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# MASLOV INDEX FOR HAMILTONIAN SYSTEMS

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Abstract. The aim of this article is to give an explicit formula for computing the Maslov index of the fundamental solutions of linear autonomous Hamiltonian systems in terms of the Conley-Zehnder index and the map time one flow.

## 1. Introduction

The Maslov index is a semi-integer homotopy invariant of paths $l$ of Lagrangian subspaces of a symplectic vector space $(V, \omega)$ which gives the algebraic counts of non transverse intersections of the family $\{l(t)\}_{t\in[0,1]}$ with a given Lagrangian subspace $l_*$. To be more precise, let us denote by $\Lambda(V) := \Lambda(V, \omega)$ the set of all Lagrangian subspaces of the symplectic space $V$ and let $\Sigma(l_*) = \{l \in \Lambda : l \cap l_* \neq (0)\}$ be the *train* or the *Maslov cycle* of $l_*$. Then, it can be proven that $\Sigma(l_*)$ is a co-oriented one codimensional algebraic subvariety of the Lagrangian Grassmannian $\Lambda(V)$ and the Maslov index counts algebraically the number of intersections of $l$ with $\Sigma(l_*)$. This is the basic invariant out of which many others are defined. For example, if $\phi\colon [a, b] \to \mathrm{Sp}(V)$ is a path of symplectic automorphisms of $V$ and $l_*$ is a fixed Lagrangian subspace, then the Maslov index of $\phi$ is by definition the number of intersections of the path $[a, b] \ni t \mapsto \phi_t(l_*) \in \Lambda(V)$ with the train of $l_*$. The aim of this paper is to explicitly compute the Maslov index of the fundamental solution associated to

$$w'(x) = Hw(x)$$

where $H$ is a (real) constant Hamiltonian matrix. The idea in order to perform our computation is to relate the Maslov index with the Conley-Zehnder index and then to compute an arising correction term which is written in terms of an invariant of a triple of Lagrangian subspaces also known in literature as Kashiwara index.

We remark that the result is not new and it was already proven in [8]. However the contribution of this paper is to provide a different and we hope a simpler proof of this formula.

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## 2. Preliminaries

The purpose of this section is to recall some well-known facts about the geometry of the Lagrangian Grassmannian and the Maslov index needed in our computation. For further details see for instance [1, 3, 4, 8, 9].

**Definition 2.1.** Let $V$ be a finite dimensional real vector space. A symplectic form $\omega$ is a non degenerate anti-symmetric bilinear form on $V$. A symplectic vector space is a pair $(V, \omega)$.

The archetypical example of symplectic space is $(\mathbb{R}^{2n}, \omega_0)$ where the symplectic structure $\omega_0$ is defined as follows. Given the splitting $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and the scalar product of $\mathbb{R}^n$ $\langle \cdot, \cdot \rangle$ then for each $z_k = (x_k, y_k) \in \mathbb{R}^n \oplus \mathbb{R}^n$ for $k = 1, 2$ we have

$$\omega_0(z_1, z_2) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle.$$

This symplectic structure $\omega_0$ can be represented against the scalar product by setting $\omega_0(z_1, z_2) = \langle J z_1, z_2 \rangle$ for all $z_i \in \mathbb{R}^{2n}$ with $i = 1, 2$ where we denoted by $J$ the *standard complex structure* of $\mathbb{R}^{2n}$ which can be written with respect to the canonical basis of $\mathbb{R}^{2n}$ as

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{2.1}$$

where $I_n$ is the $n$ by $n$ identity matrix. Given a linear subspace of the symplectic vector space $(V, \omega)$, we define the orthogonal of $W$ with respect to the symplectic form $\omega$ as the linear subspace $W^\sharp$ given by $W^\sharp = \{v \in V : \omega(u, v) = 0 \ \forall u \in W\}$.

**Definition 2.2.** Let $W$ be a linear subspace of $V$. Then

- (i) $W$ is *isotropic* if $W \subset W^\sharp$;
- (ii) $W$ is *symplectic* if $W^\sharp \cap W = 0$.
- (iii) $W$ is *Lagrangian* if $W = W^\sharp$.

**Definition 2.3.** Let $(V_1, \omega_1)$ and $(V_2, \omega_2)$ be symplectic vector spaces. A symplectic isomorphism from $(V_1, \omega_1)$ to $(V_2, \omega_2)$ is a bijective linear map $\varphi \colon V_1 \to V_2$ such that $\varphi^* \omega_2 = \omega_1$, meaning that

$$\omega_2(\varphi(u), \varphi(v)) = \omega_1(u, v), \quad \forall\, u, v \in V_1.$$

In the case $(V_1, \omega_1) = (V_2, \omega_2)$, $\varphi$ is called a symplectic automorphism or symplectomorphism.

The matrices which correspond to symplectic automorphisms of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ are called symplectic and they are characterized by the equation

$$A^T J A = J$$

where $A^T$ denotes the adjoint of $A$. The set of all symplectic automorphisms of $(V, \omega)$ forms a group, denoted by $\mathrm{Sp}(V, \omega)$. The set of the symplectic matrices is a Lie group, denoted by $\mathrm{Sp}(2n)$. Since each symplectic vector space of dimension $2n$ is symplectically isomorphic to $(\mathbb{R}^{2n}, \omega_0)$, then $\mathrm{Sp}(V, \omega)$ is isomorphic to $\mathrm{Sp}(2n)$. The Lie algebra of $\mathrm{Sp}(2n)$ is

$$\mathrm{sp}(2n) := \{H \in L(2n) : H^T J + J H = 0\}$$

where $L(2n)$ is the vector space of all real matrices of order $2n$. The matrices in $\mathrm{sp}(2n)$ are called *infinitesimally symplectic* or *Hamiltonian*.

2.1. **The Krein signature on** $\mathrm{Sp}(2n)$**.** Following the argument given in [1, Chapter 1, Section 1.3] we briefly recall the definition of Krein signature of the eigenvalues of a symplectic matrix.

In order to define the Krein signature of a symplectic matrix $A$ we shall consider $A$ as acting on $\mathbb{C}^{2n}$ in the usual way

$$A(\xi + i\eta) := A\xi + iA\eta, \quad \forall\, \xi, \eta \in \mathbb{R}^{2n}$$

and we define the Hermitian form $g(\xi, \eta) := \langle G\xi, \eta\rangle$ where $G := -iJ$. The complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$ consists of the complex matrices $A$ such that

$$A^* G A = G \tag{2.2}$$

where as usually $A^* = \bar{A}^T$ denotes the transposed conjugate of $A$.

**Definition 2.4.** Let $\lambda$ be an eigenvalue on the unit circle of a complex symplectic matrix. The *Krein signature* of $\lambda$ is the signature of the restriction of the Hermitian form $g$ to the generalized eigenspace $E_\lambda$.

If the real symplectic matrix $A$ has an eigenvalue $\lambda$ on the unit circle of Krein signature $(p, q)$, it is often convenient to say that $A$ has $p + q$ eigenvalues $\lambda$, and that $p$ of them are Krein-positive and $q$ which are Krein-negative.

Let $A$ be a semisimple symplectic matrix, meaning that the algebraic and geometric multiplicity of its eigenvalues coincides.

**Definition 2.5.** We say that $A$ is in *normal form* if $A = A_1 \oplus \cdots \oplus A_p$, where $A_i$ has one of the forms listed below:

(i) $A_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, for $\alpha \in \mathbb{R}$.

(ii) $A_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, for $\mu \in \mathbb{R}$ and $|\mu| > 1$.

(iii)
$$A_3 = \begin{pmatrix} \lambda\cos\alpha & -\lambda\sin\alpha & 0 & 0 \\ \lambda\sin\alpha & \lambda\cos\alpha & 0 & 0 \\ 0 & 0 & \lambda^{-1}\cos\alpha & -\lambda^{-1}\sin\alpha \\ 0 & 0 & \lambda^{-1}\sin\alpha & \lambda^{-1}\cos\alpha \end{pmatrix},$$
for $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$, $\mu \in \mathbb{R}$, $|\mu| > 1$.

2.2. **The Maslov index.** In this section we define the Maslov index for Lagrangian and symplectic paths. Our basic reference is [9]. Given a symplectic space $(V, \omega)$, let us consider the set of its Lagrangian subspaces $\Lambda(V, \omega)$. For any $L_0 \in \Lambda(V, \omega)$ fixed and for all $k = 0, 1, \dots, n$ we set

$$\Lambda_k(L_0) = \big\{L \in \Lambda : \dim(L \cap L_0) = k\big\}, \quad \Sigma(L_0) = \cup_{k=1}^n \Lambda_k(L_0).$$

It can be proven that each stratum $\Lambda_k(L_0)$ is connected of codimension $\frac{1}{2}k(k+1)$ in $\Lambda$. If $l\colon [a, b] \to \Lambda$ is a $C^1$-curve of Lagrangian subspaces, we say that $l$ has a *crossing* with the *train* $\Sigma(L_0)$ of $L_0$ at the instant $t = t_0$ if $l(t_0) \in \Sigma(L_0)$. At each non transverse crossing time $t_0 \in [a, b]$ we define the *crossing form* $\Gamma$ as the quadratic form

$$\Gamma(l, L_0, t_0) = l'(t_0)|_{l(t_0)\cap L_0}$$

and we say that a crossing $t$ is called *regular* if the crossing form is nonsingular. It is called *simple* if it is regular and in addition $l(t_0) \in \Lambda_1(L_0)$.

**Definition 2.6.** Let $l\colon [a,b] \to \Lambda$ be a smooth curve having only *regular crossings* we define the *Maslov index*

$$\mu(l, L_0) := \frac{1}{2}\operatorname{sign}\Gamma(l, L_0, a) + \sum_{t\in ]a,b[} \operatorname{sign}\Gamma(l, L_0, t) + \frac{1}{2}\operatorname{sign}\Gamma(l, L_0, b) \tag{2.3}$$

where the summation runs over all crossings $t$.

For the properties of this number we refer to [9]. Now let $\psi\colon [a,b] \to \mathrm{Sp}(2n)$ be a continuous path of symplectic matrices and $L \in \Lambda(n)$ where $\Lambda(n)$ denotes the set of all Lagrangian subspaces of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$. Then we define the *Maslov index* of the $\psi$ as

$$\mu_L(\psi) := \mu(\psi L, L).$$

Given the vertical Lagrangian subspace $L_0 = \{0\} \oplus \mathbb{R}^n$ and assuming that $\psi$ has the block decomposition

$$\psi(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & d(t) \end{pmatrix}, \tag{2.4}$$

then the crossing form of the path of Lagrangian subspaces $\psi L_0$ at the crossing instant $t = t_0$ is the quadratic form $\Gamma(\psi, t_0)\colon \ker b(t_0) \to \mathbb{R}$ given by

$$\Gamma(\psi, t_0)(v) = -\langle d(t_0)v, b'(t_0)v\rangle \tag{2.5}$$

where $b(t_0)$ and $d(t_0)$ are the block matrices defined in (2.4). Lemma below will be crucial in our final computation.

**Lemma 2.7.** *Consider the symplectic vector space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ equipped by the symplectic form $\overline{\omega} = -\omega_0 \times \omega_0$. Then*

$$\mu(\psi L, L_1) = \mu(\mathrm{Gr}(\psi), L \times L_1) \tag{2.6}$$

*where* Gr *denotes the graph and where $L, L_1 \in \Lambda(n)$.*

For the proof of this result see [9, Theorem 3.2].

**Definition 2.8.** Given a continuous path of symplectic matrices $\psi$, we define the *Conley-Zehnder index $\mu_{CZ}(\psi)$* as

$$\mu_{CZ}(\psi) := \mu(\mathrm{Gr}(\psi), \Delta),$$

where $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ denotes the diagonal in the product space.

Let us consider the path

$$\psi_1(x) = \begin{pmatrix} \cos\alpha x & -\sin\alpha x \\ \sin\alpha x & \cos\alpha x \end{pmatrix}. \tag{2.7}$$

Given the Lagrangian $L_0 = \{0\} \oplus \mathbb{R}$ in $\mathbb{R}^2$ let us consider the path of Lagrangian subspaces of $\mathbb{R}^2$ given by $l_1 := \psi_1 L_0$. It is easy to check that the crossing points are of the form $x \in \pi\mathbb{Z}/\alpha$ so by formula (2.5) if $x_0$ is a crossing instant then we have

$$\Gamma(\psi_1, x_0)(k) = \alpha k\ \cos(\alpha x_0) k \cos(\alpha x_0) = \alpha k^2 \cos^2(\alpha x_0).$$

Thus if $\alpha \neq 0$, we have

$$\operatorname{sign}\Gamma(\psi_1, x_0) = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha < 0. \end{cases}$$

Summing up we have the following lemma.

**Lemma 2.9.** *Let $\psi_1 : [0,1] \to \mathrm{Sp}(2)$ be the path of symplectic matrices given in* (2.7)*. Then the Maslov index is given by*

(1) *non transverse end-point $\mu(\psi_1) = \alpha/\pi$;*
(2) *transverse end-point $\mu(\psi_1) = \left[\frac{\alpha}{\pi}\right] + 1/2$.*

*where we have denoted by $[\cdot]$ the integer part.*

Let $\psi(x) = e^{xH}$ be the fundamental solution of the linear system

$$z'(x) = Hz(x) \quad x \in [0,1]$$

where $H$ is a semi-simple infinitesimally symplectic matrix and let us denote by $L_0'$ the vertical Lagrangian $\oplus_{j=1}^p L_0^j$ of the symplectic space $(V, \oplus_{j=1}^p \omega_j)$ where $\omega_j$ is the standard symplectic form in $\mathbb{R}^{2m}$ for $m = 1, 2$ corresponding to the decomposition of $V$ into 2 and 4 dimensional $\psi(1)$-invariant symplectic subspaces. Then as a direct consequence of the product property of the Maslov index the following holds.

**Proposition 2.10.** *Let $e^{i\alpha_1}, \dots, e^{i\alpha_k}$ be the Krein positive eigenvalues of $\psi$. Then the Maslov index with respect to the Lagrangian $L_0'$ is given by:*

$$\mu_{L_0'}(\psi) = \sum_{j=1}^{k} f\Big(\frac{\alpha_j}{\pi}\Big), \tag{2.8}$$

*where $f$ be the function which holds identity on semi-integer and is the closest semi-integer not integer otherwise.*

**Remark 2.11.** By using the zero property for the Maslov index (see for instance [9]) a direct computation shows that the (ii) and (iii) of Definition 2.5 do not give any non null contribution to the Maslov index.

2.3. **The Kashiwara and Hörmander index.** The aim of this section is to discuss a different notion of Maslov index. Our basic references are [3], [7], [2, Section 8] and [4, Section 3]. The Hörmander index, or four-fold index has been introduced in [5, Chapter 10, Sect. 3.3] who also gave an explicit formula in terms of a triple of Lagrangian subspaces, which is known in literature with the name of *Kashiwara index* and which we now describe.

Given the Lagrangians $L_1, L_2, L_3 \in \Lambda(V,\omega)$, the *Kashiwara index* $\tau_V(L_1, L_2, L_3)$ is defined as the signature of the (symmetric bilinear form associated to the) quadratic form $Q : L_1 \oplus L_2 \oplus L_3 \to \mathbb{R}$ given by:

$$Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1). \tag{2.9}$$

It is proven in [2, Section 8] that $\tau_V$ is the unique integer valued map on $\Lambda \times \Lambda \times \Lambda$ satisfying the following properties:

(P1) (skew symmetry) If $\sigma$ is a permutation of the set $\{1, 2, 3\}$,
$$\tau_V(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)}) = \mathrm{sign}(\sigma)\tau_V(L_1, L_2, L_3);$$
(P2) (symplectic additivity) given the symplectic spaces $(V,\omega)$, $(\widetilde{V}, \widetilde{\omega})$, and the Lagrangians $L_1, L_2, L_3 \in \Lambda(V,\omega)$, $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3 \in \Lambda(\widetilde{V}, \widetilde{\omega})$, we have
$$\tau_{V\oplus\widetilde{V}}(L_1 \oplus \widetilde{L}_1, L_2 \oplus \widetilde{L}_2, L_3 \oplus \widetilde{L}_3) = \tau_V(L_1, L_2, L_3) + \tau_{\widetilde{V}}(\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3);$$
(P3) (symplectic invariance) if $\phi : (V,\omega) \to (\widetilde{V}, \widetilde{\omega})$ is a symplectomorphism, then
$$\tau_V(L_1, L_2, L_3) = \tau_{\widetilde{V}}(\phi(L_1), \phi(L_2), \phi(L_3));$$

(P4) (normalization) if $V = \mathbb{R}^2$ is endowed with the canonical symplectic form, and $L_1 = \mathbb{R}(1,0)$, $L_2 = \mathbb{R}(1,1)$, $L_3 = \mathbb{R}(0,1)$, then $\tau_V(L_1, L_2, L_3) = 1$.

Let $(V,\omega)$ be a $2n$-dimensional symplectic vector space and let $L_1, L_2, L_3$ be three Lagrangians and let us assume that $L_3$ is transversal both to $L_1$ and $L_2$. If $L_1$ and $L_2$ are transversal we can choose coordinates $z = (x, y) \in V$ in such a way that $L_1$ is defined by the equation $y = 0$, $L_2$ by the equation $x = 0$ and consequently $L_3$ is defined by $y = Ax$ for some symmetric non-singular matrix $A$. We claim that

$$\tau_V(L_1, L_2, L_3) = \operatorname{sign} A.$$

In fact, since every symplectic vector space of dimension $2n$ is symplectically isomorphic to $(\mathbb{R}^{2n}, \omega_0)$, by property [P3] on the symplectic invariance of the Kashiwara index and by equation (2.9), it is enough to compute the signature of the quadratic form $Q$ for

$$x_1 = (x, 0), \quad x_2 = (z, Az), \quad x_3 = (0, y), \quad \text{where } x, y, z \in \mathbb{R}^n,$$

and $\omega = \omega_0$. Thus, $\omega_0(x_1, x_2) = \langle z, Az\rangle$, $\omega_0(x_3, x_1) = -\langle x, y\rangle$ and $\omega_0(x_2, x_3) = \langle x, y\rangle$ and by this we conclude that

$$Q(x_1, x_2, x_3) = \langle z, Az\rangle.$$

In the general case, let $K = L_1 \cap L_2$. Then $K$ is an isotropic linear subspace of the symplectic space $(V,\omega)$ and $K^\#/K := V^K$ is a symplectic vector space with the symplectic form induced by $(V,\omega)$. If $L$ is any Lagrangian subspace in $(V,\omega)$ then $L^K = L \cap K^\# \mod K$, is a Lagrangian subspace in $V^K$.

**Lemma 2.12.** *For an arbitrary subspace $K$ of $L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1$,*

$$\tau_V(L_1, L_2, L_3) = \tau_{V^K}(L_1^K, L_2^K, L_3^K).$$

*where for $i = 1, 2, 3$ the Lagrangian subspaces $L_i^K$ are the image of $L_i$ under the symplectic reduction*

$$(K + K^\#) \to V^K := (K + K^\#)/(K \cap K^\#).$$

For the proof of the above lemma, see [7, Proposition 1.5.10]. We will now proceed to a geometrical description of $\tau_V$ using the Maslov index for paths. To this aim we will introduce the *Hörmander index*.

**Lemma 2.13.** *Given four Lagrangians $L_0, L_1, L_0', L_1' \in \Lambda$ and any continuous curve $l : [a,b] \to \Lambda$ such that $l(a) = L_0'$ and $l(b) = L_1'$, then the value of the quantity $\mu(l, L_1) - \mu(l, L_0)$ does* not *depend on the choice of $l$.*

The proof of the above lemma can be found in [9, Theorem 3.5]. We are now ready for defining the map $s : \Lambda \times \Lambda \times \Lambda \times \Lambda \to \frac{1}{2}\mathbb{Z}$.

**Definition 2.14.** Given $L_0, L_1, L_0', L_1' \in \Lambda$, the *Hörmander index* $s(L_0, L_1; L_0', L_1')$ is the half-integer $\mu(l, L_1) - \mu(l, L_0)$, where $l : [a,b] \to \Lambda$ is any continuous curve joining $l(a) = L_0'$ with $l(b) = L_1'$.

The Hörmander's index, satisfies the following symmetries. (See, for instance [4, Proposition 3.23]). We can now establish the relation between the Hörmander index $s$ and the Kashiwara index $\tau_V$.This will be made in the same way as in [4, Section 3]. We define $\overline{s} : \Lambda \times \Lambda \times \Lambda \to \mathbb{Z}$ by:

$$\overline{s}(L_0, L_1, L_2) := 2s(L_0, L_1; L_2, L_0). \tag{2.10}$$

Observe that the function $s$ is completely determined by $\overline{s}$, because of the following identity

$$\begin{aligned}2s(L_0, L_1; L_0', L_1') &= 2s(L_0, L_1; L_0', L_0) + 2s(L_0, L_1; L_0, L_1')\\ &= \overline{s}(L_0, L_1, L_0') - \overline{s}(L_0, L_1, L_1').\end{aligned} \tag{2.11}$$

**Proposition 2.15.** *The map $\overline{s}$ defined in* (2.10) *coincides with the Kashiwara index $\tau_V$.*

*Proof.* By uniqueness, it suffices to prove that $\overline{s}$ satisfies the properties (P1), (P2), (P3) and (P4). See [4] for further details. □

As a direct consequence of Proposition 2.15 and formula (2.11) we have

$$s(L_0, L_1; L_0', L_1') = \frac{1}{2}[\tau_V(L_0, L_1, L_0') - \tau_V(L_0, L_1, L_1')]. \tag{2.12}$$

## 3. The main result

Let $\psi$ be the fundamental solution of the linear Hamiltonian system

$$w'(x) = Hw(x), \quad x \in [0, 1].$$

By Lemma 2.7 we have $\mu_{L_0}(\psi) = \mu(\mathrm{Gr}(\psi), L_0 \times L_0)$; hence

$$\begin{aligned}\mu(\mathrm{Gr}(\psi), L_0 \times L_0) &= \mu(\mathrm{Gr}(\psi), \Delta) + s(\Delta, L_0 \times L_0, \mathrm{Gr}(I), \mathrm{Gr}(\psi(1)))\\ &= \mu_{CZ}(\psi) - \frac{1}{2}\tau_V(\Delta, L_0 \times L_0, \mathrm{Gr}(\psi(1)))\end{aligned} \tag{3.1}$$

where the last equality follows by (2.12). For one periodic loop the last term in formula (3.1) vanishes identically because of the anti-symmetry of the Kashiwara index and by the fact that $\mathrm{Gr}(\psi(1)) = \Delta$. Thus in this case we conclude that

$$\mu_{L_0}(\psi) = \mu_{CZ}(\psi).$$

From now on we assume the following transversality condition:

(H) $\psi(1)L_0 \cap L_0 = \{0\}$.

Let $L = L_0 \times L_0$ and $L_2 = \mathrm{Gr}\,(\psi(1))$. Thus we only need to compute the last term in formula (3.1) which is $-\frac{1}{2}\tau_V(\Delta, L, L_2)$ where the product form can be represented with respect to the scalar product in $\mathbb{R}^{4n}$ by the matrix

$$\widetilde{J} = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

for $J$ defined in (2.1). We denote by $K$ the isotropic subspace $\Delta \cap L$; it is the set of all vectors of the form $(0, u, 0, u)$ for $u \in \mathbb{R}^n$. Moreover $K^{\#}$ is

$$\begin{aligned}K^{\#} &= \{(x, y, z, v) \in \mathbb{R}^{4n} : \overline{\omega}[(x, y, z, v), (0, u, 0, u)^T] = 0\}\\ &= \{(x, y, x, v) : x, y, v \in \mathbb{R}^n\}.\end{aligned}$$

Identifying the quotient space $K^{\#}/K$ with the orthogonal complement $S_K$ of $K$ in $K^{\#}$ we have $S_K = \{(t, w, t, -w) \colon t, w \in \mathbb{R}^n\}$; moreover $\Delta \cap K^{\#} = \Delta$, $L \cap K^{\#} = L$. Now if $\psi(1)$ has the following block decomposition

$$\psi(1) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then $L_2 \cap K^{\#} = \{\big(r, s, Ar + Bs, Cr + Ds\big) \colon Ar + Bs = r; r, s \in \mathbb{R}^n\}$. Since $K^{\#} = S_K \oplus K$ then the image in $S_K$ of an arbitrary point in $K^{\#}$ is represented by the point $[\epsilon, \eta, \epsilon, -\eta]$ where $[\cdot]$ denotes the equivalence class in the quotient space. Thus we have

(i) $\Delta^K = \{[\alpha, 0, \alpha, 0] : \alpha \in \mathbb{R}^n\}$;
(ii) $L^K = \{[0, u, 0, -u]; u \in \mathbb{R}^n\}$:
(iii) $L_2^K = \{[r, s, r, Cr+Ds] : Ar+Bs = r\} = \{[r, s-Ds, r, Cr+Ds]; Ar+Bs = r\}$.

Then we have

$$\bar{\omega}(x_1, x_2) = \bar{\omega}\big([\alpha, 0, \alpha, 0], [0, u, 0, -u]\big) = -2\langle \alpha, u\rangle.$$
$$\bar{\omega}(x_2, x_3) = \bar{\omega}\left([0, u, 0, -u], [r, s-Ds, r, Cr]\right) = 2\langle u, r\rangle.$$
$$\bar{\omega}(x_3, x_1) = \bar{\omega}\left([r, s-Ds, r, Cr], [\alpha, 0, \alpha, 0]\right) = \langle s - Ds - Cr, \alpha\rangle.$$

Hence the quadratic form $Q$ is given by

$$Q(x_1, x_2, x_3) = -2\langle \alpha, u\rangle + 2\langle u, r\rangle + \langle s - Ds - Cr, \alpha\rangle,$$

where $\alpha, u, r, s, \in \mathbb{R}^n$ and $Ar + Bs = r$. Due to the transversality condition $(H)$ we have $s = B^{-1}(I_n - A)r$ and by setting $2X = (I_n - D)B^{-1}(I_n - A) - C$ the quadratic form $Q$ can be written as follows

$$Q(x_1, x_2, x_3) = -2\langle \alpha, u\rangle + 2\langle u, r\rangle + 2\langle Xr, \alpha\rangle = \langle Yw, w\rangle$$

for $w = (\alpha, u, r)$ and $Y$ given by

$$Y = \begin{pmatrix} 0_n & -I_n & X \\ -I_n & 0_n & I_n \\ X^T & I_n & 0_n \end{pmatrix}.$$

The Cayley-Hamilton polynomial of $A$ is given by

$$p_Y(\lambda) = \lambda^3 I_n - (2I_n + XX^T)\lambda + (X + X^T).$$

In order to compute the spectrum of $Y$ we prove the following result.

**Lemma 3.1.** *For any symplectic block matrix of the form* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *the* $n$ *by* $n$ *matrix* $2X = (I_n - D)B^{-1}(I_n - A) - C$ *is symmetric.*

*Proof.* In fact

$$2X = (B^{-1} - DB^{-1})(I_n - A) - C = B^{-1} - B^{-1}A - DB^{-1} + DB^{-1}A - C$$
$$2X^T = [B^T]^{-1} - [B^T]^{-1}D^T - A^T[B^T]^{-1} + A^T[B^T]^{-1}D^T - C^T.$$

Moreover by multiplying this last equation on the right by $BB^{-1}$ we have

$$\begin{aligned} 2X^T &= ([B^T]^{-1}B - [B^T]^{-1}D^TB - A^T[B^T]^{-1}B + A^T[B^T]^{-1}D^TB - C^TB)B^{-1} \\ &= ([B^T]^{-1}B - [B^T]^{-1}B^TD - A^T[B^T]^{-1}B + A^T[B^T]^{-1}B^TD - C^TB)B^{-1} \\ &= ([B^T]^{-1}B - D - A^T[B^T]^{-1}B + A^TD - C^TB)B^{-1} \\ &= ([B^T]^{-1}B - D - A^T[B^T]^{-1}B + I_n)B^{-1} \\ &= [B^T]^{-1} - DB^{-1} - A^T[B^T]^{-1} + B^{-1}, \end{aligned}$$

where we used the relations $A^TD - C^TB = I_n$, $A^TC = C^TA$ and finally $D^TB = B^TD$. Thus by the expression for $2X$ and this last equality it follows that in order to prove the thesis it is enough to show that

$$-B^{-1}A + DB^{-1}A - C = [B^T]^{-1} - A^T[B^T]^{-1}.$$

Now we observe that by multiplying on the left the first member of the above equality by $[B^T]^{-1}B^T$, we have

$$\begin{aligned}([B^T]^{-1}B^T)(-B^{-1}A + DB^{-1}A - C) &= [B^T]^{-1}(-B^TB^{-1}A + B^TDB^{-1}A - B^TC)\\ &= [B^T]^{-1}(-B^TB^{-1}A + D^TBB^{-1}A - B^TC)\\ &= [B^T]^{-1}(-B^TB^{-1}A + D^TA - B^TC)\\ &= -B^{-1}A + [B^T]^{-1}.\end{aligned}$$

Thus we reduced to show that $-B^{-1}A + [B^T]^{-1} = [B^T]^{-1} - A^T[B^T]^{-1}$ or which is the same to $B^{-1}A = A^T[B^T]^{-1}$. Otherwise stated since $A^T[B^T]^{-1} = (B^{-1}A)^T$, it is enough to check that the $n$ by $n$ matrix $U = B^{-1}A$ is symmetric. In fact

$$\begin{aligned}U = I_n \cdot U &= (A^TD - C^TB)B^{-1}A\\ &= A^TDB^{-1}A - C^TA\\ &= A^TDB^{-1}A - A^TC\\ &= A^T(DB^{-1}A - C);\end{aligned}$$

moreover $U^T = A^T[B^T]^{-1}$. Thus the condition $U^T = U$ reduced to show that $A^T[B^T]^{-1} = A^T(DB^{-1}A - C)$ and then the only thing to prove is that $[B^T]^{-1} = DB^{-1}A - C$. In fact by multiplying the second member of this last equality on the left by $[B^T]^{-1}B^T$, it then follows that

$$\begin{aligned}[B^T]^{-1}(B^TDB^{-1}A - B^TC) &= [B^T]^{-1}(D^TBB^{-1}A - B^TC)\\ &= [B^T]^{-1}(D^TA - B^TC)\\ &= [B^T]^{-1}I_n\end{aligned}$$

and this completes the proof of the Lemma. □

Now since by Lemma 3.1 $X$ is a symmetric matrix, there exists an $n$ by $n$ orthogonal matrix such that $M^TXM = \operatorname{diag}(\lambda_1, \dots, \lambda_k)$ where the eigenvalues $\lambda_j$ are counted with multiplicity. Hence in order to compute the solutions of $p_Y(\lambda) = 0$ it is enough to compute the solutions of $M^Tp_Y(\lambda)M = 0$ which is the same to solve

$$0 = \lambda^3 - (2 + \lambda_j^2)\lambda + 2\lambda_j = (\lambda - \lambda_j)(\lambda^2 + \lambda_j\lambda - 2), \quad \text{for} \quad j = 1, \dots, k.$$

Now the solutions of the equation $\lambda^2 + \lambda_j\lambda - 2 = 0$ are one positive and one negative and therefore they do not give any contribution to the signature of $Y$. Thus we proved that $\operatorname{sign}(Y) = \operatorname{sign}(X)$ and therefore

$$-\frac{1}{2}\tau_{\mathbb{R}^{4n}}(\Delta, L, L_2) = -\frac{1}{2}\,\tau_{\mathbb{R}^{2n}}(\Delta^K, L^K, L_2^K) = -\frac{1}{2}\,\operatorname{sign} X.$$

Summing up the previous calculation we proved the following result.

**Theorem 3.2.** *Let $\psi\colon [0,1] \to \mathrm{Sp}(2n)$ be the fundamental solution of the Hamiltonian system*

$$w'(x) = Hw(x), \quad x \in [0,1]$$

*and let us assume that condition (H) holds. Then the Maslov index of $\psi$ is*

$$\mu_{L_0}(\psi) = \mu_{CZ}(\psi) + \frac{1}{2}\,\operatorname{sign}\widetilde{X} \tag{3.2}$$

*for $\widetilde{X} = C + (D - I_n)B^{-1}(I_n - A)$.*

**Corollary 3.3.** *Let $e^{i\alpha_1}, \dots, e^{i\alpha_k}$ be the Krein positive purely imaginary eigenvalues of the fundamental solution $\psi(x) = e^{xH}$ counted with algebraic multiplicity and we assume that $(H)$ and $\det(\psi(1) - I_{2n}) \neq 0$ hold. Then the Maslov index of $\psi$ is given by*

$$\mu_{L_0}(\psi) = \sum_{j=1}^{k} g\Big(\frac{\alpha_j}{\pi}\Big) + \frac{1}{2}\,\mathrm{sign}\,\widetilde{X}, \tag{3.3}$$

*where we denoted by $g$ the double integer part function which holds the identity on integers and it is the closest odd integer otherwise.*

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## Addendum posted on July 19, 2008

The main contribution of the above article [A4] (as stated in its introduction) is to provide a different proof of a result already proven in [A3]. After the publication of [A4], the author was informed by Prof. Maurice de Gosson that the main result stated in [A4, Theorem 3.2] basically can be found (up to minor details) in the proofs of [A1, Proposition 3] and [A2, Proposition 5.7].

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