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The Bass–Heller–Swan Formula for the Equivariant Topological Whitehead Group

STRATOS PRASSIDIS
Department of Mathematics, Vanderbilt University, Nashville, TN 37240, U.S.A.

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Abstract. In the first part of this paper, a geometric definition of the K-theory equivariant nilpotent groups is given. For a finite group G, the Nil-groups are defined as functors from the category of G-spaces and G-homotopy classes of G-maps to Abelian groups. In the nonequivariant case, these groups are isomorphic to the classical algebraic Nil-groups.

In the second part, the Bass–Heller–Swan formula is proved for the equivariant topological Whitehead group. The main result of this work is that if X is a compact G-ANR and G acts trivially on S^1, then

$$\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1) \approx \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus \tilde{K}_{0G}^{\mathrm{Top}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X).$$

Key words. Equivariant topological Whitehead group, equivariant nilpotent group, equivariant wrapping-up, topological Bass–Heller–Swan formula.

0. Introduction

In this paper, the Bass–Heller–Swan formula is proved for the equivariant topological Whitehead group.

The Bass–Heller–Swan formula, in the classical algebraic K-theory is a formula that calculates the K-theory of the polynomial extension of a ring in terms of the K-theory of the ring. More specifically, if R is a ring with unit, then

$$K_1(R[t, t^{-1}]) \approx K_1(R) \oplus K_0(R) \oplus \widetilde{\mathrm{Nil}}(R) \oplus \widetilde{\mathrm{Nil}}(R).$$

([4], Chapter XII, §7). This formula sometimes is called 'The fundamental theorem of algebraic K-theory', and it has been used as a tool of calculation in algebraic K-theory.

Many important applications of algebraic K-theory in topology are given through the Whitehead group, which is a quotient of the K_1 group (for example, see [29]). The Whitehead group can be considered as a functor from the category of groups to the category of Abelian groups. It associates to a group π, a quotient of $K_1(\mathbf{Z}\pi)$. The Bass–Heller–Swan formula for the Whitehead group can be stated as

$$\mathrm{Wh}(\pi \times \mathbf{Z}) \approx \mathrm{Wh}(\pi) \oplus \tilde{K}_0(\mathbf{Z}\pi) \oplus \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi) \oplus \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi). \qquad (1)$$

There are geometric interpretations of the above formula in the fibering theorem of F. T. Farrell [17], and its interpretation by L. Siebenmann [44], as well as the splitting theorems of F. T. Farrell and W. C. Hsiang [18, 20], and F. Waldhausen [53].

S. Illman and H. Hauschild extended the definition of the Whitehead group to CW-complexes with group actions [27, 24]. We restrict our attention to finite groups, but most of the constructions can be done in compact Lie group actions. For any finite group G and any finite G-CW complex X, S. Illman and H. Hauschild defined the group $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ (in their notation $\mathrm{Wh}_G(X)$) using the geometric definition of the classical Whitehead group. Also, in [1–3, 31–34], there are extensions of the definition of the $\tilde{K}_0$-term and of the theory of finiteness obstruction in the equivariant case.

The important difference between the equivariant and nonequivariant category is the s-cobordism theorem. The Whitehead group in the nonequivariant case classifies differential, piecewise linear, and topological h-cobordisms. But there are differentiably nontrivial G-h-cobordisms over G-smooth manifolds which are trivial topologically (see [7]). In order to classify G-h-cobordisms over topological locally linear G-manifolds, M. Steinberger, and J. West introduced the equivariant topological Whitehead group, defined for any locally compact G-ANR (see [46, 47]). The methods of T. A. Chapman [10] were used in the definition of this group. The connection of the two different Whitehead groups is given by an exact sequence [46, 47]:

$$\mathrm{Wh}_G^{\mathrm{PL}}(X)_c \to \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{Top}}(X) \to \tilde{K}_{0G}^{\mathrm{PL}}(X)_c \to \tilde{K}_{0G}^{\mathrm{PL}}(X), \tag{2}$$

where $\mathrm{Wh}_G^{\mathrm{PL}}(X)_c$ and $\tilde{K}_{0G}^{\mathrm{PL}}(X)_c$ are the equivariant analogues of the controlled K-groups defined by T. A. Chapman [11].

Also, M. Steinberger [46] gave a Bass–Heller–Swan splitting for the controlled equivariant Whitehead groups generalizing the results of T. A. Chapman [11]:

$$\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c \approx \mathrm{Wh}_G^{\mathrm{PL}}(X)_c \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X)_c. \tag{3}$$

In the first part of this paper, a geometric definition of the Nil-groups is given. This is done by generalizing the ideas of F. T. Farrell [17], F. T. Farrell and W. C. Hsiang [18], and A. Ranicki [37–39]. We first define a category which is the geometric analogue of the categories used in the above references and the Nil-groups are defined as the Grothendieck groups of these categories. The Nil-groups, then, become functors from the category of G-spaces and G-homotopy classes of G-maps to Abelian groups. In the nonequivariant case, $G = 1$, the Nil-groups defined geometrically are isomorphic to the classical algebraic Nil-groups. This provides a new more geometric definition of the classical algebraic Nil-groups.

In the second part of this work, the Bass–Heller–Swan formula for the equivariant topological Whitehead group is given.

MAIN THEOREM. *If X is a compact G-ANR and G acts trivially on S^1, then*

$$\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1) \approx \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus \tilde{K}_{0G}^{\mathrm{Top}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X).$$

Notice that the Nil-groups are the equivariant analogues of the Nil-groups appearing in (1). The summand $\tilde{K}_{0G}^{\mathrm{Top}}(X)$ is, by definition, the subgroup of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ consisting of the elements which are invariant under the double covering map of S^1.

It was remarked by the referee that the above formula should be called, using A. Ranicki's terminology, the geometrically significant splitting in contrast to the algebraically significant splitting which corresponds to the classical Bass–Heller–Swan formula. Since the methods of this paper are geometric, we are not going to prove an analogue of the algebraically significant splitting. The right terminology for the above formula seems to be 'geometric Bass–Heller–Swan formula'.

The proof of the main theorem is given by combining the exact sequences (2), (3), and the Bass–Heller–Swan formula for $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ (see [27, 34, 52]), after identifying $\tilde{K}_{0G}^{\mathrm{Top}}(X)$ with $\mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$.

In the appendix, we describe a geometric construction of the split monomorphism from $\mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$ to $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$. The construction of this monomorphism is given using an equivariant version of 'wrapping up' for Hilbert cube manifolds described by T. A. Chapman in [14] and its interpretation by L. Siebenmann [44].

1. Preliminaries

In this section, we establish the notation and some of the basic properties of the objects we are going to use in this work.

Space always means a topological Hausdorff space which is compactly generated in the sense of [56], p. 17. A map between spaces will always mean a continuous map.

DEFINITION 1.1. Let $f\colon Y \to Z$ be a map. The *mapping cylinder* of f, denoted $M(f)$, is defined to be the quotient space of the disjoint union:

$$M(f) = Y \times [0,1] \amalg Z/\sim,$$

where $\sim$ is the equivalence relation generated by the rule: $(y, 1) \sim f(y)$, for all $y \in Y$. Also, if A is a closed subset of Y, define the reduced mapping cylinder of f, $M_A(f)$, to be $M(f)/\sim$, where $\sim$ is the equivalence relation generated by the rule: $(a, t) \sim f(a)$ for all $a \in A$ and $t \in [0, 1]$.

DEFINITION 1.2. Let $f\colon Y \to Y$ be a map. Define the *mapping torus* of f, denoted $T(f)$, to be the space formed by identifying the top of $M(f)$ with the base by the identity map. The mapping torus can be represented as the space

$$T(f) = Y \times [0,1]/\sim,$$

where $\sim$ is the equivalence relation generated by the rule: $(f(y), 0) \sim (y, 1)$. As before, if A is a closed subset of Y and $f|_A = \mathrm{id}$, define the reduced mapping torus of f, $T_A(f)$, to be $T(f)/\sim$, where $\sim$ is the equivalence relation generated by $(a, t) \sim a$ for all $a \in A$ and $t \in [0, 1]$.

There is a natural map $\pi\colon T(f) \to S^1$ given by the projection, $\pi(y, t) = t$, for all $y \in Y$, $t \in I$ (here we identify $S^1 = [0,1]/\sim$, where $\sim$ is the equivalence relation generated by $0 \sim 1$). The universal covering map of $S^1, e\colon \mathbf{R} \to S^1$, is given by

$e(x) = x - [x]$, where $[-]$ is the greatest integer function. Pull back e using π, and get a commutative diagram:

$$\begin{array}{ccc} D(f) & \xrightarrow{\bar{\pi}} & \mathbf{R} \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\ T(f) & \xrightarrow{\pi} & S^1 \end{array}$$

DEFINITION 1.3. Define the *mapping telescope* of f, denoted $D(f)$, to be the infinite cyclic cover of $T(f)$. $D(f)$ is a countable union of copies of the mapping cylinder of f sewn together by identifying the base of one with the top of the next. $D(f)$ can be written also as $Y \times [0, 1] \times \mathbf{Z}/\sim$, where $\sim$ is the equivalence relation generated by the rule: $(f(y), 0, n + 1) \sim (y, 1, n)$ for $y \in Y, n \in \mathbf{Z}$. Each point of $D(f)$ can be written in the form $[y, t, m]$ where $(y, t, m) \in Y \times [0, 1] \times \mathbf{Z}$. This expression is unique if $0 \leqslant t < 1$. If A is a closed subset of Y and $f|_A = \mathrm{id}$, define the reduced mapping telescope of f to be $D_A(f) = D(f)/\sim$, where $\sim$ is the equivalence relation generated by $[a, t, n] = [a, 0, 0]$ for $a \in A$, $t \in [0, 1]$, $n \in \mathbf{Z}$.

If we use the above notation for $D(f)$, the map $\bar{\pi}: D(f) \to \mathbf{R}$ is given by

$$\bar{\pi}[y, t, n] = t + n \quad \text{for} \quad [y, t, n] \in D(f).$$

Notice, also, that $\mathbf{Z}$ acts on $D(f)$ on the right by translations:

$$[y, t, m]n = [y, t, n + m] \quad \text{for } n \in \mathbf{Z}, \quad (y, t, m) \in D(f).$$

Set $D(f)_K = \bar{\pi}^{-1}(K)$ for any subset K of $\mathbf{R}$. By composing mapping cylinder collapses, we get a map $C_n: D(f) \to D(f)_{[n, +\infty)}$ for each $n \in \mathbf{Z}$. More precisely, if $(y, t, m) \in Y \times [0, 1] \times \mathbf{Z}$ represents an element of $D(f)$, then

$$C_n[y, t, m] = \begin{cases} [f^{n-m}(y), 0, n], & \text{if } n - m \geqslant 0, \\ [y, t, m], & \text{if } n - m < 0. \end{cases}$$

Notice that C_n restricts to a G-map $c_n: D(f)_{(-\infty, n]} \to Y$ for every $n \in \mathbf{Z}$ (Y is identified with the subset $Y \times 0 \times \{n\}$ of the mapping telescope). We can make the same constructions in the reduced mapping telescope of f. As before, $\mathbf{Z}$ acts on $D_A(f)$. Define $D_A(f)_{(-\infty,n]} = D(f)_{(-\infty,n]}/\sim$, where $\sim$ is the equivalence relation generated by $(a, t, m) = (a, 0, 0)$ for $a \in A$, $t \in [0, 1]$, $m \in \mathbf{Z}$, $m \leqslant n$. The maps C_n and c_n above induce the maps

$$C'_n: D_A(f) \to D_A(f)_{[n, +\infty)},$$
$$c'_n: D_A(f)_{(-\infty,n]} \to Y$$

for each $n \in \mathbf{Z}$.

Let G be a finite group. A G-space, in general, is a space on which G acts on the left by homeomorphisms [6].

DEFINITION 1.4. Let X, Y be G-spaces.

(i) Let H be a subgroup of G. Set

$$X^H = \{x \in X/hx = x, \text{ for all } h \in H\},$$
$$X^{>H} = \{x \in X^H/G_x \neq H\},$$
$$X^{(H)} = \{x \in X/ \text{ a conjugate of } H \text{ is a subgroup of } G_x\}.$$

Notice that if $N(H)$ is the normalizer of H, then the group WH $= N(H)/H$ acts freely on $X^H - X^{>H}$.

(ii) A map $f: X \to Y$ between G-spaces is called *G-equivariant map* if $f(gy) = gf(y)$, for all $y \in Y$ and $g \in G$. From now on, by a G-map between G-spaces, we will mean a G-equivariant map.

The mapping cylinder of a G-map is a G-space. Similarly, the mapping torus and the mapping telescope of a self G-map is a G-space. The same is true for the reduced analogues of the above constructions.

DEFINITION 1.5. A G-space X is called a *G-ANR* if for any closed G-embedding, i: X → Y, in a metric G-space Y, i(X) is a G-retract of some open G-neighborhood of Y. (By a G-ANR in this paper we will mean a metric G-ANR)
Note: Any G-ANR has the G-homotopy type of a G-CW complex [32].

DEFINITION 1.6. (a) Let (X, A) be a pair of G-spaces such that:

(1) (X, A) is a relative CW-complex in the sense of [56], p. 48.
(2) G permutes the cells of X–A.

Then we say that (X, A) is a relative G-complex. If X–A has only finitely many cells, we call (X, A) a finite relative G-complex.

(b) Let X be a G-space. The equivariant Whitehead group of X, $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ is defined as the group of equivalence classes of pairs (Y, X), where

(i) (Y, X) is a finite relative G-complex and there is a strong G-deformation retraction of Y to X.
(ii) Two such pairs (Y, X) and (Y', X) are called equivalent if there is a sequence of G-formal deformations from Y to Y' rel$X$ [27].

(c) A G-map $f: X \to Y$ between G-spaces is called a *G-cell like map* (denoted G-CE map) if it is a proper G-map, and for each $y \in Y$ and each open G_y-neighborhood U of $f^{-1}(y)$, the inclusion of $f^{-1}(y)$ into U is G_y-nullhomotopic.

Note. Notice that a G-CE map between G-spaces is a G-homotopy equivalence ([46]).

DEFINITION 1.7. (a) Let X, Y, and Y' be G-spaces such that $Y \cap Y' \supset X$. Then Y and Y' are called G-CE equivalent rel$X$, if there are a G-space Z containing Y and Y', and G-CE maps $r: Z \to Y$, $r': Z \to Y'$ such that, if $i: Y \to Z$, $i': Y' \to Z$ are the inclusion maps, $ri = \mathrm{id}|_Y$, $r'i' = \mathrm{id}|_{Y'}$, $ir \simeq_G \mathrm{id}|_Z$ rel$X$, $i'r' \simeq_G \mathrm{id}|_Z$ rel$X$.

(b) Let X be a G-ANR. If Y and Y' are the union of X and a finite number of G-cells, then Y and Y' are called simple G-homotopy equivalent rel$X$, if there is a sequence of formal G-deformations, rel$X$, from Y to Y'.

We summarize the basic properties of the mapping cylinder and mapping torus construction. The proofs are essentially in [23], Lemmas 2.2, 2.3; [16], 5.5, 5.6, in the nonequivariant case, $G = \{e\}$, and in the case where G is a finite group in [27], §3, under the assumption that all the spaces are G-CW complexes. The general case follows as a simple generalization of arguments in [23]. Let X be a G-space. Let Y, Y', Y'' be G-spaces containing X as a closed subspace.

LEMMA 1.8. *Let $f_t : Y \to Y'$ $0 \leqslant t \leqslant 1$, be a G-homotopy such that $f_t|_X = f_0|_X$. Then $M(f_0)$ and $M(f_1)$ are G-*CE *equivalent* $\mathrm{rel}(M((f_0)|_X) \cup Y \cup Y')$ *and $M_X(f_0)$ and $M_X(f_1)$ are G-*CE *equivalent* $\mathrm{rel}(X \cup Y \cup Y')$. *If (Y, X) and (Y', X) are finite relative G-complexes, then the above equivalences are simple G-equivalences.*

LEMMA 1.9. *Let f_t: $Y \to Y$ $0 \leqslant t \leqslant 1$, be a G-homotopy such that $f_t|_X = \mathrm{id}_X$. Then, $T(f_0)$ and $T(f_1)$ are G-*CE *equivalent* $\mathrm{rel}(X \times S^1 \cup Y \times \{0\})$ *and $T_X(f_0)$ and $T_X(f_1)$ are G-*CE *equivalent* $\mathrm{rel}(X \cup (Y \times \{0\})$. *If (Y, X) and (Y', X) are finite relative G-complexes, then the above equivalences are simple G-equivalences. (This property follows from the fact that all the maps and the homotopies of 1.8 fix the ends of the mapping cylinders.)*

As before, let Y, Y', Y'' be G-spaces containing X as a closed subspace. Let $f: Y \to Y'$, f': $Y' \to Y''$ be G-maps which are the identity on X. Define $M(f, f')$ to be the space obtained from $M(f) \amalg M(f')$ by identifying the base Y' of the mapping cylinder $M(f)$ with the top $Y' \times \{0\}$ of the mapping cylinder $M(f')$ by the identity map. Define $M_X(f, f')$ to be the space obtained by the above procedure applied to the relative mapping cylinders $M_X(f)$ and $M_X(f')$. So, $M_X(f, f') = M_X(f) \amalg M_X(f')/\sim$, where $\sim$ is the equivalence relation generated by $y' \sim (y', 0)$ for *$y' \in Y'$.*

LEMMA 1.10. *$M(f, f')$ and $M(f'f)$ are G-*CE *equivalent* $\mathrm{rel}((X \times I) \cup Y \cup Y'')$ *and $M_X(f, f')$ and $M_X(f'f)$ are G-*CE *equivalent* $\mathrm{rel}(X \cup Y \cup Y'')$. *If (Y, X) and (Y', X) are finite relative G-complexes, then the above equivalences are simple G-equivalences.*

(Strictly speaking, $M(f, f')$ contains $(X \times I \amalg X \times I)/\sim$, where $\sim$ is the equivalence relation generated by $(x, 1) \sim (x, 0)$. By abuse of language, we consider this space as *$X \times I$.)*

We will give the mapping torus version of this property. Define $T(f, f') = (Y \times [0, \frac{1}{2}]) \amalg (Y' \times [\frac{1}{2}, 1])/\sim$, where $\sim$ is the equivalence relation generated by

$$(y, \tfrac{1}{2}) \sim (f(y), \tfrac{1}{2}) \quad \textit{for } y \in Y,$$
$$(f'(y'), 0) \sim (y', 1) \quad \textit{for } y' \in Y'.$$

Define $T_X(f, f') = T(f, f')/\sim$, where $\sim$ is the relation generated by $(x, t) \sim x$ for all $x \in X$. We construct also $D(f, f')$ from $T(f, f')$ as we constructed $D(f)$ from $T(f)$. More precisely, $D(f, f')$ is defined as the countable union of copies of $M(f, f')$ sewn together by identifying the base of one with the top of the next. This comes with a natural right $\mathbf{Z}$ action. There is also a collapse map defined for $D(f, f')$. Also, we can get the relative version of $D(f, f')$, $D_X(f, f')$.

LEMMA 1.11. *$T(f, f')$ and $T(f'f)$ are G-*CE *equivalent* $\mathrm{rel}(X \times S^1 \cup Y \times \{0\})$, *and $T_X(f, f')$ and $T_X(f'f)$ are G-*CE *equivalent* $\mathrm{rel}(X \cup (Y \times \{0\})$. If *$(Y, X)$ and (Y', X) are finite relative G-complexes, then the above equivalences are simple G-equivalences.*

2. The Geometric Definition of the Equivariant Nil Groups

In this section, a geometric construction of a functor is given, from the category of topological spaces with a group action, and equivariant homotopy classes of equivariant maps to the category of Abelian groups. This construction is the geometric analogue of the construction of the nilpotent K-theory groups of a ring.

Let G be a finite discrete group. For each G-space X a category is constructed. Then the equivariant Nil-group of X is defined as the 'Grothendieck' group of this category.

We start with the definition of the basic category which will be used in the construction of the geometric Nil-groups.

DEFINITION 2.1. Let X be a G-space. Define $\mathbf{n}_G(X)$ to be the category whose objects are pairs (Y,f), where (Y, X) is a relative G-complex, and $f\colon Y \to Y$ is a map such that

(i) $f|_X = \mathrm{id}_X$

(ii) The inclusion of $X \times S^1$ into the mapping torus of f is a G-homotopy equivalence.

A morphism $F\colon (Y,f) \to (Y',f')$ in $\mathbf{n}_G(X)$ is a G-map $F\colon Y \to Y'$, $F|_X = \mathrm{id}_X$, making the following diagram commutative

$$\begin{array}{ccc} Y & \xrightarrow{F} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{F} & Y' \end{array} \qquad (*)$$

Remark. Since (Y, X) is a relative G-complex, the inclusion map $X \times S^1 \to T(f)$ is a G-cofibration. So, it can be assumed that $X \times S^1$ is a strong G-deformation retraction of $T(f)$ ([45], p. 31).

DEFINITION 2.2. Let (Y_i, f_i), $i = 0, 1, 2$ and (Y, f) be objects of $\mathbf{n}_G(X)$. Assume that (Y_1, Y_0) and (Y_2, Y_0) are relative G-CW pairs, and that the restriction of f_i to Y_0 is f_0, $i = 1, 2$. Let $i'\colon Y_0 \to Y_1$ and $i\colon Y_0 \to Y_2$ be the inclusion maps. The diagram

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array}$$

is called a *push-out diagram* if

Given any commutative diagram of objects and morphisms in $\mathbf{n}_G(X)$:

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (Y_2, f_2) & \xrightarrow{\varphi'} & (Y', f') \end{array}$$

there is a unique morphism $h: (Y, f) \to (Y', f')$ such that

$$hj = \varphi, \qquad hj' = \varphi' \text{ as morphisms in } \mathbf{n}_G(X).$$

Note. The diagram of G-spaces and G-maps

$$\begin{array}{ccc} Y_0 & \xrightarrow{i'} & Y_1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Y_2 & \xrightarrow{j'} & Y \end{array}$$

is a push-out diagram of spaces and f is the push-out of the maps f_i, $i = 0, 1, 2$.

LEMMA 2.3. *Push-out diagrams exist in $\mathbf{n}_G(X)$ and they are unique up to isomorphism in $\mathbf{n}_G(X)$.*

Proof. Let (Y_i, f_i), $i = 0, 1, 2, 3$ be as in Definition 2.1. Define $Y = Y_1 \amalg Y_2/\sim$, where $\sim$ is the relation generated by $y \sim y$ for $y \in Y_0$. Then, (Y, X) is a relative G-CW pair. Also, define $f = f_1 \cup f_2$: $Y \to Y$. Then (Y, f) is an object in $\mathbf{n}_G(X)$ and it is the push-out of (Y_i, f_i). The uniqueness of the push-out is obvious from the definition.

Let (Y, f) be an object in $\mathbf{n}_G(X)$. Then f induces a map $\mathrm{Tor}(f)$: $T_X(f) \to T_X(f)$, $\mathrm{Tor}(f)(y, t) = (f(y), t)$. But, $\mathrm{Tor}(f)$ is G-homotopic rel$X$ to the identity. The homotopy is given by k_s: $T_X(f) \to T_X(f)$

$$k_s(y, t) = \begin{cases} (y, s + t) & s + t \leqslant 1 \\ (f(y), s + t - 1), & s + t \geqslant 1 \end{cases}$$

([36]). From now on, we will refer to this homotopy as Mather's trick. Hence, $T(\mathrm{Tor}(f))$ and $T(\mathrm{id}_{T_X(f)})$ are G-homotopic equivalent rel$X \times S^1$ (by 1.11). Since X is a strong G-deformation retract of $T_X(f)$, it follows that $X \times S^1$ is a strong G-deformation retract of $T(\mathrm{id}_{T_X(f)})$ and of $T(\mathrm{Tor}(f))$. Also $(T_X(f), X)$ is a relative G-CW pair. Therefore, $(T_X(f), \mathrm{Tor}(f))$ is an object in $\mathbf{n}_G(X)$.

DEFINITIONS 2.4. (a) Let (Y, f) be an object in $\mathbf{n}_G(X)$. The *torus* of (Y, f) is the object $(T_X(f), \mathrm{Tor}(f))$ of $\mathbf{n}_G(X)$.

(b) Let $\mathbf{Y}' = (Y', f')$ and $\mathbf{Y} = (Y, f)$ be two objects in $\mathbf{n}_G(X)$ such that (Y', Y) is a relative G-CW complex, and $f'|_Y = f$. Define the *quotient* $\mathbf{Y}'/\mathbf{Y}$ to be the pair (C, F) where C is given as the push-out:

$$\begin{array}{ccc} (Y, f) & \xrightarrow{\quad i \quad} & (Y', f') \\ {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle j} \\ (T_X(f), \mathrm{Tor}(f)) & \xrightarrow{j'} & (C, F) \end{array}$$

where i: $Y \to Y'$ is the inclusion map, $\imath: Y \to T_X(f)$ is the map given by $\imath(y) = (y, 0)$ for $y \in Y$.

Let (Y, X) be a pair of G-spaces. We call (Y, X) a relative finitely dominated G-pair if there is a pair of G-spaces (K, X) such that

(i) (K, X) is a finite relative G-complex.
(ii) There are G-maps $Y \xrightarrow{u} K \xrightarrow{d} Y$ such that, $u|_X = d|_X = \mathrm{id}|_X$, $du \simeq_G \mathrm{id}_Y \,\mathrm{rel} X$.

If $d: (Y, X) \to (K, X)$ is a G-homotopy equivalence rel$X$, we say (Y, X) has the G-homotopy type of a finite relative G-complex.

DEFINITIONS 2.5. (a) Write $\mathbf{nil}_G(X)$ for the full subcategory of $\mathbf{n}_G(X)$ consisting of those objects (Y, f) of $\mathbf{n}_G(X)$ such that (Y, X) is relative finitely dominated G-pair.

(b) Write $\widetilde{\mathbf{nil}}_G(X)$ for the full subcategory of $\mathbf{nil}_G(X)$ consisting of those objects (Y, f) of $\mathbf{nil}_G(X)$, with the property that (Y, X) has the G-homotopy type of a finite relative G-complex.

(c) Write $\mathbf{K}_{0G}(X)$ for the full subcategory of $\mathbf{nil}_G(X)$ consisting of objects (Y, f) of $\mathbf{nil}_G(X)$, where f is a G-retraction of Y into X.

Remark. Let $\tilde{K}_{0G}^{\mathrm{PL}}(Y)$ be the subgroup of the equivariant Whitehead group $\mathrm{Wh}_G^{\mathrm{PL}}(Y \times S^1)$, consisting of all those elements which are invariant under the transfer induced by the covers of S^1 ([46, 49]). If (Y, f) is an object of $\mathbf{nil}_G(X)$, then there is an obstruction $\sigma_G(Y, X) \in \tilde{K}_{0G}^{\mathrm{PL}}(Y)$ [21, 54, 55] for $G = 1$; [33, 34, 2, 31, 40] for finite G. The element $\sigma_G(Y, X) = 0$ if and only if (Y, X) has the G-homotopy type of a finite relative G-CW pair. In particular, $\sigma_G(Y, X) = 0$ if and only if (Y, f) is an object of $\widetilde{\mathbf{nil}}_G(X)$.

The next proposition is another characterization of objects in $\mathbf{nil}_G(X)$ sometimes more useful for the applications. Also, this proposition shows that the category $\mathbf{nil}_G(X)$ is an analogue of the categories used in [17, 19, 39] in the construction of the algebraic Nil-groups.

PROPOSITION 2.6. *Let (Y, X) be a G-complex, and $f: Y \to Y$ be a G-map such that*

(i) *the pair (Y, X) is a relative finitely dominated G-pair*
(ii) $f|_X = \mathrm{id}_X$.

Then, (Y, f) is an object in $\mathbf{nil}_G(X)$ if and only if there is an integer $n \in \mathbf{N}$ such that f^n is G-homotopic rel$X$ *to a G-retraction of Y to X.*

Proof. Suppose first that (Y, f) is an object in $\mathbf{nil}_G(X)$. So, there are

(i) A strong G-deformation retraction $f_t: T(f) \to T(f)$, $t \in I$, of $T(f)$ to $X \times S^1$ which induces a strong G-deformation retraction, $\bar{f}_t: D(f) \to D(f)$, of $D(f)$ to $X \times \mathbf{R}$.
(ii) there is a pair of G-spaces (K, X) such that
   (a) (K, X) is a finite relative G-complex.
   (b) There are G-maps $Y \xrightarrow{r} K \xrightarrow{i} Y$ such that $ir \simeq_G \mathrm{id}_Y \,\mathrm{rel} X$.

Define $f': K \to K$ as the composition rfi. There is a G-homotopy equivalence, $q: D(f') \to D(f)$ rel$(X \times \mathbf{R})$ ([23], p. 106). Therefore, there is a strong G-deformation retraction $\bar{f}'_t: D(f') \to D(f')$ of $D(f')$ to $X \times \mathbf{R}$. The maps q, $\bar{f}'_t$, $\bar{f}_t$ are $\mathbf{Z}$-equivariant. Identify K with $K \times 0 \times 0$ in $D(f')$. Let M' be the mapping cylinder $M(f')$ in $D(f')$ with top $K \times \{0\} \times \{0\}$ and $M = M(f)$ with top $Y \times 0 \times 0$ in $D(f)$. Then, M' is the union of $X \times [0, 1]$ with a finite number of G-cells. Let C be the union of the G-cells. By the compactness of C, there is a number $m > 0$ such that

$$\bar{f}'_t(M') \subset D(f')_{(-\infty,m]} \cup (X \times \mathbf{R}), \quad \text{for all } t \in I.$$

Then, since q maps M' to M and the homotopies preserve M' and M, there is a number $n > 0$ such that $\bar{f}_t(M)$ is contained in $D(f)_{(-\infty,n]} \cup (X \times \mathbf{R})$. The G-homotopy $\bar{f}_t$ induces a G-homotopy $\varphi_t: D_X(f) \to D_X(f)$ from the identity map to a retraction to X, rel$X$. Notice that the map

$$c'_n: D_X(f)_{(-\infty,n+1]} \to Y$$

is the identity on Y. The homotopy

$$(c'_n)(\varphi_t|_{Y \times \{0\} \times \{0\}}): Y \times \{0\} \times \{0\} \to Y$$

is a G-homotopy from f^{n+1} to a map $c'_n\varphi_1$ sending Y into X, rel$X$. This completes the first part of the proof.

Assume that there is $n \in \mathbf{N}$ such that f^n is G-homotopic rel$X$ to a G-retraction of Y to X, i.e. $f^n \simeq_G jr$, where $j: X \to Y$ is the inclusion map and $r: Y \to X$ is a G-retraction. Let $e^n: S^1 \to S^1$ be the cover of S^1 corresponding to the subgroup $n\mathbf{Z}$. Consider the pull back diagram

$$\begin{array}{ccc} T^n(f) & \xrightarrow{\pi'} & S^1 \\ \downarrow{\scriptstyle e'^n} & & \downarrow{\scriptstyle e^n} \\ T(f) & \xrightarrow{\pi} & S^1 \end{array}$$

The space $T^n(f)$ is given as $M(f) \times \{1, 2, \ldots, n\}/\sim$, where $\sim$ is the equivalence relation generated by

(i) $(y, m) \sim (y, 0, m+1)$ for $(y, m) \in Y \times \{m\} \subset M(f) \times \{m\}$ and $(y, 0, m+1) \in Y \times \{0\} \times \{m+1\} \subset M(f) \times \{m+1\}$ for $m < n$.
(ii) $(y, n) \sim (y, 0, 0)$ for $(y, n) \in Y \times \{n\} \subset M(f) \times \{n\}$ and $(y, 0, 0) \in Y \times \{0\} \times \{0\} \subset M(f) \times \{0\}$.

By successive applications of 1.10, $T^n(f)$ is G-homotopy equivalent, rel$X \times S^1$, to $T(f^n)$. Using the properties 1.9 and 1.11, there is a sequence of G-homotopy equivalences rel$X \times S^1$:

$$T(f^n) \simeq_G T(jr) \simeq_G T(rj) = T(\mathrm{id}_X) = X \times S^1.$$

Therefore, $T^n(f)$ strong G-deformation retracts to $X \times S^1$.

Also, consider the pull-back diagram

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{i'} & T^n(f) \\ {\scriptstyle \mathrm{id}_X \times e^n}\downarrow & & \downarrow{\scriptstyle e'^n} \\ X \times S^1 & \xrightarrow{i} & T(f) \end{array}$$

For each subgroup H of G, this induces a pull-back diagram

$$\begin{array}{ccc} X^H \times S^1 & \xrightarrow{i'^H} & T^n(f)^H \\ \downarrow & & \downarrow \\ X^H \times S^1 & \xrightarrow{i^H} & T(f)^H \end{array}$$

Since the map i'^H is a homotopy equivalence, and the above diagram is a pull-back diagram of a fibration, the map i^H induces an isomorphism on the homotopy groups. Therefore $\pi_i(T(f)^H, X^H \times S^1) = 0$ for $i > 0$. Since $(T(f)^H, X^H \times S^1)$ is a relative CW-pair, i^H is a homotopy equivalence for all finite subgroups H of G. Therefore, i is a G-homotopy equivalence. So $X \times S^1$ is a strong G-deformation retraction of $T(f)$. This completes the proof of the proposition.

We define an equivalence relation on the set of isomorphism classes of objects of $\mathbf{nil}_G(X)$, similar to the homotopy relation in [39], §9. Let (Y, f) and (Y', f') are two objects in $\mathbf{nil}_G(X)$. Define $(Y, f) \sim (Y', f')$ if and only if there is a G-homotopy equivalence $F\colon Y \to Y' \,\mathrm{rel} X$ such that Ff is G-homotopic $\mathrm{rel} X$, to $f'F$.

Denote by $[Y, f]$ the equivalence class of the object (Y, f).

From the definition of the equivalence relation, it is obvious that:

LEMMA 2.7. *Let* (Y, f) and (Y', f') be two equivalent objects of $\mathbf{nil}_G(X)$. *Then, if* (Y, f) *is an object in* $\widetilde{\mathbf{nil}}_G(X)$ *so is* (Y', f').

PROPOSITION 2.8. *Push-out diagrams exist in* $\mathbf{nil}_G(X)$ ($\widetilde{\mathbf{nil}}_G(X)$ *or* $\mathbf{K}_{0G}(X)$).

Proof. Let

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array}$$

be a push-out diagram in $\mathbf{n}_G(X)$.

(i) Let $(Y_i, f_i), i = 0, 1, 2$ be objects in $\mathbf{nil}_G(X)$. Then, since (Y_i, X) is relatively G-finitely dominated, $i = 0$, 1, 2, and Y is the push-out of Y_i, (Y, X) is relatively G-finitely dominated. So, (Y, f) is an object of $\mathbf{nil}_G(X)$.

(ii) Let $(Y_i, f_i), i = 0$, 1, 2, 3 be objects in $\widetilde{\mathbf{nil}}_G(X)$. Then,

$$\sigma_G(Y, X) = j_* \sigma_G(Y_1, X) + j'_* \sigma_G(Y_2, X) - (j'i)_* \sigma_G(Y_0, X) = 0,$$

since $\sigma_G(Y_i, X) = 0$ for $i = 0, 1, 2$. So, (Y, f) is an object of $\widetilde{\mathbf{nil}}_G(X)$.

(iii) Let (Y_i, f_i), $i = 0, 1, 2$ be objects in $\mathbf{K}_{0G}(X)$, then $f: Y \to Y$ is given as the union of two retractions. Therefore, f is a G-retraction, and the push-out is an object in $\mathbf{K}_{0G}(X)$.

COROLLARY 2.9. *Let* $\mathbf{Y}' = (Y', f')$ *and* $\mathbf{Y} = (Y, f)$ *be two objects in* $\mathbf{nil}_G(X)$ $(\widetilde{\mathbf{nil}}_G(X))$ *or* $\mathbf{K}_{0G}(X)$ *such that* (Y', Y) *is a relative G-CW pair, and* $f'|_Y = f$. *Then, the quotient* $\mathbf{Y}'/\mathbf{Y}$ *is an object in* $\mathbf{nil}_G(X)$ $(\widetilde{\mathbf{nil}}_G(X)$ *or* $\mathbf{K}_{0G}(X))$.

We can give now the definition of the geometric Nil-groups.

DEFINITIONS 2.10. (A) Define $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ to be the group $\mathbb{F}/\mathbb{N}$ where

(i) $\mathbb{F}$ is the free Abelian group generated by equivalences classes of objects in $\mathbf{nil}_G(X)$.

(ii) $\mathbb{N}$ is the subgroup of F generated by elements of the form:

(a) $[Y, f] + [Y_0, f_0] - [Y_1, f_1] - [Y_2, f_2]$, if there is a push-out diagram

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array}$$

(b) $[Y, f]$, where (Y, X) is relative G-homotopy equivalent to a relative finite G-CW pair, and f is G-homotopic rel$X$ to a G-retraction.

(B) Define $\widetilde{\mathrm{Nil}}_G(X)$ to be the group constructed $\mathbb{F}'/\mathbb{N} \cap \mathbb{F}'$, where $\mathbb{F}'$ is the free Abelian group generated by the equivalence classes of objects in $\widetilde{\mathbf{nil}}_G(X)$.

(C) Define $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ to be the group constructed $\mathbb{F}''/\mathbb{N}''$, where $\mathbb{F}''$ is the free Abelian group generated by the equivalence classes of objects in $\mathbf{K}_{0G}(X)$ and $\mathbb{N}''$ is the subgroup of F generated by elements of the form:

(1) $[Y, r] + [Y_0, r_0] - [Y_1, r_1] - [Y_2, r_2]$, if there is a push-out diagram

$$\begin{array}{ccc} (Y_0, r_0) & \xrightarrow{i'} & (Y_1, r_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, r_2) & \xrightarrow{j'} & (Y, r) \end{array}$$

(2) $[Y, r]$, where (Y, X) is relative G-homotopy equivalent to a relative finite G-CW pair.

Remarks. (1) Notice that in (B) above the groups $\widetilde{\mathrm{Nil}}_G(X)$ are not specified as PL-groups. The reason is that these groups will be the same in the PL and the topological case. This is due to the fact that the difference between the PL and the Top groups is measured by controlled groups and the Nil-elements are not controlled.

(2) Part (C) is another interpretation of the group defined in [33, 34].

LEMMA 2.11. (i) *Let* (Y, f) *be an object in* $\widetilde{\mathbf{nil}}_G(X)$. *Then the torus object* $(T_X(f), \mathrm{Tor}(f))$ *determined by* (Y, f) *is such that* $[T_X(f), \mathrm{Tor}(f)] = 0$ *in* $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

(ii) *Let* $\mathbf{Y}'=(Y',f')$ *be an object* $\mathbf{nil}_G(X)$ and $Y\subset Y'$ *be a closed G-space such that* (Y',Y) *is a relative G-CW pair, and* $\mathbf{Y}=(Y,f'|_Y)$ *is an object in* $\mathbf{nil}_G(X)$. *Let* $\mathbf{Y}'/\mathbf{Y}=(C,F)$ *be the quotient. Then,*

$$[C,F]+[Y,f]=[Y',f'] \text{ in } \mathrm{Nil}_G^{\mathrm{PL}}(X).$$

(iii) *If* (Y,f) *and* (Y',f') *are two objects of* $\mathbf{nil}_G(X)$ *then,*

$$[Y,f]+[Y',f']=[Y\cup_X Y',f\cup f'] \text{ in } \mathrm{Nil}_G^{\mathrm{PL}}(X).$$

(iv) *Let* (Y,f) *and* (Y,f') *be two objects of* $\mathbf{nil}_G(X)$ *such that* $f\simeq_G f'$, rel$X$. *Then,*

$$[Y,f]=[Y,f'] \text{ in } \mathrm{Nil}_G^{\mathrm{PL}}(X).$$

(v) *Let* (Y,f) *be an object in* $\mathbf{nil}_G(X)$. *Then the inverse of* $[Y,f]$ *in* $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ *is given by the class of the quotient of* $(T_X(f),\mathrm{Tor}(f))$ *by* (Y,f).

Proof. (i) $T_X(f)$ strong G-deformation retracts to X, and by Mather's trick, $\mathrm{Tor}(f)$ is G-homotopic to the identity on $T_X(f)$, rel$X$. Therefore, $\mathrm{Tor}(f)$ is G-homotopic to a retraction rel$X$. By (iib) Definition 2.10, $[T_X(f),\mathrm{Tor}(f)]=0$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

(ii) (C,F) is given as the push-out diagram of Definition 2.4(b), then

$$[C,F]+[Y,f]=[T_X(f),\mathrm{Tor}(f)]+[Y',f'] \text{ in } \mathrm{Nil}_G^{\mathrm{PL}}(X)$$

By (i):

$$[C,F]+[Y,f]=[Y',f']\ \mathrm{Nil}_G^{\mathrm{PL}}(X).$$

(iii) It is obvious that $(Y\cup_X Y',f\cup f')$ is an object in $\mathbf{nil}_G(X)$. Also, there is a push-out diagram in $\mathbf{nil}_G(X)$:

$$\begin{array}{ccc}(X,\mathrm{id}_X) & \xrightarrow{i'} & (Y,f)\\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j}\\ (Y',f') & \xrightarrow{j'} & (Y\cup_X Y',f\cup f').\end{array}$$

By (iia) Definition 2.10, $[Y,f]+[Y',f']=[Y\cup_X Y',f\cup f']$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

(iv) The identity map id: $Y\to Y$ induces an equivalence between (Y,f) and (Y,f'). Therefore, $[Y,f]=[Y,f']$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

(v) Obvious from (i) and (ii).

Remark. Lemma 2.11 actually proves that any element of $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ can be represented as $[Y,f]$ for some object (Y,f) of the category $\mathbf{nil}_G(X)$. The same is true for the subgroups $\widetilde{\mathrm{Nil}}_G(X)$ and $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ of $\mathrm{Nil}_G^{\mathrm{PL}}(X)$, and the categories $\widetilde{\mathbf{nil}}_G(X)$, $\mathbf{K}_{0G}(X)$.

The following Lemma characterizes equality in the group $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

LEMMA 2.12. *Let* (Y,f) *and* (Y',f') *be two objects in* $\mathbf{nil}_G(X)$ *such that*

(i) (Y',Y) *is a finite relative G-CW pair.*
(ii) $f'|_Y=f$.
(iii) $f'(Y')$ *is contained in* Y.

Then, $[Y,f]=[Y',f']$ *in* $\mathrm{Nil}_G^{\mathrm{PL}}(X)$.

Proof. Notice that, since Y' is formed by adding finitely many G-cells to $Y, i_*(\sigma_G(Y,X)) = \sigma_G(Y',X)$ in $\tilde{K}_{0G}(Y')$ by the additive property of the finiteness obstruction. Let (C,F) be the quotient. Then, (C,F) is given by the push-out diagram

$$\begin{array}{ccc} (Y,f) & \xrightarrow{i} & (Y',f') \\ \iota\downarrow & & j\downarrow \\ (T_X(f), \mathrm{Tor}(f)) & \xrightarrow{j'} & (C,F) \end{array}$$

Notice that

$$\sigma_G(C,X) = j'_*\sigma_G(T_X(f),X) + j_*\sigma_G(Y',X) - (ji)_*\sigma_G(Y,X).$$

But,

$$j_*i_*(\sigma_G(Y,X)) = j_*(\sigma_G(Y',X)) \quad \text{and} \quad \sigma_G(T_X(f),X) = 0.$$

Therefore, $\sigma_G(C,X) = 0$, and (C,X) is relatively G-homotopy equivalent to a relative finite G-CW pair. The map F maps C to $T_X(f)$. The restriction of F to $T_X(f)$ is just $\mathrm{Tor}(f)$. But, $\mathrm{Tor}(f)$ is G-homotopic, rel$X$, to a retraction $r: T_X(f) \to X$. If $k: X \to T_X(f)$ is the inclusion map:

$$F(j'k) = \mathrm{Tor}(f)k \simeq_G rk = \mathrm{id}_X.$$

But, the inclusion map $X \to C$ is a G-cofibration. So F is G-homotopic, rel$X$, to a retraction ([45], p. 29). Therefore $[C,F] = 0$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$. So $[Y,f] = [Y',f']$.

The next lemma shows that the class of an element (Y,r) of $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ does not depend on the retraction r.

LEMMA 2.13. *Let (Y,X) be a relative G-complex such that (Y,X) is relatively G-finitely dominated and $r,r': Y \to X$ be two retractions. Then $[Y,r] = [Y,r']$ in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$.*

Proof. By Lemma 2.11 (v), there is an object (Y',s) in $\mathbf{K}_{0G}(X)$ such that

$$[Y,r] + [Y',s] = 0$$

in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$. In particular, the relative finiteness obstruction $\sigma_G(Y \cup_X Y', X) = 0$. Therefore $(Y \cup_X Y', X)$ has the G-homotopy type of a finite relative G-complex. But $[Y,r'] + [Y',s] = [Y \cup_X Y', r' \cup s]$ and $(Y \cup_X Y', r' \cup s) \in \mathbb{N}''$. So,

$$[Y,r'] + [Y',s] = 0 \Rightarrow [Y,r'] = -[Y',s] = [Y,r].$$

in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$.

The next proposition is the geometric analogue of the formula $\mathrm{Nil}(R) \approx \widetilde{\mathrm{Nil}}(R) \oplus \tilde{K}_0(R)$ for any ring R ([17], Lemma 1.4; [19], Proposition 6; [39], §9]).

PROPOSITION 2.14. *For a G-space X, $\mathrm{Nil}_G^{\mathrm{PL}}(X) \approx \widetilde{\mathrm{Nil}}_G(X) \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X)$.*

Proof. The inclusion induced map $\varphi: \tilde{K}_{0G}^{\mathrm{PL}}(X) \to \mathrm{Nil}_G^{\mathrm{PL}}(X)$ is well-defined group homomorphism. We will prove that φ is a split monomorphism. We construct a left inverse for $\varphi, \rho: \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \tilde{K}_{0G}^{\mathrm{PL}}(X)$:

Let (Y, f) represent an element of $\mathrm{Nil}_G^{\mathrm{PL}}(X)$. Then by Proposition 2.6, there is a number $n \in \mathbf{N}$, such that f^n is G-homotopic rel$X$ to a G-retraction $r: Y \to X$. Let $i: X \to Y$ be the inclusion map. Define $\rho[Y, f] = [Y, ir]$.

CLAIM. *The map ρ is a well-defined group homomorphism.*

Proof. (i) Notice that the map f^{n+1} is also G-homotopic to a retraction $r': Y \to X$. But $ir' \simeq_G f^{n+1} = ff^n \simeq_G fir = ir$. So, $[Y, ir'] = [Y, ir]$ in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ and $\rho[Y, f]$ does not depend on the number n.

(ii) If $(Y, f) \sim (Y', f')$ in $\mathbf{nil}_G(X)$, there is a G-homotopy equivalence $F: Y \to Y'$, rel$X$ such that $Ff \simeq_G f'F$ rel$X$. Then, $Ff^n \simeq_G f'^n F$ rel$X$, where n is a number for which f^n and f'^n are G-homotopic rel$X$ to G-retractions $r: Y \to X$ and $r': Y' \to X$, respectively. So, $[Y, ir] = [Y', i'r']$, where $i': X \to Y$ is the inclusion. Hence, $\rho[Y, f] = \rho[Y', f']$.

(iii) If (Y, X) is G-homotopy equivalent to a relative finite G-complex (K, X) and $f: Y \to Y$ is G-homotopic rel$X$ to a G-retraction, then $\rho[Y, f] = 0$.

(iv) Let

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array}$$

be a push-out diagram in $\mathbf{nil}_G(X)$. Choose an integer $n > 0$ such that f_i^n and f^n are G-homotopic, rel$X$, to G-retractions $r_i: Y_i \to X, r: Y \to X$, respectively $(i = 0, 1, 2)$.

Write $\iota_i: X \to Y_i$, $i = 0, 1, 2$, $\iota: X \to Y$ for the inclusion maps, and $r_i': Y_i \to X$ for the restriction of r to Y_i. By Lemma 2.13, $[Y_i, \iota_i r_i] = [Y_i, \iota_i r_i']$ in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$.

Also, the following is a push-out diagram in $\tilde{K}_{0G}^{\mathrm{PL}}(X)$:

$$\begin{array}{ccc} (Y_0, r_0') & \xrightarrow{i'} & (Y_1, r_1') \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, r_2') & \xrightarrow{j'} & (Y, r) \end{array} \qquad (1)$$

Putting these observations together, we get:

$$\begin{aligned} \rho([Y, f] + [Y_0, f_0]) &= \rho([Y \cup_X Y_0, f \cup f_0]) = [Y \cup_X Y_0, r \cup r_0] \\ &= [Y, r] + [Y_0, r_0] = [Y, r] + [Y_0, r_0']. \end{aligned}$$

Similarly, $\rho([Y_1, f_1] + [Y_2, f_2]) = [Y_1, r_1'] + [Y_2, r_2']$. But from the above push-out diagram (1), we get

$$\begin{aligned} [Y, r] + [Y_0, r_0'] = [Y_1, r_1'] + [Y_2, r_2'] \Rightarrow \rho([Y, f] &+ [Y_0, f_0]) \\ = \rho([Y_1, f_1] &+ [Y_2, f_2]). \end{aligned}$$

This completes the proof that ρ is a well-defined map.

It is obvious that ρ is a group homomorphism. It is a group epimorphism since

$$\rho\varphi[Y, f] = \rho[Y, f] = [Y, f].$$

There is also an inclusion-induced group homomorphism

$$j\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Nil}_G^{\mathrm{PL}}(X).$$

j is obviously a group monomorphism.

We complete the proof of the proposition by proving that the sequence

$$0 \to \widetilde{\mathrm{Nil}}_G(X) \xrightarrow{j} \mathrm{Nil}_G^{\mathrm{PL}}(X) \xrightarrow{\rho} \widetilde{K}_{0G}^{\mathrm{PL}}(X) \to 0$$

is split exact. The only thing that remains to be proved is that $\operatorname{Ker}\rho = \operatorname{Im} j$.

$\operatorname{Ker}\rho \supset \operatorname{Im} j$: Let $[Y,f] \in \widetilde{\mathrm{Nil}}(X)$. So (Y,X) is G-homotopy relative finite G-CW pair. Also, $\rho j[Y,f] = [Y,f'']$, which belongs to $\mathbb{N}$ and so it represents the zero element of $\widetilde{K}_{0G}^{\mathrm{PL}}(X)$. Therefore $\rho j = 0$.

$\operatorname{Im} j \supset \operatorname{Ker}\rho$: Let $[Y,f] \in \operatorname{Ker}\rho$. Then, $r[Y,f] = [Y,f''] = 0$ in $\widetilde{K}_{0G}^{\mathrm{PL}}(X)$. So (Y,X) is homotopy relative finite G-CW pair. This means that $[Y,f] = j[Y,f]$, where $[Y,f] \in \widetilde{\mathrm{Nil}}_G(X)$. So $[Y,f] \in \operatorname{Im} j$.

We give a geometric description of the projection $p\colon \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \widetilde{\mathrm{Nil}}_G(X)$:

Let (Y,f) be an object in $\mathbf{nil}_G(X)$. Since (Y,X) is relatively G-finitely dominated, there is a finite relative G-complex (K,X) and G-maps $Y \xrightarrow{r} K \xrightarrow{i} Y$ which are the identity on X, such that $ir \simeq_G \mathrm{id}_Y \operatorname{rel} X$. Let $f'\colon K \to K$ as the composition rfi. The pair (K,f') is an object in $\widetilde{\mathbf{nil}}_G(X)$. Define $p'\colon \mathbb{F} \to \widetilde{\mathrm{Nil}}_G(X)$ by $p'(Y,f) = [K,f']$.

CLAIM: *p' is a well-defined group homomorphism.*

Proof. First we prove that p' is independent of the choice of the domination:

(a) Assume that $Y \xrightarrow{r'} K' \xrightarrow{i'} Y$ is another domination of (Y,X) such that

$$K' \subset K, r = r', \quad \text{and} \quad i' = i|_{K'}.$$

Set $f' = rfi$ and $f'' = r'fi'$. Then, $f'(K) \subset K'$, and the objects (K,f) and (K',f'') satisfy the hypothesis of Lemma 2.12, so $[K,f'] = [K',f'']$ in $\widetilde{\mathrm{Nil}}_G(X)$.

(b) Assume that $Y \xrightarrow{r'} K' \xrightarrow{i'} Y$ is another domination of (Y,X). Define $M = M_X(r'i)$, and $i''\colon M \to Y$ to be $i'c$, where c is the collapse map to K'. Also, define $r''\colon Y \to M$ to be jr', where $j\colon K' \to M$ is the inclusion map. Then

$$r''i'' = jr'i'c \simeq_G jr \simeq_G \mathrm{id}_Y \operatorname{rel} X.$$

By (a), $[M,m] = [K',f'']$, where $m = r''fi''$. Let $j'\colon K \to M$ be the map $j'(k) = (k,0)$. Then, $j'i'' = i'r' \simeq_G i, \operatorname{rel} X$, and $[K, ifr] = [K, j'i''fr]$.

Similarly,

$$cj'r = r'ir \simeq_G r' \Rightarrow j'r \simeq_G jcjr \simeq_G jr' = r'' \operatorname{rel} X.$$

Hence,

$$[M, r''fi''] = [M, r''fj'i] \quad \text{and} \quad [M, r''fj'i] = [K, j'r''fi] \Rightarrow [M,m] = [K,f].$$

Now, we prove that p' does not depend on the equivalence class of (Y,f):

If (Y, f) and (Y', f') are equivalent, then there is a G-homotopy equivalence $F\colon Y \to Y'$, rel$X$ such that $Ff \simeq_G f'F$ rel$X$. Let $Y \xrightarrow{r} K \xrightarrow{i} Y$ be a domination for Y. Then

$$Y' \xrightarrow{F^{-1}} Y \xrightarrow{r} K \xrightarrow{i} Y \xrightarrow{F} Y'$$

is a domination for Y', where F^{-1} is a G-homotopy inverse of F. Then

$$p'(Y, f) = [K, rfi] \quad \text{and} \quad p'(Y', f') = [K, rF^{-1}f'Fi].$$

But, $rF^{-1}f'Fi \simeq_G rfi$ rel$X$. So,

$$p'(Y, f) = p'(Y', f').$$

It is obvious that p' is a group homomorphism.

CLAIM. $p'(\mathbb{N}) = 0$.

Proof. (a) Let (Y, f) be such that (Y, X) has the G-homotopy type of a finite relative G-complex and f is G-homotopic rel$X$ to a retraction. Then $p'(Y, f) = [Y, f] = 0$.

(b) Consider the push-out diagram in $\mathbf{nil}_G(X)$:

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array} \tag{1}$$

We are going to show that

$$p'((Y, f) + (Y_0, f_0)) = p'((Y_1, f_1) + (Y_2, f_2)).$$

Let L_j dominates Y_j, $j = 0, 1, 2$ and $Y_j \xrightarrow{r'_j} L_j \xrightarrow{i'_j} Y_j$ be dominations. Define:

$$K_0 = L_0, \qquad K_1 = M_X(r_1 i i_0), \qquad K_2 = M_X(r_2 i i_0).$$

Define the dominations:

$i_j\colon K_j \to Y$ is the composition of the collapse and i'_j,

$r_j\colon Y \to K_j$ is the composition of r'_j and the inclusion map ($j = 1, 2$).

Let K be the push-out of K_j, $j = 0, 1, 2$. Then K dominates Y.

Summarizing, there is a push-out diagram

$$\begin{array}{ccc} (K_0, f'_0) & \xrightarrow{i'} & (K_1, f'_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (K_2, f'_2) & \xrightarrow{\varphi'} & (K, f') \end{array} \tag{2}$$

So,

$$\begin{aligned} p'((Y, f) + (Y_0, f_0)) &= (K, f') + (K_0, f'_0) = (K_1, f'_1) + (K_2, f'_2) \\ &= p'((Y_1, f_1) + (Y_2, f_2)). \end{aligned}$$

So, p' induces a map:

$$p\colon \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \widetilde{\mathrm{Nil}}_G(X).$$

It is obvious that $pj = \mathrm{id}$, and $ip = 0$

This completes the proof of Proposition 2.14.

The next proposition is about the naturality properties of the above construction.

PROPOSITION 2.15. *Let $k\colon X \to X'$ be a G-map between G-spaces. Then, k induces a group homomorphism*

$$k_*\colon \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \mathrm{Nil}_G^{\mathrm{PL}}(X')$$

such that k_* maps $\widetilde{\mathrm{Nil}}_G(X)$ into $\widetilde{\mathrm{Nil}}_G(X')$ and $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ into $\tilde{K}_{0G}^{\mathrm{PL}}(X')$.

Proof. Define $k_*[Y, f]$ to be $[Y', f']$, where $Y' = Y \cup_k X' = Y \cup X'/\sim$ with $x \sim k(x)$ for x in X and $f'\colon Y' \to Y'$ is just $f \cup \mathrm{id}$. First of all (Y', f') is indeed an object in $\mathbf{nil}_G(X')$, because $f'^n = f^n \cup \mathrm{id}$, the G-homotopy of f^n to a retraction of Y to X extends by the identity to a G-homotopy of f'^n to a retraction of Y' to X', and (Y', X') is an G-NDR pair. It is obvious that the map k_* is a well-defined group homomorphism and that it maps $\widetilde{\mathrm{Nil}}_G(X)$ to $\widetilde{\mathrm{Nil}}_G(X')$ and $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ to $\tilde{K}_{0G}^{\mathrm{PL}}(X')$.

PROPOSITION 2.16 *Let X, X', X'' are G-spaces. If $k\colon X \to X'$ and $k'\colon X' \to X''$ are G-maps, then $(k'k)_* = k'_* k_*$.*

Proof. Let $[Y, f] \in \mathrm{Nil}_G^{\mathrm{PL}}(X)$. Then,

$$k'_* k_*[Y, f] = [(Y \cup_k X') \cup_{k'} X'', f''] = [Y \cup_{k'k} X'', f''] = (k'k)_*[Y, f].$$

PROPOSITION 2.17. *Let X, X' be G-spaces. If $k, k'\colon X \to X'$ are two G-homotopic maps, then $k_* = k'_*\colon \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \mathrm{Nil}_G^{\mathrm{PL}}(X')$.*

Proof. Let $[Y, f]$ represent an element in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$. By the proof of the adjunction lemma of L. Siebenmann [44], p. 21, construct a G-homotopy equivalence relX' of $Y \cup_k X'$ to $Y \cup_{k'} X'$. This homotopy equivalence induces an isomorphism in $\mathbf{nil}_G(X')$ of $(Y \cup_k X', f \cup \mathrm{id})$ and $(Y \cup_{k'} X', f \cup \mathrm{id})$. So $k_* = k'_*$.

COROLLARY 2.18. *If $k\colon X \to X'$ is a G-homotopy equivalence between G-spaces, then k_* is an isomorphism.*

So, we have defined three functors from the category of G-spaces, and G-homotopy classes of G-maps, to the category of Abelian groups and group homomorphisms.

3. Nil-Groups as Summands of the Whitehead Group

In this chapter, we show that $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ is a direct summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$. The methods are similar to the ones used in the nonequivariant case [4, 5, 17, 19, 39].

Let X be a G-space. We first define two injections, $j(\pm)\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

LEMMA 3.1. *Let* (Y, f), (Y', f') *be two equivalent objects of* $\mathbf{nil}_G(X)$, *such that* (Y, X) *and* (Y', X) *are finite relative G-complexes. Then* $T(f)$ *and* $T(f')$ *are G-simply equivalent* $\mathrm{rel}(X \times S^1)$. *Also,* $T_X(f)$ *and* $T_X(f')$ *are G-simply equivalent* $\mathrm{rel}X$.

Proof. It follows from 1.9 and 1.11.

CONSTRUCTION 3.1.1. We construct a map: $j(+)\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

Let (Y, f) be an object of $\widetilde{\mathbf{nil}}_G(X)$ representing an element in $\widetilde{\mathrm{Nil}}_G(X)$. Then (Y, X) has the G-homotopy type of a finite relative G-complex (K, X). Let $a\colon Y \to K$ be the G-homotopy equivalence, and $b\colon K \to Y$ is a G-homotopy inverse $(\mathrm{rel}X)$. Then, $T(afb) \simeq_G T(baf) \simeq_G T(f)$, $\mathrm{rel}(X \times S^1)$. Since the $T(f)$ strong, G-deformation retracts to $X \times S^1$, the object (Y, f) determines an element $[T(afb), X \times S^1]$ of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$. Define $j_+(Y, f) = [T(afb), X \times S^1]$.

PROPOSITION 3.2. *The map* j_+ *induces a split injection* $j(+)\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

Proof. Lemma 3.1 shows that j_+ extends to a well-defined homomorphism

$$j_+\colon \mathbb{F}' \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1).$$

To complete the proof that j_+ induces a homomorphism as stated, we have to prove that j_+ respects the relations in $\widetilde{\mathrm{Nil}}_G(X)$.

(i) If f is G-homotopic rel$X$ to a retraction r from Y into X, then $[Y, f] = [Y, ir]$, where i is the inclusion map. We can assume that (Y, X) is a finite relative G-complex. Then, by 1.11:

$$[T(ir), X \times S^1] = [T(ri), X \times S^1] = [X \times S^1, X \times S^1] = 0 \quad \text{in } \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1).$$

Therefore, $j_+[Y, f] = 0$.

(ii) Consider the push-out diagram in $\widetilde{\mathbf{nil}}_G(X)$:

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array} \tag{1}$$

We are going to show that $j_+([Y, f] + [Y_0, f_0]) = j_+([Y_1, f_1] + [Y_2, f_2])$.

ASSERTION. *We can replace the above push-out diagram with a push-out diagram of finite relative G-complexes:*

$$\begin{array}{ccc} (K_0, f'_0) & \xrightarrow{i'} & (K_1, f'_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \varphi} \\ (K_2, f'_2) & \xrightarrow{\varphi'} & (K, f') \end{array} \tag{2}$$

where (K_i, f'_i) *is equivalent to* (Y_i, f_i), $i = 0, 1, 2$, *and* (K, f') *is equivalent to* (Y, f).

Proof. Diagram (1) provides a diagram

$$\begin{array}{ccc} (L_0, f_0'') & \xrightarrow{\kappa'} & (L_1, f_1'') \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\psi} \\ (L_2, f_2'') & \xrightarrow{\psi'} & (L, f'') \end{array} \tag{3}$$

commutative up to G-homotopy rel$X$, such that where (L_i, f_i'') is equivalent to (Y_i, f_i), $i = 0, 1, 2$, and (L, f'') is equivalent to (Y, f) and (L_i, X), $i = 0, 1, 2$, (L, X) are finite relative G-complexes. Notice that κ, κ', ψ, ψ' are morphisms in $\widetilde{\mathbf{nil}}_G(X)$ 'up to homotopy' in the sense that the corresponding diagrams commute up to homotopy.

Define:

$$(K_0, f_0') = (L_0, f_0''), \ (K_1, f_1') = (M_X(\kappa), k).$$

To define the map k, we use the G-homotopy, rel$X$, $H\colon L_0 \times I \to L_2$ between $\kappa f_0''$ and $f_2''\kappa$. Define $k\colon M_X(\kappa) \to M_X(\kappa)$ by

$$k(y, t) = \begin{cases} (f_0''(y), 2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ H_{2t-1}(y), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \quad (y, t) \in K_0 \times I,$$

$$k(y') = f_2''(y'), \quad \text{for } y' \in L_2$$

$(K_2, f_2') = (M_X(\kappa'), k')$ and if $H'\colon L_0 \times I \to L_1$ between $\kappa' f_0''$ and $f_1''\kappa'$, the map $k'\colon M_X(\kappa) \to M_X(\kappa)$ is given by

$$k'(y, t) = \begin{cases} (f_0''(y), 2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ H'_{2t-1}(y), & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \quad (y, t) \in K_0 \times I,$$

$$k'(y') = f_1''(y'), \quad \text{for } y' \in L_1.$$

Then, (K_1, f_1') and (K_2, f_2') are objects in $\widetilde{\mathbf{nil}}_G(X)$ equivalent to (L_1, f_1'') and (L_2, f_2'') (The inclusion map $j\colon L_i \to K_i$ induces the equivalence.) Also, the push-out of (K_i, f_i'), $i = 0, 1, 2$, is equivalent to (L, f'').

Then

$$\begin{aligned} &j_+([Y, f] + [Y_0, f_0]) \\ &\quad = j_+[Y \cup_X Y_0, f \cup f_0] = j_+[K \cup_X K_0, f' \cup f_0'] \\ &\quad = [T(f' \cup f_0'), X \times S^1] = [T(f') \cup_{X \times S^1} T(f_0'), X \times S^1] \\ &\quad = [T(f'), X \times S^1] + [T(f_0'), X \times S^1]. \end{aligned}$$

Similarly,

$$j_+([Y_1, f_1] + [Y_2, f_2]) = [T(f_1'), X \times S^1] + [T(f_2'), X \times S^1].$$

But, the following diagram is a push-out diagram

$$\begin{array}{ccc} T(f_0') & \longrightarrow & T(f_1') \\ \downarrow & & \downarrow \\ T(f_2') & \longrightarrow & T(f') \end{array}$$

So, by the additivity property of the Whitehead torsion

$$[T(f'), X \times S^1] + [T(f'_0), X \times S^1] = [T(f'_1), X \times S^1] + [T(f'_2), X \times S^1].$$

Therefore,

$$j_+((Y, f) + (Y_0, f_0)) = j_+((Y_1, f_1) + (Y_2, f_2)).$$

So j_+ induces a homomorphism $j(+)\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

In order to prove that $j(+)$ is a split monomorphism, we define its left inverse. First, define a map $\omega(+)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \mathrm{Nil}_G^{\mathrm{PL}}(X)$ as follows: Let $(Y, X \times S^1)$ be a finite relative G-complex. Let $f\colon Y \to X \times S^1$ be a strong G-deformation retraction from Y to $X \times S^1$. Actually, f is a proper strong G-deformation retraction. Lift f to a proper strong G-deformation retraction on the infinite cyclic covers [2], Lemma 4.2:

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{f}} & X \times \mathbf{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle (\mathrm{id} \times e)} \\ Y & \xrightarrow{f} & X \times S^1 \end{array}$$

Notice that $\mathrm{cl}(\bar{Y} - (X \times \mathbf{R}))$ has two ends $e(+)$ and $e(-)$ corresponding to the two ends of $\mathbf{R}$. Let L be a G-subset of $\bar{Y}$ such that

(i) $(L, L \cap (X \times \mathbf{R}))$ is a relative G-complex.
(ii) $\mathrm{cl}(L - (X \times \mathbf{R}))$ is a neighborhood of $e(-)$ in $\mathrm{cl}(\bar{Y} - (X \times \mathbf{R}))$.
(iii) $\mathrm{cl}((\bar{Y} - L) - (X \times \mathbf{R}))$ is a neighborhood of $e(+)$ in $\mathrm{cl}(\bar{Y} - (X \times \mathbf{R})$.
(iv) There is a covering transformation, $z\colon \bar{Y} \to \bar{Y}$ which generates the group of covering transformations, such that $L \supset z(L)$.

Define $Y(+) = L/\!\sim$ where $(x, t) \sim (x, 0)$ for all $(x, t) \in (X \times \mathbf{R}) \cap L$, and a map $f(+)\colon Y(+) \to Y(+)$, by $f(+) = z|\colon Y(+) \to Y(+)$.

CLAIM. *$(Y(+), f(+))$ is an object in* $\mathbf{nil}_G(X)$.

Proof. Notice that $(Y(+), X)$ is relative G-complex. Also, $(L, L \cap (X \times \mathbf{R}))$ is relatively G-finitely dominated pair ([23, 44, 4.4, 2]). This implies that $(Y(+), X)$ is relatively G-finitely dominated. To complete the proof of the claim, it remains to be proved that some power of $f(+)$ is G-homotopic rel$X$ to a G-retraction. The pair (L, X) is G-finitely dominated, and there is a G-homotopy $k_t\colon L \to L$, rel$X$, of the identity on L, and a compact subset K of L such that $k_1|_{(L-K)} = \bar{f}|_{(L-K)}$. Choose $n \in \mathbf{N}$ so big that $L - K \supset z^n(L)$. Then, $k_t z^n\colon Y(+) \to Y(+)$ is a G-homotopy, rel$X$, from $f(+)^n$ to a G-retraction into X. This completes the proof of the claim.

Define $\omega(+)(Y, X \times S^1) = [Y(+), f(+)]$.

(i) $\omega(+)(Y, X \times S^1)$ does not depend on the choice of L:

Repeat the above construction starting with another closed G-subset M satisfying (i)–(iv) above. Using this data, we construct $(Y'(+), f'(+))$ an object of $\mathbf{nil}_G(X)$. It is enough to prove that $[Y(+), f(+)] = [Y'(+), f'(+)]$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ in the case that $M \supset L$.

Case 1: Suppose that $L \supset z(M)$. Then by Lemma 2.12, $[Y(+), f(+)] = [Y'(+), f'(+)]$.

Case 2: Suppose that $M \supset L$. Let $(Y_m(+), f_m(+))$ be constructed using $z^m(L)$. Then, by Case 1, $[Y(+), f(+)] = [Y_m(+), f_m(+)]$, for all $m \in \mathbf{N}$. Choose $n \in \mathbf{N}$ so big that $L \supset z^n(M)$. By Case 1, $[Y_{n-1}(+), f_{n-1}(+)] = [Y'(+), f'(+)]$. Therefore, $[Y(+), f(+)] = [Y'(+), f'(+)]$.

(ii) If $[Y, X \times S^1] = (Y', X \times S^1]$ in $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$. Then, Y and Y' are connected by a sequence of equivariant formal deformations. Then it is obvious that

$$\omega(+)[Y, X \times S^1] = \omega(+)[Y', X \times S^1]$$

So, $\omega(+)$ is a group homomorphism. Define $q(+)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \widetilde{\mathrm{Nil}}_G(X)$ as the composition of $\omega(+)$ and the projection $p\colon \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \widetilde{\mathrm{Nil}}_G(X)$.

ASSERTION. *The map* $q(+)j(+)\colon \widetilde{\mathrm{Nil}}_G(X) \to \widetilde{\mathrm{Nil}}_G(X)$ *is the identity. So* $j(+)$ *is a split injection.*

Proof. Let (Y, f) be an object of $\widetilde{\mathbf{nil}}_G(X)$ with (Y, X) a finite relative G-complex. Then, $j(+)[Y, f] = [T(f), X \times S^1]$. The infinite cyclic cover of $T(f)$ is $D(f)$. L can be chosen to be $D(f)_{(-\infty, 0]}$. So, $Y(+) = D_X(f)_{(-\infty, 0]}$. But the collapse map $c_0\colon D_X(f)_{(-\infty, 0]} \to Y \times \{0\}$, is a G-homotopy equivalence rel$X$ with inverse the inclusion map $i\colon Y \times \{0\} \to D_X(f)_{(-\infty, 0]}$. Then $[Y(+), f(+)] = [Y, c_0 f(+) i]$ in $\mathrm{Nil}_G(X)$. Notice that $c_0 f(+) i = f$. So $q(+)j(+)[Y, f] = [Y, f]$.

The second injection is constructed similarly. Define $j(-)\colon \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ as the composition

$$j(-) = (\mathrm{id}_X \times \mu)_* j(+),$$

where $\mu\colon S^1 \to S^1$ is the orientation reversing homeomorphism given $\mu(s) = 1 - s$.

We give the definition of the left inverse of $j(-)$: Define $\omega(-)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \mathrm{Nil}_G^{\mathrm{PL}}(X)$ as the composition: $\omega(-) = \omega(+)(\mathrm{id}_X \times \mu)_*$.

PROPOSITION 3.3. $q(-)j(+) = q(+)j(-) = 0$.

Proof. We will give the proof of $q(-)j(+) = 0$, the other part follows similarly. Let $[Y, f]$ represent an element of $\widetilde{\mathrm{Nil}}_G(X)$ with Y a finite G-complex. Then, $j(+)[Y, f] = [T(f), X \times S^1]$. The infinite cyclic cover of $T(f)$ is $D(f)$.

$$q(-)j(+)[Y, f] = q(+)(\mathrm{id}_X \times \mu)_*[T(f), X \times S^1].$$

Notice that the map $\mathrm{id}_X \times \mu$ reverses the orientation on $X \times S^1$. So, it reverses the ends of $\mathrm{cl}(D(f) - (X \times \mathbf{R}))$. So,

$$q(+)(\mathrm{id}_X \times \mu)_*[T(f), X \times S^1] = [D_X(f)_{[0,\infty)}, f'],$$

where $f'\colon D_X(f)_{[0,\infty)} \to D_X(f)_{[0,\infty)}$ is induced by the translation.

But f^n is G-homotopic, rel$X$, to a G-retraction of $r\colon Y \to X$. Then,

$$D_X(f)_{[0,\infty)} \simeq_G D_X(f^n)_{[0,\infty)} \simeq_G D_X(r)_{[0,\infty)} \simeq_G X,$$

and all the G-homotopy equivalences are rel$X$. $[D_X(f)_{[0,\infty)}, f'] = 0$ in $\widetilde{\mathrm{Nil}}_G(X)$, and therefoe $q(-)j(+) = 0$.

Proposition 3.3 states that there are two orthogonal disjoint summands of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ each isomorphic to $\widetilde{\mathrm{Nil}}_G(X)$.

4. Nonequivariant Nil-Groups

In this section we compare the geometric Nil-groups constructed in Section 2 with the algebraic Nil-groups of a ring defined in [4, 17, 19, 39]. In the nonequivariant case, it turns out that the geometric Nil-groups of a reasonable space X are isomorphic to the algebraic Nil-groups of the group ring $\mathbf{Z}\pi_1(X)$. This result provides a new, more geometric description of the Nil-groups.

We recall the definition of the algebraic Nil-groups (see [4]). Let R be a ring with identity. By an R-module we mean a left R-module. $\mathrm{Nil}(R) = \mathbf{F}/\mathbf{N}$, where $\mathbf{F}$ is the free Abelian group generated by isomorphism classes of pair (P, f), where P is a finitely generated projective R-module, and $f: P \to P$ is a nilpotent R-map, and $\mathbf{N}$ is the subgroup generated by the elements:

(i) If $0 \to (P, f) \to (P', f') \to (P'', f'') \to 0$ is an exact sequence of pairs (i.e. exact sequence of modules with the corresponding diagrams commutative), then

$$(P', f') - (P, f) - (P'', f'').$$

(ii) $(F, 0)$, where F is a finitely generated free R-module.

In [39], §9, there is an alternative definition of $\mathrm{Nil}(R)$. $\mathrm{Nil}(R) = \mathbf{F}'/\mathbf{N}'$ where $\mathbf{F}'$ is the free Abelian group generated by chain homotopy classes of pairs (C, f), where C is a finite, finitely generated, projective R-chain complex, and $f: C \to C$ is a chain homotopy nilpotent chain map (i.e. there is an integer $n > 0$ such that f^n is chain homotopic to the zero map), and $\mathbf{N}'$ is the subgroup generated by

(i) If $0 \to (C, f) \to (C', f') \to (C'', f'') \to 0$ is an exact sequence of pairs, as before, then

$$(C', f') - (C, f) + (C'', f'').$$

(ii) $(C, f) = 0$, where C is a finitely generated free R-chain complex and f is chain homotopic to the zero map.

In [39], Proposition 9.3, it is proved that both definitions produce the same group. In comparing the algebraic with the geometric Nil-group, we use both definitions.

For X a path-connected space a map $a: \mathrm{Nil}(X) \to \mathrm{Nil}(\mathbf{Z}\pi_1(X))$ is defined. An element of $\mathrm{Nil}(X)$ is represented by an object (Y, f) of $\mathbf{nil}(X)$. Then, there is an integer $n > 0$ such that f^n is homotopic to a retraction $r: Y \to X$, rel$X$. Let $\tilde{X}$ be the universal cover of X. Let $\tilde{Y}$ be the pull back of $\tilde{X}$ under the retraction r. Then the relative chain complex $C_*(\tilde{Y}, \tilde{X})$ is a finitely dominated chain complex over $\mathbf{Z}\pi_1(X)$.

Since Y is finitely dominated rel$X$, there is a finitely generated projective $\mathbf{Z}\pi_1(X)$ chain complex P_* chain homotopy equivalent to $C_*(\bar{Y}, \tilde{X})$. Let $q\colon P_* \to C_*(\bar{Y}, \tilde{X})$ be the chain homotopy equivalence. Define

$$a[Y, f] = [P_*, q^{-1}f_*q].$$

(here q^{-1} denotes any homotopy inverse of q). Notice that since f^n is homotopic rel$X$ to a retraction, $(q^{-1}f_*q)^n$ is chain homotopic to the zero map. So $[P_*, q^{-1}f_*q]$ represents an element in $\mathrm{Nil}(\mathbf{Z}\pi_1(X))$ ([38]).

PROPOSITION 4.1. *If X is a path-connected space, then a: $\mathrm{Nil}(X) \to \mathrm{Nil}(\mathbf{Z}\pi_1(X))$ is a group epimorphism.*

Proof. It is obvious that a does not depend on the choices of n, P_*, and q above. Also, $a[Y, f]$ depends only on the equivalence class of the object (Y, f) in $\mathbf{nil}(X)$. Let (Y, f) be an object in $\mathbf{nil}(X)$, such that (Y, X) is homotopy equivalent, rel$X$, to a finite relative CW-pair and f is homotopic rel$X$ to a retraction of Y into X. Notice that $C_*(\bar{Y}, \tilde{X})$ is chain homotopy equivalent to a finitely generated free $\mathbf{Z}\pi_1(X)$ chain complex, C_*, and f_* is chain homotopic to the zero map. Then the pair (C_*, f_*) represents the zero element in $\mathrm{Nil}(\mathbf{Z}\pi)$ and $a[Y, f] = 0$ in $\mathrm{Nil}(\mathbf{Z}\pi)$.

Consider the push-out diagram in $\mathbf{nil}(X)$:

$$\begin{array}{ccc} (Y_0, f_0) & \xrightarrow{i'} & (Y_1, f_1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ (Y_2, f_2) & \xrightarrow{j'} & (Y, f) \end{array}$$

The relative Mayer–Vietoris sequence gives an exact sequence of chain complexes, and homotopy nilpotent maps

$$0 \to (C_*(Y_0, X), f_{0*}) \to (C_*(Y_1, X), f_{1*}) \oplus (C_*(Y_2, X), f_{2*}) \to (C_*(Y, X), f_*) \to 0$$

The chain complexes are considered with $\mathbf{Z}\pi_1(X)$ coefficients. This implies

$$(C_*(Y, X), f_*) + (C_*(Y_0, X), f_{0*} = (C_*(Y_1, X), f_{1*}) + (C_*(Y_2, X), f_{2*})$$

in $\mathrm{Nil}(\mathbf{Z}\pi_1(X))$. So $a([Y, f] + [Y_0, f_0]) = a([Y_1, f_1] + [Y_2, f_2])$. Therefore, a is a well-defined group homomorphism.

Now we prove that a is onto. For this we use the original definition of $\mathrm{Nil}(\mathbf{Z}\pi_1(X))$ given in [4] and [17]. Given a pair (P, k), where P is a left finitely generated projective $\mathbf{Z}\pi_1(X)$-module and k a nilpotent endomorphism, we construct an object (Y, f) of $\mathbf{nil}(X)$ such that $[P, k] = a[Y, f]$ in $\mathrm{Nil}(\mathbf{Z}\pi_1(X))$. P can be represented as a pair (F, p) where F is a finitely generated free $\mathbf{Z}\pi_1(X)$-module and $p\colon F \to F$ is a projection ($p^2 = p$). Then (P, k) can be represented by triples (F, p, n), where $n\colon F \to F$ is a nilpotent endomorphism and $pn = np$, $\mathrm{Im} p = P$. Suppose that F is generated by m elements. Define Y' to be $X \vee (m(S^2))$, the wedge of X with the wedge of m copies of S^2. Notice that $\pi_2(Y', X)$ and F are isomorphic as $\mathbf{Z}\pi_1(X)$-modules, and $\pi_2(Y', X)$ is generated by the classes of the wedged spheres. Then the maps $p, n\colon F \to F$ induce maps $p, n\colon \pi_2(Y', X) \to \pi_2(Y', X)$ and so maps $p, n\colon Y' \to Y'$

extending the identity on X. Notice also that $q_t\colon p^2 \simeq p, pn \simeq np$, and n^s is homotopic rel$X$ to a retraction of Y' to X. Define $Y = D(p)_{[0,+\infty)}/\sim$, where $(x,t) \sim x$ for $x \in X$ and $t \in [0,\infty)$. Then (Y,X) is relatively dominated by (Y',X). The map $d\colon Y \to Y'$ is defined as $d(y',t,m) = q_t(y')$, and $i\colon Y' \to Y$ is defined as $i(y') = (y',0,0)$ [35]. We define $f\colon Y \to Y$ by $f = $ ind. Then f^s is homotopic, rel$X$, to a retraction of Y to X. So (Y,f) is an object in **nil**(X). We want to compute $a(Y,f)$. Notice that $\pi_1(X) \approx \pi_1(Y)$. Also, $C_*(Y,X)$ as $\mathbf{Z}\pi_1(X)$-chain complex is consentrated in dimensions 2 and 3, and $C_2(Y,X) \approx C_3(Y,X) \approx F_\infty$, where F_∞ means the direct sum of infinite copies of F. The boundary map $\partial\colon C_3(Y,X) \to C_2(Y,X)$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & . & . \\ p & -1 & 0 & . \\ 0 & p & -1 & . \\ . & . & . & . \end{pmatrix}$$

There are chain maps

$$d_*\colon C_*(Y,X) \to C_*(Y',X) \quad \text{and} \quad i_*\colon C_*(Y',X) \to C_*(Y,X),$$

induced by the maps d and i, respectively. Then $d_* i_* = p_*$ and $i_* d_* \simeq \text{id}$, by construction. The maps are nonzero only at dimension 2 because $C_*(Y',X) = C_2(Y',X) = F$. The map $i_*\colon F \to F_\infty$ is given by the matrix

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \end{pmatrix}$$

and the map $d_*\colon F_\infty \to F$ is given by $(p\ p\ p \ldots)$. So, $i_* d_*\colon F_\infty \to F_\infty$ is given by the matrix

$$\begin{pmatrix} p & p & \cdots & . \\ 0 & 0 & \cdots & . \\ . & . & \cdots & . \end{pmatrix}$$

We show that $C_*(Y,X)$ is chain homotopy equivalent to a chain complex P_* which is 0 for $* \neq 2$ and $P_2 = P$. The maps d and i actually induce maps $d_*\colon C_*(Y,X) \to P_*$

and $i_*: P_* \to C_*(Y, X)$. Then $i_* d_* \simeq \mathrm{id}$ as before and $d_* i_*: P_* \to P_*$ is given by an element of P has the form $p(x)$ for some x in F then

$$d_* i_*(p(x)) = p(p(x)) = p^2(x) = p(x),$$

and $d_* i_*$ is the identity on P_*. Therefore $C_*(Y, X)$ is a chain homotopy equivalent to P_*. By definition, $a[Y, f] = (P_*, d_* i_* n_* d_* i_*) = (P_*, n_*)$ where $n: P \to P$ is just the map k. This shows that $a[Y, f] = (P, k)$. So a is an epimorphism.

COROLLARY 4.2. *If X is a path-connected space, then a induces epimorphisms*:

$$\tilde{a}: \widetilde{\mathrm{Nil}}(X) \to \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X)) \quad \text{and} \quad a_0: \tilde{K}_0(X) \to \tilde{K}_0(\mathbf{Z}\pi_1(X)).$$

PROPOSITION 4.3. *If X is path-connected. Then $\tilde{a}: \widetilde{\mathrm{Nil}}(X) \to \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X))$ is an isomorphism.*

Proof. It remains to be proved that $\tilde{a}$ is a monomorphism. To accomplish this, we compare the image of $\widetilde{\mathrm{Nil}}(X)$ into $\mathrm{Wh}(X \times S^1)$, with the image of a monomorphism of $\widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X))$ into $\mathrm{Wh}(\pi_1(X \times S^1))$. The map $j'(+): \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X)) \to \mathrm{Wh}(\pi_1(X) \times \mathbf{Z})$ is given by $j'(+)(C, f) = (C[t, t^{-1}], 1 - tf)$ ([37], §9), where C is a finite, finitely generated free $\mathbf{Z}\pi_1(X)$-chain complex and $f: C \to C$ is a chain homotopy nilpotent chain map.

CLAIM. *The following diagram commutes:*

$$\begin{array}{ccc} \widetilde{\mathrm{Nil}}(X) & \xrightarrow{\tilde{a}} & \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X)) \\ {\scriptstyle j(+)}\downarrow & & \downarrow{\scriptstyle j'(+)} \\ \mathrm{Wh}(X \times S^1) & \xrightarrow{e} & \mathrm{Wh}(\pi_1(X) \times \mathbf{Z}) \end{array}$$

where e is the isomorphism e given in [51].

Proof. Let $[Y, f]$ represent an element of $\widetilde{\mathrm{Nil}}(X)$, with (Y, X) a relative finite CW-complex

$$j'(+)\tilde{a}[Y, f] = j'(+)(C_*(\bar{Y}, \tilde{X}), \bar{f}_*) = (C_*(\bar{Y}, \tilde{X})[t, t^{-1}], 1 - t\bar{f}_*),$$

where $\bar{Y}$ is the pull-back of the universal cover of X induced by the retraction determined by f and $\bar{f}$ is the lift of f to $\bar{Y}$, and

$$ej(+)[Y, f] = e(T(f), X \times S^1) = (C_*(T(\bar{f})), \tilde{X} \times S^1)).$$

By [38], the chain complex of the mapping torus of $f: Y \to Y$ is given by $\hat{C}(1 - t\bar{f}_*[t, t^{-1}])$, where $\hat{C}(1 - t\bar{f}_*[t, t^{-1}])$ is the modified mapping cone of $1 - t\bar{f}_*[t, t^{-1}]$. Then

$$e(T(f), X \times S^1) = (\hat{C}(1 - t\bar{f}_*[t, t^{-1}])).$$

But

$$(\hat{C}(1 - t\bar{f}_*[t, t^{-1}])) = (C(1 - t\bar{f}_*[t, t^{-1}])) = (C_*(\bar{Y}, \tilde{X}), 1 - t\bar{f}_*))$$
$$= j'(+)(C_*(\bar{Y}, \tilde{X}), \bar{f}_*).$$

Since $j'(+)$, $j(+)$ are monomorphisms and e is an isomorphism, we conclude that $\tilde{a}$ is a monomorphism and, therefore, an isomorphism.

In [39] one more injection, $j'(-)\colon \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X)) \to \mathrm{Wh}(\pi_1(X) \times \mathbf{Z})$, is defined by $j'(-)(C, f) = (C[t, t^{-1}], 1 - t^{-1} f)$, where C is a finite, finitely generated free $\mathbf{Z}\pi_1(X)$-chain complex and $f\colon C \to C$ is a chain homotopy nilpotent chain map. We compare $j'(-)$ with the geometrically defined injection $j(-)\colon \widetilde{\mathrm{Nil}}(X) \to \mathrm{Wh}(X \times S^1)$.

COROLLARY 4.4. *The following diagram commutes*:

$$\begin{array}{ccc} \widetilde{\mathrm{Nil}}(X) & \xrightarrow{\ \tilde{a}\ } & \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X)) \\ {\scriptstyle j(-)}\downarrow & & \downarrow{\scriptstyle j'(-)} \\ \mathrm{Wh}(X \times S^1) & \xrightarrow{\ e\ } & \mathrm{Wh}(\pi_1(X) \times \mathbf{Z}) \end{array}$$

(*The notation is the same as in the proof of Proposition* 4.1.)

Proof. Let $[Y, f]$ represent an element of $\widetilde{\mathrm{Nil}}(X)$, and $h\colon T(f) \to X \times S^1$ is a strong deformation retract. Then $ej(-)[Y, f] = [(\mathrm{id}_X \times \mu)h]$, where $\mu\colon S^1 \to S^1$ is given by $\mu(s) = 1 - s$. By the sum formula for the Whitehead torsion, we get

$$[(\mathrm{id}_X \times \mu)h] = (\mathrm{id}_X \times \mu)_*[h] + [(\mathrm{id}_X \times \mu)].$$

But $[(\mathrm{id}_X \times \mu)]$ is a homeomorphism, so $[(\mathrm{id}_X \times \mu)] = 0$ in $\mathrm{Wh}(X \times S^1)$. Therefore,

$$ej(-)[Y, f] = (\mathrm{id}_X \times \mu)_*[h] = (\mathrm{id}_X \times \mu)_*[1 - t\bar{f}_*],$$

where $\bar{f}_*\colon C_*(\bar{Y}, \tilde{X}) \to C_*(\bar{Y}, \tilde{X})$ is as in proof of Proposition 4.1. But μ induces the map $\mu_\#\colon \mathbf{Z} = \langle t \rangle \to \mathbf{Z} = \langle t \rangle$, $\mu_\#(t) = t^{-1}$. Therefore $(\mathrm{id}_X \times \mu)_*[1 - t\bar{f}_*] = [1 - t^{-1}\bar{f}_*]$, where

$$1 - t^{-1}\bar{f}_*\colon C_*(\bar{Y}, \tilde{X})[t, t^{-1}] \to C_*(\bar{Y}, \tilde{X})[t, t^{-1}].$$

On the other hand,

$$j'(-)\tilde{a}[Y, f] = j(-)(C_*(\bar{Y}, \tilde{X}), \bar{f}_*) = (C_*(\bar{Y}, \tilde{X})[t, t^{-1}], 1 - t\bar{f}_*.$$

Therefore, $ej(-) = j'(-)\,\tilde{a}$.

COROLLARY 4.5. *Assume that X is a path connected such that $\pi_1(X)$ is finitely presented. Then the map $a_0\colon \tilde{K}_0(X) \to \tilde{K}_0(\mathbf{Z}\pi_1(X))$ is an isomorphism.*

Proof. We must prove that a_0 is a monomorphism. Let $[Y, r]$ represent an element of $\tilde{K}_0(X)$. Since X is path connected and (Y, X) is relatively finitely dominated Y has finitely many path components. By adding finite many 1-cells to Y, we obtain a space Y' which is connected and a retraction $r'Y' \to X$. By Lemma 2.12, $[Y, r] = [Y', r']$ in $\tilde{K}_0(X)$. So we can assume that an element of $\tilde{K}_0(X)$ can be represented by a pair $[Y, r]$, were Y is path connected. The next step is to improve the connectivity of the map r without changing the class of $[Y, r]$ in $\tilde{K}_0(X)$.

CLAIM. *There is an object (Y', r') of $\mathbf{K}_0(X)$ such that*

(i) *$[Y, r] = [Y', r']$ in $\tilde{K}_0(X)$,*
(ii) *r' induces an isomorphism on the fundamental groups.*

Proof. Since r is a retraction, it induces an epimorphism on the fundamental groups. Let $r_*: \pi_1(Y) \to \pi_1(X)$ be the map induced by r on the fundamental groups. Since $\pi_1(X)$ is finitely presented and (Y, X) is relatively finitely dominated, $\pi_1(Y)$ is finitely presented. Then $\mathrm{Ker}(r_*)$ is finitely generated as a normal subgroup of $\pi_1(Y)$ [34], Lemma 14.8, [43]. By attaching finitely many cells on Y, we replace (Y, r) by an object (Y', r') such that $[Y, r] = [Y', r']$ in $\tilde{K}_0(X)$ and r' induces a monomorphism on the fundamental groups. So r' induces an isomorphism on the fundamental groups.

Let $[Y, r]$ is an element in the kernel of a_0. By the claim, we can assume that r induces an isomorphism on the fundamental groups. Then the fact $a_0[Y, r] = 0$ means that (Y, X) is homotopy equivalent to a relative finite complex (K, X). In particular, $[Y, f] = 0$ in $\tilde{K}_0(X)$. Therefore, a_0 is a monomorphism.

So we have proved

PROPOSITION 4.6. *If X is path connected and $\pi_1(X)$ is finitely presented, then $a: \mathrm{Nil}(X) \to \mathrm{Nil}(\mathbf{Z}\pi_1(X))$ is an isomorphism.*

Assume that X is a space with finitely many path components $\{X_i\}_{i \in I}$, where I is a finite index set. We define a map

$$d: \mathrm{Nil}(X) \to \bigoplus_{i \in I} \mathrm{Nil}(X_i).$$

Let $[Y, f]$ represent an element in $\mathrm{Nil}(X)$. Since $T(f)$ strong deformation retracts to $X \times S^1$, $T(f)$ has the same number of path components with $X \times S^1$. Also, any component of $T(f)$ has the form $T(f')$ where f' is the restriction of f to a subset Y' of Y. Let $T(f_i)$, $f_i: Y_i \to Y_i$, be the component of $T(f)$ corresponding to $X_i \times S^1$, for each $i \in I$. Define

$$d[Y, f] = ([Y_i, f_i])_{i \in I}.$$

This is obviously a well-defined group homomorphism.

We construct the inverse of d. Denote by $\varphi_i: X_i \to X$ the inclusion map. Define

$$d': \bigoplus_{i \in I} \mathrm{Nil}(X_i) \to \mathrm{Nil}(X)$$

by

$$d'([Y_i, f_i])_{i \in I} = \sum_{i \in I} (\varphi_i)_*[Y_i, f_i].$$

So, we have proved the following.

LEMMA 4.7. *Let X be as above. The map $d: \mathrm{Nil}(X) \to \oplus_{i \in I} \mathrm{Nil}(X_i)$ defined above is an isomorphism.*

COROLLARY 4.8. *Let X be as before. Then, the restriction of d induces isomorphisms*

$$\tilde{d}: \widetilde{\mathrm{Nil}}(X) \to \bigoplus_{i \in I} \widetilde{\mathrm{Nil}}(X_i)$$

$$d_0: \tilde{K}_0(X) \to \bigoplus_{i \in I} \tilde{K}_0(X_i).$$

By combining Proposition 4.6 and Lemma 4.7, we get the following proposition.

PROPOSITION 4.9. *Let X be a space with finitely many path components $\{X_i\}_{i\in I}$, where I is a finite index set. Furthermore, assume that $\pi_1(X_i)$ is a finitely presented group for each $i \in I$. Then, there is an isomorphism*

$$\eta\colon \mathrm{Nil}(X) \to \bigoplus_{i\in I} \mathrm{Nil}(\mathbf{Z}\pi_1(X_i)).$$

Similarly, there are isomorphisms

$$\tilde{\eta}\colon \widetilde{\mathrm{Nil}}(X) \to \bigoplus_{i\in I} \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(X_i)),$$

$$\eta_0\colon \tilde{K}_0(X) \to \bigoplus_{i\in I} \tilde{K}_0(\mathbf{Z}\pi_1(X_i)).$$

5. Equivariant Nil-Groups

In the equivariant case, the geometric Nil-groups split as a direct sum with one summand for each conjugacy class of subgroups of G and each component of the fixed-point set. This decomposition is consistent with the decomposition of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ given in [34] and [27].

Let X be a G-space. For any group H, let EH denote a contractible free H-complex (EH is unique up to H-homotopy equivalence). For each subgroup H of G, WH acts on $E\mathrm{WH} \times X^H$ diagonally. Let $E\mathrm{WH} \times_{\mathrm{WH}} X^H$ be the orbit space of this action. We want to compare the groups $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ and $\oplus_{(H)} \mathrm{Nil}(E\mathrm{WH} \times_{\mathrm{WH}} X^H)$, where H ranges over the conjugacy classes of subgroups of G.

Notation. Let Ω be a collection of subgroups of G. Ω is called a family if $H \in \Omega$, then all the conjugates of H belong to Ω. If H is a subgroup of G, write $[H]$ for the family consisting of all the conjugates of H.

Let Ω be a family of subgroups of G. Write $\mathrm{Nil}_G^{\mathrm{PL}}[\Omega](X)$ for the subgroup of $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ consisting of pairs $[Y, f]$, where Y is obtained by attaching cells of types $\{H_i\}_{i\in I}$, where $H_i \in \Omega$ for each $i \in I$ (I is an index set).

Let $\mathrm{Con}(G)$ denote the set of conjugacy classes of subgroups of G. Then $\mathrm{Con}(G)$ is a partially ordered set: $(H) \leqslant (K)$ if there is $g \in G$ such that $gHg^{-1} \subset K$.

The argument will follow the lines of the argument given in [24].

STEP 1. *The map*

$$Z\colon \bigoplus_{H\in \mathrm{Con}(G)} \mathrm{Nil}_G^{\mathrm{PL}}[H](X) \to \mathrm{Nil}_G^{\mathrm{PL}}(X)$$

defined by

$$Z([Y(H), f(H)]_{H\in \mathrm{Con}(G)}) = \sum_{H\in \mathrm{Con}(G)} [Y(H), f(H)]$$

is an isomorphism.

Proof. It is obvious that Z is a group homomorphism.

(i) Z is injective: Assume that $([Y(H), f(H)]_{H\in \mathrm{Con}(G)})$ belongs to the kernel to Z. So, $[Y, f] = \sum_{H\in \mathrm{Con}(G)}[Y(H), f(H)] = 0$ in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$. Let H_i be maximal in $\mathrm{Con}(G)$ so that $[Y(H_i), f(H_i)] \neq 0$ in $\mathrm{Nil}_G^{\mathrm{PL}}[H_i](X)$. Then $[Y^{(H_i)}, f^{(H_i)}] = 0$ in $\mathrm{Nil}_G^{\mathrm{PL}}[H_i](X^{(H_i)})$. Also, if $j: X^{(H_i)} \to X$ is the inclusion map, $[Y(H_i), f(H_i)] = j*[Y^{(H_i)}, f^{(H_i)}] = 0$. Therefore $\ker(Z) = 0$ and Z is an injection.

(ii) Z is surjective: Let $[Y, f] = y$ represent an element in $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ and let H_i be maximal in $\mathrm{Con}(G)$ such that $Y - X$ contains type H_i-cells. Then, with the above notation, $a(i) = j_*[Y^{(H_i)}, f^{(H_i)}] \in \mathrm{Nil}_G^{\mathrm{PL}}[H_i](X)$. Then the element $y - a(i)$ can be represented by an element $[Y', f']$ so that Y' is obtained from X by attaching cells of a type smaller than H_i or which are not compatible with H_i. Repeating this process, we get that the sum of the elements is of the form $a(k)$. So Z is surjective.

STEP 2. *The map*

$$BH: \mathrm{Nil}_G^{\mathrm{PL}}[H](X) \to \mathrm{Nil}_{\mathrm{WH}}^{\mathrm{PL}}[e](X^H)$$

given by $BH[Y, f] = [Y^H, f^H]$ *is an isomorphism.*

Proof. We define the inverse of BH, $B'H: \mathrm{Nil}_{\mathrm{WH}}^{\mathrm{PL}}[e](X^H) \to \mathrm{Nil}_G^{\mathrm{PL}}[H](X)$. Notice that the inclusion $j: X^H \to X$ induces a map $j*: \mathrm{Nil}_{\mathrm{WH}}^{\mathrm{PL}}[e](X^H) \to \mathrm{Nil}_{\mathrm{NH}}^{\mathrm{PL}}[H](X)$. Let $[Y, f]$ represent an element of $\mathrm{Nil}_{\mathrm{WH}}^{\mathrm{PL}}[e](X^H)$. Set $[Y', f'] = j*[Y, f]$. Y' is obtained from X by attaching NH-cells of type H. The attaching maps of these cells induce attaching maps for G-cells of type H. Let Y'' be the space obtained from X by attaching these cells. In a similar way, f' extends to a G-map $f'': Y'' \to Y''$. Notice that $T(f'')$ is G-homotopy equivalent to $X \times S^1$ since $T(f')$ is ([24], Satz III.3). So, $[Y'', f'']$ represents an element in $\mathrm{Nil}_G^{\mathrm{PL}}[H](X)$. Define $B'H[Y, f] = [Y'', f'']$. $B'H$ is the inverse of BH.

STEP 3. *We construct a map* FG: $\mathrm{Nil}_G^{\mathrm{PL}}[e](X) \to \mathrm{Nil}_G^{\mathrm{PL}}(EG \times X)$. *Let* $[Y, f]$ *represent an element in* $\mathrm{Nil}_G^{\mathrm{PL}}[H](X)$. *Then* (Y, X) *is a relative G-complex which is relative free. In* [26], *a construction is given of a relative G-complex* $(Y', EG \times X)$ *so that* Y' *is G-homotopy equivalent* $\mathrm{rel}(EG \times X)$ *to* $EG \times Y$ *and the number of equivariant n-cells of* (Y, X) *(this number can be* ∞*) is equal to the number of equivariant n-cells of* $(Y', EG \times X)$. *Let* $f': Y' \to Y'$ *be the map given by the composition:*

$$Y' \xrightarrow{a} EG \times Y \xrightarrow{\mathrm{id} \times f} EG \times Y \xrightarrow{b} Y',$$

where a and b are G-homotopy inverses. Then $[Y', f']$ *represents an element in* $\mathrm{Nil}_G^{\mathrm{PL}}(EG \times X)$. *The map FG defined above is an isomorphism whose inverse is induced by the projection* $p: EG \times X \to X$.

STEP 4. *Let X be a free G-space, then the map* $c: \mathrm{Nil}_G^{\mathrm{PL}}(X) \to \mathrm{Nil}(X/G)$ *by* $c[Y, f] = [Y/G, f/G]$ *is an isomorphism.*

Proof. The inverse of c is given by pull back. Let $[Z, k]$ represent an element of $\mathrm{Nil}(X/G)$. Then some power of k is homotopic rel$X$ to a retraction $r: Z \to X/G$. Define Y by the pull-back

$$\begin{array}{ccc} Y & \xrightarrow{r'} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{r} & X/G \end{array}$$

Also k can be lifted to a G-map f: Y. Then (Y, f) is an object of $\mathbf{nil}_G(X)$. Define $c': \mathrm{Nil}(X/G) \to \mathrm{Nil}_G^{\mathrm{PL}}(X)$ by $c'[Z, k] = [Y, f]$. This is a well-defined group homomorphism which is a two-sided inverse of c. So c is an isomorphism.

PROPOSITION 5.1. *There is an isomorphism*

$$\mathrm{Nil}_G^{\mathrm{PL}}(X) \to \bigoplus_{(H)} \mathrm{Nil}(E\,\mathrm{WH} \times_{\mathrm{WH}} X^H)$$

and it restricts to isomorphisms

$$\widetilde{\mathrm{Nil}}(X) \to \bigoplus_{(H)} \widetilde{\mathrm{Nil}}(E\,\mathrm{WH} \times_{\mathrm{WH}} X^H), \quad \tilde{K}_{0G}^{\mathrm{PL}}(X) \to \bigoplus_{(H)} \tilde{K}_0(E\,\mathrm{WH} \times_{\mathrm{WH}} X^H),$$

where (H) ranges over the conjugacy classes of subgroups of G.

Combining Propositions 4.9 and 5.1 we get

PROPOSITION 5.2. *Let X be a G-space so that $\pi_1(C)$ is finitely presented for each path component of X^H and for each subgroup H of G. Then, there are isomorphisms*

(i) $e: \mathrm{Nil}_G(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \mathrm{Nil}(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C))$,

(ii) $\tilde{e}: \widetilde{\mathrm{Nil}}_G(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C))$,

(iii) $e': \tilde{K}_{0G}^{\mathrm{PL}}(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \tilde{K}_0(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C))$,

where (H) ranges over the conjugacy classes of subgroups of G and WH(C) *is the subgroup of* WH *which fixes the component C of X^H.*

Proof. In [1], it is shown that $\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)$ is a finite extension of $\pi_1(C)$. So $\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)$ is finitely presented and we can apply Propositions 4.9 and 5.1. Notice that the hypothesis of Proposition 5.2 is satisfied when X is a compact G-ANR.

There is a similar isomorphism given in [24, 27, 34] for any G-space X:

$$\alpha: \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \mathrm{Wh}(\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C) \times \mathbf{Z}).$$

In [37], for any group π, a split injection $j': \tilde{K}_0(\mathbf{Z}\pi) \to \mathrm{Wh}(\pi \times \mathbf{Z})$ is defined called the geometric injection. By combining the map j' with the isomorphisms given above, we obtain a split injection (X is as in Proposition 5.2):

$$j: \tilde{K}_{0G}^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1).$$

Geometrically, the injection j is given in [21, 12, 30, 2, 31, 33, 34, 40]. We repeat it here for the reader's convenience.

CONSTRUCTION 5.3.1. Let (Y, r) be an object of $\mathbf{K}_{0G}(X)$. Then, Y is G-finitely dominated rel$X$. So there is a G-pair (K, X) and G-maps $Y \xrightarrow{d} K \xrightarrow{t} Y$ such that

(i) (K, X) is a finite relative G-complex,
(ii) $td \simeq_G \mathrm{id}_Y$, rel$X$.

Then, the map $dt\colon K \to K$ satisfies $T(dt) \simeq_G Y \times S^1$ rel$(X \times S^1)$ (by 1.9, 1.11, and [34], §7, [31, 2, 40]). Define a G-homotopy equivalence $w\colon T(dt) \to T(dt)$ as the composition $v(\mathrm{id}_Y \times \mu)u$, where $u\colon T(e) \to Y \times S^1$ is a G-homotopy equivalence v is a G-homotopy inverse, and $\mu\colon S^1 \to S^1$ is given by $\mu(s) = 1 - s$. Define $j(Y, r) = (r \times \mathrm{id})_* u_* \tau(w)$, where r is a retraction such that f is G-homotopic to ιr, ι the inclusion of X into Y. Notice that $u * \tau(w)$ is just the relative finiteness obstruction of (Y, X), i.e. $j(Y, r) = (r \times \mathrm{id})_* \sigma_G(Y, X)$. The proof that the map j corresponds to the algebraically defined map is given in [37, 38]. This map is a group monomorphism onto the elements of $\mathrm{Wh}_G^{PL}(X \times S^1)$ invariant under the transfer induced by double cover of S^1. ([12], §7 in the nonequivariant case, [34], Proposition 10.52 in the equivariant case).

The splitting of j geometrically can be described as follows: Define a map $s\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \tilde{K}_{0G}^{\mathrm{PL}}(X)$ as the composition $\rho\omega(+)$. In [2], the proof is given that this corresponds to the algebraically defined splitting of j. Therefore $\tilde{K}_{0G}^{\mathrm{PL}}(X)$ is a direct summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

LEMMA 5.3. *$\tilde{K}_{0G}^{\mathrm{PL}}(X)$ is isomorphic to a summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ which is orthogonal to the two summands isomorphic to $\widetilde{\mathrm{Nil}}_G(X)$.*

Proof. Lemma 5.3 follows from the fact that the splitting of $\widetilde{\mathrm{Nil}}_G(X)$ and $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ is natural with respect to the injections $j(\pm)$ and the projections $q(\pm)$.

In summary, we state the following theorem.

THEOREM 1. *Let X be a G-space so that $\pi_1(C)$ is finitely presented for each path component of X^H and for each subgroup H of G. Then there are isomorphisms*

(i) $\tilde{e}\colon \widetilde{\mathrm{Nil}}_G(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C))$,

(ii) $e'\colon \tilde{K}_{0G}^{\mathrm{PL}}(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \tilde{K}_0(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C))$,

such that

$$\alpha(j(-) \oplus j(+) \oplus j) = (\tilde{e} \oplus \tilde{e} \oplus e') \times \left[\bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} (j'(-)(H, C) \oplus j'(+)(H, C) \oplus j'(H, C)) \right]$$

where α *is the inverse of the isomorphism given in* [34], *and*

$$j'(\pm)(H, C)\colon \widetilde{\mathrm{Nil}}(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)) \to \mathrm{Wh}(\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C) \times \mathbf{Z}),$$
$$j'(H, C)\colon \widetilde{K}_0(\mathbf{Z}\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)) \to \mathrm{Wh}(\pi_1(E\,\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C) \times \mathbf{Z}),$$

are the injections defined in [39] *for the geometric splitting of* $\mathrm{Wh}(- \times \mathbf{Z})$.

6. The Bass–Heller–Swan Formula for $\mathrm{Wh}_G^{\mathrm{PL}}$

In this section we compare the equivariant K-groups defined in the previous two sections and the maps between them with the classical equivariant K-groups and the maps which prove the Bass–Heller–Swan formula for the PL-equivariant Whitehead group. We do not use the equivariant analogues of the maps given by Bass [4] but rather the equivariant analogues of Ranicki [39], §10. In this section, X will always be a compact G-ANR. The formula we want to prove is:

THEOREM 2. *If* X *is a compact* G-ANR, *and* G *acts trivially on* S^1, *then the map*

$$(i_*, j, j(+), j(-))\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus \widetilde{K}_{0G}^{\mathrm{PL}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$$

is an isomorphism, where i_* *is the map induced by the inclusion* $i\colon X \to X \times S^1$.

In Section 3, the proof that $\widetilde{K}_{0G}^{\mathrm{PL}}(X)$ and $\widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X)$ inject to direct summands of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ is given. It remains to be studied how $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ injects as a direct summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ whose intersection with the other summands is $\{0\}$.

There is a natural map $i_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$, induced by the inclusion $i\colon X \to X \times S^1$, $i(x) = (x, 0)$.

LEMMA 6.1. *The map* $q(-)i_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \widetilde{\mathrm{Nil}}_G(X)$ *is the zero map.*

Proof. Let (K, X) be a finite relative G-complex and $f\colon K \to X$ be a strong G-deformation retraction from K to X, representing an element of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$. Then $i_*(K, X)$ is given by

$$f \cup \mathrm{id}\colon K' = K \cup_X (X \times S^1) \to X \times S^1$$

The infinite cyclic cover of $K \cup_X (X \times S^1)$ is $K'' = (K \times \mathbf{Z}) \cup_{X \times \mathbf{Z}} (X \times \mathbf{R})$, and $f \cup \mathrm{id}$ lifts to a proper G-deformation retraction, $f' = (f \times \mathrm{id}_{\mathbf{Z}}) \cup \mathrm{id}_{X \times \mathbf{R}}\colon K'' \to X \times \mathbf{R}$. A neighborhood of the negative end of K'' is $L = (K \times \mathbf{Z}^-) \cup_{X \times \mathbf{Z}^-} (X \times (-\infty, 0])$, where $\mathbf{Z}^-$ is the set of nonpositive integers. Then, $K'(-) = L/\!\sim$ where $(x, n) = (x, 0)$ for each $(x, n) \in X \times \mathbf{Z}^-$, and the map $f \times \mathrm{id}_{\mathbf{Z}^-}\colon K'(-) \to X$, is a strong G-deformation retraction. Notice that $q(-)i_*(K, X) = p([K'(-), f'(-)])$, where $f'(-)$ is induced by the translation and so $[K'(-), f'(-)] = 0$ in $\mathrm{Nil}_G(X)$ since $K'(-)$ is G-homotopy equivalent to a finite G-complex, namely X, and $f'(-)$ is G-homotopic to a retraction to X, since $K'(-)$ strong G-deformation retracts to X. So $q(-)i_* = 0$.

Similarly, there is the following result.

COROLLARY 6.2. *The maps* $q(+)i_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \widetilde{\mathrm{Nil}}_G(X)$ *and* $si_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \tilde{K}_{0G}^{\mathrm{PL}}(X)$ *are the zero maps.*

We must prove that $i_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ is a split monomorphism so that $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ is a direct summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$.

LEMMA 6.3. $i_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ *is a split monomorphism and* $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ *is isomorphic to a direct summand of* $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ *whose intersection with the summand isomorphic to* $\tilde{K}_{0G}^{\mathrm{PL}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X)$ *is* $\{0\}$.

Proof. Define a map

$$k = p_*(1 - js - j(-)q(-) - j(+)q(+))\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \mathrm{Wh}_G^{\mathrm{PL}}(X),$$

where $p_*\colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \mathrm{Wh}_G^{\mathrm{PL}}(X)$ is the natural map induced by the projection $p\colon X \times S^1 \to X$. Then k is the left inverse of i_*.

If π is a group, $i'_*\colon \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi \times \mathbf{Z})$ is the map induced by the natural group inclusion $i'\colon \pi \to \pi \times \mathbf{Z}$.

LEMMA 6.4. *The following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{Wh}_G^{\mathrm{PL}}(X) & \xrightarrow{\varepsilon'} & \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \mathrm{Wh}(\pi_1(E\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)) \\
\Big\downarrow i_* & & \Big\downarrow \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} i'_*(H, C). \\
\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) & \xrightarrow{\alpha'} & \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \mathrm{Wh}(\pi_1(E\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C) \times \mathbf{Z}),
\end{array}$$

where ε' *and* α' *are the isomorphisms given in* [34].

Now we have all the machinery to prove Theorem 2.

Proof of Theorem 2. The homomorphism

$$i_* \oplus j \oplus j(-) \oplus j(+)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$$

is such that

$$\alpha'(i_* \oplus j \oplus j(-) \oplus j(+)) = (\varepsilon' \oplus e(-) \oplus e(+) \oplus e')\left[\bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} (i'_*(H, C) \oplus j'(H, C) \oplus j'(-)(H, C) \oplus j'(+)(H, C))\right].$$

But the fundamental theorem of algebraic K-theory ([5]) asserts that for each H and for each C, $i'_*(H, C) \oplus j'(H, C) \oplus j'(-)(H, C) \oplus j'(+)(H, C))$ is an isomorphism. By

Theorem 1 and [34], the maps $\alpha', \varepsilon' \oplus e(-) \oplus e(+) \oplus e'$ are isomorphisms. So the map $i_* \oplus j \oplus j(-) \oplus j(+)$ is an isomorphism, with inverse given by

$$(k, s, q(-), q(+)) \colon \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \to \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X) \oplus \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X).$$

In [8, 46, 41] a Whitehead group is defined for a compact G-ANR X using isovariance whenever equivariance occurs (a G-map between G-spaces is called isovariant if it preserves the isotropy groups). The resulting Whitehead group is denoted by $\mathrm{Wh}_G^{\mathrm{PL,ISO}}(X)$. Then, the splitting of $\mathrm{Wh}_G^{\mathrm{PL,ISO}}(X)$ is given by

$$\gamma \colon \mathrm{Wh}_G^{\mathrm{PL,ISO}}(X) \to \bigoplus_{(H)} \bigoplus_{C \in \pi_0(X^H/\mathrm{WH})} \mathrm{Wh}(\pi_1(E\mathrm{WH}(C) \times_{\mathrm{WH}(C)} C)),$$

where H varies over the conjugacy classes of the isotropy subgroups of G.

The same observations apply for $\mathrm{Nil}_G^{\mathrm{PL}}(X)$. By replacing the equivariant maps in the definition of $\mathrm{Nil}_G^{\mathrm{PL}}(X)$ by isovariant and we get $\mathrm{Nil}_G^{\mathrm{PL,ISO}}(X)$. The isomorphisms (i), (ii), (iii) in Theorem 1 remain valid for the ISO-groups if H ranges over the conjugacy classes of the isotropy subgroups of G. In particular, we get:

LEMMA 6.5. *If X is a compact G*-ANR, *and G acts trivially on S^1, then*

$$\mathrm{Wh}_G^{\mathrm{PL,ISO}}(X \times S^1) \approx \mathrm{Wh}_G^{\mathrm{PL,ISO}}(X) \oplus \tilde{K}_{0G}^{\mathrm{PL,ISO}}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\mathrm{ISO}}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\mathrm{ISO}}(X).$$

In what follows, by a G-manifold we mean a locally linear G-manifold, except if otherwise stated. Let M be a G-manifold. A G-h-cobordism, $(W; M, M')$, is a G-manifold, W with boundary $\partial W = M \amalg M'$, the disjoint union of M and M', such that the inclusion maps $i \colon M \to W$, and $i' \colon M'' \to W$ are proper equivariant homotopy equivalences.

DEFINITION 6.6. A G-manifolds has codimension $\geqslant 3$ gaps if each inclusion $M_\beta^K \to M_\alpha^H$, of components of fixed-point sets of M under $G \supset K \supset H$ is either the identity or has codimension at least 3.

For a finite G-complex X define $\mathrm{Wh}_G^{\mathrm{PL},\rho}(X)$ to be the subgroup of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ generated by pairs (Y, X) such that $Y_\alpha^H - Y_\alpha^{>H} = \emptyset$, whenever $X_\alpha^H - X_\alpha^{>H} = \emptyset$. It turns out [46] that $\mathrm{Wh}_G^{\mathrm{PL},\rho}(M)$ is a summand of $\mathrm{Wh}_G^{\mathrm{PL}}(M)$. Also, if M is a compact G-manifold with codimension $\geqslant 3$ gaps, $\mathrm{Wh}_G^{\mathrm{PL},\rho}(M)$ is isomorphic to $\mathrm{Wh}_G^{\mathrm{PL,ISO}}(M)$.

COROLLARY 6.7. *If M is a compact G-manifold with codimension $\geqslant 3$ gaps and G acts trivially on S^1, then*

$$\mathrm{Wh}_G^{\mathrm{PL},\rho}(M \times S^1) \approx \mathrm{Wh}_G^{\mathrm{PL},\rho}(M) \oplus \tilde{K}_{0G}^{\mathrm{PL},\rho}(M) \oplus \widetilde{\mathrm{Nil}}_G^{\rho}(M) \oplus \widetilde{\mathrm{Nil}}_G^{\rho}(M).$$

7. The Bass–Heller–Swan Formula for $\mathrm{Wh}_G^{\mathrm{Top}}$

In this section, we complete the proof of the Bass–Heller–Swan formula for the equivariant topological Whitehead group.

If X is a locally compact G-ANR, the topological equivariant Whitehead group, $\mathrm{Wh}_G^{\mathrm{Top}}(X)$, of X was defined by M. Steinberger and J. West [47, 46] as follows:

Let $D(X)$ be the set of all pairs (Y, X) where Y is a locally compact G-ANR and X is a proper strong G-deformation retract of Y. An equivalence relation is defined on $D(X)$: $(Y, X) \sim (Y', X)$ if there is a locally compact G-ANR, Z, and G-CE maps $r: Z \to Y$, $r': Z \to Y'$ such that $fr \simeq_G f'r'$ rel$X$ where $f: Y \to X$ and $f': Y' \to X$ are strong G-deformation retractions. The elements of $\mathrm{Wh}_G^{\mathrm{Top}}(X)$ are equivalence classes of objects of $D(X)$. The operation on $\mathrm{Wh}_G^{\mathrm{Top}}(X)$ is induced by push-outs.

In [46], the connection is given of the topological and the PL Whitehead group of a locally compact G-ANR X which is G-dominated by a finite-dimensional G-complex in a 5-term exact sequence

$$\mathrm{Wh}_G^{\mathrm{PL}}(X)_c \xrightarrow{f} \mathrm{Wh}_G^{\mathrm{PL}}(X) \xrightarrow{\phi} \mathrm{Wh}_G^{\mathrm{Top}}(X) \xrightarrow{\nu} \tilde{K}_{0G}^{\mathrm{PL}}(X)_c \xrightarrow{\psi} \tilde{K}_{0G}^{\mathrm{PL}}(X). \tag{1}$$

The group $\mathrm{Wh}_G^{\mathrm{PL}}(X)_c$ is the controlled Whitehead group [49]. The group $\tilde{K}_{0G}^{\mathrm{PL}}(X)_c$ is the subgroup of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c$ consisting of elements which are invariant under the transfer map induced by the finite covers of S^1. The maps f and ψ are 'forget control' maps, the map ϕ is 'forget cell-structure'. If (Y, X) represents an element of $\mathrm{Wh}_G^{\mathrm{Top}}(X)$, then $\nu(Y, X)$ is defined to be the controlled relative propriety obstruction of the pair (Y, X), i.e. the obstruction that (Y, X) is a properly controlled G-homotopy equivalent rel$X$ to a relative G-CW pair [46].

In [46], Chapter 10, Bass–Heller–Swan formulas are given for the controlled groups: If X is a finite G-complex and G acts trivially on S^1 then

$$\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c \approx \mathrm{Wh}_G^{\mathrm{PL}}(X)_c \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X)_c, \tag{2}$$

$$\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c \approx \tilde{K}_{0G}^{\mathrm{PL}}(X)_c \oplus K_{-1G}^{\mathrm{PL}}(X)_c, \tag{3}$$

where $K_{-1G}^{\mathrm{PL}}(X)_c$ is the subgroup of the transfer invariant elements of $\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$. It is observed in [46] that the above isomorphisms are natural with respect to 'forget control'. This observation together with the equivariant version of the work of L. C. Siebenmann [42] gives an exact sequence for any compact G-ANR X:

$$\tilde{K}_{0G}^{\mathrm{PL}}(X)_c \xrightarrow{f'} \tilde{K}_{0G}^{\mathrm{PL}}(X) \xrightarrow{\phi'} \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R}) \xrightarrow{\nu'} K_{-1G}^{\mathrm{PL}}(X)_c \xrightarrow{\psi'} K_{-1G}^{\mathrm{PL}}(X). \tag{4}$$

From now on, X will denote a compact G-ANR. Using the information provided by the above exact sequences, we study the direct sum decomposition of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$. First of all, notice that there is an exact sequence

$$\begin{aligned}&\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c \xrightarrow{f''} \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) \xrightarrow{\phi''} \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)\\ &\xrightarrow{\nu''} \tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c \xrightarrow{\psi''} \tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1).\end{aligned} \tag{5}$$

We start with the summands corresponding to the Nil-groups. Let $j(\pm)$: $\widetilde{\mathrm{Nil}}_G(X) \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ be the injections constructed in Section 2.

LEMMA 7.1. $\mathrm{Im}(j(\pm)) \cap \mathrm{Im}(f'') = \{0\}$.

Proof. We prove the lemma for $j(+)$. The proof for $j(-)$ is similar. From (2), $\mathrm{Im}(f'') \subset \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X)$. Alsp $\mathrm{Im}(j(+)) \cap (\mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus \tilde{K}_{0G}^{\mathrm{PL}}(X)) = \{0\}$.

COROLLARY 7.2. *The restriction of* ϕ'' *to* $\mathrm{Im}(j(+)) \oplus \mathrm{Im}(j(-))$ *is a monomorphism.*

Proof. This follows from Lemma 7.1 and the exact sequence (5).

Corollary 7.2 states that the map $\phi''(j(+)\oplus j(-))\colon \widetilde{\mathrm{Nil}}_G(X)\oplus\widetilde{\mathrm{Nil}}_G(X)\to \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ is injective.

Notice that there is a natural map $\imath\colon \mathrm{Wh}_G^{\mathrm{Top}}(X)\to \mathrm{Wh}_G^{\mathrm{Top}}(X\times S^1)$ induced by the inclusion $i\colon X\to X\times S^1$. This map is injective (the projection $X\times S^1\to X$ induces a left inverse for $\imath$).

Let $\mathrm{tr}(n)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X\times S^1)\to \mathrm{Wh}_G^{\mathrm{PL}}(X\times S^1)$ be the transfer map induced by the nth cover of S^1 (the cover corresponding to the subgroup $n\mathbf{Z}$ of $\mathbf{Z}$).

LEMMA 7.3. *If* $x\in \mathrm{Wh}_G^{\mathrm{PL}}(X)$, *then* $\mathrm{tr}(n)i_*(x) = ni_*(x)$.

Proof. We will prove the lemma for $n = 2$. The general case follows similarly. Let x be represented by a strong G-deformation retraction $k\colon K\to X$ where K is a finite G-complex. Then $i_*(x)$ is represented by the strong G-deformation retraction $k' = k\cup \mathrm{id}\colon K' = K\cup_X (X\times S^1)\to X\times S^1$. The double transfer of k' is $k''\colon K''\to X\times S^1$ where $K'' = (K\cup_X(X\times S^1))\cup_X K$ and identify x with $(x,0)$ in the first union and x with $(x,\frac{1}{2})$ in the second union for all $x\in X$. The map $k'' = k\cup\mathrm{id}\cup k$. It is obvious that

$$(K'', X\times S^1) = (K', X\times S^1) + (K', X\times S^1) = 2i_*(x).$$

LEMMA 7.4. *Let* $x\in\widetilde{\mathrm{Nil}}_G(X)$. *Then, there is a positive integer* n *such that* $\mathrm{tr}(n)(j(+)(x)) = 0$. *Similarly, there is a negative integer* n' *such that* $\mathrm{tr}(n')(j(-)(x)) = 0$.

Proof. Let (Y, f) represent the element x of $\widetilde{\mathrm{Nil}}_G(X)$ so that Y is a finite G-complex. The $j(+)(x) = (T(f), X\times S^1)$. For each positive number, k $\mathrm{tr}(k)(j(+)(x)) = (T(f^k), X\times S^1)$ (For the proof of this fact for $k = 2$, see [12], Lemma 8.1. The general case follows similarly). But there is a positive number n so that f^n is G-homotopic rel$X$ to a retraction of Y into X. For this number n, $\mathrm{tr}(n)(j(+)(x)) = (T(f^n), X\times S^1) = 0$.

For $j(-)$, notice that for each integer k, $\mathrm{tr}(k)(j(-)(x)) = \mathrm{tr}(-k)(j(+)(x))$. So, this case follows from the previous observations.

There is also transfer maps on $\mathrm{Wh}_G^{\mathrm{Top}}(X\times S^1)$. We denote these maps by $t(n)$.

COROLLARY 7.5. (a) *If* $x\in \mathrm{Wh}_G^{\mathrm{Top}}(X)$, *then* $t(n)\imath(x) = n\imath(x)$.

(b) *If* $x\in\widetilde{\mathrm{Nil}}_G(X)$, *then there is a positive integer* n *such that* $t(n)(\phi'' j(+)(x)) = 0$.

(c) *If* $x\in\widetilde{\mathrm{Nil}}_G(X)$, *then there is a negative integer* n' *such that* $t(n')(\phi'' j(-)(x)) = 0$.

Proof. The proof is exactly as in the PL case. Alternatively, the result is obvious from the fact that the maps in the exact sequence (5) commute with the transfer maps.

Let $e\colon \mathbf{R}\to S^1$ be the universal cover of S^1. This defines a map $\mathrm{tr}\colon \mathrm{Wh}_G^{\mathrm{Top}}(X\times S^1)\to \mathrm{Wh}_G^{\mathrm{Top}}(X\times\mathbf{R})$. More precisely, let $x\in \mathrm{Wh}_G^{\mathrm{Top}}(X\times S^1)$ be represented be a strong G-deformation retraction $f\colon Y \to X \times S^1$, where Y is a compact G-ANR. Form the pull back

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{f}} & X \times \mathbf{R} \\ \downarrow{\scriptstyle e'} & & \downarrow{\scriptstyle e} \\ Y & \xrightarrow{f} & X \times S^1 \end{array}$$

Then $\bar{f}$ is a proper strong G-deformation retraction. Define $\mathrm{tr}(x) = (\bar{Y}, X \times \mathbf{R})$. This is a well-defined homomorphism.

There is a commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Wh}_G^{\mathrm{PL}}(X)_c & \xrightarrow{f} & \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus N & \xrightarrow{\phi \oplus \mathrm{id}} & \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus N & \xrightarrow{\nu \oplus 0} & \tilde{K}_{0G}^{\mathrm{PL}}(X)_c & \xrightarrow{\psi} & \tilde{K}_{0G}^{\mathrm{PL}}(X) \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle i_c} & & \downarrow{\scriptstyle i_0} \\
\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c & \xrightarrow{f''} & \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1) & \xrightarrow{\phi''} & \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1) & \xrightarrow{\nu''} & \tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c & \xrightarrow{\psi''} & \tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1) \\
\downarrow{\scriptstyle s_1} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle s_c} & & \downarrow{\scriptstyle s_0} \\
\tilde{K}_{0G}^{\mathrm{PL}}(X)_c & \xrightarrow{f'} & \tilde{K}_{0G}^{\mathrm{PL}}(X) & \xrightarrow{\phi'} & \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R}) & \xrightarrow{\nu'} & K_{-1G}^{\mathrm{PL}}(X)_c & \xrightarrow{\psi'} & K_{-1G}^{\mathrm{PL}}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

Here

$$N = \widetilde{\mathrm{Nil}}_G(X) \oplus \widetilde{\mathrm{Nil}}_G(X),$$
$$i = i_* \oplus j(+) \oplus j(-)\colon \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus N \to \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$$

and

$$\chi = \imath \oplus \phi'' j(+) \oplus \phi'' j(-)\colon \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus N \to \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1).$$

In the above diagram, the first and the fourth columns are split short exact sequences given by the isomorphisms (2) and (3), and the second column is the split exact sequence which proves the Bass–Heller–Swan formula for $\mathrm{Wh}_G^{\mathrm{PL}}$. The fifth column is such that i_0 is a split monomorphism, s_0 is a split epimorphism, $s_0 i_0 = 0$. The maps i_0 and s_0 are defined similarly to i_* and s. In the third column, we know that χ is a monomorphism. To complete the proof of the Bass–Heller–Swan formula for the equivariant topological Whitehead group, we have to prove that the third column is a split short exact sequence. We have already proved that χ is a monomorphism.

CLAIM. $\mathrm{Ker}(\mathrm{tr}) \subset \mathrm{Im}(\chi)$.

Proof. Let

$$x \in \mathrm{Ker}(\mathrm{tr}) \Rightarrow \mathrm{tr}(x) = 0 \Rightarrow \nu'(\mathrm{tr}(x)) = 0 \Rightarrow s_c(\nu''(x)) = 0 \Rightarrow \nu''(x) \in \mathrm{Ker}(s_c) = \mathrm{Im}(i_c),$$

so there is $a \in \tilde{K}_{0G}^{\mathrm{PL}}(X)_c$ such that $\nu''(x) = i_c(a)$.

$$0 = \psi''(\nu''(x)) = \psi''(i_c(a)) = i_0(\psi(a)).$$

Since i_0 is a monomorphism, $\psi(a) = 0$. This means $a \in \operatorname{Ker}(\psi) = \operatorname{Im}(v \oplus 0)$ and there is $b \in \mathrm{Wh}_G^{\mathrm{Top}}(X)$ such that $v(b) = a$.

$$v''(\imath(b)) = i_c(v(b)) = i_c(a) = v''(x), \qquad \text{i.e. } x - \imath(b) \in \operatorname{Ker}(v'') = \operatorname{Im}(\phi'').$$

Therefore, there is $a' \in \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ such that $x - \imath(b) = \phi''(a')$.

$$\phi'(s(a')) = \operatorname{tr}(\phi''(a')) = \operatorname{tr}(x) - \operatorname{tr}(\imath(x)) = 0 \Rightarrow s(a') \in \operatorname{Ker}(\phi') = \operatorname{Im}(f')$$

and there is $b' \in \tilde{K}_{0G}^{\mathrm{PL}}(X)_c$ such that $s(a') = f'(b')$. Since s_1 is an epimorphism, there is $a'' \in \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)_c$ such that

$$s(a') = f'(s_1(a'')) = s(f''(a'')) \Rightarrow a' - f''(a'') \in \operatorname{Ker}(s) = \operatorname{Im}(i)$$

and there is $c \in \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus N$ such that $a' - f''(a'') = i(c)$.

We now compute $\chi(b + (\phi \oplus \mathrm{id})(c)$:

$$\begin{aligned}\chi(b + (\phi \oplus \mathrm{id})(c)) &= \chi(b) + \chi(\phi \oplus \mathrm{id})(c) = \imath(b) + \phi''(i(c)) \\ &= x - \phi''(a') + \phi''(a' - f''(a'')) = x.\end{aligned}$$

This means that $x \in \operatorname{Im}(\chi)$.

CLAIM 2. $\operatorname{Im}(\chi) \subset \operatorname{Ker}(\operatorname{tr})$.

Proof. We will show that $\operatorname{tr}\chi = 0$. Let $x \in \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus N$. Then

$$v'\operatorname{tr}\chi(x) = s_c i_c v(x) = 0 \Rightarrow \operatorname{tr}\chi(x) \in \operatorname{Ker} v' = \operatorname{Im} \phi'$$

and there is $a \in \tilde{K}_{0G}^{\mathrm{PL}}(X)$ such that $\operatorname{tr}\chi(x) = \phi'(a)$. Notice that s is an isomorphism from the transfer invariant elements of $\mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ to $\tilde{K}_{0G}^{\mathrm{PL}}(X)$. Therefore, there is a transfer invariant element $b \in \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ so that $a = s(b)$. Then

$$\operatorname{tr}\chi(x) = \phi' s(b) = \operatorname{tr}\phi''(b) \Rightarrow \chi(x) - \phi''(b) \in \operatorname{Ker}(\operatorname{tr}) \subset \operatorname{Im}(\chi)$$

and there is

$$y \in \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus N \quad \text{such that} \quad \chi(x) - \phi''(b) = \chi(y) \Rightarrow \chi(x - y) = \phi''(b).$$

In other words, $\phi''(b)$ is a transfer invariant element of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ and belongs to $\operatorname{Im}(\chi)$. By 7.5, $\phi''(b) = 0$. Therefore, $\operatorname{tr}\chi(x) = 0$.

If $x \in N$ there is an easy geometric way of showing that $\operatorname{tr}\chi(x) = 0$.

(i) Let $x \in \widetilde{\mathrm{Nil}}_G(X)$ be represented by a pair (Y, f) where Y is a finite G-complex and $f: Y \to Y$ is a map which is the identity on X and there is $n \in \mathbf{N}$ such that $f^n \simeq_G ir$, where $i: X \to Y$ is the inclusion map and $r: Y \to X$ is a retraction. Then

$$\chi(x) = j(+)(x) = (T(f), X \times S^1) \quad \text{and} \quad \operatorname{tr}(\chi(x)) = (D(f), X \times \mathbf{R}).$$

There is a sequence of G-CE equivalences rel$(X \times \mathbf{R})$:

$$D(f) \sim D(f^n) \sim D(ir) \sim D(ri) = X \times \mathbf{R}.$$

These G-CE equivalences are the infinite cyclic covers of the G-CE equivalences constructed in 1.9 and 1.11. This shows that $\operatorname{tr}(\chi(x)) = 0$.

(iii) If x, (Y, f) are as before and $\chi(x) = j(-)(x)$, then $\mathrm{tr}(\chi(x))$ is represented by the telescope of f with the ends reversed. So, the argument in (ii) applies and we have $\mathrm{tr}(\chi(x)) = 0$.

CLAIM 3. *The map* tr: $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1) \to \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$ *is a split epimorphism.*

Before we give the proof of Claim 3, we need the following

ASSERTION. *Let* $y \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$. *Assume that* $v''(y)$ *is a transfer invariant element of* $\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$. *Then, there is an element* $z \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ *such that* y' *is invariant under* $t(2)$ *the double transfer map and* $v''(z) = v''(y)$.

Proof of the Assertion. We assume that y is not invariant under $t(2)$. Write $y' = t(2)(y)$. Then, since $v''(y)$ is transfer invariant, $v''(y) = v''(y') \Rightarrow y - y' \in \mathrm{Ker}(v'') = \mathrm{Im}(\phi'')$ and there is $a \in \mathrm{Wh}_G^{\mathrm{PL}}(X \times S^1)$ such that $y - y' = \phi''(a)$. Notice that

$$\mathrm{tr}(y) = \mathrm{tr}(y') \Rightarrow y - y' \in \mathrm{Ker}(\mathrm{tr}) = \mathrm{Im}(\chi) \Rightarrow$$

there is $a' \in \mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus N$ such that $y - y' = \chi(a')$.

$$0 = v''(y) - v''(y') = v''(\chi(a')) = i_c((v \oplus 0)(a'))$$

and i_c is a monomorphism. So

$$(v \oplus 0)(a') = 0 \Rightarrow a' \in \mathrm{Ker}(v \oplus 0) = \mathrm{Im}(\phi \oplus \mathrm{id}) \Rightarrow$$

there is $b \in \mathrm{Wh}_G^{\mathrm{PL}}(X) \oplus N$ such that $(\phi \oplus \mathrm{id})(b) = a'$.

Also, $\phi''(i(b)) = \chi((\phi \oplus \mathrm{id})(b)) = \chi(a') = y - y'$. The element b has three components $b = (k, n_+, n_-)$. Then from 7.5, $\mathrm{tr}(m)(i(b)) = mi_*(k) + \mathrm{tr}(m)(j(+)(n_+))$ for some negative integer m. So we can assume that the element y we started with has the property that $y - y' = \phi''(i(b))$ and $b = (k, n_+, 0)$. This can be done by replacing y with $t(m)(y)$. Under these assumptions set

$$z = y' + 2\phi'' i_*(k) - t(2)(\phi'' j(+)(n_+)) - t(4)(\phi'' j(+)(n_+)) - \cdots \\ - t(2^p)(\phi'' j(+)(n_+)),$$

where p is a positive number such that $t(2^{p+1})(j(+)(n_+)) = 0$. Then

$$t(2)(z) = t(2)(y') + 4\phi'' i_*(k) - t(4)(\phi'' j(+)(n_+)) - t(8)(\phi'' j(+)(n_+)) - \cdots \\ - t(2^p)(\phi'' j(+)(n_+).$$

But

$$y - y' = \phi''(i_*(k)) + \phi''(j(+)(n_+)) \Rightarrow t(2)(y') = t(2)(y) - 2\phi''(i_*(k)) - \\ - t(2)(\phi'' j(+)(n_+)).$$

From this it follows that $t(2)(z) = z$. Notice that $v''(z) = v''(y') = v''(y)$. This completes the proof of the assertion.

Proof of Claim 3. Let $i'_c\colon K^{\mathrm{PL}}_{-1G}(X)_c \to \tilde{K}^{\mathrm{PL}}_{0G}(X \times S^1)_c$ be the right inverse of s_c and i'_0: $K^{\mathrm{PL}}_{-1G}(X) \to \tilde{K}^{\mathrm{PL}}_{0G}(X \times S^1)$ be the right inverse of s_0. The images of these maps are the transfer invariant elements of the corresponding groups and the maps are natural with respect to 'forget control' (see the appendix and [46], §10). Let $x \in \mathrm{Wh}^{\mathrm{Top}}_G(X \times \mathbf{R})$. Then,

$$\psi'v'(x) = 0 \Rightarrow i'_0\psi'v'(x) = 0 \Rightarrow \psi''i'_cv'(x) = 0 \Rightarrow i'_cv'(x) \in \mathrm{Ker}(\psi'') = \mathrm{Im}(v'') \Rightarrow$$

there is $y \in \mathrm{Wh}^{\mathrm{Top}}_G(X \times S^1)$ such that $v''(y) = i'_cv'(x)$. Since $i'_cv'(x)$ is transfer invariant, we can choose y to be variant under double transfers using the assertion. Choose $y \in \mathrm{Wh}^{\mathrm{Top}}_G(X \times S^1)$ invariant under double transfers such that $v''(y) = i'_cv'(x)$. Notice

$$v'\mathrm{tr}(y) = s_cv''(y) = s_ci'_cv'(x) = v'(x) \Rightarrow \mathrm{tr}(y) - x \in \mathrm{Ker}(v') = \mathrm{Im}(\phi') \Rightarrow$$

there is $b \in \tilde{K}^{\mathrm{PL}}_{0G}(X)$ such that $\mathrm{tr}(y) - x = \phi'(b)$. Define a map $j'\colon \mathrm{Wh}^{\mathrm{Top}}_G(X \times \mathbf{R}) \to \mathrm{Wh}^{\mathrm{Top}}_G(X \times S^1)$ by $j'(x) = y - \phi''j(b)$ with y and b as above and $j\colon \tilde{K}^{\mathrm{PL}}_{0G}(X) \to \mathrm{Wh}^{\mathrm{PL}}_G(X \times S^1)$ is the right inverse of s discussed in Section 2.

(i) *j' is well defined.* Let y' be another invariant under double transfers element of $\mathrm{Wh}^{\mathrm{Top}}_G(X \times S^1)$ such that $v''(y) = v''(y') = i'_cv'(x)$ and $b' \in \tilde{K}^{\mathrm{PL}}_{0G}(X)$ be such that $\mathrm{tr}(y') - x = \phi'(b')$. Then,

$$v''(y) = v''(y') \Rightarrow y - y' \in \mathrm{Ker}(v'') = \mathrm{Im}(\phi'')$$

and there is $a \in \mathrm{Wh}^{\mathrm{PL}}_G(X \times S^1)$ such that $y - y' = \phi''(a)$.

$$x = \mathrm{tr}(y) - \phi'(b) = \mathrm{tr}(y') - \phi'(b') \Rightarrow \mathrm{tr}(y - y') = \phi'(b' - b).$$

But, $\mathrm{tr}(y - y') = \mathrm{tr}\phi''(a) = \phi's(a)$. $\phi's(a) = \phi'(b' - b)$ and there is $\alpha \in \tilde{K}^{\mathrm{PL}}_{0G}(X)_c$ such that $f'(\alpha) + s(a) = b' - b$. Then,

$$y - \phi''j(b) - y' + \phi''j(b') = \phi''(a) + \phi''jf'(\alpha) + \phi''js(a).$$

But, $jf'(\alpha) = \phi''j$\s\do2(ı)(α), where $j_1\colon \tilde{K}^{\mathrm{PL}}_{0G}(X)_c \to \mathrm{Wh}^{\mathrm{PL}}_G(X \times S^1)_c$ is the right inverse of the projection s_1, and

$$y - \phi''j(b) - y' + \phi''j(b') = \phi''(a) + \phi''js(a).$$

Case 1. If a is a transfer invariant element of $\mathrm{Wh}^{\mathrm{PL}}_G(X \times S^1)$, then there is $a' \in \tilde{K}^{\mathrm{PL}}_{0G}(X)$ such that $a = j(a')$, and $js(a) = jsj(a') = j(a') = a$. In this case

$$y - \phi''j(b) - y' + \phi''j(b') = 0.$$

Case 2. If a is not transfer invariant, then there is $a'' \in \mathrm{Wh}^{\mathrm{PL}}_G(X) \oplus N$ such that $a = i(a'')$. Then $\phi''(b) = \phi''i(a'') = \chi(\phi \oplus \mathrm{id})(a'') = y - y'$. But, $y - y'$ is invariant under double transfers. By 7.5, the only invariant under double transfer element in $\mathrm{Im}(\chi)$ is 0. So,

$$\chi(\phi \oplus \mathrm{id})(a'') = 0 \Rightarrow (\phi \oplus \mathrm{id})(a'') = 0 \Rightarrow$$

there is $\alpha' \in \mathrm{Wh}_G^{\mathrm{PL}}(X)_c$ such that $f(\alpha') = a''$. Then,

$$\phi''(a) = \phi''i(a'') = \chi(\phi \oplus \mathrm{id})(a'') = \chi(\phi \oplus \mathrm{id})f(\alpha') = 0.$$

Also,

$$\phi''js(a) = \phi''jsi(a'') = 0 \quad \text{and} \quad y - \phi''j(b) - y' + \phi''j(b') = 0.$$

This proves that $j'(x)$ is independent of the choices of y and b.

(ii) *j' is a group homomorphism:* This is obvious from (i).

(iii) *j' is a right inverse of tr:* If $x \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$, then $j'(x) = y - \phi''j(b)$, where $y \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ is a double transfer invariant element such that $v''(y) = i_c'v'(x)$ and $b \in \tilde{K}_{0G}^{\mathrm{PL}}(X)$ such that $\mathrm{tr}(y) - x = \phi'(b)$. Then

$$\mathrm{tr}j'(x) = \mathrm{tr}(y) - \mathrm{tr}\phi''j(b) = x + \phi'(b) - \phi'sj(b) = x.$$

(iv) *j' is an isomorphism onto the group of the transfer invariant elements of* $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$: If $y \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ is transfer invariant, then we compute $j'\mathrm{tr}(y)$. Notice that

$$v'\mathrm{tr}(y) = s_c v''(y) \Rightarrow i_c'v'\mathrm{tr}(y) = i_c's_c v''(y) = v''(y) \quad \text{and} \quad \mathrm{tr}(y) = \mathrm{tr}(y) + 0.$$

We can choose $b = 0$ in the definition of $j'\mathrm{tr}(y)$. This proves that $j'\mathrm{tr}(y) = y$. Therefore, the map j' from the transfer invariant elements of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ to $\mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$ is an isomorphism.

This completes the proof of the assertion that the third column in the above commutative diagram is split short exact. By (iv) above, we can identify $\mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$ with the transfer invariant elements of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$. This completes the proof of the main theorem.

For a locally compact G-ANR X, define $\mathrm{Wh}_G^{\mathrm{Top},\rho}(X)$ to be the subgroup of $\mathrm{Wh}_G^{\mathrm{Top}}(X)$ generated by pairs (Y, X) such that $Y_a^H - Y_a^{>H} = \emptyset$ whenever $X_a^H - X_a^{>H} = \emptyset$. It turns out (see [46]) that $\mathrm{Wh}_G^{\mathrm{Top},\rho}(X)$ is a summand of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$. Then, using the main theorem, we get

COROLLARY 7.6. *If X is a compact G-ANR and G acts trivially on S^1, then*

$$\mathrm{Wh}_G^{\mathrm{Top},\rho}(X \times S^1) \approx \mathrm{Wh}_G^{\mathrm{Top},\rho}(X) \oplus \tilde{K}_{0G}^{\mathrm{Top},\rho}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\rho}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\rho}(X).$$

In what follows, by a G-manifold we mean a locally linear G-manifold. Let M be a G-manifold. A G-h-cobordism, $(W; M, M')$, is a G-manifold W with boundary $\partial W = M \cup M'$, the disjoint union of M and M', such that the inclusion maps $i\colon M \to W$, and $i'\colon M'' \to W$ are proper equivariant homotopy equivalences.

The connection of $\mathrm{Wh}_G^{\mathrm{Top},\rho}(M)$, for a G-manifold M, with the geometry is given by the equivariant G-h-cobordism theorem. We call two G-h-cobordisms, $(W; M, N)$ and $(W'; M, N')$ equivalent if there is a G-homeomorphism $F\colon W \to W'$ which is the identity on M.

EQUIVARIANT *G*-*h*-COBORDISM THEOREM [46]. *Let M be a compact smooth G-manifold and M has codimension ⩾3 gaps, and such that all the fixed points of M have dimensions ⩾5. Then, there is a bijection set of the equivalence classes of G-h-cobordisms over M and elements of* $\mathrm{Wh}_G^{\mathrm{Top},\rho}(M)$ *in such a way that the trivial G-h-cobordism* $(M \times I; M, M)$ *corresponds to the zero element of* $\mathrm{Wh}_G^{\mathrm{Top},\rho}(M)$.

In [46], $\mathrm{Wh}_G^{\mathrm{Top,ISO}}(X)$ is defined for any locally compact G-ANR X. The definition of this group is given by using isovariance whenever equivariance occurs. Then $\mathrm{Wh}_G^{\mathrm{Top,ISO}}(X) \approx \mathrm{Wh}_G^{\mathrm{Top},\rho}(X)$. So

COROLLARY 7.7. *If X is a compact G*-ANR, *and G acts trivially on* S^1 *then*

$$\mathrm{Wh}_G^{\mathrm{Top,ISO}}(X \times S^1) \approx \mathrm{Wh}_G^{\mathrm{Top,ISO}}(X) \oplus \tilde{K}_{0G}^{\mathrm{Top,ISO}}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\mathrm{ISO}}(X) \oplus \widetilde{\mathrm{Nil}}_G^{\mathrm{ISO}}(X).$$

Appendix: Equivariant Wrapping-up

We give an equivariant version of 'wrapping up' over S^1 for Hilbert cube manifolds. This construction can be used as an alternative definition of the split monomorphism j': $\mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R}) \to \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$.

In the nonequivariant case, this method was developed in [44], Chapter 5, and in [13], Chapter 4, for topological finite-dimensional manifolds, and in [15], §4, for Hilbert cube manifolds. The input for this process is a proper homotopy equivalence, $f: M \to X \times \mathbf{R}$, from a finite-dimensional manifold (or Hilbert cube manifold) M to $X \times \mathbf{R}$, where X is a finite CW-complex (or a compact ANR) and the output is a homotopy equivalence $f': M' \to X \times S^1$ whose infinite cyclic cover is f. It turns out that the element of $\mathrm{Wh}(\pi_1(X) \times \mathbf{Z})$ determined by f' is invariant under the transfer maps induced by the double covers of S^1.

In [50], there is an extension of the methods in [14] in the equivariant case. We use this work to extend the 'wrapping up' in [15], §4, to the equivariant case. We start with a proper G-homotopy equivalence $f: M \to X \times \mathbf{R}$, from an equivariant Hilbert cube manifold M to $X \times \mathbf{R}$, where X is a compact G-ANR, and we produce a G-homotopy equivalence $f': M' \to X \times S^1$ whose infinite cyclic cover is f.

The main reason that we use infinite-dimensional manifolds is the engulfing arguments. Engulfing arguments are simpler in the infinite-dimensional case because we do not really need any codimension conditions. Engulfing arguments in the finite-dimensional case have been developed by Steinberger and West [48] and most probably a variation of these arguments will be enough for the case under consideration. Once the engulfing arguments have been established, the rest of the construction works in the finite-dimensional case.

We start with the definitions needed in this chapter. Let $|G| = n$ and let $I^n = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n / -1 \leqslant x_i \leqslant 1\}$ be the unit hypercube of the regular real representation of G.

DEFINITION A.1. (1) A G-map $f: A \to B$ between G-spaces is called a near homeomorphism if for each open G-cover α of A there is a G-homeomorphism $f': B \to A$ which is α-close to f.

(2) The G-Hilbert cube Q_G is the countable product of I^n,

$$Q_G = \prod_{i=0}^{\infty} I_i^n,$$

where each $I_i^n = I^n$.

(3) A Q_G-manifold M is a separable metric G-space, all of whose orbits have neighborhoods which are G-homeomorphic to open subspaces of Q_G.

In [50], it is observed that the results for Hilbert cube manifolds in [14], which do not use the topological invariance of the Whitehead torsion, can be generalized in the equivariant case.

(A.2) (Equivariant Edwards' Theorem). *A G-space X is a G-ANR if and only if $X \times Q_G$ is a Q_G-manifold.*

(A.3) (α-Approximation Theorem for Q_G-manifolds). *Let α be an open cover of the Q_G-manifold M. Then, there is an open G-cover β such that if $f: N \to M$ is a G-β-homotopy equivalence from a Q_G-manifold N, then f is α-close to a G-homeomorphism.*

Notes: (i) In particular, if the map f in (A.3) is a G-CE map, then f is a near G-homeomorphism ([46], Proposition 4.5).

(ii) In [48], the α-approximation theorem is proved for finite-dimensional locally linear G-manifolds.

(A.4) *If M is a Q_G-manifold, then $M \times Q_G \approx_G M$. Moreover, the projection map $p: M \times Q_G \to M$ is a near G-homeomorphism.*

Note: In particular, $M \times I^n \approx_G M$ and the projection map is a near G-homeomorphism.

DEFINITION A.5. (1) Let X be a G-space. A closed G-subset A of X is called a G-Z-set if for any open G-cover α of X, there is a G-map of X into $X - A$ which is α-close to the identity ([14] §3, for the nonequivariant analogue).

(2) A G-Z-embedding $f: A \to X$ is a proper G-embedding such that $f(A)$ is a G-Z-set in X.

The basic properties of Z-sets in Hilbert cube manifolds can be generalized in the equivariant case [50].

(A.6) (G-Z-embedding Approximation Theorem). *Let (A, A') be a pair so that A is a locally compact G-space and A' is a closed G-subset of A. Let M be a Q_G-manifold and let $f: A \to M$ be a proper G-map such that $f|_{A'}$ is a G-Z-embedding. Then, for each open G-cover α of M, there is an α G-homotopy from f to a G-Z-embeddings $f': A \to M$ so that $f'|_{A'} = f|_{A'}$.*

(A.7) (G-Z-set Unknotting Theorem) ([46] Theorem 4.3, [50]). *Let M be a Q_G-manifold and let A be a G-ANR. Then any two properly G-homotopic G-Z-embeddings of A in M are ambiently G-isotopic.*

Note: If the G-homotopy between the two G-Z-embeddings in (A.5) is α-homotopy, then the ambient isotopy can be chosen to be α-isotopy ([14], Theorem 19.4).

We start with an equivariant engulfing result for Q_G-manifolds.

Notation: (i) If $\alpha: A \to B \times \mathbf{R}$ is a continuous map between topological spaces, K is a subset of A, and C is a subset of $\mathbf{R}$, write K_C for $K \cap \alpha^{-1}(B \times C)$.

(ii) In what follows, M is a Q_G-manifold and X is a compact G-ANR and the group G acts trivially on $\mathbf{R}$ or on S^1.

LEMMA A.9. *Let $f: M \to X \times \mathbf{R}$ be a proper G-homotopy equivalence. Then, there is a G-isotopy, $h_t: M \to M$, with compact support, from the identity on M such that*

$$h_1 f^{-1}(X \times (-\infty, 0)) \supset f^{-1}(X \times (-\infty, 1]).$$

Proof. Using the note following (A.4), the projection map $\mathrm{pr}: M \times I^n \to M$ is a near G-homeomorphism, i.e. there is a G-homeomorphism $k: M \times I^n \to M$ which is close to the projection, which induces a proper G-homotopy equivalence $f' = fk: M \times I^n \to X \times \mathbf{R}$. Let d be a G-homotopy inverse of f', $f'_t: M \times I^n \to M \times I^n$ be a proper G-homotopy from the identity on $M \times I^n$ to df', and $d_t: X \times \mathbf{R} \to X \times \mathbf{R}$ be a proper G-homotopy from the identity on $X \times \mathbf{R}$ to $f'd$. Fix a number $a \in \mathbf{R}$, such that $M \times I^n_{(-\infty,a)} \supset f'_t((M \times I^n)_{\{1.5\}})$ and $X \times (-\infty, a) \supset d_t(X \times \{1.5\})$, for all $0 \leqslant t \leqslant 1$. Notice that the main diagonal (L) in I^n, consisting of the points whose coordinates are all equal, is fixed pointwise under the action of G. Let $A = (1, 1, \ldots, 1) \in (L)$, and $B = (-1, -1, \ldots, -1) \in (L)$.

The first step of the proof is to construct a G-isotopy, $u_t: M \times I^n \to M \times I^n$, with compact support, from the identity on $M \times I^n$, such that

$$u_1((M \times I^n)_{(-\infty,0)}) \supset (M \times \{A\})_{(-\infty,a+1]} \tag{1}$$

To construct u_t we apply the G-Z-set unknotting theorem as follows:

(i) The inclusion map $i: M \times \{A\} \to M \times I^n$ is a G-Z-embedding.

(ii) Choose a G-homotopy $q_t: X \times \mathbf{R} \to X \times \mathbf{R}$ from the identity on $X \times \mathbf{R}$, with compact support, such that $X \times (-\infty, 0) \supset q_1(X \times (-\infty, a+1])$. Define a G-map $\imath: M \times \{A\} \to M \times [0,1]$ as the composition $dq_1(f'|_{M \times \{A\}})$. Then, there is a G-homotopy with compact support from the inclusion map to $\imath$ and $M \times I^n_{(-\infty,0)} \supset \imath((M \times \{A\})_{(-\infty,a+1]})$. The map $\imath$ can be approximated by a G-Z-embedding $i': M \times \{A\} \to M \times I^n$. Then, i' is still G-homotopic to i and it can be chosen to satisfy $(M \times I^n)_{(-\infty,0)} \supset i'(M \times \{A\})_{(-\infty,a+1]})$.

Two G-homotopic G-Z-embeddings have been constructed from $M \times \{A\}$ to $M \times I^n$. By the G-Z-unknotting theorem, there is a G-isotopy $F_t: M \times I^n \to M \times I^n$ from the identity on $M \times I^n$, such that $F_1 i = i'$. In particular

$$(M \times I^n)_{(-\infty,0)} \supset F_1((M \times \{A\})_{(-\infty,a+1]}).$$

Also, there are real numbers b and c so that the isotopy F_t can be chosen with compact support in $(M \times I^n)_{[b,c]}$.

Set $u_t = F_t^{-1}$. This completes the first construction.

The second step is the construction of a G-isotopy, $v_t: M \times I^n \to M \times I^n$, with compact support in $(M \times I^n)_{(-\infty,a]}$, from the identity on $M \times I^n$, such that

$$v_1^{-1}((M \times I^n)_{(1,+\infty)}) \supset (M \times \{B\})_{[b,+\infty)}. \tag{2}$$

This isotopy is constructed as in the first step. The isotopy v_t can be chosen to have compact support in $(M \times I^n)_{(-\infty,a]}$. By continuity, there is a G-neighborhood U of $M \times \{B\}$ in $M \times I^n$ such that

$$v_1^{-1}((M \times I^n)_{(1,+\infty)}) \supset U_{[b,+\infty)} \tag{3}$$

The third step is the construction of a G-isotopy $w_t: M \times I^n \to M \times I^n$ with compact support from the identity on $M \times I^n$ such that

$$U_{(b,+\infty)} \cup (M \times I^n)_{(a+1,+\infty)}) \supset w_1 u_1((M \times I^n)_{[0,+\infty)}). \tag{4}$$

The construction of w_t will complete the proof of the theorem because $h_t = v_t w_t u_t$ is a G-isotopy from the identity on $M \times I^n$ with compact support and

$$\begin{aligned} h_1((M \times I^n)_{(-\infty,0)}) &\supset v_1(M - (U_{(b,+\infty)} \cup (M \times I^n)_{(a+1,+\infty)})) \\ &= v_1((M \times I^n)_{(-\infty,a+1]} - U_{(b,+\infty)}) = v_1((M \times I^n)_{(-\infty,a+1]}) - v_1(U_{(b,+\infty)}). \end{aligned}$$

Since v_t has support in $(M \times I^n)_{(-\infty,a]}$,

$$v_1((M \times I^n)_{(-\infty,a+1]}) = M \times I^n_{(-\infty,a+1]} \quad \text{and} \quad (M \times I^n)_{(1,+\infty)} \supset v_1(U_{(b,+\infty)})$$

by (3). Then

$$h_1((M \times I^n)_{(-\infty,0)}) \supset (M \times I^n)_{(-\infty,a+1]} - (M \times I^n)_{(1,+\infty)} = (M \times I^n)_{(-\infty,1]}.$$

Now, we give the construction of w_t. There is a real number a' such that

$$(M \times I^n)_{(-\infty,a')} \supset u_1(M \times I^n_{\{0\}}).$$

Choose a G-isotopy

$$a_t: (M \times I^n)_{[-1,a']} \to (M \times I^n)_{[-1,a']}$$

from the identity on $(M \times I^n)_{[-1,a']}$ such that $U \cap (M \times I^n_{[-1,a']}) \supset a_1(M \times I^n)_{[-1,a']})$. This isotopy can be constructed by moving A on (L) towards B, and moving the sides of I^n, which intersect in A, parallel towards the sides of I^n which intersect in B. In this way, I^n is isotoped to the sets

$$I^n(s) = \{(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n / -1 \leqslant x_i \leqslant s\}, \quad \text{for all } s \in (-1, 1].$$

Define w_t to be the extension of a_t to $M \times I^n$, such that w_t is the identity in the complement of $(M \times I^n)_{[-2,a'+1]}$. Then, w_t has compact support and satisfies (4).

Let M be as in Lemma A.9. Then M has two ends $e(+)$ and $e(-)$ corresponding to the two ends of $\mathbf{R}$.

LEMMA A.10. *There is a G-neighborhood* U_+ *of* $e(+)$ *and a G-isotopy of open embeddings* $h_t^+ : U_+ \to M$ *such that*

(i) h_0^+ *is the inclusion,*
(ii) h_t^+ *fixes pointwise a smaller neighborhood of* $e(+)$,
(iii) $h_1^+ = h_+$ *is a G-homeomorphism.*

Proof. The proof of this lemma is essentially in [9], Lemma 2.7. Set $a_0 = 0$, $a_1 = -1$. Let $h_t^1 : M \to M$ be a G-isotopy, with compact support, constructed as in Lemma A.1 such that $h_1 f^{-1}(X \times (-\infty, -1)) \supset f^{-1}(X \times (-\infty, 0])$. Let a_0' be a real positive number such that $M_{(a_0', +\infty)}$ contains the support of h_t^1. Using Lemma A.9, we can construct two sequences of real numbers, $\{a_i\}_{i=0}^{+\infty}$ and $\{a_i'\}_{i=0}^{+\infty}$, such that

(i) a_0, a_1, a_0' are as above.
(ii) For each $i \geqslant 1$ there is a G-isotopy, $h_t^i : M \to M$, from the identity on M with compact support in $M_{(-\infty, a_{i-1}')}$ such that $h_1^i(M_{(a_{i-1}, +\infty)}) \supset M_{[a_i, +\infty)}$.

Set

$$P_i = M_{(a_i', +\infty]},\ A_i = M_{[a_i, a_{i-1}]}, \qquad W_i = M_{[a_{i-1}, +\infty)} \quad \text{for } i \geqslant 1.$$

Define also a sequence of G-subsets of M as follows

$$V_0 = M_{(a_0', +\infty)},$$
$$V_i = V_{i-1} \cup (h_1^i h_1^{i-1} \dots h_1^1)^{-1}(P_i) \quad \text{for } i \geqslant 1.$$

CLAIM 1. $V_i \subset W_1$ *for all* $i \geqslant 1$.

Proof. The proof is by induction on i: For $i = 0$, the claim is obvious from the definition. Assume the claim is true for $i - 1$. Notice that

$$h_1^i h_1^{i-1} \dots h_1^1(W_1) \supset W_1 \cup A_1 \cup A_i \supset W_i \cup A_i \supset P_i.$$

This proves that $V_i \subset W_1$.

Set $U_+ = \cup_i V_i$ and define a G-isotopy $h_t^+ : U_+ \to M$ as follows

$$h_t^+(x) = \begin{cases} h^k_{k(k+1)t-(k+1)(k-1)} h_1^{k-1} \dots h_1^1(x) & \text{for } t \in \left[\dfrac{k-1}{k}, \dfrac{k}{k+1}\right] \\ \lim\limits_{k \to +\infty} h_1^k \dots h_1^1(x) & \text{for } t = 1. \end{cases} \quad \text{and } k \geqslant 1,$$

CLAIM 2. h_t^+ *is well-defined.*

Proof. Notice that if $j > i$, the restriction of h_t^j to P_i is the identity. This implies that

$$h_1^i \dots h_1^1(V_i) = V_0 \cup P_i \cup \dots \cup P_1$$

and so the restriction of h_t^j to $h_1^i \dots h_1^1(V_i)$ is the identity for $j > i$. Therefore, if

$$t \in \left[\frac{i}{i+1}, 1\right] \quad \text{and} \quad x \in V_i,$$

then $h_t^-(x) = h_1^i \dots h_1^1(x)$. This completes the proof of the claim.

The isotopy h_t^+ satisfies Lemma A.10 because

(i) h_0^+ is the inclusion map,
(ii) h_t^+ fixes $M_{(-\infty, a_0')}$,
(iii) h_1^+ is an injection and the image of h_1^+ is M. So h_1^+ is a G-homeomorphism.

This completes the proof of Lemma A.10.

There is nothing special about $e(+)$. The same proof applies for $e(-)$.

COROLLARY A.11. *There is a G-neighborhood U_- of $e(-)$ and a G-isotopy $h_t^-: U_- \to M$ such that*

(i) h_0^- *is the inclusion,*
(ii) h_t^- *fixes pointwise a smaller neighborhood of* $e(-)$,
(iii) $h_1^- = h_-$ *is a G-homeomorphism.*

Remark. The proof of Lemma A.10 really proves that there are arbitrarily small G-neighborhoods of $e(+)$ with the property satisfied by U_+ in Lemma A.10. This can be done by choosing a_0, a_1 large enough.

The above remark suggests that we can apply an equivariant version of the twist gluing construction given in [44]. We can combine Siebenmann's approach with the ideas in Chapman's wrapping up to construct a G-homotopy equivalence $f': M' \to X \times S^1$, where M' is a compact Q_G-manifold. The lifting of f' to the infinite cyclic cover induced by the cover $X \times \mathbf{R} \xrightarrow{\text{id} \times e} X \times S^1$, where e is the universal cover of S^1 will be essentially f.

CONSTRUCTION. The input for the construction is a proper G-homotopy equivalence $f: M \to X \times \mathbf{R}$, where M is a Q_G-manifold and X is a compact G-ANR. Then M has two ends, $e(\pm)$, corresponding to the two ends of $\mathbf{R}$. Using Lemma A.10 and Corollary A.11, we can find G-neighborhoods $U_\pm$ of $e(\pm)$ and G-isotopies $h_t^\pm: U_\pm \to M$ which fixes smaller G-neighborhoods of $e(\pm)$, respectively, from the inclusion maps to G-homeomorphisms $h_\pm: U_\pm \to M$. From the remark following Corollary A.11, we can assume that $U_+ \cap U_- = \emptyset$. Define $M(h_+, h_-) = M/\sim$, where $\sim$ is the relation generated by $x \sim h_+^{-1} h_-(x)$ for $x \in U_-$. Write $q: M \to M(h_+, h_-)$ for the quotient map.

The next step in the construction is the definition of a G-map, $f(h_+, h_-): M(h_+, h_-) \to X \times S^1$. Choose a number $a \in \mathbf{R}$ such that $M_{(-\infty, a]}$ is fixed pointwise by h_t^-.

LEMMA A.12. *There is a G-isotopy through G-homeomorphisms $h_t: M \to M$ such that*

(i) $h_0 = \text{id}_M$,
(ii) h_t *has compact support,*
(iii) $h_t | h_+^{-1}(M_{[a, +\infty)}) = h_t^+ | h_+^{-1}(M_{[a, +\infty)})$.

Proof. We use the notation of Lemma A.10. Notice that there is an integer $m \geqslant 1$ such that $h_+^{-1}(M_{[a,+\infty)}) \subset V_m$. Following the proof of Lemma A.10, we define

$$h_t(x) = \begin{cases} h^k_{k(k+1)t-(k+1)(k-1)} h_1^{k-1} \ldots h_1^1(x) & \text{for } t \in \left[\dfrac{k-1}{k}\, \dfrac{k}{k+1}\right] \text{ and } k \leqslant m, \\ h_1^m \ldots h_1^1(x) \quad \text{for } t \geqslant \dfrac{m}{m+1}. \end{cases}$$

It follows from the construction of h_t^+ that

$$h_t|_{V_m} = h_t^+|_{V_m}.$$

So $h_t|h_+^{-1}(M_{[a,+\infty)}) = h_t^+|h_+^{-1}(M_{[a,+\infty)})$, and h_t satisfies the requirements of Lemma A.3.

Choose a number $b \in \mathbf{R}$ such that $M_{(-\infty,b)} \subset h_1^{-1}(M_{(-\infty,a]})$. Let $Y = h_1^{-1}(M_{(-\infty,a]}) - M_{(-\infty,a)}$. Define a G-map $\phi(h_+, h_-)\colon Y \to X \times [a,b]$ as follows

(i) If $p_{\mathbf{R}}\colon X \times \mathbf{R} \to \mathbf{R}$ is the projection define

$$p_{\mathbf{R}}\phi(h_+,h_-)| = f|\colon M_{[a,b]} \to [a,b] \quad \text{and} \quad p_{\mathbf{R}}\phi(h_+,h_-)(Y - M_{[a,b]}) = b.$$

(ii) Let $p_X\colon X \times \mathbf{R} \to X$ is the projection to X. Define a G-homotopy

$$\alpha_t\colon h_1^{-1}(M_{\{a\}}) \cup (M_{\{a\}}) \to X, \alpha_t = p_X f \text{ on } M_{\{a\}} \text{ and } \alpha_t = p_X f h_t \text{ on } h_1^{-1}(M_{\{a\}}).$$

Notice that $\alpha_0 = p_X f$. Using the Homotopy Extension Theorem ([28]), we can extend α_t to a G-homotopy $e_t\colon Y \to X$. Define $p_X\phi(h_+,h_-) = e_1$.

Notice that $\phi(h_+,h_-) = f$ on $M_{\{a\}}$ and $p_X\phi(h_+,h_-) = p_X f h_1$ on $h_1^{-1}(M_{\{a\}})$. In particular,

$$p_X\phi(h_+,h_-)(x) = p_X\phi(h_+,h_-)(h_1^{-1}(x)).$$

Extend $\phi(h_+,h_-)$ to a G-map $\Phi(h_+,h_-)\colon M \to X \times [a,b]$ as follows:

If $x \in M_{(-\infty,a)}$, then $h_+^{-1}h_-(x) = h_+^{-1}(x) \in Y$. Define

$$\Phi(h_+,h_-)(x) = \phi(h_+,h_-)(h_+^{-1}h_-(x)).$$

If $x \in h_+^{-1}(M_{(a,+\infty)})$ then $h_-^{-1}h_+(x) \in h_-^{-1}(M_{(a,+\infty)}) \subset U_- - M_{(-\infty,a)} \subset Y$. Define

$$\Phi(h_+,h_-)(x) = \phi(h_+,h_-)(h_-^{-1}h_+(x)).$$

This is a well defined G-map and induces a G-map $f(h_+,h_-)\colon M(h_+,h_-) \to X \times ([a,b]/\sim)$ where $\sim$ is generated by $a \sim b$. So we have constructed a G-map

$$f(h_+,h_-)\colon M(h_+,h_-) \to X \times S^1.$$

We can construct $f(h_+,h_-)$ to be the restriction of f in $M_{[a,\infty)} - M_{(a',\infty)}$, where $M_{(a',\infty)} \supset \mathrm{cl}(U_+)$.

Remark. Following T. A. Chapman, we define $M' = Y/\sim$ where $x \sim h_1^{-1}(x)$ for $x \in M_{\{a\}}$. Then, the restriction of $\Phi(h_+,h_-)$ to Y induces a G-map $f'\colon M' \to X \times S^1$.

CLAIM. *There is a G-homeomorphism $k: M' \to M(h_+, h_-)$ such that $f' = kf(h_+, h_-)$.*

Proof. The map $k: M' \to M(h_+, h_-)$ given by $k(x) = x$ for all $x \in Y$ is a G-homeomorphism ([9] Theorem 3.3). By definition, $f' = kf(h_+, h_-)$.

LEMMA A.13. *The G-space $M(h_+, h_-)$ is a compact Q_G-manifold.*

Proof. Since M is a Q_G-manifold by [44], Proposition 5.5, $M(h_+, h_-)$ is a metric space. Since $M \cup \{e(+), e(-)\}$ is compact, the above proposition asserts that $M(h_+, h_-)$ is compact. Notice that in [25], Assertion 5, it is proved that for each $x \in M'$ there are G-neighborhoods N' of x in M' and N of M that are G-homeomorphic. In particular, this proves that M' is a Q_G-manifold. Using the Claim above, we can see that $M(h_+, h_-)$ is also a Q_G-manifold.

LEMMA A.14. *The map $f(h_+, h_-): M(h_+, h_-) \to X \times S^1$ is a G-homotopy equivalence.*

Proof. The proof of [15], Lemma 4.1, generalizes to the equivariant setting and it gives that $f': M' \to X \times S^1$ is a G-homotopy equivalence.

Remarks. (i) In [13], chapter 7, it is proved that (M', f') does not depend on the G-isotopy h_t chosen above in the sense that if h'_t is an other G-isotopy as in Lemma A.3 and (M'', f'') is constructed using h'_t, then there is a G-homeomorphism $K: M' \to M''$ such that $Kf'' \simeq_G f'$.

(ii) The same proof gives that the construction is independent of a, b, and the proper G-homotopy class of f.

(iii) Also, in [44], Theorem 5.2, it is proved that the above construction is independent of the G-neighborhoods $U_\pm$ and the G-isomorphisms $h_\pm$.

(iv) If there is a G-CE map from a Q_G-manifold N, $c: N \to M$, then by A.2, c can be approximated by a G-homeomorphism. If we apply the above construction to the composition $fc: N \to X \times \mathbf{R}$, we get a compact Q_G-manifold N' and a G-homotopy equivalence $f'': N' \to X \times S^1$. Then, by (iii), there is a G-homeomorphism $L: N' \to M'$ such that $Lf' \simeq_G f'$.

(v) In [13], chapter 7, it is proved that (M', f') is invariant under the transfers induced by the finite covers of S^1.

There is a natural infinite cyclic cover of $M(h_+, h_-)$. Following [9], theorem 3.4, and [44], §5, we define $\bar{M}(h_+, h_-) = M \times \mathbf{Z}/\sim$, where $(x, k) \sim (h_-^{-1}h_+(x)\, k + 1)$. Write $\tau: U_+ \times \mathbf{Z} \to U_- \times \mathbf{Z}$ for the G-map $\tau(x, k) = (h_-^{-1}h_+(x), k + 1)$. Using this, we define a G-map $t: M \times \mathbf{Z} \to M \times \mathbf{Z}$ by $t = (h_- \times 1)\tau(h_+^{-1} \times 1)$. The map t induces a G-homeomorphism $T: \bar{M}(h_+, h_-) \to \bar{M}(h_+, h_-)$. The map T is given by $T(x, k) = (x, k + 1)$. Write $\theta: M \times \mathbf{Z} \to \bar{M}(h_+, h_-)$. There is a natural map $i: M \to \bar{M}(h_+, h_-)$ given by $i(x) = (x, 0)$ for all $x \in M$. Now we compare M with $\bar{M}(h_+, h_-)$ ([44] Proposition 7.8). For this we need the following Lemma.

LEMMA A.15. *Let $A \subset U_-$ be a closed subset of Y. Then there is a G-isotopy $f_t: U_- \to M$ from the inclusion map to a G-homeomorphism through G-embeddings such that f_t fixes A pointwise.*

Proof. In Corollary A.11, we constructed a G-isotopy $h_t^-: U_- \to M$ from the inclusion map to a G-homeomorphism which fixes a G-neighborhood U' of $e(-)$. Since A is closed in Y, there is a G-neighborhood U'' of the positive end of U_- such that $A \subset (U'')_c$. By applying Lemma A.10 to U_-, we construct a G-isotopy $\phi_t: U_- \to U_-$ from the identity with compact support in U_- such that $\phi_1(U'') \cup U' = U_-$. Then $\phi_1(A) \subset U'$. Extend ϕ_t to a G-isotopy $\Phi_t: M \to M$ by the identity outside U_-. Define $f_t = \Phi_1^{-1} h_1^{-1} \phi_1$. Then, f_0 is the inclusion map, f_t fixes A pointwise because $h_t^- \phi_1(a) = \phi_1(a)$ for all $a \in A$ and f_1 is a G-homeomorphism.

Define $M_+ = M \times \mathbf{N}/\sim$ where $\sim$ is the restriction of the relation we used to define $\bar{M}(h_+, h_-)$. Similarly, define $M_- = M \times \mathbf{Z}_-/\sim$, where $\mathbf{Z}_-$ is the set of the nonpositive integers. Also, we write M_0 for $i(M) \subset \bar{M}(h_+, h_-)$ and $U_n = \theta(M \times \{n\}) \cap \theta(M \times \{n+1\}) \subset \bar{M}(h_+, h_-)$.

PROPOSITION A.16. *Let $A \subset M_0$ be closed in $M_\pm$. Then there is a G-isotopy through G-embeddings $\phi_t^\pm: M_0 \to M_\pm$ to a G-homeomorphism fixing A pointwise.*

Proof. This is given in [44], Proposition 7.8. We give the proof for M_+. Consider a sequence $A = A_1 \subset A_2 \subset A_3 \subset \ldots$ of closed subsets of M_+, each contained in M_0 such that $\cup_n(\mathrm{Int}(A_n)) = M_0$. We construct ϕ_t^+ inductively. Set $\phi_t^1 = i$. Suppose that inductively we constructed a G-isotopy $\phi_t^n: M_0 \to \theta(M \times \{0\}) \cup \theta(M \times \{1\}) \cup \ldots \cup \theta(M \times \{n\})$ of the map i, through G-embeddings fixing A, such that ϕ_1^n is a G-homeomorphism. Notice that there is a G-homeomorphism of M to $\theta(M \times \{n+1\})$ mapping U_- to U_n. Then, we can apply Lemma A.15 to find a G-isotopy $f_t: U_n \to \theta(M \times \{n+1\})$, from the inclusion map to a G-homeomorphism, through G-embeddings such that f_t fixes a neighborhood of the negative end of U_n containing $U_n \cap (\cup_{t \in I} \phi_t^n(A_n))$. Then, f_t extends as the identity outside U_n to a G-isotopy

$$\begin{aligned} f_t'\colon & \theta(M \times \{0\}) \cup \theta(M \times \{1\}) \cup \ldots \cup \theta(M \times \{n\}) \\ & \to \theta(M \times \{0\}) \cup \theta(M \times \{1\}) \cup \ldots \cup \theta(M \times \{n+1\}). \end{aligned}$$

Define ϕ_t^{n+1} by $f_t'\phi_t^n$. This completes the construction of the maps $\phi_t^1, \phi_t^2, \phi_t^3, \ldots$. Define $\phi_t^+|_{\mathrm{Int}(A_n)} = \phi_t^n|_{\mathrm{Int}(A_n)}$. This extends to a G-isotopy on M which fixes A.

COROLLARY A.17. *Let $B \subset M_0$ be closed in $\bar{M}(h_+, h_-)$. Then there is a G-isotopy through G-embeddings $\Phi_t^\pm: M_0 \to \bar{M}(h_+, h_-)$ to a G-homeomorphism fixing B pointwise.*

In particular, this means that (M, f) is an infinite cyclic cover of $(M(h_+, h_-), f(h_+, h_-))$ ([9], Theorem 3.4).

Let X be a locally compact G-ANR. In [46], an alternative definition of the equivalence relation in $\mathrm{Wh}_G^{\mathrm{Top}}(X)$ is given: Let Y and Y' be locally compact G-ANR's and X is proper strong G-deformation retraction of Y and Y'. Then $(Y, X) \sim (Y', X)$ if there is a G-homeomorphism $Y \times Q_G \cong Y' \times Q_G$ which commutes up to proper G-homotopy with the natural inclusions of X. In [46], it is proved that the relation generated by G-CE maps and the above relation produce the same group. Using Theorem A.1, we can assume that any element of $\mathrm{Wh}_G^{\mathrm{Top}}(X)$ can be represented by a Q_G-manifold.

In what follows, we use the notation from Section 7. Let X be a compact G-ANR. Let $j''\colon \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R}) \to \mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$ be defined as follows: If $x \in \mathrm{Wh}_G^{\mathrm{Top}}(X \times \mathbf{R})$ is represented by a proper strong G-deformation retraction, $f\colon M \to X \times \mathbf{R}$, where M is a Q_G-manifold. Define $j''(x) = (M', X \times S^1)$. By the remark above, this is a well-defined map. Also $j''(x)$ is a transfer invariant element of $\mathrm{Wh}_G^{\mathrm{Top}}(X \times S^1)$, and $\mathrm{tr}j''(x) = x$. From the definition of $i_c'\colon K_{-1G}^{\mathrm{PL}}(X)_c \to \tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$, we can prove that $i_c'v'(x) = v''j''(x)$: notice, first of all, that

$$s_c i_c' v'(x) = v'(x) \quad \text{and} \quad s_c v'' j''(x) = v'\mathrm{tr}j''(x) = v'(x)$$

and so

$$s_c i_c' v'(x) = s_c v'' j''(x) \Rightarrow i_c' v'(x) - v'' j''(x) \in \mathrm{Ker}(s_c) = \mathrm{Im}(i_c)$$

and there is $b \in \tilde{K}_{0G}^{\mathrm{PL}}(X)_c$ such that $i_c'v'(x) - v''j''(x) = i_c(b)$. But $i_c'v'(x) - v''j''(x)$ is a transfer invariant element of $\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$ and so $i_c(b)$ is a transfer invariant element of $\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$. But the only transfer invariant element of $\tilde{K}_{0G}^{\mathrm{PL}}(X \times S^1)_c$ which in $\mathrm{Im}(i_c)$ is the zero element. So,

$$i_c'v'(x) - v''j''(x) = 0 \Rightarrow i_c'v'(x) = v''j''(x).$$

Following the definition of j' in Section 7, we see that $j'(x) = j''(x)$.

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