Let an orientable and an oriented r-dimensional h-manifold K be a triangulation. Let m and n be independent distributions of masses on the vertices of K (see section 4:3). Let the chain

$$\xi_1 \in L^{r-p}(K, \mathfrak{A})$$

correspond to the m-chain

$$x_m = \sum_{i=1}^{\alpha^{r-p}} \xi_1 (t_i^{r-p})_m s_i^p \in L^p_{\square}(K, \mathfrak{A}),$$

and let the chain $\xi_2 \in L^{r-q}(K, \mathfrak{A})$ correspond to the n-chain

$$x_n = \sum_{j=1}^{\alpha^{r-p}} \xi_2 (t_j^{r-q})_n s_j^q \in L^q_{\square}(K, \mathfrak{A}).$$

By [4:44], the chains x_m and x_n are in general position in K; consequently, the intersection

$$x_m \times x_n = x \in L_{\square}^{p+q-r}(K, \mathfrak{A})$$

is defined. Let the chain $\xi = \sum_{k=1}^{\alpha^{p+q}} \phi(x)_k^{p+q-r} t_k^{p+q}$ correspond to the chain x.

We will call this chain the product of the chains ξ_1 and ξ_2:

(5:41) $$\xi = \xi_1 \times \xi_2.$$

If the chains ξ_1 and ξ_2 are ∇-cycles, then, as easily follows from the above, ξ is also a ∇-cycle and the ∇-class of the cycle ξ is the product of ∇-classes corresponding to the ∇-cycles ξ_1 and ξ_2 as defined by formula (5:4).

The indices of the traces $(x)_k^{p+q-r}$ of the intersection of the chains x_m and x_n can be easily found on the basis of the calculations of section 4:4. From these calculations it follows immediately that the product (5:41) coincides with the product of Alexander, and consequently, the ring $\nabla(K, \mathfrak{A})$, with the multiplication (5:41), coincides with the ring of Alexander.

Since Alexander's ring is topologically invariant [see Whitney Ann. of Math. (2) 39, 397-432 (1938)], the topological invariance of the ring of Lefschetz follows.

Characteristic cycles on differentiable manifolds

[Matematičeskiĭ Sbornik (N.S.) 21(63), 233–284 (1947)]

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Translated from

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Характеристические циклы дифференцируемых многообразий

[Математический Сборник (Н.С.) 21(63), 233-284 (1947)]

by A. A. Brown

First published by the American Mathematical Society in 1950 as Translation Number 32. Reprinted without revision.

Introduction

In the present work I give a full account of a part of results published earlier in the Doklady Akademiĭ Nauk [1].

The method of spherical mappings plays an essential role in geometry, as can be concluded from the following.

Let M^k be a differentiable orientable manifold of dimension k with continuously turning tangents, embedded in Euclidean space R^{k+1} of dimension $k + 1$. At the point $x \in M^k$ we erect a unit normal N_x to M^k, the direction being chosen in accordance with the orientation to M^k. We displace the normal N_x, by parallel displacement in R^{k+1}, in such a way that its origin falls on a fixed point O of the space R^{k+1}; then the end of the normal falls on the point $N(x)$ of the unit sphere S^k centered at O.

In this way we obtain a spherical representation N of the manifold M^k, mapping each point $x \in M^k$ on the corresponding point $N(x) \in S^k$.

The study of the mapping N leads to the discovery of several invariants of the manifold M^k, both topological and differential-geometric. In particular, if the manifold M^k is closed, and k is finite, the degree of the mapping N represents in itself a topological invariant of the manifold M^k, equal to half the Euler characteristic [2].

It is known that not every manifold M^k of dimension k can be properly embedded in a $k+1$-dimensional space. In view of this there arises naturally the notion of giving a construction, analogous to the spherical mapping, for manifolds M^k embedded in a Euclidean space R^{k+l} of dimension $k + l$, where l is an arbitrary integer.

The present work is devoted to the construction of several invariants of closed differentiable orientable manifolds, based on construction of a mapping of the type mentioned. It is still an open question whether these invariants are new or whether they can be computed from those already known, for example the ring of intersections. It is indubitable, however, that these invariants are closely connected with many geometric questions; in particular, they are bound up with the problem of classification of mappings of a sphere of high dimension on spheres of lower dimensionality.

The means by which I construct the present invariants are contained in the well-known cycles of Stiefel [3], and, incidentally, the present paper closely adjoins a series of papers of Whitney [4], in which Stiefel cycles are studied in detail. Stiefel cycles represent, however, only a part of the whole set of invariants considered here. The connection with Stiefel cycles and related objects is given in my paper [5] which will be published later.

The general course of the present investigations is as follows:

Two differentiable manifolds are called *equivalent*, or simply *homeomorphic*, if there exists a homeomorphic continuously differentiable mapping of one on the other and the Jacobian of the mapping nowhere vanishes. The invariants which we shall consider are invariants from the point of view of this equivalence.

The dimension of a given space P will be denoted by $D(P)$.

Let R^{k+l} denote the Euclidean space of dimension $k + l$, and O a certain fixed point of this space.

By $H(k,l)$ we denote the manifold of all orientable k-dimensional linear subspaces of R^{k+l} containing O.

The dimension of $H(k,l)$ is defined correspondingly:

$$D(H(k, l)) = k \cdot l. \tag{1}$$

It is clear that if $R^{k+l'} \subset R^{k+l}$, then

$$H(k, l') \subset H(k, l). \tag{2}$$

This inclusion relation provides a connection between the various manifolds $H(k,l)$ for fixed k.

Now let M^k be an abstractly defined differentiable orientable manifold and f a homeomorphic mapping of it on the manifold $f(M^k) \subset R^{k+l}$ with continuously turning tangent.

At the point $f(x) \in f(M^k)$ we construct the orientable linear space T_x of dimension k tangent to $f(M^k)$ and denote by $T(x)$ the element of the manifold $H(k,l)$ parallel to T_x.

In this way there arises a mapping T of the manifold M on the manifold $H(k,l)$. I call the mapping T the *tangential representation* of the manifold M^k; it depends on the imbedding f of the manifold M^k in the space R^{k+l}, and plays the role of the spherical representation N.

If f_0 and f_1 are two distinct imbeddings of M^k in R^{k+l}, the tangential representations T_0 and T_1 corresponding to them are distinct. It is easily shown that for $l \geq k + 1$, T_0 and T_1 are always homotopic in $H(k,l)$.

Thus, with precision up to a homotopy, the tangential representation is independent of the imbedding of the manifold M^k in the Euclidean space R^{k+l}. The inclusion relation (2) frees the tangential representation from dependence even on the number l.

In the present work we construct a complete system of homology invariants of the tangential representation of a closed manifold M^k on $H(k,l)$. The construction proceeds essentially as follows.

Let Z be any cycle of dimension $kl - r$ of the manifold $H(k,l)$, with $r \leq k$ (cf. [1]). Since the dimension of the manifold $H(k,l)$ is large compared with the dimension of the manifold M^k, there subsists a homeomorphism T_1 of the manifold M^k on $H(k,l)$, approximating the tangential mapping T and such that the cycle Z and the manifold $T_1(M^k)$ are in a general position in $H(k,l)$.

The algebraic intersection [6] of Z_1 with the manifold $T_1(M^k)$, $Z \times T_1(M^k)$, represents a $k-r$-dimensional cycle on $T_1(M^k)$. Let us denote by X the image of this cycle in M^k under the mapping T_1^{-1}. Since the cycle X is independent, up to a homology, of the approximation T_1 to the mapping T, we may write simply

$$X \sim T^{-1}(Z \times T(M^k)), \qquad (3)$$

although this formula has no immediate sense.

Since the tangential mapping T, up to a homotopy, is uniquely defined by the manifold M^k, the homology class of the cycle X depends only on the homology class of the cycle Z and on the manifold M^k. The orientation of the manifold $H(k,l)$ can be chosen arbitrarily.

Choosing the cycle Z in different ways, we construct a complete system of homology invariants of the tangential mapping T. The inclusion relation (2) allows us independence of the number l.

In order to decide what cycles Z to choose in $H(k,l)$, we must

study the homologies of dimension $kl - r$, $r \leq k$, of the manifold $H(k,l)$. This study occupies an important place in the present work; it is derived from the method of Ehresmann [7]. Since reference to Ehresmann's definitive results is not possible here, I will give an account which is independent of Ehresmann's papers.

An adequate stock of cycles can be constructed by the follow- procedure.

Let ω be a monotone non-decreasing integer-valued function of the integer argument $i = 1, 2, \ldots, k$, satisfying the condition

$$0 \leqslant \omega(i) \leqslant l. \qquad (4)$$

We consider in the space R^{k+l} the sequence

$$R_1 \subset R_2 \subset \ldots \subset R_k \qquad (5)$$

of linear subspaces each containing O, with dimensions satisfying the conditions

$$D(R_i) = \omega(i) + i, \qquad i = 1, 2, \ldots, k. \qquad (6)$$

We will denote by $Z(\omega)$ the set of all elements $R^k \in H(k,l)$ satisfying the conditions

$$D(R^k \times R_i) \geqslant i, \quad i = 1, 2, \ldots, k. \qquad (7)$$

It should be noted that $Z(\omega)$ represents a closed pseudomanifold the dimension of which is defined by

$$D(Z(\omega)) = r(\omega) = \sum_{i=1}^{k} \omega(i). \qquad (8)$$

In the case where $Z(\omega)$ is orientable, it can be regarded, with a given orientation, as an integer cycle on $H(k,l)$.

If $Z(\omega)$ is non-orientable, it can be looked on as a cycle modulo 2. In this case, we can derive from it an integer cycle, making use of the following general method. Let Y be a cycle mod 2. If we consider all the simplexes of the cycle Y, each with a given orientation, we have an integer chain Y', for which

$$\Delta Y' = 2\Gamma Y, \qquad (9)$$

where ΓY is an integer cycle whose homology class is uniquely defined by the homology class of the cycle Y.

In place of the function ω it is desirable to introduce the function χ, defined by the relation

$$\omega+\chi\equiv l,\ \text{i.e.}\ \omega(i)+\chi(i)=l,\quad i=1,2,\ldots,k. \tag{10}$$

The function χ is an integer-valued monotone non-increasing function of the integer argument i, satisfying the condition

$$l\geqslant\chi(i)\geqslant 0. \tag{11}$$

By making use of the relation (10), we can always construct from a function χ satisfying (11) a function ω satisfying (4).

The pseudomanifold $Z(\omega)$, and the corresponding cycle, will be denoted by Z_χ; its dimension, in view of the relations (8) and (10), will be given by the formula

$$D(Z_\chi)=kl-r(\chi),\quad r(\chi)=\sum_{i=1}^{k}\chi(i). \tag{12}$$

We now strengthen the condition (11), as applied to the function χ; we suppose, in fact, that

$$l-1\geqslant\chi(i)\geqslant 0. \tag{13}$$

It should be noted that if this condition is satisfied the question of the orientability of the pseudomanifold Z_χ is soluble on the basis of the properties of the function χ, and does not depend on the number l.

If in the relation (3) we consider the cycle Z_χ instead of the arbitrary cycle Z, we are led to the following definition.

Definition. Let M^k be a closed orientable differentiable manifold of dimension k; let T be a tangential mapping of it on $H(k,l)$, and let χ be a function satisfying condition (13) and also the condition $r(\chi)\leq k$.

We set

$$X_\chi(M^k)=X_\chi\sim T^{-1}(Z_\chi\times T(M^k))\quad (D(X_\chi)=k-r(\chi)). \tag{14}$$

If the pseudomanifold Z_χ is non-orientable, then the intersection is taken mod 2, and X_χ is a cycle mod 2.

If the pseudomanifold Z_χ is orientable, then both it and the manifold $H(k,l)$ have a certain definite sense of orientation, and X_χ is an integer cycle.

It turns out that the homology class of the cycle X_χ in the manifold M^k is uniquely defined by the oriented differentiable manifold M^k and the function χ; we will call the cycle X_χ the *characteristic cycle of type* χ of the oriented manifold M^k.

In the case in which Z_χ is non-orientable, we will call the homology class of the cycle

$$\Gamma X_\chi\sim T^{-1}(\Gamma X_\chi\times T(M^k))\quad (D(\Gamma X_\chi)=k-r(\chi)-1 \tag{15}$$

the secondary characteristic cycle of type χ of the oriented manifold M^k.

If $r(\chi)=k$, then $D(X_\chi)=0$, and the homology class of the characteristic cycle X_χ is uniquely connected with an integer or with a remainder mod 2, depending on whether the pseudomanifold Z_χ is orientable or not. In this case we will denote by X_χ not only the 0-dimensional cycle, but also the corresponding integer or remainder. The characteristic number or remainder is an invariant of the oriented manifold M^k.

For a more detailed study of characteristic cycles let us look at certain properties of the function χ.

By a jump point of the function χ we will mean a value i of its argument for which $\chi(i+1)<\chi(i)$. Let $i_1, i_2, \ldots, i_{n-1}$ be the set of all jump points of the function χ, written in increasing order. We will suppose, further that $i_0=0$, $i_n=k$, and we will set

$$\left.\begin{aligned}&\alpha_h=i_h-i_{h-1},\quad h=1,2,\ldots,n;\\&\beta_h=\chi(i_h)-\chi(i_h+1),\quad h=1,2,\ldots,n-1;\quad \beta_n=\chi(k).\end{aligned}\right\} \tag{16}$$

We now condider the equations

$$\alpha_1+\beta_1\equiv\alpha_2+\beta_2\equiv\ldots\equiv\alpha_{n-1}+\beta_{n-1}\equiv 0\ (\mathrm{mod}\ 2). \tag{17}$$

The set of functions χ for which (13) and (17) are satisfied will be denoted by X_o.

The set of functions χ for which (13) is satisfied and (17) is not, we will denote by X_2.

It turns out that for $\chi \in X_o$ the pseudomanifold Z_χ is orientable, and that Z_χ is non-orientable for $\chi \in X_2$.

We denote by X' the set of functions χ satisfying (13) and also the relation

$$\alpha_1 \geqslant 2. \tag{18}$$

For a given function χ we consider two sequences of numbers

$$\left.\begin{array}{l}\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}, \alpha_n; \\ \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{n-1}, \beta_{n-1}.\end{array}\right\} \tag{19}$$

We let χ correspond to the first of these sequences if $\beta_n > 0$, and to the second if $\beta_n = 0$.

We assign a function $\chi \in X'$ to:

the set X, if all numbers corresponding to it in (19) are even;

the set X_α, if there are odd numbers in the sequence of (19) corresponding to it, and the first of these is some α;

the set X_β, if there are odd numbers in the sequence of (19) corresponding to it, and the first of these is some β.

In this way the set X' is decomposed into the sum of the three disjunct sets X, X_α, and X_β. It is clear that $X \subset X_o$ and $X_\beta \subset X_2$.

It is to be noted also that for $\chi \in X_o - X$,

$$2Z_\chi \sim 0. \tag{20}$$

The theorem stated below provides a homology basis for $H(k, l)$, for the dimensions of interest to us.

Theorem 1. *A canonical basis for the homologies of dimension $kl - r$, $r \leq l - 1$, of the manifold $H(k, l)$ can be constructed from the cycles*

$$Z_\chi, \chi \in X, r(\chi) = r; \quad \Gamma Z_{\chi'}, \chi' \in X_\beta, r(\chi') = r - 1. \tag{21}$$

Here the cycles Z_χ are free, and the cycles $\Gamma Z_{\chi'}$ have order two.

From theorem 1 it follows that all characteristic cycles of tha manifold M^k can be expressed by the cycles X_χ, $\chi \in X$, and $X_{\chi'}$, $\chi' \in X_\beta$. These characteristic cycles we will call *basic*. The cycles X_χ, $\chi \in X$ are integer cycles, and the $X_{\chi'}$, $\chi' \in X_\beta$, are cycles mod two. If $\chi^* \in X$, the cycles X_{χ^*} is itself basic. If $\chi^* \in X_o - X$, then

$$X_{\chi^*} \sim \sum_{\chi \in X_\beta} b_\chi \Gamma X_\chi, \tag{22}$$

where b_χ is a remainder mod two, uniquely determined by the function χ^*.

If $\chi^* \in X_2$, then

$$X_{\chi^*} \sim \sum_{\chi \in X} a_\chi X_\chi + \sum_{\chi \in X_\beta} b_\chi \Gamma X_\chi + \sum_{\chi \in X_\beta} c_\chi X_\chi \pmod 2, \tag{23}$$

where a_χ, b_χ, and c_χ are remainders mod two, uniquely determined by the function χ^*.

Further, for $\chi^* \in X_2$ we have

$$\Gamma X_{\chi^*} \sim \sum_{\chi \in X_\beta} c_\chi \Gamma X_\chi. \tag{24}$$

where c_χ is a remainder mod two, uniquely determined by the function χ^*.

For characteristic numbers, the results we have obtained mean that the characteristic number X_χ, $\chi \in X_o$, can be different from 0 only for $\chi \in X$. In fact, the zero-dimensional Betti group for an integer coefficient field has no torsion and therefore, if a zero-dimensional cycle X_χ satisfies the condition $2X_\chi \sim 0$ (cf. (20)), then $X_\chi \sim 0$, that is the characteristic number X_χ reduces to zero. For the same reason the zero-dimensional integer cycles of the form ΓX_χ do not give characteristic numbers different from zero.

Thus, it is meaningful to consider only those characteristic numbers X_χ corresponding to $\chi \in X$. As for the characteristic remainders, relation (23) yields,

$$X_{\chi^*} = \sum_{\chi \in X} a_\chi X_\chi + \sum_{\chi \in X_\beta} c_\chi X_\chi \pmod 2, \tag{25}$$

where a_χ and c_χ are remainders mod two, uniquely defined by the function χ.

Among the characteristic numbers we distinguish one, namely $X_\chi = X_1$, for which $\chi \equiv 1$. This function χ belongs to the set X_0, but belongs to X only if k is even. Therefore, for odd k, $X_1 = 0$. In my paper [5] it was shown that the characteristic number X_1 is equal to the Euler characteristic of the manifold under consideration. This fact corresponds to the theorem of Hopf, recalled above, to the effect that the degree of a spherical mapping for even-dimensional manifolds is one-half the Euler characteristic.

If the orientation of the manifold M^k is reversed, that is, if we consider the manifold $-M^k$, the tangential representation of it is essentially changed, since we are considering oriented tangents.

The connection between the characteristic cycles of the manifolds M^k and $-M^k$ is given by the following theorem.

Theorem 2. *If we denote by $-M^k$ the manifold which coincides geometrically with M^k but is oppositely oriented, then*

$$X_1(-M^k) \sim X_1(M^k). \tag{26}$$

For $\chi \not\equiv 1$, *but* $\chi \in X$: $\quad X_\chi(-M^k) \sim -X_\chi(M^k)$, (27)

and for the remaining characteristic cycles

$$X_\chi(-M^k) \sim X_\chi(M^k). \tag{28}$$

From theorem 2 it follows immediately that if a characteristic cycle X_χ, $\chi \neq 1$, $\chi \in X$, has the property that, taken twice, it is not homologous to zero, then the manifold is asymmetric--that is, it cannot be mapped on itself by a sense-reversing homeomorphism.

The theorem below displays a certain geometrical property of characteristic numbers and remainders.

Theorem 3. *If the oriented closed manifold M^k can form the boundary of an oriented bounded manifold M^{k+1}, then all characteristic numbers and remainders of M^k vanish, with the possible exception of X_1, which is even.*

That the Euler characteristic of M^k is even can be shown very simply by doubling the manifold M^{k+1}.

Thus, we observe an interesting geometrical fact: not every closed oriented manifold can be the boundary of an oriented manifold. The simplest example of this is the complex projective plane, which, considered as a four-dimensional manifold, is orientable and has an Euler characteristic equal to three.

Theorem 3 gives us nothing in the case of two-dimensional and three-dimensional manifolds.

It is known that every oriented surface can form the boundary of an oriented three-dimensional manifold. It would be interesting, and for some purposes important, to show that the same thing is true of three-dimensional manifolds.

One further property of characteristic cycles is worth noting.

If the manifold M^k can be imbedded in Euclidean space of dimension $k+1$, then all its characteristic cycles are homologous to zero, with the possible exception of X_1, and in this case the characteristic number X_1 is even.

For two and three dimensional manifolds the characteristic cycles yield no new invariants.

For two-dimensional manifolds M^2 we obtain only the characteristic number X_1, which is equal to the Euler characteristic.

For three-dimensional manifolds M^3, all characteristic cycles can be reduced to the Stiefel cycle, which, as he showed, is homologous to zero.

For four-dimensional manifolds M^4 there are three basic characteristic cycles. The first, already considered, namely X_1, yields the Euler characteristic. The second, also zero-dimensional, X_{22}, corresponds to the function χ with the values $\chi(1) = \chi(2) = 2$, $\chi(3) = \chi(4) = 0$.

The characteristic numbers X_1 and X_{22} are equal mod two, but to a considerable extent are independent, as is shown in my paper [5]. The characteristic cycle X_{22} does not belong to the class of Stiefel cycles.

Beyond the two already indicated, there exists another basic cycle X_{21} for four-dimensional manifolds, defined by the function χ with values $\chi(1) = \chi(2) = 1$, $\chi(3) = \chi(4) = 0$. The cycle X_{21} is defined mod two, is two-dimensional, and is a Stiefel cycle.

If we were able to discover all the connections among the three cycles X_1, X_{22}, X_{21} of a four-dimensional manifold M^4, we would have made significant progress toward solving the problem of classifying the mappings of an S^{n+3}--sphere of dimension $n + 3$ on an n-dimensional sphere S^n. For example, if it could be shown that from $X_{21} \sim 0$ it would follow that $X_{22} \sim 0$, then we could show that the number of distinct homotopy classes of mappings of S^{n+3} on S^n is countable.

In conclusion it should be remarked that the actual calculation of characteristic cycles on the basis of the definitions given here does present great difficulty. In view of this an alternative construction is of interest, if it leads to characteristic cycles. Such a construction is given in my paper [5], where the study of vector fields on a manifold leads to characteristic cycles. The question, however, of the extent to which this process lightens the task of calculations is still not free of obscurity.

It would be most important to give a definition of characteristic cycles which would be applicable to combinatorial, though not to differentiable, manifolds, since there exists an algorithm for the calculation of characteristic cycles which depends on the combinatorial concept of the manifold.

§ 1. The Manifold $H(k,l)$

Here we construct the manifold $H(k,l)$ and a certain pseudo-manifold situated in it; later we construct a homology basis for $H(k,l)$.

We will consider only finite-dimensional real vector spaces.

The orientation of an n-dimensional vector space R^n can be specified by a sequence $e_1, \ldots, e_n$ of vectors of a basis for R^n, taking account of the fact that the positive orientation is given by the orientation of the simplex $(0, e_1, e_2, \ldots, e_n)$.

The dimensionality of the set M will be denoted by $D(M)$.

Definition 1. Let R^{k+l} be a vector space of dimension $k + l$, $(k \geq 1,\ l \geq 1)$. The manifold of all oriented k-dimensional vector subspaces of the space R^{k+l} we will denote by $H(k,l)$. It will be shown below (see (A)), that $D(H(k,l)) = kl$, and that for a suitably chosen local coordinate system $H(k,l)$ becomes an analytic manifold.

If R^k is an arbitrary element of $H(k,l)$, the symbol $\hat{R}^k$ will denote an element of $H(k,l)$ distinguished from R^k only as to orientation: $\hat{R}^k = -R^k$. Clearly, the correspondence $R^k \rightleftarrows \hat{R}^k$ is a homeomorphism, and it is evident from (A) that it is also analytic. If M is an arbitrary set of elements of $H(k,l)$, we denote by $\hat{M}$ the set which is the image of M under the transformation $R^k \to \hat{R}^k$.

Let $R^{k+l'}$ be a vector subspace of dimension $k + l'$, and $H'(k,l')$ the manifold of all its k-dimensional oriented vector subspaces. If a is a non-degenerate linear mapping of $R^{k+l'}$ on R^{k+l}, then we will map every $R^k \in H'(k,l')$ into the corresponding $a(R^k) \in H(k,l)$; thus, there arises a mapping a of the manifold $H'(k,l')$ on the manifold $H(k,l)$.

(A) Suppose $R_o^k \in H(k,l)$, and let $e_1, \ldots, e_k$ be a basis for R_o^k specifying its orientation. We will denote by $f_1, \ldots, f_l$ a system of vectors in R^{k+l} such that the vectors

$$e_1, \ldots, e_k, f_1, \ldots, f_l \tag{1}$$

form a basis for the space R^{k+l}, and we will denote by P the linear span of the vectors $f_1, \ldots, f_l$. Then R^{k+l} is the direct sum of its subspaces R_o^k and P, so that for every $x \in R^{k+l}$ we have $x = u + v$, where $u \in R_o^k$ and $v \in P$.

Let us set $u = \phi(x)$; then ϕ is a projection of the space R^{k+l} on R_o^k in the direction P. The mapping ϕ will play a very important role in the later portions of this paper.

We denote by U the collection of all elements of $H(k,l)$ which under projection by the mapping ϕ are mapped into R_o^k without degeneration and with preservation of orientation. Clearly, U is a region in $H(k,l)$ containing R_0^k. The regions U and $\hat{U}$ do not intersect; we will denote the complement of their sum, in $H(k,l)$, by V.

We now introduce in U a coordinate system depending on the system (1).

If $R_\xi^k \in U$, there exists in R_ξ^k a basis $e'_1, \ldots, e'_k$ specifying the orientation of R_ξ^k and such that

$$\varphi(e'_i) = e_i, \quad i = 1, \ldots, k;$$

we have

$$e'_i = e_i + \sum_{j=1}^{l} \xi_i^j f_j, \quad i = 1, \ldots, k. \tag{2}$$

Here $\|\xi_i^j\| = \xi$ is a real numerical matrix, whose elements (there are kl of them) we take as coordinates of the element $R_\xi^k \in U$. If we arbitrarily specify a matrix $\|\xi_i^j\| = \xi$ and define the vectors $e'_1, \ldots, e'_k$ by the relation (2), and then span a subspace R_ξ^k on them, to so-defined space R_ξ^k is an element of U.

Thus, in the region U containing R_0^k we have introduced coordinates with R_0^k as origin. Accordingly, U is homeomorphic to a kl-dimensional Euclidean space. The system of coordinates which we have selected in U depends on the basis (1) of the space R^{k+l}; the analyticity of the transformation from one coordinate system to another is demonstrated without difficulty.

Definition 2. Let $\omega(i)$ be an integral-valued monotone non-decreasing function of the integer argument $i = 1, \ldots, k$; $0 \leq \omega(i) \leq l$. In the space R^{k+l} (see definition 1) we choose an increasing sequence

$$R_1 \subset R_2 \subset \ldots \subset R_k \tag{3}$$

of vector subspaces, such that

$$D(R_i) = i + \omega(i), \quad i = 1, \ldots, k.$$

By $Z(\omega)$ we denote the set of those elements R^k of $H(k,l)$ which are such that

$$D(R^k \cap R_i) \geqslant i, \quad i = 1, \ldots, k.$$

It is clear that $Z(\omega)$ is compact, and that $Z(\omega) = \hat{Z}(\omega)$. It will be shown below, (see (G)), that $Z(\omega)$ is a pseudomanifold of dimension $r(\omega) = \sum_{i=1}^{k} \omega(i)$.

If the pseudomanifold $Z(\omega)$ is orientable, then assigning to it one of the two possible orientations, we have a cycle in $H(k,l)$ in the field of integer coefficients.

If the pseudomanifold $Z(\omega)$ is non-orientable, then we may consider it as a cycle in $H(k,l)$ modulo two.

In place of the sequence (3) let us now consider in R^{k+l} a sequence

$$R'_1 \subset R'_2 \subset \ldots \subset R'_k \tag{4}$$

of vector subspaces of the same dimensions:

$$D(R'_i) = D(R_i), \quad i = 1, \ldots, k.$$

Just as we defined the pseudomanifold $Z(\omega)$ with the aid of (3), so we define a $Z'(\omega)$ with the aid of (4). Clearly, there exists a continuous rotation of R^{k+l} carrying the sequence (3) into the sequence (4). Since $H(k,l)$ goes into itself under this rotation, the pseudomanifold $Z(\omega)$ moves in $H(k,l)$ and at the completion of the rotation occupies the position $Z'(\omega)$.

Thus the pseudomanifolds $Z(\omega)$ and $Z'(\omega)$, considered as cycles and taken with suitable orientation, are homologous in $H(k,l)$. This explains why the notation for $Z(\omega)$ does not involve the sequence (3) but only the function ω.

(B) By a point of increase of the function ω (see definition 2) we will mean a value i of its argument for which $\omega(i+1) \neq \omega(i)$.

Let $i_1, \ldots, i_{n-1}$ be the set of all points of increase of the function ω, taken in increasing order; and write $i_0 = 0$, $i_n = k$. Let us denote the space R_{i_h} of the sequence (3) by S_h; we consider the increasing sequence

$$S_1 \subset S_2 \subset \ldots \subset S_n \tag{5}$$

of vector subspaces of the space R^{k+l}. It turns out that if we impose on the element R^k of $H(k,l)$ the condition

$$D(R^k \cap S_h) \geq i_h, \quad h = 1, \ldots, n.$$

we again have the set $Z(\omega)$. Thus, $Z(\omega)$ is uniquely defined by the subsequence (5) of the sequence (3).

We will demonstrate (B). Suppose that i is not a point of increase of the function ω and is not equal to k. We remark that if $D(R^k \cap R_{i+1}) \geq i + 1$, then $D(R^k \cap R_i) \geq i$; from this it follows that we can delete the space R_i from the sequence (3) without ill consequences. By this means, the possibility of passing from the sequence (3) to the subsequence (5) is guaranteed.

We put

$$R^k \cap R_{i+1} = R^s, \quad R^k \cap R_i = R^r.$$

The dimensions of the spaces R^s and R^r are equal to s and r respectively. Evidently, $R^r = R^s \cap R_i$, and since R^s and R_i are both contained in R_{i+1},

$$r \geq s + (i + \omega(i)) - (i + 1 + \omega(i+1)) = s - 1.$$

But $s \geq i + 1$. Therefore, $r \geq i$.

Thus, the assertion (B) is proved.

(C) We will say that the element R^k of the set $Z(\omega)$ is in general position in $Z(\omega)$, if

$$D(R^k \cap S_h) = i_h, \quad h = 1, \ldots, n. \text{ (See (B))}.$$

If R^k_o is in general position in $Z(\omega)$, there exists a basis

$$e_1, \ldots, e_k, f_1, \ldots, f_l \tag{6}$$

of the space R^{k+l} such that the vectors

$$e_1, \ldots, e_k \tag{7}$$

from a basis for R^k_o, defining its orientation, and the vectors

$$e_1, \ldots, e_{i_h}, f_1, \ldots, f_{\omega(i_h)}, \quad h = 1, \ldots, n, \tag{8}$$

for a basis for the space S_h.

Now let U be a coordinate region in $H(k, l)$ constructed on the basis (6) in the same way as was done in (A). It turns out that all the elements of $U \cap Z(\omega)$ are in general position in $Z(\omega)$, and that $R^k_\xi \in U$ belongs to $Z(\omega)$ if and only if the matrix $\xi = \|\xi^j_i\|$ satisfies the condition:

$$\xi^j_i = 0 \text{ for } j > \omega(i), \quad i = 1, \ldots, k. \tag{9}$$

Thus, the neighborhood $U \cap Z(\omega)$ of the element R^k_o in the set $Z(\omega)$ is homeomorphic to the Euclidean space of dimension $r(\omega) = \sum_{i=1}^{k} \omega(i)$.

We begin the proof of (C) by choosing in R^{k+l} a basis of the type (6) indicated in (C).

We set $R^k_o \cap S_h = R^{i_h}$. Since the spaces R^{i_h}, $h = 1, \ldots, n$, form an increasing system, the vectors $e_1, \ldots, e_k$ can easily be chosen so that the system $e_1, \ldots, e_{i_h}$ forms a basis for R^{i_h}, $h = 1, \ldots, n$. Since the vectors $e_1, \ldots, e_{i_1}$ are independent, and lie in S_1, the basis for S_1 can be constructed by completing the system $e_1, \ldots, e_{i_1}$ with an arbitrary system, independent of these, of vectors $f_1, \ldots, f_{\omega(i_1)}$ lying in S_1. Thus we will have constructed a system of the type (8) for $h = 1$, such that it forms a basis for S_1.

Now suppose that the system (8) has been constructed for some given h in such a way that it forms a basis for S_h. The vectors of the system

$$e_1, \ldots, e_{i_h}, f_1, \ldots, f_{\omega(i_h)}$$

are linearly independent, and the vectors $e_{i_h+1}, \ldots, e_{i_{h+1}}$ are independent of them in view of the fact that $D(R^k_o \cap S_h) = i_h$. Since the vectors $e_1, \ldots, e_{i_{h+1}}, f_1, \ldots, f_{\omega(i_h)}$ are independent and lie in S_{h+1}, we can complete this system by means of vectors $f_{\omega(i_h)+1}, \ldots, f_{\omega(i_{h+1})}$ to a basis for the space S_{h+1}, and we have then the system (8) for a larger value of h.

In this way the system (8) is inductively constructed. If $\omega(k) < l$, we complete the system (8) for $h = n$ to form a system (6).

Let ξ be a matrix satisfying condition (9); we will show that

R_ξ^k belongs to $Z(\omega)$ and is in general position in $Z(\omega)$. We remember that R_ξ^k is spanned by the vectors $e'_1, \ldots, e'_k$, and the vector e'_i is given by formula (2). In view of condition (9), the vector e'_i belongs to S_h for $i \leq i_h$, and accordingly

$$D(R_\xi^k \cap S_h) \geq i_h.$$

One easily sees, then, that in fact

$$D(R_\xi^k \cap S_h) = i_h.$$

Therefore, R_ξ^k belongs to $Z(\omega)$ and is in general position in $Z(\omega)$.

We now suppose that $R_\xi^k \in Z(\omega)$, and we show that the matrix ξ satisfies condition (9).

Let x be an arbitrary vector belonging to $R_\xi^k \cap S_h$. Since $x \in R_\xi^k$, we can write $x = \sum_{i=1}^{k} x^i e'_i$. Since $x \in S_h$, it follows that $x^i = 0$ for $i > i_h$, so that

$$x = \sum_{i=1}^{i_h} x^i e'_i. \tag{10}$$

Since $D(R_\xi^k \cap S_h) \geq i_h$, and the expression on the right side of (10) contains exactly i_h parameters, these parameters must be given arbitrary values; in particular, we see that the vector e'_i belongs to S_h for $i \leq i_h$. Since the vector e'_i is expressed by formula (2) and belongs to the space S_h, for which the system (8) forms a basis, we must have

$$\xi_i^j = 0 \quad \text{for} \quad i \leq i_h, \ j > \omega(i_h). \tag{11}$$

We suppose that $i_{h-1} < i \leq i_h$; then $\omega(i) = \omega(i_h)$, and the relation (11) can be written in the form

$$\xi_i^j = 0 \quad \text{for} \quad i_{h-1} < i \leq i_h, \quad j > \omega(i), \quad h = 1, \ldots, n,$$

and this means that the matrix ξ satisfies condition (9).

Thus, assertion (C) is fully demonstrated.

(D) Suppose that $\omega \not\equiv 0$. We assume that the space R^{k+l} and the spaces making up the system (5) are oriented. If $\omega(k) = l$, the space R^{k+l} and the space S_n coincide; in this case we will assume that their orientations also coincide. It turns out that the basis (6) for the space R^{k+l}, as constructed in (C), can be so chosen that the orientations of the spaces R^{k+l}, R_o^k, S_h are determined in accordance with their bases (6), (7), (8).

In order to fulfil this requirement, it is sufficient to change the signs of certain vectors of (6). This operation would not be permissible if $\omega \equiv 0$, since the spaces R_o^k and S_1 would coincide and we might encounter conflict between their orientations, which are given in advance. If, of course, these orientations agree with one another, the given requirement may be fulfilled even if $\omega \equiv 0$.

(E) For $\omega \not\equiv 0$ the set M of all elements in general position in $Z(\omega)$ is connected; M is then a manifold of dimension $r(\omega)$. (See (C)).

For $\omega \equiv 0$ the set $Z(\omega)$ consists of two elements R_o^k and $\hat{R}_o^k$, and these are in general position in $Z(\omega)$.

We will show that for $\omega \not\equiv 0$, M is connected. We suppose that the space S_h of the system (5) and the space R^{k+l} are oriented as in (D). We set

$$S_0 = \{0\}, \quad S_{n+1} = R^{k+l}.$$

Further, we denote by T_{h+1} a vector subspace of S_{h+1} such that S_{h+1} is the direct sum of S_h and T_{h+1}, for $h = 0, 1, \ldots, n$.

We denote by A the group of all linear transformations of the space R^{k+l} which carry every space S_h into itself with preservation of orientation, and we show that A is connected.

Let A_1 be the group of all linear transformations of R^{k+l} which carry the spaces T_h into themselves with preservation of orientation; then $A_1 \subset A$. Since R^{k+l} is the direct sum of all spaces T_h, the group A_1 is the direct product of a group of positive (i.e., sense-preserving) transformations of the spaces T_h.

Therefore the group A_1 is connected. Suppose that

$$a \in A, \quad x = \sum_{h=1}^{n+1} x_h, \quad x_h \in T_h.$$

We set

$$a(x_h) = b(x_h) + a_1(x_h),$$

where $b(x_h) \in S_{h-1}$, $a_1(x_h) \in T_h$. Now suppose that t is a real number, $0 \le t \le 1$, and let

$$a_t(x) = \sum_{h=1}^{n+1} (1-t)\, b(x_h) + a_1(x_h).$$

We have

$$a_t \in A, \quad a_0 = a, \quad a_1 \in A_1.$$

Therefore, A is a connected group.

Let R_o^k and $\overline{R}_o^k$ be two elements of M, and $e_1, \ldots, e_k, f_1, \ldots, f_l$ and $\overline{e}_1, \ldots, \overline{e}_k, \overline{f}_1, \ldots, \overline{f}_l$ be two bases for the space R^{k+l}, constructed for the elements R_o^k and $\overline{R}_o^k$ by the process indicated in (C) and (D).

We define a linear transformation a, putting

$$a(e_i) = \overline{e}_i, \quad i = 1, \ldots, k, \qquad a(f_j) = \overline{f}_j, \quad j = 1, \ldots, l.$$

Then $a \in A$, and $a(R_o^k) = \overline{R}_o^k$.

Thus M admits a connected transitive group A of transformations, and is therefore connected.

(F) Put $N = Z(\omega) - M$. (See (E).) We show that

$$D(N) \leqslant r(\omega) - 3, \quad D(Z(\omega)) = r(\omega).$$

Suppose R^k is not in general position in $Z(\omega)$; then for some number q $(1 \le q \le n-1)$,

$$D(R^k \cap S_q) \geqslant i_q + 1.$$

We introduce a new function $\omega_{(q)}(i)$, defined by the conditions:

$$\begin{aligned} &\text{for } i \leqslant i_{q-1}: \quad \omega_{(q)}(i) = \omega(i), \\ &\text{for } i_{q-1} < i \leqslant i_q + 1: \quad \omega_{(q)}(i) = \omega(i_q) - 1, \\ &\text{for } i > i_q + 1: \quad \omega_{(q)}(i) = \omega(i). \end{aligned}$$

The function $\omega_{(q)}(i)$ has no jump at the point i_q; instead, there appears a jump at the point $i_q + 1 = i'_q$. For the remaining values of h we take $i'_h = i_h$. Then R^k satisfies the condition

$$D(R^k \cap S_h) \geqslant i'_h, \quad h = 1, \ldots, n,$$

and therefore $R^k \in Z(\omega_{(q)})$.

We have

$$r(\omega) - r(\omega_{(q)}) = \sum_{i=1}^{k} (\omega(i) - \omega_{(q)}(i)) = i_q - i_{q-1} + 1 + \omega(i_q + 1) - \omega(i_q) \geqslant 3. \tag{12}$$

Further, it is evident that

$$N \subset \sum_{q=1}^{n-1} Z(\omega_{(q)}). \tag{13}$$

If we now note that $D(Z(\omega')) = r(\omega')$ for any function ω' satisfying the condition $r(\omega') < r(\omega)$, the relations (12) and (13) yield:

$$D(N) \leqslant r(\omega) - 3.$$

Since, further, $Z(\omega) = M + N$, $D(Z(\omega))$ is equal to the larger of the numbers $D(M)$ and $D(N)$, that is, $r(\omega)$.

Thus, assertion (F) is proved.

(G) It follows immediately from propositions (E) and (F) that the space $Z(\omega)$, represented in the form of a complex, is a pseudo-manifold of dimension $r(\omega)$.

(H) For $kl > 1$ the manifold $H(k,l)$ is simply connected; that is, it has a trivial fundamental group, and since $H(1,1)$ is homeomorphic to a circle, $H(k,l)$ is always orientable. Further, for $k \ge 2$, $l \ge 2$, the region $H(k,l) - Z(\omega)$ is simply connected.

Suppose R^{k+l} is an oriented Euclidean vector space. Then to every oriented k-dimensional subspace R^k of the space R^{k+l} corresponds uniquely its oriented complement R^l, $R^k \to R^l$. This correspondence yields, evidently, a homeomorphic mapping of the manifold $H(k,l)$ on the manifold $H(l,k)$; we denote this by ψ.

It is clear that the manifold $H(1,l)$ is homeomorphic to an l-dimensional sphere, and by applying the mapping ψ we can convince ourselves that $H(k,1)$ is homeomorphic to a k-dimensional sphere.

Therefore, if $kl > 1$, and one or other of k, l is equal to unity, then $H(k,l)$ is simply connected.

Let P^s be an s-dimensional subspace of the space R^{k+l}, and let Q^{k+l-s} be its orthogonal complement in R^{k+l}. We set

$$\omega_1(1) = l-1, \quad \omega_1(2) = \dots = \omega_1(k) = l;$$
$$\omega_2(1) = l-2, \quad \omega_2(2) = \dots = \omega_2(k) = l;$$
$$\omega_3(1) = \omega_3(2) = l-1, \quad \omega_3(3) = \dots = \omega_3(k) = l.$$

The pseudomanifolds $Z(\omega_1)$, $Z(\omega_2)$, $Z(\omega_3)$ are defined, respectively, by the conditions

$$D(R^k \cap P^l) \geqslant 1, \quad D(R^k \cap P^{l-1}) \geqslant 1, \quad D(R^k \cap P^{l+1}) \geqslant 2. \quad \text{(See (B).)}$$

We have

$$r(\omega_1) = kl - 1, \quad r(\omega_2) = r(\omega_3) = kl - 2.$$

It is easily seen that ω_1, ω_2, ω_3 exhaust the set of functions ω, for which $r(\omega) \geq kl - 2$, with the exception of $\omega \equiv l$.

It is not difficult to see that under the mapping ψ the pseudomanifold $Z(\omega_2) \subset H(k,l)$ goes into the pseudomanifold $Z(\omega_3) \subset H(l,k)$.

The region $H(k,l) - Z(\omega_1)$ consists of two regions U and $\hat{U}$ (see (A)) and is therefore simply connected.

The region $H(k,l) - Z(\omega_2)$ consists of all $R^k \in H(k,l)$ whose intersection with P^{l-1} contains the zero point only; therefore the orthogonal projection of R^k on Q^{k+1} can be carried out without degeneration. If we carry R^k by a definite motion into its projection $R^k_o \subset Q^{k+1}$, we have a continuous deformation of the region $H(k,l) - Z(\omega_2)$ into the manifold $H(k,l)$ consisting of all R^k_o lying in Q^{k+1}. Thus, the fundamental group of $H(k,l) - Z(\omega_2)$ is isomorphic to the fundamental group of the manifold $H(k,l)$, and for $k > 1$ is trivial.

Since $r(\omega_2) = kl - 2$, an arbitrary curve in $H(k,l)$ can be deformed into a curve lying in $Z(\omega_2)$; and in $H(k,l) - Z(\omega_2)$ this curve can be shrunk to a point if $k > 1$. Thus, in this case, the fundamental group of $H(k,l)$ is trivial.

The region $H(k,l) - Z(\omega_3)$ is carried by the mapping ψ into the region $H(l,k) - Z(\omega_2)$, and therefore is simply connected for $l > 1$.

Now if $r(\omega) \leq kl - 3$, the region $H(k,l) - Z(\omega)$ is simply connected, because the manifold $H(k,l)$ is. In fact, not only every curve, but also every two-dimensional surface in $H(k,l)$, can be continuously deformed into $Z(\omega)$.

Thus, proposition (H) is proved.

§ 2. Orientation of the Pseudomanifold $Z(\omega)$

We now consider that to every orientable pseudomanifold $Z(\omega)$ there is pre-assigned a certain definite orientation.

(A) Let R^n be an n-dimensional vector space, in which a certain system of cartesian coordinates has been introduced; that is, every $x \in R^n$ has been put into correspondence with a sequence of numbers $x^1, \dots, x^n$, $x = (x^1, \dots, x^n)$. To the chosen system of coordinates in R^n corresponds a definte basis $e_1, \dots, e_n$ such that $x = \sum_{i=1}^{n} x^i e_i$, and this basis assigns a definite orientation to the space R^n, as was pointed out at the beginning of paragraph 1. Thus, to a definite coordinate system in R^n corresponds a definite orientation of R^n.

We will denote by $\{R^n;\ x^{j_1} = 0, \dots, x^{j_r} = 0\}$ a coordinate plane of the space R^n, defined by the equations $x^{j_1} = 0, \dots, x^{j_r} = 0$; in this plane the coordinate system is $x^{i_1}, \dots, x^{i_{n-r}}$ and therefore a definite orientation is given.

A subspace of the space R^n, given by the inequality $x^m < 0$, will be denoted by $\{R^n,\ x^m < 0\}$; we can take it as oriented in the same way as R^n. It is easy to see that

$$\Delta\{R^n, x^m < 0\} = (-1)^{m-1}\{R^n, x^m = 0\}. \tag{1}$$

In the coordinate neighborhood U (see § 1, (A)) there are given coordinates ξ^j_i which are elements of the matrix ξ. In order that these coordinates specify an orientation in U and in the coordinate planes of U, it is sufficient to number them in some definite way. We will suppose that the subscript indicates the number of the row and the superscript the number of the column; we will order the elements by columns; that is, we write them in a sequence as follows:

$$\xi^1_1, \xi^1_2, \ldots, \xi^1_k, \quad \xi^2_1, \xi^2_2, \ldots, \xi^2_k, \ldots, \xi^l_1, \xi^l_2, \ldots, \xi^l_k. \tag{2}$$

The coordinate surface $U \cap Z(\omega)$, given by equations (9) § 1, has a definite orientation, consistent with our condition; we will later assign this to the neighborhood $U \cap Z(\omega)$; this orientation depends, therefore, on the bases (6) § 1 of the space R^{k+l}.

(B) Let $\omega \neq 0$ be a function such that the pseudomanifold $Z(\omega)$ is orientable. We will suppose that the spaces $S_1, \ldots, S_n$ of the system (5) § 1, defining $Z(\omega)$, are all oriented, and we assign the orientation of $Z(\omega)$ in accordance with the chosen orientations of the spaces S_h. We assign the space R^{k+l} any orientation, except that if $\omega(k) = l$ this orientation is to coincide with the chosen orientation of S_n. We so choose the basis (6) § 1 of the space R^{k+l} that condition (D) § 1 shall be satisfied. This basis determines the orientation of the neighborhood $U \cap Z(\omega)$ of the element $R^k_o \in Z(\omega)$ in the same way as was done in (A). Since the pseudomanifold $Z(\omega)$ is orientable, the orientation of the neighborhood $U \cap Z(\omega)$ induces an orientation of the whole of $Z(\omega)$. It turns out that the orientation of $Z(\omega)$ so determined depends only on the orientations chosen for the spaces of the system (5) § 1. We will clarify the nature of this dependence in the case where $\omega(1) > 0$.

Let us introduce the notation:

$$\left.\begin{array}{l} \alpha_h = i_h - i_{h-1}, \quad h = 1, \ldots, n; \\ \beta_h = \omega(i_h + 1) - \omega(i_h), \quad h = 1, \ldots, n-1; \quad \beta_n = l - \omega(k) \\ (\text{see § 1, (B)}); \end{array}\right\} \tag{3}$$

the numbers β_h will be used later. Now suppose

$$S'_h = \varepsilon_h S_h, \quad \varepsilon_h = \pm 1, \quad h = 1, \ldots, n.$$

The pseudomanifold $Z'(\omega)$, constructed with the aid of the spaces S'_h, can differ from $Z(\omega)$ only as to orientation. It turns out that

$$\text{for } \omega(1) > 0: Z'(\omega) = \varepsilon_1^{\alpha_1} \cdot \varepsilon_2^{\alpha_2} \cdots \varepsilon_n^{\alpha_n} Z(\omega). \tag{4}$$

Suppose R^k_o and $\bar{R}^k_o$ are two elements in general position in $Z(\omega)$, and

$$e_1, \ldots, e_k, \quad f_1, \ldots, f_l, \tag{5}$$

$$\bar{e}_1, \ldots, \bar{e}_k, \quad \bar{f}_1, \ldots, \bar{f}_l, \tag{6}$$

are bases for the space R^{k+l} constructed from these elements by the process indicated in (C) and (D), § 1, i.e., taking account of the orientations of the spaces S'_h and R^{k+l}. We show that the orientations induced in $Z(\omega)$ by the bases (5) and (6) coincide.

We define a linear mapping a of the space R^{k+l}, setting:

$$a(e_i) = \bar{e}_i, \quad i = 1, \ldots, k; \qquad a(f_j) = \bar{f}_j, \quad j = 1, \ldots, l.$$

By virtue of the construction given in (E), § 1, there exists a one-parameter family a_t of linear mappings of the space R^{k+l}, such that $a_t(S_h) = S_h$, $h = 1, \ldots, n$, and $a_1 = a$ and a_o is the identity mapping. The basis

$$a_t(e_1), \ldots, a_t(e_k), \quad a_t(f_1), \ldots, a_t(f_l)$$

of the space R^{k+l} defines an orientation in the neighborhood $a_t(U \cap Z(\omega)) \subset Z(\omega)$. From continuity considerations it follows that the orientation so defined on the pseudomanifold $Z(\omega)$ always coincides with the orientation induced by the basis (5). For $t = 1$, this yields the desired result.

Let us demonstrate (4). We remark, first of all, that if in the basis (5) we change the orientation of the vector f_q, the orientation of the neighborhood $U \cap Z(\omega)$, given in (B), is multiplied by the factor $(-1)^\gamma$, where γ is the number of coordinates ξ^q_i which can be distinct from zero in $U \cap Z(\omega)$ (cf. (9) § 1). For $q = \omega(i_{h-1} + 1)$ we have $\gamma = k - i_{h-1}$; for $q = \omega(i_{h-1} + 1) + 1$, we have $\gamma = k - i_h$. Let us now change the sign of each of the indicated vectors; then the orientation of the neighborhood $U \cap Z(\omega)$ is multiplied by $(-1)^{\alpha_h}$ (cf. (3)). On the other hand, the indicated changes of sign of two vectors change the orientation of the space S_h, and do not change the orientation of the remaining spaces of the sequence (5), § 1. The orientation of the space R^{k+l} changes under this operation only in the case that $h = n$ and $S_n = R^{k+l}$. From what has been said, formula (4) follows. If $\omega(k) < l$, the orientation of $Z(\omega)$ does not depend on the orientation of R^{k+l}, since, changing the sign of the vector f_q only, for $q = \omega(i_{n-1} + 1) + 1$, we have $\gamma = k - i_n = 0$.

(C) Let ω be a function such that $\omega(1) > 0$, $Z(\omega)$ is orientable, and $2Z(\omega)$ not homologous to zero in $H(k, l)$; then

$$\text{for} \quad \omega(k) < l: \quad \alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_n \equiv 0 \pmod 2; \tag{7}$$

$$\text{for} \quad \omega(k) = l: \quad \alpha_1 \equiv \alpha_2 \equiv \ldots \equiv \alpha_{n-1} \equiv 0 \pmod 2. \tag{8}$$

Therefore, in the case $\omega(k) < l$ the orientation of $Z(\omega)$, given in (B), does not depend on the orientation of the spaces of the system (5) § 1 (cf. 4); in the case $\omega(k) = l$ the orientation of $Z(\omega)$ depends only on the orientation of the space $S_n = R^{k+l}$. Both these cases can be summed up in the single formula

$$Z'(\omega) = \epsilon^k Z(\omega), \quad \omega(1) > 0, \tag{9}$$

where $Z(\omega)$ is constructed by starting from the oriented space R^{k+l}, and $Z'(\omega)$ by starting from the oriented space ϵR^{k+l}, $\epsilon = \pm 1$. It turns out, moreover, that if we fix the orientation of the space R^{k+l}, and construct the oriented pseudomanifold $Z(\omega)$, starting from the system of spaces (5) § 1, and also construct the oriented pseudomanifold $Z^*(\omega)$, starting from some other system S_h^*, $h = 1, \ldots, n$, then

$$Z^*(\omega) \sim Z(\omega) \quad \text{in} \quad H(k, l). \tag{10}$$

Let us demonstrate (7) and (8). Suppose

$$S'_h = \epsilon_h S_h, \quad \epsilon_h = \pm 1, \quad h = 1, \ldots, n.$$

If $\omega(k) = l$, we set $\epsilon_n = +1$. Then it is easy to construct a linear mapping a of the space R^{k+l}, conserving its orientation, and such that

$$a(S_h) = S'_h, \quad h = 1, \ldots, n.$$

Since a conserves the orientation of R^{k+l}, there exists a one-

parameter family of linear mappings a_t of the space R^{k+l}, such that $a_1 = a$ and a_0 is the identity mapping. The pseudomanifold $a_t(Z(\omega))$ depends on the parameter t; for $t = 0$ it becomes $Z(\omega)$, and for $t = 1$ it becomes $Z'(\omega)$; thus, $Z'(\omega) \sim Z(\omega)$. Hence, from relation (4) follows

$$(1 - \epsilon_1^{\alpha_1} \cdot \epsilon_2^{\alpha_2} \cdots \epsilon_n^{\alpha_n}) Z(\omega) \sim 0.$$

Since the relation $2Z(\omega) \sim 0$ is inadmissible by hypothesis, we see that (7) and (8) are in fact true.

Let us prove (9). In virtue of (3), $k = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. From this, because of the relations (4), (7), and (8), we derive (9).

Let us demonstrate (10). Let a be a linear orientation-preserving mapping of the space R^{k+l}, such that

$$a(S_h) = S_h^*, \quad h = 1, \ldots, n.$$

The spaces S_h and S_h^* are here regarded as not oriented, since their orientation does not determine that of the pseudomanifolds $Z(\omega)$ and $Z^*(\omega)$. Since a conserves the orientation of R^{k+l}, there exists a one-parameter family a_t of linear mappings of the space R^{k+l}, such that $a_1 = a$ and a_0 is the identity mapping. The oriented pseudomanifold $a_t(Z(\omega))$ depends on the parameter t. Its value is $Z(\omega)$ for $t = 0$ and $Z^*(\omega)$ for $t = 1$. Therefore, $Z(\omega) \sim Z^*(\omega)$.

Thus, the assertion (C) is completely proved.

(D) Let $\omega \equiv c$, where c is a natural number. It is easy to see that the pseudomanifold $Z(c)$ is homeomorphic to the manifold $H(k, c)$, which is orientable (cf. § 1, (H)); thus, $Z(c)$ is also orientable.

Since the function $\omega \equiv c$ has no jumps, we have $n = 1$, $\alpha_1 = k$, and the system (5) § 1 consists of the single space $S_1 = R^{k+c}$, the orientation of which defines the orientation of $Z(c)$ (cf. (B)). Also

$$Z'(c) = \epsilon_1^k Z(c) \quad \text{(cf. (4))}. \tag{11}$$

(E) Let R^{k+l} and $R^{k+l''} \subset R^{k+l}$, $l - l'' = m > 0$, be two oriented vector spaces; then $H(k, l'') \subset H(k, l)$, and both these manifolds are oriented by virtue of (B) (cf. (D)). Further, let $Z(\omega)$ be the pseudomanifold given by the system $\{S_h\}$, $h = 1, \ldots, n$. If $Z(\omega)$ is orientable, we will suppose that the spaces S_h are oriented and give the orientation of $Z(\omega)$.

We will study the algebraic intersection $Z(\omega) \times H(k, l'')$--modulo 2 if $Z(\omega)$ is non-orientable, and with the computed orientation otherwise.

Let us set $\omega'' = \omega - m$; the function ω'' is suitable for the definition of a pseudomanifold $Z(\omega_2)$ only if $\omega(1) \geq m$.

We will suppose that the spaces S_h and $R^{k+l''}$ are in general position in R^{k+l}, and we set $S''_h = S_h \times R^{k+l''}$ with the calculated orientation for orientable $Z(\omega)$. We have

$$D(S''_h) = i_h + \omega(i_h) - m.$$

Therefore, for $\omega(1) \geq m$, the spaces S''_h, $h = 1, \ldots, n$, can serve as a foundation for the construction of a pseudomanifold $Z(\omega'') \subset H(k, l'')$. It turns out that

$$\text{for } \omega(1) < m: \quad Z(\omega) \cap H(k, l'') \text{ is empty;} \qquad (12)$$

$$\text{for } \omega(1) \geq m, \quad \omega \not\equiv m: \quad Z(\omega) \times H(k, l'') = Z(\omega''); \qquad (13)$$

$$\text{for } \omega \equiv m: \quad I(Z(\omega), H(k, l'')) = 1 + (-1)^k. \qquad (14)$$

In (14) we have the index of intersection; in (13) the intersection mod 2 if $Z(\omega)$ is non-orientable, and with the calculated orientation otherwise.

Let us proceed to the demonstration. If $R^k \in Z(\omega)$, $R^k \in H(k, l'')$, then

$$D(R^k \cap S_h) > i_h, \quad R^k \subset R^{k+l''},$$

and therefore $D(R^k \cap S''_h) \geq i_h$. In the case $\omega(1) < m$, we have, therefore, $i_1 \leq i_1 + \omega(i_1) - m \leq i_1 - 1$; this is impossible, and so (12) is true. In the case $\omega(1) \geq m$, we have $Z(\omega) \cap H(k, l'') \subset Z(\omega'')$. The inverse inclusion relationship obviously holds, and therefore,

$$\text{for } \omega(1) \geq m: \quad Z(\omega) \cap H(k, l'') = Z(\omega''). \qquad (15)$$

We will now suppose that $\omega(1) \geq m$. If $\omega \equiv m$, we set $R^k_o = S''_1$. If $\omega \not\equiv m$, we choose for R^k_o an arbitrary element in general position in $Z(\omega'')$. Suppose that

$$e_1, \ldots, e_k, f''_1, \ldots, f''_{l''} \qquad (16)$$

is a basis for the space $R^{k+l''}$, constructed as in (C) § 1 for the elements $R^k_o \in Z(\omega'')$; that is, such that the vectors

$$e_1, \ldots, e_k \qquad (17)$$

represent a basis for the space R^k_o, and the vectors

$$e_1, \ldots, e_k, f''_1, \ldots, f''_{\omega''(i_h)}, \quad h = 1, \ldots, n, \qquad (18)$$

represent a basis for the space S''_h. The orientations of the spaces R^{k+l}, R^k_o, S''_h are defined in accordance with the bases (16), (17), and (18). In the space S_1 we choose vectors $f_1, \ldots, f_m$, linearly independent of S''_1, and make up a basis for the space R^{k+l} from the vectors

$$e_1, \ldots, e_k, f_1, \ldots, f_m, f_{m+1} = f''_1, \ldots, f''_l = f''_{l''}. \qquad (19)$$

Then the basis of the space S_h will consist of the vectors

$$e_1, \ldots, e_k, f_1, \ldots, f_{\omega(i_h)}. \qquad (20)$$

We choose the vectors $f_1, \ldots, f_m$ in such a way that the basis (19) determines the orientation of the space R^{k+l}. Then, from the condition $S''_h = S_h \times R^{k+l''}$ it follows without difficulty that the basis (20) gives the orientation of the space S_h.

The basis (19) defines a coordinate neighborhood U of the element R^k_o in $H(k, l)$ (see § 1, (A)). The neighborhood $U' = U \cap Z(\omega)$ of the element R^k_o in $Z(\omega)$ is distinguished in U by the equations

$$\xi^j_i = 0 \quad \text{for } j > \omega(i), \quad i = 1, \ldots, k. \qquad (21)$$

The neighborhood $U'' = U \cap H(k, l'')$ of the element R^k_o in $H(k, l'')$ is distinguished by the equations

$$\xi^j_i = 0 \quad \text{for } j \leq m. \qquad (22)$$

The neighborhood $U''' = U \cap Z(\omega'')$ of the element R_o^k in $Z(\omega'')$ arises as the union of the systems (21) and (22). The orientation of the coordinate spaces U, U', U'', U''' is given by the rule stated in (A), i.e., with the aid of an enumeration of the elements of the matrix ξ by columns.

Let A be the sequence of all unit coordinate vectors of U, written in that order. We will denote the portions of the sequence A which are in U', U'', U''' by A', A'', A''', respectively. Then the orientations of the neighborhoods U, U', U'', U''' are given by the respective sequences A, A', A'', A'''. The sequence derived from A' by removing the elements contained in A''' we will denote by B'. Similarly, the sequence derived from A'' by removing the elements of A'' which are contained in A''' we will denote by B''. Since the numeration was carried out by columns, it follows that in the sequence A every element of B' precedes every element of B''. Let

$$A = \epsilon(A'''B'B''), \quad A' = \epsilon'(A'''B'), \quad A'' = \epsilon''(A'''B''). \tag{23}$$

Here $\epsilon = \pm 1$, $\epsilon' = \pm 1$, $\epsilon'' = \pm 1$, and the relationships (23) disclose the connection between the orientations of the spaces under consideration, as given with the help of the several sequences of vectors. In view of the fact that in the sequence A every element of B' precedes every element of B'', we have $\epsilon = \epsilon' \cdot \epsilon''$, and this, in virtue of Lefschetz' rule, shows that $U''' = U' \times U''$ with orientations as computed.

Thus, the relationship (13) is demonstrated.

For $\omega \equiv m$, the result so obtained yields $I(U', U'') = +1$. Therefore, at the point R_o^k the index of intersection of $Z(\omega)$ and $H(k, l'')$ is equal to $+1$. For $\omega \equiv m$, the intersection $Z(\omega) \cap H(k, l'') = Z(\omega'') = Z(0)$ (see (15)) contains only two points, the one already considered R_o^k, and another $\hat{R}_o^k$. In order to calculate the index of intersection of $Z(\omega)$ and $H(k, l'')$ at the point $\hat{R}_o^k$, we reverse the orientation of S_1, i.e., take $S_1' = -S_1$. We will denote by $Z'(\omega)$ the pseudomanifold defined by the space S_1'. Then

$$Z'(\omega) = (-1)^k Z(\omega) \quad \text{(see (11))}. \tag{24}$$

The roles of the elements R_o^k and $\hat{R}_o^k$ are here interchanged: R_o^k was defined as $S_1'' = S_1 \cap R^{k+l''}$. Therefore, the index of intersection of $Z'(\omega)$ and $H(k, l'')$ at the point $\hat{R}_o^k$ is equal to $+1$, and from (24) we see that the index of intersection of $Z(\omega)$ and $H(k, l'')$ at the point $\hat{R}_o^k$ is equal to $(-1)^k$. Accordingly, the relationship (14) does in fact hold.

Thus assertion (E) is demonstrated.

The pseudomanifolds $Z(\omega)$ yield a significant collection of cycles of the manifold $H(k, l)$. The problem of constructing from them a homology basis will be resolved in a later section (see § 5). We will now give a general operation for the construction of cycles.

(F) Let Z be an r-dimensional cycle modulo two. Assigning to each of its simplexes an orientation, we have a chain Y with integer coefficients, such that $\Delta Y = 2X$, where X is an integer-cycle of dimension $r - 1$, the homology class of which is uniquely defined by the homology class of the cycle Z. We set $X = \Gamma Z$. The operation Γ, defined up to a homology, will be applied later to the non-orientable pseudomanifold $Z(\omega)$.

If Z is an r-dimensional ∇-cycle modulo two, then, assigning to its simplexes an arbitrary orientation, we have a whole-number chain Y, such that $\nabla Y = 2X$, where X is an (integer-coefficient) ∇-cycle of dimension $r + 1$, the homology class of which is uniquely defined by the homology class of the ∇-cycle Z. We write $X = \Gamma Z$.

§ 3. Tangential Representations and Characteristic Cycles

Here we introduce a concept which is fundamental to the work as a whole: the concept of characteristic cycle of a closed differentiable oriented k-dimensional manifold M^k.

Let U^k be some coordinate neighborhood of the point a of the differentiable manifold M^k, in which are introduced local coordinates $x^1, \ldots, x^k$. The mapping f of the manifold M^k on the vector space R^{k+l} with coordinates $y^1, \ldots, y^{k+l}$ is called by Whitney *regular in the point* a if near the point a the mapping can be written in coordinate form as

$$y^j = y^j(x^1, \ldots, x^k) = f^j(x), \quad j = 1, \ldots, k + l, \tag{1}$$

and the functional matrix

$$F = \left\| \frac{\partial f^j(x)}{\partial x^i} \right\|, \quad i = 1, \ldots, k, \quad j = 1, \ldots, k + l.$$

is of rank k at the point a and also depends continuously on x there. If the mapping f is regular at every point $a \in M^k$, then it is called *regular*. Whitney showed that a differentiable manifold M^k can be regularly mapped on the vector space R^{2k}. The manifold M^k can be mapped on the vector space R^{2k+1} both regularly and homeomorphically at the same time. We will call such a mapping the inclusion of the differentiable manifold M^k in the vector space R^{k+l} and we will sometimes identify the image $f(M^k)$ with the original manifold M^k and write $M^k \subset R^{k+l}$.

We shall have occasion below to consider mappings of the manifold M^k on various manifolds $H(k,l)$ (cf. definition 1); in view of this it is worth while to establish the concept of equivalence of such mappings.

(A) Let $R'^{k+l'}$ and $R''^{k+l''}$ be two vector spaces, and $H'(k,l')$ and $H''(k,l'')$ the manifolds corresponding to these (cf. definition 1).

Two mappings θ' and θ'' of the p-dimensional complex K^p on the manifolds $H'(k,l')$ and $H''(k,l'')$, respectively, will be called *equivalent* if there exist non-degenerate linear mappings a' and a'' of the vector spaces $R'^{k+l'}$ and $R''^{k+l''}$ on some vector space R^{k+l}, such that the mappings $a'\theta'$ and $a''\theta''$ of the complex K^p on $H(k,l)$ are homotopic to one another in $H(k,l)$. Here we keep in mind that to the linear mappings a' and a'' correspond the mappings a' and a'' of the manifolds $H'(k,l')$ and $H''(k,l'')$ in $H(k,l)$. The reflexivity and symmetry of this definition is evident; the transitivity will be demonstrated below. It turns out that every mapping θ of the complex K^p in some $H(k,l)$ is equivalent to some mapping of it in $H(k,p)$, and, accordingly, to some mapping in $H(k, p+1)$; since $H(k,p) \subset H(k, p+1)$. Further, two mappings θ' and θ'' of the complex K^p in the manifold $H(k, p+1)$ are equivalent in the given sense of the word when and only when they are homotopic to each other in $H(k, p+1)$. Therefore, in order to classify the mappings of the complex K^p from the point of view of the equivalence relationship just introduced, it is sufficient to classify all mappings of K^p in the manifold $h(k, p+1)$ according to the usual homotopy theory. From this the transitivity of the equivalence follows.

For the proof of the assertions made in (A), we prove (B).

(B) Let R^{k+l} and $R^{k+l-1} \subset R^{k+l}$ be two vector spaces, and $H(k, l-1) \subset H(k,l)$ manifolds corresponding to them (see definition 1). It turns out that if $p \leq l - 1$, then every mapping θ of the complex K^p in $H(k,l)$ is homotopic to some mapping θ' of the complex K^p in $H(k, l-1)$. Further, if $p \leq l - 2$, and θ_o and θ_1 are two mappings of the complex K^p in $H(k, l-1)$ which are homotopic in $H(k,l)$, they are also homotopic in $H(k, l-1)$.

We will suppose that the space R^{k+l} is Euclidian, and denote by e the unit vector in R^{k+l} normal to R^{k+l-1}. We will denote the set of all elements $R^k \in H(k,l)$ which contain e by H. It is easy to see that H is homeomorphic to the manifold $H(k-1, l)$ and therefore has dimension $(k-1)l$.

Now let R^k_o be an element of $H(k,l)$, belonging neither to H nor to $H(k, l-1)$. We will denote the projection of the vector e on R^k_o by g, and the projection of the vector g on R^{k+l-1} by h. We will denote the unit vectors in the directions of g and h by e_o and e_1, respectively. We will denote the linear subspace of the space R^k_o which is orthogonal to e_o by R^{k-1}_o. Let e_t, $0 \leq t \leq 1$, be a unit vector, uniformly rotating from the position e_o to the position e_1 in the plane of the vectors e_o, e_1. We denote by R^k_t the linear span of the vector e_t and the space R^{k-1}_o.

If $R^k_o \in H(k, l-1)$, we will suppose that $R^k_t = R^k_o$. Thus, to every element R^k_o not belonging to H, we have defined a process by which it moves from the position R^k_o to the position $R^k_1 \in H(k, l-1)$, while the elements of $H(k, l-1)$ do not move. The deformation so defined we will denote by ψ.

Suppose $p \leq l - 1$, and let θ be a mapping of the complex K^p in $H(k,l)$. Since the difference between the dimensions of $H(k,l)$ and H is equal to $l > p$, we can by a slight deformation of the mapping θ obtain a mapping θ'' such that $\theta''(K^p)$ does not intersect H. Applying the deformation ψ to the mapping θ'', we obtain the desired mapping θ'.

Now suppose $p \leq l - 2$, and that θ_t, $0 \leq t \leq 1$, is a family of mappings of the complex K^p in $H(k,l)$ such that

$$\theta_o(K^p) \subset H(k, l-1), \quad \theta_1(K^p) \subset H(k, l-1);$$

this means that the mappings θ_o and θ_1 are homotopic to one another in $H(k, l)$. We will denote by K^{p+1} the product of the complex K^p by the unit segment of the real number line. To the family θ_t corresponds in a known way a mapping θ of the complex K^{p+1} in $H(k, l)$. Just as before, by a slight deformation of the mapping θ we can transform it into a mapping θ'' such that $\theta''(K^{p+1})$ does not intersect H. Applying the deformation ψ to the mapping θ'', we obtain a mapping θ' of the complex K^{p+1} in $H(k, l-1)$ which realizes the homotopy between θ_0 and θ_1 in $H(k, l-1)$.

Thus assertion (B) is demonstrated.

Let us now demonstrate (A). It follows immediately from (B) that every mapping θ of the complex K^p on an $H(k, l)$ is equivalent to some mapping of the complex K^p on the manifold $H(k,p)$.

Now, let θ' and θ'' be two equivalent mappings of the complex K^p in $H(k, p+1)$. This means that there exist non-degenerate linear mappings a' and a'' of the space R^{k+p+1} on some space R^{k+l}, such that the mappings $a'\theta'$ and $a''\theta''$ of the complex K^p in $H(k,p)$ are homotopic in $H(k,l)$. Without loss of generality we may assume that $p + 1 < l$, so that there exists a one-parameter family a_t, $0 \leq t \leq 1$, of non-degenerate linear mappings of the space R^{k+p+1} on the space R^{k+l}, such that $a_0 = a'$, and $a_1 = a''$. Thus, the mappings $a_0\theta'$ and $a_1\theta'$ are homotopic in $H(k,l)$. Since, by hypothesis, the mappings $a_0\theta'$ and $a_1\theta''$ are homotopic in $H(k,l)$, the mappings $a_1\theta'$ and $a_1\theta''$ are homotopic in $H(k,l)$. It follows immediately from this, on the basis of (B), that the mappings $a_1\theta'$ and $a_1\theta''$ of the complex K^p on $a_1(H(k, p+1))$ are homotopic in the manifold $a_1(H(k, p+1))$ itself. Applying the mapping a_1^{-1}, the inverse of a_1, we see that the mappings θ' and θ'' are homotopic in $H(k, p+1)$. Thus assertion (A) is demonstrated.

Definition 3. Suppose f is a regular (although not necessarily homeomorphic) mapping of the differentiable oriented manifold M^k on the vector space R^{k+l}. (See (4).) Then to the point $x \in M^k$ there corresponds a well-defined oriented T_x, tangent to $f(U^k)$ at the point $f(x)$, where U^k is a small neighborhood of x in M^k. We will denote the oriented k-dimensional vector subspace of the space R^{k+l} which is parallel to T_x by $T(x)$. Since $T(x) \in H(k, l)$, we obtain a continuous mapping T of the manifold M^k on $H(k,l)$, which we will call the *tangential representation*. It turns out that all

tangential representations of a closed oriented manifold M^k are equivalent to one another in the sense developed in (A).

Let f' and f'' be two regular mappings of the manifold M^k on the vector spaces $R'^{k+l'}$ and $R''^{k+l''}$, and let T' and T'' be the corresponding tangential representations of M^k in the manifolds $H'(k, l')$ and $H''(k, l'')$. We now show that the mappings T' and T'' are equivalent to one another (see (A)).

We will denote the direct sum of the spaces $R'^{k+l'}$ and $R''^{k+l''}$ by R^{k+l}, with $l = k + l' + l''$. We will take $R'^{k+l'}$ and $R''^{k+l''}$ as subspaces of the space R^{k+l}. To the inclusion relations $R'^{k+l'} \subset R^{k+l}$ and $R''^{k+l''} \subset R^{k+l}$ correspond the relations $H'(k,l') \subset H(k,l)$ and $H''(k,l'') \subset H(k,l)$. It suffices for us to show that the mappings T' and T'' of the manifold M^k into $H(k,l)$ are homotopic in $H(k,l)$.

We set

$$f_t(x) = (1 - t)f'(x) + tf''(x), \quad x \in M^k.$$

The regularity of the mapping f_t follows easily from the regularity of the mappings f' and f''. The tangential representation corresponding to the regular mapping f_t will be denoted by T_t. Therefore T_t is a continuous deformation which carries the representation T' into the representation T'', and these two are therefore homotopic in $H(k,l)$.

With this, the equivalence of all tangential representations of a closed oriented differentiable manifold is demonstrated.

We must remember that the tangential representation T depends on the orientation of the manifold M^k. Alteration of the orientation of the manifold gives another tangential representation, not, in general, equivalent to the first.

Among all possible mappings of an oriented manifold M^k on the manifold $H(k, k+1)$ (see (A)) we distinguish a special homotopy class of mappings, equivalent to a tangential representation. This tangential class of mappings undoubtedly expresses deep properties of a differential manifold M^k. One must keep in mind the fact that a tangential class of mappings does not depend on the process by which a topological manifold is rendered differentiable; it is very likely that such a class can be defined for a topological manifold.

Our task is seen to be the study of the homology properties of the tangential class of mappings; these can be fully expressed by the characteristic cycles of the manifold M^k (see definition 4).

We introduce some notations:

(C) Let $\chi(i)$ be a monotone non-increasing integer-valued function of the integer-valued argument $i = 1, \ldots, k$, satisfying the condition $0 \leq \chi(i) \leq l$. Then the function $\omega \equiv l - \chi$, defined by the equations

$$\omega(i) = l - \chi(i), \quad i = 1, \ldots, k,$$

satisfies the conditions of definition 2 and can therefore serve as the basis for constructing a set $Z(\omega)$ in $H(k, l)$. We put $Z_\chi = Z(l - \chi)$; then

$$D(Z_\chi) = kl - r(\chi), \quad r(\chi) = \sum_{i=1}^{k} \chi(i). \tag{2}$$

The numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, corresponding to the function $\omega \equiv l - \chi$ (see § 2, (3)), can be computed with the aid of the function χ. For this it is sufficient to determine the jump points of the function χ, as was done for the function ω in § 2, (B); then the jump points of both functions coincide, and the numbers $\alpha_1, \ldots, \alpha_n$ are expressed by means of the function χ. The numbers $\beta_1, \ldots, \beta_n$ are given by the formula

$$\beta_h = \chi(i_h) - \chi(i_h + 1), \quad h = 1, \ldots, n-1, \quad \beta_n = \chi(k).$$

The condition $\omega(1) > 0$ is equivalent to the condition $\chi(1) < l$, and the condition $\omega(k) < l$ is equivalent to the condition $\chi(k) > 0$. It is easily seen that the numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ determine the function χ. We will show later (see § 5, (B)) that, for $\omega(1) > 0$, the orientability of $Z(\omega)$ depends solely on the numbers $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n$; for this reason, the pseudomanifolds Z_χ, for $l > \chi(1)$ are either all orientable, or all non-orientable, independently of the value of l.

We now rewrite the relations (12), (13), (14) of the preceding paragraph in a new form. To this end, we remark that if $\omega = l - \chi$, then $\omega'' = l'' - \chi$; therefore we adopt the notation Z''_χ for the pseudomanifolds $Z(\omega'')$ in $H(k, l'')$. Then in place of (12), (13), (14) of § 2 we will have:

$$\text{for } \chi(1) > l'': \quad Z_\chi \cap H(k, l'') \text{ is empty;} \tag{3}$$

$$\text{for } \chi(1) \leqslant l'', \quad \chi \not\equiv l'': \quad Z_\chi \times H(k, l'') = Z_\chi; \tag{4}$$

$$\text{for } \chi(1) \equiv l'': \quad I(Z_\chi, H(k, l'')) = 1 + (-1)^k. \tag{5}$$

(D) Let θ be a mapping of the p-dimensional complex K^p on the manifold $H(k, l)$, let χ be such a function that $r(\chi) \leq p$, $\chi(1) < l$ (see (C)), and let E be an arbitrary oriented simplex of dimension $r(\chi)$ of the complex K^p. We set

$$Y^\theta_\chi(E) = I(Z_\chi, \theta(E)). \tag{6}$$

On the right of this equation is the index of intersection of the pseudomanifold Z_χ with the image of the simplex E in $H(k, l)$; it is computed modulo two if Z_χ is non-orientable, and in the field of the integers otherwise. Thus, Y^θ_χ is a function of the oriented simplexes of the complex K^p. As is known, Y^θ_χ is a ∇-cycle of the complex K^p. It will be shown below that the homology class of the ∇-cycle Y^θ_χ depends only on the function χ and the class $\{\theta\}$ of mappings homotopic to θ (see (A)). We will call the ∇-cycle Y^θ_χ the *characteristic ∇-cycle of type* χ of the mapping θ.

If the pseudomanifold Z_χ is non-orientable, we have

$$\Gamma Y^\theta_\chi(E') = I(\Gamma Z_\chi, \theta(E')) \quad \text{(see § 2, (F))}. \tag{7}$$

We will call the ∇-cycle ΓY^θ_χ the *second characteristic ∇-cycle of type* χ of the mapping θ.

If the complex K^p is an oriented manifold M^k, then, as is known, there corresponds to the ∇-cycle Y^θ_χ a cycle X^θ_χ of dimension $k - r(\chi)$, defined up to a homology, and X^θ_χ satisfies the condition

$$Y^\theta_\chi(E) = I(X^\theta_\chi, E), \tag{8}$$

where E is an arbitrary oriented $r(\chi)$-dimensional simplex of M^k, and on the right-hand side of (8) we have the index of intersection taken in M^k.

The cycle X^θ_χ is defined immediately by the relationship

$$X_\chi^\theta = \theta^{-1}(Z_\chi \times \theta(M^k)). \tag{9}$$

Here $Z_\chi \times (M^k)$ is the algebraic intersection, calculated in $H(k,l)$, and the operation θ^{-1} carries this intersection back again into M^k. For the calculation it is necessary that Z_χ and $\theta(M^k)$ be in general position in $H(k,l)$. This is easily achieved by replacing the mapping θ by another which approximates it. If the dimension of $H(k,l)$ is great enough, which is always true below, we can always secure an imbedding of M^k in $H(k,l)$ without having M^k intersect itself. In this case the inverse mapping θ^{-1} has a visualizable character. We will write the relation (9) without presupposing that Z_χ and $\theta(M^k)$ are in general position, remembering that for actual calculations θ must be replaced by a neighboring mapping.

We will call the cycle X_χ^θ the *characteristic cycle of type* χ of the mapping θ. If the pseudomanifold Z_χ is non-orientable, the cycle ΓX_χ^θ (cf. § 2, (F)) will be called the *second characteristic cycle of type* χ of the mapping θ. Clearly, we have

$$\Gamma X_\chi^\theta = \theta^{-1}(\Gamma Z_\chi \times \theta(M^k)). \tag{10}$$

We now show that the homology class of the ∇-cycle Y_χ^θ depends only on the function χ and the class of the mapping $\{\theta\}$.

First, let us consider the question of orientation. If Z_χ is orientable, then formula (6) presupposes given orientations of Z_χ and $H(k,l)$.

If $2Z_\chi \sim 0$, then $2Y_\chi^\theta \sim 0$, that is $-Y_\chi^\theta \sim Y_\chi^\theta$. Thus, the homology class of Y_χ^θ does not depend on the choice of orientation. If, however, $2Z_\chi \nsim 0$, the orientation of Z_χ and $H(k,l)$ is prescribed by the orientation of the space R^{k+l}, and when the latter is reversed, both the former are multiplied by the coefficient $(-1)^k$, (cf. § 2, (C), (D)). Thus, even in this case, arbitrary choice of orientation has no effect on the homology class of Y_χ^θ.

Now suppose that θ' and θ'' are two equivalent mappings of the complex K^p on the manifolds $H'(k,l')$ and $H''(k,l'')$ (cf. (A)). This means that there exist two non-degenerate linear mappings a' and a'' of the spaces $R'^{k+l'}$ and $R''^{k+l''}$ on the space R^{k+l}, such that the mappings $a'\theta'$ and $a''\theta''$ are homotopic in $H(k,l)$. Let θ be an arbitrary mapping of K^p on $H(k,l)$, which is homotopic to both of the mappings $a'\theta'$ and $a''\theta''$. We will denote the characteristic ∇-cycles, calculated for the mappings θ, θ', and θ'', by Y_χ^θ, $Y_\chi^{\theta'}$, and $Y_\chi^{\theta''}$, respectively. We will show that $Y_\chi^\theta \sim Y_\chi^{\theta'}$ and $Y_\chi^\theta \sim Y_\chi^{\theta''}$. Thus, it will be shown that $Y_\chi^{\theta'} \sim Y_\chi^{\theta''}$. Since θ' and θ'' are fully equivalent, it will be sufficient to show that $Y_\chi^\theta \sim Y_\chi^{\theta''}$.

The ∇-cycle Y_χ^θ calculated for the mapping θ by formula (6), is defined up to a homology by a class of mutually homotopic mappings θ, and therefore we may choose for our mapping θ the mapping $a''\theta''$. We set

$$a''(R''^{k+l''}) = R^{k+l''}, \quad a''(H''(k,l'')) = H(k,l'').$$

Then

$$Y_\chi^\theta(E) = I(Z_\chi, \theta(E)), \quad Y_\chi^{\theta''}(E) = I(Z''_\chi, \theta(E)).$$

Since the pseudomanifolds Z_χ and Z''_χ may be chosen arbitrarily in the manifolds $H(k,l)$ and $H(k,l'')$, we may suppose that Z''_χ is defined by Z_χ according to formula (4), and this formula then shows that

$$I(Z_\chi, \theta(E)) = I(Z''_\chi, \theta(E)).$$

Thus the invariance of the characteristic ∇-cycle is demonstrated.

Let us apply the result just derived to the tangential representation.

Definition 4. Let M^k be a closed differentiable oriented manifold, and T its tangential representation in $H(k,l)$ (cf. definition 3).

We will call the characteristic ∇-cycle Y_χ^T (cf. (D)) the *characteristic ∇-cycle of type* χ of the manifold M^k. We will denote it by $Y_\chi(M^k)$ or by Y_χ.

We will call the characteristic cycle X_χ^T the *characteristic cycle of type* χ of the manifold M^k. We will denote it by $X_\chi(M^k)$ or by X_χ. Thus, for $\chi(1) < l$:

$$X_\chi(M^k) = X_\chi = T^{-1}(Z_\chi \times T(M^k)), \quad D(X_\chi) = k - r(\chi). \tag{11}$$

The cycle X_χ is a cycle mod 2 if the pseudomanifold Z_χ is non-orientable, and is an integer cycle otherwise.

If Z_χ is orientable, and $2Z_\chi \nsim 0$, then the orientations of $H(k,l)$ and Z_χ should be so chosen that they follow from one and the same orientation of the space R^{k+l} (cf. § 2, (C), (D)).

For non-orientable Z_χ the second characteristic cycle ΓX_χ of type χ of the manifold M^k is defined by the relation

$$\Gamma X_\chi = T^{-1}(\Gamma Z_\chi \times T(M^k)), \quad D(\Gamma X_\chi) = k - r(\chi) - 1. \tag{12}$$

We remark that for the calculation of the characteristic cycle X_χ according to formula (11) the condition $\chi(1) < l$ can be waived (cf. (4)), but for $\chi(1) > l$,

$$X_\chi \sim 0 \quad \text{(cf. (3))}. \tag{13}$$

For $\chi(1) = l$ the question whether to choose X_χ as a cycle mod 2 or a cycle in the field of integer coefficients is to be decided according as Z_χ is non-orientable or orientable for $l > \chi(1)$.

(E) The characteristic cycle X_χ is of special interest if $r(\chi) = k$. In this case its dimension is zero, and its homology class is defined by an integer or by a residue mod 2, depending on whether the pseudomanifold Z_χ is orientable or not. In this case we will mean by X_χ either the integer or the residue, respectively. If Z_χ is orientable and $2Z_\chi \sim 0$, the number X_χ is 0, since $2X_\chi \sim 0$. In this case the characteristic number X_χ is of no interest. For the same reason there is no need to consider the characteristic numbers ΓX_χ $(r(\chi) = k - 1)$.

A study of characteristic cycles must include an investigation of the relations between them. Such relations can be of two distinct types. Relations of the first kind refer to the characteristic cycles Y_χ^θ and hold for arbitrary choice of the mapping θ of the complex K^p. Relations of the second kind refer to the characteristic cycle Y_χ of the manifold M^k and take into account the specific tangential representations. It cannot be doubted that these are deeper than relationships of the first kind. For the development of relationships of the first kind one must at the outset study homology relationships among the pseudomanifolds Z_χ in $H(k,l)$.

For the investigation of the mappings of the complex K^p we are interested only in functions χ satisfying the condition $r(\chi) \leq p$; and at the same time we may suppose that $p \leq l - 1$ (cf. (A)), so that $r(\chi) \leq l - 1$. For the functions $\omega \equiv l - \chi$ this yields

$$r(\omega) \geq kl - l + 1. \tag{14}$$

In view of this, it is sufficient for our present purposes to study those homology bases of the manifold $H(k,l)$ which are of dimension r, with r satisfying the condition

$$r \geq kl - l + 1. \tag{15}$$

The next two paragraphs will be devoted to a study of this question.

§ 4. Cellular Decomposition of the Manifold $H(k,l)$

I will calculate homologies in the manifold $H(k,l)$ by Ehresmann's method, with the aid of a decomposition of $H(k,l)$ into cells of a very general type. Ehresmann applied his method, in particular, to a Grassmann manifold, for which $H(k,l)$ serves as a two-layered universal covering space. Thus, for $H(k,l)$ a cell counts for twice as much as on the Grassmann manifold, and the whole construction is rather complicated. In view of the impossibility of citing Ehresmann's definitive results, and for the convenience of the reader, I will go through the whole development from the beginning.

(A) In the vector space R^{k+l}, we choose a fixed basis

$$g_1, g_2, \ldots, g_{k+l}, \tag{1}$$

and we denote by Q^m a space with the basis

$$g_1, g_2, \ldots, g_m, \quad m \leq k + l. \tag{2}$$

We denote by $Z_o(\omega)$ the pseudomanifold $Z(\omega)$ (cf. definition 2) consisting of all those $R^k \in H(k,l)$ satisfying the relationship

$$D(R^k \cap Q^{\omega(i)+1}) \geq i, \quad i = 1, \ldots, k.$$

Further, we set

$$e_1 = g_{\omega(1)+1}, \ \ldots, \ e_k = g_{\omega(k)+k}. \tag{3}$$

That subsequence of the sequence (1) which consists of the vectors not in (3), we will write in the form $f_1, \ldots, f_l$. The vectors

$$e_1, \ \ldots, \ e_k, \ f_1, \ \ldots, \ f_l \tag{4}$$

form a basis for the space R^{k+l}. We denote by R^k_o the oriented linear span of the vectors of the system (3). Then R^k_o is an element in general position of $Z_o(\omega)$, and the basis (4) satisfies the requirement introduced in § 1 (C) with respect to $R^k_o \in Z_o(\omega)$ (cf. § 1, (6)).

Thus, to the basis (4) corresponds an oriented neighborhood $U \cap Z_o(\omega)$ of the element R^k_o in $Z_o(\omega)$, which we denote by $U(\omega)$ (cf. § 1, (C); § 2, (A)). We denote by $\hat{U}(\omega)$ the oriented image of $U(\omega)$ under the mapping $R^k \to \hat{R}^k$. Therefore, $U(\omega)$ and $\hat{U}(\omega)$ are oriented cells of dimension $r(\omega)$ of the manifold $H(k, l)$. It will be shown below (see the auxiliary theorem) that these cells constitute a cellular decomposition of the manifold $H(k, l)$.

(B) Let ω' and ω be two admissible functions (cf. definition 2). We will write $\omega' \le \omega$, if $\omega'(i) \le \omega(i)$ for $i = 1, \ldots, k$. If, further, $\omega'(i) \not\equiv \omega(i)$, we will write $\omega' < \omega$.

Suppose $\omega'' < \omega$, and $r(\omega'') = r(\omega) - 1$. Then it is evident that ω'' coincides with ω for all values of the argument with the exception of one, say p, for which ω'' is one less than ω. The function ω'' being non-decreasing, the exceptional value p of the argument cannot be chosen arbitrarily, and, in fact, $p = i_h + 1$, $h = 0, 1, \ldots, n - 1$ (cf. § 1, (B)). We set $\omega'' = \gamma^p(\omega) = \omega^h$. Thus,

$$\text{for } i \neq i_h + 1: \quad \omega^h(i) = \omega(i), \quad \omega^h(i+1) = \omega(i_h + 1) - 1.$$

If $\omega(1) = 0$, the function ω^o has a negative value $(\omega^o(1) = -1)$ and therefore cannot be used for our purposes. In order not to have to treat this case separately, from now on we will take $U(\omega^o)$, $\hat{U}(\omega^o)$, $Z_o(\omega^o)$, and $Z(\omega^o)$ as equal to zero whenever $\omega(1) = 0$. It is easy to see that if $\omega' < \omega$, by a series of applications of the operator γ^p to the function ω for different values of p, we can obtain the function ω'.

(C) For every $R^k \in H(k, l)$ there exists one and only one function ω such that R^k belongs to the sum $U(\omega) \cup \hat{U}(\omega)$. Since $U(\omega)$ and $\hat{U}(\omega)$ do not intersect one another (cf. § 1, (A)), we see that $H(k, l)$ splits into the sum of all cells constructed in (A). Further, the pseudomanifold $Z_o(\omega)$ splits into the sum of all $U(\omega')$ and $\hat{U}(\omega')$, where $\omega' \le \omega$. Therefore, we have

$$Z_0(\omega) = \sum_{\omega' \le \omega} U(\omega') \cup \hat{U}(\omega') \tag{5}$$

Let us prove (C). We consider an integer-valued function $D(m)$ defined by the relation

$$D(m) = D(R^k \cap Q^m), \quad m = 1, \ \ldots, \ k + l. \tag{6}$$

It is easy to see that this function satisfies the following conditions:

$$\left.\begin{array}{l} 0 \le D(1) \le 1, \quad D(k+l) = k; \\ 0 \le D(m+1) - D(m) \le 1, \quad m = 1, \ldots, k+l-1. \end{array}\right\} \tag{7}$$

The last of these relations follows from the fact that the dimensions of Q^{m+1} and Q^m differ by one. It follows from (7) that the set of all positive values of the function $D(m)$ consists of the numbers $1, \ldots, k$.

We denote by m_i the smallest value of the number m for which $D(m) = i$; $i = 1, \ldots, k$. Thus,

$$\left.\begin{array}{l} D(R^k \cap Q^{m_i}) = i; \\ \text{for } m < m_i, \ D(R^k \cap Q^m) < i; \\ \text{for } m \ge m_i, \ D(R^k \cap Q^m) \ge i. \end{array}\right\} \tag{8}$$

The relations (8) set up a correspondence between the elements R^k of $H(k, l)$ and the sequence

$$m_1, \ \ldots, \ m_k, \quad 0 < m_1 < \ldots < m_k \le k + l. \tag{9}$$

We now consider the relations

$$R^k \in Z_o(\omega), \tag{10}$$

and

$$m_i \le \omega(i) + i, \quad i = 1, \ \ldots, \ k, \tag{11}$$

and show that they are equivalent.

If (10) holds, then

$$D(R^k \cap Q^{\omega(i)+i}) \geq i, \quad i = 1, \dots, k,$$

and (11) follows from (8). But, if (11) holds, we conclude by (8) that

$$D(R^k \cap Q^{\omega(i)+i}) \geq i, \quad i = 1, \dots, k,$$

that is, (10) holds. Thus (10) and (11) are equivalent.

We now consider the relations

$$R^k \in U(\omega) \cup \hat{U}(\omega), \tag{12}$$

and

$$m_i = \omega(i) + i, \quad i = 1, \dots, k, \tag{13}$$

and show that they are equivalent.

By (A), the basis (4) of the space R^{k+l} corresponds to the function ω. The linear span of the vectors $f_1, \dots, f_l$ will be denoted by P; we define ϕ as the projection of the space R^{k+l} on R^k_o in the direction P (cf. § 1, (A)). Let x_i be a vector of $R^k \cap Q^{m_i}$, not belonging to $R^k \cap Q^{m_i-1}$. It is evident that the system $x_1, \dots, x_k$ is a basis for the space R^k. We have

$$x_i = \sum_{m=1}^{m_i} b_i^m g_m, \quad i = 1, \dots, k. \tag{14}$$

Since $U(\omega) \cup \hat{U}(\omega) \subset Z_o(\omega)$, we can derive (10) from (12) and consequently, can derive (11) from (12). Thus, if either (12) or (13) holds, then (11) holds, and therefore

$$x_i = \sum_{m=1}^{\omega(i)+i} b_i^m g_m, \quad i = 1, \dots, k. \tag{15}$$

Thus we have

$$\varphi(x_i) = \sum_{s=1}^{i} b_i^{\omega(s)+s} e_s, \quad i = 1, \dots, k. \text{ (cf. (3))}. \tag{16}$$

We now set $a_i^s = b_i^{\omega(s)+s}$; the matrix $\|a_i^s\|$ has triangular form, and (16) is valid if one of the relations (12) or (13) holds.

Let us now suppose that (12) holds. Then the determinant of the matrix $\|a_i^s\|$ is different from zero, and accordingly all terms $a_i^i = b_i^{\omega(i)+i}$, $i = 1, \dots, k$, are different from zero. This means that the vector x_i belongs to $Q^{\omega(i)+i}$, but not to $Q^{\omega(i)+i-1}$. From this it follows immediately, because of the choice of the vector x_i, that $m_i = \omega(i) + 1$, that is, that (13) holds.

Let us now suppose that (13) holds. Then the vector x_i, belonging to $Q^{\omega(i)+i}$, cannot belong to $Q^{\omega(i)+i-1}$, and therefore the numbers $b^{\omega(i)+i}$ are different from zero (cf. (15)). This means that the determinant of the matrix $\|a_i^s\|$ is different from zero, which in turn means that the projection ϕ of R^k on R^k_o is non-degenerate. Taking into account the relation $R^k \in Z_o(\omega)$, this proves (12).

Thus the relations (12) and (13) are equivalent.

In view of the equivalence of (12) and (13), the relationship (13) uniquely defines the function ω for which (12) holds.

Because of this, (5) is an immediate consequence of the equivalence of (10) and (11).

Thus (C) is completely proved.

<u>Auxiliary Theorem.</u> *The cells $U(\omega)$ and $U(\omega)$ defined in (A) constitute a cellular decomposition of the manifold $H(k,l)$. The set-theoretical boundaries $V(\omega) = \overline{U}(\omega) - U(\omega)$ and $\hat{V}(\omega) = \overline{\hat{U}}(\omega) - \hat{U}(\omega)$ of the cells $U(\omega)$ and $\hat{U}(\omega)$ are defined by the relation*

$$V(\omega) = \hat{V}(\omega) = \sum_{\omega' < \omega} U(\omega') \cup \hat{U}(\omega'). \tag{17}$$

The algebraic boundaries $\Delta U(\omega)$ and $\Delta\hat{U}(\omega)$ of the same cells are given by the relations

$$\Delta U(\omega) = \sum_{h=0}^{n-1} (U(\omega^h) + (-1)^{s(\omega,h)}\hat{U}(\omega^h))(-1)^{t(\omega,h)}, \tag{18}$$

$$\Delta \hat{U}(\omega) = \sum_{h=0}^{n-1} (\hat{U}(\omega^h) + (-1)^{s(\omega,h)} U(\omega^h))(-1)^{t(\omega,h)}, \tag{19}$$

$$\begin{aligned} s(\omega, h) &= \omega(i_h + 1) + i_h + 1 + k \\ &= \omega(1) + \alpha_1 + \beta_1 + \dots + \alpha_h + \beta_h + k + 1 \text{ (cf. §2, (3));} \end{aligned} \tag{20}$$

and we do not write out $t(\omega, h)$ (cf. (53)).

Proof. By (A), to the function ω there corresponds the basis

$$e_1, \dots, e_k, f_1, \dots, f_l \tag{21}$$

of the space R^{k+l}. In the same way, to the function ω^h (cf. (B)) corresponds the basis

$$e_1^h, \dots, e_k^h, f_1^h, \dots, f_l^h \tag{22}$$

of the space R^{k+l}. Both bases (21) and (22) are obtained by relabelling the elements of the basis (1). In order to show the connection between the bases (21) and (22), we introduce the notation

$$p = i_h + 1, \quad q = \omega(i_h + 1), \quad h = 0, \dots, n-1. \tag{23}$$

Then we have:

$$\text{for } i \neq p\colon e_i^h = e_i,\ e_p^h = f_q; \text{ for } j \neq q\colon f_j^h = f_j,\ f_q^h = e_p. \tag{24}$$

In order to prove (24), it is only necessary to spell out the process of going from a basis (1) to a basis (4) with the aid of some function ω. This process consists in first selecting the vectors $g_{\omega(i)+i}$ for $i = 1, \dots, k$ from the sequence (1) and denoting these by e_i, $i = 1, \dots, k$, respectively, and then denoting the remaining, unselected vectors by $f_1, \dots, f_l$. Now, since ω and ω^h differ only for $i = p$, where $\omega^h(p) = \omega(p) - 1$, it is clear that after the first step of our process the vector e_p^h precedes the vector e_p, while at the same time all other vectors e_i^h coincide with the corresponding vectors e_i. In order to see what takes place in the second step, it is necessary to investigate the index-number of the vector f_j which immediately precedes e_p in the sequence (1). It is easy to see that $j = p + q - p = q$. Thus, in going from the function ω to the function ω^h the vector f_q moves one place to the right and is renumbered to become f_q^h. The relation (24) follows.

Now we consider a linear transformation a_t, $0 \leq t \leq 2$, of the space R^{k+l}, which we define by the relations (e_i being elements of

the basis (21)):

$$\text{for } i \neq p\colon a_t(e_i) = e_i; \text{ for } j \neq q\colon a_t(f_j) = f_j; \tag{25}$$

$$\left.\begin{aligned} a_t(e_p) &= \left(\cos\tfrac{\pi}{2}t\right)e_p + \left(\sin\tfrac{\pi}{2}t\right)f_q, \\ a_t(f_q) &= -\left(\sin\tfrac{\pi}{2}t\right)e_p + \left(\cos\tfrac{\pi}{2}t\right)f_q. \end{aligned}\right\} \tag{26}$$

For $t = 2$ and $t = 1$ we have:

$$\left.\begin{aligned} &\text{for } i \neq p\colon a_2(e_i) = e_i,\ a_2(e_p) = -e_p; \\ &\text{for } j \neq q\colon a_2(f_j) = f_j,\ a_2(f_q) = -f_q; \end{aligned}\right\} \tag{27}$$

$$a_1(e_i) = e_i^h; \text{ for } j \neq q\colon a_1(f_j) = f_j^h,\ a_1(f_q) = -f_q^h \text{ (cf. (24)).} \tag{28}$$

It is easy to see that $a_t(Q^{\omega(i)+i}) = Q^{\omega(i)+i}$, $i = 1, \dots, k$, and, therefore,

$$a_t(Z_o(\omega)) = Z(\omega). \tag{29}$$

Just as in § 1, (A), we denote by P the linear span of the vectors $f_1, \dots, f_l$, and by ϕ the operation of projecting R^{k+l} on R_o^k in the direction P. Then

$$\text{for } i \neq p\colon \varphi(a_t(e_i)) = e_i,\ \varphi(a_t(e_p)) = \left(\cos\tfrac{\pi}{2}t\right)e_p \text{ (cf. (25), (26));} \tag{30}$$

$$\text{for } j \neq q\colon \varphi(a_t(f_j)) = 0;\ \varphi(a_t(f_q)) = -\left(\sin\tfrac{\pi}{2}t\right)e_p \text{ (cf. (25), (26)).} \tag{31}$$

In the cell $U(\omega)$ we introduce coordinates in the same way as we did in § 1, (C); namely, in the element $R_\xi^k \in U(\omega)$ there exists a basis $e'_1, \dots, e'_k$, specifying its orientation, and defined by the relations

$$e'_i = e_i + \sum_{j=1}^{l} \xi_i^j f_j, \quad i = 1, \dots, k, \tag{32}$$

$$\xi_i^j = 0 \text{ for } j > \omega(i). \tag{33}$$

Here the numbers ξ_i^j, satisfying (33), are coordinates of $R_\xi^k \in U(\omega)$. These coordinates specify the orientation of $U(\omega)$ in the way described in § 1, (A). We denote by $a_t(U(\omega))$ the oriented image of the cell $U(\omega)$ under the mapping a_t.

We now consider the conditions under which $a_t(R^k)$ belongs to either $U(\omega)$ or $\hat{U}(\omega)$. For the resolution of this question we study the mapping ϕ of the space $a_t(R_\xi^k)$.

We have

$$\left.\begin{aligned} &\text{for } i < p: \varphi(a_t(e_i')) = e_i, \\ &\varphi(a_t(e_p')) = \left(\cos\frac{\pi}{2}t\right)e_p - \left(\sin\frac{\pi}{2}t\right)\xi_p^q e_p. \\ &\text{for } i > p: \varphi(a_t(e_i')) = e_i - \left(\sin\frac{\pi}{2}t\right)\xi_i^q e_p. \end{aligned}\right\} \text{(cf. (30), (31), (32), (33))}, \quad (34)$$

Thus, the determinant of the mapping ϕ of the space $a_t(R_\xi^k)$ on R_o^k is equal to

$$\cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q,$$

and, accordingly, (cf. (29)),

$$\left.\begin{aligned} &\text{for } \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q > 0: \quad \varphi(a_t(R_\xi^k)) \in U(\omega), \\ &\text{for } \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q < 0: \quad \varphi(a_t(R_\xi^k)) \in \hat{U}(\omega), \\ &\text{for } \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q = 0: \\ &\qquad \varphi(a_t(R_\xi^k)) \in Z_0(\omega) - (U(\omega) \cup \hat{U}(\omega)). \end{aligned}\right\} \text{(cf. (29))} \quad (35)$$

From this it follows that

$$a_t(U(\omega)) \cap U(\omega) = \left\{a_t(U(\omega)),\ \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p > 0\right\}, \quad (36)$$

$$a_t(U(\omega)) \cap \hat{U}(\omega) = \left\{a_t(U(\omega)),\ \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q < 0\right\}. \quad (37)$$

Here the right-hand sides of the relationships denote the regions of $a_t(U(\omega))$ defined by the inequalities

$$\cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q > 0, \quad \cos\frac{\pi}{2}t - \left(\sin\frac{\pi}{2}t\right)\xi_p^q < 0,$$

where the numbers ξ_i^j, satisfying (33), are the coordinates of the element $a_t(R_\xi^k)$ in the cell $a_t(U(\omega))$. Thus, the intersections (36) and (37) are either connected regions in the cell $a_t(U(\omega))$ or are empty, depending on the value of the parameter t.

It is an immediate consequence of (35) and (37) that the cells $U(\omega)$ and $a_2(U(\omega))$ coincide as sets, but may have different orientations. We now examine the connection between their orientations.

We denote by $\hat{a}_2(R_\xi^k)$ the image of the element $a_2(R_\xi^k)$ under the mapping $R^k \to \hat{R}^k$. Since $\hat{a}_2(R_\xi^k) \in U(\omega)$, $\hat{a}_2(R_\xi^k) = R_{\xi'}^k$. Let us examine the connection between ξ and ξ'. The element $a_2(R_\xi^k)$ has a basis $a_2(e_1'), \ldots, a_2(e_k')$ (cf. (32)).

We write,

$$\text{for } i \neq p: \hat{e}_i = a_2(e_i'), \quad \hat{e}_p = -a_2(e_p'). \quad (38)$$

Then the vectors $\hat{e}_1, \ldots, \hat{e}_k$ form a basis for the element $\hat{a}_2(R_\xi^k)$. From the relations (27), (32), and (38) follows:

$$\hat{e}_i = e_i + \sum_{j=1}^{l} \xi'^j_i f_j, \quad (39)$$

$$\left.\begin{aligned} &\text{for } i \neq p,\ j \neq q: \xi'^j_i = \xi_i^j; \text{ for } i \neq p: \xi'^q_i = -\xi_i^q; \\ &\text{for } i \neq q: \xi'^j_p = -\xi_p^j,\ \xi'^q_p = \xi_p^q. \end{aligned}\right\} \quad (40)$$

We see that the transformation from the matrix ξ to the matrix ξ' is given by the relations (40). The determinant of the mapping $\xi \to \xi'$ is, accordingly, equal to $(-1)^{p+q+k+1}$, and we have

$$a_2(U(\omega)) = (-1)^{p+q+k+1}\hat{U}(\omega). \quad (41)$$

Let us now consider the cell $a_1(U(\omega))$. The element $a_1(R_\xi^k)$ has the basis

$$a_1(e_1'), \ldots, a_1(e_k').$$

By (28) and (32), this basis can be written in the form

$$a_1(e_i') = e_i^h + \sum_{j=1}^{l} \eta_i^j f_j^h, \quad (42)$$

$$\text{for } j \neq q: \eta_i^j = \xi_i^j;\ \eta_i^q = -\xi_i^q. \quad (43)$$

Since the quantities ξ_i^j satisfy condition (33), the quantities η_i^j satisfy the analogous condition

$$\eta_i^j = 0 \text{ for } j > \omega(i). \quad (44)$$

The equation $\xi_p^q = 0$ provides the equation

$$\eta_p^q = 0. \quad (45)$$

Equations (44) and (45) give us

$$\eta_i^j = 0 \text{ for } j > \omega^h(i). \quad (46)$$

Thus, in the cell $a_1(U(\omega))$ the equation $\xi_p^q = 0$ distinguishes the cell $U(\omega^h)$, and we may write

$$\{a_1(U(\omega)),\ \xi_p^q = 0\} = U(\omega^h). \tag{47}$$

This relation takes no account of orientation. The cell $a_1(U(\omega))$ is decomposed, therefore, by the cell $U(\omega^h)$ into two subspaces, and, taking account of orientation, we may write

$$\Delta\{a_1(U(\omega)),\xi_p^q < 0\} = \epsilon U(\omega^h);\ \Delta\{a_1(U(\omega)),\xi_p^q > 0\} = -\epsilon U(\omega^h), \tag{48}$$

where $\epsilon = \pm 1$, and we do not calculate it here explicitly.

Fixing our attention on the relations (36) and (37) for $t = 1$, we see that the two parts into which the cell $U(\omega^h)$ divides the cell $a_1(U(\omega))$ coincide respectively with the intersections $a_1(U(\omega)) \cap U(\omega)$ and $a_1(U(\omega)) \cap \hat{U}(\omega)$.

Since $a_o(U(\omega)) = U(\omega)$, and since for $0 \le t \le 1$ the intersection (36) is connected, the orientations induced in the intersection $a_1(U(\omega)) \cap U(\omega)$ by $U(\omega)$ and by $a_1(U(\omega))$ are identical; therefore,

$$\Delta U(\omega) = \epsilon U(\omega^h) + \dots \text{ (cf. (48))}. \tag{49}$$

Since $a_2(U(\omega)) = (-1)^{p+k+q+1}\hat{U}(\omega)$ (cf. (41)), and since for $1 \le t \le 2$ the intersection (37) is connected, the orientations induced in $a_1(U(\omega)) \cap \hat{U}(\omega)$ by $(-1)^{p+q+k+1}\hat{U}(\omega)$ and by $a_1(U(\omega))$ are identical; therefore,

$$\Delta\hat{U}(\omega) = \epsilon(-1)^{p+q+k}U(\omega^h) + \dots \text{ (cf. (48))}.$$

Applying the mapping $R^k \to \hat{R}^k$ to this latter relationship, we obtain

$$\Delta U(\omega) = \epsilon(-1)^{p+q+k}\hat{U}(\omega^h) + \dots. \tag{50}$$

Taken together, (49) and (50) yield

$$\Delta U(\omega) = \epsilon(U(\omega^h) + (-1)^{p+q+k}\hat{U}(\omega^h)) + \dots. \tag{51}$$

Relation (51) shows that the set-theoretical boundary $V(\omega)$ of the cell $U(\omega)$ contains both cells $U(\omega^h)$ and $\hat{U}(\omega^h)$. Since $V(\omega)$ is closed, we conclude by (B) that $V(\omega)$ contains all cells $U(\omega')$ and $\hat{U}(\omega')$ for $\omega' < \omega$. On the other hand, it is clear that $V(\omega) \subset Z_o(\omega)$; whence, in view of (5), (17) follows.

By virtue of (17), the algebraic boundary of the cell $U(\omega)$ can contain only those cells $U(\omega'')$ and $\hat{U}(\omega'')$ for which $\omega'' < \omega$, and $r(\omega'') = r(\omega) - 1$. Because of (B), we conclude that the algebraic boundary of the cell $U(\omega)$ can contain only terms of the type of (51), which are all written out in (18), and so (18) holds (cf. (23)).

The relation (19) is derived from (18) by the mapping $R^k \to \hat{R}^k$.

With this, the proof of the auxiliary theorem is complete.

It should be said that the theorem just proven follows easily from Ehresmann's results. The same is not true, as I see it, of the results of the following section.

Let us define the function $\bar{\omega}$ by the relations

$$\left.\begin{aligned} &\text{for } i \leqslant i_h+1:\quad \bar{\omega}(i) = \omega(i),\\ &\text{for } i \geqslant i_h+1:\quad \bar{\omega}(i) = \omega(i_h+1). \end{aligned}\right\} \tag{52}$$

Then

$$t(\omega, h) = r(\bar{\omega}) = \sum_{i=1}^{k} \bar{\omega}(i) \quad \text{(cf, (20))}; \tag{53}$$

this relationship can be derived without difficulty from (1) § 1 and (43) § 4, but it will not be used in the present work.

§ 5. Homology in $H(k,l)$

Here we will construct the r-dimensional canonical homology basis of the manifold $H(k,l)$ for arbitrary r satisfying the inequality

$$r \ge kl - l + 1. \tag{1}$$

Construction of the homology basis of lower dimensions is troublesome, and is not needed for what we are doing now (cf. § 3, (15)).

The cellular decomposition of the manifold $H(k,l)$, as set forth in the preceding section, is fundamental to the study of the homology theory of $H(k,l)$, as we shall consider chains which are linear forms with integer coefficients on the oriented cells constructed in § 4, (A). It is easy to see that for the function ω we can derive from (1) the condition

$$\omega(1) \geq 1, \tag{2}$$

which plays a most important role in what follows.

(A) The cellular decomposition of the pseudomanifold $Z_o(\omega)$ contains only two cells of maximal dimension, namely $U(\omega)$ and $\hat{U}(\omega)$ (cf. § 4, (C)): We will now consider $Z_o(\omega)$, as a chain, putting

$$Z_0(\omega) = U(\omega) + (-1)^{\omega(1)+k}\hat{U}(\omega). \tag{3}$$

Then

$$\hat{Z}_0(\omega) = (-1)^{\omega(1)+k} Z_0(\omega), \tag{4}$$

$$\Delta U(\omega) = Z_0(\omega^0)(-1)^{t(\omega,0)} + \sum_{h=1}^{n-1} (U(\omega^h) + (-1)^{s(\omega,h)}\hat{U}(\omega^h))(-1)^{t(\omega,h)}, \tag{5}$$

$$\Delta\hat{U}(\omega) = Z_0(\omega^0)(-1)^{t(\omega,0)+\omega(1)+k-1} + \sum_{h=1}^{n-1} (\hat{U}(\omega^h) + (-1)^{s(\omega,h)}U(\omega^h))(-1)^{t(\omega,h)}, \tag{6}$$

$$\Delta Z_0(\omega) = \sum_{h=1}^{n-1} (1-(-1)^{\alpha_1+\beta_1+\dots+\alpha_h+\beta_h}) Z_0(\omega^h)(-1)^{t(\omega,h)}. \tag{7}$$

The relation (4) follows immediately from (3). The relations (5) and (6) follow immediately from (18), (19), and (20) of § 4, and from (3) § 5. Relation (7) follows from the same, but one must take account of the fact that for $h \geq 1$, $\omega^h(1) = \omega(1)$. Thus, (4), (5), (6), and (7) are true.

(B) Suppose $\omega(1) \geq 1$. If the chain $aU(\omega) + b\hat{U}(\omega) = X$ has the property that its boundary ΔX contains the cells $U(\omega^o)$ and $\hat{U}(\omega^o)$ with zero coefficients, then

$$X = aZ_o(\omega). \tag{8}$$

Further, the pseudomanifold $Z(\omega)$ (see definition 2) is orientable if and only if

$$\alpha_1 + \beta_1 \equiv \alpha_2 + \beta_2 \equiv \dots \equiv \alpha_{n-1} + \beta_{n-1} \equiv 0 \pmod 2. \tag{9}$$

If the function ω is constant, $\omega \equiv c$, then it has no jumps (that is, $n = 1$) and (9) is certainly satisfied. Therefore, the pseudomanifold $Z(c)$ is orientable.

It follows from (5) and (6) that

$$\Delta X = \pm(a + b(-1)^{\omega(1)+k+1}) Z_0(\omega^0) + \dots,$$

so that if the cells $U(\omega^o)$ and $\hat{U}(\omega^o)$ do not enter ΔX, we have $b = a(-1)^{\omega^o(1)+k+1}$, and (8) holds. Now let us suppose that the pseudomanifold $Z_o(\omega)$ is orientable; this means that for a suitable choice of signs, the chain $\pm U(\omega) \pm \hat{U}(\omega)$ is a cycle. Since it is a cycle, the cells $U(\omega^o)$ and $\hat{U}(\omega^o)$ do not enter its boundary, and so this chain is equal to $\pm Z_o(\omega)$ (cf. (8)); and the latter is a cycle if and only if (9) is satisfied.

An arbitrary linear form with integer coefficients on chains of the type of (3), i.e., $\sum_\omega c_\omega Z_o(\omega)$ for $r(\omega) = r$ will be called a Z-chain of dimension r. Relation (7) shows that the boundary of a Z-chain is again a Z-chain.

The fact that the boundary of a Z-chain is a Z-chain leads naturally to the hypothesis that in computing the homologies of $H(k, l)$ one need take into account only Z-chains. This hypothesis turns out to be correct for dimensions satisfying (1), and the class of chains considered can be even further narrowed.

The chain

$$\sum_\omega (a_\omega U(\omega) + b_\omega \hat{U}(\omega)),$$

where the summation is extended over only those ω for which the first jump point is at not less than 2 ($i_1 \geq 2$), will be called *special*. It follows from (7) that the boundary of a special Z-chain is again a special Z-chain.

Lemma. *Let X be a chain whose dimension satisfies condition (1) and whose boundary is a special chain; then there exists a special Z-chain Y such that $X - Y$ is a cycle homologous to zero.*

This lemma shows that for the study of homology in dimensions satisfying condition (1), it is sufficient to consider special Z-chains.

Proof of the lemma. Let us arrange in lexicographical order the set of all functions ω satisfying the requirements of definition 2. In other words, we will say that $\omega_1 \prec \omega_2$ if for the least value of i for which $\omega_1(i) \neq \omega_2(i)$ we have $\omega_1(i) < \omega_2(i)$.

If C be taken as an arbitrary chain, it can be written in the form

$$C = \sum_{p=1}^{q} (a_p U(\omega_p) + b_p \hat{U}(\omega_p)), \tag{10}$$

where for each p the coefficients a_p and b_p do not vanish simultaneously and $\omega_1 \prec \omega_2 \prec \dots \prec \omega_q$. Now let

$$C' = \sum_{p=1}^{q'} (a'_p U(\omega'_p) + b'_p \hat{U}(\omega'_p))$$

be a second chain written in the same fashion. If $\omega_1 \prec \omega'_1$, we will say that $C \prec C'$. Further, if $\omega_p = \omega'_p$, for $p < t$, but $\omega_t \prec \omega'_t$, we will also say that $C \prec C'$. Finally, if $q' < q$ and $\omega_p = \omega'_p$ for $p \leq q'$, we will also say that $C \prec C'$. In this way, the set of all chains is partially ordered. The ordering is not of a customary kind, since it may be that $q = q'$ and $\omega_p = \omega'_p$ for $p = 1, \dots, q$, while C and C', as chains, are distinct.

Let M be some set of chains. We will call the chain $C \in M$ maximal in M if there is no chain $C' \in M$ such that $C \prec C'$.

Among all chains C satisfying the condition $X - C \sim 0$, we choose a maximal chain, and we denote it by Y. Let

$$Y = \sum_{p=1}^{q} (a_p U(\omega_p) + b_p \hat{U}(\omega_p)),$$

according to the form (10). We will show that Y is a special Z-chain.

Let us note, first of all, that because of the condition imposed on the dimension of the chain Y (cf. (1)), $\omega_p(1) \geq 1$ (cf. (2)).

Suppose that for $p < t$ the first jump point of the function

ω_p is not less than 2 ($i_1 \geq 2$), and that

$$a_p U(\omega_p) + b_p \hat{U}(\omega_p) = Y_p = c_p Z_0(\omega_p)$$

(for $t = 1$ this condition is vacuous). We show that then the first jump point of ω_t is not less than 2, and that

$$a_t U(\omega_t) + b_t \hat{U}(\omega_t) = Y_t = c_t Z_0(\omega_t).$$

We have

$$\Delta Y_t = d\,U(\omega_t^0) + e\,\hat{U}(\omega_t^0) + \dots .$$

We will now show that $d = e = 0$, from which it follows (cf. (B)) that $Y_t = c_t Z_0(\omega_t)$. Since $\omega_t(1) \geq 1$, the first jump point of the function ω_t^0 is equal to unity, and therefore ΔY_t is a special chain only if $a = e = 0$. Since ΔY is a special chain, the terms $dU(\omega_t^0) + e\hat{U}(\omega_t^0)$ disappear in computing ΔY. They cannot, however, cancel terms of ΔY_p for $p < t$, since for $p < t$, ΔY_p is a special chain (cf. (D)). On the other hand, they cannot cancel terms of ΔY_p for $p > t$, since $\omega_t^0 \prec \omega_p^h$ for $p > t$. Therefore, $d = e = 0$, and $Y_t = c_t Z_0(\omega_t)$.

Let us now suppose that the initial jump point of the function ω_t is at $1 (i_1 = 1)$; then there exists a function ω such that $\omega_t = \omega^0$, and therefore

$$\Delta U(\omega) = \epsilon Z_0(\omega_t) + \dots, \quad \epsilon = \pm 1 \quad \text{(cf. (5))}.$$

It is clear that

$$Y - \epsilon c_t \Delta U(\omega_t) \succ Y,$$

that is, Y is not, as postulated, a maximal chain. The first jump point of the function ω_t is therefore not less than 2.

We now introduce certain notations which will be useful later.

(C) Let us denote by Ω' the set of all functions ω (cf. definition 2) for which the initial jump point is not less than 2 ($i_1 \geq 2$) (cf. § 1, (B)).

The functions ω were put into correspondence with the numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ (cf. § 2, (3)). Let us now consider the sequences

$$\left.\begin{array}{l}\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \ldots,\ \alpha_{n-1},\ \beta_{n-1},\ \alpha_n,\\ \alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ \ldots,\ \alpha_{n-1},\ \beta_{n-1}.\end{array}\right\} \tag{11}$$

The functions ω will be put into correspondence with the first of the sequences if $\omega(k) < l$, and with the second, if $\omega(k) = l$.

We will assign to the set Ω all functions $\omega \in \Omega'$ for which the corresponding sequence (11) contains no odd number. In particular, we will assign to Ω the function $\omega \equiv l$, since for this function $n = 1$ and $\omega(k) = l$.

We will assign to the set Ω_α all functions $\omega \in \Omega'$ for which the corresponding sequence (11) contains odd numbers, and the first odd number is some α_h.

We will assign to the set Ω_β all functions $\omega \in \Omega'$ for which the corresponding sequence (11) contains odd numbers, and the first odd number is some β_h.

In this fashion the set Ω' is broken down into three non-intersecting sets Ω, Ω_α, Ω_β.

It follows from (B) that for $\omega \in \Omega$ the pseudomanifold $Z(\omega)$ is orientable, and for $\omega' \in \Omega_\beta$ the pseudomanifold $Z(\omega')$ is non-orientable. If $\omega \equiv l - \chi$ (cf. § 3, (C)), then, knowing χ, we can determine whether $\omega \in \Omega'$. We can also determine, for $\omega \in \Omega'$, whether ω belongs to Ω, Ω_α, or Ω_β. We can thus decompose the set of functions χ, and we denote by X, X_α, X_β the sets corresponding to Ω, Ω_α, Ω_β.

Theorem 1. *The pseudomanifold $Z(\omega)$, $\omega \in \Omega$, is orientable* (cf. (C)), *and, taken in some orientation, can be considered as a cycle of dimension $r(\omega)$. The pseudomanifold $Z(\omega')$, $\omega' \in \Omega_\beta$, is non-orientable* (cf. (C)), *and consequently* $\Gamma Z(\omega')$ (cf. § 2, (F)) *is a cycle of dimension $r(\omega') - 1$. The canonical homology basis of dimension $r \geq kl - l + 1$ of the manifold $H(k,l)$ is made up of the cycles:*

$$Z(\omega),\ \omega \in \Omega,\ r(\omega) = r;\ \Gamma Z(\omega'),\ \omega' \in \Omega_\beta,\ r(\omega') - 1 = r. \tag{12}$$

Here $Z(\omega)$ is a free cycle, and $\Gamma Z(\omega')$ is a cycle of order two.

Demonstration. We divide the proof into several sections.

(a) Suppose $\omega \in \Omega_\beta$; then we have

$$\Delta Z_0(\omega) = \sum_{h=1}^{n-1} a_h Z_0(\omega^h), \tag{13}$$

where the coefficients a_h can have the values ± 2 and 0 (cf. (7)). It turns out that in (13) there are non-zero coefficients; the first of these we denote by a_q, and we put $\omega^q = \Psi(\omega)$. It is true, further, that

$$\text{for } m > q:\ \omega^m \in \Omega_\beta,\ \Psi(\omega) = \omega^q \in \Omega_\alpha. \tag{14}$$

Finally, the mapping Ψ of Ω_β into Ω_α so determined is a one-to-one mapping of Ω_β on Ω_α.

Let us prove (a). The numbers α and β (cf. § 2, (3)), corresponding to the function ω will be denoted by $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$, and the same numbers corresponding to ω^m will be denoted by $\alpha_1^m, \ldots, \alpha_n^m, \beta_1^m, \ldots, \beta_n^m$. Then

$$\text{for } h < m:\ \alpha_h^m = \alpha_h,\ \beta_h^m = \beta_h, \tag{15}$$

$$\text{for } \beta_m = 1:\ \alpha_m^m = \alpha_m + 1, \tag{16}$$

$$\text{for } \beta_m > 1:\ \alpha_m^m = \alpha_m,\ \beta_m^m = \beta_m - 1,\ \alpha_{m+1}^m = 1. \tag{17}$$

In view of (7), the number a_h is different from zero if and only if the number

$$(\alpha_1 + \beta_1) + \ldots + (\alpha_h + \beta_h) \tag{18}$$

is odd. Therefore, q is to be defined as the smallest number h for which (18) is odd; by virtue of the fact that $\omega \in \Omega_\beta$, β_q is the first odd number in the sequence (11) corresponding to the function ω.

If $m > q$, then it follows from (15) that in the sequence (11) corresponding to the function ω^m, the first odd number is β_q^m, that is, for $m > q$, $\omega^m \in \Omega_\beta$.

Now suppose $m = q$; then for $\beta_q = 1$ the first odd number in the sequence (11) corresponding to the function ω^q will be α_q^q (cf. (15), (16)), that is, $\omega^q \in \Omega_\alpha$; but if $\beta_q > 1$, the first odd number will be α_{q+1}^q (cf. (15), (17)), that is, $\omega^q \in \Omega_\alpha$. That is, (14) is now proved.

Let ω^* be an arbitrary function from Ω_α; we will denote the numbers α and β, corresponding to this function by $\alpha_1^*, \ldots, \alpha_{n^*}^*$, $\beta_1^*, \ldots, \beta_{n^*}^*$. We will so choose the function ω that for some $h > 0$, $\omega^* = \omega^h$; the function ω is, as is easily seen, given by the following expressions:

$$\left.\begin{array}{ll} \text{for } i \neq \alpha_1^* + \ldots + \alpha_p^*: & \omega(i) = \omega^*(i), \\ \text{for } i = \alpha_1^* + \ldots + \alpha_p^*: & \omega(i) = \omega^*(i) + 1. \end{array}\right\} \tag{19}$$

(Here the equation $p = n^*$ is possible only if $\omega^*(k) < l$.) In fact, we have

$$\text{for } \alpha_p^* = 1\colon \omega^* = \omega^{p-1}; \text{ for } \alpha_p^* > 1\colon \omega^* = \omega^p. \tag{20}$$

Denoting by $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ the numbers α and β, corresponding to ω, we have

$$\text{for } h < p-1\colon \alpha_h = \alpha_h^*, \ \beta_h = \beta_h^*; \ \alpha_{p-1} = \alpha_{p-1}^*, \tag{21}$$

$$\text{for } \alpha_p^* = 1\colon \beta_{p-1} = \beta_{p-1}^* + 1, \tag{22}$$

$$\text{for } \alpha_p^* > 1\colon \beta_{p-1} = \beta_{p-1}^*, \ \alpha_p = \alpha_p^* - 1, \ \beta_p = 1. \tag{23}$$

Now let α_t^* be the first odd number in the sequence (11) corresponding to the function ω^*. It is easy to determine that for $p < t$ the chain $Z_o(\omega^*)$ enters $\Delta Z_o(\omega)$ with coefficient zero. For $p > t$ the function ω belongs to Ω_α, and only for $p = t$ do we have $\omega \in \Omega_\beta$ and $\omega^* = \Psi(\omega)$. Thus, Ψ is a one-to one mapping of Ω_β on Ω_α.

(b) For $\omega' \in \Omega_\beta$ we define the cycle $\Gamma Z(\omega')$ of dimension $r(\omega') - 1$, putting

$$\Delta Z_o(\omega') = 2\Gamma Z_o(\omega') \quad (\text{cf. } (13)). \tag{24}$$

It then turns out that taken together, the chains

$$\begin{array}{c} Z_o(\omega),\ \omega \in \Omega,\ r(\omega) = r;\ \Gamma Z_o(\omega'),\ \omega' \in \Omega_\beta,\ r(\omega') - 1 = r; \\ Z_o(\omega''),\ \omega'' \in \Omega_\beta,\ r(\omega'') = r \end{array} \tag{25}$$

form a linearly independent basis of r-dimensional special Z-chains for arbitrary r, and at the same time that the chains

$$Z_o(\omega),\ \omega \in \Omega,\ r(\omega) = r;\ \Gamma Z_o(\omega'),\ \omega' \in \Omega_\beta,\ r(\omega') - 1 = r, \tag{26}$$

form a linearly independent basis of all r-dimensional special Z-cycles.

Let us first show that the system (26) is linearly independent. Suppose

$$\sum_{\omega \in \Omega} a_\omega Z_0(\omega) + \sum_{\omega' \in \Omega_\beta} b_{\omega'} \Gamma Z_0(\omega') = 0. \tag{27}$$

If $b_{\omega_1'} \neq 0$, then the term $\pm b_{\omega_1'} Z_o(\psi(\omega_1'))$ enters (27), (cf. (13)), and this term cannot cancel the corresponding term $\pm b_{\omega_2'} Z_o(\Psi(\omega_2'))$ of (27) for $\omega_1' \neq \omega_2'$, since in this case $\Psi(\omega_1') \neq \Psi(\omega_2')$ (cf. (a)). Further, the term $+ b_{\omega_1'} Z_o(\Psi(\omega_1'))$ cannot cancel other terms of $b_{\omega_2'} \Gamma Z_o(\omega_2')$ (cf. (14) and (24)). Similarly, $\pm b_{\omega_1'} Z_o(\Psi(\omega_1'))$ cannot cancel terms of the form $a_\omega Z_o(\omega)$, since $\omega \in \Omega$, $\psi(\omega_1') \in \Omega_\alpha$.

Thus, all coefficients $b_{\omega'}$ are equal to zero. Since the system $Z_o(\omega)$, $\omega \in \Omega$, is linearly independent, all the coefficients a_ω are equal to zero. Thus, the independence of the system (26) is demonstrated.

Let us now show that the system (25) is linearly independent. Suppose

$$\sum_{\omega \in \Omega} a_\omega Z_0(\omega) + \sum_{\omega' \in \Omega_\beta} b_{\omega'} \Gamma Z_0(\omega') + \sum_{\omega'' \in \Omega_\beta} c_{\omega''} Z_0(\omega'') = 0. \tag{28}$$

Taking the boundary of the left-hand side of (28) we have

$$\sum_{\omega'' \in \Omega_\beta} 2c_{\omega''} \Gamma Z_o(\omega'') = 0. \tag{29}$$

Since the system (26) is linearly independent, it follows from (29) that the coefficients $c_{\omega''}$ are all zero. Hence, in particular, it is evident that the linear form (25) can be a cycle only when it contains cycles of the system (26) alone.

Since all $c_{\omega''}$ are equal to zero, the relation (28) coincides with (27), and therefore all a_ω and $b_{\omega'}$ are also equal to zero. Thus, the system (25) is linearly independent.

Let us now show that an arbitrary special Z-chain X of dimension r is expressible linearly in terms of the chains (25). We consider the chain $X + Y$, where Y is a linear sum of chains (26). Among all chains of the form $X + Y$, where X is given and Y a linear form of the indicated type, we choose a maximal chain $X' = X + Y'$ (maximal in the sense defined in the proof of our lemma). We show that

$$X' = \sum_{\omega'' \in \Omega_\beta} c_{\omega''} Z_0(\omega''). \tag{30}$$

We write the chain X' in the form

$$X' = \sum_{p=1}^{q} c_p Z_0(\omega_p), \quad \omega_p \in \Omega', \quad p = 1, \ldots, q, \tag{31}$$

where all coefficients c_p are different from 0 and $\omega_1 \precsim \omega_2 \precsim \ldots \precsim \omega_q$, (the ordering of the functions $\omega \in \Omega'$ being that used in the proof of the lemma).

Let us show that $\omega_p \in \Omega_\beta$. We suppose that $\omega_p \in \Omega$; then, clearly,

$$X' - c_p Z_0(\omega_p) \succsim X',$$

and, accordingly, X' does not satisfy the condition of maximality. Therefore, ω_p cannot belong to Ω. We suppose that $\omega_p \in \Omega_\alpha$; then there exists a function $\omega' \in \Omega_\beta$ such that $\omega_p = \Psi(\omega')$ (cf. (a)), and if

$$\Gamma Z_0(\omega') = \varepsilon Z_0(\omega_p) + \ldots, \quad \varepsilon = \pm 1 \quad \text{(cf. (13), (24))},$$

then it is clear that

$$X' - \varepsilon c_p \Gamma Z_0(\omega') \succsim X',$$

that is, X' likewise fails to be maximal. Therefore, there remains only the possibility $\omega_p \in \Omega_\beta$, and relation (30) is demonstrated.

We now have $X = -Y' + X'$, where $-Y' + X'$ is a linear sum of chains of the type (25).

Thus, assertion (b) is proved.

(c) Since, in view of the lemma, every cycle of dimension $r \geq kl - l + 1$ is homologous to a special Z-cycle, and a special Z-cycle of dimension $r \geq kl - l + 1$ which is homologous to zero bounds a special Z-chain, it follows from (b) that the system (26) forms a canonical homology basis for the manifold $H(k, l)$ for dimensions $r \geq kl - l + 1$ (cf. (24)), and moreover the cycles $Z_0(\omega)$, $\omega \in \Omega$, are free cycles and the cycles $\Gamma Z_0(\omega')$, $\omega' \in \Omega_\beta$, are of order two.

(d) Instead of the specially chosen cycle $Z_0(\omega)$ one may take an arbitrary cycle $Z(\omega)$, $\omega \in \Omega$ (cf. definition 2). Similarly, in place of the cycle $\Gamma Z_0(\omega')$ one may choose $\Gamma Z(\omega')$, $\omega' \in \Omega_\beta$.

With this, the theorem is completely proved.

It follows from theorem 1 that the torsions of dimension $r \geq kl - l + 1$ of the manifold $H(k, l)$ are equal to two. In view of this, it is hardly necessary to consider modular cycles in $H(k, l)$, of the dimensions indicated, of modulus other than two.

(D) For $r \geq kl - l + 1$ the r-dimensional homology basis mod two of the manifold $H(k, l)$ can be constructed of the cycles

$$\left.\begin{array}{c} Z(\omega),\ \omega \in \Omega,\ r(\omega) = r; \quad \Gamma Z(\omega'),\ \omega' \in \Omega_\beta,\ r(\omega') - 1 = r; \\ Z(\omega''),\ \omega'' \in \Omega_\beta,\ r(\omega'') = r, \end{array}\right\} \tag{32}$$

taken mod two (cf. theorem 1); it can also be constructed of the pseudomanifolds

$$Z(\omega), \quad \omega \in \Omega', \quad r(\omega) = r, \tag{33}$$

considered as cycles mod two.

The fact that (32) constitutes a homology basis mod two follows from theorem 1 on the basis of well-known results.

In order to convince ourselves that the system (33) also represents a homology basis mod two, we consider instead the system

$$Z_0(\omega), \quad \omega \in \Omega', \quad r(\omega) = r, \tag{34}$$

which is equivalent to it:

$$Z_0(\omega) \sim Z(\omega) \pmod 2.$$

It follows immediately from our lemma, taken modulo two, and relation (7) that (34) is a homology basis for the manifold $H(k,l)$, mod two.

§ 6. Certain Properties of Characteristic Cycles

Here we will consider characteristic cycles of a closed orientable differentiable manifold M^k, that is, cycles X_χ, $r(\chi) \le k$, and $\Gamma X_{\chi'}$, $r(\chi') + 1 \le k$ (cf. definition 4).

The cycle X_χ is an integer cycle or a cycle modulo two according as the pseudomanifold $Z(l-\chi) = Z$ is or is not orientable. Using proposition (B) § 5, we can determine which case holds on the basis of properties of the function χ.

(A) The function χ is determined with respect to the numbers $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$ (cf. § 3, (C)). Let us consider the relations

$$\alpha_1+\beta_1\equiv\alpha_2+\beta_2\equiv\ldots\equiv\alpha_{n-1}+\beta_{n-1}\equiv 0 \pmod 2. \tag{1}$$

We will denote by X_o the set of all functions χ for which (1) is satisfied. We denote by X_2 the set of all remaining functions χ.

It follows immediately from (B) § 5 that for $\chi \in X_o$, X_χ is an integer cycle, and for $\chi \in X_2$, X_χ is a cycle mod two.

It follows immediately, from proposition (C) § 5, that

$$\mathrm{X}\subset\mathrm{X}_0,\quad \mathrm{X}_\beta\subset\mathrm{X}_2. \tag{2}$$

Furthermore, it turns out that if $\chi \in X_o$, (that is, if Z_χ for $l > \chi(1)$ is orientable), but $2Z_\chi \not\sim 0$ in $H(k,l)$, then

$$\chi\in\mathrm{X}. \tag{3}$$

This follows immediately from (B) § 5 and (C) § 2. Thus, if $\chi \in X_o - X$, then

$$2X_\chi \sim 0. \tag{4}$$

Theorem 1 allows us to select out of the class of all characteristic cycles a part in terms of which all remaining characteristic cycles can be expressed.

(B) We will call the characteristic cycles X_χ, $\chi \in X$, and

$X_{\chi'}$, $\chi' \in X_\beta$, (cf. § 5, (C)), basic cycles. All remaining characteristic cycles can be expressed in terms of these.

If $\chi^* \in X$, then X_{χ^*} is a basic characteristic cycle. If $\chi^* \in X_o - X$, then

$$X_{\chi^*}\sim\sum_{\chi\in\mathrm{X}_\beta} b_\chi\Gamma X_\chi, \tag{5}$$

where b_χ are remainders modulo two, determined by the function χ^*.

If $\chi^* \in X_2$, then

$$X_{\chi^*}\sim\sum_{\chi\in\mathrm{X}} a_\chi X_\chi+\sum_{\chi\in\mathrm{X}_\beta} b_\chi\Gamma X_\chi+\sum_{\chi\in\mathrm{X}_\beta} c_\chi X_\chi \pmod 2, \tag{6}$$

where a_χ, b_χ, c_χ are remainders modulo two, determined by the function χ^*. Finally, if $\chi^* \in X_2$, it follows from (6) that

$$\Gamma X_{\chi^*}\sim\sum_{\chi\in\mathrm{X}_\beta} c_\chi\Gamma X_\chi. \tag{7}$$

Let us prove (5). It follows from theorem 1 that

$$Z_{\chi^*}\sim\sum_{\chi\in\mathrm{X}} a_\chi Z_\chi+\sum_{\chi\in\mathrm{X}_\beta} b_\chi\Gamma Z_\chi, \tag{8}$$

where the a_χ are whole numbers and b_χ are remainders mod 2. Since theorem 1 gives us a canonical basis, the a_χ and b_χ are uniquely determined by the function $l - \chi^*$. Relation (4) § 3 shows that a_χ and b_χ do not depend on the number l, but are defined by the function χ^* alone. Since $\chi^* \in \mathrm{X}_o - X$, we have $2Z_{\chi^*} \sim 0$ (cf. (A)), and therefore all the numbers a_χ in (8) are equal to zero. Then (8) yields (5) (cf. § 3, (11), (12)).

Let us prove (6). By virtue of (D) § 5 we have

$$Z_{\chi^*}\sim\sum_{\chi\in\mathrm{X}} a_\chi Z_\chi+\sum_{\chi\in\mathrm{X}_\beta} b_\chi\Gamma Z_\chi+\sum_{\chi\in\mathrm{X}_\beta} c_\chi Z_\chi \pmod 2; \tag{9}$$

where a_χ, b_χ, and c_χ are remainders mod two. Since the basis derived in (D) § 5 is independent mod two, the a_χ, b_χ, and c_χ are uniquely defined by the function χ^*; but, in view of (4) § 3, a_χ,

b_χ, and c_χ do not depend on l, but are defined by the function χ^* alone. Relation (9) yields (6) (cf. § 3, (11), (12)).

(C) If $\chi(k) > 0$, then from the condition $r(\chi) \leq k$ it follows immediately that $\chi \equiv 1$. Thus, for $\chi(k) > 0$ we always have $\chi \equiv 1$. The cycle $X_1 = X_\chi$ ($\chi \equiv 1$) has dimension zero, and therefore defines a characteristic number X_1 (cf. § 3, (F)). In reference [5] it will be shown that the characteristic number $X_1(M^k)$ is the Euler characteristic $\mathcal{E}(M^k)$ of the manifold M^k. The function $\chi \equiv 1$ belongs to X_0; it belongs to X if and only if k is even. Thus, for uneven k, the number X_1 is equal to zero (cf. § 3, (E)), and therefore its identity with the Euler characteristic is patent in this case.

(D) Among all integer characteristic cycles X_χ, those are of especial interest for which $\chi \in X$, since it is only for these that the relation $2X_\chi \sim 0$ fails in general to hold. We have already considered (in (C)) one such cycle, namely X_1 for even k. If $\chi \in X$, $\chi \not\equiv 1$, then $\chi(k) = 0$, and therefore for the function χ all the numbers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{n-1}$ are even (cf. § 5, (C)). Thus, for $\chi \in X$ and $\chi \not\equiv 1$,

$$\left.\begin{array}{c}\chi(1) = \chi(2) \geqslant \ldots \geqslant \chi(2p-1) = \chi(2p) \geqslant \chi(2p+1) = \ldots = \chi(k) = 0; \\ \chi(i) \equiv 0 \pmod 2;\ r(\chi) \leqslant k.\end{array}\right\} \tag{10}$$

From (10) it is clear that $r(\chi)$ is divisible by four. All the characteristic numbers X_χ are determined by the conditions $\chi \in X$, $r(\chi) = k$. The characteristic number X_1 can be non-zero only for even k, and the other characteristic numbers X determined by (10) and the equation $r(\chi) = k$, are different from zero only for k divisible by four. For $k = 2$ there exists only the single characteristic number X_1. For $k = 4$ there exist X_1 and one more, X_χ, defined by the formulae

$$\chi(1) = \chi(2) = 2,\ \chi(3), = \chi(4) = 0. \tag{11}$$

Let us now clear up the question of the behavior of the characteristic cycles under a change of orientation of the manifold M^k, that is, when we go from M^k to $-M^k$.

<u>Theorem 2.</u> *Under a passage from the oriented manifold M^k to the oriented manifold $-M^k$ having the opposite orientation, we have*

$$\textit{for } \chi \textit{ not belonging to } X\textit{:}\ X_\chi(-M^k) \sim X_\chi(M^k), \tag{12}$$

$$\textit{for } \chi \equiv 1\textit{:}\ X_1(-M^k) \sim X_1(M^k), \tag{13}$$

$$\textit{for } \chi \in X,\ \chi \not\equiv 1\textit{:}\ X_\chi(-M^k) \sim -X_\chi(M^k). \tag{14}$$

<u>Proof.</u> Let $M^k \subset R^{k+l}$, and let T be a tangential representation of the manifold M^k on $H(k, l)$. We denote by $\hat{T}(\chi)$ the vector subspace of the space R^{k+l} which is distinguished from $T(\chi)$ only as to orientation. Then $\hat{T}$ is a tangential representation of the manifold $-M^k$ on $H(k, l)$.

Let us write $H = H(k, l)$. The relation (4) of § 5 was proved for chains $Z_0(\omega)$, but it is true also of the pseudomanifolds $Z_0(\omega)$. In fact, if $Z_0(\omega)$ is non-orientable, the relation in question is vacuous; if, however, $Z_0(\omega)$ is orientable, then, taken with a suitable orientation and considered as a cycle, it coincides with the chain $Z_0(\omega)$. Since we can pass from the pseudomanifold $Z_0(\omega)$ to an arbitrary pseudomanifold $Z(\omega)$ by a rotation of R^{k+l}, the relationship under consideration also holds for arbitrary pseudomanifolds $Z(\omega)$. Thus, from (4) § 5 we obtain

$$\hat{Z}_\chi = (-1)^{k+l+\chi(1)} Z_\chi;\ \hat{H} = (-1)^{k+l} H. \tag{15}$$

For the proof of the theorem we will make use of characteristic ∇-cycles. We denote by $\hat{Y}_\chi$ the characteristic ∇-cycle of the manifold $-M^k$. Then

$$Y_\chi(E) = I_H(Z_\chi, T(E));\ \hat{Y}_\chi(E) = I_H(Z_\chi, \hat{T}(E)), \tag{16}$$

where E is an arbitrary oriented simplex of dimension $r(\chi)$ of the manifold M^k. In order to calculate the right-hand side of the second of the equations (16) we subject the manifold $H(k, l)$ to the homeomorphic mapping $R^k \to \hat{R}^k$. Then

$$I_H(Z_\chi, \hat{T}(E)) = I_{\hat{H}}(\hat{Z}_\chi, T(E)) = (-1)^{\chi(1)} I_H(Z_\chi, T(E)) \text{ (cf. (15)).} \tag{17}$$

From (16) and (17) we obtain

$$\hat{Y}_\chi = (-1)^{\chi(1)} Y_\chi. \tag{18}$$

In going from the ∇-cycle to the corresponding Δ-cycle the orientation of the manifold M^k is taken into account (cf. § 3, (8)). From

(18) we obtain

$$X_\chi(-M^k) = (-1)^{\chi(1)+1} X_\chi(M^k). \tag{19}$$

If $\chi \in X_2$, then X_χ is a cycle mod two, and (19) yields (12). If $\chi \in X_o - X$, then $X_\chi \sim -X_\chi$ (cf. (4)), and again (19) yields (12). If $\chi \equiv 1$, then $\chi(1) = 1$, and (19) yields (13). Finally, if $\chi \in X$, $\chi \not\equiv 1$, then $\chi(k) = 0$ (cf. (C)), and all the numbers $\beta_1, \ldots, \beta_{n-1}$ are even (cf. § 5, (C)). Since $\chi(1) = \beta_1 + \ldots + \beta_{n-1} + \chi(k)$, (19) yields (14).

Thus, theorem 2 is proved.

As an immediate consequence of theorem 2, we have:

(E) If the manifold M^k admits a homeomorphic differentiable mapping on itself which reverses orientation, then every characteristic cycle X_χ, $\chi \not\equiv 1$, satisfies the condition $2X_\chi \sim 0$. In particular, every characteristic number X_χ, $\chi \not\equiv 1$, of the manifold M^k reduces to 0.

Theorem 3. *If the manifold M^k is the boundary of an oriented differentiable bounded manifold M^{k+1}, the characteristic number X_1 of M^k is even, and all other characteristic numbers and remainders are zero.*

Proof. We will suppose that $M^{k+1} \subset R^{k+l}$.

It is easy to define a differentiable numerical function $f(x)$, $x \in M^{k+1}$, satisfying the following conditions:

(a) $f(x) \geq 0$;

(b) the equation $f(x) = 0$ defines the boundary M^k of the manifold M^{k+1};

(c) the critical points of the function f are isolated and lie in M^k.

Suppose a is an arbitrary non-critical point of M^{k+1}. We will so orient the level surface defined by $f(x) = f(a)$ that it is the boundary of the piece $f(x) \leq f(a)$, and we construct at the point a an oriented tangent T'_a to this level surface. We denote by $T'(a)$ the oriented vector subspace of the space R^{k+l} which is parallel to the oriented hyperplane T'_a. Then $T'(a) \in H(k, l)$, and we have a representation T', defined and continuous for all non-critical points $a \in M^{k+1}$, and T' is a tangential representation of M^k.

Let $a_1, \ldots, a_t$ be the set of all critical points of the function f. We will denote by P_j^{k+1} the tangent to M^{k+1} at the point a_j, and by R_j^{k+1} the vector subspace of R^{k+l} parallel to it.

Further, let δ be a small positive number. We denote by E_j^{k+1} the spherical neighborhood of the point a_j in M^{k+1}, of radius δ, oriented in accord with M^{k+1}, and we denote by S_j^k its boundary, suitably oriented.

Let us further set

$$M_\delta^{k+1} = M^{k+1} - (E_1^{k+1} \cup \ldots \cup E_t^{k+1}).$$

Then the representation T' is defined on the whole of the bounded manifold M_δ^{k+1}, and, accordingly,

$$T'(M^k) \sim T'(S_1^k) + \ldots + T'(S_t^k) \quad \text{in} \quad H(k, l). \tag{20}$$

Thus, in order to calculate the characteristic numbers and remainders X_χ of the manifold M^k it is sufficient to calculate the index of intersection $I(Z_\chi, T'(S_j^k))$ for all $j = 1, \ldots, t$.

If δ is small enough, then for $a \in S_j^k$ the surface $T'(a)$ is close to its projection $T''(a)$ on R_j^{k+1}, in the topology of $H(k, l)$. Thus, the representation T' of the sphere S_j^k in $H(k, l)$ can by a small deformation be carried into the representation T'' which is such that for $a \in S_j^k$ we have $T''(a) \subset R_j^{k+1}$.

If we denote by $H_j(k, 1)$ the manifold of all k-dimensional oriented subspaces of the space R_j^{k+1}, the representation T'' carries S_j^k into $H_j(k, 1)$. Since the dimension of $H_j(k, 1)$ is equal to k, the representation T'' of the sphere S_j^k in $H_j(k, 1)$ has some degree γ_j, and we obtain

$$I(Z_\chi, T'(S_j^k)) = \gamma_j I(Z_\chi, H_j(k, 1)). \tag{21}$$

Let us now make use of relations (3) and (5) of § 3, putting $R_j^{k+1} = R^{k+l''}$, that is, setting $l'' = 1$. If $\chi \equiv 1$, then from (5) § 3 we have

$$I(Z_1, H_j(k, 1)) = 1 + (-1)^k. \tag{22}$$

If $\chi \not\equiv 1$, then from $r(\chi) = k$ there follows $\chi(1) > 1$, and relation

(3) § 3 yields

$$I(Z_\chi, H_j(k, 1)) = 0. \tag{23}$$

Theorem 3 follows from relations (20), (21), (22), and (23).

Since the characteristic number X_1 is equal to the Euler characteristic of the manifold, it follows from theorem 3 that the manifold M^k, as the boundary of an orientable manifold M^{k+1}, has an even-valued Euler characteristic. This fact can be easily demonstrated directly for combinatorial manifolds M^k and M^{k+1} by matching two copies of the manifold M^{k+1} along their boundaries and counting simplexes to get the Euler characteristic.

Thus, there exists a four-dimensional orientable manifold which cannot be the boundary of a five-dimensional orientable manifold. The simplest example of this is the complex projective plane, for which the Euler characteristic has the value 3.

It would be interesting to show that every three-dimensional orientable manifold is the boundary of a four-dimensional orientable bounded manifold.

The following propositions (F) and (G) can be used for the calculation of characteristic cycles of certain manifolds.

(F) If there exists a regular mapping f of the manifold M^k on the vector space R^{k+1} (cf. § 3, (1)), then all characteristic cycles of M^k with the possible exception of X_1 (cf. (D)), are homologous to zero in M^k, and X_1 is, as a characteristic number, even.

Let $R^{k+1} \subset R^{k+l}$, where l is sufficiently large; then $H(k,1) \subset H(k,l)$.

The manifold $H(k,1)$ is homeomorphic to the k-dimensional sphere. The tangential representation T of the manifold M^k, constructed on the basis of the regular mapping f (cf. definition 3), has the property that $T(M^k) \subset H(k,1)$.

If $\chi(1) > 1$, then by (3) § 3, the pseudomanifold Z_χ can be so chosen that the intersection $Z_\chi \cap H(k,1)$ is empty. If $\chi(1) = 1$, but $r(\chi) < k$, then $Z_\chi \times H(k,1) \sim 0$ in $H(k,1)$, and therefore there exists a cycle Z^*_χ homologous to the cycle Z_χ in $H(k,l)$ and such that the intersection $Z^*_\chi \cap H(k,1)$ is empty. If $\chi(1) = 1$ and $r(\chi) = k$, we have $\chi \equiv 1$, and then, in view of (5) § 3,

$$I(Z_\chi, H(k, 1)) = 1 + (-1)^k.$$

From this and from the fact that $T(M^k) \subset H(k,1)$, proposition (F) follows at once.

(G) Suppose M^k is oriented, and, possibly, disconnected, and let U^k_1 and U^k_2 be two small spherical neighborhoods in it, the closures of which do not intersect: $\overline{U}^k_1 \cap \overline{U}^k_2 = 0$. We denote the boundaries of these neighborhoods by S^{k-1}_1 and S^{k-1}_2; both of these are homeomorphic to the $(k-1)$-dimensional sphere. We cut the regions U^k_1 and U^k_2 out of M^k, and in the bounded manifold so obtained we match the domains S^{k-1}_1 and S^{k-1}_2, preserving their orientations. We denote by M^k_1 the closed oriented manifold which we obtain as a result.

It is easily seen that in going from M^k to M^k_1 only the Betti groups of dimension 0, 1, $k-1$, k can be altered. Thus, for all other dimensions we may speak of the coincidence of the cycles of M^k and M^k_1. As far as the null-dimensional cycles are concerned, we can speak of their coincidence in the sense of the coincidence of indices. It turns out that all characteristic cycles, with the possible exception of X_1 (cf. (D)), coincide for the manifolds M^k and M^k_1.

Suppose that R^{k+l} is a vector space of sufficiently high dimension, and suppose $R^{k+1} \subset R^{k+l}$. Then $H(k,1) \subset H(k,l)$. It is easy to construct a regular mapping of the manifold M^k into R^{k+l}, say f, which satisfies the following conditions:

(a) $f(U^k_1) \subset R^{k+1}$, $f(U^k_2) \subset R^{k+1}$,

(b) the mapping f of the bounded manifold $M^k - U^k_1 - U^k_2$ constitutes a regular mapping of the manifold M^k_1 into R^{k+l}.

Choosing the cycles Z_χ and Z^*_χ just as we did in (F), we can convince ourselves of the truth of proposition (G).

Now let us consider certain very simple characteristic cycles.

(H) Let χ be a function having at most one jump and satisfying the condition $r(\chi) \leq k$. Then it is defined by the following relations:

$$\chi(1) = \ldots = \chi(p) = q, \quad \chi(p+1) = \ldots = \chi(k) = 0, \quad r(\chi) = pq \leqslant k. \tag{24}$$

Since a function χ so defined is determined by the two numbers p and q, we denote it by $\chi_{p,q}$. We denote the pseudomanifold Z_χ, defined by this function, by $Z_{p,q}$, and the characteristic cycle X_χ by $X_{p,q}$. In the case when χ has no jumps at all, that is, $p = k$, the relation $pq \leq k$ implies either $q = 1$ or $q = 0$. For $q = 1$ we have the characteristic cycle $X_{k,1} = X_1$ with which we are already familiar, (cf. (C)). For $q = 0$ we have $Z_{k,o} = H(k,l)$ and $X_{k,o} = M^k$.

If χ has one jump, then $p < k$, and $\chi_{p,q}(k) = 0$. Further, $\alpha_1 = p$, $\beta_1 = q$, and therefore

$$\left.\begin{array}{ll} \text{for } p+q\equiv 0 \pmod 2: & \chi_{p,q}\in X_0, \\ \text{for } p+q\not\equiv 0 \pmod 2: & \chi_{p,q}\in X_2, \\ \text{for } p\equiv q\equiv 0 \pmod 2: & \chi_{p,q}\in X. \end{array}\right\} \tag{25}$$

In all cases considered the pseudomanifold $Z_{p,q}$ is defined by a single surface S_1 (cf. § 1, (B)) of dimension $l + p - q$; in fact, $Z_{p,q}$ consists of all $R^k \in H(k,l)$ satisfying the condition

$$D(R^k \cap S_1) \geqslant p. \tag{26}$$

(Received by the Editors 6 June 1947)

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