From Goeritz matrices to quasi-alternating links

by

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Introduction

Knot Theory is currently a very broad field. Even a long survey can only cover a narrow area. Here we concentrate on the path from Goeritz matrices to quasi-alternating links. On the way, we often stray from the main road and tell related stories, especially if they allow as to place the main topic in a historical context. For example, we mention that the Goeritz matrix was preceded by the Kirchhoff matrix of an electrical network. The network complexity extracted from the matrix corresponds to the determinant of a link. We assume basic knowledge of knot theory and graph theory, however, we offer a short introduction under the guise of a historical perspective.

1 Short historical introduction

Combinatorics, graph theory, and knot theory have their common roots in Gottfried Wilhelm Leibniz' (1646-1716) ideas of Ars Combinatoria, and Geometria Situs. In Ars Combinatora, Leibniz was influenced by Ramon Llull (1232 – 1315) and his combinatorial machines (Figure 1.1; [Bon]).



Fig. 1.1; Combinatorial machine of Ramon Llull from his Ars Generalis Ultima

Geometria (or Analysis) Situs seems to be an invention of Leibniz. I am not aware of any Ancient or Renaissance influence (compare however [P-21]. The first convincing example of geometria situs was proposed by Heinrich Kuhn in a letter written in 1735 to Leonard Euler (1707-1783). Kuhn (1690-1769) was a Danzig (Gdańsk) mathematician born in Königsberg, studied at the Pedagogicum there, and in 1733 settled in Danzig as a mathematics professor at the Academic Gymnasium (he was also a co-founder of the Nature Society) [Janus]. Kuhn communicated to Euler the puzzle of bridges of Königsberg, suggesting that this may be an example of geometria situs. Kuhn was communicating, in fact, through his friend Carl Leonhard Gottlieb Ehler (1685-1753), correspondent of Leibniz and future mayor of Danzig. The first letter by Ehler did not survive but in the letter of March 9, 1736 he writes: "You would render to me and our friend Köhn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of Calculi Situs, worthy of your great genius. I have added a sketch of the said bridges ..." In the reply of April 3, 1736 Euler writes "... Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others.

In the meantime, most noble Sir, you have assigned this question to the **geometry of position**, but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way ... " [H-W]. However when composing his famous paper on bridges of Königsberg, Euler already agrees with Kuhn suggestion. The geometry of position figures even in the title of the paper *Solutio problematis ad geometriam situs pertinentis*.¹

The first paper mentioning knots from the mathematical point of view is that of Alexandre-Theophile Vandermonde (1735-1796) *Remarques sur les problèmes de situation* [Va]. Carl Friedrich Gauss (1777-1855) had interest in Knot Theory whole his life, starting from 1794 drawings of knots, the drawing of a braid with complex coordinates (c. 1820), several drawing of knots with "Gaussian codes", and Gauss' linking number of 1833. He did not publish anything however; this was left to his student Johann Benedict Listing (1808-1882) who in 1847 published his monograph (Vorstudien zur Topologie, [Lis]). The monograph is mostly devoted to knots, graphs and combinatorics.

In the XIX century Knot Theory was an experimental science. Topology (or geometria situs) had not developed enough to offer tools allowing precise definitions and proofs² (here Gaussian linking number is an exception). Furthermore, in the second half of that century Knot Theory was developed mostly by physicists (William Thomson (Lord Kelvin)(1824-1907), James

¹In the paper, Euler writes: "The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the "geometry of position" (geometria situs). This branch of geometry deals with relations dependent on position; it does not take magnitudes into considerations, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them. Hence, when a problem was recently mentioned, which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position–especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position. 2. The problem, which I am told is widely known, is as follows: in Königsberg in Prussia, there is... "[Eu, B-L-W].

²Listing writes in [Lis]: In order to reach the level of exact science, topology will have to translate facts of spatial contemplation into easier notion which, using corresponding symbols analogous to mathematical ones, we will be able to do corresponding operations following some simple rules.

Clerk Maxwell (1831-1879), Peter Guthrie Tait(1831-1901)) and one can argue that the high level of precision was not appreciated³. We outline the global history of the Knot Theory in [P-21] and in the second chapter of my book on Knot Theory [P-Book]. In the next subsection we deal with the mathematics developed in order to understand precisely the phenomenon of knotting.

1.1 Precision comes to Knot Theory

Throughout the XIX century knots were understood as closed curves in a space up to a natural deformation, which was described as a movement in space without cutting and pasting. This understanding allowed scientists (Tait, Thomas Penyngton Kirkman, Charles Newton Little, Mary Gertrude Haseman) to build tables of knots but didn't lead to precise methods allowing one to distinguish knots which could not be practically deformed from one to another. In a letter to O. Veblen, written in 1919, young J. Alexander expressed his disappointment⁴: "When looking over Tait On Knots among other things, He really doesn't get very far. He merely writes down all the plane projections of knots with a limited number of crossings, tries out a few transformations that he happen to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant ... for links is an invariant merely of the particular projection of the knot that you are dealing with, - the very thing I kept running up against in trying to get an integral that would apply. The same is true of his 'Beknottednes'."

In the famous Mathematical Encyclopedia Max Dehn and Poul Heegaard outlined a systematic approach to topology, in particular they precisely formulated the subject of the Knot Theory [D-H], in 1907. To bypass the notion of deformation of a curve in a space (then not yet well defined) they introduced lattice knots and the precise definition of their (lattice) equivalence.

³This may be a controversial statement. The precision of Maxwell was different than that of Tait and both were physicists.

⁴We should remember that it was written by a young revolutionary mathematician forgetting that he is "standing on the shoulders of giants." [New]. In fact the invariant Alexander outlined in the letter is closely related to Kirchhoff matrix, and extracted numerical invariant is equivalent to complexity of a signed graph corresponding to the link via Tait translation; see Subsection 1.4.

Later Reidemeister and Alexander considered more general polygonal knots in a space with equivalent knots related by a sequence of Δ -moves; they also explained Δ -moves by elementary moves on link diagrams – Reidemeister moves (see Subsection 1.6). The definition of Dehn and Heegaard was long ignored and only recently lattice knots are again studied. It is a folklore result, probably never written down in detail⁵, that the two concepts, *lattice knots* and *polygonal knots*, are equivalent.

1.2 Lattice knots and Polygonal knots

In this part we discuss two early XX century definitions of knots and their equivalence, by Dehn-Heegaard and by Reidemeister. In the XIX century knots were treated from the intuitive point of view and was P. Heegaard in his 1898 thesis who came close to a formal proof that there are nontrivial knots.

Dehn and Heegaard gave the following definition of a knot (or curve in their terminology) and of equivalence of knots (which they call isotopy of curves)⁶.

Definition 1.1 ([D-H])

A curve is a simple closed polygon on a cubical lattice. It has coordinates x_i, y_i, z_i and an isotopy of these curves is given by:

- (i) Multiplication of every coordinate by a natural number,
- (ii) Insertion of an elementary square, when it does not interfere with the rest of the polygon.
- (iii) Deletion of the elementary square.

Elementary moves of Dehn and Heegaard can be summarized/explained as follows:

- (DH_0) Rescaling. We show in [P-Book] that this move is a consequence of other Dehn-Heegaard moves.
- (DH_1) If a unit square intersects the lattice knot in exactly two neighboring edges then we replace this edges by two other edges of the square, as illustrated in Fig. 1.2 (DH_1) .

 $^{^{5}}$ It is however long routine exercise

⁶Translation from German due to Chris Lamm.

 (DH_2) If a unit square intersects the lattice knot in exactly one edge then we replace this edges by three other edges of the square, as illustrated in Fig. 1.2 (DH_2) .



Fig. 1.2; Lattice moves DH_1 and DH_2

In this language, lattice knots (or links) and lattice isotopy are defined as follows.

Definition 1.2 A lattice knot is a simple closed polygon on a cubical lattice. Its vertices have integer coordinates x_i, y_i, z_i and edges, of length one, are parallel to one of the coordinate axis. We say that two lattice knots are lattice isotopic if they are related by a finite sequence of elementary lattice ("square") moves as illustrated in Fig. 1.2 (we allow DH_1 -move, DH_2 -move and its inverse DH_2^{-1} -move). These are moves (ii) and (iii) of Dehn and Heegaard.

Below we give a few examples of lattice knots.

They can be easily coded as (cyclic) words over the alphabet $\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$. For example the trivial knot can be represented by $xyx^{-1}y^{-1}$, the trefoil knot by $x^2z^3y^2x^{-1}z^{-2}y^{-3}zx^2y^2x^{-3}y^{-1}z^{-2}$, and the figure-eight knot by $y^2z^2xy^{-3}x^2y^2z^{-1}x^{-4}y^{-2}x^3yz^2x^{-2}z^{-3}$; see Figure 1.3.



Fig. 1.3; A trivial lattice knot, with 4 edges, 4 right angles and no changes of planes. A lattice trefoil with 24 edges, 12 right angles and 8 changes of

planes. A lattice figure-eight knot with 30 edges, 14 right angles and 8 changes of planes. The numbers are the z-levels and the dots are the sticks in z-direction.

1.3 Early invariants of links

The fundamental problem in knot theory is⁷ to be able to distinguish nonequivalent knots. It was not achieved (even in the simple case of the unknot and the trefoil knot) until Jules Henri Poincaré (1854-1912) in his "Analysis Situs" paper ([Po-1] 1895) laid foundations for algebraic topology. According to W.Magnus wrote [Mag]: Today, it appears to be a hopeless task to assign priorities for the definition and the use of fundamental groups in the study of knots, particularly since Dehn had announced [De] one of the important results of his 1910 paper (the construction of Poincaré spaces with the help of knots) already in 1907.. Wilhelm Wirtinger (1865-1945) in his lecture delivered at a meeting of the German Mathematical Society in 1905 outlined a method of finding a knot group presentation (it is called now the Wirtinger presentation of a knot group) [Wir], but examples using his method were given after the work of Dehn.

1.4 Kirchhoff's complexity of a graph

Gustav Robert Kirchhoff (1824-1887) in his fundamental paper on electrical circuits [Kir]. published in 1847, defined the complexity of a circuit. In the language of graph theory, this complexity of a graph, $\tau(G)$, is the number of spanning trees of G, that is trees in G which contain all vertices of G. It was noted in [BSST] that if e is an edge of G that is not a loop then $\tau(G)$ satisfies the deleting-contracting relation:

$$\tau(G) = \tau(G - e) + \tau(G/e),$$

where G - e is the graph obtained from G by deleting the edge e, and G/e is obtained from G by contracting e, that is identifying endpoints of e in G - e. The deleting-contracting relation has an important analogue in knot theory, usually called a skein relation (e.g. Kauffman bracket skein relation). Connections were discovered only about a hundred years later (e.g. the Kirchhoff complexity of a circuit corresponds to the determinant of the knot or link yielded by the circuit, see the next subsection).

For completeness, and to be later to see clearly connection to Goeritz matrix in knot theory, let us defined the (version of) the Kirchhoff matrix of

⁷One should rather say "was"; there are algorithms allowing recognition of any knots, even if very slow. Modern Knot Theory looks rather for structures on a space of knots or for a mathematical or physical meaning of knot invariants.

a graph, G, determinant of which is the complexity $\tau(G)$.

Definition 1.3 Consider a graph G with vertices $\{v_0, v_1, \ldots, v_n\}$ possibly with multiple-edges and loops (however loops are ignored in definitions which follows).

- (1) The adjacency matrix of the graph G is the $(n + 1) \times (n + 1)$ matrix A(G) whose entries, a_{ij} are equal to the number of edges connecting v_i with v_j ; we set $v_{i,i} = 0$.
- (2) The degree matrix $\Delta(G)$ is the diagonal $(n+1) \times (n+1)$ matrix whose ith entry is the degree of the vertex v_i (loops are ignored). Thus the ith entry is equal to $-\sum_{j=0}^n a_{ij}$.
- (3) The Laplacian matrix Q'(G) is defined to be $\Delta(G) A(G)$; [Big]. Notice, that the sum of rows of Q'(G) is equal to zero and that Q'(G) is a symmetric matrix.
- (4) The Kirchoff matrix (or reduced Laplacian matrix) Q(G) of G is obtained from Q'(G) by deleting the first row and the first column from Q'(G).

Theorem 1.4 $det(Q(G)) = \tau(G)$.

Proof: The shortest proof, I am aware of, is by direct checking that det(Q(G)) satisfies deleting-contracting relation for any edge e, not a loop, that is

$$det(Q(G)) = det(Q(G - e)) + det(Q(G/e)).$$

The above equation plays an important role in showing in Section 7 that an alternating link is a quasi-alternating as well. \Box

Example 1.5 Consider the graph $v_2 v_1$. For this graph we have:

$$A(\bigcirc) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad \Delta(\bigcirc) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$
$$Q'(\bigcirc) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}; \quad Q(\bigcirc) = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$det(Q(\bigcirc)) = det \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} = 5 = \tau(\bigcirc).$$

As we will see in the next subsection the corresponding knot is the figure eight knot (Fig. 1.4).

1.5 Tait's relation between knots and graphs.

Tait was the first to notice the relation between knots and planar graphs. He colored the regions of the knot diagram alternately white and black (following Listing) and constructed the graph by placing a vertex inside each white region, and then connecting vertices by edges going through the crossing points of the diagram (see Figure 1.4)[D-H].



Figure 1.4; Tait's construction of graphs from link diagrams, according to Dehn-Heegaard

It is useful to mention the Tait construction going in the opposite direction, from a signed planar graph, G to a link diagram D(G). We replace every edge of a graph by a crossing according to the convention of Figure 1.5 and connect endpoints along edges as in Figures 1.6 and 1.7.



Fig. 1.5; convention for crossings of signed edges (edges without markers are assumed to be positive)



Fig. 1.6; The knot 8_{19} and its Tait graph (8_{19} is the first in tables non-alternating knot)



Fig. 1.7; Octahedral graph (with all positive edges) and the associated link diagram

We should mention here one important observation known already to Tait (and in explicit form to Listing):

Proposition 1.6 The diagram D(G) of a connected graph G is alternating if and only if G is positive (i.e. all edges of G are positive) or G is negative.

A proof is illustrated in Figure 1.8.



Fig. 1.8; Alternating and non-alternating parts of a diagram

1.6 Link diagrams and Reidemeister moves

In this part we define, after Reidemeister, a polygonal knot and link, and Δ -equivalence of knots and links. A Δ -move is an elementary deformation of a polygonal knot which intuitively agrees with the notion of "deforming without cutting and glueing" which is the first underlining principle of topology.

Definition 1.7 (Polygonal knot, Δ -equivalence).

- (a) A polygonal knot is a simple closed polygonal curve in \mathbb{R}^3 .
- (b) Let us assume that u is a line segment (edge) in a polygonal knot K in R³. Let Δ be a triangle in R³ whose boundary consists of three line segments u, v, w and such that Δ∩L = u. The polygonal curve defined as K' = (K - u) ∪ v ∪ w is a new polygonal knot in R³. We say that the knot K' was obtained from K by a Δ-move. Conversely, we say that L is obtained from L' by a Δ⁻¹-move (Fig. 1.9). We allow the triangle

 Δ to be degenerate so that the vertex $v \cap w$ is on the side u; in other words we allow subdivision of the line segment u.⁸

(c) We say that two polygonal knots are Δ -equivalent (or combinatorially equivalent) if one can be obtained from the other by a finite sequence of Δ - and Δ^{-1} -moves.



Fig. 1.9

Polygonal links are usually presented by their projections to a plane. Let $p: R^3 \to R^2$ be a projection and let $L \subset R^3$ be a link. Then a point $P \in p(L)$ is called a multiple point (of p) if $p^{-1}(P)$ contains more than one point (the number of points in $p^{-1}(P)$ is called the multiplicity of P).

Definition 1.8 The projection p is called regular if

- (1) p has only a finite number of multiple points and all of them are of multiplicity two,
- (2) no vertex of the polygonal link is an inverse image of a multiple point of p.

Thus in case of a regular projection the parts of a diagram, illustrated in the figure below, are not allowed.



⁸Notice, that any subdivision is a combination of three non-degenerate Δ -moves, or more precisely two Δ -moves and the inverse to a Δ -move:



Maxwell was the first person to consider the question when two projections represent equivalent knots. He considered some elementary moves (reminding future Reidemeister moves), but never published his findings.

The formal interpretation of Δ -equivalence of knots in terms of diagrams. Was done by Reidemeister [Re-1], 1927, and Alexander and Briggs [A-B], 1927.

Theorem 1.9 (Reidemeister theorem)

Two link diagrams are Δ -equivalent⁹ if and only if they are connected by a finite sequence of Reidemeister moves $R_i^{\pm 1}$, i = 1, 2, 3 (see Fig. 1.10) and isotopy (deformation) of the plane of the diagram. The theorem holds also for oriented links and diagrams. One then has to take into account all possible coherent orientations of diagrams involved in the moves.

⁹In modern Knot Theory, especially after the work of R. Fox, we use usually the equivalent notion of ambient isotopy in \mathbb{R}^3 or \mathbb{S}^3 . Two links in a 3-manifold M are ambient isotopic if there is an isotopy of M sending one link into another.



Figure 1.10; Reidemeister moves; we draw two versions of the first and the third moves which are related by a mirror symmetry in the plane of the projection

2 Goeritz matrix and signature of a link

In the first half of XX-century combinatorial methods ruled over knot theory, even if more topological approach was possible, for example, Reidemeister moves were used to prove existence of the Alexander polynomial even if purely topological prove using the fundamental group was possible and probably well understood by Alexander himself. Later, after the Second World War, to great extend under influence of Ralph Hartzler Fox (1913 -1973), Knot Theory was considered to be a part of algebraic topology with fundamental group and coverings playing an important role. The renaissance of combinatorial methods in Knot Theory can be traced back to Conway's paper [Co-1] and bloomed after the Jones breakthrough [Jo-1] with Conway type invariants and Kauffman approach (compare Chapter III of [P-Book]). As we already mentioned, these had their predecessors in 1930th [Goe, Se]. Goeritz matrix of a link can be defined purely combinatorially and is closely related to Kirchhoff matrix of an electrical network. Seifert matrix is a generalization of the Goeritz matrix and, even historically, its development was mixing combinatorial and topological methods.

In this section we start from the work of L. Goeritz. He showed [Goe] how to associate a quadratic form to a diagram of a link and moreover how to use this form to get algebraic invariants of the knot (the signature of this form, however, is not an invariant of the knot). Later, H. F. Trotter [Tro-1], using Seifert form (see Section 3), introduced another quadratic form, the signature of which was an invariant of links.

C. McA. Gordon and R. A. Litherland [G-L] provided a unified approach to Goeritz and Trotter forms. They showed how to use the form of Goeritz to get (after adding a correcting factor) the signature of a link (this signature is often called a classical or Trotter, or Murasugi [M-10] signature of a link).

We begin with a purely combinatorial description of the matrix of Goeritz and of the signature of a link. This description is based on [G-L] and [Tral-1].

Definition 2.1 Let L be a diagram of a link. Let us checkerboard color the complement of the diagram in the projection plane \mathbb{R}^2 , that is, color in black and white the regions into which the plane is divided by the diagram¹⁰. We assume that the unbounded region of $\mathbb{R}^2 \setminus L$ is colored white and it is denoted by X_0 while the other white regions are denoted by X_1, \ldots, X_n . Now, to any crossing, p, of L we associate the number $\eta(p)$ which is either +1 or -1 according to the convention described in Fig. 2.1.

¹⁰This (checkerboard) coloring was first used by P. G. Tait in 1876/7, compare Chapter II of [P-Book], however we switched, after C. Gordon, the role of white and black. We can say that Tait convention worked well with a blackboard, while our convention with white-board.



Fig. 2.1

Let $G' = \{g_{i,j}\}_{i,j=0}^n$, where

$$g_{i,j} = \begin{cases} -\sum_{p} \eta(p) & \text{for } i \neq j, \text{ where the summation extends} \\ & \text{over crossings which connect } X_i \text{ and } X_j \\ -\sum_{k=0,1,\dots,n; k \neq i} g_{i,k} & \text{if } i = j \end{cases}$$

The matrix G' = G'(L) is called the unreduced Goeritz matrix of the diagram L. The reduced Goeritz matrix (or shortly Goeritz matrix) associated to the diagram L is the matrix G = G(L) obtained by removing the first row and the first column of G'.

Theorem 2.2 ([Goe, K-P, Ky].) Let us assume that L_1 and L_2 are two diagrams of a given link. Then the matrices $G(L_1)$ and $G(L_2)$ can be obtained one from the other in a finite number of the following elementary operations on matrices:

1. $G \Leftrightarrow PGP^T$, where P is a matrix with integer entries and det $P = \pm 1$.

2.

$$G \Leftrightarrow \left[\begin{array}{cc} G & 0 \\ 0 & \pm 1 \end{array} \right]$$

3.

$$G \Leftrightarrow \left[\begin{array}{cc} G & 0 \\ 0 & 0 \end{array} \right]$$

Moreover, if L is a diagrams of a knot¹¹, then operations (1) and (2) are sufficient.

 $^{^{11}\}mathrm{It}$ suffices to assume that L represent a non-split link, that is a link all projections of which are connected.

Corollary 2.3 $|\det G|$ is an invariant of isotopy of knots called the determinant of a knot¹².

A sketch of a proof of Theorem 2.2

We have to examine how a Goeritz matrix changes under Reidemeister moves. The matrix does not depend on the orientation of the link, let us assume, however, that the diagram L is oriented. We introduce new notation: a crossing is called of type I or II according to Fig. 2.2. Moreover, we define $\mu(L) = \sum \eta(p)$, where the summation is taken over crossings of type II.



Fig. 2.2

Now let us construct a graph with vertices representing black regions (this is the Tait's construction, however, the choice of black and white regions is reversed) and edges in bijection with crossings of L. Edges of the graph are in bijection with crossings of L: two vertices of the graph are joined if and only if the respective regions meet in a crossing¹³. Let B(L) denote the number of components of such a graph. From now on, let R be a Reidemeister move. We denote by G_1 the Goeritz matrix of L, and by G_2 the matrix of R(L). Similarly we set $\mu_1 = \mu(L), \ \mu_2 = \mu(R(L))$ and also $\beta_1 = B(L)$,

¹²Often, by the determinant of a knot one understands the more delicate invariant whose absolute value is equal to $|\det G|$; see Corollary 2.7. This determinant can be defined, also, as the Alexander-Conway or Jones polynomial at t = -1; compare Corollary 6.18.

¹³This construction of Tait is an important motivation for material in Chapter V of [P-Book]. The constructed graph, which we denote by $G_b(L)$, is usually called the Tait graph of L (see the first section). For an alternating diagram L this graph is the same as the graph $G_{s_+}(L)$ considered in Chapter V of [P-Book]. We often equip the edges of $G_b(L)$ with signs: the edge corresponding to a vertex p has the sign $\eta(p)$ (see Figure 2.1). The signed graph $G_b(L)$ is considered in Chapter V of [P-Book]; compare also Definition 7.4.

 $\beta_2 = B(R(L))$. We will write $G_1 \approx G_2$ if G_1 and G_2 are in relation (1) and $G_1 \sim G_2$ if G_2 can be obtained from G_1 by a sequence of relations (1)–(3).

- 1. Let us consider the first Reidemeister move R_1 .
 - (a) In the case shown in Fig. 2.3 we have: $\beta_1 = \beta_2$, $\mu_1 = \mu_2$ and $G_1 \approx G_2$.



(b) In the case shown in Fig. 2.4 we have:

$$\beta_1 = \beta_2, \ \mu_2 = \mu_1 + \eta(p), \ G_2 = \begin{bmatrix} G_1 & 0 \\ 0 & \eta(p) \end{bmatrix}$$



Fig. 2.4

- 2. Let us consider the second Reidemeister move R_2 .
 - (a) In the case described in Fig. 2.5 we get immediately that $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$ (either both crossings are of type I or of type II and always of opposite signs), $G_1 \approx G_2$.



(b) In the case described in Fig. 2.6 we have to consider two subcases. In each of them $\mu_1 = \mu_2$, since the two new crossings are either both of type I or both of type II and always of opposite signs:



Fig. 2.6

(i)
$$\beta_1 = \beta_2$$
. Then

$$G_{2} \approx \begin{bmatrix} G_{1} & 0 \\ 1 & \\ 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} G_{2} & 0 \\ 0 & 1 \end{bmatrix} \approx \begin{bmatrix} G_{1} & 0 \\ 1 & \\ 0 & -1 \end{bmatrix}$$

We leave it for the reader to check, c.f. [K-P]. Both possibilities give $G_1 \sim G_2$.

(ii) $\beta_2 = \beta_1 - 1$. Then we see immediately that

$$G_2 \approx \left[\begin{array}{cc} G_1 & \mathbf{0} \\ \mathbf{0} & 0 \end{array} \right].$$

3. Let us consider the Reidemeister move R_3 (Fig. 2.7).



Fig. 2.7

We see immediately that $\beta_1 = \beta_2$. Next we should consider different orientations of arcs participating in R_3 and two possibilities for the crossing p. However, we will get always $\mu_2 = \mu_1 + \eta(p)$ and

$$G_2 \approx \begin{bmatrix} G_1 & 0 \\ 0 & \eta(p) \end{bmatrix}.$$

We leave it for the reader to check (c.f. [Goe] and [Re-2]).

This concludes the proof of Theorem 2.2.

- **Corollary 2.4** (1) For a link L let us define $\sigma(L) = \sigma(G(L)) \mu(L)$, where $\sigma(G(L))$ is the signature of the Goeritz matrix of L. Then $\sigma(L)$ is an invariant of the link L, called the signature of the link; compare Corollary 2.7 and Definition 6.7.
 - (2) Let us define nul $(L) = nul (G(L)) + \beta(L) 1$, where nul (G(L)) is the nullity (i.e. the difference between the dimension and the rank) of the matrix G(L). Then nul (L) is an invariant of the link L and we call it the nullity (or defect) of the link.

Proof. It is enough to apply Theorem 2.2 to see that $\sigma(L)$ and nul (L) are invariant with respect to Reidemeister moves.

L. Traldi [Tral-1] introduced a modified matrix of an oriented link, the signature and the nullity of which are invariants of the link.

Definition 2.5 Let L be a diagram of an oriented link. Then we define the generalized Goeritz matrix

$$H(L) = \begin{bmatrix} G & \bigcirc \\ & A & \\ \bigcirc & B \end{bmatrix},$$

where G is a Goeritz matrix of L, and the matrices A and B are defined as follows. The matrix A is diagonal of dimension equal to the number of type II crossings and the diagonal entries equal to $-\eta(p)$, where p's are crossings of type II. The matrix B is of dimension $\beta(L) - 1$ with all entries equal to 0.

Lemma 2.6 ([Tral-1].) If L_1 and L_2 are diagrams of two isotopic oriented links then $H(L_1)$ can be obtained from $H(L_2)$ by a sequence of the following elementary equivalence operations:

- 1. $H \Leftrightarrow PHP^T$, where P is a matrix with integer entries and with det $P = \pm 1$,
- 2.

$$H \Leftrightarrow \left[\begin{array}{cc} H & \bigcirc \\ & 1 & \\ \bigcirc & -1 \end{array} \right].$$

Proof. Lemma 2.6 follows immediately from the proof of Theorem 2.2.

Corollary 2.7 The determinant det(iH(L)) $(i = \sqrt{-1})$ is an isotopy invariant of a link L, called the determinant of the link, Det_L . Moreover, $\sigma(H(L)) = \sigma(L)$ and nul (H(L)) = nul (L).

The proof follows immediately from Lemma 2.6 and from the proof of Theorem 2.2.

Example 2.8 Consider a torus link of type (2, k), we denote it by $T_{2,k}$. It is a knot for odd k and a link of two components for k even; see Fig. 2.8.



The matrix G' of $T_{2,k}$ is then equal to $\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$, and thus Goeritz matrix of the link is G = [k]. Moreover, $\beta = 1$ and $\mu = k$ because all crossings are of type II. Therefore, for $k \neq 0$, $\sigma(T_{2,k}) = \sigma(G) - \mu = 1 - k$ and nul $(T_{2,k}) = nul (G) = 0$. The generalized Goeritz matrix H of the knot $T_{2,k}$ is of dimension k + 1 and it is equal to

$$H = \begin{bmatrix} k & & \bigcirc \\ & -1 & \\ & & -1 & \\ & & & \ddots & \\ \bigcirc & & & & -1 \end{bmatrix}$$

Therefore $Det_L = \det(iH) = (-1)^k i^{k+1}k = i^{1-k}k$. Notice also that $i^{\sigma(T_{2,k})} = \frac{Det_{T_{2,k}}}{|Det_{T_{2,k}}|}$; compare Exercise 2.10.

Let us note that if we connect black regions of the plane divided by the diagram of the link by half-twisted bands (as indicated in Fig. 2.9) then we get a surface in \mathbb{R}^3 (and in \mathbb{S}^3), the boundary of which is the given link; we denote this surface by F_b , and call the Tait surface of a link diagram; compare Definition 7.4. If, for some checkerboard coloring of the plane, the constructed surface has an orientation which yields the given orientation of the link then this oriented diagram is called a special diagram.



Exercise 2.9 Prove that an oriented diagram of a link is special if and only if all crossings are of type I for some checkerboard coloring of the plane. Conclude from this that for a special diagram D, we have $\sigma(D) = \sigma(G(D))$.

Exercise 2.10 Show that any oriented link has a special diagram. Conclude from this that for any oriented link L one has $Det_L = i^{\sigma(L)} |Det_L|$; compare Lemma 6.16 and Corollary 6.18.

Assume now that L_0 is a sublink of an oriented link L. Let L' be an oriented link obtained from L by changing the orientation of L_0 to the opposite orientation. Let D_L be a diagram of L and define $lk(L - L_0, L_0)$ be defined a $\sum_p \text{ sgn } p$ where the sum is taken over all crossings of the diagram of $L - L_0$ and L_0 (as subdiagrams of L_D . This definition does not depend on the choice of D_L , as checked using Reidemeister moves and agrees with the standard notion of linking number as defined recalled in the next section.

From Corollary 2.4 and Corollary 2.7, we obtain.

Proposition 2.11 ([M-11])

- (i) $Det_{L'} = (-1)^{lk(L-L_0,L_0)} Det_L.$
- (*ii*) $\sigma(L') = \sigma(L) + 2lk(L L_0, L_0).$

(ii) $\sigma(L) + lk(L)$ is independent on orientation of L.

Proof: The derivation of formulas is immediate but it is still instructive to see how Corollary 2.11(ii) follows from Corollary 2.4(1):

$$\sigma(L') = \sigma(G(L')) - \mu(L') = \sigma(G(L)) - \mu(L') = \sigma(L) + \mu(L) - \mu(L') = \sigma(L) + 2lk(L - L_0, L_0).$$

Recall ([P-2]) that an *n*-move is a local change of an unoriented link diagram described in Figure 2.10.



Fig. 2.10; L_n obtained from $L = L_0$ by an *n*-move, and L_∞

When computing and comparing Goeritz matrices of $L = L_0$, L_n and L_∞ we can assume that black regions are chosen as in Figure 2.10 and that the white region X in $R^2 - L_\infty$ is divided into two regions X_0 and X_1 in $R^2 - L$.

Lemma 2.12 $G(L_n) = \begin{bmatrix} G(L_\infty) & \alpha \\ \alpha^T & q+n \end{bmatrix}$,

Corollary 2.13

(i) $DetG(L_n) - DetG(L_0) = nDetG(L_\infty),$

(*ii*) $\sigma(G(L_0)) \le \sigma(G(L_n)) \le \sigma(G(L_0)) + 2, n \ge 0.$

(iii) $|\sigma(G(L_n)) - \sigma(G(L_\infty))| \leq 1$. Furthermore, $\sigma(G(L_n)) = \sigma(G(L_\infty))$ if and only if $rankG(L_n) = rankG(L_\infty)$ or $rankG(L_n) = rankG(L_\infty) + 2$.

If we orient $L = L_0$ we can use Corollary 2.13(ii) to obtain very useful properties of signature of L and L_n .

Corollary 2.14 ([P-2]) (i) Assume that L_0 is oriented in such a way that its strings are parallel. L_n is said to be obtained from L_0 by a t_n -move $(\longrightarrow \rightarrow) :)$; then

$$n-2 \le \sigma(L_0)) - \sigma(L_n) \le n$$

$$0 \le \sigma(L_{2k}) - \sigma(L_0) \le 2.$$

(iii) (Giller [Gi])

$$0 \leq \sigma(L_{\swarrow}) - \sigma(L_{\swarrow}) \leq 2$$

Proof: (i) All new crossings of L_n are of type II (we use shading of Figure 2.10), thus $\mu(L_n) - \mu(L_0) = n$. Therefore by Corollary 2.13(ii) we have $n-2 \leq \sigma(G_{L_0}) - \mu(L_0) - (\sigma(G_{L_n}) - \mu(L_n) \leq n$, and Corollary 2.14(i) follows by Corollary 2.4.

(ii) In this case $\mu(L_{2k}) = \mu(L_0)$ thus (ii) follows from Corollary 2.13(ii). The generalization of Corollary 2.14(ii) to Tristram-Levine signatures is given in Corollary 6.9(ii).

(iii) follows from (i), or (ii) for n = 2. \Box

We finish the section with an example of computing a close form for the determinant of the family of links called Turk-head links. We define the *n*th Turk-head link, Th_n as the closure of the 3-braid $(\sigma_1 \sigma_2^{-1})^n$ (see Figure 2.11 for Th_6).¹⁴.

Example 2.15 We compute that

$$Det_{Th_n} = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2,$$

 $^{^{14}}Th_0$ is the trivial link of 3 components, Th_1 the trivial knot, Th_2 the figure eight knot (4_1) , Th_3 the Borromean rings (6_2^3) , Th_4 , the knot 8_{18} , Th_5 the knot 10_{123} , Th_6 the link 12_{474}^3 (that is 474th link of 12 crossings and 3 components in unpublished M. Thistleth-waite tables; compare [This-1]), and Th_7 and Th_8 are the knots 14_{a19470} and $16_{a275159}$, respectively, in Thistlethwaite (Knotscape) list.

or it can be written as $Det_{Th_n} = T_n(3) - 2$, where $T_i(z)$ is the Chebyshev (Tchebycheff) polynomial of the first kind¹⁵:

$$T_0 = 2, \quad T_1 = z, \quad T_i = z T_{i-1} - T_{i-2}.$$

In particular, $Det_{Th_2} = 5$, $Det_{Th_3} = 16$, $Det_{Th_4} = 45$, $Det_{Th_5} = 121$, $Det_{Th_6} = 320$, $Det_{Th_7} = 841$, and $Det_{Th_8} = 2205$; compare [Sed, Mye].

To show the above formulas, consider the (unreduced) Goeritz matrix related to the checkerboard coloring of the diagram of Th_n as shown in Figure 2.11 (we have here z = 3 and we draw the case of n = 6).

$$G'(Th_6) = \begin{bmatrix} -n & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -z & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & -z & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -z & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -z & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -z & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & -z \end{bmatrix},$$

By crossing the first row and column of $G'(Th_n)$ we obtain the Goeritz matrix of Th_n which is also the circulant matrix with the first row (-z, 1, 0, ..., 0, 1) (z = 3 and n = 6 in our concrete case):

$$G(Th_6) = \begin{bmatrix} -z & 1 & 0 & 0 & 0 & 1 \\ 1 & -z & 1 & 0 & 0 & 0 \\ 0 & 1 & -z & 1 & 0 & 0 \\ 0 & 0 & 1 & -z & 1 & 0 \\ 0 & 0 & 0 & 1 & -z & 1 \\ 1 & 0 & 0 & 0 & 1 & -z \end{bmatrix},$$

To compute the determinant of the circulant matrix $CM_n(z)$ of the size $n \times n$ and the first row (-z, 1, 0, ..., 0, 1) we treat each row as a relation and find the structure of the Z[z] module generated by columns (indexed by $(e_0, e_1, ..., e_n)$). Thus we have n relations of the form $e_k = ze_{k-1} - e_{k-2}$, where k is taken modulo n. The relation recalls the relation of Chebyshev

 $[\]overline{{}^{15}T_n(3)}$ is often named the Lucas number; more precisely $T_n(3) = L_{2n}$, where $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ as $L_n = 3L_{n-2} - L_{n-4}$.

polynomials, and in fact we easily check that $e_k = S_{k-1}(z)e_1 - S_{k-2}(z)e_0$, where $S_k(z)$ is the Chebyshev polynomial of the second kind:

$$S_0 = 1, \quad S_1 = z, \quad S_i = zS_{i-1} - S_{i-2}.$$

Thus we can eliminate all vectors (columns) e_k except e_0 and e_1 , and we are left with two equations $e_0 = e_n = S_{n-1}e_1 - S_{n-2}e_0$, and $e_1 = e_{n+1} = S_ne_1 - S_{n-1}e_0$. Thus, our module can be represented by the 2 × 2 matrix

$$\begin{bmatrix} S_{n-1} & 1-S_n \\ S_{n-2}+1 & -S_{n-1} \end{bmatrix},$$

We conclude that, up to a sign, $detCM_n(z)$ is equal to the determinant of our 2×2 matrix, that is $S_n - S_{n-2} - 1 - S_{n1}^2 + S_n S_{n-2}$. To simplify this expression let us use the substitution $z = a + a^{-1}$. Then $S_n(z) = a^n + a^{n-2} + \dots a^{2-n} + a^{-n} = \frac{a^{n+1} - a^{-n-1}}{a - a^{-1}}$, and $T_n(z) = a^n + a^{-n}$. Therefore, $S_n - S_{n-2} - 1 - S_{n-1}^2 + S_n S_{n-2} = S_n - S_{n-2} - 1 - ((\frac{a^n - a^{-n}}{a - a^{-1}})^2 - (\frac{a^{n+1} - a^{-n-1}}{a - a^{-1}})(\frac{a^{n-1} - a^{-n+1}}{a - a^{-1}})) = S_n - S_{n-2} - 1 - (\frac{(a^n - a^{-n})^2 - (a^{n+1} - a^{-n-1})(a^{n-1} - a^{-n+1})}{(a - a^{-1})^2}) = S_n - S_{n-2} - 2 = a^n + a^{-n} - 2 = T_n(z) - 2.$

By comparing the maximal power of z in $detCM_n(z)$ and $T_2(z) - 2$, we get that $detCM_n(z) = (-1)^n(T_n(z) - 2)$. For z = 3 we have $a + a^{-1} = 3$, thus $a = \frac{3\pm\sqrt{5}}{2}$ so we can choose $a = \frac{3\pm\sqrt{5}}{2}$ and $a^{-1} = \frac{3-\sqrt{5}}{2}$, and thus $T_n(3) = (\frac{3\pm\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n$.

Because, Th_n is an amphicheiral link, its signature is equal to 0 and

$$Det_{Th_n} = i^{\sigma(Th_n)} |detCM_n(3)| = T_n(3) - 2 = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$



Figure 2.11; The Turk-head link Th_6 and its checkerboard coloring

We computed the determinant of the circulant¹⁶ matrix CM_n for a general variable z and till now used it only for z = 3, we see in the next exercise that the matrix has knot theory interpretation for any rational number z.

Exercise 2.16 Consider the "braid like" closure of the tangle $(\sigma_2^{-\frac{1}{a}}\sigma_1^b)^n$ for any integers a and b, (see Figure 2.12 for $(\sigma_2^{-\frac{1}{3}}\sigma_1^3)^4$). Show that the determinant of the link satisfies the formula¹⁷



$$|Det_{(\sigma_2^{-\frac{1}{a}}\sigma_1^b)^n}| = |b^n det CM_n(2 + \frac{a}{b})| = |b^n(T_n(2 + \frac{a}{b}) - 2)|.$$

Fig. 2.12; The closure of the tangle $(\sigma_2^{-\frac{1}{3}}\sigma_1^3)^4$

3 Seifert surfaces

It was first demonstrated by P. Frankl and L. Pontrjagin in 1930 [F-P] that any knot bounds an oriented surface¹⁸. H. Seifert found a very simple construction of such a surface [Se] and developed several applications of the

¹⁶Recall, that the circulant $n \times n$ matrix, satisfies $a_{i,j} = a_{i-1,j-1} = \dots a_{1,j-i+1}$, $0 \leq i, j \leq n-1$. Such a matrix has (over *C*) *n* different eigenvectors: $(1, \omega, \omega^2, \dots, \omega^{n-1})$, where ω is any *n*th root of unity ($\omega^n = 1$). The corresponding eigenvalues are $\lambda_{\omega} = \sum_{i=0}^{n-1} \omega^i a_{1,i}$. Thus Example 2.15 leads to a curious identity $\prod_{i=0}^{n-1} (\omega^i + \omega^{-i} - z) = detCM_n = (-1)^n (T_n(z) - 2)$ for any primitive *n*th root of unity ω .

¹⁷It is also the formula for the number of spanning trees of the generalized wheel, $W_{a,b,n}$, which is the Tait graph of the closure of $(\sigma_2^{-\frac{1}{a}}\sigma_1^b)^n$ $(W_{3,3,4} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j$

¹⁸According to [F-P]: "The Theorem... [was] found by both authors independently from each other. In what follows, the Frankl's form of the proof is presented." One should add that Seifert refers in [Se] to the Frankl-Pontrjagin paper and says that they use a different method.

surface, named now *Seifert surface* (also, infrequently, Frankl-Pontrjagin surface)¹⁹.

Definition 3.1 A Seifert surface of a link $L \subset S^3$ is a compact, connected, orientable 2-manifold $S \subset S^3$ such that $\partial S = L$.

For example: a Seifert surface of a trefoil knot is pictured in Fig. 3.1. If the link L is oriented then its Seifert surface S is assumed to be oriented so that its orientation agrees with that of L.



Fig. 3.1

Definition 3.2 The genus of a link $L \subset S^3$ is the minimal genus of a Seifert surface of L.

The genus is an invariant (of ambient isotopy classes) of knots and links. The following theorem provides that it is well defined.

Theorem 3.3 (Frankl-Pontrjagin-Seifert) Every link in S^3 bounds a Seifert surface. If, moreover, the link is oriented then there exists a Seifert surface, an orientation of which determines the orientation of its boundary coinciding with that of L.

Construction 3.4 (Seifert) Consider a fixed diagram D of an an oriented link L in S^3 . In the diagram there are two types of crossings, in a neighborhood of each of crossings we make a modification of the link (called smoothing) according to Fig. 3.2.

¹⁹Kauffman in [K-3, K-8] uses the term *Seifert surface* to describes the surface obtained from an oriented link diagram by the Seifert algorithm (Construction 3.4), and the term spanning surface for an oriented surface bounding a link (our Seifert surface of Definition 3.1). In [Bol] the name *Frankl surface* is used for any, oriented or unoriented spanning surface.



After smoothing all crossings of D we obtain a family of disjoint oriented simple closed curves in the plane, called by R. Fox, Seifert circles and denoted by $D_{\vec{s}}$. Each of the curves of $D_{\vec{s}}$ bounds a disk in the plane; the disks do not have to be disjoint (they can be nested). Now we make the disks disjoint by pushing them slightly up above the projection plane. We start with the innermost disks (that is disks without any other disks inside) and proceed outwards (i.e. if $D' \subset D$ then D' is pushed above D; see Fig. 3.3.



Fig. 3.3

The disks are two-sided so we can assign the sings + and - to each of the sides of a disk according to the following convention: the sign of the "upper" side of the disk is + (respectively, -) if its boundary is oriented counterclockwise (respectively, clockwise), see Fig. 3.4.



Now we connect the disks together at the original crossings of the diagram D by half-twisted bands so that the 2-manifold which we obtain has L as its boundary, see Fig. 3.5.



Fig. 3.5; Seifert surface around a crossing

Since the "+ side" is connected to another "+ side" it follows that the resulting surface is orientable. Moreover, this surface is connected if the projection of the link is connected (for example if L is a knot). If the surface is not connected then we join its components by tubes (see Fig. 3.6) in such a way that the orientation of components is preserved.



Fig. 3.6

Remark 3.5 If the link L has more than one component then the Seifert surface, which we constructed above, depends on the orientation of components of L. This can be seen on the example of a torus link of type (2, 4), see Fig. 3.7.



Fig. 3.7; different orientations result in different Seifert surfaces

The Seifert surface from Fig. 3.7(a) has genus 1 while the surface from Fig. 3.7(b) has genus 0. Therefore the link L has genus 0 (as an unoriented link).

Corollary 3.6 If a projection of a link L is connected (e.g. if L is a knot) then the surface, from the Seifert Construction 3.4, is unknotted, that is, its complement in S^3 is a handlebody. The genus of the handlebody is equal to c + 1 - s and the Euler characteristic is equal to s - c, where c denotes the number of crossings of the projection and s the number of Seifert circles.

Proof: The complement in S^3 of the plane projection of L is a 3-disk with c + 1 handles (the projection of L cuts the projection plane (or 2-sphere) into c + 2 regions). Furthermore adding s 2-disks in the construction of the Seifert surface we cut s of the handles thus the result remains a 3-disk with c + 1 - s handles. The Euler characteristic of obtained handlebody is equal to 1 - (c + 1 - s) = s - c. \Box

Corollary 3.7 A knot K in S^3 is trivial if and only if its genus is equal to 0.

Exercise 3.8 Let L be a link with n components and D_L its diagram. Moreover, let c denote the number of crossings in D_L and let s be the number of Seifert circles. Prove that the genus of the resulting Seifert surface is equal to:

$$genus(S) = p - \frac{s+n-c}{2},$$

where p is the number of connected components of the projection of L. Check that the Euler characteristic of S, for p = 1, is equal to s - c so it agrees with the Euler characteristic of handlebody described in Corollary 3.6.

Suppose that the solid torus V_K is a closure of a regular neighborhood of a knot K in S^3 and set $M_K = S^3 - \operatorname{int} V_K$ (note that M_K is homotopy equivalent to the knot complement). Let us write Mayer-Vietoris sequence for the pair (M_K, V_K) :

$$0 = H_2(S^3) \to H_1(\partial M_K) \to H_1(M_K) \oplus H_1(V_K) \to H_1(S^3) = 0.$$

For a torus ∂M_K and the solid torus V_K homology are $Z \oplus Z$ and Z, respectively. Therefore $H_1(M_K) = Z$ and it is generated by a meridian in $\partial M_K = \partial V_K$, where by the meridian we understand a simple closed curve in ∂V_k which bounds a disk in V_K . We denote the meridian by m. A simple closed curve on ∂M_K which generates ker $(H_1(\partial M_K) \to H_1(M_K))$ is called longitude and it is denoted by l. If S^3 and K are oriented then the longitude is orientated in agreement with the orientation of K. Subsequently, the meridian is given the orientation in such a way that the pair (m, l) induces on ∂V_K the same orientation as the one induced by the solid torus V_K , which inherits its orientation from S^3 . Equivalently, the linking number of m and K is equal to 1 (compare Section 5). Similar reasoning allows us also to conclude:

Proposition 3.9 For any link L in S^3 the first homology of the exterior of L in S^3 is freely generated by meridians of components of L. In particular, $H_1(S^3 - L) = Z^{com(L)}$.

We also can use the Mayer-Vietoris sequence to find the homology of the exterior the Seifert surface in S^3 . Let F_L be a Seifert surface of a link L and F' its restriction to $M_L = S^3 - intV_L$. Let $V_{F'}$ be a regular neighborhood of F' in M_L . Because F' is orientable $V_{F'}$ is a product $F' \times [-1, 1]$ with $F^+ = F' \times \{1\}$ and $F^- = F' \times \{-1\}$. The boundary, $\partial V_{F'}$ is homeomorphic to F^+ and F^- glued together naturally along their boundary. Now let us apply the Mayer-Vietoris sequence to $V_{F'}$ and $S^3 - intV_{F'}$. We get:

$$0 = H_2(S^3) \to H_1(\partial V_{F'}) \xrightarrow{(i_1, -i_2)} H_1(S^3 - intV_{F'}) \oplus H_1(V_{F'}) \to H_1(S^3) = 0.$$

where i_1 and i_2 are induced by embeddings. Clearly, $H_1(S^3 - intV_{F'})$ is isomorphic to the kernel of i_2 . We can easily identify the elements $x^+ - x^-$ as elements of the kernel, for any x a cycle in F'. In the case of L being a knot, these elements generate the kernel.

Corollary 3.10 The homology groups, $H_1(F_L)$ and $H_1(S^3 - F_L)$ are isomorphic to $Z^{2g+com(L)-1}$, where g is the genus of F_L and com(L) is the number of components of L thus also the number of boundary components of F_L . Compare Theorem 3.12.

Corollary 3.11 Let $x_1, ..., x_{2g}$ be a basis of $H_1(F_K)$ where F_K is the Seifert surface of a knot K. Then $x_1^+ - x_1^-, ..., x_{2g}^+ - x_{2g}^-$ form a basis of $H_1(S^3 - K)$.

With some effort we can generalize Corollary 3.10 to get the following result which is a version of Alexander-Lefschetz duality²⁰ (see [Li-12] for an elementary proof).

Theorem 3.12 *i* Let F be a Seifert surface of a link, then $H_1(S^3 - F)$ is isomorphic to $H_1(F)$ and there is a nonsingular bilinear form

$$\beta: H_1(S^3 - F) \times H_1(F) \to Z$$

given by $\beta(a,b) = lk(a,b)$, where lk(a,b) is defined to be the intersection number of a and a 2-chain whose boundary is b (see Chapter 5).

4 Connected sum of links.

Definition 4.1 Assume that K_1 and K_2 are oriented knots in S^3 . A connected sum of knots, $K = K_1 \# K_2$, is a knot K in S^3 obtained in the following way:

First, for i = 1, 2, choose a point $x_i \in K_i$ and its regular neighborhood C_i in the pair (S^3, K_i) . Then, consider a pair $((S^3 - int C_1 \cup_{\varphi} S^3 - int C_2), (K_1 - int C_1 \cup_{\varphi} K_2 - int C_2))$, where φ is an orientation reversing homeomorphism $\partial C_1 \rightarrow \partial C_2$ which maps the end of $K_1 \cap (S^3 - int C_1)$ to the beginning of $K_2 \cap (S^3 - int C_2)$ (and vice versa). (Notice that notions of beginning and end are well defined because K_1 and K_2 are oriented.) We see that $(S^3 - int C_1) \cup_{\varphi} (S^3 - int C_2)$ is a 3-dimensional sphere and $(K_1 - int C_1) \cup_{\varphi} (K_2 - int C_2)$ is an oriented knot.

²⁰Let us recall that Alexander duality gives us an isomorphism $\tilde{H}^i(S^n - X) = \tilde{H}_{n-i-1}(X)$ for a compact subcomplex X of S^n and that on the free parts of homology the Alexander isomorphism induces a nonsingular form $\beta : \tilde{H}_i(S^n - X) \times \tilde{H}_{n-i-1}(X) \to Z$, where \tilde{H} denotes reduced (co)homology.

Lemma 4.2 The connected sum of knots is a well defined, commutative and associative operation in the category of oriented knots in S^3 (up to ambient isotopy).

A proof of the lemma follows from two theorems in PL topology which we quote without proofs.

Theorem 4.3 Let (C, I) be a pair consisting of a 3-cell C and 1-cell I which is properly embedded and unknotted in C (i.e. the pair (C, I) is homeomorphic to $(\overline{B}(0, 1), [-1, 1])$ where $(\overline{B}(0, 1)$ is the closed unit ball in \mathbb{R}^3 and [-1, 1]is the interval (x, 0, 0) parameterized by $x \in [-1, 1]$. Respectively, let (S^2, S^0) be a pair consisting of the 2-dimensional sphere and two points on it. Then any orientation preserving homeomorphism of C (respectively, S^2) which preserves I and is constant on ∂I (respectively, it is constant on S^0) is isotopic to the identity.

Theorem 4.4 Let K be a knot in S^3 and let C' and C'' be two regular neighborhoods in the pair (S^3, K) of two points on K. Then there exists an isotopy F of the pair (S^3, K) which is constant outside of a regular neighborhood of K and such that $F_0 = Id$ and $F_1(C') = C''$.

Remark 4.5 In the definition of the connected sum of knots we assumed that the homeomorphism φ reverses the orientation. This assumption is significant as the following example shows.

Let us consider the right-handed trefoil knot K (i.e. a torus knot of type (2,3)), Fig. 3.1. Let \overline{K} be the mirror image of K (i.e. a torus knot of type (2,-3)). Then $K\#\overline{K}$ is the square knot while K#K is the knot "Granny" and these two knots are not equivalent. To distinguish them it is enough to compute their signature or the Jones polynomial, or Homflypt (Jones-Conway) polynomial.

Theorem 4.6 (Schubert [Sch-2]) Genus of knots in S^3 is additive, that is

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

A proof of the Schubert theorem can be found in e.g. [J-P, Li-12].

Corollary 4.7 Any knot in S^3 admits a decomposition into a finite connected sum of prime knots, i.e. knots which are not connected sums of non-trivial knots.

In fact Schubert [Sch-1] showed that the prime decomposition of knots is unique up to order of factors; in other worlds, knots with connected sum form a unique factorization commutative monoid.

Corollary 4.8 The trefoil knot is non-trivial and prime.

Proof. The trefoil knot is non-trivial therefore its genus is positive (Corollary 3.7). Fig. 3.1 demonstrates that the genus is equal to 1. Now primeness follows from Theorem 4.6 and Corollary 4.7.

Similarly as for knots, the notion of connected sum can be extended to oriented links. It, however, depends on the choice of the components which we glue together. The weak version of the unique factorization of links with respect to connected sum was proven by Youko Hashizume [Hash].

5 Linking number; Seifert forms and matrices.

We start this Section by introducing the linking number lk(J, K) for any pair of disjoint oriented knots J and K. Our definition is topological and we will show that it agrees with the diagrammatic definition considered before. We use the notation introduced right after the Exercise 3.8.

Definition 5.1 The linking number lk(J, K) is an integer such that [J] = lk(J, K)[m], where [J] and [m] are homology classes of the oriented curve J and the meridian m of the oriented knot K, respectively.

Lemma 5.2 Let $S \subset S^3 - int V_K$ be a Seifert surface of a knot K (more precisely, its restriction to $S^3 - int V_K$), such that its orientation determines the orientation of ∂S compatible with that of the longitude l. Then lk(J, K) is equal to the algebraic intersection number of J and S.

Proof. First, let us recall the convention for the orientation of the boundary of an oriented manifold M. For $x \in \partial M$ we consider a basis (v_2, \ldots, v_n) of the tangent space $T_x \partial M$ together with the normal \overline{n} of ∂M in M which is directed outwards. Then, v_2, \ldots, v_n defines orientation of $T_x \partial M$ if $\overline{n}, v_2, \ldots, v_n$ defines the orientation of $T_x M$. Returning to the proof of 5.2 we note that the meridian m intersects the Seifert surface S exactly at one point. Moreover, the algebraic intersection number of m and S is +1, according to our definition of the orientation of S. Thus, if the algebraic intersection number of J and S is equal to i, then [J] = i[m], that is i = lk(J, M), which concludes the proof.

Lemma 5.3 Let us consider a diagram of a link $J \cup K$ consisting of two disjoint oriented knots J and K. We assume that the orientation of $S^3 = R^3 \cup \infty$ is induced by the orientation of the plane containing the diagram of $J \cup K$ and the third axis which is directed upwards Now, to any crossing of the diagram where J passes under K we assign +1 in the case of $\stackrel{1}{\longrightarrow} \stackrel{\kappa}{\rightarrow}$ and -1 in the case of $\stackrel{1}{\longrightarrow} \stackrel{1}{\longrightarrow} \stackrel{\kappa}{\longrightarrow}$. Then the sum of all numbers assigned to such crossings is equal to linking number lk(J, K).

Proof. Let us consider a Seifert surface of the knot K constructed from the diagram of K, as described in Construction 3.4. We may assume that the knot J is placed above this surface, except small neighborhoods of the crossings where J passes under K. We check now that the sign of the intersection of J with this surface coincides with the number that we have just assigned to such a crossing.

Exercise 5.4 Show that lk(J, K) = lk(K, J) = -lk(-K, J), where -K denotes the knot K with reversed orientation. Hint. Apply Lemma 5.3.

The linking number may be defined for any two disjoint 1-cycles in S^3 . For example, as a definition we may use the condition from Lemma 5.2. That is, if α and β are disjoint 1-cycles in S^3 then $\text{lk}(\alpha, \beta)$ is defined as the intersection number of α with a 2-chain in S^3 whose boundary is equal β .

Exercise 5.5 Prove that $lk(\alpha,\beta)$ is well defined, that is, it does not depend on the 2-chain whose boundary is β .

Exercise 5.6 Show that lk is symmetric and bilinear, i.e. $lk(\alpha, \beta) = lk(\beta, \alpha)$ and $lk(\alpha, n\beta) = n \cdot lk(\alpha, \beta)$, and if a cycle β' is disjoint from α then $lk(\alpha, \beta + \beta') = lk(\alpha, \beta) + lk(\alpha, \beta')$.

Exercise 5.7 Prove that, if β and β' are homologous in the complement of α , then $lk(\alpha, \beta) = lk(\alpha, \beta')$.
Now we define a Seifert form of a knot or a link. Let S be a Seifert surface of a knot or a link K, then S is a two-sided surface in S^3 . Let $S \times [-1, 1]$ be a regular neighborhood of S in S^3 . For a 1-cycle x in int S we can consider a cycle x^+ (respectively x^-) in $S \times \{1\}$ (respectively $S \times \{-1\}$) which is obtained by pushing the cycle x to $S \times \{1\}$ (respectively, to $S \times \{-1\}$). (We note that the sides of S are uniquely defined by the orientations of K and S^3 .)

Definition 5.8 The Seifert form of the knot K is a function $f : H_1(int S) \times H_1(int S) \to Z$ such that $f(x, y) = lk(x^+, y)$. Similarly we define a Seifert form of an oriented link L using an oriented Seifert surface S of L.

Lemma 5.9 The function f is a well defined bilinear form on the Z-module (*i.e.* abelian group) $H_1(int S)$.

Proof. The result follows from Exercises 5.6 and 5.7.

Definition 5.10

By a Seifert matrix $V = \{v_{i,j}\}$ in a basis $e_1, e_2, ..., e_{2g+com(L)-1}$ of $H_1(S)$ we understand the matrix of f in this basis, that is

$$v_{i,j} = lk (e_i^+, e_j).$$

Then, for $x, y \in H_1(S)$ we have $f(x, y) = x^T V y$. We use the convention that coefficients of a vector are written as a column matrix²¹.

Notice, that a change of the basis in $H_1(S)$ results in the change of the matrix V to a similar matrix $P^T V P$, where $det P = \pm 1$.

Example 5.11 The Seifert matrix of a Seifert surface of the right-handed trefoil knot, computed in the basis $[\alpha], [\beta]$, is equal to $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ (see Fig. 5.1).

²¹Our notation agrees with that of Kauffman [K-3], Kawauchi [Kaw-4], and [J-P] but in the books by Burde and Zieschang [B-Z], Lickorish [Li-12], Rolfsen [Ro-1], Livingston [Liv], and Murasugi [M-9] the convention is the opposite, that is $f(x, y) = lk(x, y^+)$.



Example 5.12 The Seifert matrix of a Seifert surface of the figure-eight knot, computed in the basis $[\alpha], [\beta]$ is equal to $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ (see Fig. 5.2).



Fig. 5.2

With some practice one should be able to find Seifert form efficiently and we encourage a reader to compute more examples and develop some rules; for example if α is a simple closed curve on S and on the plane then $lk(\alpha^+, \alpha) = -\frac{1}{2}\sum$ sgn p where the sum is taken over all crossings of the diagram traversed by α . We illustrate it by one more example, the Seifert matrix of a pretzel knot. The computation is almost the same as in the trefoil case as the genus of the surface is equal to 1 and three crossings of the right-handed trefoil knot $\bar{3}_1$ are replaced by $2k_1 + 1$, $2k_2 + 1$, and $2k_3 + 1$, respectively.

Example 5.13 Let P_{n_1,n_2,n_3} denote the pretzel link of type (n_1, n_2, n_3) (compare Fig. 5.3). The Seifert matrix of a Seifert surface of the pretzel knot

 $P_{2k_{1}+1,2k_{2}+1,2k_{3}+1}, \text{ computed in the basis } [\alpha], [\beta], \text{ is equal to } \begin{bmatrix} -k_{1}-k_{2} & k_{2} \\ k_{2}+1 & -k_{1}-k_{2} \end{bmatrix}$ (see Fig. 5.3).



Fig. 5.3; $P_{1,3,5}$ – the pretzel knot of type (1,3,5)

There is a classical skew-symmetric form on a homology group of an oriented surface, called an intersection form, which is related to the Seifert form f.

Definition 5.14 Let S be an oriented surface. For two homology classes $x, y \in H_1(S)$ represented by transversal cycles we define their algebraic intersection number $\tau(x, y)$ as the sum of the signed intersection points where the sign is defined in the following way: if x meet y transversally at a point p then the sign of the intersection at p is equal +1 if $\stackrel{+}{\xrightarrow{}}$ and -1 if $\stackrel{+}{\xrightarrow{}}$.

Exercise 5.15 Prove that $\tau : H_1(S) \times H_1(S) \to Z$ is bilinear and skew-symmetric (i.e. $\tau(x, y) = -\tau(y, x)$).

Exercise 5.16 Prove that the determinant of a matrix of τ is equal to 1 if $\partial S = S^1$, or $\partial S = \emptyset$ and it is equal to 0 otherwise.

Solution. Assume that S has more then one boundary component and ∂_1 is one of them. Then ∂_1 is a nontrivial element in $H_1(S)$ with trivial intersection number with any element of $H_1(S)$. Thus the matrix of τ is singular and its determinant is equal to zero.

Assume now that $\partial S = S^1$, or $\partial S = \emptyset$. Let us choose loops representing a basis of $H_1(S)$ such as in Fig. 5.4. In this basis the matrix of τ is as follows



Thus, its determinant is equal to 1. Notice also that the determinant of a matrix changing a basis of $H_1(S)$ is equal to 1 or -1, thus the determinant of the form does not depend on the choice of a basis.



Exercise 5.17 Prove that, if S is a Seifert surface of a link then $\tau(x, y) =$ f(x,y) - f(y,x).

Solution. It follows from Lemma 5.3 that the crossing change between two oriented disjoint curves J and K in S^3 changes the linking number between them by 1 or -1, diagrammatically we have: $lk(\stackrel{I}{\longrightarrow} \stackrel{K}{\longrightarrow}) - lk(\stackrel{I}{\longrightarrow}) = 1.$ If J and K are two, possibly intersecting, oriented curves on an oriented surface we see that the pair (J^+, K) differ from the pair (J^-, K) by crossing changes at crossings of J and K. Furthermore the convention we use is that $sgn(\xrightarrow{J}{K}) = -1.$

 $Thus f(J,K) - f(K,J) = lk (J^+,K) - lk (K^+,J) = lk (J^+,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^+,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^-,K) = lk (J^-,K) - lk (J^$

 $\frac{\sum_{p \in J \cap K)} sgn \ p = \tau(J, K). \ The \ solution \ is \ completed^{22}.$ $2^{22} \text{The equality} \xrightarrow{|} \rightarrow - \xrightarrow{|} \rightarrow = \xrightarrow{|} \text{ is a defining relation of Vassiliev-Gusarov invariants}$ or skein modules of links; compare Chapter IX of [P-Book]. This relation, combined with \rightarrow \rightarrow = 0 (that is, the value of a link with at least two singular crossings is equal to zero), leads to the (global) linking number, described as Vassiliev-Gusarov invariant of degree 1.

Corollary 5.18 The Seifert matrix V of a knot K in S^3 satisfies the following equation:

$$\det(V - V^T) = 1.$$

Proof. We note that $V - V^T$ is a matrix of τ (Exercise 5.17) and its determinant is equal to 1 (Exercise 5.16).

A Seifert matrix is not an invariant of a knot or a link, but it can be used to define some well known invariants, including the Alexander polynomial.

Now we describe relation between Seifert matrices of (possibly different) Seifert surfaces of a given link.

Definition 5.19 We call matrices S-equivalent if one can be obtained from the other by a finite number of the following modifications:

- (1) $A \Leftrightarrow PAP^T$ where det $P = \mp 1$.
- (2)

$$A \Leftrightarrow \left[\begin{array}{ccc} A & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \text{ and } A \Leftrightarrow \left[\begin{array}{ccc} A & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

where α is a column and β is a row.

Theorem 5.20 Let us assume that L_1 and L_2 are isotopic links and F_1 , respectively, F_2 , are their Seifert surfaces. If A_1 and A_2 are their Seifert matrices computed in some basis B_1 and, respectively, B_2 then A_1 is S-equivalent to A_2 .

We perform the proof in two steps. Namely, we will prove the following two claims:

- (1) If we attach a handle to F_1 then the resulting surface (boundary of which is again L_1) has its Seifert surface S-equivalent to the Seifert surface of F_1 .
- (2) We can assume that $L_1 = L_2$. Then there exists a Seifert surface for L_1 which can be reached (modulo isotopy) from both F_1 and F_2 by the operation of attaching handles.

First we prove (1).

Let A_1 be a Seifert matrix of F_1 (in some basis of $H_1(F)$). By γ and μ let us denote two new generators of $H_1(F \cup \text{handle})$ — see Fig. 5.5.



Let us recall that the Seifert form $f: H_1(F) \times H_1(F) \to Z$ was defined by the formula $f(x, y) = \text{lk}(x^+, y)$, where lk denotes linking number in S^3 and x^+ is obtained by pushing the cycle x out of F in the normal direction of F.

If the pushing moves the cycle μ outside of the handle (that is μ^+ is outside the handle) then the resulting Seifert matrix is

$$\left[\begin{array}{ccc} A & \alpha & 0 \\ \beta & w_0 & 0 \\ 0 & \pm 1 & 0 \end{array}\right]$$

which is S-equivalent to the matrix A (α and ω_0 can be converted to 0 matrices by type (1) operations; similarly, ± 1 can be converted to 1 by a type (1) operation). In the matrix, β is a row vector determined by linking numbers of λ^+ with the basis of $H_1(F)$, α is a column vector determined by linking numbers of the basis $H_1(F)$ with λ^- and $\omega_0 = lk(\lambda^+, \lambda^-)$.

Otherwise (i.e. μ^+ is inside the handle) we get the matrix:

$$\left[\begin{array}{ccc} A & \alpha & 0 \\ \beta & \omega_0 & \pm 1 \\ 0 & 0 & 0 \end{array}\right]$$

which is S-equivalent to A as well.

Proof of (2). Assume that the Seifert surface F_1 intersects F_2 transversally (modulo the boundary L_1 ; in the neighborhood of L_1 they may be assumed to be disjoint outside L_1). Now we will use the following

Lemma 5.21 Let M be compact connected 3-manifold and let F_1 , F_2 be such submanifolds of ∂M that $\partial M = F_1 \cup F_2$ and $F_1 \cap F_2 = \partial F_1 = \partial F_2$. Then there exists a surface F in M such that $\partial F = \partial F_1$ and F can be obtained from F_1

as well as from F_2 by attaching 1-handles. More precisely: F cuts M into two 3-submanifolds M_1 , containing F_1 and M_2 containing F_2 . Furthermore M_i can be obtained from F_i (more precisely $F_i \times [0,1]$) by attaching 1-handles²³ to $int(F_i)$. We have $F_i \cup F = \partial M_i$; in particular F is obtained from F_i by 1-surgeries.

Sketch of the proof. The presented proof is based on the proof of existence of Heegaard decomposition of a 3-manifold from triangulation (e.g. [Hemp, J-P]). Let X be a triangulation of (M, F_1, F_2) . In particular L is in the 1-skeleton of triangulation Γ_1 . Let Γ_2 denote the dual 1-skeleton. That is, Γ_2 is the maximal 1-subcomplex of the first baricentric subdivision X' of X, such that Γ_2 is disjoint with Γ_1 . Let V_i (i = 1, 2) be a regular neighborhood of Γ_i associated to the second baricentric subdivision of X. Then X = $V_1 \cup V_2$ and V_i is obtained from F_i by attaching (solid) 1-handles. Therefore $(F_1 \cup V_1) \cap (F_2 \cup V_2)$ is the surface F that we look for.

The proof of claim (2) is inductive with respect to the number of circles in the intersection $F_1 \cap F_2$:

- (1) Suppose that $F_1 \cap F_2 = L_1$. Then we apply Lemma 5.21 to a part of S^3 which is bounded by the closed surface $F_1 \cup F_2$.
- (n+1) Inductive step. Suppose that (2) holds if the number of components of $F_1 \cap F_2$ is smaller than n + 1.

Now, assume that $F_1 \cap F_2$ consists of n+1 circles. Then $F_1 \cup F_2$ cuts S^3 into a number of connected components and moreover different "sides" of F_1 and F_2 bound different components. Let M be a component such that $F'_1 = F_1 \cap \partial M$ and $F'_2 = F_2 \cap \partial M$. Now we apply Lemma 5.21 to the triple (M, F'_1, F'_2) and consequently let F'_0 be the surface provided by the lemma. That is, F'_0 is obtained by attaching solid 1-handles to either F'_1 or F'_2 .

Let F_1^0 and F_2^0 be obtained from F_1 and F_2 by replacing F'_1 and F'_2 by F'_0 . Then by moving slightly surfaces F_1^0 and F_2^0 we can obtain a smaller number of components of their intersection and thus we can apply the inductive assumption. This concludes the proof of (2) and of Theorem 5.20.

²³We attach k-handle to an (n + 1)-dimensional manifold M along an open subset, N, of the boundary by choosing a disk $D^{n+1} = D^k \times D^{n+1-k}$ and the embedding ϕ : $\partial D^k \times D^{n+1-k} \to N$, and gluing M with D^{n+1} using ϕ . In our case n = 2.

An elementary, diagrammatic proof of Theorem 5.20, based on Reidemeister moves and the fact that any link has a special diagram (compare Exercises 2.9 and 2.10 or Proposition 13.15 of [B-Z]), is given in [BFK].

6 From Seifert form to Alexander polynomial and signatures

The Conway's potential function is defined as a normalized version of the Alexander polynomial using Seifert matrix, as follows [K-1]:

Lemma 6.1 Let A be a Seifert matrix of an oriented link L and define the potential function $\Omega_L(x) = \det(xA - x^{-1}A^T)$. Then $\Omega_L(x)$ does not depend on the choice of a Seifert surface and its Seifert matrix. In particular, if T_1 is the trivial knot then $\Omega_{T_1}(x) = 1$.

Proof. The result follows from Lemma 5.20. Indeed, simple computations show that if we replace the matrix A with another S-equivalent matrix then $\Omega_L(x)$ remains the same. We use the following identity

$$\det\left(x \begin{bmatrix} 0 & 1\\ 0 & 0 \end{bmatrix} - x^{-1} \begin{bmatrix} 0 & 0\\ 1 & 0 \end{bmatrix}\right) = \det\left[\begin{array}{cc} 0 & x\\ -x^{-1} & 0 \end{array}\right] = 1.$$

The same identity is used in the computations for the trivial knot.

If we choose $x = -\sqrt{t}$ then the potential function is the normalized Alexander polynomial (i.e. Alexander-Conway polynomial). The transposition of a matrix is preserving its determinant thus the substitution $x \to -x^{-1}$ (or $\sqrt{t} \to \frac{1}{\sqrt{t}}$ is preserving the potential function and Alexander-Conway polynomial. Furthermore, we can put $z = x^{-1} - x = \sqrt{t} - \frac{1}{\sqrt{t}}$. As follows from Theorem 6.2, we obtain, after the substitution, the Conway polynomial $\nabla_L(z)$ (terminology maybe sometimes confused, as $\nabla_L(z)$ is also often called Alexander-Conway polynomial).

Theorem 6.2 (Kauffman [K-1]) $\Omega_L(x) = \Delta_L(t) = \nabla_L(z)$, where $x = -\sqrt{t}$, $z = x^{-1} - x = \sqrt{t} - \frac{1}{\sqrt{t}}$.

Proof (hint). We have to show that $\Omega_{L_+}(x) - \Omega_{L_-}(x) = (x^{-1} - x)\Omega_{L_0}(x)$. In order to demonstrate it we use the properly chosen Seifert surfaces F_+, F_-, F_0 for L_+, L_- and L_0 respectively.

We give all details in the analysis of the more general case of the behavior of Seifert matrices under \bar{t}_{2k} -moves, which generalize the crossing change, which is $\bar{t}_{\pm 2}$ -move.

Definition 6.3 ([P-2]) The \bar{t}_{2k} -move (introducing k full twists on anti-parallel oriented arcs) is the elementary operation on an oriented diagram L resulting in $\bar{t}_{2k}(L)$ as illustrated in Figure 6.1.

Notice that \bar{t}_2 -move is a crossing change from a positive to negative crossing $(L_- = \bar{t}_2(L_+))$. We can choose Seifert surfaces F(L), $F(\bar{t}_{2k}(L))$, and $F(L_{\infty})$ for $L = L_{i}$, $\bar{t}_{2k}(L)$, and $L_{\infty} = L_{i}$, respectively, to look locally as in Figure 6.1.



Fig. 6.1; Oriented links L, $\bar{t}_{2k}(L)$, and L_{∞} , and their Seifert surfaces

Let us choose a basis for $H_1(F(L_{\hat{\lambda}} \zeta))$ and add one, standard, element, $e_{\hat{\lambda}}$ to obtain a basis for $H_1(F(L_{\hat{\lambda}}))$, and $e_{\bar{t}_{2k}(L)}$ to get a basis of $H_1(F(\bar{t}_{2k}(L)))$. Denote the Seifert matrix of $L_{\hat{\lambda}} \zeta$ in the chosen basis by $A_{L_{\hat{\lambda}}} \zeta$. In these bases we have immediately.

Lemma 6.4

$$A_{L_{i}} = \begin{bmatrix} A_{L_{i}} & \alpha \\ \beta & q \end{bmatrix},$$
$$A_{\bar{t}_{2k}(L)} = \begin{bmatrix} A_{L_{i}} & \alpha \\ \beta & q+k \end{bmatrix}$$

where α is a column given by linking numbers of $e_{\overleftarrow{t}_{2k}(L)}^+$ (or $e_{\overline{t}_{2k}(L)}^+$) with basis elements of $H_1(F(L_{\backslash \zeta}))$, β is a row given by linking numbers of basis elements of $H_1(F(L_{j\zeta}))$ with e_{\smile}^- (or $e_{\overline{t}_{2k}(L)}^-$), and q is a number equal to $lk(e_{\smile}^+, e_{\smile})$ (compare [K-1, P-T-2] or [P-1]).

- **Corollary 6.5** (i) If two oriented links are \bar{t}_{2k} equivalent (that is they differ by a finite number of \bar{t}_{2k} -moves) then their Seifert matrices are S-equivalent modulo k.
 - (ii) The potential function satisfies:

$$\Omega_{\bar{t}_{2k}(L)} - \Omega_{i} = k(x - x^{-1})\Omega_{i}$$

In particular the case k = -1 gives: $\Omega_{L_{+}}(x) - \Omega_{L_{-}}(x) = (x^{-1} - x)\Omega_{L_{0}}(x)$.

Proof: (i) It follows from the fact we noted in Lemma 6.4 that for properly chosen Seifert surfaces and basis of their homology, the entries of Seifert matrices for \bar{t}_{2k} and \approx are congruent modulo k.

(*ii*)
$$\Omega_{\bar{t}_{2k}(L)} = \det(xA_{\bar{t}_{2k}(L)} - x^{-1}A_{\bar{t}_{2k}(L)}^T) = \det \begin{bmatrix} A_L & x\alpha - x^{-1}\beta^T \\ x\beta - x^{-1}\alpha^T & (x - x^{-1})(q + k) \end{bmatrix},$$

and $\Omega_{\swarrow} = \det \begin{bmatrix} A_L & x\alpha - x^{-1}\beta^T \\ x\beta - x^{-1}\alpha^T & (x - x^{-1})q \end{bmatrix}.$

Thus the difference is equal to $k(x - x^{-1})\Omega_{\searrow}$.

Example 6.6 We can use Corollary 6.5 to compute the potential (and Alexander-Conway) polynomial of the pretzel link $L = P_{2k_1+1,2k_2+1,...,2k_m+1}$ (see Fig. 5.3 or 6.2). Namely, we apply the formula of Corollary 6.5(ii) for any column of a pretzel link. For $z = x^{-1} - x$ we get

$$\Omega_L(x) = \nabla_L(z) = \sum_{j=0}^{m-1} s_{m,j} z^j \nabla_{T_{2,m-j}}(z) = z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-1}{0} + s_{m,1} \binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-2}{0} + s_{m,2} \binom{m-3}{0} + \ldots \right) + z^{m-1} \left(\binom{m-3}{0} + \ldots \right) + z$$

$$z^{m-3} \begin{pmatrix} m-2\\1 \end{pmatrix} + s_{m,1} \begin{pmatrix} m-3\\1 \end{pmatrix} + s_{m,2} \begin{pmatrix} m-4\\1 \end{pmatrix} + \dots \end{pmatrix} + \dots = \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-2j}{2} \binom{m-1-j-i}{j} s_{m,j} z^{m-1-2j},$$

Where $s_{m,j}$ is an elementary symmetric polynomial in variables $k_1, ..., k_m$ of degree j, that is $\prod_{i=1}^m (z+k_i) = \sum_{j=0}^m s_{m,j} z^{m-j}$ and $\nabla_{T_{2,m-j}}(z) = \nabla_{P_{1,1,...,1}}(z)$ is the Alexander-Conway polynomials of the torus links of type (2, m-j), in particular, it satisfies Chebyshev type²⁴ (compare Example 2.15) relations $\nabla_{T_{2,n}}(z) = z \nabla_{T_{2,n-1}}(z) + \nabla_{T_{2,n-2}}(z)$ (with initial data $\nabla_{T_{2,0}}(z) = 0$ and $\nabla_{T_{2,1}}(z) = 1$). In particular, $\Omega_{T_{2,n}}(x) = \nabla_{T_{2,n}}(x^{-1} - x) = \frac{x^{-n} - (-1)^n x^n}{x^{-1} + x} = {\binom{n-1}{0}} z^{n-1} + {\binom{n-2}{1}} z^{n-3} + \ldots + {\binom{n-1-i}{i}} z^{n-1-2i} + \ldots = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} {\binom{n-1-2i}{i}} z^{n-1-2i}.$



Fig. 6.2; $P_{5,7,-3}$ – the pretzel knot with the trivial Alexander-Conway polynomial

6.1 Tristram- Levine signature

We generalize definition of the classical (Trotter-Murasugi) signature after Tristram and Levine (see [Gor, Lev, P-T-2, Tr]).

Recall that a symmetric Hermitian form $h : C^n \times C^n \to C$ is a map which satisfies h(a+b,c) = h(a,c) + h(b,c), $h(\lambda a,b) = \lambda h(a,b)$, and $h(a,b) = \overline{h(b,a)}$. The matrix H of a symmetric Hermitian form in any basis is called

²⁴We have $\nabla_{T_{2,n}}(z) = i^{1-n} S_{n-1}(iz).$

a Hermitian matrix (i.e. $H = \overline{H}^T$). A symmetric Hermitian form has a basis in which the matrix is diagonal with 1, -1 or 0 entries. The numbers, n_1 of 1's, n_{-1} of -1's and n_0 of 0's form a complete invariant of a symmetric Hermitian form (the Sylvester law of inertia). The number n_0 is called the nullity of the form and $\sigma = n_1 - n_{-1}$ is called the signature of the form. Recall also that if we count eigenvalues of H (with multiplicities) then n_1 is the number of positive eigenvalues of H and n_{-1} is the number of negative eigenvalues.

Definition 6.7 ([Tr, Lev]) Let A_L be a Seifert matrix of a link L. For each complex number ξ ($\xi \neq 1$) consider the Hermitian matrix $H_L(\xi) =$ $(1 - \bar{\xi})A_L + (1 - \xi)A_L^T$. The signature of this matrix is called the Tristram-Levine signature of the link L. If the parameter ξ is considered, we denote the signature by $\sigma_L(\xi)$, if we consider $\psi = 1 - \xi$ as a parameter, we use notation $\sigma_{\psi}(L)$ The classical signature σ satisfies $\sigma(L) = \sigma_1(L) = \sigma_L(0) = \sigma_L(-1)$. Also, by well justified convention, we put $\sigma_L(1) = 0$ (see Remark 6.8).

Tristram-Levine signature is a well defined link invariant as it is an invariant of S-equivalence of Seifert matrices. Checking this is similar to the calculation for the potential function (we leave a pleasure exercise of verifying it to the reader).

Remark 6.8 The signature of a Hermitian matrix is unchanged when matrix is multiplied by a positive number²⁵, we can (and will) often assume that ξ in $\sigma_L(\xi)$ and ψ in $\sigma_{\psi}(L)$ are of unit length. With such assumptions we have Tristram-Levin signature functions, $\sigma_L(\xi), \sigma_{\psi}(L) : S^1 \to Z$. $\sigma_L(\xi)$ is the signature function tabulated in [Ch-L], and $\sigma_{\psi}(L)$ is used in Examples in this book. S^1 will be usually parameterized by $\arg(\psi) \in [-\pi, \pi]$.²⁶ Generally, we have $\sigma_L(\xi) = \sigma_{1-\xi}(L)$ but when restricted to the unit circle, we have to write $\sigma_L(\xi) = \sigma_{(1-\xi)/(|1-\xi|}(L)$. Notice that for $\psi = \frac{1-\xi}{|1-\xi|}$, we have $\psi^2 = \frac{(1-\xi)(1-\xi)}{(1-\xi)(1-\xi)} = \frac{1-\xi}{1-\xi} = -\xi$ (and $(i\psi)^2 = \xi$). Therefore, $\sigma_{\psi}(L) = \sigma_L((i\psi)^2) = \sigma_L(\xi)$, for $Re(\psi) \geq 0$. As we show in Corollary 6.13, $\sigma_i(L) = 0$, which justifies the convention²⁷ that $\sigma_L(1) = 0$.

²⁵The Hermitian matrix H is Hermitian similar to λH for any real positive number λ ; $\lambda H = (\sqrt{\lambda}Id)H(\sqrt{\lambda}Id)$.

²⁶In [Ch-L], S^1 is parameterized by $\frac{arg\xi}{\pi}$.

²⁷In the literature on Tristram-Levine signature of knots, often used normalization of

Corollary 6.9 ([P-2])

(i) For any \bar{t}_{2k} -move and $Re(1-\xi) \ge 0$ (i.e. $|arg(\psi)| \le \pi/2$) we have:

$$0 \le \sigma_{\bar{t}_{2k}(L)}(\xi) - \sigma_L(\xi) \le 2.$$

In particular ([P-T-2], for $Re(1-\xi) \ge 0$, we have $-2 \le \sigma_{L_+}(\xi) - \sigma_{L_-}(\xi) \le 0$.

(ii) Furthermore, for any ξ and k we have:

$$0 \le |\sigma_{L_{\lambda}}(\xi) - \sigma_{\bar{t}_{2k}(L)}(\xi)| \le 1.$$

In particular, $0 \leq |\sigma_{L_+}(\xi) - \sigma_{L_0}(\xi)| \leq 1$.

Proof: Applying Lemma 6.4 we obtain

$$H_{\bar{t}_{2k}(L}(\xi) = \begin{bmatrix} H_{L_{\tilde{\zeta}}}(\xi) & a \\ a^{-T} & m + k(2 - \xi - \bar{\xi}) \end{bmatrix},$$
$$H_{L}(\xi) = \begin{bmatrix} H_{L_{\tilde{\zeta}}}(\xi) & a \\ a^{-T} & m \end{bmatrix},$$

where $a = (1 - \bar{\xi})\alpha + (1 - \xi)\beta^T$ and $m = ((1 - \bar{\xi}) + (1 - \xi))q$. Because $2 - \xi - \bar{\xi} \ge 0$, so $0 \le \sigma(H_{\bar{t}_{2k}(L)}(\xi)) - \sigma(H_L(\xi)) \le 2$, and the proof of (i) is finished²⁸. Part (ii) follows from the easy observation that deleting the last row and column of a Hermitian matrix can change the signature at most by ± 1 . \Box

We can use results of computations in Examples 5.11, 5.12, and 5.13 to find the Tristram-Levine signature for the trefoil knot, the figure eight knot, and the pretzel knot $P_{2k_1+1,2k_2+1,2k_3+1}$.

the Hermitian matrix $(1 - \bar{\xi})A_L + (1 - \xi)A_L^T$ is to take $|\xi| = 1$ $(\xi \neq 1)$. When one writes the function $\sigma_K(\xi)$ then usual assumption about the parameter ξ is that it is on the unit circle. Then one has $\det((1 - \bar{\xi})A_L + (1 - \xi)A_L^T) = \det((\xi - 1)(\frac{1 - \bar{\xi}}{\xi - 1}A - A^T)) =$ $\det((\xi - 1)(\bar{\xi}A - A^T)) \doteq (\xi - 1)^n \Delta(\bar{\xi})$, where \doteq denotes equality up to $\pm t^i$, [Gor] (compare Lemma 6.16). When dealing with links, we found more convenient (see [P-T-2, P-1] to consider $\psi = 1 - \xi$ and assume that $|\psi| = 1$. Then we have $\det(i(\bar{\psi}A + \psi A^T)) =$ $\det(i\bar{\psi}A - i\psi A^T) = \Omega(i\bar{\psi}) = \Omega(i\psi) = \nabla(-i(\bar{\psi} + \psi))$ (compare Lemma 6.16). Therefore, for any knot $\sigma_{\psi}(K) = \sigma_K((i\psi)^2) = \sigma_K((\xi)$.

²⁸It holds, in general, that if two $n \times n$ Hermitian matrices H and H' differ only at one entry, $a'_{nn} > a_{nn}$ then $0 \le \sigma(H') - \sigma(H) \le 2$. Furthermore, if det H det H' > 0 then $\sigma(H') = \sigma(H)$ and if det H det H' < 0 then $\sigma(H') = \sigma(H) + 2$.

Example 6.10 Using the Seifert matrix for the right-handed trefoil knot $(\bar{3}_1)$ computed in Example 5.11 we find that:

$$H_{\bar{3}_{1}}(\xi) = \begin{bmatrix} \xi + \bar{\xi} - 2 & 1 - \xi \\ 1 - \bar{\xi} & \xi + \bar{\xi} - 2 \end{bmatrix}$$

Therefore

$$\sigma_{\bar{3}_1}(\xi) = \begin{cases} -2 & if \quad Re(1-\xi) > \frac{1}{2} \\ -1 & if \quad Re(1-\xi) = \frac{1}{2} \\ 0 & if \quad -\frac{1}{2} < Re(1-\xi) < \frac{1}{2} \\ 1 & if \quad Re(1-\xi) = -\frac{1}{2} \\ 2 & if \quad Re(1-\xi) < \frac{1}{2} \end{cases}$$

Part of the regularity of the Tristram-Levine signature can be explained by the observation that for $\xi_2 = 2 - \xi_1$ (i.e. $1 - \xi_2 = -(1 - \xi_1)$) we have $H_L(\xi_2) = -H_L(\xi_1)$ and $\sigma_L(\xi_2) = -\sigma_L(\xi_1)$.

Example 6.11 Using the Seifert matrix for the figure eight knot (4_1) computed in Example 5.12 we find that:

$$H_{4_1}(\xi) = \begin{bmatrix} 2-\xi-\bar{\xi} & \bar{\xi}-1\\ \xi-1 & \xi+\bar{\xi}-2 \end{bmatrix}$$

For any $\xi \neq 1$, we have det $H_{4_1}(\xi) = -(2-\xi-\bar{\xi})^2 - (1-\xi)(1-\bar{\xi}) < 0$, thus $\sigma_{4_1}(\xi) = 0$.

The observation that for the figure eight knot the Tristram-Levin signature is always equal to zero is not that unexpected because the figure eight knot is an ampchiheiral knot $(4_1 = \bar{4}_1)$ and we have:

Corollary 6.12 If \overline{L} is the mirror image of a link L then the Seifert matrix $A_{\overline{L}} = -A_L$, $H_{\overline{L}}(\xi) = -H_L(\xi)$, $\sigma_{\overline{L}}(\xi) = -\sigma_L(\xi)$, and $\sigma_{\psi}(\overline{L}) = -\sigma_{\psi}(L)$. In particular, the Tristram-Levine signature of an ampchiheiral link is equal to zero.

We can also observe that $i \ (i = \sqrt{-1})$ times the matrix of τ from Exercise 5.15 is a Hermitian matrix of the signature equal to 0 thus for a knot, $\sigma_i(K) = 0$. This holds also for links as signature is unchanged by adding to the matrix rows and columns of zeros:

Corollary 6.13 For any link L we have $\sigma_i(L) = \sigma_{-i}(L) = 0$.

It is useful to summarize our observations about the Tristram-Levin signature of links using $\psi = 1 - \xi$ and $|\psi| = 1$.

Corollary 6.14 When we change ψ from 1 to i, the signature $\sigma_{\psi}(L)$ changes from the classical (Trotter-Murasugi) $\sigma(L)$ to 0 (equivalently, if ξ changes from 1 to -1, then $\sigma_L(\xi)$ changes from 0 to $\sigma(L)$). Furthermore, $\sigma_{\psi}(L) = \sigma_{\bar{\psi}}(L) = -\sigma_{-\psi}(L) = -\sigma_{\psi}(\bar{L})$.

Example 6.15 Using the Seifert matrix of the pretzel knot $P_{2k_1+1,2k_2+1,2k_3+1}$ computed in Example 5.13 we find that:

$$H_{P_{2k_1+1,2k_2+1,2k_3+1}} = \begin{bmatrix} -(\psi + \bar{\psi})(k_1 + k_2 + 1) & k_2\bar{\psi} + (k_2 + 1)\psi \\ (k_2 + 1)\bar{\psi} + k_2\psi & -(\psi + \bar{\psi})(k_2 + k_3 + 1) \end{bmatrix}$$

Furthermore,

$$\det H_{P_{2k_1+1,2k_2+1,2k_3+1}} = (\psi + \bar{\psi})^2 (1 + k_1 + k_2 + k_3 + k_1 k_2 + k_1 k_3 + k_2 k_3) - 1.$$

Therefore the Tristram-Levine signature of a pretzel knot with $1 + k_1 + k_2 + k_3 + k_1k_2 + k_1k_3 + k_2k_3 > 0$ (e.g. a positive pretzel knot) satisfies (in lieu of Corollary 6.12 we consider only $Re(\psi) \ge 0$):

$$\sigma_{\psi}(P_{2k_{1}+1,2k_{2}+1,2k_{3}+1})) = \begin{cases} -2 & if \quad Re(\psi) > \frac{1}{2\sqrt{1+k_{1}+k_{2}+k_{3}+k_{1}k_{2}+k_{1}k_{3}+k_{2}k_{3}}} \\ -1 & if \quad Re(\psi) = \frac{1}{2\sqrt{1+k_{1}+k_{2}+k_{3}+k_{1}k_{2}+k_{1}k_{3}+k_{2}k_{3}}} \\ 0 & if \quad 0 \le Re(\psi) < \frac{1}{2\sqrt{1+k_{1}+k_{2}+k_{3}+k_{1}k_{2}+k_{1}k_{3}+k_{2}k_{3}}} \end{cases}$$

Notice, that in the example of Seifert of $P_{5,7,-3}$, Figure 6.2, we have det $H_{P_{5,7,-3}} = -1$ and $\sigma_{\psi}(P_{5,7,-3}) \equiv 0$. We utilize the result of this calculation in *[P-Ta]*.

6.2 Potential function and Tristram-Levine signature

Lemma 6.1 and Definition 6.7 suggest that there is a relation between the potential function and the Tristram-Levine signature of links. In fact we have:

Lemma 6.16 Assume that the potential function at $i\psi$ is different from zero. Then

$$i^{\sigma_{\psi}(L)} = \frac{\Omega_L(i\psi)}{|\Omega_L(i\psi)|} = \frac{\Delta_L(t_0)}{|\Delta_L(t_0)|} = \frac{\nabla_L(-i(\psi+\psi))}{|\nabla_L(-i(\psi+\bar{\psi}))|}$$

where $\Delta_L(t_0)$ is the Alexander-Conway polynomial and $t_0 = -\psi^2$ ($\sqrt{t_0} = -i\psi$). In particular, the Tristram-Levine signature is determined modulo 4 by the appropriate value of the potential function (or Alexander-Conway polynomial); compare Chapter III of [P-Book].

Proof: The idea is to compare the formulas for the potential functions and the signature, that is:

$$\Omega_L(i\psi) = \det(i\psi A_L - (i\psi)^{-1}A_L^T) =$$

$$i^n \det(\psi A_L + \bar{\psi}A_L^T) \quad and$$

$$\sigma_{\psi}(L) = \sigma(\bar{\psi}A_L + \psi A_L^T)$$

In more detail, we write our proof as follows:

Let H be a non-singular Hermitian matrix of dimension n and $\lambda_1, \lambda_2, ..., \lambda_n$ its eigenvalues (with multiplicities). Then $det(iH) = i^n \det H = i^n \lambda_1 \lambda_2 \cdots \lambda_n =$ $i^n(-1)^{n-1} |\det H| = i^{n-2n-1} |detH| = i^{n_1-n_{-1}} |detH| = i^{\sigma(H)} |detH|$. Therefore, $\frac{det(iH)}{|det(iH)|} = i^{\sigma(H)}$. By applying this formula for $H = \psi A_L + \bar{\psi} A_L^T$, $|\psi| = 1$, and remembering that $\sigma(\bar{H}) = \sigma(H)$, we obtain the formula of Lemma 6.16.

Example 6.17 We can use Lemma 6.16 to compute quickly Tristram-Levine signature²⁹ of the torus link of type (2, n), $T_{2,n}$. We use the fact that we already computed the classical signature, and Alexander-Conway (and potential) polynomial to be (for $k \neq 0$):

$$\sigma(T_{2,n}) = 1 - n, \quad \Delta_{T_{2,n}}(z) = \Omega_{T_{2,n}}(x) = \frac{x^{-n} - (-1)^n x^n}{x^{-1} + x} = t^{\frac{1-n}{2}} \frac{t^n + (-1)^{n+1}}{t+1}$$

where $z = x^{-1} - x = t^{1/2} - t^{-1/2}$. In particular $\sigma_{\psi}(T_{2,n})$ can change only if $x = i\psi$ is a root of the potential function, and because $\Omega_{T_{2,n}}(i\psi) = i^{1-n}\frac{\psi^n - \psi^{-n}}{\psi - \psi - 1}$, the only changes holds at ψ satisfying $\psi^{2n} = 1$ and $\psi \neq \pm 1$. We have for $Re\psi \ge 0$, $k \ne 0$, $0 \le j \le n - 1$:

$$\sigma_{\psi}(T_{2,n}) = \begin{cases} 1-n & if & Re(\psi) > Re(e^{\pi/n}) \\ 1-n+2j & if & Re(e^{j\pi/n}) > Re(\psi) > Re(e^{(j+1)\pi/n}), \ j > 0 \\ 2-n+2j & if & Re(\psi) = Re(e^{j\pi/n}), \ j > 0. \end{cases}$$

²⁹It is, essentially, the same proof we used in Chapter III of [P-Book] to show that a signature is a skein equivalence invariant: The Alexander-Conway polynomial determines signature modulo 4 and the Murasugi type inequalities $(|\sigma_{\psi}(L_{+}) - \sigma_{\psi}(L_{0})| \leq 1$ and for $Re(\psi) \geq 0, 0 \leq (\sigma_{\psi}(L_{-}) - \sigma_{\psi}(L_{+}) \leq 2)$ gives the direction, and limit the size of the signature change, compare also Corollary 6.5.

Corollary 6.18 The classical (Trotter-Murasugi) signature $\sigma(L) = \sigma_1(L) = \sigma_L(-1)$, satisfies:

$$i^{\sigma(L)} = i^{\sigma(A_L + A_L^T)} = \frac{\Omega_L(i)}{|\Omega_L(i)|} = \frac{\Delta_L(-1)}{|\Delta_L(-1)|} = \frac{Det_L}{|Det_L|} = \frac{\nabla(-2i)}{|\nabla(-2i)|}, \quad assuming \quad Det_L \neq 0;$$

here $\Delta_L(-1)$ denotes $\Delta_L(t)$ for $\sqrt{t} = -i$. Recall, that $Det_L = \Delta_L(-1) = \Omega_L(i) = det(i(A_L + A_L^T))$ is called the determinant³⁰ of a link L.

Example 6.19 We compute here the Tristram-Levine signature of the knot $K = 6_2$ using Lemma 6.16 and discuss the standard convention and notation. We have:

$$\sigma_{\psi}(6_2) = \begin{cases} -2 & if \qquad Re(\psi) > \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}} \\ -1 & if \qquad Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}} \approx 0.636... \\ 0 & if \qquad 0 \le Re(\psi) < \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}. \end{cases}$$

Step 1. We compute the the Conway polynomial $\nabla_{6_2}(z) = 1 - z^2 + z^4$; we use resolution in Figure 6.3 to find this value and also observe that changing a crossing at p results in the trivial knot and smoothing at p results in a connected sum of the right handed trefoil knot and the left handed Hopf link $(K_0^p = \bar{3}_1 \# H_-)$. In particular the unknotting number $u(6_2) = 1$.

Step 2. $Det_K = \nabla_K(-2i) = -11$, thus $\delta(K) \equiv 2 \mod 4$, and because K can be unknotted by changing one positive crossing, thus $-2 \leq \sigma(K) \leq 0$, and finally $\sigma(K) = -2$.

Step 2. Roots of $\nabla_{6_2}(z)$ are at $z^2 = \frac{-1\pm\sqrt{5}}{2}$. Thus for $t_0 = \xi = -\psi^2$, $z = -i(\psi + \bar{\psi})$, we have $\xi + \bar{\xi} = (i\psi)^2 + (i\bar{\psi})^2 = z^2 + 2 = \frac{3\pm\sqrt{5}}{2}$. Because $|\psi| = |\xi| = 1$, therefore $-2 \le \xi + \bar{\xi} \le 2$ and $\xi + \bar{\xi} = \frac{3-\sqrt{5}}{2}$ (Re $(\xi) = \frac{3-\sqrt{5}}{4}$). Finally, assuming Re $(\psi) \ge 0$ we get $\psi = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$.

Step 3. For $Re(\psi) \ge 0$, the value $Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$ is the only place where the Tristram-Levine signature $\sigma_{\psi}(6_2)$ can be changing, and because we know

³⁰We should mention here that $|Det_L|$ is equal to $|det(G_L)|$ where G_L is a Goeritz matrix of L. Furthermore, if D_L is a special diagram of an oriented link L then $G_{D_L} = A_L + A_L^T$ for a properly chosen basis of $H_1(S)$ where S is the Seifert surface of D_L constructed according to Seifert algorithm. Thus not only $Det_L = det(iG_{D_L})$ but also $\sigma(L) = \sigma(G_L)$; compare Corollary 2.7.

already that $\sigma_1(6_2) = -2$ and $\sigma_i(6_2) = 0$ we conclude that $\sigma_{\psi}(6_2) = -2$ if $Re(\psi) > \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$ and $\sigma_{\psi}(6_2) = 0$ if $0 \le Re(\psi) < \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$.

Step 4. It remains to show that $\sigma_{\psi}(6_2) = -1$ for $Re(\psi) = \frac{1}{2}\sqrt{\frac{1+\sqrt{5}}{2}}$. Here we argue that, because the considered ψ is the singular root of the Alexander polynomial (precisely $t_0 = -\psi^2$), therefore the value of signature at this point cannot differ by more than one from the neighboring values (so from 0 and from -2). More detailed analysis of the Hermitian matrix $\bar{\psi}A + \psi A^T$, leads to the conclusion that if $t_0 = -\psi^2$ is a singular root of the Alexander polynomial of a knot K then $\sigma_{\psi}(K) = \frac{\sigma_{\psi_-}(K) + \sigma_{\psi_+}(K)}{2}$, where ψ_- and ψ_+ are parameters just before ψ and just after ψ on the unit circle [Mat].

In the convention of [Gor, Ch-L] one defines the Tristram-Levine signature function of variable ξ ($|\xi| = 1$) as $\sigma_L(\xi) = \sigma((1 - \bar{\xi})A + (1 - \xi)A^T)$). For $Re(\psi) \ge 0$, one has $\sigma_{\psi}(L) = \sigma_L(\xi)$, where $\xi = -\psi^2$ ($\psi = \frac{1-\xi}{|1-\xi|}$). In knotinfo Web page [Ch-L], the parameter s satisfying $\xi = e^{\pi i s}$ is used. In particular, $\sigma_{6_2}(\xi) = -1$ for $Re(\xi) = \frac{3-\sqrt{5}}{4} = 1 - \cos(\pi/5) \approx 0.191$, and $s = \arccos(1 - \cos(\pi/5))/\pi \approx 0.44$ (compare Remark 6.8).



Fig. 6.3; Computing the Conway polynomial of the knot $K = 6_2$. $\nabla_{6_2}(z) = \nabla_{T_1}(z) + \nabla_{\bar{3}_1 \# H_-}(z) = 1 + (1+z^2)(-z) = 1 - z^2 - z^4$

Example 6.20 The knot $9_{42} =$ is the smallest knot which is not

amphicheiral but the Jones, Homflypt, and Kauffman polynomials are symmetric (e.g. $V_{9_{42}}(t) = V_{9_{42}}(t^{-1})$). The non-amphicheirality of 9_{42} is detected by signature: $\sigma(9_{42}) = -2 = -\sigma(\overline{9_{42}})$. This description can leave however an impression that the fact that 9_{42} is not ambient isotopic to its

mirror image cannot be checked by the Jones polynomial alone. However, it follows from Corollary 5.17 that $(-1)^{\sigma(K)/2} = sign(V_K(-1))$ for any knot K, thus if a knot is amphicheiral then $V_K(-1) > 0$. For 9_{42} we have $V_{9_{42}}(-1) = Det_{9_{42}} = -7 < 0$ thus 9_{42} is not amphicheiral. Furthermore, because 9_{42} can be unknotted by changing one positive crossing, we can deduce that $\sigma(9_{42}) = -2$.

In fact, the absolute value of the determinant $|Det_K| = |V_K(-1)| = |\Delta_K(-1)| = |\nabla(-2i)|$ suffices to show that the knot 9_{42} is not amphicheiral. K. Murasugi proved in [M-10] (Theorem 5.6), the following result:

Theorem 6.21 For any knot K

$$\sigma_K \equiv |Det_K| - 1 \mod 4$$

Proof: We use the fact that $Det_K = \nabla(-2i) \equiv 1 \mod 4$. Therefore, $|Det_K| \equiv 1 \mod 4$ if $Det_K > 0$ and $|Det_K| \equiv -Det_K \equiv -1 \mod 4$ if $Det_K < 0$. Furthermore, from Corollary 6.18 follows that $Det_K = (-1)^{\sigma(K)/2} |Det_K|$. Therefore,

$$\sigma_K \stackrel{\text{mod } 4}{\equiv} \begin{cases} 0 & if \quad |Det_K| \equiv 1 \mod 4\\ 2 & if \quad |Det_K| \equiv 3 \mod 4 \end{cases}$$

and Theorem 6.21 follows. \Box

Murasugi's Theorem leads to a curious formula:

Corollary 6.22 For any knot K

$$Det_K = (-1)^{(|Det_K| - 1)/2} |Det_K|.$$

J. Milnor proved that the signature of a knot with the Alexander polynomial equal to one is equal to zero [Mil]. In fact, it follows directly from Lemma 5.16 that the Tristram-Levin signature can change only at roots of unit length of Alexander polynomial; therefore a link which has the Alexander polynomial without any root on the unit circle has constant Tristram-Levin signature function. Thus:

Corollary 6.23 ([Mil]) If the Alexander polynomial $\Delta_L(t)$ is different from zero on the unit circle then for any ψ , $(|\psi| = 1)$, we have $\sigma_{\psi}(L) = 0$.

If we assume only that the determinant of a knot is equal to 1 then we get as a conclusion that the signature is divisible by eight (compare [M-9], page 149 after Exercise 7.5.4):

Proposition 6.24 If the determinant of a knot K is equal to 1 then $\sigma(K) \equiv 0 \mod 8$.

Proof: $Det_K = 1$ means that for a Seifert matrix A of a knot K, $det(A + A^T) = 1$; The matrix/form $A + A^T$ is often called the Trotter form. The diagonal entries of the Trotter form are even because the diagonal of $A + A^T$ is twice a diagonal of A. We can summarize these conditions by saying that the Trotter form is even and unimodular; recall that unimodularity means that $det(A + A^T)$ is invertible (here equal to ±1). The form is even if $x(A + A^T)x^T$ is always an even number. Finally, every even unimodular form over Z has its signature divisible by 8; see Theorem II.5.1 in [M-H]. □

7 A combinatorial formula for the signature of alternating diagrams; Quasi-alternating links

Corollary 6.18 has various interesting consequences. P. Traczyk used it back in 1987 [Tra] to find the combinatorial formula for the signature of alternating links, starting from analysis of the condition $\sigma(L_+) = \sigma(L_0) - 1$ (and $\sigma(L_-) = \sigma(L_0) + 1$) and observing that it holds for any essential crossing of an alternating diagram. The property was refined by Manolescu, Ozsvath, and Szabo and used to define quasi-alternating links [O-S], whose Khovanov [Kho] and Heegaard Floer homology share with alternating links many interesting properties [M-O, C-K] (compare Chapter X of [P-Book]). The property, of links which Manolescu, Ozsvath, and Szabo observe to be important, and which always holds for alternating links, is the following (compare Subsection 1.4):

$$|Det_{i}| = |Det_{i}| + |Det_{i}|$$

The following result combines the above properties (compare [M-O]):

Theorem 7.1

The following two conditions are equivalent, providing that determinants of L_0 and L_{∞} are not equal to zero³¹

³¹In (a) one deals with a Kauffman skein triple of unoriented links; in (b) one chooses any orientation of L_+ (e.g. \searrow) and related orientation of L_0 (\bowtie), and any orientation of L_{∞} (e.g. $\backslash \zeta$ or $\backslash \zeta$).

 $\begin{array}{ll} (a) & |Det_{L_{+}}| = |Det_{L_{0}}| + |Det_{L_{\infty}}| \\ (b) & \sigma(L_{+}) = \sigma(L_{0}) - 1 \ and \ \sigma(L_{+}) = \sigma(L_{\infty}) - \frac{1}{2}(w(L_{0}) - w(L_{\infty})). \\ The similar equivalence also holds for a negative crossing: \\ (a') & |Det_{L_{-}}| = |Det_{L_{0}}| + |Det_{L_{\infty}}| \\ (b') & \sigma(L_{-}) = \sigma(L_{0}) + 1 \ and \ \sigma(L_{-}) = \sigma(L_{\infty}) + \frac{1}{2}(w(L_{0}) - w(L_{\infty})). \end{array}$

Proof: ((a) \Leftrightarrow (b)): We apply the formula $Det(L) = i^{\sigma(L)} |Det(L)|$ and use the relation between the Jones polynomial, and its Kauffman bracket variant, with the signature. Recall, that the Jones polynomial $V_L(t)$ of an oriented link L is normalized to be one for the trivial knot and satisfies the skein relation $t^{-1}V_{\checkmark}(t) - tV_{\checkmark}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{\bigstar}(t)$. For t = -1 (or, more precisely, $\sqrt{t} = i$) we obtain exactly the skein relation of the determinant: $Det_{\checkmark} - Det_{\checkmark} = -2iDet_{\bigstar}$. Thus $Det_L = V_L(-1); \sqrt{t} = i$. Recall also that the Kauffman bracket polynomial of unoriented link diagrams, $\langle D \rangle \in Z[A^{\pm 1}]$, is defined by the following properties [K-6]:

- (i) $\langle \bigcirc \rangle = 1$
- (ii) $\langle \bigcirc \sqcup D \rangle = -(A^2 + A^{-2}) \langle D \rangle$
- (iii) $\langle \) = A \langle \simeq \rangle + A^{-1} \langle \rangle \langle \rangle$

Furthermore, if D is an oriented diagram with underlying unoriented diagram D then $V_{\vec{D}}(t) = (-A^3)^{w(\vec{D})} \langle D \rangle$. Thus for $A^2 = -i (A^4 = -1)$ we get: $Det(D) = (-A^3)^{-w(D)} \langle D \rangle = A^{w(D)} \langle D \rangle$. Recursive formula for the Kauffman bracket $\langle D_+ \rangle = A \langle D_0 \rangle + A^{-1} \langle D_\infty \rangle$ leads to $(-A^3)^{w(D_+)}Det(D_+) = A(-A^3)^{w(D_0)}Det(D_0) + A^{-1}(-A^3)^{w(D_\infty)}Det(D_\infty)$ then leads to $A^{-w(D_+)}Det(D_+) = A^{1-w(D_0)}Det(D_0) + A^{-1-w(D_\infty)}Det(D_\infty)$, then leads to $A^{-w(D_+)}i^{\sigma(D_+)}|Det(D_+)| = A^{1-w(D_0)}i^{\sigma(D_0)}|Det(D_0)| + A^{-1-w(D_\infty)}i^{\sigma(D_\infty)}|Det(D_\infty)|$ and eventually to $|Det(D_+)| = A^{w(D_+)-w(D_0)+1}i^{\sigma(D_0)-\sigma(D_+)}|Det(D_0)| + A^{w(D_+)-w(D_\infty)-1}i^{\sigma(D_\infty)-\sigma(D_+)}|Det(D_\infty)| = i^{\sigma(D_0)-\sigma(D_+)-1}|Det(D_0)| + i^{\sigma(D_\infty)-\sigma(D_+)-1/2(w(D_0)-w(D_\infty))}|Det(D_\infty)|$.

When we compare this formula with that of Theorem 7.1(a) we see that (a) holds iff $i^{\sigma(D_0)-\sigma(D_+)-1} = 1$ and $i^{\sigma(D_\infty)-\sigma(D_+)-1/2(w(D_0)-w(D_\infty))} = 1$ and these

conditions are equivalent to conditions

 $\sigma(D_0) - \sigma(D_+) \equiv 1 \mod 4 \text{ and } \sigma(D_\infty) - \sigma(D_+) - \frac{1}{2}(w(D_0) - w(D_\infty)) \equiv 0 \mod 4.$ These conditions are equivalent to (b) because by Corollary 2.14(i), we have generally that $|\sigma(D_+) - \sigma(D_0)| \leq 1$). Furthermore, in general, we have that $|\sigma(D_+) - \sigma(D_\infty) + \frac{1}{2}(w(D_0) - w(D_\infty))| \leq 2$. The last inequality require some explanation and consideration of two cases in which \searrow is either a mixed crossing or a self-crossing.

- (m) If \searrow is a mixed crossing then let D_j be a component of D_+ such that the change of the orientation of D_j results in the link $D'_- = \swarrow$. Then by Corollary 2.14 $|\sigma(\bigotimes) - \sigma(\Diamond \zeta)| \leq 1$. Further, by Proposition 2.11(ii), $|\sigma(\bigotimes + 2lk(D_j, D_+ - D_j) - \sigma(\Diamond \zeta)| \leq 1$. Because $4lk(D_j, D_+ - D_j) =$ $w(D_+) - w(D'_-) = w(\bigotimes) - w(\Diamond \zeta) + 2$ we obtain $|\sigma(D_+) - \sigma(D_\infty) + \frac{1}{2}(w(D_0) - w(D_\infty)) + 1| \leq 1$ and finally $-2 \leq \sigma(D_+) - \sigma(D_\infty) - \frac{1}{2}(w(D_0) - w(D_\infty)) \leq 0$.
 - (s) If \nearrow is a self-crossing then in $D_0 = \$ the two parallel arcs belong to different link components. Let D_j component contain the lower arc and let $D'_0 = \$ be obtained from D_0 by changing the orientation of D_j . After performing the second Reidemeister move on D'_0 we obtain a diagram $\$ which has two mixed crossings. We use Corollary 2.14(i) on one of them to get $|\sigma(\) - \sigma(\rangle \langle)| \le 1$. Because $\sigma(D'_0) =$ $\sigma(D_0) + 2lk(D_j, D_0 - D_j) = \sigma(D_0) - \frac{1}{2}(w(D_0) - w(D'_0)) = \sigma(D_0) - \frac{1}{2}(w(D_0) - w(D_\infty))$, we obtain $|\sigma(\) - \sigma(\rangle \langle) + \frac{1}{2}(w(\) - w(\rangle \langle))| \le 1$, then $|(\sigma(\) - \sigma(\)) + (\sigma(\) - \sigma(\rangle \langle) + \frac{1}{2}(w(\) - w(\rangle \langle)))| \le 1$, and finally $|\sigma(\) - \sigma(\rangle \langle) + \frac{1}{2}(w(\) - w(\rangle \langle))| \le 2$ as required.

The equivalence $(a') \Leftrightarrow (b')$ follows from (a) \Leftrightarrow (b) by considering mirror images of diagrams from (a) and (b). In particular, for the diagram \overline{D} being the mirror image of D, we always have that $\sigma(\overline{D}) = -\sigma(D)$, and $w(\overline{D}) = -w(D)$).

It is not difficult to see that any crossing of an alternating diagram satisfies properties (a),(a') of Theorem 7.1. This follows from the fact that if D is an alternating diagram then also D_0 and D_{∞} are alternating, and for an alternating diagram $|Det_D|$ can be interpreted as the number of spanning trees of the underlying Tait graph, $G_b(D)$, and the number of spanning trees is additive under deleting contracting rule; see Subsection 1.4. These ideas are developed in Chapter V of [P-Book]. Without referring to it, the properties (a) and (a') of alternating links follow from the proof of Traczyk formula for the signature of alternating diagrams which we present below. First, we have to recall the necessary terminology. In fact, we use this as an opportunity for introducing basic language which unifies the notion of Tait surface and Tait graph (Footnote 13) with that of Seifert surface and Seifert graph [Crom]. Before general definition let us recall the definition of the Seifert graph.

Definition 7.2 [Crom]

The Seifert graph of an oriented diagram \vec{D} is a signed (planar) graph $\Gamma(\vec{D})$ whose vertices correspond to Seifert circles of the diagram and edges correspond to crossings of the diagram. The sign of an edge is determined by the sign of the corresponding crossing.

In the more general setting we allow arbitrary smoothings of crossings of (not necessary oriented) diagram D.

Definition 7.3 A Kauffman state s of D is a function from the set of crossings of D to the set $\{+1, -1\}$. Equivalently, we assign to each crossing of D a marker according to the following convention:



Fig. 7.1; markers and associated smoothings

By D_s we denote the system of circles in the diagram obtained by smoothing all crossings of D according to the markers of the state s, Fig. 7.1. $|D_s|$ denotes the number of circles in D_s . In this notation the Kauffman bracket polynomial of D is given by the state sum formula:

$$< D > = \sum_{s} A^{\sigma(s)} (-A^2 - A^{-2})^{|D_s|-1},$$

where $\sigma(s) = \sum_{p} s(p)$ is the number of positive markers minus the number of negative markers in the state s.

The state sum formula looks like a useful but not necessarily sophisticated tool, however, state sums (and their limits) are basic and deep concepts in the statistical physics and very likely the next breakthrough in Knot Theory (and more) will utilize a connection (still to be discovered) between phase transition of a physical system and Khovanov type homology based on closeness of states of the system (possibly persistent homology [E-Ha] will play a role here).

But we are straying too far from our local goal of associating graphs and surfaces to any Kauffman state s.

Definition 7.4 ([PPS])

- (i) Let D be a diagram of a link and s its Kauffman state. We form a graph, $G_s(D)$, associated to D and s as follows. Vertices of $G_s(D)$ correspond to circles of D_s . Edges of $G_s(D)$ are in bijection with crossings of D and an edge connects given vertices if the corresponding crossing connects circles of D_s corresponding to the vertices. As in the case of the Tait graph, $G_s(D)$ is a signed graph where the sign of an edge e(p) is s(p), that is the sign of the marker of the Kauffman state s at the crossing p.
- (ii) In the language of associated graphs we can state the definition of an sadequate diagrams as follows: the diagram D is s-adequate if the graph $G_s(D)$ has no loops (adequacy is studied and utilized in Chapter V of [P-Book]).
- (iii) We associate with every Kauffman state s of a diagram D, a surface $F_s(D)$ embedded in \mathbb{R}^3 and with $\partial F_s(D) = D$, in a manner similar to Construction 3.4 of a Seifert surface. That is, we start from the collection of circles D_s . Each of the circles bounds a disk in the projection plane. We make the disks disjoint by pushing them slightly up above the plane of projection, starting from the innermost disks. We connect the

disks together at the original crossings of the diagram D by half-twisted bands so that the 2-manifold which we obtain has D as its boundary, see Figure 3.5 (we ignore orientation of the diagram, and the resulted surface can be unorientable). Equivalently, we can start a construction of $F_s(D)$ from the graph $G_s(D)$ as a spine (strong deformation retract) of the constructed surface and proceed as follows: The graph $G_s(D)$ possesses an additional structure, that is a cyclic ordering of edges at every vertex following the ordering of crossings at any circle of D_s . The sign of each edge is the label of the corresponding crossing. In short, we can assume that $G_s(D)$ is a ribbon (or framed) graph, and that with every state we associate a surface $F_s(G)$ whose core is the graph $G_s(D)$. $F_s(G)$ is naturally embedded in \mathbb{R}^3 with $\partial F_s(G) = D$. If s is the state separating black regions of checkerboard coloring of $R^2 - D$ then $F_s(G)$ is the Tait surface of the diagram described in Exercise 2.8. For $s = \vec{s}$, that is, D is oriented and markers of \vec{s} agree with orientation of D, $G_s(D)$ is the Seifert graph of D and $F_s(G)$ is the Seifert surface of D obtained by Seifert construction. We do not use this additional data in this survey but it may be of great use in analysis of Khovanov homology (compare [A-P] or Chapter X of [P-Book]).

The surface $F_s(G)$ is not the only surface associated with the graph $G_s(D)$, another such surface is Turaev surface, M(s) [Tu], which for positive (s_{+}) or negative (s_{-}) states of an alternating diagram is a planar surface. With some justification Turaev surface can be called a background surface of a diagram. The construction of M(s) for a given state s of D is illustrated, after [Tu], in Figure 7.2. That is, M(s) is obtained from a regular neighborhood of a projection of a link by modifying (by half-twists) neighborhoods of s-wrong edges (see Figure 7.2 and compare it to Figure 1.8 to see that any alternating digram has only s_{+} -true edges). Notice, that M(s) depends on s and the link projection but not over-under information of a link diagram. Alternatively, we can say that M(s) is a surface realizing the natural cobordism between circles of D_s and circles of D_{-s} . In [DFKLS] the Turaev genus of a link is defined to be the minimal genus of Turaev surface over all diagrams D of a link with $s_{+}(D)$ states. The immediate consequence is that alternating link has the Turaev genus equal to zero. Notice also, that if we cup off the circles of D_s in M(s) by 2-discs we obtain the surface, $M^+(s)$ with boundary D_{-s} and the graph $G_s(D)$ as its spine.



Surface along s-true edge



Surface along s-wrong edge

Fig. 7.2; Turaev surface M(s) is composed of squares along every crossing of D connected by ribbons according to convention illustrated in this Figure. s-true edge and s-wrong edge are arcs of the diagram D connecting crossings and the name depends on the label given by s to boundary crossings [Tu]

Going back to Traczyk's combinatorial formula, we recall the convention for checkerboard shading of the projection plane. In an alternating diagram we choose the *standard shading* as in Figure 7.3(a) complementary to the shading given in Figure 7.3(b) (this essentially agrees with Tait's convention of checkerboard coloring, however we do not assume that the outside region is white or black).



Fig. 7.3; Checkerboard shading of the plane of the projection: (a) Tait's, (b) dual to Tait's

We denote by B the number of black (shaded) areas and by W the number of white areas (for an alternating diagram D we have $B = |D_{s_-}|$ and $W = |D_{s_+}|$). Furthermore for an oriented diagram \vec{D} let $\Gamma(\vec{D})$ denote its Seifert graph (Definition 7.2), T its (signed) spanning tree and $d_+(T)$ (resp. $d_-(T)$) the number of positive (resp. negative) edges in T. For an alternating diagram the numbers $d_+(T), d_-(T)$ do not depend on T so we can write $d_+(\vec{D})$, and $d_-(\vec{D})$ in this case³²

³²This is the case for more general class of homogeneous diagrams introduced in [Crom] and defined as diagrams for which 2-connected components of the Seifert graph have all

Lemma 7.5 If \vec{D} is an oriented connected alternating diagram of a link then

$$\frac{1}{2}(w(\vec{D}) + |D_{s_+}| - |D_{s_-}|) = d_+(\vec{D}) - d_-(\vec{D})$$

In particular, the left hand side of the equation is unchanged when one goes from \vec{D} to \vec{D}_0^p for a non-nugatory crossing p (in \vec{D}_0^p the crossing p is smoothed according to orientation of \vec{D}).

Proof: One can easily proof Lemma 7.5 by induction on the number of nonnugatory crossings of \vec{D} . First one observes that if \vec{D} has only nugatory crossings then $\Gamma(\vec{D})$ is a tree and $d_+(\vec{D}) = c_+(\vec{D}) = s_+(\vec{D}) - 1$ (and $d_-(\vec{D}) = c_-(\vec{D}) = s_-(\vec{D}) - 1$), thus the formula in Lemma 7.5 holds. In the inductive step we consider a non-nugatory crossing p of \vec{D} and compare ingredients of the formula for \vec{D} and \vec{D}_0^p , and having the formula for \vec{D}_0^p deduct it for \vec{D} . It is worth however to compare d_+ , d_- , c_+ , c_- , $|D_{s_+}|$, and $|D_{s_-}|$) in more detail. \Box

Lemma 7.6 Let p be any crossing of an oriented diagram \vec{D} . Then

(i)

$$\vec{s}(p) = \begin{cases} s_+(p) & if \quad p \text{ is positive} \\ s_-(p) & if \quad p \text{ is negative} \end{cases}$$

In particular if \vec{D} is a positive diagram then $\vec{s} = s_+$, and if \vec{D} is a negative diagram then $\vec{s} = s_-$.

(*ii*)
$$|(D_0^p)_{\vec{s}}| = |D_{\vec{s}}|$$

(iii)

$$\begin{split} |(\vec{D}_0^p)_{s_+}| &= \begin{cases} |\vec{D}_{s_+}| & if \quad p \text{ is positive} \\ |\vec{D}_{s_+}| - \varepsilon_+ & if \quad p \text{ is negative} \end{cases} \\ |(\vec{D}_0^p)_{s_-}| &= \begin{cases} |\vec{D}_{s_-}| - \varepsilon_- & if \quad p \text{ is positive} \\ |\vec{D}_{s_-}| & if \quad p \text{ is negative} \end{cases} \end{split}$$

Here ε_+ and ε_- are +1 or -1. If p is a non-nugatory crossing of an alternating diagram then $\varepsilon_+ = \varepsilon_- = 1$.

edges of the same sign (i.e. they are homogeneous). Alternating diagrams are special cases of homogeneous diagrams; this well known fact follows also from Lemma 7.5 as the lemma can be proved for a fixed choice of a spanning tree and the left side of the equation does not depend on the choice of a spanning tree.

Proof: (i) The proof is illustrated in Figure 7.4.

The other parts are equally elementary and we leave them as exercise for the reader. \Box



Fig. 7.4; $\vec{s}(p) = s_+(p)$ if sgn(p) = 1, and $\vec{s}(p) = s_-(p)$ if sgn(p) = -1

Lemma 7.7 If D is a connected alternating diagram, then for a complex number A such that $A^4 = -1$, we have:

 $(i) < D >_{A^4 = -1} = A^{B-W} | < D >_{A^4 = -1} |$

(ii) For any crossing p of an alternating diagram D one has:

$$| < D >_{A^4=-1} | = | < D_0^p >_{A^4=-1} | + | < D_\infty^p >_{A^4=-1} |$$

in other words the absolute value of the determinant of a diagram is additive under the Kauffman bracket skein triple.

Proof: If all crossings of D are nugatory, then D represents the trivial knot. Choose an orientation of D. The orientation defines signs of crossings, which are however independent on chosen orientation. As we noticed in Lemma XX in this case $c_+ = W - 1$ and $c_- = B - 1$. Thus $\langle D \rangle = (-A^3)^{w(D)} = (-A^3)^{W-B}$ (for a knot w(D) does not depend on the orientation of D). For $A^4 = -1$, $\langle D \rangle_{A^4=-1} = (-A^4)^{W-B} (A)^{B-W} = A^{B-W}$ as required. The inductive step follows easily: If p is a non-nugatory crossing of D, then from the Kauffman bracket skein relation

$$\langle D \rangle = A \langle D_0 \rangle + A^{-1} \langle D_{0+} \rangle$$

and by the inductive assumption, for $A^4 = -1$, follows that:

$$\langle D \rangle_{A^4 = -1} = A A^{B - W - 1} |\langle D_0^p \rangle_{A^4 = -1}| + A^{-1} A^{B - W + 1} |\langle D_\infty^p \rangle_{A^4 = -1}| =$$

$$A^{B-W}(|\langle D_0^p \rangle_{A^4=-1}| + |\langle D_\infty^p \rangle_{A^4=-1}|) = A^{B-W}|\langle D \rangle_{A^4=-1}|$$

which completes the proof of Lemma 7.7(i). It also establishes Lemma 7.7(ii) for a non-nugatory crossing p of a connected diagram D. If p is a nugatory crossing, then $\langle D_0^p \rangle_{A^4=-1}|$ or $|\langle D_{\infty}^p \rangle_{A^4=-1}|$ is equal to zero and (ii) holds immediately. If D is not connected diagram then (ii) holds for any connected component of D and (ii) follows because Kauffman bracket (and signature) is multiplicative under disjoint sum.

As a corollary of Theorem 7.1, 7.5, and 7.7, we have Traczyk's result.

Theorem 7.8 [Tra] If D is a reduced³³ alternating diagram of an oriented link, then

- (1) $\sigma(D) = -(c_+ c_-) + d_+ d_- = -w + d_+ d_-$
- (2) $\sigma(D) = -\frac{1}{2}(c_{+} c_{-}) + \frac{1}{2}(W B) = -\frac{1}{2}w + \frac{1}{2}(W B) = -\frac{1}{2}(w + |D_{s_{+}}| |D_{s_{-}}|)$

(3)
$$\sigma(D) = \sigma(D_0^p) - sign(p)$$

7.1 Quasi-alternating links

Quasi-alternating links introduced by Manolescu, Ozsvath, and Szabo in [O-S, M-O, C-K] are motivated by properties (a),(a') of Theorem 7.1, described in the theorem relations to signature, and applications of these properties to the thinness of Khovanov and Heegaard Floer homology:

Definition 7.9 [O-S] The family of quasi-alternating links is the smallest family of links which satisfies:

- (i) The trivial knot is quasi-alternating.
- (ii) If L is a link which admits a crossing such that (1) both smoothings $(L_0 \text{ and } L_\infty)$ are quasi-alternating, and (2) $|Det_L| = |Det_{L_0}| + |Det_{L_\infty}|$, then L is quasi-alternating.

³³*Reduced* means that no crossing of D is nugatory and the crossing p of D is called nugatory if D_0^p has more (graph) component from D.

The crossing used in the definition is called a quasi-alternating crossing of L.

Notice that a split link has its determinant equal to 0 so it cannot be quasi-alternating (determinants of quasi-alternating links are always positive as easily follows by induction from Definition 7.9). Therefore, we can use condition (b) of Theorem 7.1 as alternative definition of the family of quasialternating links.

One can ask why we choose condition (2) in the definition of quasialternating links and not a weaker first part of conditions (b), (b') from the Theorem 7.1 ($\sigma(D_+) = \sigma(D_0) - 1$ or $\sigma(D_-) = \sigma(D_0) + 1$). The first answer is purely practical: this is exactly what is needed to have thin Khovanov (and Heegaard) homology (see Chapter X of [P-Book]). One can also argue that condition which refers only to unoriented links is sometimes a plus.

We already have proved that non-split alternating links satisfy properties which make them quasi-alternating: if D is an alternating diagram then also D_0 and D_{∞} are alternating, and every non-nugatory crossing of an alternating diagram is quasi-alternating (satisfies property (ii)(2)) as long as D is a nonsplit link.

According to [M-O] among the 85 prime knots with up to nine crossings, 82 are quasi-alternating (71 are alternating), 2 are not quasi-alternating (8_{19} and 9_{42}), and the knot 9_{46} still remains undecided. It was showed by A. Schumakovitch using odd Khovanov homology that 9_{46} is not quasi-alternating. The classification of quasi-alternating knots up to 11 crossings was completed by J. Greene in [Gr].

It was also determined which pretzel links are quasi-alternating (partial classification of quasi-alternating Montesinos links is advanced in [C-K, Gr, J-S, Wid]:

Theorem 7.10 [C-K, Gr] [Characterization of quasi-alternating pretzel links] The pretzel link $P_{(1,...,1,p_1,...,p_n,-q_1,...,q_m)}$ with e 1th, $e + n + m \ge 3$, and $p_i \ge 2$, $q_i \ge 3$ is quasi-alternating if and only if one of the conditions below holds: (1) $e \ge m$, (2) e = m - 1 > 0, (3) e = 0, n = 1, and $p_1 > min(q_1, ..., q_m)$, (4) e = 0, m = 1, and $q_1 > min(p_1, ..., p_n)$, The same is true on permuting parameters³⁴ p_i and q_j .

³⁴Thus all pretzel links are covered in the theorem.

The importance of quasi-alternating links rests in the following results of Manolescu and Ozsvath:

(1) Quasi-alternating links are Khovanov homologically σ -thin (over Z).

(2) Quasi-alternating links are Floer homologically σ -thin (over Z_2).

We explain the meaning of the first result in Chapter X of [P-Book] showing also how to generalize it to Khovanov homologically k-almost thin links.

To have some measure of complexity or depth of quasi-alternating links we introduce the quasi-alternating computational tree index QACTI(L) is defined inductively from the definition of quasi-alternating link as follows:

Definition 7.11 For the trivial knot T_1 , $QACTI(T_1) = 0$. QACTI(L) is the minimum over all quasi-alternating crossings p (of any diagram) of L of $max(QACTI(L_0^p), QACTI(L_\infty^p)) + 1$.

In other words, QACTI(L) is the minimal depth of any binary computational resolving tree of L using only quasi-alternating crossings and having the trivial knot in leaves.

From Definitions 7.9 and 7.11, and Theorem 7.1 we get approximation on QACTI(L):

Corollary 7.12 Let L be a quasi-alternating link then:

- (i) $|Det(L)| 1 \ge QACTI(L) \ge log_2(|Det(L)|)$
- (ii) $QACTI(L) \geq |\sigma(\vec{L})|$, for every orientation of L.
- (iii) If p is a quasi-alternating crossing of L then $QACTI(L) \leq QACTI(L_0^p) + 1$, and $QACTI(L) \leq QACTI(L_\infty^p) + 1$.

Let us finish this survey with a nice example of a quasi-alternating knot of 13 crossings due to S. Jablan and R. Sazdanovic [J-S].



Figure 7.5; A quasi-alternating knot $13_{n_{1659}}$ with 2 diagrams of (minimal number) 13 crossings. The first diagram is (Conway) algebraic but no crossing is quasi-alternating. The second diagram, which bases on Conway's polyhedron 6^{*}, has the circled crossing quasi-alternating. The determinant of $13_{n_{1659}}$ is equal to 51 while smoothings of quasi-alternating crossing gives the trivial knot and a quasi-alternating link with determinant 50, [J-S].

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