2. S. V. Kerov, Interlacing measures, in *Kirillov's Seminar on Representation Theory*, Amer. Math. Soc. Transl. Ser. 2, no. 181, American Mathematical Society, Providence, 1998, pp. 35–83.
3. D. Romik, Explicit formulas for hook walks on continual Young diagrams, *Adv. Appl. Math.* **32** (2004) 625–654.

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Quaternions and Rotations in $\mathbb{E}^4$

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1. INTRODUCTION. In 1843, Sir William Rowan Hamilton invented the quaternion algebra, which is customarily denoted $\mathbb{H}$ in his honor. Soon after, people recognized that quaternions could be used to represent rotations in $\mathbb{E}^3$. In 1855, Arthur Cayley discovered that quaternions could also be used to represent rotations in $\mathbb{E}^4$. This note explores Cayley's representation. Ultimately we use it to show that any rotation in $\mathbb{E}^4$ is a product of rotations in a pair of orthogonal two-dimensional subspaces, a result first proved by Edouard Goursat [**3**].

In section 2 we review the algebraic structure of $\mathbb{H}$ and show that $\mathbb{H}$ has a natural inner product that allows us to identify it with four-dimensional Euclidean space $\mathbb{E}^4$. In section 3 we show that a pair $\mathbf{p}$ and $\mathbf{q}$ of unit vectors (also called unit quaternions) in $\mathbb{H}$ determines a rotation $C_{\mathbf{p},\mathbf{q}} : \mathbb{H} \to \mathbb{H}$. According to Goursat's result, $C_{\mathbf{p},\mathbf{q}}$ is a product of rotations in a pair of orthogonal planes. By this we mean the following: there exist rotations $R_1, R_2 : \mathbb{H} \to \mathbb{H}$ and a pair of orthogonal planes V_1 and V_2 in $\mathbb{H}$, such that the restrictions $R_1|_{V_2}$ and $R_2|_{V_1}$ are identities on their respective planes and

$$C_{\mathbf{p},\mathbf{q}} = R_1 \circ R_2 = R_2 \circ R_1.$$

Thus, $\mathbb{H} = V_1 \oplus V_2$, where $V_1 \perp V_2$, and $C_{\mathbf{p},\mathbf{q}}$ rotates vectors in the plane V_1 through a determined angle α_1 and vectors in the plane V_2 through a determined angle α_2.

The principal goals of this note are to prove Theorems 1 and 2, which are stated precisely in section 5. Theorem 1 not only proves Goursat's result for $C_{\mathbf{p},\mathbf{q}}$, but also shows that one can easily determine the planes V_1 and V_2 and the angles α_1 and α_2 in terms of $\mathbf{p}$ and $\mathbf{q}$. Theorem 2 establishes that every rotation in $\mathbb{E}^4$ can be represented by some $C_{\mathbf{p},\mathbf{q}}$. Together, these theorems prove Goursat's result for every four-dimensional rotation.

The observation that $C_{\mathbf{p},\mathbf{q}}(V_i) = V_i$ $(i = 1, 2)$ motivates the method of proof. The V_i are known as invariant subspaces for $C_{\mathbf{p},\mathbf{q}}$. If we wish to see that $C_{\mathbf{p},\mathbf{q}}$ is indeed a product of rotations, it is natural to look first for invariant subspaces of that transformation. In section 4 we recall some elementary results from the theory of ordinary differential equations that are related to subspaces and two-dimensional rotations. Finally, in section 5, we apply these results to find the $C_{\mathbf{p},\mathbf{q}}$-invariant subspaces and the rotation angles.

2. THE QUATERNION ALGEBRA. Let $\mathbb{H}$ denote

$$\{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{R}\},$$

and define addition and multiplication by a real scalar in component-wise fashion. In so doing, $\mathbb{H}$ becomes a four-dimensional real vector space and $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis for $\mathbb{H}$. To define the product of two points in $\mathbb{H}$, one simply asserts the following: multiplication distributes over addition; $\mathbf{1}$ is the multiplicative identity; and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

These operations of addition and multiplication on $\mathbb{H}$ satisfy all the axioms for a field, except the commutativity of multiplication.

It is convenient to decompose a quaternion into two parts that are traditionally called its scalar and vector parts. If $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, then we write

$$\mathbf{q} = q_0 + \vec{q},$$

where $q_0 = q_0\mathbf{1}$ and $\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. We call q_0 the *scalar part* and $\vec{q}$ the *vector part* of $\mathbf{q}$. It is straightforward to check that the product

$$\mathbf{pq} = (p_0 + \vec{p})(q_0 + \vec{q}) = (p_0q_0 - \vec{p} \cdot \vec{q}) + (p_0\,\vec{q} + q_0\,\vec{p} + \vec{p} \times \vec{q}), \tag{1}$$

where $\vec{p} \cdot \vec{q}$ and $\vec{p} \times \vec{q}$ are, respectively, the usual inner product and vector cross product in $\mathbb{E}^3$.

There is another important operation on $\mathbb{H}$; it is called *conjugation*. If $\mathbf{q} = q_0 + \vec{q}$ belongs to $\mathbb{H}$ then $\bar{\mathbf{q}} = q_0 - \vec{q}$ is called the *conjugate* of $\mathbf{q}$. Conjugation has several nice properties, the most important of which is the following:

$$\overline{\mathbf{pq}} = \bar{\mathbf{q}}\,\bar{\mathbf{p}}$$

(note the change in order).

From equation (1) it also follows that

$$\mathbf{q}\bar{\mathbf{q}} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = \bar{\mathbf{q}}\mathbf{q}.$$

Thus, if we identify $\mathbb{H}$ with Euclidean four-space $\mathbb{E}^4$ by associating $\mathbf{q}$ with the vector (q_0, q_1, q_2, q_3) and denote the Euclidean inner product of $\mathbf{p}$ and $\mathbf{q}$ by $\langle \mathbf{p}, \mathbf{q} \rangle$, then $\mathbf{q}\bar{\mathbf{q}} = \langle \mathbf{q}, \mathbf{q} \rangle$. Using the fact that $\mathbf{q}\bar{\mathbf{q}}$ and $\langle \mathbf{q}, \mathbf{q} \rangle$ are quadratic forms (i.e., each is $\mathbb{R}$-linear in the two "slots" that appear in these expressions), it is a simple matter to verify that

$$\mathbf{p}\bar{\mathbf{q}} + \mathbf{q}\bar{\mathbf{p}} = 2\langle \mathbf{p}, \mathbf{q} \rangle. \tag{2}$$

Note, in particular, that $\mathbf{p}$ is orthogonal to $\mathbf{q}$ if and only if $\mathbf{p}\bar{\mathbf{q}} + \mathbf{q}\bar{\mathbf{p}} = \mathbf{0}$.

We denote the Euclidean norm of a quaternion $\mathbf{q}$ by $|\mathbf{q}|$. Since scalars commute with every quaternion,

$$|\mathbf{pq}|^2 = \langle \mathbf{pq}, \mathbf{pq} \rangle = \mathbf{pq}\overline{\mathbf{pq}} = \mathbf{pq}\bar{\mathbf{q}}\bar{\mathbf{p}} = \mathbf{p}|\mathbf{q}|^2\bar{\mathbf{p}} = |\mathbf{p}|^2|\mathbf{q}|^2.$$

This gives the following important result:

$$|\mathbf{pq}| = |\mathbf{p}||\mathbf{q}|. \tag{3}$$

Once the norm of a quaternion is available, we can obtain a formula for the inverse of a quaternion that is reminiscent of what occurs with complex numbers. It is easy to show that if $\mathbf{q} \neq \mathbf{0}$, then

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{|\mathbf{q}|^2}.$$

If $\mathbf{q}$ in $\mathbb{H}$ has $|\mathbf{q}| = 1$, then we call $\mathbf{q}$ a *unit quaternion*. For a unit quaternion $\mathbf{q}$ we see that $\mathbf{q}^{-1} = \bar{\mathbf{q}}$. If, in particular, $\mathbf{u}$ is a *pure unit quaternion* (i.e., if $\mathbf{u} = \vec{u}$), then $\mathbf{u}^{-1} = -\mathbf{u}$. Thus every pure unit quaternion is a square root of $-\mathbf{1}$. Also, by equation (2), two pure unit quaternions $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if it is the case that $\mathbf{uv} + \mathbf{vu} = \mathbf{0}$.

If $\mathbf{q} = q_0 + \vec{q}$ is a unit quaternion, then $q_0^2 + |\vec{q}\,|^2 = 1$. Hence there is a real number θ and a pure unit quaternion $\mathbf{u}$ such that $\mathbf{q} = \mathbf{1}\cos\theta + \mathbf{u}\sin\theta$. Since $\mathbf{u}^2 = -\mathbf{1}$, the power series expansion of e^t leads to

$$e^{\mathbf{u}\theta} = \sum_{n=0}^{\infty} \frac{(\mathbf{u}\theta)^n}{n!} = \mathbf{1}\cos\theta + \mathbf{u}\sin\theta, \tag{4}$$

providing equivalent representations for a unit quaternion $\mathbf{q} = q_0 + \vec{q} = \mathbf{1}\cos\theta + \mathbf{u}\sin\theta = e^{\mathbf{u}\theta}$. Note that neither $\mathbf{u}$ nor θ is uniquely determined by $\mathbf{q}$. When $\mathbf{q} \neq \pm\mathbf{1}$, $\sin\theta = \pm|\vec{q}\,|$ and $\mathbf{u} = \pm\vec{q}/|\vec{q}\,|$; when $\mathbf{q} = \pm\mathbf{1}$, $\mathbf{u}$ can be any pure unit quaternion.

We note that $e^{\mathbf{u}\theta}$ acts like the usual exponential as a function of a complex variable. However, since the multiplication in $\mathbb{H}$ is not commutative, if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent pure unit quaternions, it is not the case that $e^{\mathbf{u}\theta}e^{\mathbf{v}\phi}$ is the same as either $e^{\mathbf{v}\phi}e^{\mathbf{u}\theta}$ or $e^{\mathbf{u}\theta+\mathbf{v}\phi}$. However, since each component of $e^{\mathbf{u}\theta}$ is a differentiable function of θ, it is not difficult to verify that

$$\frac{d}{d\theta}e^{\mathbf{u}\theta} = -\mathbf{1}\sin\theta + \mathbf{u}\cos\theta = \mathbf{u}e^{\mathbf{u}\theta} = e^{\mathbf{u}\theta}\mathbf{u}.$$

3. ROTATIONS IN $\mathbb{E}^3$. We introduce the $\mathbb{R}$-linear transformations representing left and right multiplication in $\mathbb{H}$. Let $\mathbf{q}$ be a quaternion. Then $L_{\mathbf{q}} : \mathbb{H} \to \mathbb{H}$ and $R_{\mathbf{q}} : \mathbb{H} \to \mathbb{H}$ are defined as follows:

$$L_{\mathbf{q}}(\mathbf{x}) = \mathbf{qx}, \quad R_{\mathbf{q}}(\mathbf{x}) = \mathbf{xq} \qquad (\mathbf{x} \in \mathbb{H}).$$

If $\mathbf{q}$ is a unit quaternion, then both $L_{\mathbf{q}}$ and $R_{\mathbf{q}}$ are orthogonal transformations of $\mathbb{H}$. This is an easy consequence of equation (3). Specifically, when $|\mathbf{q}| = 1$

$$|L_{\mathbf{q}}(\mathbf{x})| = |\mathbf{qx}| = |\mathbf{q}||\mathbf{x}| = |\mathbf{x}|.$$

Thus, for unit quaternions $\mathbf{p}$ and $\mathbf{q}$, the mapping $C_{\mathbf{p},\mathbf{q}} : \mathbb{H} \to \mathbb{H}$ defined by

$$C_{\mathbf{p},\mathbf{q}} = L_{\mathbf{p}} \circ R_{\mathbf{q}} = R_{\mathbf{q}} \circ L_{\mathbf{p}}$$

is also an orthogonal transformation of $\mathbb{H}$. It is worth noting for later applications (in Theorems 2 and 3) that

$$C_{\mathbf{p}_1,\mathbf{q}_1} \circ C_{\mathbf{p}_2,\mathbf{q}_2} = C_{\mathbf{p}_1\mathbf{p}_2,\mathbf{q}_2\mathbf{q}_1}.$$

We examine briefly the transformation $C_{\mathbf{q},\bar{\mathbf{q}}}$, where $\mathbf{q}$ is a unit quaternion. For the time being, we simply denote it by C. If we write $\mathbf{q} = e^{\mathbf{u}\theta}$, where $\mathbf{u}$ is a pure unit quaternion, then $C(\mathbf{x}) = \mathbf{q}\mathbf{x}\bar{\mathbf{q}} = e^{\mathbf{u}\theta}\mathbf{x}e^{-\mathbf{u}\theta}$. First, observe that C preserves scalar quaternions because $C(\mathbf{1}) = \mathbf{q}\mathbf{1}\bar{\mathbf{q}} = \mathbf{q}\bar{\mathbf{q}}\mathbf{1} = \mathbf{1}\mathbf{1} = \mathbf{1}$. Since C is an orthogonal transformation of $\mathbb{H}$, it must also preserve the orthogonal complement to the scalars, the space of pure quaternions that we henceforth denote by $\mathbb{E}^3$. We restrict C to $\mathbb{E}^3$ and call the resulting map C as well. Note that $\mathbf{u} = \vec{u}$ is a member of $\mathbb{E}^3$.

Proposition 1. *If* $\mathbf{q}$ *is a unit quaternion, then there exist a pure unit quaternion* $\mathbf{u}$ *and a real scalar* θ *such that* $\mathbf{q} = e^{\mathbf{u}\theta}$. *The transformation* $C : \mathbb{E}^3 \to \mathbb{E}^3$ *defined by* $C(\mathbf{x}) = \mathbf{q}\mathbf{x}\bar{\mathbf{q}}$ *is a rotation in the plane orthogonal to* $\vec{u}$ *through an angle* 2θ.

Proof. We have already shown that every unit quaternion has an exponential representation. Choose $\mathbf{u}$ and θ so that $\mathbf{q} = e^{\mathbf{u}\theta}$. Observe that $C(\vec{u}) = e^{\mathbf{u}\theta}\mathbf{u}e^{-\mathbf{u}\theta} = \vec{u}$, because $\mathbf{u}$ commutes with $e^{\mathbf{u}\theta}$. Thus C fixes the one-dimensional subspace L spanned by $\vec{u}$, hence fixes its orthogonal complement $L^{\perp}$ $(\subset \mathbb{E}^3)$ as well. Let $\mathbf{v} = \vec{v}$ be a unit vector in $L^{\perp}$ and set $\mathbf{w} = \mathbf{u}\mathbf{v} = \vec{u} \times \vec{v}$. Notice that $\mathbf{u}\mathbf{v} = -\mathbf{v}\mathbf{u}$, since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. This implies that $\mathbf{v}e^{-\mathbf{u}\theta} = e^{\mathbf{u}\theta}\mathbf{v}$. Accordingly,

$$C(\vec{v}) = e^{\mathbf{u}\theta}\mathbf{v}e^{-\mathbf{u}\theta} = e^{2\mathbf{u}\theta}\mathbf{v} = \cos(2\theta)\vec{v} + \sin(2\theta)\vec{w}. \qquad \blacksquare$$

Notice that we can represent every rotation (i.e., every proper orthogonal transformation of $\mathbb{E}^3$) as $C_{\mathbf{q},\bar{\mathbf{q}}}$ for an appropriate unit quaternion $\mathbf{q}$. A transformation is *proper* if it is orientation-preserving or, in other words, if it has positive determinant.

For an elaboration of the topics presented so far the reader can refer to chapters 17 and 18 of the text by Michael Henle [**4**].

4. SOME FACTS ABOUT ORDINARY DIFFERENTIAL EQUATIONS. We now turn our attention to $C_{\mathbf{p},\mathbf{q}}$ for arbitrary unit quaternions $\mathbf{p}$ and $\mathbf{q}$. It was Cayley who first noticed that these are proper orthogonal transformations of $\mathbb{E}^4$. As has been known for some time (see [**3**]), such transformations must be the product of two rotations in a pair of orthogonal two-dimensional subspaces of $\mathbb{E}^4$. We would like to see how these rotations and subspaces are related to $\mathbf{p}$ and $\mathbf{q}$. Coxeter elucidated this relation in an earlier paper in this journal [**2**]. We intend to do the same using distinctly different methods and, in fact, we will show from first principles that $C_{\mathbf{p},\mathbf{q}}$ is a product. To do that we call upon one tool from the theory of ordinary differential equations.

Proposition 2. *Let* $\tilde{\mathbf{x}} : \mathbb{R} \to \mathbb{R}^n$ *satisfy a kth-order linear homogeneous differential equation, where* $1 \le k \le n$. *Then the image of* $\tilde{\mathbf{x}}$ *lies in a* k*-dimensional subspace of* $\mathbb{R}^n$.

Proof. We present the proof for the case $k = 2$, which suits our application. The reader should be able to generalize this to any k.

Let $\mathbf{x}_0 = \tilde{\mathbf{x}}(0)$ and $\mathbf{x}_0' = d\tilde{\mathbf{x}}/dt(0)$ be the initial position and initial velocity for the given curve $\tilde{\mathbf{x}}$. Additionally, suppose that $\tilde{\mathbf{x}}$ satisfies the second-order linear homogeneous differential equation

$$\frac{d^2\mathbf{x}}{dt^2} + \alpha\frac{d\mathbf{x}}{dt} + \beta\mathbf{x} = \mathbf{0},$$

where α and β are differentiable real-valued functions of t. From standard ODE theory, we know that when two solutions of this differential equation have the same initial position and the same initial velocity, the two solutions are identical.

Now suppose that f_i ($i = 0, 1$) are real-valued functions that satisfy the differential equation $f'' + \alpha f' + \beta f = 0$ and, in addition, $f_0(0) = 1$, $f_0'(0) = 0$, $f_1(0) = 0$, and $f_1'(0) = 1$. Then the curve $\mathbf{x}(t) = f_0(t)\mathbf{x}_0 + f_1(t)\mathbf{x}_0'$ satisfies the same ODE as $\tilde{\mathbf{x}}$ and has the same initial position and initial velocity. Thus $\tilde{\mathbf{x}} = f_0(t)\mathbf{x}_0 + f_1(t)\mathbf{x}_0'$, so we see that the image of $\tilde{\mathbf{x}}$ lies in the subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}_0$ and $\mathbf{x}_0'$. ∎

That the functions f_0 and f_1 of the preceding proof exist is guaranteed by the standard theory for linear ordinary differential equations [**1**]. The following proposition is an easy consequence of the proof of Proposition 2. It will prove to be quite useful.

Proposition 3. *Suppose that $\tilde{\mathbf{x}} : \mathbb{R} \to \mathbb{H}$ satisfies the differential equation*

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2} + \omega^2\tilde{\mathbf{x}} = \mathbf{0},$$

where $\omega > 0$ is a constant, and that the initial position vector $\tilde{\mathbf{x}}(0)$ and the initial velocity vector $d\tilde{\mathbf{x}}/dt(0)$ satisfy the conditions $|\tilde{\mathbf{x}}(0)| = \omega^{-1}|d\tilde{\mathbf{x}}/dt(0)|$ and $\langle\tilde{\mathbf{x}}(0), d\tilde{\mathbf{x}}/dt(0)\rangle = 0$. Then $\tilde{\mathbf{x}}(1) = R(\tilde{\mathbf{x}}(0))$, where R is a rotation in the plane of the image of $\tilde{\mathbf{x}}$ through an angle ω in the direction that turns $\tilde{\mathbf{x}}(0)$ toward $\omega^{-1}d\tilde{\mathbf{x}}/dt(0)$.

Proof. Following the construction in the proof of Proposition 2, we choose $f_0(t) = \cos(\omega t)$ and $f_1(t) = \omega^{-1}\sin(\omega t)$. Then

$$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{x}}(0)\cos(\omega t) + \omega^{-1}\frac{d\tilde{\mathbf{x}}(0)}{dt}\sin(\omega t),$$

which shows that

$$\tilde{\mathbf{x}}(1) = \tilde{\mathbf{x}}(0)\cos(\omega) + \omega^{-1}\frac{d\tilde{\mathbf{x}}(0)}{dt}\sin(\omega).$$

That the rotation R exists follows from the assumptions that $\tilde{\mathbf{x}}(0)$ and $\omega^{-1}d\tilde{\mathbf{x}}/dt(0)$ are orthogonal vectors and have the same length. ∎

5. PROPER ORTHOGONAL TRANSFORMATIONS OF $\mathbb{E}^4$. Let C be shorthand for $C_{\mathbf{p},\mathbf{q}}$, where for suitable choices of pure unit quaternions $\mathbf{u}$ and $\mathbf{v}$ and corresponding real numbers θ and ϕ, $\mathbf{p} = e^{\mathbf{u}\theta}$ and $\mathbf{q} = e^{\mathbf{v}\phi}$. We seek two-dimensional invariant subspaces for C. If $\mathbf{x}$ in $\mathbb{H}$ lies in some C-invariant subspace S, so does $C^n(\mathbf{x})$ for all integers n. Moreover, if C^t made sense for arbitrary real t, we would expect the same to be true of $C^t(\mathbf{x})$. It is this observation that motivates what we do next.

First, notice that C^t does make sense; in fact, for any real t let C^t be defined by

$$C^t(\mathbf{x}) = e^{\mathbf{u}\theta t}\mathbf{x}e^{\mathbf{v}\phi t}.$$

To each quaternion $\mathbf{x}$ we associate a curve $\tilde{\mathbf{x}} : \mathbb{R} \to \mathbb{H}$ defined by $\tilde{\mathbf{x}}(t) = C^t(\mathbf{x})$. We will compute two derivatives of $\tilde{\mathbf{x}}$. Note that the usual formulas for differentiating a product or a composition apply, as the reader can check by examining the components in these formulas or by considering equation (1) with the real constants replaced with real-valued functions. Keep in mind, however, that the order of terms in products is important. As the first derivative of $\tilde{\mathbf{x}}$ we obtain

$$\frac{d\tilde{\mathbf{x}}}{dt}(t) = \mathbf{u}\theta\, e^{\mathbf{u}\theta t}\,\mathbf{x}\,e^{\mathbf{v}\phi t} + e^{\mathbf{u}\theta t}\,\mathbf{x}\,e^{\mathbf{v}\phi t}\,\mathbf{v}\phi = \mathbf{u}\theta\,\tilde{\mathbf{x}}(t) + \tilde{\mathbf{x}}(t)\,\mathbf{v}\phi. \tag{5}$$

Differentiating the left-hand and right-hand sides of (5), at the same time using (5) to eliminate first-order derivatives of $\tilde{\mathbf{x}}(t)$, we get

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2}(t) = -(\theta^2+\phi^2)\tilde{\mathbf{x}}(t) + 2\theta\phi\mathbf{u}\tilde{\mathbf{x}}(t)\mathbf{v}. \tag{6}$$

Now if it happened that

$$\mathbf{u}\tilde{\mathbf{x}}(t)\mathbf{v} = \lambda(t)\tilde{\mathbf{x}}(t), \tag{7}$$

where λ is a real-valued function, then $\tilde{\mathbf{x}}$ would satisfy a linear homogeneous second-order ordinary differential equation with real coefficients. By Proposition 2, the image of $\tilde{\mathbf{x}}$ would lie in a two-dimensional subspace and necessarily the span of $\mathbf{x} = \tilde{\mathbf{x}}(0)$ and $C(\mathbf{x}) = \tilde{\mathbf{x}}(1)$ would be an invariant subspace. Note however that $|\mathbf{u}\tilde{\mathbf{x}}(t)\mathbf{v}| = |\tilde{\mathbf{x}}(t)|$, because $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Thus if equation (7) were to hold, then λ would have to be either the constant function 1 or the constant function -1 (since all the functions we consider are continuous). In fact, we can simplify the condition of equation (7) with $\lambda = \pm 1$.

Lemma 1. *For* $\mathbf{x}$ *in* $\mathbb{H}$, $\mathbf{uxv} = \pm\mathbf{x}$ *if and only if* $\mathbf{u}\tilde{\mathbf{x}}(t)\mathbf{v} = \pm\tilde{\mathbf{x}}(t)$ *holds for all* t.

Proof. Since $\mathbf{x} = \tilde{\mathbf{x}}(0)$, the *if* direction is obvious. To prove the *only if* direction, assume that $\mathbf{uxv} = \pm\mathbf{x}$. Apply C^t to both sides of this equation to get

$$\pm C^t(\mathbf{x}) = C^t(\mathbf{uxv}) = e^{\mathbf{u}\theta t}\mathbf{uxv}e^{\mathbf{v}\phi t} = \mathbf{u}C^t(\mathbf{x})\mathbf{v}$$

(i.e., $\pm\tilde{\mathbf{x}}(t) = \mathbf{u}\tilde{\mathbf{x}}(t)\mathbf{v}$). ■

Thus we look for those $\mathbf{x}$ in $\mathbb{H}$ that satisfy one of the linear equations $\mathbf{ux} \pm \mathbf{xv} = \mathbf{0}$. To do this we introduce a basis for $\mathbb{H}$. A natural choice is the set consisting of $\mathbf{1}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{uv}$. Of course, this is not a basis if $\mathbf{u} = \pm\mathbf{v}$, but it is otherwise. (Note: when $\mathbf{p} = \pm\mathbf{1}$ or $\mathbf{q} = \pm\mathbf{1}$, at least one of $\mathbf{u}$ or $\mathbf{v}$ may be chosen arbitrarily. When this occurs, we always choose $\mathbf{u}$ and $\mathbf{v}$ to be orthogonal.)

We first consider the case where $\mathbf{u} = \pm\mathbf{v}$ and look for solutions to $\mathbf{ux} \pm \mathbf{xu} = \mathbf{0}$. It is easy to see that $\mathbf{1}$ and $\mathbf{u}$ are solutions to $\mathbf{ux} - \mathbf{xu} = \mathbf{0}$; in fact, they form an orthonormal basis for the solutions to this equation. On the other hand, the solutions to $\mathbf{ux} + \mathbf{xu} = \mathbf{0}$ are the pure quaternions $\mathbf{x}$ that are orthogonal to $\mathbf{u}$. In a minor abuse of notation we use $\vec{u}^{\perp}$ (recall that $\vec{u}$ is the vector part of $\mathbf{u}$) to signify this set of solutions. Since any pure quaternion is necessarily orthogonal to $\mathbf{1}$, the space $\vec{u}^{\perp}$ is the orthogonal complement of the space spanned by $\mathbf{1}$ and $\mathbf{u}$. Thus the solution spaces to the two equations $\mathbf{ux} \pm \mathbf{xu} = \mathbf{0}$ give a decomposition of $\mathbb{H}$ into the sum of two two-dimensional orthogonal subspaces.

Now assume that $\mathbf{u} \neq \pm\mathbf{v}$. One can then introduce the basis $\{\mathbf{1}, \mathbf{u}, \mathbf{v}, \mathbf{uv}\}$ and in a straightforward fashion find solutions. However, it is easier to guess solutions. For example, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ satisfies $\mathbf{ux} - \mathbf{xv} = \mathbf{0}$. For this $\mathbf{x}$, the curve $\tilde{\mathbf{x}}$ lies in a two-dimensional invariant subspace; hence $d\tilde{\mathbf{x}}/dt(0)$ is also in that subspace. Using (5), we see that for $\mathbf{x} = \mathbf{u} + \mathbf{v}$

$$\frac{d\tilde{\mathbf{x}}}{dt}(0) = \mathbf{u}\theta\,(\mathbf{u}+\mathbf{v}) + (\mathbf{u}+\mathbf{v})\,\mathbf{v}\phi = (\theta+\phi)(\mathbf{uv}-\mathbf{1}).$$

A direct calculation confirms that $\mathbf{x} = \mathbf{uv} - \mathbf{1}$ is another solution of $\mathbf{ux} - \mathbf{xv} = \mathbf{0}$. Moreover, since C^t is a rotation for every t, $|\tilde{\mathbf{x}}(t)| = |C^t(\mathbf{u}+\mathbf{v})| = |\mathbf{u}+\mathbf{v}|$ is a constant. It follows that $\tilde{\mathbf{x}}(0) = \mathbf{u}+\mathbf{v}$ is orthogonal to $\tilde{\mathbf{x}}'(0)$ and thus to $\mathbf{uv} - \mathbf{1}$. Observe also that, since $\mathbf{uv} - \mathbf{1} = (\mathbf{u} + \mathbf{v})\mathbf{v}$, equation (3) implies that these two quaternions have the same norm. A direct calculation gives the common value of these norms: $\sqrt{2(1 + \cos\alpha)}$, where α $(0 < \alpha < \pi)$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. In a similar fashion, we can show that $\mathbf{v} - \mathbf{u}$ and $\mathbf{uv} + \mathbf{1}$ are orthogonal solutions of $\mathbf{ux} + \mathbf{xv} = \mathbf{0}$, each having norm $\sqrt{2(1 - \cos\alpha)}$. Finally, it is easy to check that each pair of vectors is orthogonal to the other pair. Thus the vectors $\mathbf{u} + \mathbf{v}$, $\mathbf{uv} - \mathbf{1}$, $\mathbf{v} - \mathbf{u}$, and $\mathbf{uv} + \mathbf{1}$ constitute an orthogonal basis for $\mathbb{H}$.

Assume that $\mathbf{u} \neq -\mathbf{v}$, let $\mathbf{x} = \mathbf{u} + \mathbf{v}$, and recall that this $\mathbf{x}$ satisfies $\mathbf{ux} - \mathbf{xv} = \mathbf{0}$ (or equivalently $\mathbf{uxv} = -\mathbf{x}$). Here $|\mathbf{u} + \mathbf{v}| = |\mathbf{uv} - \mathbf{1}| = \sqrt{2(1 + \cos\alpha)}$ even when $\alpha = 0$. For this $\mathbf{x}$, (6) becomes

$$\frac{d^2\tilde{\mathbf{x}}}{dt^2}(t) + (\theta + \phi)^2\tilde{\mathbf{x}}(t) = \mathbf{0}. \tag{8}$$

Also, by (5) evaluated at $t = 0$,

$$\frac{d\tilde{\mathbf{x}}}{dt}(0) = \theta\mathbf{ux} + \phi\mathbf{xv} = (\theta + \phi)(\mathbf{ux}) = (\theta + \phi)(\mathbf{uv} - \mathbf{1}).$$

Hence $|\theta + \phi|^{-1}|d\tilde{\mathbf{x}}/dt(0)| = |\tilde{\mathbf{x}}(0)|$ and $\langle\tilde{\mathbf{x}}(0), d\tilde{\mathbf{x}}/dt(0)\rangle = 0$. Thus we can invoke Proposition 3 and conclude that C restricted to the plane spanned by $\mathbf{u} + \mathbf{v}$ and $\mathbf{uv} - \mathbf{1}$ is a rotation through the angle $|\theta + \phi|$ in the direction that turns $\mathbf{u} + \mathbf{v}$ toward $\operatorname{sign}(\theta + \phi)(\mathbf{uv} - \mathbf{1})$. Stated more simply, C restricted to the plane spanned by $\mathbf{u} + \mathbf{v}$ and $\mathbf{uv} - \mathbf{1}$ is a rotation through $\theta + \phi$ in the direction that turns $\mathbf{u} + \mathbf{v}$ toward $\mathbf{uv} - \mathbf{1}$. In a similar fashion, when $\mathbf{u} \neq \mathbf{v}$ we can show that C restricted to the plane spanned by $\mathbf{v} - \mathbf{u}$ and $\mathbf{uv} + \mathbf{1}$ is a rotation by $\theta - \phi$ in the direction that turns $\mathbf{v} - \mathbf{u}$ towards $\mathbf{uv} + \mathbf{1}$. The same kind of results hold in the remaining cases when $\mathbf{u} = \pm\mathbf{v}$.

We consolidate what we have learned into our first theorem:

Theorem 1. *Let* $\mathbf{p} = e^{\mathbf{u}\theta}$ *and* $\mathbf{q} = e^{\mathbf{v}\phi}$, *where* $\mathbf{u}$ *and* $\mathbf{v}$ *are pure unit quaternions. The orthogonal transformation* $C_{\mathbf{p},\mathbf{q}}$ *of* $\mathbb{H}$ *is a product of two rotations in orthogonal planes. If* $\mathbf{u} \neq \pm\mathbf{v}$, *then* $C_{\mathbf{p},\mathbf{q}}$ *rotates the plane spanned by* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{uv} - \mathbf{1}$ *through the angle* $|\theta + \phi|$ *and the plane spanned by* $\mathbf{v} - \mathbf{u}$ *and* $\mathbf{uv} + \mathbf{1}$ *through the angle* $|\theta - \phi|$. *If* $\mathbf{u} = \pm\mathbf{v}$, *then the invariant planes are the span of* $\mathbf{1}$ *and* $\mathbf{u}$ *and its orthogonal complement, and the rotation angles in appropriate planes are still* $|\theta + \phi|$ *and* $|\theta - \phi|$.

As an aside, we note the following special case of Theorem 1. When $\theta\phi = 0$, which implies that $\mathbf{p} = \mathbf{1}$ or $\mathbf{q} = \mathbf{1}$, the differential equation (8) is equivalent to (6). Then for every $\mathbf{x}$ the associated curve $\tilde{\mathbf{x}}$ satisfies (8), and we see that every nonzero $\mathbf{x}$ lies in a C-invariant plane and is rotated through the same angle, namely, $|\theta + \phi|$.

We can now prove that every proper orthogonal transformation of $\mathbb{H}$ is of the form $C_{\mathbf{p},\mathbf{q}}$ and so can be described by Theorem 1 as well. First note that $C = C_{\mathbf{p},\mathbf{q}}$ is proper since its determinant is 1. This follows from the fact that $\det(C^t)$ is a continuous function of t and can only take on the values ± 1 (each C^t is orthogonal). However, C^0 is the identity, which has determinant 1.

Theorem 2. *If* A *is a proper orthogonal transformation of* $\mathbb{H}$, *then there exist unit quaternions* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $A = C_{\mathbf{p},\mathbf{q}}$.

Proof. Let A be a proper orthogonal transformation of $\mathbb{H}$, and let $\mathbf{p} = A(\mathbf{1})$. Clearly, $\mathbf{p}$ is a unit quaternion. Observe that $C_{\mathbf{p},\mathbf{1}}$ maps $\mathbf{1}$ to $\mathbf{p}$. Then $C_{\mathbf{p},\mathbf{1}}^{-1} \circ A$ fixes $\mathbf{1}$ and thus defines a proper orthogonal transformation in $\mathbb{E}^3$. By Proposition 1, there exists a unit quaternion $\mathbf{q}$ such that $C_{\mathbf{p},1}^{-1} \circ A = C_{\mathbf{q},\bar{\mathbf{q}}}$. It follows that $A = C_{\mathbf{p},1} \circ C_{\mathbf{q},\bar{\mathbf{q}}} = C_{\mathbf{pq},\bar{\mathbf{q}}}$. ■

It is not the case that each A is uniquely represented as $C_{\mathbf{p},\mathbf{q}}$. Our final theorem shows precisely to what extent this representation is not unique.

Theorem 3. *Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, and $\mathbf{q}_2$ be unit quaternions. The transformations $C_{\mathbf{p}_1,\mathbf{q}_1}$ and $C_{\mathbf{p}_2,\mathbf{q}_2}$ are equal if and only if $\mathbf{p}_2/\mathbf{p}_1 = \mathbf{q}_2/\mathbf{q}_1 = \pm\mathbf{1}$.*

Proof. The theorem follows from the observation that $C_{\mathbf{p},\mathbf{q}}$ is the identity transformation if and only if $\mathbf{p} = \mathbf{q} = \pm\mathbf{1}$. One direction of this equivalence is obvious, and the proof of the other direction is an easy application of Theorem 1, which we leave to the reader. ■

REFERENCES

1. G. Birkhoff and G.-C. Rota, *Ordinary Differential Equations*, 4th ed., John Wiley & Sons, New York, 1989.
2. H. S. M. Coxeter, Quaternions and reflections, this MONTHLY **53** (1946) 136–146.
3. E. Goursat, Sur les substitutions orthogonales et les divisions régulières de l'espace, *Annales Scientifiques de l'École Normale Supérieure* **6** (1889) 2–102.
4. M. Henle, *Modern Geometries: Non-Euclidean, Projective, and Discrete*, 2nd ed., Prentice-Hall, Upper Saddle River, NJ, 2001.

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Another Proof of the Fundamental Theorem of Algebra

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The goal of this note is to prove the fundamental theorem of algebra. To be more precise, we show that the degree of an irreducible polynomial in $\mathbb{R}[X]$ is either 1 or 2. The same method can be used to prove that the degree of an irreducible polynomial in $\mathbb{C}[X]$ is always 1.

Let n be an integer larger than 1, and let P be an irreducible polynomial in $\mathbb{R}[X]$ of degree n. We assert that $n = 2$. Denote by $\langle P \rangle$ the ideal generated by P in the ring $\mathbb{R}[X]$. Since P is irreducible, the quotient of the ring $\mathbb{R}[X]$ by $\langle P \rangle$ is a field. If we define $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}[X]/\langle P \rangle$ by

$$(a_0, a_1, \dots, a_{n-1}) \mapsto a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} + \langle P \rangle,$$

then ψ is a group isomorphism from $(\mathbb{R}^n, +)$ onto $(\mathbb{R}[X]/\langle P \rangle, +)$. This isomorphism induces in the obvious way a field structure in $\mathbb{R}^n$, the addition being the usual one. The