Tendril of the Hop and Tendril of the Vine: Peter Guthrie Tait and the Promotion of Quaternions, Part I

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Peter Guthrie Tait and the Promotion of Quaternions, Part I

CHRIS PRITCHARD

Two days before his death in the summer of 1901 Peter Guthrie Tait gave his son some handwritten notes on quaternions for safekeeping. Though he had distinguished himself in many areas of mathematical physics and had influenced the work of Thomson and Maxwell it was his evangelical promotion of quaternions which would be remembered in the years to follow and it was fitting that his last energies be devoted to the cause.

The story of quaternions begins on the banks of the River Liffey in Dublin on 16 October 1843 with William Rowan Hamilton's sudden inspired realisation that a vector can be converted into another vector using one number to adjust the magnitude and three to effect the rotation. The general form of a quaternion q is:

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$$

where a, b, c and d are real numbers and **i**, **j** and **k** are vectors of magnitude $\sqrt{-1}$ in the x, y and z directions. Hamilton had ingeniously dropped the standard algebraic requirement that multiplication be commutative, replacing it with the combination law

$$i^2 = j^2 = k^2 = ijk = -1.$$

After a ten-year gestation his massive book, *Lectures on Quaternions*, came to press and Tait, then at Peterhouse, Cambridge, purchased a copy on the sole basis of an advertisement in the *Athenaeum*. He digested the first six chapters without difficulty but faltered at Hamilton's attempts in the seventh to apply quaternions to the solution of physical problems. The impasse was so great that Tait put the book aside for five years. Then, in July 1858, he read in *Crelle's Journal* a paper by Helmholtz on vortex motion which appeared to open up the possibility of applying quaternions to potential theory. He turned to Hamilton's *Lectures* once more, determined to master the sections on quaternion differentials which had defeated him on first reading.

At this time Hamilton's reputation as a mathematician was probably second to none in the English-speaking world though he was ostensibly engaged in astronomical duties at the Dunsink Observatory, outside Dublin. Tait was by now Professor of Mathematics at Queen's College, Belfast. In August 1858 Tait's colleague, Thomas Andrews, wrote to Hamilton to introduce the 27 year-old Tait and to prepare the ground for the remarkable correspondence into which they soon entered. From the outset, Tait demonstrated unusual familiarity with quaternions but earnestly sought the support of the master in untangling some of the complexities. Following

Hamilton's opening letter of 19 August they communicated with each other on a weekly basis for eleven months and thereafter less frequently. Many of the letters were of phenomenal length and were posted off a few pages at a time. One of Hamilton's letters ran to 88 pages followed by eight postscripts! The correspondence reawakened Hamilton's interest in quaternions and enabled Tait to become proficient in their manipulation and application.

There were three problems which the proponents of quaternions faced during this period. Firstly, Hamilton's highly theoretical exposition had proved totally unsuitable as a text for the quaternion novice. Indeed, the astronomer, Sir John Herschel, had himself tried to come to terms with Hamilton's tome but had given up after just three chapters, lamenting that if it was too difficult for a former Senior Wrangler then what chance had undergraduates. Someone would have to sit down and write a comprehensible, elementary text which would give Cambridge dons and their students a grounding in the novel methods. From the correspondence it is clear that at some point in 1859 both Hamilton and Tait resolved to do just that. Secondly, as yet the special fitness of quaternions for research in the physical sciences had not been demonstrated in original papers. With Hamilton already in his mid-fifties and increasingly turning to the bottle, if this were to be achieved it would only be achieved by Tait. And thirdly, at least one leading physicist other than Tait would have to endorse quaternions. It soon became clear that there were no circumstances under which the obvious candidate, William Thomson, would be that physicist.

With the Board of Trinity College, Dublin promising a publication grant of £100 Hamilton set about writing a simple and practical 300 page guide to quaternions, terming it in his correspondence with Tait a 'working volume'. Tait had decided on the same course, but as the recipient of many of Hamilton's unpublished results he felt honour-bound to seek express permission. In the summer of 1859 they struck a deal. Hamilton would reserve the right to publish first and would concentrate on the underlying principles of quaternions, Tait would emphasise their physical applications. Aware of the Irishman's publication schedule, Tait planned his own schedule accordingly. Unfortunately, advance publicity for Tait's proposed book, put out by the publishers Macmillan, found its way into Hamilton's hands and, as his biographer, Seán O'Donnell, puts it, 'a distinct chill set in for a while' [1]. Indeed the matter was only resolved by Tait's coming out to Dunsink in the summer of 1861.

Before the chill, Tait, with Hamilton's encouragement, was undertaking original quaternionic research on the Fresnel wave surface. When, in 1860, the Chair of Natural Philosophy in Edinburgh University fell vacant as a result of the retirement of Forbes, Hamilton was eager to champion Tait's candidature, attesting that,

> ' in consequence of a rather copious correspondence .... on mathematical and physical subjects, including Quaternions, and the Wave-surface of Fresnel, my opinion of the energy and other

> capabilities of Professor Tait for any such appointment is very favourable indeed.'

It is surely no coincidence that in his inaugural public lecture, delivered in the University of Edinburgh on 7 November 1860, the successful candidate spoke for some time on quaternions, and in very positive terms:

> ' Newton, the first great invader of this domain of unexplored truth, the application of Mathematical Physical Science, had to forge his own weapon, and, though suffering under such disadvantages, he wielded it with a giant's grasp .... now that Britain is once more taking her proper place in this honourable and glorious struggle, she has a second time furnished the means; reflecting men look to the "wondrous machinery of Quaternions," the last great step which has been made in fitting mathematics to deal adequately with force and motion; they look, I say, to the yet infant invention of Quaternions, as destined to aid us to a degree unsuspected in the interrogation of Nature.'

It appears that as Tait moved back to the city in which he grew up he was in the process of assuming Hamilton's mantle as the prime promoter of quaternions.

Progress on both texts proved painfully slow, and when the books did, finally, see the light of day – Hamilton's *Elements of Quaternions* posthumously in 1866 and Tait's *Elementary Treatise on Quaternions* [2] the following year – they furnished neither profit nor enlightenment. Quaternions offered a revolutionary cleaving together of physics and mathematics but these rather advanced expositions proved far from popular with Cambridge dons. *Introduction to Quaternions*, written in the main by Kelland but with advice from Tait and published in 1873, fitted the bill far better and its accessibility ensured that it was in demand for many years.

The question that must be answered at this stage is: why did Tait's *Elementary Treatise* fail to strike the right note? Certainly, Tait was distracted by the stop-start publishing saga but there were two greater barriers to success; firstly, the time and effort which he had to put into his collaboration with William Thomson on their magnificent physics text *Treatise on Natural Philosophy* – he and not Thomson was the driving force behind that project and it cost him time and energy when he could have been preparing the *Elementary Treatise* – and secondly, the enduring hostility of Thomson towards quaternions. From first to last, Thomson's physics was locked into Cartesians. In later life he would write of quaternions that they 'came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell'. And later still he would describe the vector as a 'useless survival, or offshoot, from quaternions' that 'has never been of the slightest use to any creature'. Though Thomson never attacked quaternion or vector methods in print it is likely that the views of Britain's most influential scientist were well known. Michael Crowe has written, in his definitive *History of Vector Analysis* [3], 'This was the golden opportunity for quaternions, for their inclusion in such an important work on mathematical physics would have acquainted numerous readers with quaternion methods.'

On the plus side the second edition of the *Elementary Treatise* provided both Gibbs and Heaviside with their first taste of quaternions and, with the appearance of German and French translations in the 1880s, quaternions were made accessible, at least in terms of language, to European physicists. Tait's great success had been in applying quaternions to geometry, to kinematics, to homogeneous strains, to the rotation of a rigid body and to dynamics, with treatments of the Foucault pendulum and the Fresnel wave surface, the effects of electric currents on magnets and on each other and the theorems of Gauss, Green and Stokes. But we are getting a little ahead of ourselves for many of these applications owe a good deal to the fertile mind of one James Clerk Maxwell.

As a schoolboy at the Edinburgh Academy back in the 1840s Maxwell had struck up a close and lifelong friendship with Tait and much of their correspondence has recently been made available by Harman [4]. Maxwell's letters, postcards and draft papers reveal one strand of his intellectual journey from complete novice in quaternions in 1865 to mastery by 1871. We can see too the strong influence of Tait in prompting, encouraging, even exhorting his friend to immerse himself in quaternions, and just as plainly, Maxwell's ambivalence towards them.

We know that Maxwell was aware of quaternions by March 1865 because on the seventh of that month he enquired of Tait, 'Does anyone write quaternions but Sir W. Hamilton and you?' He followed with increasing interest his friend's progress towards producing a book on the subject but at this stage took few steps to become familiar with quaternions himself. Towards the end of 1867, he checked with Tait when the book was due and sought to clarify the name of the differential operator

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

given in Tait's 1862 paper 'Note on a quaternion transformation' [5]. Maxwell was already familiar with its usage both by Lamé and by Betti but was keen to establish a consistency of terminology. This theme of ensuring that all scientists use the same jargon, the same symbols and the same three-dimensional representation was one to which he returned time and time again.

Tait responded by return of post, confirming that the book had been published but making it plain that its success would rest not so much on the number of copies it sold but rather on the number of scientists sold on quaternions by reading it. He went on to explain that the operator is used for the flux or rate of change and expressed interest in checking Lamé's paper, not for the terminology used, but out of concern that Hamilton's priority might be compromised. Then he implored Maxwell to 'read the last 20 or 30 pages of my book. I think you will see that 4^{ions} are worth getting up, for there it is shown that they go into that [operator] business like greased lightning.'

As the 1860s came to a close Tait was turning out his best work on quaternions. Two papers, in particular, brought acclaim in the form of the Keith Prize of the Royal Society of Edinburgh: 'On the rotation of a rigid body about a fixed axis' (1868) and 'On Green's and other allied theorems' (1870) [5]. A laudatory note, penned by Maxwell in the autumn of 1870 for the occasion of the prize-giving, described the former paper as 'very powerful' but the latter as 'really great'. Though Maxwell harboured many reservations, he was clear about the merits of quaternions. His eulogy went on:

> ' The work of mathematicians is of two kinds, one is counting, the other is thinking. Now these two operations help each other very much, but in a great many investigations the counting is such long and such hard work, that the mathematician girds himself to it as if he had contracted for a heavy job, and thinks no more that day. Now Tait is the man to enable him to do it by thinking, a nobler though more expensive occupation, and in a way by which he will not make so many mistakes as if he had pages of equations to work out.'

Maxwell was especially interested in the way physical quantities related to one another. He was conscious of Thomson's analogy between Fourier's work on heat and Poisson's work on electricity, an analogy which Thomson had been able to perceive only because the same mathematical representation was being used in the two apparently disparate disciplines. Quaternions, it seemed to Maxwell, were uniquely equipped to bring out such links.

So, what was it about Tait's papers, particularly the second, that excited Maxwell so much? Firstly, it was the possibility of using Hamilton's differential operator and Tait's major quaternion innovation, its square

$$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$$

to formulate the as yet unchristened space relations Grad, Convergence and Curl in such a way that their central position and epiphytic nature within electricity theory could be discerned. In an appendix to the 1870 paper, Tait dispensed with Cartesian representation altogether, yet succeeded in finding volumes, areas and lengths of arcs via quaternion integration, with the theorems of Gauss, Green and Stokes elicited as special cases.

Secondly, it was a property of quaternions themselves, brought out by Tait and intrinsically attractive to Maxwell, a man who enjoyed an extraordinary capability to think geometrically or visually. As he would later write in *Nature*, the quaternion offered a new

> ' method of thinking, and not, at least for the present generation, a method of saving thought..... It calls upon us at every step to form

a mental image of the geometrical features represented by the symbols, so that in studying geometry by this method we have our minds engaged with geometrical ideas, and are not permitted to fancy ourselves geometers when we are only arithmeticians.'

The opportunity was there for Maxwell to describe electromagnetic forces with clarity, power, expressiveness and concision. Yet, in deciding to incorporate quaternion results alongside co-ordinates in his *Treatise on Electricity and Magnetism* [6] the significant factor was his valued, life-long friendship with Tait.

But where was Maxwell to begin? After all, there was no common, agreed terminology. On 7 November 1870, Maxwell wrote to Tait asking for guidance especially with regard to the differential operator which he knew would play a central role:

' What do you call this? Atled? [i.e. delta inverted] I want to get a name or names for the result of it on scalar or vector functions of the vector of a point. Here are some rough hewn names. Will you, like a good Divinity shape their ends so as to make them stick.

(1) The result of ∇ applied to a scalar function might be called the slope of the function. Lamé would call it the differential parameter, but the thing itself is a vector, now slope is a vector word, whereas parameter has, to say the least a scalar sound.

2 If the original function is a vector then ∇ applied to it may give two parts. The scalar part I would call the twist of the vector function. Here the word twist has nothing to do with a screw or helix. (If the words turn or version would do they would be better than twist for twist suggests a screw.)

' Twirl is free from the screw motion and is sufficiently racy. Perhaps it is too dynamical for pure mathematicians so for Cayleys sake I might say Curl (after the fashion of Scroll).

' Hence the effect of ∇ on a scalar function is to give the slope of that scalar and its effect on a vector function is to give the convergence and the twirl of that vector.

' The result of ∇^2 applied to any function may be called the concentration of that function because it indicates the mode in which the value of the function at a given point exceeds (in the Hamiltonian sense) the average value of the function in a little spherical surface drawn around it.

....

' What I want is to ascertain from you if there are any better names for these things, or if these names are inconsistent with anything in Quaternions, for I am unlearned in quaternion idioms and may make solecisms.

' I want phrases of this kind to make statements in

> electromagnetism and I do not wish to expose either myself to the contempt of the initiated, or Quaternions to the scorn of the profane.'

Now Tait's assistant, William Robertson Smith was, singularly, both physicist and Hebrew scholar. He suggested that since the symbol for the differential operator resembled an Assyrian harp the name *nabla* be adopted. This found immediate favour with Maxwell, who in a letter to Tait of 14 November 1870 went on to make his intentions explicit:

> ' With regard to my dabbling in Hamilton I want to leaven my book with Hamiltonian ideas without casting the operations into a Hamiltonian form for which neither I nor, I think, the public are ripe.
>
> ' Now the value of Hamiltons idea of a vector is unspeakable and so are those of the addition and multiplication of vectors. I consider the form into which he put these ideas, such as the names of Tensor, Versor, Quaternion &c important and useful but subject to the approval of the mathematical world.....'

(Maxwell appears to use the word 'unspeakable' here to mean 'goes without saying'.)

With the terminology clarified Maxwell began to apply quaternions to electromagnetism. Or rather, he began to express electromagnetic quantities using just the vector part of quaternions, dispensing with the scalar part altogether. In his manuscript on the 'Application of Quaternions to Electromagnetism', written in that key month of November 1870, he began by arguing that,

> ' The invention of the Calculus of Quaternions by Hamilton is a step towards the knowledge of quantities related to space which can only be compared for its importance with the invention of triple coordinates by Descartes......"

He proceeded to define vectors and scalars, to add and subtract vectors and to show that the product of two vectors is part scalar, part vector. Notably, he defined the scalar product as $S.(OA.OB) = OA.OB \cos AOB$, a positive quantity for acute angles. In justification of this departure from the definition of Hamilton and Tait he cited the case in which OA represents a force and OB the displacement of a body acted on by the force, with the work done being the scalar product. He introduced the notion of the point vector and argued that the potential of a point is a scalar function of it while the resultant force is a vector function of it. With the preliminaries over he unveiled his vector functions of the electromagnetic field. They were labelled using German capital letters, ostensibly out of deference to Bismarck, though more plausibly as a mark of respect to Helmholtz.

Electromagnetic momentum	***Mechanical force***
Magnetic force	***Velocity***
Electric current	***Magnetization***
Electric displacement	***Magnetic induction***
Electromotive force	***Conduction current***

Maxwell introduced the differential operator which, when applied to a scalar function of a position vector, yields the *slope* and when applied to a vector function of a position vector gives the sum of a scalar product (dubbed *convergence*) and a vector product (dubbed *curl*). For a vector function which represents the velocity of a fluid the rate at which the motion converges to a given point is its *convergence* while its *curl* is its rotation. The square of the differential operator gives a function's *concentration*, the value of the function at a specific point in excess of the mean value in the immediate locale.

Maxwell recognised that free of their scalar parts these vector functions could be represented in three-dimensional space without displacing the origin. Unfortunately, scientists were split over the orientation of the three axes, so in May 1871 Maxwell and Tait, now writing to each other on postcards which required only a ha'penny stamp, debated the issue of which was the best direction for them. Maxwell wrote:

> ' I am desolated! I am like the Ninevites! Which is my right hand? Am I perverted? a mere man in a mirror, walking in a vain show? What saith the Master of Quaternions? i to the South, j to the West and k to the Heavens above. .... Lay hold of one of these and turn screw wise and you rotate +. To this agree the words of my text. But what say T and T' §234. They are perverted. If a man at Dublin finds a watch, he lays it on the ground with its face up, and its hand go round from S to W and he says this is + rotation about an axis looking upwards. If the watch goes to Edin^{h} or Glasgow T' or T carefully lays it down on its face, and after observing the gold case he utters the remarkable aid to memory contained in §234 of the book. Please put me out of my suspense..... I must get hold of the Math. Society and get a consensus on the craft.'

In short, Maxwell's argument is this: We have two models for the axes. In the one, rotation in the **i-j** plane is defined by the rotation of the hands of a watch placed face up; in the other the rotation is defined by the rotation of the hands of a watch placed face down. We must all work to the same system or confusion will reign.

It is clear that Tait was already in agreement that the orientation of the axes should be the second of these two models. In a postcard to Maxwell the very next day (9 May 1871) he made it clear that he had adopted Hamilton's convention to avoid confusion but had deferred to Thomson when it came to the *Treatise*. The advantage was its consonance with the Earth's rotation to anyone in the northern hemisphere. Two days later Maxwell did indeed seek a definitive ruling at the London Mathematical Society. At this meeting of 11 May 1871 he argued:

> ' In pure mathematics little inconvenience is felt from this want of uniformity; but in astronomy, electromagnetics, and all physical sciences, it is of the greatest importance that one or other system should be specified and persevered in.'

He also introduced an allusion to the tendrils of the hop and the vine, following a suggestion from the crystallographer, William Hallowes Miller, and noted that the one system is a mirror image of the other. Following the meeting he reported to Tait that

> ' No arguments in favour of the opposite system being given, the right-handed system, symbolised by a corkscrew or the tendril of the vine, was adopted by the Society.'

The very next day Maxwell sent another postcard, confirming his intention to use the right-handed system in his *Treatise on Electricity and Magnetism* and pleading, almost childlike:

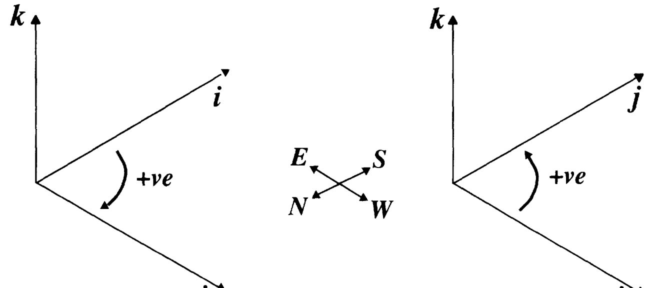
> ' Tell me that I may print.'

And yet another postcard towards the end of May closed with Maxwell's thanks:

> ' Finally, I thank you and praise you for turning me from the system of the hop to that of the vine. I have perverted the whole of electro-magnetics to suit.'

During 1871 and 1872, as *Treatise on Electricity and Magnetism* was being readied for the presses, Maxwell consulted Tait time and time again. Maxwell was pleased that a basic terminology and geometrical representation had been agreed but right up to the time of publication in

TENDRIL OF THE HOP (left-handed screw) | TENDRIL OF THE VINE (right-handed screw)



Hamilton
Listing
Tait (Quaternions)

Linnæus
Thomson & Tait
Maxwell
L. M. S.

February-March 1873 he argued for the adoption of a standardised notation and style. In mid-June 1871 he wrote to Tait:

> ' I think you should make a supplementary book on Quaternions explaining the true principles of dots and brackets and defining the limits of the sway of symbols as the Spaniards define the end of an interrogation or we that of a quotation.'

And again in October 1872:

> ' The great want of the day is a Grammar of 4^{nions} in the form of dry rules as to notation and interpretation not only of S, T, U, V but of . ( ) and the proper position of $d\sigma$ &c. Contents, Notation, Syntax, Prosody, Nablody.'

With the publication of *Treatise on Electricity and Magnetism* in 1873 Maxwell sent a rather mixed message to the scientific community. On the face of it, it was a message of qualified support for quaternions. For some scientists, no doubt, the very fact that such an eminent physicist had chosen to immerse himself in quaternions was reason enough to follow suit. Others must have taken a prompt from the line Maxwell adopted in the book. He began by explaining the advantage in physics of using the 'more primitive and more natural' quaternions rather than Cartesians. Forces are more readily represented in terms of magnitude and direction than by attaching a triplet of co-ordinates. However, in ending some sections of the book by summarising the important results in quaternion form, Maxwell was making more of a gesture than a genuine attempt to break new ground. The main body of the text was presented in Cartesians. Though there were a number of promotional references to the methods of Hamilton and Tait, readers were left in no doubt about the distinction Maxwell made between quaternion ideas, with which he was sympathetic, and their operations and methods, with which he had no truck. Nevertheless, it is likely that both Gibbs and Heaviside were persuaded to study the second edition of Tait's *Elements of Quaternions* by reading Maxwell's *Treatise* and this in itself is of some moment in the history of vector analysis.

And what better way for Tait to fulfil his role of arch promoter than to supply the journal *Nature* with an anonymous, adulatory review in which much was made of the fact that Maxwell had used quaternions. In alluding to Maxwell's name as one 'which requires only the stamp of antiquity to raise it almost to the level of that of Newton' Tait sought to bolster quaternions whilst paying a fitting tribute to a personal friend and outstanding scientist.

In Part II of this paper, 'Flaming Swords and Hermaphrodite Monsters', Tait locks horns with Arthur Cayley, supporter of co-ordinate geometry, and Willard Gibbs, promoter of vectors. It will appear in the next issue of the *Gazette*.

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The pointless perils of subediting

In Robert L. Weber's delightful *Science with a Smile* (Institute of Physics, Bristol, 1992), the following occurs on p. 303 as a quote from John E. Littlewood's *A Mathematician's Miscellany* (Methuen, London, 1960).

I once challenged Hardy to find a misprint on a certain page of a joint paper: he failed. It was his own name: 'G H Hardy'.

Not being able to see anything wrong in the above, I got out my copy of Littlewood, which first appeared in 1953. On p. 38, we find the following.

I once challenged Hardy to find a misprint on a certain page of a joint paper: he failed. It was his own name: 'G, H. Hardy'.

Bollobás's edition, *Littlewood's Miscellany* (CUP, 1986), has the correct form on p. 56.

Sent in by David Singmaster.

An unknown quantity

Corbett, however, was a different matter, a personal symbol. Let Corbett represent x and the equation would then be $x - x = 0$, the equation representing the preferable state. Or representing himself as y, $x / y = 0$. And Klein was testing him here, of course, not really calling him off but pretending to, in order to determine whether ...

From *Temple Dogs* by Robert L. Duncan, sent in by Frank Tapson.