manuscripta math. 12, 113 - 120 (1974)

A REMARK ON "HOMOTOPY FIBRATIONS"

Volker Puppe

We define a category Top^F of "homotopy fibrations with fibre F" (or rather "maps with homotopy fibre F") and show that this category is closed under certain colimits and "homotopy colimits". It follows that the geometric realization of a semisimplicial object in Top^F is again in Top^F. As a corollary we show that for a homotopy everything H-space A_*(i.e. a (special) Γ-space in the sense of G. Segal (s.[9],[10])) with homotopy inverse the loop space of the classifying space of A_* is homotopy equivalent (not only weakly s. [9],[10]) to A_1 even without assuming that all spaces involved have the homotopy type of CW-complexes (compare [8]).

This note reflects part of the attempt to understand the main theorem (p.2) in G. Segal's famous preprint [9]. I wish to thank E.E. Floyd for stimulating conversations on this subject.

To say that the map $EA_* \longrightarrow BA_*$ of the contractible space EA_* to the classifying space of a homotopy everything H-space A_* (i.e. a (special) Γ-space in the sense of [9],[10]) with homotopy inverse is a "homotopy fibration" might be ambiguous (s. [4], p. 110 and [10], p.7 for two different definitions of "homotopy fibration"). But it should be unmistakable to state that the homotopy fibre of the above map (at each point $b \in BA_*$) is homotopy equivalent to A_1(s.[9]). To prove this last statement we propose the use of the following category (for other possibilities s. [10],[5],[7]): Let F be a fixed topological space. The category Top^F has as its objects maps $p : X \longrightarrow A$ such that the homotopy fibre of p at each point $a \in A$ (i.e. the actual fibre of the "associated" Hurewicz fibration $W_p \longrightarrow A$ (s. for example [4], 5.3 - 5.4)) is homotopy equivalent to F. A morphism between two objects $p : X \longrightarrow A$ and $p' : X' \longrightarrow A'$ in Top^F is a pair of maps $f : A \longrightarrow A'$,

$\tilde{f} : X \longrightarrow X'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & X' \\ p\downarrow & & \downarrow p' \\ A & \xrightarrow{f} & A' \end{array}$$

commutes and the canonical map $X \longrightarrow f^*(W_{p'})$ is a homotopy equivalence. ($f^*(W_{p'})$ denotes the total space of the fibration induced by the Hurewicz fibration $W_{p'} \longrightarrow A'$ via the map $f : A \longrightarrow A'$, i.e. the pullback (fibre product) of the diagram

$$\begin{array}{ccc} & & W_{p'} \\ & & \downarrow \\ A & \longrightarrow & A' \end{array} \quad .)$$

<u>REMARK</u>. If $p : X \longrightarrow A$ is a Hurewicz fibration (or Dold fibration, i.e. has the WCHP (s. [1])), then p is an object in Top^F if and only if the actual fibre at each point $a \in A$ is homotopy equivalent to F (s. [1],(6.1)). Let

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & X' \\ p\downarrow & & \downarrow p' \\ A & \xrightarrow{f} & A' \end{array}$$

be a commutative diagram and p,p' Hurewicz (or Dold) fibrations with fibre homotopy equivalent to F . If $(\tilde{f},f)$ is a morphism in Top^F, then the restriction to each fibre is a homotopy equivalence (s. [1],(6.1)). If A is numerably contractible (s. [8]), then also the converse holds (s. [1],(6.3)).

The category Top^F is closed under certain colimits and "homotopy colimits (s. Lemma 2 and 3 below). As a consequence one has the following theorem (compare [10], (1.6)).

THEOREM. If $p_*: \Delta \longrightarrow Top^F$ is a semisimplicial object in Top^F, then the geometric realization $\|p_*\|$(defined without using the degeneracies s.[10]) is an object in Top^F. (p_* can be viewed as a morphism between two semisimplicial objects in the category Top of topological spaces. Hence $\|p_*\|$ is a map between the geometric realizations of these objects).

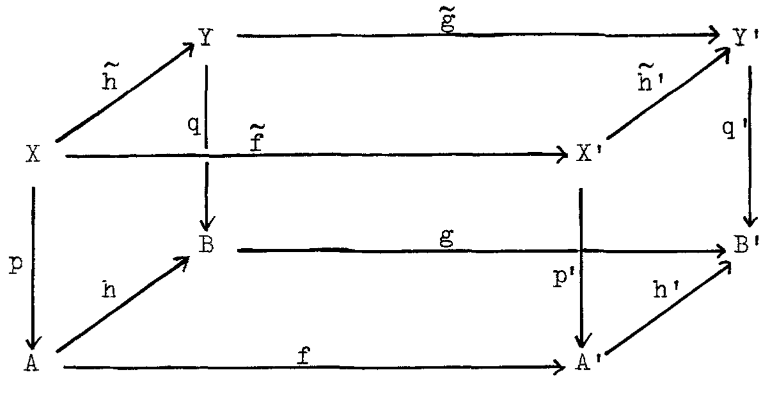
COROLLARY. If A_* is a homotopy everything H-space (i.e. a (special) Γ-space in the sense of [9],[10] with homotopy inverse, then the geometric realization of the projection of the semisimplicial path space PA_*(s. [10]) of A_* to the semisimplicial space A_* has the homotopy fibre A_1. In particular A_1 is homotopy equivalent to $\Omega\|A_*\|$.

Proof of the corollary. By [10], p. 8 the morphism $PA_* \underset{A_1}{\longrightarrow} A_*$ can be considered a semisimplicial object in Top^{A_1}.

REMARK. Under certain conditions (e.g. if all degeneracies of A_* are closed cofibrations) the canonical map $\|A_*\| \longrightarrow |A_*|$ is a homotopy equivalence (s.[10],[5],[12],prop.1).

The rest of this note is concerned with the proof of the above theorem.

LEMMA 1. If



is a commutative diagram with $h, \tilde{h}, h', \tilde{h}'$ being homotopy equivalences (in Top) then $(\tilde{f}, f)$ is a morphism in Top^F if and only if $(\tilde{g}, g)$ is in Top^F.

Proof. Replacing all vertical maps in the above diagram by the associated Hurewicz fibrations, the lemma can be verified easily (use e.g. [4],(7.22) or the "dual" of [11], (5.13)p.60).

LEMMA 2. (compare [10],(1.7)). Let

$$\begin{array}{ccccc} X_1 & \xleftarrow{\tilde{f}_1} & X_o & \xrightarrow{\tilde{f}_2} & X_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_o} & & \downarrow{\scriptstyle p_2} \\ A_1 & \xleftarrow{f_1} & A_o & \xrightarrow{f_2} & A_2 \end{array}$$

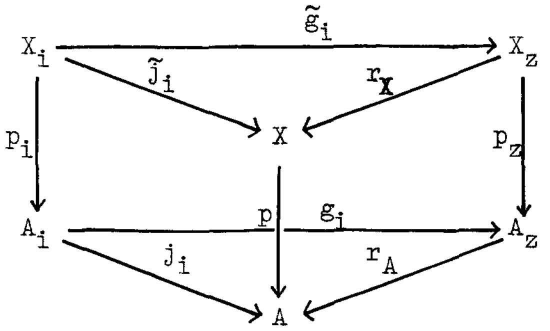
be a diagram in Top^F. The induced map $p_z : X_z \longrightarrow A_z$ between the row-wise double mapping cylinders is an object in Top^F and the canonical pairs of inclusions $\tilde{g}_i : X_i \longrightarrow X_z, g_i : A_i \longrightarrow A_z$ are morphisms in Top^F for $i = 0,1,2$. If the projections $r_X : X_z \longrightarrow X$ and $r_A : A_z \longrightarrow A$ of the row-wise double mapping cylinders to the row-wise colimits (fibre sums, pushouts) are homotopy equivalences (e.g. if at least one horizontal map in each row is a cofibration (s.e.g. [3], Lemma 1)), then the induced map $p : X \longrightarrow A$ between the row-wise colimits is (as an object in Top^F) the colimit of the above diagram in Top^F.

Proof. By Lemma 1 and [3], Lemma 1 one can assume (for the proof of the first part) all $p_i, i = 0,1,2$, to be Hurewicz fibrations (or Dold fibrations). Immitating M. Fuchs' construction (s.[6], §§ 2-3) one can construct a Dold Fibration $p' : E \longrightarrow A_z$ and a homotopy equivalence $l : X_z \longrightarrow E$ such that the diagram

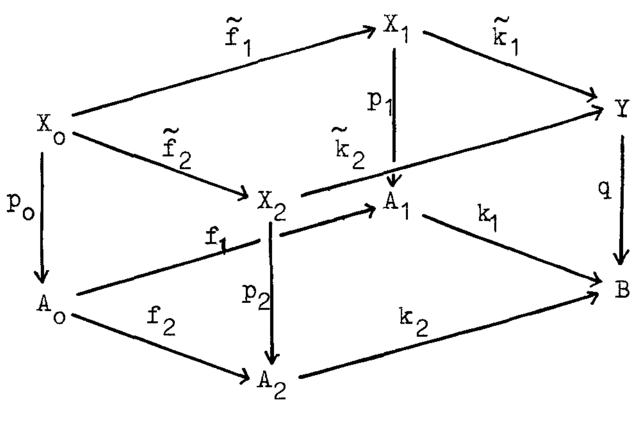
$$\begin{array}{ccc} X_z & \xrightarrow{l} & E \\ & {\scriptstyle p_z}\searrow \quad \swarrow{\scriptstyle p'} & \\ & A_z & \end{array}$$

commutes and the canonical maps $X_i \longrightarrow g_i^*(E)$ are homeomorphisms (compare [4],(17.8)). (Fuchs actually gives a "modified mapping cylinder" construction for certain morphisms between locally homotopy trivial fibrations. But the "same" construction applies to the double mapping cylinder in our situation. It follows from [6] and [1], (5.12) that this construction gives a Dold fibration in our case.) Therefore $(1\tilde{g}_i, g_i)$ and by Lemma 1 also $(\tilde{g}_i, g_i)$ are morphisms in Top^F. (In particular p' and p_Z are objects in Top^F).

To prove the second part of Lemma 2 we do not assume any more that the $p_i, i = 0,1,2,$ are Dold fibrations. Applying Lemma 1 to the commutative diagram



(where $\tilde{j}_i, j_i$ denote the canonical maps into the colimit) shows that (j_i, j_i) is a morphism in Top^F (assuming r_X and r_A are homotopy equivalences). Let



be a commutative diagram in Top^F. To show that

$$\begin{array}{ccc} X & \xrightarrow{\tilde{k}} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{k} & B \end{array}$$

is a morphism in Top^F (where k (resp. $\tilde{k}$) are given by k_1 and k_2 (resp. $\tilde{k}_1$ and $\tilde{k}_2$)) it suffices (again by Lemma 1) to prove that

$$\begin{array}{ccc} X_z & \xrightarrow{\tilde{k}r_X} & Y \\ {\scriptstyle p_z}\downarrow & & \downarrow{\scriptstyle q} \\ A_z & \xrightarrow{kr_A} & B \end{array}$$

is in Top^F, i.e. that this diagram induces a homotopy equivalence $X_z \xrightarrow{\simeq} (kr_A)^*(W_q)$. By assumption the canonical maps $X_i \longrightarrow (kr_A g_i)^*(W_q) = k_i^*(W_q)$ are homotopy equivalences. Since the canonical map from the double mapping cylinder of the diagram

$$k_1^*(W_q) \longleftarrow k_o^*(W_q) \longrightarrow k_2^*(W_q) \quad \text{to} \quad (kr_A)^*(W_q)$$

is a homotopy equivalence the assertion follows (s.[3], Theorem 1).

LEMMA 3. Let

$$\begin{array}{ccccccccc} X_o & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots\cdots \\ {\scriptstyle p_o}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_3}\downarrow & & \\ A_o & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots\cdots \end{array}$$

be a sequence of morphisms in Top^F. The induced map $p_T : X_T \longrightarrow A_T$ between the row-wise telescopes is an object in Top^F and the canonical pairs of inclusions

$\tilde{g}_i : X_i \longrightarrow X_T$, $g_i : A_i \longrightarrow A_T$ are morphisms in Top^F for $i = 0,1,2 \ldots$. If the projections of the row-wise telescopes to the row-wise colimits are homotopy equiva-

lences (e.g. if all horizontal maps are cofibrations) *then the induced map between the row-wise colimits is* (as an object in Top^F) *the colimit of the above diagram in* Top^F.

Proof. The proof is very similar to that of the previous lemma. Assuming again all p_i, $i = 0,1,2 \ldots$ being Dold fibrations one can construct a "modified telescope" using the "modified mapping cylinder" of Fuchs, i.e. there is a Dold fibration $p' : E \longrightarrow A_T$ and a homotopy equivalence $l : X_T \longrightarrow E$ such that the diagram

$$\begin{array}{ccc} X_T & \xrightarrow{\ l\ } & E \\ & {\scriptstyle p_T}\searrow \quad \swarrow{\scriptstyle p'} & \\ & A_T & \end{array}$$

commutes and the canonical maps $X_i \longrightarrow g_i^*(E)$ are isomorphisms. Using this construction the proof proceeds analogously to that of Lemma 2. Details are left to the reader.

Proof of the Theorem. The geometric realization $\|A_*\|$ of a semisimplicial space A_* is defined by $\|A_*\| = \varinjlim_n \|A_*\|_{(n)}$ where $\|A_*\|_{(n)}$ is given inductively as the colimit of the diagram

$$\|A_*\|_{(n-1)} \longleftarrow \dot{\Delta}^n \times A_n \longrightarrow \Delta^n \times A_n \qquad \text{(s.[10])}.$$

The right hand map in this diagram and therefore the inclusion $\|A_*\|_{(n-1)} \rightarrow \|A_*\|_{(n)}$ are cofibrations. Hence an iterated application of Lemma 2 followed by an application of Lemma 3 proves the theorem.

REMARK. If in the above Corollary one is only interested in showing that A_1 and $\Omega\|A_*\|$ are weakly homotopy equivalent, one can use [2], (2.2),(2.15) and (2.10) rather than the construction of Fuchs to prove the analogs of Lemma 2 and Lemma 3 above for the appropriate category of maps having the property that the homotopy fibre at each point is weakly homotopy equivalent to F (s. [10],[5]).

References

[1] DOLD, A.: Partitions of unity in the theory of fibrations. Ann. of Math. 78, 223-255 (1963).

[2] DOLD, A. und R. THOM: Quasifaserungen und unendliche symmetrische Produkte. Ann. of Math. 67, 239-281 (1958).

[3] tom DIECK, T.: Partitions of unity in homotopy theory. Compositio math. 23, 159-167 (1971).

[4] tom DIECK, T., K.H. KAMPS, D. PUPPE: Homotopietheorie, Springer Lecture Notes Vol. 157 (1970).

[5] FLOYD, E.E.: Lecture course, Univ. of Virginia, Charlottesville (1972/73).

[6] FUCHS, M.: A modified Dold-Lashof contruction that does clyssify H-principal fibrations. Math. Ann. 192, 328-340 (1971).

[7] KRAUS, J.: (in preparation)

[8] PUPPE, D.: Some well-known weak homotopy equivalences are genuine homotopy equivalences. Istituto Nazionale di Alta Matematica, Symposia Mathematica, Vol. V (1970).

[9] SEGAL, G.: Homotopy-Everything H-spaces. (Preprint).

[10] SEGAL, G.: Categories and cohomology theories. (Preprint).

[11] BROCKER, TH., tom DIECK, T.: Kobordismentheorie, Springer Lecture Notes Vol. 178 (1970).

[12] tom DIECK, T.: On the homotopy type of classifying spaces, manuscripta math. (to appear).

Volker Puppe
Mathematisches Institut der Universität
69 Heidelberg
Im Neuenheimer Feld 9

(Received September 7, in revised form October 31, 1973)