TODD'S CANONICAL CLASSES

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The aim of this note is to recover Todd's original definition (1937) of his canonical classes in terms of singularities of maps.

For simplicity we put ourselves in the category of complex analytic manifolds and maps. Characteristic classes of vector bundles will be Chern classes. The singular set of a map $f : X \to Y$, the set of points where the kernel rank of its differential is less than $\inf\{\dim X, \dim Y\}$, will be denoted by Σf.

Let X be a complex manifold of dimension n, let L be a complex line bundle over X and let $h : L \to \underset{\sim}{C}_X^{r+1}$ be a bundle map over X ($\underset{\sim}{C}_X^{r+1}$ denoting the trivial bundle over X with fibre $\underset{\sim}{C}^{r+1}$). The dual map $h^{\vee} : \underset{\sim}{C}^{r+1} \to L^{\vee}$ induces sections of $L^{\vee}$, one induced by each vector in $\underset{\sim}{C}^{r+1}$. Suppose that S is the manifold of zeros of some section of $L^{\vee}$, transversal to the zero section. Then the dual cohomology class carried by S is $c_1(L^{\vee})$. We denote this class by s. So $c(L^{\vee}) = 1 + s$, $c(L) = 1 - s$.

Consider $h' : L \to \underset{\sim}{C}^{r+1}$, the composite of h with the projection map $\underset{\sim}{C}_X^{r+1} \to \underset{\sim}{C}^{r+1}$. Then the differential of h' maps each tangent vector to the manifold L to a vector of $\underset{\sim}{C}^{r+1}$. Now $\underset{\sim}{C}$ acts on L and therefore on TL by multiplication along the fibre. Let QL denote the vector bundle of tangent vector fields on L invariant under this action (cf. [1]). This bundle has rank $n + 1$, for there is an exact sequence

$$\{0\} \to \underset{\sim}{C}_X \xrightarrow{i} QL \xrightarrow{p} TX \to \{0\},$$

the image of i being the invariant tangent fields along the fibres of L and p being induced by the differential of the projection map $L \to X$. Moreover, the differential of h' induces a bundle map

$$\chi : QL \to \mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1}),$$

in the obvious way. The section of $\mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1})$ induced by the composite $\chi i : \underset{\sim}{C}_X \to \mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1})$ and $1 \in \underset{\sim}{C}$ is just that induced by h .

Now suppose that $\Sigma h = \emptyset$. Then, since the image by h' of each fibre of L is a line through 0 in $\underset{\sim}{C}^{r+1}$, we obtain a map $f : X \to \underset{\sim}{C}P^{r+1}$. Let K denote the Hopf bundle on $\underset{\sim}{C}P^{r+1}$ associating to each point of $\underset{\sim}{C}P^{r+1}$ itself as a line in $\underset{\sim}{C}^{r+1}$. Then $L = f^*K$. Moreover there is a natural identification $QK = \mathrm{Hom}(K, \underset{\sim}{C}^{r+1})$ and a commutative diagram

$$\begin{array}{ccccccccc}
\{0\} & \to & \underset{\sim}{C}_X & \to & QL & \to & TX & \to & \{0\} \\
 & & \| & & \downarrow \chi & & \downarrow Tf & & \\
\{0\} & \to & \underset{\sim}{C}_X & \to & \mathrm{Hom}(L, \underset{\sim}{C}^{r+1}) & \to & f^*T(\underset{\sim}{C}P^{r+1}) & \to & \{0\} \\
 & / \!/ & & & \| & & & & \\
f^*\underset{\sim}{C}_{\underset{\sim}{C}P^{r+1}} & & & & f^*\mathrm{Hom}(K, \underset{\sim}{C}^{r+1}) & & & &
\end{array}$$

with exact rows inducing a natural isomorphism between $\ker \chi$ and $\ker Tf$. As we have seen, χ is defined even where $\Sigma h \neq \emptyset$. We define $J(h) = \Sigma\chi$ to be the <u>Jacobian</u> of the <u>linear system</u> h . This coincides with the singular set of f together with some, possibly, of the points of Σh , the <u>base locus</u> of h , where f is not defined.

Now I have shown elsewhere [9] that, for any bundle map $h : E \to F$ over a manifold X with $\mathrm{rk}\,E \leqslant \mathrm{rk}\,F$, the dual cohomology class of Σh is $c_{\mathrm{rk}\,F - \mathrm{rk}\,E + 1}(F - E)$, provided that the induced section of the bundle $\mathrm{Hom}(E, F)$ is transversal to the zero section.

As a first example consider $h : L \to \underset{\sim}{C}_X^{r+1}$. Then, if transversality is satisfied,

$$\begin{aligned}
D\Sigma(h) &= c_{r+1}(\underset{\sim}{C}_X^{r+1} - L) \\
&= (1 / c(L))_{r+1} \\
&= (1 / 1 - s)_{r+1} \\
&= s^{r+1} \quad ,
\end{aligned}$$

which makes sense in terms of the interpretation of Σh as the base locus of h , and of s as the dual class of the set of zeros S of a section of L (transversal to the zero section, of course).

Secondly, consider $\chi : QL \to \mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1})$, or rather its dual $\chi^\vee : \mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1})^\vee \to (QL)^\vee$, in the case that $r \leqslant n$. Then $J(h) = \Sigma\chi = \Sigma\chi^\vee$, so that, with the transversality assumption fulfilled,

$$\begin{aligned} D(J(h)) &= c_{n-r+1}((QL)^\vee - \mathrm{Hom}(L, \underset{\sim}{C}_X^{r+1})^\vee) \\ &= \left(\frac{c((TX)^\vee)}{(1-s)^{r+1}}\right)_{n-r+1} \\ &= ((1 + \frac{s}{1-s})^{r+1} \cdot c((TX)^\vee))_{n-r+1} \quad . \qquad (*) \end{aligned}$$

Our final task is to give a geometrical interpretation to each term in the expansion of the right hand side of this equation.

Let S be the set of zeros of a section of $L^\vee$ transversal to the zero section and let $j : S \to X$ be the inclusion map. Then $s = j_* 1$. Now, from the exact sequence

$$\{0\} \to TS \to j^*TX \to NS \to \{0\} ,$$

or rather its dual, and the elementary fact that $NS = j^*L^\vee$, we find that

$$c(j^*L^\vee)\, c((TS)^\vee) = c(j^*(TX)^\vee),$$

that is

$$j^*(1 - s) \cdot c(TS)^\vee = j^*c(TX)^\vee \quad ,$$

from which it follows, by applying j_* to both sides, and using the relation $j_*(a \cdot j^*b) = (j_*a) \cdot b$, that

$$(1 - s) \cdot j_*c(TS)^\vee = s \cdot c(TX)^\vee \quad ,$$

or

$$j_*c(TS)^\vee = \left(\frac{s}{1-s}\right) \cdot c(TX)^\vee \quad ,$$

a formula known classically as the adjunction formula. This is the interpretation we required. If we now suppose the existence of $r + 1$ sections of L fully transversal to each other (Todd emphasises that Geometry is an experimental science!) then formula (*) can be turned to provide an inductive definition of the Chern classes of $(TX)^\vee$ in terms of the corresponding classes of lower-dimensional manifolds (the intersections of the zero manifolds of the various sections) and the Jacobians of the linear systems involved.

This is how these classes were first defined by Todd [10], except that Todd worked in an algebraic category, the cohomology ring available in the case of complex analytic manifolds being replaced by a ring of rational equivalence [2] whose status at that time was still somewhat in doubt. He termed his classes the <u>canonical classes</u> of X . About the same time a similar definition was given of the classes by Eger [3], [11] , who used r bundle maps $L \to \underset{\sim}{C}_X^2$ in place of the single map $L \to \underset{\sim}{C}_X^{r+1}$ that Todd used. A generalisation of both methods was given later by Monk [7] . The methods of the present paper can easily be generalised to cover these cases. The relationship between the canonical classes of Todd and Eger and the characteristic classes of Chern was proved by Hodge [5] and Nakano [8] using other formulas than those discussed here. The line bundle presentation of linear systems is due to Hirzebruch [4] (where an excellent introduction to characteristic classes is to be found). Jacobians of bundle maps $h : L \to \underset{\sim}{C}_X^{r+1}$ in the case that $r > n$ have been studied from a classical point of view more recently by Ingleton and Scott [6] .

(This paper is a simplified version of a previously unpublished part of the author's thesis (Cambridge 1960). The results were announced in a short talk at the I.C.M., Stockholm, 1962.)

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