Maslov Index and Symplectic Sturm Theorems*

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Introduction

The Sturm theorems on zeros of solutions of a second-order ordinary differential equation describe the rotation of a line in the phase plane of the equation [3, 13]. In the symplectic version of these theorems, lines are replaced by Lagrangian planes and the instants of intersection with a given line are replaced by instants of nontransversality with a given plane [2]. The train of a given Lagrangian plane is a hypersurface (with singularities), in the Lagrangian Grassmannian, that consists of the Lagrangian planes not transversal to the given plane.

By the Maslov index we mean the index of intersection of a curve on a Lagrangian Grassmannian with the train. The symplectic Sturm theorems [2] describe some properties of the Maslov index. The symplectic Sturm theory was developed by Morse [10, 11], Lidskii [9], and Arnold [2], and its Hermitian version by Bott [4] and Edwards [5]. The properties of (non)oscillation of Hamiltonian equations, and cocycles representing a generator of the first cohomology group of the symplectic group, were studied by Yakubovich [15–17]. The symplectic Sturm theory was used by Givental [7] who proved the Lagrangian nonoscillation of the Picard–Fuchs equation for hyperelliptic integrals. In the present paper we determine a class of hypersurfaces in the Lagrangian Grassmannian on each of which the symplectic Sturm theorems can be extended. The author wishes to thank V. I. Arnold and A. G. Khovanskii for fruitful discussions.

§1. Necessary Definitions

In this section we construct transversally oriented hypersurfaces in the Lagrange–Grassmann manifold that determine a one-dimensional cocycle coinciding with the Maslov index.

Let us consider a symplectic vector space $(\mathbb{R}^{2n}, \omega)$. Denote by Λ_n the manifold of all Lagrangian subspaces in $(\mathbb{R}^{2n}, \omega)$, which we call the Lagrange-Grassmann manifold. We choose Darboux coordinates $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$ in $(\mathbb{R}^{2n}, \omega)$, $\omega = \sum dp_i \wedge dq_i = dp \wedge dq$. The plane p = 0 is called the *q*-plane and the plane q = 0 the *p*-plane.

Let us consider the set $X \subset \Lambda_n$ of all Lagrangian planes transversal to the plane p = 0. The set X is open and dense in Λ_n (a chart of Λ_n) that is diffeomorphic to $\mathbb{R}^{n(n+1)/2}$. A Lagrangian plane belonging to X can be identified with a symmetric matrix, namely, to a matrix A, the plane q = Ap corresponds.

Definition [2]. The set of all Lagrangian planes that are not transversal to a given Lagrangian plane is called the *train* of this plane. In the chart X, the train of the *p*-plane is given by the equation det A = 0.

We identify each element of the tangent space at a point of a vector space, as well as each translationinvariant vector fields on the vector space, with an element of the vector space. In our case, we identify translation-invariant vector fields in the chart X with matrices.

Let us consider a hypersurface in X specified by the equation

$$(L_{A_1}\cdots L_{A_k}\det)(A)=0,$$

where A_1, \ldots, A_k are positive-definite matrices $(0 \le k < n)$, L_v is the derivative along the vector field v, and det is the determinant.

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Definition. The closure of the hypersurface $\{A : (L_{A_1} \cdots L_{A_k} \det)(A) = 0\} \subset X$ in the Lagrange-Grassmann manifold Λ_n is called a *generalized train* and is denoted by $\Sigma_{\mathscr{A}}$. Sometimes we denote the generalized train by Σ_{A_1,\ldots,A_k} ; in this case we assume that some Darboux coordinates are chosen.

Thus, a generalized train is constructed from the following data: Darboux coordinates in $(\mathbb{R}^{2n}, \omega)$ and k positive-definite $(0 \le k < n)$ matrices $n \times n$.

Remark. This set of data is excessive, namely, instead of Darboux coordinates, it suffices to choose k + 2 Lagrangian planes satisfying the following conditions:

(1) the first plane is transversal to the other planes;

(2) if some Darboux coordinates are chosen so that the first plane is the q-plane and the second is the p-plane, then the remaining k Lagrangian planes must be represented by positive-definite matrices in the corresponding chart X.

After this we can repeat the above definition of the generalized train for the chosen Darboux coordinates and for the matrices thus obtained. The resulting hypersurface and the validity of condition (2) do not depend on the choice of Darboux coordinates.

The train of a Lagrangian plane is a special case of the generalized train (k = 0). As is shown below, for n > 2, all generalized trains in Λ_n are hypersurfaces with singularities. For n = 2, the generalized train constructed for a single matrix (the first nontrivial example of a generalized train) is a smooth hypersurface in Λ_2 diffeomorphic to the two-dimensional sphere.

Definition [2]. By *positive vectors* on the Lagrange–Grassmann manifold we mean vectors of the velocities of motion of Lagrangian planes under the action of positive-definite Hamiltonians.

At nonsingular points, the train of any plane is transversally oriented by positive vectors. The index of intersection of oriented loops with the train determines the canonical generator of the group $H^1(\Lambda_n, \mathbb{Z}) = \mathbb{Z}$. [1, 2] which is the Maslov class.

A generalized train is a hypersurface (with singularities) in Λ_n . The properties of a generalized train are mainly similar to those of the train of a Lagrangian plane. The following theorem holds.

Theorem 1.1. A generalized train Σ_{α} is a hypersurface in the Lagrange-Grassmann manifold Λ_n such that

(1) positive vectors are transversal to $\Sigma_{\mathscr{A}}$ at nonsingular points;

(2) the transversal orientation of the generalized train by means of positive vectors specifies an element $\alpha_{\mathscr{A}} \in H^1(\Lambda_n, \mathbb{Z})$ which is the index of intersection of the oriented loops with $\Sigma_{\mathscr{A}}$;

(3) the element $\alpha_{\mathscr{A}}$ coincides with the Maslov class.

Definition. By a *positive path* we mean a smooth path in the Lagrange–Grassmann manifold whose velocity vector at any point is positive.

Corollary 1.2. The index of intersection of a positive path with a generalized train is nonnegative.

The proof of Theorem 1.1 is given in §3. The intersection of a generalized train with the chart X is a zero-level surface of a hyperbolic polynomial. We need some statements concerning hyperbolic polynomials, which we present in the next section.

§2. Hyperbolic Polynomials

Recall that a real polynomial of degree n is said to be hyperbolic along a vector a if its restriction to any line parallel to a has exactly n real roots (counted according to their multiplicities). In this case, the vector a is called a hyperbolic vector of this polynomial or, by misuse of language, it is simply said to be hyperbolic.

The vectors along which a given homogeneous polynomial is hyperbolic form open convex cones [6, 14]. We need the following statements on homogeneous hyperbolic polynomials.

Let K be an open cone that forms a connected component of the domain of hyperbolic vectors of a homogeneous polynomial f.

Proposition 2.1 [2]. The parallel translation of the cone K to a point of the surface f = 0 does not intersect this surface in a sufficiently small neighborhood of the new vertex.

As in $\S1$, we identify translation-invariant (constant) vector fields on a vector space with vectors of this space.

Proposition 2.2 [14]. The derivative of the polynomial f along a constant hyperbolic vector field that is equal to a vector of the cone K is a homogeneous hyperbolic polynomial with respect to the vectors of the cone K.

Proposition 2.3. A path whose velocity vector belongs to the cone K at each intersection point with the surface f = 0 intersects the surface f = 0 at finitely many points.

Proof. Denote by g(x) the number of nonpositive roots of a polynomial f(x + ta) in t (counted according to their multiplicities) for $a \in K$. The function g is locally constant on the complement to the hypersurface f = 0 because f is hyperbolic. The value of g does not depend on the choice of $a \in K$. The function g restricted to the path increases at the intersection points of the path with the surface f = 0, which can readily be seen from Proposition 2.1. This proves our proposition.

Proposition 2.4. The set of critical points of the derivative of a homogeneous polynomial f along a constant hyperbolic vector field is contained in the set of critical points of the polynomial f.

Proof. Let $a \in K$ be a constant hyperbolic vector field. Let x be a critical point of the hyperbolic polynomial $L_a f$. Since the polynomial $L_a f$ is homogeneous, we have $(L_a f)(x) = 0$. The multiplicity of the root t = 0 of the polynomial $(L_a f)(x + tb)$ in the variable t is constant for $b \in K$ by Propositions 2.1 and 2.2 because all roots of the polynomial $(L_a f)(x + tb)$ in the variable t are real and continuously depend on the vectors $b \in K$. Thus, the point x is a critical point of the hyperbolic polynomial $L_a f$ if and only if t = 0 is a multiple root of the polynomial $l(t) = (L_a f)(x + ta)$. The polynomial l(t) is equal to the derivative of the polynomial f(x + ta) in the variable t. The polynomials f(x + ta) and l(t) of the variable t are real.) According to Rolle's lemma, a root of the derivative of a hyperbolic polynomial is multiple if and only if it is a multiple root (of multiplicity at least three) of the polynomial itself. Hence, t = 0 is a multiple root of the polynomial f(x + ta) in t, and x is a critical point of the polynomial itself. Hence, t = 0 is a multiple root of the polynomial f(x + ta) in t, and x is a critical point of the polynomial itself. Hence, the polynomial is multiple root of the polynomial itself.

The following statement is well known. This is a consequence of the theorem on the reduction to principal axes for a pair of forms one of which is positive definite.

Proposition 2.5. The determinant of a symmetric matrix is a hyperbolic homogeneous polynomial on the space of symmetric matrices with respect to the cone of positive-definite matrices.

Corollary 2.6. Let A_1, \ldots, A_k be positive-definite matrices. The polynomial $L_{A_1} \cdots L_{A_k}$ det is a hyperbolic homogeneous polynomial on the space of symmetric matrices with respect to the cone of positive-definite matrices.

§3. Properties of a Generalized Train

Recall that by X we denote the chart of Λ_n that consists of all Lagrangian planes transversal to the q-plane. By means of the linear structure, we identify the tangent vectors at different points of X with constant vector fields on X and with matrices that are elements of X.

Lemma 3.1. Positive vectors in the chart X of a Lagrange-Grassmann manifold are represented by positive-definite matrices.

Proof. Let H be a positive-definite Hamiltonian. Consider the Lagrangian plane q = Ap. Under the action of the phase flow of the Hamiltonian vector field $\dot{p} = -H_q$, $\dot{q} = H_p$ with Hamiltonian H at time ε , its image is the plane $q = (A + \varepsilon B)p + o(\varepsilon)$. Let us prove that the matrix B is positive definite. Indeed,

$$Ap + \varepsilon H_p(p, Ap) = (A + \varepsilon B)(p - \varepsilon H_q(p, Ap)) + o(\varepsilon),$$

and hence $H_p(p, Ap) = Bp - AH_q(p, Ap)$. Therefore,

$$\langle Bp, p \rangle = \langle H_p(p, Ap), p \rangle + \langle AH_q(p, Ap), p \rangle = \langle H_p(p, Ap), p \rangle + \langle H_q(p, Ap), Ap \rangle = 2H(p, Ap)$$

by the Euler theorem. Thus, the matrix B is positively definite. The converse is also true, namely any positive-definite matrix represents a positive vector. This proves the lemma.

Let us consider a chart Y of the Lagrange-Grassmann manifold Λ_n that consists of all Lagrangian planes transversal to the plane p = 0 (q-plane). As in the case of X, in the chart Y we identify a Lagrangian plane with a symmetric matrix B, namely, the plane p = Bq corresponds to the matrix B. The coordinates of a plane $\lambda \in X \cap Y$ in the charts X and Y are related by the formula $A = B^{-1}$ (A (B, respectively) is the matrix corresponding to the plane λ in the chart X (Y, respectively)).

Lemma 3.2. (a) In the chart Y, the equation $(L_{A_1} \cdots L_{A_k} \det)(A) = 0$ has the form

$$(L_{-BA_1B}\cdots L_{-BA_kB}\frac{1}{\det})(B)=0$$

(b) $(L_{-BA_1B}\cdots L_{-BA_kB}\frac{1}{\det})(B) = P_{A_1,\ldots,A_k}(B)/\det B$, where $P_{A_1,\ldots,A_k}(B)$ is a homogeneous polynomial of degree k that is equal to the sum of terms of the form

$$\pm \operatorname{tr}(A_{i_1}B\cdots A_{i_{m_1}})\operatorname{tr}(A_{i_{m_1+1}}B\cdots A_{i_{m_2}}B)\cdots \operatorname{tr}(A_{i_{m_{l-1}+1}}B\cdots A_{i_{m_l}}B)$$

 $((i_1,\ldots,i_{m_l})$ is a permutation of the elements $(1,\ldots,k)$.

Proof. (a) For any matrix M (det $M \neq 0$) we have $(M + \epsilon A_i)^{-1} = M^{-1} - \epsilon M^{-1} A_i M^{-1} + o(\epsilon)$.

(b) We can readily see that $(L_{-BA_1B}\frac{1}{\det})(B) = \operatorname{tr} A_1B/\det B$. The remaining part of the proof can be performed by induction. For instance, $P_{A_1,A_2}(B) = \operatorname{tr} A_1B\operatorname{tr} A_2B - \operatorname{tr} A_1BA_2B$. This completes the proof.

For convenience, we sometimes denote the operator $L_{A_1} \cdots L_{A_k}$ by $L_{\mathscr{A}}$ and the polynomial P_{A_1,\ldots,A_k} by $P_{\mathscr{A}}$.

Lemma 3.3. The set of singular points of the hypersurface $L_{\alpha} \det = 0$ is of codimension not less than three in the chart X.

The set of singular points of the hypersurface $\{B : P_{\mathscr{A}}(B) = 0\}$ is of codimension not less than three in the chart Y, and the hypersurface $\{B : P_{\mathscr{A}}(B) = 0\}$ intersects the hypersurface $\{B : \det B = 0\}$ transversally outside a set of codimension not less than three in the chart Y.

Proof. The first assertion of the lemma follows from Proposition 2.4 because the set of singular points of the hypersurface $L_{\mathscr{A}} \det = 0$ is contained in the set of singular points of the hypersurface $\det = 0$ whose codimension is equal to three. Let us prove the other assertion. The intersection of the set of singular points of the hypersurface $\{B : P_{\mathscr{A}}(B) = 0\}$ with the set $\{B : \det B \neq 0\}$ is of codimension not less than three in the chart Y because this is a representation of the set of singular points of the hypersurface $L_{\mathscr{A}} \det = 0$ in the chart Y.

The set of singular points of the hypersurface $\{B : \det B = 0\}$ is of codimension three in the chart Y. Hence, it suffices to show that the hypersurface $\{B : P_{\mathscr{A}}(B) = 0\}$ and the manifold of nonsingular points of the hypersurface $\{B : \det(B) = 0\}$ intersect transversally outside a set of codimension not less than three in the chart Y.

The set of nonsingular points of the hypersurface $\{B : \det(B) = 0\}$ is the set of matrices with onedimensional kernel. This is a bundle over the corresponding projective space (the kernel corresponds to a matrix). Let us prove that the set of singular roots of the equation $P_{\mathscr{A}}(B) = 0$ that is restricted to any fiber of this bundle is of codimension not less than two in the fiber (this fact yields the other assertion of the lemma).

We consider a fiber of this bundle over the line $q_1 = \cdots = q_{n-1} = 0$. The rightmost column and the bottom row of the matrices from this fiber are zero and the determinant of the upper left minor of size $(n-1) \times (n-1)$ is nonzero. Denote this minor by \tilde{B} . In this fiber, the equation $P_{\mathscr{A}}(B) = P_{A_1,\ldots,A_k}(B) = 0$ has the form $P_{\tilde{A}_1,\ldots,\tilde{A}_k}(\tilde{B}) = 0$ (by Lemma 3.2), where \tilde{A}_i is the upper left minor of size $(n-1) \times (n-1)$ of the matrix A_i . The matrices \tilde{A}_i are positive definite, and hence, as was shown above, for k < n-1, the equation $P_{\mathscr{A}}(B) = 0$ determines a hypersurface in the fiber under consideration such that the set of singular points of this hypersurface is of codimension not less than three in this fiber. If k = n-1, then $P_{\tilde{A}_1,\ldots,\tilde{A}_k}(\tilde{B})/\det \tilde{B} = (L_{\tilde{A}_1}\cdots L_{\tilde{A}_k}\det)(\tilde{B}^{-1}) \neq 0$. Therefore, the fiber under consideration is not contained in the hypersurface $\{B: P_{\mathscr{A}}(B) = 0\}$. The case of other fibers can be reduced to that treated above by changing the Darboux coordinates that preserve p- and q-planes. This proves the lemma.

Denote by O the union, over all fibers of the bundle introduced in the proof of Lemma 3.3, of the sets of singular roots of the equation $P_{A_1,\ldots,A_k}(B) = 0$ on a fiber of this bundle. By Lemma 3.3, the codimension of the set O is not less than three.

Corollary 3.4. The intersection of a generalized train with the union of the charts X and Y is a hypersurface that is smooth outside a set of codimension not less than three.

Proof. Indeed, the intersection of a generalized train with the chart X is nonsingular outside a set of codimension not less than three, which is the first assertion of the above lemma. The singular points of the intersection of a generalized train with the chart Y that belong to $Y \setminus X$ are contained in the set of singular points of the hypersurface $\{B : \det(B) = 0\}$ and in the set O.

Lemma 3.5. A positive vector is transversal to a generalized train at any of its nonsingular points.

Proof. Let a Lagrangian plane λ be a nonsingular point of a generalized train. In this case, λ is the limit of a sequence of nonsingular points of the intersection of the generalized train with the chart X. At the nonsingular points of the intersection of the generalized train with the chart X, any positive vector is transversal to the generalized train by Proposition 2.1 and Lemma 3.1. Let us chose Darboux coordinates (\tilde{p}, \tilde{q}) so that λ is the \tilde{p} -plane. By Lemma 3.1, in the chart Z of Lagrangian planes transversal to the \tilde{q} -plane, the positive vectors are represented by positively definite matrices. The lemma follows from the next simple statement.

Proposition 3.6. Let K be an open cone in \mathbb{R}^N and let Γ be a smooth hypersurface in \mathbb{R}^N . In this case, the set of points of the hypersurface Γ at which all vectors of K are transversal to Γ is closed in Γ .

Proof. Indeed, the set of points of Γ at which there exists a vector from K that is tangent to Γ is obviously open in Γ .

Lemma 3.7. The index of intersection of oriented closed curves with a generalized train $\Sigma_{\mathscr{A}} = \Sigma_{A_1,\ldots,A_k}$ (transversally oriented by positive vectors) determines an element $\widetilde{\alpha}_{A_1,\ldots,A_k} \in H^1(X \cup Y, \mathbb{Z})$. The element $\widetilde{\alpha}_{A_1,\ldots,A_k}$ does not depend on the choice of positive-definite matrices A_1,\ldots,A_k .

Proof. Indeed, the first statement of the lemma follows from Corollary 3.4 and Lemma 3.5. Let us prove the other statement. Let us consider a loop in $X \cup Y$ and bring it to general position with respect to Σ_{A_1,\ldots,A_k} and to the train of the *q*-plane. The loop thus obtained intersects Σ_{A_1,\ldots,A_k} and the train of the *q*-plane transversally at nonsingular points of these hypersurfaces. In this case, by Lemma 3.3, we can assume that the points of intersection of the loop with Σ_{A_1,\ldots,A_k} are contained in the chart X. If matrices B_1,\ldots,B_k are close to the matrices A_1,\ldots,A_k , respectively, then the resulting loop intersects Σ_{B_1,\ldots,B_k} transversally at close points. Let us prove this fact.

Let $\gamma(t)$, $t \in \mathbb{R}/\mathbb{Z}$, be the resulting loop and let t_1, \ldots, t_l be the instants of intersection of this loop with the train of the q-plane. In this case, for a sufficiently small ε , the paths $\gamma(t)$, $t \in [t_j - \varepsilon, t_j + \varepsilon]$, are contained in the chart Y and do not intersect the hypersurface $\{B : P_{A_1,\ldots,A_k}(B) = 0\}$. Hence, these paths do not intersect the hypersurface $\{B : P_{B_1,\ldots,B_k}(B) = 0\}$ (if the matrices B_1,\ldots,B_k are close to the matrices A_1, \ldots, A_k because the equation $P_{A_1, \ldots, A_k}(B) = 0$ depends continuously on A_1, \ldots, A_k . All the more, the paths $\gamma(t), t \in [t_j - \varepsilon, t_j + \varepsilon]$, do not intersect $\Sigma_{B_1, \ldots, B_k}$.

On the other hand, the paths $\gamma(t)$, $t \in [t_j + \varepsilon, t_{j+1} - \varepsilon]$, belong to the chart X. The equation $L_{A_1} \cdots L_{A_k}(A) = 0$ depends on A_1, \ldots, A_k smoothly, and hence the paths $\gamma(t)$, $t \in [t_j + \varepsilon, t_{j+1} - \varepsilon]$ intersect Σ_{B_1,\ldots,B_k} transversally at close points by the implicit function theorem. It remains to note that the contribution of each intersection point to the index of intersection is preserved under a small modification of the matrices A_1, \ldots, A_k . We have proved that the element $\tilde{\alpha}_{A_1,\ldots,A_k}$ is locally constant, and therefore it is constant because the positive-definite matrices form a connected set. This proves the lemma.

Introduce a function g_{α} on the chart X as follows: $g_{\alpha}(B)$ is equal to the number of nonpositive roots of the polynomial $L_{\alpha}(B+tP)$ in the variable t (where P is a positive-definite matrix). The function g_{α} does not depend on the matrix P used in its definition. The results of §2 readily imply the following assertion.

Proposition 3.8. The index of intersection of a path belonging to the chart X with Σ_{α} is equal to the increment of the function g_{α} . If the starting point and the terminating point of this path are negative-and positive-definite matrices, respectively, then the index of intersection of this path with Σ_{α} is equal to n-k.

Denote by E the identity matrix and by Σ_E^k the generalized train Σ_{A_1,\ldots,A_k} with $A_1 = \cdots = A_k = E$.

Lemma 3.9. The generalized train Σ_E^k is a hypersurface that is smooth outside of a set of codimension not less than three in Λ_n . The index of intersection of oriented closed curves with Σ_E^k (transversally oriented by positive vectors) determines an element in $H^1(\Lambda_n, \mathbb{Z})$ that coincides with the Maslov class.

Proof. If k = 0, then our generalized train is simply the train of the *p*-plane, and the assertion of the lemma was proved in [1]. Assume that k > 0. Let us prove that the intersection of Σ_E^k and $\Lambda_n \setminus (X \cup Y)$ is a set of codimension not less than three (for k = 0, this is not the case). By Corollary 3.4, this implies the first assertion of the lemma.

We have the following obvious statement:

Proposition 3.10. Let $A_1 = \cdots = A_k = E$ and let μ_1, \ldots, μ_n be the eigenvalues of a symmetric matrix M. In this case,

$$L_{\mathscr{A}}\det(M) = k! \sum_{i_1 < \cdots < i_k} \mu_{i_1} \cdots \mu_{i_k}, \qquad P_{\mathscr{A}}(M) = k! \sum_{i_1 < \cdots < i_{n-k}} \mu_{i_1} \cdots \mu_{i_{n-k}}.$$

Let us continue the proof of Lemma 3.9. The set $\Lambda_n \setminus (X \cup Y)$ is a submanifold (with singularities) of codimension two in Λ_n . The set of Lagrangian planes that intersect the *p*- and *q*-planes along a onedimension subspace forms an open submanifold *L* of codimension two in Λ_n (for example, for n = 2, this is an unknotted circle in Λ_2). The complement to *L* in $\Lambda_n \setminus (X \cup Y)$ forms a submanifold (with singularities) of codimension at least three in Λ_n . To a plane $\lambda \in L$, a system of n-2 nonzero "eigenvalues" corresponds; namely, we must consider a sequence of planes (matrices) in *X* convergent to λ and take the limits of the eigenvalues. One of these limits is zero and another is infinite, and the remained eigenvalues form the desired system. By Proposition 3.10, we can readily show that the condition "the plane belongs to the generalized train Σ_E^{kn} is nontrivial for this system. Hence, Σ_E^k intersects *L* along a hypersurface. The first statement of the lemma is proved.

Let us prove the other statement of the lemma. The generalized train Σ_E^k , transversally oriented by positive vectors, determines an element α_{kE} in $H^1(\Lambda_n, \mathbb{Z})$ because the set of its singular points is of codimension three in Λ_n . Let us show that α_{kE} coincides with the Maslov class.

Since $\pi_1(\Lambda_n) = \mathbb{Z}$ [1], it suffices to show that the values of the class α_k and of the Maslov class coincide on a noncontractible loop. Consider a loop γ formed by the following paths γ_1 and γ_2 . The path $\gamma_1(t) = (t-1)A + tB$, $t \in [0,1]$, belongs to the chart X and the path $\gamma_2(t) = (1-t)B^{-1} - tA^{-1}$, $t \in [0,1]$, to the chart Y, where A and B are positive-definite matrices. The endpoints of the paths γ_1 and γ_2 does not belong to Σ_E^k . The Maslov index of the loop γ is equal to n because the path γ_2 does not

intersect the train of the *p*-plane, and the index of intersection of the path γ_1 with the train of the *p*-plane is equal to n [2]. Hence, the index of intersection of the loop γ with Σ_E^k is divisible by n. According to 3.8, the index of intersection of the path γ_1 with Σ_E^k is equal to n-k. On one hand, the index of intersection of the path γ_2 with Σ_E^k is nonnegative because the path γ_2 is positive (note that, in the chart Y, the positive vectors are represented by *negative-definite* matrices), and, on the other hand, this index does not exceed k because the intersection of Σ_E^k with the chart Y is contained in the zero-level surface of the polynomial $P_{E,...,E}$ of degree k. Thus, the index of intersection of the loop γ with Σ_E^k is positive and does not exceed n, and therefore it is equal to n. This proves Lemma 3.9.

Lemma 3.11. The index of intersection of oriented closed curves with $\Sigma_{\mathscr{A}}$ (transversally oriented by positive vectors) determines an element in $H^1(\Lambda_n, \mathbb{Z})$ coinciding with the Maslov class.

Proof. Take a loop in Λ_n and consider its small perturbation that belongs to $X \cup Y$. By Lemma 3.7, on the resulting loop, the value of $\tilde{\alpha}_{A_1,\ldots,A_k}$ is equal to the value of $\tilde{\alpha}_{E,\ldots,E}$. According to Lemma 3.9, this value coincides with the Maslov index. Hence, the index of intersection is well defined, and this proves the lemma.

Proof of Theorem 1.1. Theorem 1.1 follows from Lemmas 3.5 and 3.11.

Remarks. 1) The intersection of a generalized train $\Sigma_{\mathscr{A}}$ with the chart Y coincides with the surface $\{B: P_{\mathscr{A}}(B) = 0\}$. The polynomial $P_{\mathscr{A}}$ is hyperbolic with respect to the cone of positive-definite matrices. These assertions readily follow from the fact that the index of intersection of the path γ_2 introduced in the proof of Lemma 3.9 with $\Sigma_{\mathscr{A}}$ is equal to k.

2) Theorem 1.1 could be proved in another way if we were able to show that the singularities of the generalized train are of codimension not less than three in the Lagrange-Grassmann manifold. The author knows no proof of this fact.

§4. Construction of a Cocycle

The cone of positive vectors is related to a generalized train in the same way as the cone of hyperbolic vectors of a polynomial f is related to the surface f = 0 (see Proposition 2.1). This allows one to define a one-dimensional cocycle on Λ_n . The index of intersection of nonclosed paths with a (generalized) train is useful for different aims. For the train of a Lagrangian plane, this index was defined in [12].

Lemma 4.1. The instants of intersection of a positive path with the train of any Lagrangian plane λ are isolated.

Proof. Let $\gamma(t), t \in [0, 1]$, be a positive path and let $\gamma(t_0)$ belong to the train of the plane λ .

We can choose Darboux coordinates (\tilde{p}, \tilde{q}) so that the plane λ is the \tilde{p} -plane and the plane $\gamma(t_0)$ is transversal to the \tilde{q} -plane. In the chart of the manifold Λ_n that is formed by the planes transversal to the \tilde{q} -plane, the train of the plane λ is specified by the equation det(A) = 0. Applying Propositions 2.3 and 2.5 and Lemma 3.1, we complete the proof.

Lemma 4.2. The instants of intersection of a positive path with a generalized train are isolated.

Proof. The instants of intersection of a positive path with the train of the q-plane are isolated. Hence, the path is divided into finitely many parts that belong to the chart X. By 2.6, 2.3, and 3.1, on each such part, the number of instants of intersection with the generalized train is finite. This proves the lemma.

Let us consider the Darboux coordinates (\tilde{p}, \tilde{q}) and the chart $X_{\tilde{p}}$ of the Grassmannian Λ_n that consists of all Lagrangian planes transversal to the \tilde{q} -plane. As above, we identify the Lagrangian planes from $X_{\tilde{p}}$ with symmetric matrices.

Theorem 4.3. The parallel translation of the cone of positive-definite matrices in the chart $X_{\bar{p}}$ to a point of a generalized train does not intersect the generalized train in a sufficiently small neighborhood of the new vertex.

Proof. Let us consider a point $x_0 \in X_{\bar{p}}$ of the generalized train and draw a positive path (in the chart $X_{\bar{p}}$) through x_0 that intersects the generalized train at the point x_0 only. This can be done by Lemma 4.2. If the assertion of the theorem is false, then there exists another positive path (in the chart $X_{\bar{p}}$) with the same endpoints (as for the first one) that passes through the point x_0 and through arbitrarily many nonsingular points of the generalized train. The indices of intersection of these paths with the generalized train coincide because these paths are homotopic (with fixed endpoints). Hence, the index of intersection of the first path is arbitrarily large (see Corollary 1.2). A contradiction.

Remark. A similar statement holds for the cone of negative-definite matrices.

The train of a Lagrangian plane is naturally stratified by the dimension of the intersection with this plane. A similar stratification exists for a generalized train.

Let us consider a point x_0 of a generalized train and draw a (short) positive path γ through x_0 that does not intersect the generalized train at other points.

Definition. By the *multiplicity* of the point x_0 we mean the index of intersection of the path γ with the generalized train.

Proposition 4.4. The multiplicity is well defined and positive.

Proof. Let us consider two positive paths γ_1 and γ_2 that pass through a point x_0 of a generalized train (and have no other intersections with the generalized train). Assume that these paths are contained in the chart $X_{\bar{p}}$ of the Grassmannian Λ_n related to the Darboux coordinates (\tilde{p}, \tilde{q}) . By Theorem 4.3 and by the remark after it, the path γ_1 is homotopically equivalent to a path γ_2 such that, in the corresponding homotopy, the endpoints do not intersect the generalized train. This proves that the multiplicity is well defined. Since the nonsingular points are dense in a generalized train, it follows that the other part of the proposition is also valid.

Remark. The multiplicity of a point β of the train of a Lagrangian plane α is equal to the dimension of intersection of the planes α and β . For n > 2, all generalized trains in Λ_n are hypersurfaces with singularities because, for the generalized train Σ_{α} constructed on the basis of the Darboux coordinates (p,q) and matrices A_1, \ldots, A_k , the multiplicity of the *p*-plane is equal to n-k and the multiplicity of the *q*-plane is equal to *k*. The multiplicity of a nonsingular point of the train is equal to 1. Seemingly, if the multiplicity is equal to 1, then the point is nonsingular.

Let Σ be a generalized train. Let us define (similarly to [2, 8]) a cocycle ind_{Σ} that coincides with the index of intersection of a curve with Σ for curves with endpoints outside Σ .

Consider a path $\gamma(t)$, $t \in [0, 1]$, in Λ_n . Let γ_0 be a negative path that enters the starting point $\gamma(0)$ of the path γ and let γ_1 be a positive path issuing from the terminating point $\gamma(1)$ of the path γ . According to Theorem 4.3 and to the remark after it, the paths γ_0 and γ_1 can be chosen so that they intersect the generalized train only at the terminating point and at the starting point, respectively.

Definition. Let $\operatorname{ind}_{\Sigma}(\gamma)$ be the index of intersection of the path $\gamma_0 \cup \gamma \cup \gamma_1$ with Σ .

Proposition 4.5. The value $\operatorname{ind}_{\Sigma}(\gamma)$ is well defined. The function $\operatorname{ind}_{\Sigma}$ is a cocycle. The cohomological class of the cocycle $\operatorname{ind}_{\Sigma}$ is the Maslov class.

Proof. The fact that $\operatorname{ind}_{\Sigma}(\gamma)$ is well defined can be verified in the same way as that of the multiplicity. We can readily see that $\operatorname{ind}_{\Sigma}$ is cochain. The fact that $\operatorname{ind}_{\Sigma}$ is a cocycle follows from Theorem 1.1.

We can readily verify the validity of the following statement.

Proposition 4.6. Let γ be a positive path. The value of the cocycle $\operatorname{ind}_{\Sigma}$ on the path γ is equal to the number of intersection points of the path γ with the generalized train Σ , except for the starting point of the path γ , counted according to the multiplicities ascribed to the points of Σ .

Let g_H^{ϵ} be a mapping of the phase flow of a Hamiltonian vector field with the positive-definite (possibly nonautonomous) Hamiltonian H at time ϵ .

Proposition 4.7. Let $\gamma(t)$, $t \in [0,1]$, be a path in Λ_n . The value of the cocycle $\operatorname{ind}_{\Sigma}$ on the path γ is equal to the value of $\operatorname{ind}_{\Sigma}$ on the path $g_H^{\varepsilon}(\gamma)$ for any sufficiently small positive ε (depending on H and γ).

Proof. In fact, for a small ε , the path constructed in the definition of $\operatorname{ind}_{\Sigma}(\gamma)$ is homotopic to the path $g_{H}^{\varepsilon}(\gamma)$ in the class of paths with endpoints outside Σ .

§5. Symplectic Sturm Theorems

In this section we prove generalizations of symplectic Sturm theorems [2]. In our consideration, the following simple lemma is of particular importance.

Lemma 5.1. Any two transversal Lagrangian planes in Λ_n can be joined by a positive path γ such that $0 \leq \operatorname{ind}_{\Sigma}(\gamma) \leq n$ for any generalized train Σ .

Proof. Choose Darboux coordinates in $(\mathbb{R}^{2n}, \omega)$ such that the first plane coincides with the *p*-plane and the second with the *q*-plane. Let us identify \mathbb{R}^{2n} with \mathbb{C}^n as follows: $p_k + iq_k = z_k$. In this case, the path $\gamma(\varphi) = e^{i\varphi} \{q = 0\}, \ 0 \le \varphi \le \pi/2$, is the desired one. Indeed, the path γ thus constructed is positive because $\gamma(\varphi)$ is the image under the action of the phase flow at time φ with the Hamiltonian $\sum p_k^2 + q_k^2$. On the other hand, the path γ is a part of the positive loop $l(\varphi) = e^{i\varphi} \{q = 0\}, \ 0 \le \varphi \le \pi$, whose Maslov class is equal to n [1].

The following theorem is an alternation theorem.

Theorem 5.2. Let Σ_1 and Σ_2 be generalized trains and let l be a path in the Lagrange-Grassmann manifold Λ_n . In this case,

$$|\operatorname{ind}_{\Sigma_1}(l) - \operatorname{ind}_{\Sigma_2}(l)| \leq n$$

Proof. By Proposition 4.7, we may assume that the endpoints of the path l do not belong to Σ_1 and Σ_2 . Hence, we can assume that these endpoints are transversal Lagrangian planes. We can close them by means of the positive path γ constructed in Lemma 5.1. According to Proposition 4.5, the Maslov class of the loop thus obtained is equal to $\operatorname{ind}_{\Sigma_j}(l) + \operatorname{ind}_{\Sigma_j}(\gamma)$, j = 1, 2. Hence, $\operatorname{ind}_{\Sigma_1}(l) - \operatorname{ind}_{\Sigma_2}(l) = \operatorname{ind}_{\Sigma_2}(\gamma) - \operatorname{ind}_{\Sigma_1}(\gamma)$. Applying the estimate in Lemma 5.1 to the path γ , we obtain the desired equality.

Corollary 5.3. If a Lagrangian plane evolves under the action of the phase flow with a positivedefinite Hamiltonian, then, on any interval, the difference between the numbers of intersections with any two generalized trains (counted according to their multiplicities) does not exceed the number of degrees of freedom. On a closed interval that contains n + 1 points of intersection with a generalized train, there is a point of intersection with any other generalized train.

Corollary 5.4. The multiplicity of any point of a generalized train $\Sigma \subset \Lambda_n$ is at most n.

Let us prove the theorem on zeros.

Theorem 5.5. Let g^t be a phase flow of a linear nonautonomous Hamiltonian vector field in $(\mathbb{R}^{2n}, \omega)$, let Σ be a generalized train, let δ_i (i = 1, 2) be Lagrangian planes, and let $g^t \delta_i = g^t(\delta_i)$, $t \in [0, 1]$, be paths in the Lagrange-Grassmann manifold Λ_n . In this case,

$$|\operatorname{ind}_{\Sigma}(g^t \delta_1) - \operatorname{ind}_{\Sigma}(g^t \delta_2)| \leq n$$
.

Proof. By Proposition 4.7 we may assume that the Lagrangian planes δ_i and $g^1(\delta_i)$ (i = 1, 2) do not belong to the generalized train Σ and that the planes δ_1 and δ_2 are transversal. We join the planes δ_1 and δ_2 by the path m_1 constructed in the proof of Lemma 5.1. The paths $g^t\delta_1$ and $m_1 \cup g^t\delta_2 \cup -g^1(m_1)$ are homotopic $(-g^1(m_1))$ is the path $g^1(m_1)$ with opposite orientation). Hence, $\operatorname{ind}_{\Sigma}(g^t\delta_1) - \operatorname{ind}_{\Sigma}(g^t\delta_2) =$ $\operatorname{ind}_{\Sigma}(m_1) - \operatorname{ind}_{\Sigma}(g^1(m_1))$. We have $\operatorname{ind}_{\Sigma}(g^1(m_1)) = \operatorname{ind}_{(g^1)^{-1}(\Sigma)}(m_1)$, where $(g^1)^{-1}(\Sigma)$ is a generalized train. By the estimate in Lemma 5.1, we obtain the desired inequality. **Corollary 5.6.** If two Lagrangian planes evolve in a system with a positive-definite Hamiltonian, then the difference between the numbers of points of intersection with a generalized train (counted according to their multiplicities) does not exceed s.

Denote by $g_H^t \sigma = g_H^t(\sigma)$, $t \in [0, 1]$, the path in the Lagrange-Grassmann manifold Λ_n formed by the evolution of a Lagrangian plane σ under the action of the phase flow of a Hamiltonian vector field with Hamiltonian H (g_H^t is the mapping of the phase flow at time [0, t]). Let us prove the symplectic version of the comparison theorem.

Theorem 5.7. If $H_1 \ge H_0$, then, for generalized trains Σ_1 and Σ_2 and Lagrangian planes σ and λ , the following inequalities hold:

$$\operatorname{ind}_{\Sigma_1}(g_{H_1}^t\sigma) \ge \operatorname{ind}_{\Sigma_1}(g_{H_0}^t\sigma), \qquad \operatorname{ind}_{\Sigma_1}(g_{H_1}^t\sigma) \ge \operatorname{ind}_{\Sigma_1}(g_{H_0}^t\lambda) - n, \\ \operatorname{ind}_{\Sigma_1}(g_{H_1}^t\sigma) \ge \operatorname{ind}_{\Sigma_2}(g_{H_0}^t\lambda) - 2n.$$

Proof. By Proposition 4.7 we may assume that the planes σ , $g_{H_1}^1(\sigma)$, $g_{H_0}^1(\sigma)$, λ , $g_{H_1}^1(\lambda)$, and $g_{H_0}^1(\lambda)$ do not belong to the generalized trains Σ_1 and Σ_2 . Thus, it suffices to prove the theorem for $H_1 > H_0$. Let us consider the homotopy $H_m = (1 - m)H_0 + mH_1$ of the Hamiltonians.

Lemma 5.8. The path $g_{H_m}^1 \sigma = g_{H_m}^1(\sigma), m \in [0, 1]$, is positive.

We continue the proof of the theorem. The paths $g_{H_0}^t \sigma \cup g_{H_m}^1 \sigma$ and $g_{H_1}^t \sigma$ are homotopic. Hence, by Corollary 1.2, $\operatorname{ind}_{\Sigma_1}(g_{H_1}^t \sigma) \ge \operatorname{ind}_{\Sigma_1}(g_{H_0}^t \sigma)$. The first inequality of the theorem is thus proved. According to Theorem 5.5, we have $\operatorname{ind}_{\Sigma_1}(g_{H_1}^t \sigma) \ge \operatorname{ind}_{\Sigma_1}(g_{H_1}^t \lambda) - n$. This, together with the first inequality, proves the second inequality. By Theorem 5.2 we have $\operatorname{ind}_{\Sigma_1}(g_{H_1}^t \sigma) \ge \operatorname{ind}_{\Sigma_2}(g_{H_1}^t \sigma) - n$. This, together with the second inequality, proves the third inequality.

Proof of Lemma 5.8. Lemma 5.8 follows from the assertion below.

Let a nonautonomous quadratic Hamiltonian H depend on a parameter s so that $\tilde{H} = dH/ds > 0$. Then the family of symplectomorphisms $\psi_s = g^1_{H(s)}(g^1_{H(0)})^{-1}$ is specified by a positive-definite Hamiltonian.

The derivative of the solution (p(t, s), q(t, s)) of the Hamiltonian equations $\dot{p} = -H_q$, $\dot{q} = H_p$ with respect to the parameter s (we denote this derivative by $(h_1(t, s), h_2(t, s))$) satisfies the following system of equations in variations:

$$\dot{h}_1 = -\frac{\partial^2 H}{\partial q \,\partial p} \,h_1 - \frac{\partial^2 H}{\partial q \,\partial q} \,h_2 - \frac{\partial \widetilde{H}}{\partial q}, \qquad \dot{h}_2 = \frac{\partial^2 H}{\partial p \,\partial p} \,h_1 + \frac{\partial^2 H}{\partial p \,\partial q} \,h_2 + \frac{\partial \widetilde{H}}{\partial p}$$

with initial conditions $h_1(0, s) = 0$ and $h_2(0, s) = 0$.

According to the Euler theorem, $\omega(x, v_K(x)) = 2K(x)$ for a Hamiltonian vector field v_K with the Hamiltonian K. Thus, we must show that $\omega((p(1, s), q(1, s)), (h_1(1, s), h_2(1, s))) > 0$. We have

$$\begin{aligned} \frac{\partial}{\partial t} \,\omega((p(t), q(t)), (h_1(t), h_2(t))) &= \dot{p}h_2 - \dot{q}h_1 + p\dot{h}_2 - q\dot{h}_1 \\ &= -\frac{\partial H}{\partial q} \,h_2 - \frac{\partial H}{\partial p} \,h_1 + p \left(\frac{\partial^2 H}{\partial p \partial p} \,h_1 + \frac{\partial^2 H}{\partial p \partial q} \,h_2 + \frac{\partial \widetilde{H}}{\partial p}\right) \\ &+ q \left(\frac{\partial^2 H}{\partial q \partial p} \,h_1 + \frac{\partial^2 H}{\partial q \partial q} \,h_2 + \frac{\partial \widetilde{H}}{\partial q}\right) \end{aligned}$$

(we omit the parameter s, and the dot denotes the derivative with respect to t). Moreover,

$$\frac{\partial H}{\partial p} = p \frac{\partial^2 H}{\partial p \partial p} + q \frac{\partial^2 H}{\partial q \partial p}, \qquad \frac{\partial H}{\partial q} = p \frac{\partial^2 H}{\partial p \partial q} + q \frac{\partial^2 H}{\partial q \partial q}$$

because $\partial H/\partial p$ and $\partial H/\partial q$ are homogeneous of degree one. Hence, by the Euler theorem, we have

$$\frac{\partial}{\partial t}\omega((p(t),q(t)),(h_1(t),h_2(t)))=p\frac{\partial \widetilde{H}}{\partial p}+q\frac{\partial \widetilde{H}}{\partial q}=2\widetilde{H}>0.$$

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By integrating and taking account of the initial conditions, we obtain the desired inequality. This proves the lemma.

Assume that the Hamiltonian functions are positive definite.

Corollary 5.9. On a closed interval containing the n + 1 instants of intersection of a Lagrangian plane evolving in a Hamiltonian system with a generalized train, there exists at least one instant of intersection of a Lagrangian plane evolving in a system with a not lesser Hamiltonian function with the same train. Moreover, the difference between the numbers of instants of intersection (counted according to their multiplicities) for the systems with lesser and greater Hamiltonian functions does not exceed n.

Remark. All estimates in the theorems of this section are exact. For any inequality of any theorem, we can find generalized trains (even with a prescribed number of matrices from which the generalized train is constructed), paths in the Lagrange-Grassmann manifold, Lagrangian planes, and Hamiltonians such that the inequalities turn out to be equalities. It is of interest that no proof of the Sturm theorems that is known to the author uses the geometry of mutual displacement of two (generalized) trains and, in particular, the number of parts into which these generalized trains divide the Lagrange-Grassmann manifold, which is not known for generalized trains (the trains of two transversal Lagrangian planes divide the Lagrange-Grassmann manifold into n + 1 parts).

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