 1$, 3, or 7, it is reasonable to impose the condition $\Delta(-1) \equiv 1 \bmod 8$. A proof of the necessity of this condition, though, will be seen to imply a proof of the *Arf invariant conjecture* (see [7, p. 536]). It is probably equivalent to the *Arf invariant conjecture*.

3.2. For $n = 2q + 1$, $q$ odd, let $K$ be an $n$-knot and $V$ a submanifold of $S^n$ bounded by $K$. We define a function

$$\chi\colon H_q(V) \longrightarrow Z$$

by $\chi(\alpha) = L(\alpha \otimes i_{1*}(\alpha))$. We check immediately from (2-(2), (3)), that

$$\chi(\alpha + \beta) \equiv \chi(\alpha) + \chi(\beta) + \alpha \cdot \beta \qquad \bmod 2 .$$

We can, therefore, form the Arf invariant of $\chi$, $a(\chi)$, in the usual way (see [7, §8]). Choose a symplectic basis $\{\alpha_1, \cdots, \alpha_r; \beta_1, \cdots, \beta_r\}$ of $B_q(V)$; the residue class

$$a(\chi) = \sum_{i=1}^{r} \chi(\alpha_i)\chi(\beta_i) \in Z_2$$

does not depend on the choice of symplectic basis.

3.3. If $v$ is the positive normal field on $V$ in $S^n$, the Kervaire invariant $c(V, v)$ is defined (see [9, §4]). We would expect that $c(V, v) = a(\chi)$, and will prove

PROPOSITION. *If $V$ is $(q-1)$-connected, $c(V, v) = a(\chi)$.*

PROOF. Let $w$ be a normal field to $S^n$ in $D^{n+1}$ and $\mathcal{F} = (w, v)$ so that $(V, \mathcal{F})$ is a framed submanifold of $D^{n+1}$. By [9, 4.4], $c(V, \mathcal{F}) = c(V, v)$. The proposition will be proved if we show that $\chi = \varphi(V, \mathcal{F})$ (see [9, 4.1]), reduced mod 2.

Let $\alpha \in H_q(V)$ be represented by an imbedded sphere $s$. Then $s$ bounds an imbedded disk $d$ in $D^{n+1}$ which meets $S^n$ transversely at $\partial d = s$. The obstruction to extending $v$ to a normal frame on $d$ is an element of $\pi_q(S^q) \approx Z$, and can be identified with $\chi(\alpha)$. But the residue class mod 2 can also be identified with $\varphi(V, \mathcal{F}) \cdot \alpha$ (see e.g. [9, 4.10]).

3.4. In (3.3) we have seen that $a(\chi)$ is related to the differential structure on $K$. We now see that it is also related to the invariant $\Delta$.

PROPOSITION. *If $\Delta$ is normal,* $\Delta(-1) \equiv 1 + 4a(\chi) \mod 8$.

Let us consider the presentation matrix $P_q(t)$ constructed in §2; then $\Delta \sim \det P_q(t)$, $n = 2q + 1$. Since $P_q(t)$ has an even number of rows, say $2r$, because the intersection pairing is a unimodular skew-symmetric bilinear form (see [5, 9.3]), it follows from (2-(6)) that $\det P_q(t) = t^{2r} \det P_q(t^{-1})$. Furthermore, it follows from (2-(7)) that $\det P_q(1) = 1$, since a unimodular skew-symmetric form must have determinant 1 (compare (2.9)). Thus

$$\Delta = t^{-r} \det P_q(t) ,$$

and we want to prove

$$\det P_q(-1) \equiv (-1)^r + 4a(\chi) \qquad \mod 8 .$$

3.5. We will examine the residue class mod 8 of each of the summands in the determinant of $P_q(-1) = (a_{ij})$. Note, by (2-(16)), that $(a_{ij})$ is symmetric. Assume the basis $\alpha_1, \cdots, \alpha_{2r}$ of $B_q(V)$ is symplectic, i.e., $\alpha_i \cdot \alpha_j = \delta_{i+r,j}$ for $i \leqq j$. It follows from (2-(7)) that

(i) $a_{ij} = a_{ji}$ is *even* if $|i - j| \neq r$; we call these terms *non-special*; and

(ii) $a_{ij} = a_{ji}$ is *odd* if $|i - j| = r$; we call these terms *special*.

Clearly any summand of $\det(a_{ij})$ which contains more than two non-special terms is a multiple of 8. On the other hand there is exactly one summand made up entirely of special terms

$$(-1)^r a_{1,r+1} a_{2,r+2} \cdots a_{r,2r} a_{r+1,1} \cdots a_{2r,r} .$$

Since any odd square is $\equiv 1 \mod 8$, this summand is $\equiv (-1)^r \mod 8$.

We now consider the summands containing exactly two non-special terms. It is not hard to see that there are precisely $r^2$ summands of this type; for every pair of integers $1 \leqq p, s \leqq r$, there is a summand

$$S_{p,s} = (-1)^{r-1} a_{p,s} a_{s+r,p+r} \prod_{i \neq p} a_{i,i+r} \prod_{i \neq s} a_{i+r,i} ,$$

where $i$ ranges from 1 to $r$ except for the value indicated. Notice that $S_{p,s} = S_{s,p}$, since $(a_{ij})$ is symmetric. If $p \neq s$, $S_{p,s} + S_{s,p}$ is a multiple of 8, since $S_{p,s}$ is a multiple of 4 (by (i)).

Since there are no summands with an odd number of non-special terms, we have

$$\det(a_{ij}) \equiv (-1)^r + \sum_{p=1}^{r} S_{p,p} \qquad \mod 8$$

By (ii), $S_{pp} = (-1)^{r-1} a_{p,p} a_{p+r,p+r} b^2$, for some odd integer $b$. Thus, since $b^2 \equiv 1 \mod 8$,

$$S_{pp} \equiv (-1)^{r-1} a_{p,p} a_{p+r,p+r} \qquad \mod 8 .$$

Now $a_{p,p} = -(\mu^q_{pp} + \lambda^q_{pp}) = -2\lambda^q_{pp} = -2\chi(\alpha_p)$ by (4) and (5). Therefore,

$$\sum_{p=1}^{r} S_{pp} \equiv (-1)^{r-1} 4 \sum_{p=1}^{r} \chi(\alpha_p)\chi(\alpha_{p+r}) \qquad \mod 8$$

By the definition of $a(\chi)$, this completes the proof of the proposition.

3.6. Since we are dealing only with knots diffeomorphic to the standard sphere, the preceding two propositions, together with the Arf invariant conjecture ([7, p. 536]) might lead us to expect that, when $n = 2q + 1$, $q$ odd, $q \neq 1, 3, 7$, $\Delta$, when normalized, should satisfy the condition $\Delta(-1) \equiv 1 \mod 8$.

## 4. Knots with given Alexander invariants

4.1. The remainder of this paper will be devoted to proving

THEOREM 2. *Let* $\{\lambda_i^q\}$, $0 < q < n-1$, $0 < i < k$, *be a set of primitive elements satisfying* (a)-(d) *of Theorem* 1. *Suppose, in addition, when* $n = 2q+1$, $q$ *odd,* $q \neq 1, 3, 7$, $\Delta$ *(in normal form) satisfies*

(e) $\Delta(-1) \equiv 1 \bmod 8$.

*Then, if* $n \geqq 3$, *there exists an n-knot with* $\{\lambda_i^q\}$ *as its Alexander invariants.*

In the exceptional case $n = 2q+1$, $q$ odd, $q \neq 1, 3, 7$, if condition (e) is removed, it can be proved that the conclusion of the theorem still holds where the knot may be an *exotic* sphere. This can be proved using the techniques of [6, Th. II. 3], and matrix construction similar to those of [13]. Because of the exceptional nature of the proof, further details are, for the present, omitted.

For $n = 3$, Theorem 2 is proved in [11]; we will assume $n > 3$. The case $n = 4$ contains [8, § 6] as a consequence.

We will deal with certain special cases of the theorem and demonstrate, in (4.16), how the general result follows by use of the *connected sum* operation.

4.2. Suppose $n > 2q+1$, and $\lambda \in \Lambda$ satisfies $\lambda(1) = 1$. We construct an $n$-knot with Alexander invariants

$$\lambda_1^q = \lambda\ , \qquad \lambda_1^{n-q-1} = \lambda(t^{-1})\ , \qquad \lambda_i^p = 1 \quad \text{if } (i, p) \neq (1, q) \text{ or } (1, n-q-1)\ .$$

By Theorem 1, it suffices to construct an $n$-knot $K$ such that:

$$H_q(\tilde{X}) \approx M_\Lambda(\lambda)\ , \qquad H_i(\tilde{X}) = 0 \qquad \text{for } 0 < 2i < n,\ i \neq q\ .$$

The construction is similar to one in [6, p. 243]. We also refer the reader to [15] for related results.

4.3. Let $K$ be an imbedded null-isotopic (i.e., bounding a disk) oriented $(n-2)$-sphere in $S^q \times S^{n-q}$. Let $X_0 = S^q \times S^{n-q} - K$, and $\tilde{X}_0$ the infinite cyclic covering of $X_0$ associated with the kernel of the homomorphism $\pi_1(X_0) \to Z$ defined by intersection number with any disk bounded by $K$. It follows readily that $H_q(\tilde{X}_0)$, $H_{n-q}(\tilde{X}_0)$ are free $\Lambda$-modules of rank one, and $H_i(\tilde{X}_0) = 0$ if $i \neq 0, q, n-q$.

Notice that we have a natural isomorphism of $H_q(\tilde{X}_0) \approx \pi_q(\tilde{X}_0)$ with a subgroup of $\pi_q(X_0)$ (base-points are understood) by projection. It follows that any element of $H_q(\tilde{X}_0)$ can be represented by a lift of an imbedded sphere in $X_0$ $(n \geqq 2q)$. Choose a generator (as a $\Lambda$-module) $\alpha$ of $H_q(\tilde{X}_0)$, and let $S$ be an imbedded sphere in $X_0$ with a lift to $\tilde{X}_0$ representing $\lambda \cdot \alpha$. Since $X_0$ is a $\pi$-

manifold and $n > 2q$, there exists an imbedding $\varphi: S^q \times D^{n-q} \to X_0$ with $S = \varphi(S^q \times 0)$, and we can perform the *spherical modification* $\chi(\varphi)$ ([7, p. 513]). This converts the pair $(S^q \times S^{n-q}, X_0)$ to a new pair $(\Sigma, X)$, where $K$ is an imbedded $(n-2)$-sphere in $\Sigma$ and $X = \Sigma - K$.

4.4. With a little care we can insure that $\Sigma$ is diffeomorphic to $S^n$. In fact, the condition $\lambda(1) = 1$ implies that $S$ represents a generator of

$$\pi_q(S^q \times S^{n-q}) \approx Z \ (n - q > q) .$$

Since $n > 2q + 1$, $S$ is isotopic to $S^q \times x_0$, $x_0 \in S^{n-q}$. By the *tubular neighborhood theorem*, we may choose $\varphi$ isotopic to the standard imbedding

$$S^q \times D^{n-q} \longrightarrow S^q \times D_0^{n-q} \subset S^q \times S^{n-q} ,$$

where $D_0^{n-q}$ is a hemisphere of $S^{n-q}$. It follows immediately that $\Sigma$ is diffeomorphic to $S^n$.

4.5. We now compute $H_*(\tilde{X})$, and show it is as desired. Notice that $\varphi$ lifts to a family of imbeddings

$$\varphi_i: S^q \times D^{n-q} \longrightarrow \tilde{X}_0 , \qquad -\infty < i < \infty .$$

If we perform the spherical modifications $\chi(\varphi_i)$, $-\infty < i < \infty$, we obtain an infinite cyclic covering of $X$; by the definition of $\tilde{X}_0$ this is, in fact $\tilde{X}$.

To compute $H_*(\tilde{X})$ we need the preliminary space

$$\tilde{X}_1 = \overline{\tilde{X}_0 - \bigcup_i \varphi_i(S^q \times D^{n-q})} .$$

By excision, we have

$$H_i(\tilde{X}_0, \tilde{X}_1) \approx \begin{cases} \Lambda & i = n - q, n , \\ 0 & i \neq n - q, n . \end{cases}$$

$$H_i(\tilde{X}, \tilde{X}_1) \approx \begin{cases} \Lambda & i = q + 1, n , \\ 0 & i \neq q + 1, n . \end{cases}$$

It follows immediately from the exact homology sequences of $(\tilde{X}_0, \tilde{X}_1)$ and $(\tilde{X}, \tilde{X}_1)$ that, in the range $0 < 2i < n$, $H_i(\tilde{X}) = 0$ except, possibly, for $i = q$ or $q + 1$.

To compute $H_q(\tilde{X})$ consider the diagram

$$\begin{array}{ccccccc} H_{q+1}(\tilde{X}, \tilde{X}_1) & \longrightarrow & H_q(\tilde{X}_1) & \longrightarrow & H_q(\tilde{X}) & \longrightarrow & 0 \\ & & \downarrow \wr & & & & \\ & & H_q(\tilde{X}_0) & & & & \end{array}$$

where the horizontal sequence is exact and the vertical isomorphism is valid because $0 < q < n - q - 1$. $H_{q+1}(\tilde{X}, \tilde{X}_1)$ and $H_q(\tilde{X}_0)$ are free $\Lambda$-modules of rank 1 and the former has a generator represented by a disk whose boundary is, say, $\varphi_0(S^q \times x_0)$ for some $x_0 \in S^{n-q-1}$. But $\varphi_0(S^q \times 0)$ represents $\lambda \cdot \alpha$, for some generator $\alpha$ of $H_q(\tilde{X}_0)$; as a consequence, $H_q(\tilde{X}) \approx M_\Lambda(\lambda)$.

If $n = 2q + 2$, then $2(q + 1) \geqq n$, and our verifications are complete. If $n > 2q + 2$, we must show $H_{q+1}(\tilde{X}) = 0$. But we have an exact sequence

$$H_{q+1}(\tilde{X}_1) \longrightarrow H_{q+1}(\tilde{X}) \longrightarrow H_{q+1}(\tilde{X}, \tilde{X}_1) \longrightarrow H_q(\tilde{X}_1) .$$

Since $n - q > q + 2$, $H_{q+1}(\tilde{X}_1) \approx H_{q+1}(\tilde{X}_0) = 0$, and, since $\lambda \neq 0$, the right-hand homomorphism is injective. By exactness, $H_{q+1}(\tilde{X}) = 0$.

4.6. The case $n = 2q + 1$ will use the following:

LEMMA. *Let $(\lambda_{ij})$ be a square matrix with entries in $\Lambda$ satisfying, for some integer $q > 1$,*

( i ) $\lambda_{ij}(t) = (-1)^{q+1}\lambda_{ji}(t^{-1})$,

(ii) *if $q$ is odd, $q \neq 3, 7$, then $\lambda_{ii}(1)$ is even, and*

(iii) *$(\lambda_{ij}(1))$ is unimodular.*

*Then, there exists a $(2q + 1)$-knot such that $H_i(\tilde{X}) = 0$, $i \neq 0, q$, and $H_q(\tilde{X})$ has $(\lambda_{ij})$ as a presentation matrix.*

This lemma represents the natural extension of the main construction of [11] to higher dimensions.

4.7. Set

$$\lambda_{ij} = \sum_{k=-m}^{m} c_{ijk} t^k \qquad \text{for some } m > 0 .$$

Let $K$ be a null-isotopic oriented $(n - 2)$-sphere imbedded in $S^n$, where $n = 2q + 1$. If $(\lambda_{ij})$ is an $(r \times r)$-matrix, let us imbed in an $n$-ball $B$ in $X_0 = S^n - K$ a family $\{S_{i,k}\}$, $1 \leqq i \leqq r$, $0 \leqq k \leqq m$, of disjoint null-isotopic oriented $q$-spheres. We may arbitrarily specify the linking numbers $l(S_{i,k}, S_{j,h})$, for $(i, k) \neq (j, h)$, except for the demands of commutativity ([14, p. 278]),

$$l(S_{i,k}, S_{j,h}) = (-1)^{q+1} l(S_{j,h}, S_{i,k}) .$$

We shall specify them by

$$l(S_{i,k}, S_{j,h}) = \begin{cases} c_{ij,k-h} & \text{if } k = 0, h \geqq 0, \text{ or } k \geqq 0, h = 0 , \\ 0 & \text{otherwise.} \end{cases}$$

That the commutativity relations hold follows from (i).

For each $i$, $1 \leqq i \leqq r$, define a new oriented imbedded sphere $S_i$ in $X_0$ by forming the *connected sum*

$$S_i = S_{i,-m} \# S_{i,1-m} \# \cdots \# S_{i,0} \# S_{i,1} \# \cdots \# S_{i,m} .$$

More precisely, connect $S_{i,k}$ to $S_{i,k+1}$ with an arc $A_k$, $-m \leqq k \leqq m - 1$, such that the interiors of the $A_k$ are mutually disjoint and disjoint from all $\{S_{i,h}\}$ $(n - q > 1)$. We can thicken these arcs to *tubes* diffeomorphic to $A_k \times D^q$ meeting $S_{i,k} \cup S_{i,k+1}$ along $\partial A_k \times D^q$, so that the orientations agree (assuming

$A_k$ oriented from $S_{i,k}$ to $S_{i,k+1}$). Then define $S_i$ by

$$S_i = (\bigcup_k S_{i,k} - \bigcup_k A_k \times D^q) \cup \bigcup_k A_k \times S^{q-1}$$

with orientation induced (consistently) by the $\{S_{i,k}\}$.

We will also ask that the arcs $A_k$ pass once around $X_0$ in the positive direction, i.e., for some disk $d$ in $S^n - B$ bounded by $K$, the intersection number of $A_k$ and $d$ is 1.

4.8. Let $\tilde{X}_0$ be the universal (infinite cyclic) covering of $X_0$; note $\tilde{X}_0$ is contractible. Then each $S_i$ lifts to an infinite family $\{\sigma_{i,k}\}$, $-\infty < k < \infty$, of imbedded spheres; the $\sigma_{i,k}$ can be indexed so that the positive generator of the group of covering transformations maps $\sigma_{i,k}$ onto $\sigma_{i,k+1}$.

The linking numbers of the $\{\sigma_{i,k}\}$ are given by

$$l(\sigma_{i,k}, \sigma_{j,h}) = \sum_s l(S_{i,k+s}, S_{j,h+s}) = c_{ij,k-h}$$

(see the argument in [11, p. 139–40]).

4.9. We now put a normal frame $\mathcal{F}_i$ on $S_i$ so that the translate $S'_i$ of $S_i$ along the first vector field of $\mathcal{F}_i$ satisfies

$$l(S_i, S'_i) = \lambda_{ii}(1) .$$

Condition (ii) is needed to assure this. In fact, if $\lambda_{ii}(1) = 0$, $\mathcal{F}_i$ exists, since $S_i$ is null-isotopic in $S^n$ ($n = 2q + 1$, $q > 1$). But if $q$ is even, condition (i) assures that $\lambda_{ii}(1) = 0$. If $q$ is odd, we can alter $\mathcal{F}_i$ by any element $\alpha \in \pi_q(\mathrm{SO}_{q+1})$, and the resultant change in $l(S_i, S'_i)$ is $p_*(\alpha)$, where

$$p\colon \mathrm{SO}_{q+1} \longrightarrow S^q$$

is the standard fibration, for a proper isomorphism $\pi_q(S^q) \approx Z$. But it is well-known that $p_*$ is onto if $q = 1, 3$ or $7$, while

$$\text{Image } p_* = 2\pi_q(S^q)$$

if $q$ is odd, $q \neq 1, 3, 7$. With condition (ii), the existence of $\mathcal{F}_i$ is, thereby, assured.

4.10. At this time we point out

$$(1) \qquad l(S_i, S'_j) = \lambda_{ij}(1) = \sum_k l(\sigma_{ik}, \sigma'_{jh})$$

for any value of $h$, where $\sigma'_{jh}$ is the translate of $\sigma_{jh}$ lying over $S'_j$.

First note that $l(S_i, S'_j) = \sum_k l(\sigma_{ik}, \sigma'_{jh})$, since, if $\tilde{c}$ is any chain in $\tilde{X}_0$ bounded by $\sigma'_{jh}$, then its projection $c$ in $X_0$ is bounded by $S'_j$, and the intersections of $\tilde{c}$ with all the $\{\sigma_{ik}\}$, $-\infty < k < \infty$, correspond, in a one-one manner, with the intersections of $c$ with $S_i$.

If $i = j$, $l(S_i, S'_j) = \lambda_{ij}(1)$ by choice of $\mathcal{F}_i$. If $i \neq j$, then

$$l(S_i, S'_j) = \sum_k l(\sigma_{ik}, \sigma'_{jh}) = \sum_k l(\sigma_{ik}, \sigma_{jh}) = \sum_k c_{ij,k-h} = \lambda_{ij}(1) .$$

4.11. We can use the framed spheres $\{(S_i, \mathcal{F}_i)\}$ to perform spherical modifications and convert the pair $(S^n, X_0)$ to a new pair $(\Sigma, X)$; then $K$ is an imbedded $(n-2)$-sphere in $\Sigma$ and $X = \Sigma - K$. We will show that $\Sigma$ is a homotopy $n$-sphere; therefore, since $n \geqq 5$, we can change the differential structure at a point in $X$ to make $\Sigma$ diffeomorphic to $S^n$.

Let $Y$ be the preliminary space, as in (4.5), obtained by removing open tubular neighborhoods of the $\{S_i\}$ from $S^n$. By an argument using excision on the pairs $(S^n, Y)$ and $(\Sigma, Y)$, we see that $H_i(\Sigma) = 0$ for $i \neq 0, q, q+1$ and $n$; since $q > 1$, a similar use of the van Kampen theorem implies $\Sigma$ is 1-connected.

Finally we have a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{q+1}(\Sigma) & \longrightarrow & H_{q+1}(\Sigma, Y) & \xrightarrow{d} & H_q(Y) & \longrightarrow & H_q(\Sigma) & \longrightarrow 0 \\ & & & & & \uparrow \approx & & & \\ & & & & & H_{q+1}(S^n, Y) & & & \end{array}$$

where the row is a portion of the exact homology sequence of $(\Sigma, Y)$, and the vertical isomorphism comes from that of the pair $(S^n, Y)$. It follows, by excision, that $H_{q+1}(\Sigma, Y)$ and $H_{q+1}(S^n, Y)$ are free abelian of rank $r$. Then $H_q(Y)$ has a basis $\alpha_1, \cdots, \alpha_r$, where $\alpha_i$ is represented by a sphere $u_i$ such that the linking number $l(S_i, u_j) = \delta_{ij}$; $H_{q+1}(\Sigma, Y)$ has a basis $\beta_1, \cdots, \beta_r$, where $\beta_i$ is represented by a disk whose boundary is $S'_i$.

Now suppose $d(\beta_j) = \sum_k a_{kj}\alpha_k$. This is represented by $S'_j$. Recall that the linking number $l(S_i, S'_j) = \lambda_{ij}(1)$, by (1). On the other hand,

$$l(S_i, S'_j) = \sum_k a_{kj} l(S_i, u_k) = a_{ij} .$$

Thus $d$ has a matrix representation $(\lambda_{ij}(1))$; by condition (iii), $d$ is an isomorphism. This shows $\Sigma$ is a homotopy $n$-sphere.

4.12. A procedure similar to that in (4.11), but more complicated, will serve to compute $H_*(\tilde{X})$. First notice that $\tilde{X}$ may be constructed from $\tilde{X}_0$ by spherical modification using the $\sigma_{ik}$ with the normal frame $\mathcal{F}_{ik}$ lifted from $\mathcal{F}_i$. Let $\tilde{X}_1$ be the complement in $\tilde{X}_0$ of open tubular neighborhoods of all the $\{\sigma_{ik}\}$.

By excision $H_i(\tilde{X}_0, \tilde{X}_1)$ and $H_i(\tilde{X}, \tilde{X}_1)$ are free $\Lambda$-modules of rank $r$, if $i = q+1$, but zero for $i \neq q+1, n$. As a consequence, $H_i(\tilde{X}) = 0$ for $i \neq 0$, $q, q+1$, and we have a diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H_{q+1}(\tilde{X}) & \longrightarrow & H_{q+1}(\tilde{X}, \tilde{X}_1) & \xrightarrow{d} & H_q(\tilde{X}_1) & \longrightarrow & H_q(\tilde{X}) & \longrightarrow 0 \\ & & & & & \uparrow \approx & & & \\ & & & & & H_{q+1}(\tilde{X}_0, \tilde{X}_1) & & & \end{array}$$

where the row is exact. Now $H_{q+1}(\tilde{X}, \tilde{X}_1)$ has a basis $\beta_1, \cdots, \beta_r$, where $\beta_i$ is represented by a disk whose boundary is $\sigma'_{i0}$. $H_q(\tilde{X}_1)$ has a basis $\alpha_1, \cdots, \alpha_r$, where $\alpha_i$ is represented by a sphere $\tau_{i0}$ satisfying $l(\sigma_{jh}, \tau_{i0}) = \delta_{ij}\delta_{0h}$; consequently, $t^k\alpha_i$ is represented by a sphere $\tau_{ik}$ satisfying

$$l(\sigma_{jh}, \tau_{ik}) = \delta_{ij}\delta_{hk} .$$

Suppose $d(\beta_j) = \sum_i \mu_{ij}\cdot\alpha_i$, where $\mu_{ij} = \sum_k a_{ijk}t^k$. Since $d(\beta_j)$ is represented by $\sigma'_{j0}$, we have

$$l(\sigma_{sh}, \sigma'_{j0}) = \sum_{i,k} a_{ijk} l(\sigma_{sh}, \tau_{ik}) = a_{sjh} .$$

But if $(s, h) \neq (j, 0)$, this is the same as $l(\sigma_{sh}, \sigma_{j0}) = c_{sjh}$. It remains to prove $a_{jj0} = c_{jj0}$, which now follows from $\lambda_{jj}(1) = \mu_{jj}(1)$. In fact, using (1),

$$\mu_{jj}(1) = \sum_k a_{jjk} = \sum_k l(\sigma_{jk}, \sigma'_{j0}) = \lambda_{jj}(1) .$$

Thus $\mu_{ij} = \lambda_{ij}$, and $(\lambda_{ij})$ is a presentation matrix for $H_q(\tilde{X})$. Finally, since $(\lambda_{ij}(1))$ is unimodular, $(\lambda_{ij})$ is non-singular, and $d$ is a monomorphism; thus $H_{q+1}(\tilde{X}) = 0$.

4.13. Given $n = 2q + 1$, $q > 1$, we wish to construct an $n$-knot $K$ with complement $X$ such that $H_i(\tilde{X}) = 0$, $i \neq 0, q$, and $H_q(\tilde{X}; Q)$ has a diagonal presentation matrix whose entries $\{\lambda_i^q\}$ satisfy (a)-(e) of Theorems 1 and 2. This will be accomplished by showing that the desired $H_q(\tilde{X}; Q)$ must also have a presentation matrix $(\lambda_{ij})$ satisfying the conditions of Lemma (4.6), and then applying the lemma. Note that for $q = 3$ or 7, the matrix with entries $\{\lambda_i^q\}$ already is of this type.

We begin by expanding each $\lambda_i^q$ as a product of prime powers in $\Lambda$; since $\Gamma$ is a principal ideal domain, the diagonal matrix whose entries are the collection $\{p_i^{e_i}\}$ of prime powers obtained, is also a presentation matrix of $H_q(\tilde{X}; Q)$. Using (b) and (c) of Theorem 1, and making proper choices of $p_i$, using Lemma (1.5) also, it is not difficult to see that the $\{p_i^{e_i}\}$ may be ordered so that $p_i(t) = p_{i+k}(t^{-1})$, $e_i = e_{i+k}$, but $p_i$, $p_{i+k}$ are relatively prime if $1 \leqq i \leqq k$, $p_i(t) = p_i(t^{-1})$ if $i > 2k$, and $p_i(1) = 1$, all $i$.

For $1 \leqq i \leqq k$, set $\mu_i = (p_i p_{i+k})^{e_i}$. For $i > 2k$, consider $p_i(-1)^{e_i}$; we may assume the $\{p_i\}$ arranged so that this quantity is square for $2k < i \leqq s$, but non-square for $i > s$. Set $\mu_i = p_{i+k}^{e_{i+k}}$ for $k < i \leqq s - k$. If $i > s$, we may assume $p_i = p_{i+1}$ for $i - s$ odd, $s < i \leqq s + 2u$, while, for $i > s + 2u$, $\{p_i\}$ are pairwise relatively prime. Note that $e_i$ is odd for $i > s$. Set $\gamma_i = p_{s+2i-1}^{e_{s+2i-1}}$, $\delta_i = p_{s+2i}^{e_{s+2i}}$ for $1 \leqq i \leqq u$, and $\mu = \prod_{i>s+2u} p_i^{e_i}$.

The choices of $\mu_i$, $\gamma_i$, $\delta_i$, are made so that they constitute the entries of a diagonal matrix presenting $H_q(\tilde{X}; Q)$. Note also that these elements are normal (see (1.5)) and, in addition, $\mu_i(-1)$, $\gamma_i(-1)$, $\delta_i(-1)$ are odd squares.

It follows from (d) or (e) that $\mu(-1)$ must be an odd square or $\equiv 1 \bmod 8$, as $q$ is even or odd ($\neq 3,7$), respectively, since

$$\Delta = \prod_i \mu_i \cdot \prod_i \gamma_i \cdot \prod_i \delta_i \cdot \mu$$

4.14. Summarizing, we may conclude that $H_q(\tilde{X}; Q)$ is the direct sum of modules with one of the following presentation matrices

$$\begin{pmatrix} \mu_i & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \gamma_i & 0 \\ 0 & \delta_i \end{pmatrix}, \quad \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$$

These matrices are of the general form

$$(2) \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

where $\lambda$, $\mu$ are normal, $\mu \mid \lambda$ in $\Lambda$ and $\mu(-1)\lambda(-1)$ is square or $\equiv 1 \bmod 8$, as $q$ is even or odd ($\neq 3,7$) respectively. It suffices to show that such a matrix has the same elementary ideals (in $\Gamma$) as a matrix of the form of Lemma (4.6) (see (2.3)).

Define $\rho \in \Lambda$ by $\lambda = \rho \cdot \mu$; then $\rho$ is normal, and $\rho(-1)$ is square or $\equiv 1 \bmod 8$, as $q$ is even or odd ($\neq 3,7$).

4.15. If $q$ is even, let $\rho(-1) = (2a + 1)^2$ and set

$$\theta(t) = 1 + a(1 - t) .$$

Then $\rho$ and $\theta^2$ agree when $t = \pm 1$; so $\rho(t) - \theta(t)\theta(t^{-1})$ has $\pm 1$ as roots. Thus there exists $\nu \in \Lambda$ satisfying

$$\rho(t) = \theta(t)\theta(t^{-1}) + \nu(t)(t - t^{-1}) .$$

We check immediately that $\nu(t) = -\nu(t^{-1})$. Consider the matrix:

$$\begin{pmatrix} \nu(t)\mu(t) & \theta(t)\mu(t) \\ -\theta(t^{-1})\mu(t) & (t - t^{-1})\mu(t) \end{pmatrix}$$

This has the form of Lemma (4.6), and has the same elementary ideals as (2).

If $q$ is odd ($\neq 3,7$), note that $\rho(t) - 1$ has 1 as a root. In fact, 1 is a multiple root, for, if $\rho(t) - 1 = (t - 1)\sigma(t)$, then, because $\rho(t) = \rho(t^{-1})$, we check that $\sigma(t) = -t^{-1}\sigma(t^{-1})$. But this implies $\sigma(1) = 0$. So we can write

$$\rho(t) = 1 - (t - 1)(t^{-1} - 1)\theta(t) .$$

Then $\theta(t) = \theta(t^{-1})$ and, since $\rho(-1) \equiv 1 \bmod 8$, it follows that $\theta(-1)$ is even. This implies $\theta(1)$ is also even. Thus the following matrix has the form of Lemma (4.6) for $q$ odd ($\neq 3,7$)

$$\begin{pmatrix} \theta(t)\mu(t) & \mu(t) \\ \mu(t) & (t - 1)(t^{-1} - 1)\mu(t) \end{pmatrix},$$

and is easily seen to have the same elementary ideals as (2).

4.16. Using the knots constructed above, we can complete the proof of Theorem 2 if, given $n$-knots $K_1, K_2$, a new knot can be constructed whose Alexander invariants are the *sums* of the corresponding Alexander invariants of $K_1$ and $K_2$. For this purpose, we use the *connected sum.*

Let $S^{n-2} \subset S^n$ be the standard trivial $n$-knot. By an isotopy, we may assume $K_1 \cap D^n_- = S^{n-2} \cap D^n_-$, and $K_2 \cap D^n_+ = S^{n-2} \cap D^n_+$. Then define $K = (K_1 \cap D^n_+) \cup (K_2 \cap D^n_-)$. If $X_1, X_2$ and $X$ are the complements of $K_1, K_2$ and $K$, respectively, then $X = (X_1 \cap D^n_+) \cup (X_2 \cap D^n_-)$. Notice that

$$(X_1 \cap D^n_+) \cap (X_2 \cap D^n_-) = S^{n-1} - S^{n-3} ,$$

which is a homotopy circle, and $(X_1, X_1 \cap D^n_+)$, $(X_2, X_2 \cap D^n_-)$ yield, by excising $D^n_+, D^n_-$, respectively, pairs in which the subspace is a deformation retract of the larger space and both are homotopy circles. These facts imply that the inclusions $\widetilde{X_1 \cap D^n_+} \to \widetilde{X}_1$, $\widetilde{X_2 \cap D^n_-} \to \widetilde{X}_2$ induce homology isomorphisms, and that $\widetilde{X_1 \cap D^n_+} \cap \widetilde{X_2 \cap D^n_-}$ is contractible. By the Mayer-Vietoris theorem, we have

$$H_q(\widetilde{X}) \approx H_q(\widetilde{X}_1) + H_q(\widetilde{X}_2) , \qquad q > 0 ,$$

which, with rational coefficients, is what we mean by saying that the Alexander invariants of $K$ are the sums of those of $K_1$ and $K_2$.

This completes the proof of Theorem 2.

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