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Quadratic and Hermitian Forms in Additive and Abelian Categories

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During the last few years several papers concerned with the foundations of the theory of quadratic forms over arbitrary rings with involution have appeared. It is not necessary to give detailed references, in particular one thinks of the well known work of Bak [1], Bass [3], Karoubi, Knebusch [11, 12], Ranicki, Vaserstein, and C. T. C. Wall. During the same period a number of problems quite similar to those occuring in the theory of quadratic forms were discussed, which, however, did not fit in the formalism developed so far. For example, one thinks of problems like the classification of pairs of forms, of sesquilinear forms, isometries , quadratic spaces with systems of subspaces, and also of quadratic forms over schemes, see e.g. [12, 13, 20, 22, 23]. This situation called for a more general foundation of the theory of quadratic and hermitian forms.

In this paper we try to give this foundation. Our basic object is an additive category $\mathscr{M}$ together wit a duality functor $*: \mathscr{M} \to \mathscr{M}$. In this situation one can define the most important notions of the theory of quadratic forms. Under suitable finiteness conditions one can prove a Krull-Schmidt theorem which is a sharpening of the classical Witt theorem. This result is basic for applications to the problems mentioned above. A preliminary version of this material is contained in [17]. As just one application we discuss the classification of quadratic spaces with four subspaces. We hope that this disucssion will show clearly how one can solve a number of important classification problems of linear algebra. More applications can be found in [15, 16, 17, 22, 24].

In the second part of the paper we discuss hermitian (not quadratic) forms in an abelian category. In an abelian category one has more structure, in particular one can introduce the notion of orthogonality. This allows one to introduce Grothendieck and Witt groups analogous to the G-groups in linear algebraic K-theory which are obtained by factoring out exact sequences. As a basic result a Jordan-Hölder theorem is proved for categories where all objects are of finite length. Using this theorem the computation of the Grothendieck group is

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reduced to the subcategory of semisimple objects, and here one can use the Krull-Schmidt theorem.

This work was begun in cooperation with R. Scharlau. We are indebted to him for useful discussions and remarks. Also, his paper [22] suggested many of the problems discussed in this paper.

1. Definitions

We consider the following situation: $\mathcal{M}$ is an *additive category* and $*: \mathcal{M} \to \mathcal{M}$ is a duality *functor*, i.e. an additive contravariant functor with a natural isomorphism $(i_M)_{M \in \mathcal{M}} : id \to **$ such that $i_M^* i_{M^*} = id_{M^*}$ for all $M \in \mathcal{M}$. We call the objects of $\mathcal{M}$ *modules* and we identify each module M with M^{**} and each morphism f with f^{**}.

The category $H(\mathcal{M})$ of *sesquilinear modules* over $\mathcal{M}$ is defined as follows: The objects are pairs (M, h) with $M \in \mathcal{M}$, and $h: M \to M^*$ a morphism. We call h a *form* on M. If h is an isomorphism we call (M, h) *nonsingular*. The morphisms in $H(\mathcal{M})$ are defined in the obvious way; isomorphisms in $H(\mathcal{M})$ are called *isometries*.

For $\epsilon = \pm 1$, a form h is called *ϵ-hermitian* if $h = \epsilon h^*$; it is called *even ϵ-hermitian* if $h = g + \epsilon g^*$ for some g. We denote by $H^\epsilon(\mathcal{M})$ the full subcategory of hermitian modules (M, h) with h nonsingular ϵ-hermitian. $H_+{}^\epsilon(\mathcal{M})$ is the full subcategory of nonsingular even ϵ-hermitian modules.

For modules M and M' we identify $(M \oplus M')^*$ with $M^* \oplus M'^*$, and we define the orthogonal sum in the obvious way.

If 2 is not a unit in $\mathrm{End}(M)$, it is well known that for many questions it is more natural to consider quadratic forms (in the sense of Bak) instead of hermitian forms.

A *form parameter* (ϵ, Λ) in $\mathcal{M}$ is the assignment of a subgroup Λ_M of $\mathrm{Hom}(M, M^*)$ to every module M such that

(i) $\Lambda_{\min} \subset \Lambda_M \subset \Lambda_{\max}$

where

$$\Lambda_{\min} = \{h - \epsilon h^* \mid h \in \mathrm{Hom}(M, M^*)\}$$
$$\Lambda_{\max} = \{h \in \mathrm{Hom}(M, M^*) \mid h = -\epsilon h^*\}$$

(ii) $f^* \Lambda_{M'} f \subset \Lambda_M$ for all $f \in \mathrm{Hom}(M, M')$.

A *(ϵ, Λ)-quadratic form* $[h]$ on M is an element $[h]$ of $\mathrm{Hom}(M, M^*)/\Lambda_M$. The pair $(M, [h])$ is called a *(ϵ, Λ)-quadratic module*. A morphism $f: (M, [h]) \to (M', [h'])$ is a morphism in $\mathcal{M}$ such that $[f^* h' f] = [h]$. Isomorphisms of quadratic modules are called *isometries*. The group of all isometries of $(M, [h])$ is denoted by $\mathrm{Aut}(M, [h])$. For a (ϵ, Λ)-quadratic form $[h]$ the even ϵ-hermitian form $h + \epsilon h^*$ is well-defined. $[h]$ is called *nonsingular* if $h + \epsilon h^*$ is nonsingular.

Let $Q^{\epsilon,\Lambda}(\mathscr{M})$ denote the category of nonsingular (ϵ, Λ)-quadratic modules over $\mathscr{M}$. With the obvious orthogonal sum, the category $Q^{\epsilon,\Lambda}(\mathscr{M})$ (as well as the categories introduced before) is a category with product in the sense of Bass [2, Chap. VII].

For the maximal choice $\Lambda_{\max}$ of Λ, an equivalence

$$Q^{\epsilon,\Lambda}(\mathscr{M}) \cong H_+^{\epsilon}(\mathscr{M})$$

is induced from $[h] \mapsto h + \epsilon h^*$. If 2 is invertible in $\mathrm{End}(M)$, every ϵ-hermitian form is even, Λ is uniquely determined, and $Q^{\epsilon,\Lambda}(\mathscr{M})$ and $H^{\epsilon}(\mathscr{M})$ are equivalent categories.

For a category $\mathscr{C}$ let $\mathrm{Aut}(\mathscr{C})$ denote the category of automorphisms in $\mathscr{C}$. If $\mathscr{M}$ is a category with duality $*$, a duality functor on $\mathrm{Aut}(\mathscr{M})$ is defined by

$$(M, f)^* = (M^*, f^{*-1}).$$

For a form h on $(M, f) \in \mathrm{Aut}(\mathscr{M})$, the automorphism f is an isometry of (M, h) and one obtains an equivalence

$$H^{\epsilon}(\mathrm{Aut}(\mathscr{M})) \cong \mathrm{Aut}(H^{\epsilon}(\mathscr{M})).$$

However, in general, $\mathrm{Aut}(Q^{\epsilon,\Lambda}(\mathscr{M}))$ *cannot be interpreted as a category of quadratic forms in* $\mathrm{Aut}(\mathscr{M})$.

1.1. EXAMPLE. The most important example is the following one: Let A be a ring with involution ^{0} and a form parameter (ϵ, Λ) in the sense of Bak [1], $\epsilon = \pm 1$. On the category $\mathscr{P}(A)$ of finitely generated projective right A-modules there is the duality functor $P^* := \mathrm{Hom}_A(P, A)$, where P^* is a right-module via $(fa)(x) = a^0 f(x)$ for $f \in P^*$, $a \in A$, $x \in P$. A form h corresponds to the usual sesquilinear form $(x, y) \to h(x)(y)$. A form parameter Λ' on $\mathscr{P}(A)$ is defined by

$$\Lambda'_P = \{h \in \mathrm{Hom}(P, P^*) \mid h = -\epsilon h^*,\ h(x, x) \in \Lambda\}.$$

With these definitions we have translated Bak's definition of quadratic forms into our categorical framework: Quadratic forms in Bak's sense correspond canonically to ours, etc.

1.2 LEMMA. *Let* $\mathscr{M}$ *be an additive category with duality* $*$ *and form parameter* (ϵ, Λ). *If* $(M_0, h_0) \in H^{\epsilon_0}(\mathscr{M})$, *then an involution is defined on* $E = \mathrm{End}(M_0)$ *by*

$$e^0 := h_0^{-1} e^* h_0, \qquad e \in E,$$

and a form parameter $(\epsilon\epsilon_0, \Lambda_E)$ *on* E *is defined by* $\Lambda_E = \{h_0^{-1} h \mid h \in \Lambda_{M_0}\}$. ∎

For $M \in \mathscr{M}$ the *hyperbolic* (ϵ, Λ)-quadratic module $\mathbb{H}(M)$ is defined by

$$\mathbb{H}(M) = \mathbb{H}^{\epsilon,\Lambda}(M) = \left(M \oplus M^*, \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]\right).$$

Every isomorphism $f: M \to M'$ defines the *hyperbolic isometry*

$$\mathbb{H}(f) = \mathbb{H}^{\epsilon,\Lambda}(f) = f \oplus f^{*-1}: \mathbb{H}(M) \to \mathbb{H}(M').$$

This defines a functor $\mathbb{H}: \mathscr{M}' \to Q^{\epsilon,\Lambda}(\mathscr{M})$. (Here, the category $\mathscr{M}'$ has the same objects as $\mathscr{M}$ and as morphisms only the isomorphisms of $\mathscr{M}$.) We identify $\mathbb{H}(M) \perp \mathbb{H}(M')$ and $\mathbb{H}(M \oplus M')$. Hence, the full subcategory $\mathbb{H}(\mathscr{M})$ of hyperbolic modules is closed with respect to orthogonal sums.

For a category $\mathscr{C}$ with product $\perp$ let $K_i(\mathscr{C})$ denote the Grothendieck groups of $\mathscr{C}$ (in the sense of Bass [2, Chap. VII)]:

$K_0(\mathscr{C})$ is generated by the isomorphism classes $[C]$ of objects $C \in \mathscr{C}$ with respect to the relations $[C \perp C'] = [C] + [C']$.

$K_1(\mathscr{C}) = K_0(\mathrm{Aut}(\mathscr{C}))/R$ where R is the subgroup generated by all elements

$$[(M, fg)] - [(M, f)] - [(M, g)]$$

with $f, g \in \mathrm{Aut}(M)$.

We define $KQ_i(\mathscr{M}) = KQ_i^{\epsilon,\Lambda}(\mathscr{M}) = K_i(Q^{\epsilon,\Lambda}(\mathscr{M}))$ for $i = 0, 1$. The hyperbolic functor $\mathbb{H}$ defines homomorphisms

$$\mathbb{H}_i: K_i(\mathscr{M}) \to KQ_i(\mathscr{M}), \qquad i = 0, 1.$$

The *Witt groups* are defined as the cokernels

$$WQ_i(\mathscr{M}) = WQ_i^{\epsilon,\Lambda}(\mathscr{M}) = \mathrm{coker}(\mathbb{H}_i).$$

From now on, we shall omit ϵ, Λ whereever possible.

For $(M, [h]) \in Q(\mathscr{M})$ one proves easily (Bass [3] I, 4.8)

$$(M, [h]) \perp (M, [-h]) \cong \mathbb{H}(M).$$

Hence, every element of $KQ_1(\mathscr{M})$ can be represented by $(\mathbb{H}(M), f)$ with $f \in \mathrm{Aut}(\mathbb{H}(M))$. Therefore, the inclusion $\mathbb{H}(\mathscr{M}) \hookrightarrow Q(\mathscr{M})$ induces a canonical isomorphism

$$K_1(\mathbb{H}(\mathscr{M})) \cong KQ_1(\mathscr{M}).$$

(See Bass [2, VII 2.2].)

For $M \in \mathscr{M}$ there are group homomorphisms

$$X_+ : \Lambda_{M^*} \to \mathrm{Aut}(\mathbb{H}(M)), \qquad b \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$X_- : \Lambda_M \to \mathrm{Aut}(\mathbb{H}(M)), \qquad c \mapsto \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.$$

The elements of $X_+(\Lambda_{M^*})$ and $X_-(\Lambda_M)$ generate a subgroup of $\mathrm{Aut}(\mathbb{H}(M))$ the elements of which are called *elementary* isometries of $\mathbb{H}(M)$. In the same way as in Bass [3, II, Sect. 5] one can show that elementary isometries represent the trivial element of $KQ_1(\mathscr{M})$. Moreover, we shall need the following simple statement [3, II, 2.5]:

1.3 Lemma. *If $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an isometry of $\mathbb{H}(M)$ and $a \in \mathrm{End}(M)$ is invertible, then $ca^{-1} \in \Lambda_M$, $a^{-1}b \in \Lambda_{M^*}$ and $f = X_-(ca^{-1})\, \mathbb{H}(a)\, X_+(a^{-1}b)$.* ∎

For $M = M_1 \oplus \cdots \oplus M_n$ and $a \in \mathrm{Hom}(M_j, M_i)$ let ae_{ij} be the endomorphism of M, whose matrix has a in the (i, j)-th position and 0 elsewhere. Let $E(M_1, ..., M_n)$ denote the subgroup of $\mathrm{Aut}(M)$ generated by all elements $1 + ae_{ij}$, $i \neq j$. The elements of this subgroup are called *elementary automorphisms*. Since elementary automorphisms vanish in $K_1(\mathscr{M})$, the *hyperbolic elementary isometries*, i.e. the elements of $\mathbb{H}(E(M_1, ..., M_n))$, vanish in $KQ_1(\mathscr{M})$.

2. Reduction and Transfer

Let $\mathscr{M}$, $\mathscr{M}'$ be additive categories with duality functors * and form parameters (ϵ, Λ), (ϵ', Λ'), respectively. A *duality preserving functor* is an additive functor F: $\mathscr{M} \to \mathscr{M}'$ with a natural isomorphism $\phi = (\phi_M)_{M \in \mathscr{M}} : F^* \to {}^*F$ such that $\phi_M F(\Lambda_M) \subset \Lambda'_{F(M)}$ for all $M \in \mathscr{M}$. We call F *strictly duality preserving* if $\phi_M F(\Lambda_M) = \Lambda'_{F(M)}$. If Λ and Λ' are both maximal or minimal, then $\phi_M F(\Lambda_M) \subset \Lambda'_{F(M)}$ if $\phi_{M^*} = \epsilon\epsilon' \phi_M^*$. One proves easily

2.1 Lemma. *A duality preserving functor (F, ϕ) induces a functor*

$$Q(F) : Q^{\epsilon,\Lambda}(\mathscr{M}) \to Q^{\epsilon',\Lambda'}(\mathscr{M}'), \; (M, [h]) \mapsto (F(M), [\phi_M F(h)]).$$

$Q(F)$ preserves orthogonal sums, and $Q(F)\mathbb{H} = \mathbb{H}F$. If F is strictly duality preserving and fully faithful (an equivalence), then $Q(F)$ is fully faithful (an equivalence).

An ideal I of an additive category is a collection of subgroups $I(M, N)$ of $\mathrm{Hom}(M, N)$ for all $M, N \in \mathscr{M}$ such that $gfh \in I(M', N')$ for $f \in I(M, N)$, $g \in \mathrm{Hom}(N, N')$, $h \in \mathrm{Hom}(M', M)$. The factor category $\mathscr{M}/I$ has the same objects as $\mathscr{M}$ and $\mathrm{Hom}_{\mathscr{M}/I}(M, N) := \mathrm{Hom}(M, N)/I(M, N)$. If * is a duality in $\mathscr{M}$, the ideal I is *invariant* if $I(M, N)^* = I(N^*, M^*)$. Then $*$ induces a duality on the factor category. If (ϵ, Λ) is a form parameter we get in a canonical way a form parameter $(\epsilon, \bar{\Lambda})$ in $\mathscr{M}/I$. The canonical reduction functor $^-: \mathscr{M} \to \mathscr{M}/I$ is strictly duality preserving.

According to Kelly [9] there exists exactly one ideal J in an additive category $\mathscr{M}$ such that $J(M, M) = \operatorname{rad} \operatorname{End}(M)$ is the Jacobson radical for all $M \in \mathscr{M}$. We call J the *radical* of $\mathscr{M}$. (See also Roberts [21].) The radical J is invariant.

We shall now formulate and prove a reduction theorem of G. E. Wall [26], C. T. C. Wall [25], and Bak in the situation of an additive category $\mathscr{M}$ with duality $*$ and form parameter (ϵ, Λ). Let I be an invariant ideal contained in the radical J. Consider $\mathscr{M}/I$ with the induced duality and form parameter.

2.2 THEOREM. *The reduction functor*

$$Q^{\epsilon,\Lambda}(\mathscr{M}) \to Q^{\epsilon,\bar{\Lambda}}(\mathscr{M}/I)$$

satisfies:

(1) *Every nonsingular quadratic module over $\mathscr{M}/I$ is the image of a nonsingular quadratic module over $\mathscr{M}$.*

(2) *Assume that* $\operatorname{End}(M)$ *is $I(M, M)$-adically complete for all M. If for $(M, [h]), (M', [h']) \in Q(\mathscr{M})$ there exists an isometry*

$$\bar{f}: (M, [\bar{h}]) \to (M', [\bar{h}'])$$

in $Q(\mathscr{M}/I)$, then there exists an isometry $g: (M, [h]) \to (M', [h'])$ in $Q(\mathscr{M})$ with $\bar{g} = \bar{f}$.

Proof. A morphism h in $\mathscr{M}$ is an isomorphism if and only if $\bar{h}$ is an isomorphism. This proves (1).

(2) By assumption we have

$$f^*h'f \equiv h \pmod{\Lambda_M + I(M, M^*)}.$$

As in 1.2 we consider $E = \operatorname{End}(M)$ with the involution ^{0} defined by $h_0 = h + \epsilon h^*$, and the form parameter $(1, \Lambda_E)$ with $\Lambda_E = \{h_0^{-1}h \mid h \in \Lambda_M\}$. For $a = h_0^{-1}h$ and $a' = h_0^{-1}f^*h'f$ we have

$$a' \equiv a \pmod{\Lambda_E + I}, \quad I = I(M, M).$$

Let $e_1 = 1$. We construct inductively a sequence $e_1, e_2, \ldots$ of units in E such that

$$e_n{}^0 a' e_n \equiv a \pmod{\Lambda_E + I^n}, \qquad e_{n+1} \equiv e_n \pmod{I^n}.$$

We assume that we have found e_n, and hence $d = a + s + r$ for $d = e_n{}^0 a' e_n$ with $s \in \Lambda_E$ and $r \in I^n$. Let

$$t = (d + d^0)^{-1} r \qquad \text{and} \qquad e_{n+1} := e_n(1 - t).$$

Then $t \in I^n$, $e_{n+1} \equiv e_n \pmod{I^n}$ and using $dt + d^0 t = r$ and $t^0 dt \in I^{n+1}$ we get

$$\begin{aligned} e^0_{n+1} a' e_{n+1} &= (1 - t^0)\, d(1 - t) = d - t^0 d - dt + t^0\, dt \\ &= a + s + (d^0 t - t^0 d) + t^0\, dt \equiv a \pmod{\Lambda_E + I^{n+1}}. \end{aligned}$$

Since E is I-adically complete, $e = \lim e_n$ exists and

$$e^0 a' e = a \pmod{\Lambda_E}, \qquad e \equiv 1 \pmod{I}.$$

$g = fe$ satisfies the required conditions:

$$g \equiv f \pmod{I}, \qquad g^* h' g \equiv h \pmod{\Lambda_M}. \quad ▮$$

2.3 Corollary. *Under the assumptions of 2.2 (2) the reduction functor* $^-$: $\mathcal{M} \to \mathcal{M}/I$ *induces a bijection between the isomorphism classes of objects in* $Q(\mathcal{M})$ *and* $Q(\mathcal{M}/I)$. *In particular, there are canonical isomorphisms*

$$KQ_0(\mathcal{M}) \cong KQ_0(\mathcal{M}/I); \qquad WQ_0(\mathcal{M}) \cong WQ_0(\mathcal{M}/I).$$

Furthermore $\alpha: KQ_1(\mathcal{M}) \to KQ_1(\mathcal{M}/I)$ *is an epimorphism, and* $\bar{\alpha}: WQ_1(\mathcal{M}) \to WQ_1(\mathcal{M}/I)$ *is an isomorphism.*

Proof. It remains to show the injectivity for the WQ_1-groups. Because $K_1(\mathcal{M}) \to K_1(\mathcal{M}/I)$ is surjective, every element $\bar{x} \in \ker(\bar{\alpha})$ can be lifted to an element $x \in \ker(\alpha)$. Hence $\bar{x}$ can be represented by $(\mathbb{H}(M), f)$ with

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pmod{I}.$$

Therefore a is invertible, and by 1.3 the isometry f is a product of elementary and hyperbolic isometries. ▮

For explicit computations (as in the proof of 2.2) it is often convenient to pass from the abstract category to the endomorphism ring $\mathrm{End}(M)$ of some object. This is the principle of transfer.

Let $(N, h_0) \in H^{\epsilon_0}(\mathcal{M})$, and let $\mathcal{M}|_N$ denote the full subcategory of $\mathcal{M}$ of all objects isomorphic to some direct summand of $N \oplus \cdots \oplus N$. Since $N \cong N^*$ one has by restriction a duality and a form parameter on $\mathcal{M}|_N$.

An idempotent $e \in \operatorname{End}(M)$ *splits* if there is a diagram $M' \to^{i} M \to^{j} M'$ with $ji = id$ and $ij = e$. In $E = \operatorname{End}(N)$ we have the involution ^{0} induced from h_0 (see 1.2) and the form parameter $(\epsilon\epsilon_0, \Lambda_E)$ with $\Lambda_E = \{h_0^{-1}h \mid h \in \Lambda_N\}$. We consider $\mathscr{P}(E)$ with the corresponding duality * abd the form parameter $(\epsilon\epsilon_0, \Lambda'_E)$ (see 1.1).

2.4 Proposition (Transfer). *The functor*

$$F = \operatorname{Hom}(N, -): \mathscr{M}\,|_N \to \mathscr{P}(E)$$

is fully faithful and strictly duality preserving with respect to

$$(\phi_M)_{M\in\mathscr{M}|_N}: F^* \to {}^*F, \qquad \phi_M(f) = F(h_0^{-1}f^*).$$

If all idempotents in $\mathscr{M}\,|_N$ split, then F is an equivalence. Hence

$$Q(F): Q^{\epsilon,\Lambda}(\mathscr{M}\,|_N) \to Q^{\epsilon\epsilon_0,\Lambda_E}(E)$$

is an equivalence.

Proof. By computation. ▮

For applications of the principle of transfer, *it is crucial to know, which selfdual objects N admit a nonsingular ϵ-hermitian form.*

2.5 Proposition. *Let $N \in \mathscr{M}$ be selfdual, i.e. $N \cong N^*$. Assume that $E =$* $\operatorname{End}(N)$ *is a local ring with maximal ideal J and residue skew field $D = E/J$.*

(1) *If 2 is invertible in E or $J = 0$, then there exists a nonsingular ϵ_0-hermitian form h_0 on N for a suitable $\epsilon_0 \in \{\pm 1\}$.*

(2) *Assume $(N, h_0) \in H^{\epsilon_0}(\mathscr{M})$ and 2 is invertible in E. If there exists an $(N. h) \in H^{-\epsilon_0}(\mathscr{M})$, then the involution on D induced from h_0 is non-trivial. If E is J-adically complete, the converse is true also.*

Proof. (1) Let $g: N \to N^*$ be an isomorphism. If E is local and 2 a unit it follows from $2g = (g + g^*) + (g - g^*)$ that one of the summands is an isomorphism. If E is a skew-field, then $g = g^*$ or $g - g^*$ is an isomorphism.

(2) The existence of an isomorphism $h: N \to N^*$ with $h = -\epsilon_0 h^*$ is equivalent with the existence of an unit $e \in E$ such that $e = -e^0$, where ^{0} is the involution induced by h_0. If E is J-adically complete one can lift such an e from a $d = -d^0 \neq 0$ in D using 2.2. ▮

3. The Krull-Schmidt Theorem

We assume in this section that the additive category $\mathscr{M}$ satisfies the following conditions:

(i) All idempotents in $\mathcal{M}$ split.

(ii) Every $M \in \mathcal{M}$ has a decomposition $M \simeq N_1 \oplus \cdots \oplus N_r$ with N_i indecomposable and $\operatorname{End}(N_i)$ local.

Then the Krull-Schmidt theorem holds in $\mathcal{M}$, i.e. the decomposition of M in (ii) is unique up to isomorphism and permuting the N_i. (See Bass [2, I, 3.6].)

Let Σ be a set of indecomposable modules in $\mathcal{M}$. A module M is called of type Σ (of type Σ') if every indecomposable direct summand of M is isomorphic to some element of Σ (if no indecomposable direct summand is isomorphic to some element of Σ).

Given a decomposition of M into indecomposable direct summands, one obtains a unique decomposition $M \simeq M_1 \oplus \cdots \oplus M_s$ with M_i *isotypic*, i.e. M_i of type $\{N_i\}$ with the N_i indecomposable and pairwise non-isomorphic.

To prove a corresponding result for nonsingular quadratic modules, one needs the following result about $J = J(\mathcal{M})$.

3.1 Lemma. (1) *If $N, N' \in \mathcal{M}$ are indecomposable and not isomorphic, then* $J(N, N') = \operatorname{Hom}(N, N')$, *and every morphism $N \to^f N' \to^g N$ lies in the radical of* $\operatorname{End}(N)$.

(2) *If M_1, M_2 are of type Σ and M_1', M_2' are of type Σ', then*

$$\operatorname{Hom}_{\mathcal{M}/J}(M_1 \oplus M_1', M_2 \oplus M_2') = \operatorname{Hom}_{\mathcal{M}/J}(M_1, M_2) \oplus \operatorname{Hom}_{\mathcal{M}/J}(M_1', M_2')$$

and if $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$: $M_1 \oplus M_1' \to M_2 \oplus M_2'$ is an isomorphism, then a and d are isomorphisms.

Proof. (1) If gf were not in the radical of the local ring $\operatorname{End}(N)$, gf would be an isomorphism and N' a direct summand of N. The rest of (1) follows from [9], Lemma 6. (2) is an obvious consequence of (1). ∎

3.2 Theorem. *Every nonsingular quadratic module $(M, [h]) \in Q(\mathcal{M})$ has an orthogonal decomposition*

$$(M, [h]) \simeq \mathop{\perp}_{i=1}^{n} (M_i, [h_i])$$

with M_i of type $\{N_i, N_i^*\}$, N_i indecomposable, and $N_i \oplus N_i^* \not\simeq N_j \oplus N_j^*$ for $i \neq j$.

Proof. See the proof of 3.2 in [17]. ∎

It is natural to ask, whether this decomposition is unique. In order to prove such a result, we need another condition on $\mathcal{M}$:

(iii) For every M the ring $\operatorname{End}(M)$ is $J(M)$-adically complete, $J(M) = \operatorname{rad} \operatorname{End}(M)$.

3.3 THEOREM. *Assume that* $\mathcal{M}$ *satisfies conditions* (i), (ii), (iii). *Then the following is true:*

(1) *If*

$$(M_1, [h_1]) \perp (M_1', [h_1']) \cong (M_2, [h_2]) \perp (M_2', [h_2'])$$

with M_i *of type* Σ *and* M_i' *of type* Σ', *then*

$$(M_1, [h_1]) \cong (M_2, [h_2]) \quad \textit{and} \quad (M_1', [h_1']) \cong (M_2', [h_2']).$$

In particular, the decomposition in 3.2 *is unique up to isometry.*

(2) *If* N *is indecomposable and selfdual, then there exists a bijection between the isomorphism classes of objects in the categories* $Q^{\epsilon,\Lambda}(\mathcal{M}\,|_N)$ *and* $Q^{\epsilon',\Lambda'}(E/J)$ *for* $E = \operatorname{End}(N)$, $J = \operatorname{rad} E$, *and* $\epsilon' = \pm 1$, Λ' *suitable chosen.*

(3) *If* N *is indecomposable and* $N \not\cong N^*$ (*or* $N \cong N^*$, $\frac{1}{2} \in E$, *there is no* ϵ*-hermitian isomorphism* $N \to N^*$), *then every form* $(M, [h])$ *of type* $\{N, N^*\}$ *is hyperbolic.*

Proof (see also [17, proof of 3.4]). By reduction modulo the radical J of $\mathcal{M}$, one can reduce everything to $\mathcal{M}/J$. The statement (1) becomes trivial because an isomorphism $f: M_1 \oplus M_1' \to M_2 \oplus M_2'$ in $\mathcal{M}/J$ has matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. Statement (2) follows from 2.5 (1) and transfer. If $N \not\cong N^*$ and $M = N^r \oplus N^{*r}$ the matrix of h (in $\mathcal{M}/J$) is of form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. This easily implies the assertion. In the second case of (3), there exists a $(-\epsilon)$-hermitian isomorphism $h_0: N \to N^*$. Using transfer with $\overline{h_0}$ one gets an equivalence

$$Q^{\epsilon,\Lambda}((\mathcal{M}/J)\,|_N) \to Q^{-1,\Lambda_D}(D), \qquad D = \operatorname{End}_{\mathcal{M}/J}(N).$$

By 2.5 (2), D is a field with trivial involution and characteristic $\neq 2$. Hence we are in the symplectic case: all forms are hyperbolic. ∎

3.4 APPLICATIONS. Using splitting into isotypic components and reduction modulo the radical the classification problem of quadratic modules is reduced to the classification problem over skew fields. Therefore we get e.g. the following statements:

(1) *The Witt cancellation theorem is true in* $Q(\mathcal{M})$ (because the cancellation theorem holds over skew fields [4]). In fact, the cancellation theorem follows by 2.4 using the cancellation theorem of Reiter [19] without the completeness assumption (iii).

(2) We call a submodule $i: N \hookrightarrow M$ totally isotropic if $[i^*hi] = 0$. *Every quadratic module* $(M, [h])$ *has a decomposition—unique up to isometry—*

$$(M, [h]) \cong (M', [h']) \perp \mathbb{H}(M_F)$$

where M' *has no totally isotropic direct summand.* (Witt decomposition)

(3) *If $\mathcal{M}$ is a k-category, k an algebraically closed field, then* $(M, [h]) \simeq (M', [h'])$ *if and only if* $M \simeq M'$. (See [17, 3.5].)

(4) The last result is an extremely useful one: Application to the category of vector bundles over a complete k-scheme gives a well-known result of Grothendieck. Application to the category $\mathrm{Aut}(\mathcal{M})$ gives: *If* $\mathrm{char}(k) \neq 2$, *then two elements in* $\mathrm{Aut}(M, [h])$ *are conjugate if and only if they are conjugate in* $\mathrm{Aut}(M)$.

(5) For the Witt group one obtains a decomposition

$$WQ_0^{\epsilon,\Lambda}(\mathcal{M}) = \bigoplus_i WQ_0^{\epsilon\epsilon_i,\Lambda_i}(D_i)$$

where i runs through the isomorphism classes of selfdual indecomposable modules N_i and $D_i = \mathrm{End}(N_i)/\mathrm{rad}$.

We prove now a result giving a decomposition of isometries into isotypic components.

3.5 THEOREM. *Assume that the conditions* (i), (ii), (iii) *are satisfied.*

(1) *Let Σ be a selfdual set of indecomposable modules. If M is of type Σ and M' is of type Σ', then every isometry of $\mathbb{H}(M \oplus M')$ is a product of elementary isometries, hyperbolic-elementary isometries, and of isometries of*

$$\mathrm{Aut}(\mathbb{H}(M)) \times \mathrm{Aut}(\mathbb{H}(M')) \hookrightarrow \mathrm{Aut}(\mathbb{H}(M \oplus M')).$$

(2) *If M and M^* do not admit isomorphic indecomposable direct summands, then every isometry of $\mathbb{H}(M)$ is a product of elementary and hyperbolic isometries.*

Proof. (1) Let $^-$ denote reduction modulo J. We have

$$\mathrm{Aut}(\mathbb{H}(\bar{M} \oplus \bar{M}')) = \mathrm{Aut}(\mathbb{H}(\bar{M})) \times \mathrm{Aut}(\mathbb{H}(\bar{M}'))$$

and by 2.3 we have a surjection

$$\mathrm{Aut}(\mathbb{H}(M)) \to \mathrm{Aut}(\mathbb{H}(\bar{M})).$$

Therefore, for every $f \in \mathrm{Aut}(\mathbb{H}(M \oplus M'))$ there exists a $g \in \mathrm{Aut}(\mathbb{H}(M)) \times \mathrm{Aut}(\mathbb{H}(M'))$ with $\bar{f} = \bar{g}$. Hence $f_1 := fg^{-1} \equiv id (\mathrm{mod}\ J)$. If $f_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Aut}(\mathbb{H}(M \oplus M'))$, then $a \equiv 1 \pmod J$. Hence $a \in \mathrm{End}(M \oplus M')$ is invertible, and by 1.3 we get

$$f_1 = X_-(ca^{-1})\, \mathbb{H}(a)\, X_+(a^{-1}b).$$

For $a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{Aut}(M \oplus M')$ then a_1, a_4 must be invertible. Therefore a is a product of a diagonal matrix and elementary matrices. Hence $\mathbb{H}(a)$ is product of hyperbolic elementary matrices and an element of $\mathrm{Aut}(\mathbb{H}(M)) \times \mathrm{Aut}(\mathbb{H}(M'))$.

(2) If $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then a is invertible and the assertion follows from 1.3. ∎

4. Bilinear Spaces with Systems of Subspaces

Let I be a finite partially ordered set, k a field, and $\mathscr{V}$ the category of finite-dimensional k-vector spaces. An *I-space* $V = (V, V_i, i \in I)$ is a finite-dimensional k-vector space together with a family of subspaces V_i such that $i \leqslant j$ implies $V_i \subset V_j$. Morphisms $f: V \to V'$ are linear maps satisfying $f(V_i) \subset V'_i$. The category $\mathscr{V}(I)$ of I-spaces is a k-category and satisfies the assumptions (i), (ii), (iii) of Section 3.

Now let $\perp$ be an involution on I, i.e. a map $\perp: I \to I$ such that $i \leqslant j$ implies $j^\perp \leqslant i^\perp$ and $i^{\perp\perp} = i$. Define the dual of an I-space as

$$(V, V_i)^* := (V^*, V^0_{i^\perp})$$

where $U^0 = \{f \in V^* \mid f|_U = 0\}$ for a subspace U. We consider the category $H^\epsilon(\mathscr{V}(I), *) =: H^\epsilon(I, \perp)$.

I is of *finite representation type* over k if there exist up to isomorphism only finitely many indecomposable I-spaces. For char$(k) \neq 2$ we call $(I, \perp, \epsilon)$ of finite representation type if there exist up to isometry and multiplication by a scalar only finitely many indecomposable modules in $H^\epsilon(I, \perp)$.

The determination of the sets of finite representation type—due to Nazarova—Roiter [14], Kleiner [10], and Gabriel [6]— is achieved by a reduction process which can be outlined as follows: Assume that I does not contain four unrelated elements. If a is a maximal element of I, one can construct a new partially ordered set $I(a)$ and if $V_a \neq 0$ one can associate with every I-space (V, V_i) an $I(a)$-space $F_a(V)$ with V_a as underlying vector space. The additive functor F_a satisfies the following lemma (Gabriel [6]):

4.1 Lemma. (1) *Every $I(a)$-space is of the form $F_a(V)$ for a suitable I-space V.*

(2) *The canonical map*

$$\mathrm{Hom}_I(U, V) \to \mathrm{Hom}_{I(a)}(F_a(U), F_a(V))$$

is an epimorphism.

(3) *A morphism f between I-spaces is an isomorphism if and only if $F_a(f)$ is an isomorphism.*

It follows from this lemma that I (with no four unrelated elements) is of finite representation type if and only if the reduction process $I \to I(a)$ eventually leads to the empty set. If there are four unrelated elements, I is (rather trivially) of infinite representation type.

4.2 Lemma. *Assume I is of finite representation type. Then,* $\mathrm{End}_I(V) = k$ *for all indecomposable V.*

Proof. If $V_i = 0$ for all $i \in I$, then $\dim(V) = 1$ and the assertion is trivial. Otherwise, we have $V = \sum V_a$, summed over a maximal, since V is indecomposable. It follows that for every $f \neq 0$ in $\mathrm{End}_I(V)$ there is a maximal element a_1 with $f_1 = f \mid V_{a_1} \neq 0$. Then, V_{a_1} is indecomposable and after finitely many steps one is reduced to the trivial case above, and f_n is an isomorphism. By 4.1 (3), f is an isomorphism, i.e. $\mathrm{End}(V)$ is a skew field. Using 4.1 (2) we get $\mathrm{End}(V) = k$. ∎

We can now answer a question posed in [23].

4.3 THEOREM. *Assume* $\mathrm{char}(k) \neq 2$. $(I, \perp, \epsilon)$ *is of finite representation type if and only if I is of finite representation type.*

Proof. If I is of infinite representation type, then $(I, \perp, \epsilon)$ is of infinite representation type (apply the hyperbolic functor). If $(V, h) \in H^\epsilon(I, \perp)$ is indecomposable, then by 3.2, (V, h) is hyperbolic or V is indecomposable. If I is of finite representation type, it suffices to show that there exists up to multiplicative equivalence at most one (skew-) symmetric form on every indecomposable selfdual I-space. This follows from the last lemma. ∎

As a typical example of infinite (and tame) representation type we consider now the set I consisting of four unrelated points with different involutions. We want to determine

(i) the selfdual indecomposable objects N,

(ii) a nonsingular (skew-) symmetric form h on N,

(iii) the involution on $\mathrm{End}(N)/\mathrm{rad} =: \overline{\mathrm{End}(N)}$ induced by h.

Using this information and the results of Section 3, the classification problem will be reduced to skew fields. We continue to assume $\mathrm{char}(k) \neq 2$.

The category of I-spaces is closely related to the category $\mathrm{End}(\mathscr{V})$ which we shall discuss first. The objects are pairs (V, f), $f \in \mathrm{End}_k(V)$, where morphisms and duality are defined in the obvious way. $H^\epsilon(\mathrm{End}(\mathscr{V}))$ is the category of ϵ-symmetric bilinear spaces with a selfadjoint endomorphism. $\mathrm{End}(\mathscr{V})$ is equivalent to the category of finitely generated $k[X]$-torsion modules. Hence a complete system of indecomposables is given by

$$k[X]/(p^n) \qquad p \text{ monic irreducible polynomial.}$$

For a polynomial q of degree $d > 0$ let $N_q = k[X]/(q)$ and define $s_q\colon N_q \to k$ by $s_q(x^j) = 0$ for $j = 0, \ldots, d-2$, $s_q(x^{d-1}) = 1$. We get the symmetric bilinear module $(N_q, h_q) \in H^1(\mathrm{End}(\mathscr{V}))$ with

$$h_q(n, n') := s_q(nn') \qquad \text{for} \quad n, n' \in N_q.$$

We have $\mathrm{End}(N_q) \cong k[X]/(q)$ and h_q induces the trivial involution. Application of 3.2 gives the following result:

4.4 PROPOSITION. *Every element of* $\mathrm{End}(\mathscr{V})$ *admits a nonsingular* 1-*hermitian form*; $H^{-1}(\mathrm{End}(\mathscr{V}))$ *has only hyperbolic modules.*

For a four subspace system $(V; V_1, \ldots, V_4)$ the *defect* is defined as $2\dim(V) - \Sigma \dim(V_i)$. Selfdual I-spaces have defect 0. There is a fully faithful embedding

$$E\colon \mathrm{End}(\mathscr{V}) \to \mathscr{V}(I)$$
$$(V,f) \to (V \oplus V;\ V \oplus 0,\ 0 \oplus V,\ \mathrm{graph}(id),\ \mathrm{graph}(f)).$$

Furthermore, for every n there is the indecomposable space U_n with $\mathrm{End}(U_n) = k[X]/(X^{n+1})$,

$$U_n = (k^{n+1} \oplus k^n;\ k^{n+1} \oplus 0,\ 0 \oplus k^n,\ \mathrm{graph}(s_n),\ \mathrm{graph}^t(t_n))$$

with

$$s_n = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \ddots \\ 0 & & & 1 \end{pmatrix}, \qquad t_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{pmatrix},$$

$\mathrm{graph}^t(f) = \{(f(x), x)\}$.

It is well known that an indecomposable space of defect 0 is isomorphic to some space from $E(\mathrm{End}(\mathscr{V}))$ or to a space obtained from some U_n by permuting the subspaces ([8], [7], [5]).

Assume now the the involution $^\perp$ leaves all four points of I fixed. Then the U_n are not selfdual, and E is duality preserving with the natural isomorphism

$$\phi_{(V,f)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

between $E(V^*, f^*)$ and

$$E(V,f)^* = (V^* \oplus V^*;\ 0 \oplus V^*,\ V^* \oplus 0,\ \mathrm{graph}(-id),\ \mathrm{graph}^t(-f^*)).$$

ϕ is skew-symmetric, i.e. $\phi_{(V^*,f^*)} = -\phi^*_{(V,f)}$. Therefore E defines an embedding

$$H^1(\mathrm{End}(\mathscr{V})) \to H^{-1}(I, \perp).$$

Using 4.4 we get

4.5 THEOREM. *For* $I = \circ\circ\circ\circ$ *with the trivial involution, every selfdual I-space admits a nonsingular skew-symmetric form.* $H^1(I)$ *consists only of hyperbolic elements.* ▮

By a similar discussion one can solve another case:

4.6 THEOREM. *For $I = \bullet\bullet\bullet\bullet$ and the involution switching the first and the last two points, every selfdual I-space admits a nonsingular symmetric form, and $H^{-1}(I)$ consists only of hyperbolic modules.*

Proof. Let $\mathscr{V}_{\text{aut}}$ ($\mathscr{V}_{\text{nil}}$) be the subcategories of End($\mathscr{V}$) of all bijective (nilpotent) endomorphisms, so that one has a decomposition End($\mathscr{V}$) $= \mathscr{V}_{\text{aut}} \times \mathscr{V}_{\text{nil}}$. Let E: $\mathscr{V}_{\text{aut}} \to \mathscr{V}(I)$ be as defined before, and define E': $\mathscr{V}_{\text{nil}} \to \mathscr{V}(I)$ by

$$E'(V,f) := (V \oplus V;\ V \oplus 0,\ \text{graph}(f),\ 0 \oplus V,\ \text{graph}(id)).$$

These embeddings are duality preserving with respect to the natural symmetric isomorphisms

$$\phi_{(V,f)} = \begin{pmatrix} -f^* & 0 \\ 0 & 1 \end{pmatrix}: E(V^*,f^*) \to E(V,f)^*$$

$$\phi'_{(V,f)} = \begin{pmatrix} -f^* & 1 \\ 1 & -1 \end{pmatrix}: E'(V^*,f^*) \to E'(V,f)^*.$$

U_n and $U'_n = (k^{n+1} \oplus k^n;\ k^{n+1} \oplus 0,\ \text{graph}^t(t_n),\ 0 \oplus k^n,\ \text{graph}(s_n))$ are selfdual; other permutations of the subspaces give nonselfdual spaces. Nonsingular symmetric forms on U_n, U'_n are given by

$$h = \begin{pmatrix} b_{n+1} & 0 \\ 0 & -b_n \end{pmatrix}, \qquad h' = \begin{pmatrix} b_{n+1} & -b_{n+1}t_n \\ -b_n s_n & b_n \end{pmatrix}, \qquad \text{where } b_n = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

because $b_n s_n = t_n^* b_{n+1}$. Since $\overline{\text{End}(U_n)} = k$, the forms h, h' are uniquely determined up to multiplicative equivalence, and the theorem follows. ∎

The last case where the involution switches two points and leaves two points fixed is slightly more complicated. E defines a duality preserving embedding of Aut($\mathscr{V}$) (with duality $(V,f) \mapsto (V^*, f^{*-1})$) in $\mathscr{V}(I)$. The classification of hermitian modules is then obtained from the classification of ϵ-symmetric bilinear spaces with isometry. (See Milnor [13]). We refer to [24] for more details.

5. Hermitian Forms in Abelian Categories

Let $\mathscr{A}$ be an abelian category and $\mathscr{M}$ an *admissable* subcategory, i.e. a full additive subcategory such that for every exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathscr{A}$ with M_2, $M_3 \in \mathscr{M}$, also $M_1 \in \mathscr{M}$. (See Bass [2, Chap. VII].) Let * be a duality functor on $\mathscr{M}$ which preserves short exact sequences.

Let (M, h) be a (not necessarily nonsingular) ϵ-hermitian module over $\mathscr{M}$. If $i: N \hookrightarrow M$ is a submodule in $\mathscr{M}$, the *orthogonal space* $N^\perp$ is defined as $N^\perp = \ker(i^*h)$. We call N *admissable* if $\operatorname{coker}(i) \in \mathscr{M}$. We call N *totally isotropic* if $N \subset N^\perp$ and N is admissable.

5.1 LEMMA. *For (M, h) as above assume* $\operatorname{coker}(h) \in \mathscr{M}$. *Then we have in $\mathscr{M}$ the short exact sequence $0 \to M^\perp \to^j M \to^k M/M^\perp \to 0$, and there is exactly one nonsingular ϵ-hermitian form $\bar{h}$ on $M/M^\perp$ such that $k^*\bar{h}k = h$.*

Proof. $\operatorname{Coker}(h) \in \mathscr{M}$ implies that $M/M^\perp \cong \operatorname{Im}(h)$ lies in $\mathscr{M}$, hence $M^\perp$ lies in $\mathscr{M}$ also. We have an injection $h_0: M/M^\perp \to M^*$ with $h = h_0 k$. Since $j^*h_0k = \epsilon(hj)^* = 0$ we have $j^*h_0 = 0$. From the exact sequence

$$0 \longrightarrow (M/M^\perp)^* \xrightarrow{k^*} M^* \xrightarrow{j^*} M^{\perp *} \longrightarrow 0$$

we obtain $\bar{h}: M/M^\perp \to (M/M^\perp)^*$ with $k^*\bar{h} = h_0$, hence $k^*\bar{h}k = h$. This equation determines $\bar{h}$ uniquely and $\bar{h} = \epsilon\bar{h}^*$. Since h_0 is an injection, $\bar{h}$ is an injection, $\bar{h}$ is an injection. Because $\operatorname{coker}(h) \in \mathscr{M}$ and $h = \epsilon h^*$ we have an exact sequence

$$0 \longrightarrow M^\perp \xrightarrow{j} M \xrightarrow{h} M^* \xrightarrow{j^*} M^{\perp *} \longrightarrow 0.$$

Therefore, $\bar{h}$ is surjective also. ∎

5.2 LEMMA. *For $(M, h) \in H^\epsilon(\mathscr{M})$ and an admissable submodule $i: N \hookrightarrow M$ the following is true*:

(1) $N^\perp \cong (M/N)^* \in \mathscr{M}$ *and* $N^{\perp\perp} = N$.

(2) *If $h\,|_N$ is nonsingular, then*

$$(M, h) \cong (N, h\,|_N) \perp (N^\perp, h\,|_{N^\perp}).$$

(3) *If N is totally-isotropic, then $N^\perp/N \in \mathscr{M}$ and h induces a uniquely determined ϵ-hermitian form $\bar{h}_{N^\perp}$ on $N^\perp/N$.*

Proof. The easy proof of (1) and (2) is omitted. (3) follows from the last lemma. ∎

$(M, h) \in H^\epsilon(\mathscr{M})$ is called *metabolic*, if there exists an admissable submodule N with $N = N^\perp$. For reasons which we do not know, N is called a *Lagrangian submodule* and $((M, h), N)$ a *metabolic pair*. For $N \in \mathscr{M}$ and $a: N^* \to N$ ϵ-hermitian the module

$$\mathbb{H}^\epsilon(N, a) := \left(N \oplus N^*, \begin{pmatrix} 0 & 1 \\ \epsilon & a \end{pmatrix}\right)$$

is called a *split metabolic module*. (Obviously, N is a Lagrangian submodule.)

5.3 LEMMA. (1) *If N is a totally isotropic submodule of (M, h), then*

$$(M, h) \perp (N^{\perp}/N, -\bar{h}_{N^{\perp}})$$

is metabolic with $N^{\perp}$ diagonally embedded as a Lagrangian submodule.

(2) *If $0 \to N_1 \to^{i} N \to^{j} N_2 \to 0$ is an exact sequence in $\mathscr{M}$, then $N_1 \oplus N_2^*$ is a Lagrangian submodule of $\mathbb{H}^{\epsilon}(N) := \mathbb{H}^{\epsilon}(N, 0)$.*

Proof. (1) Let $N \to^{i} N^{\perp} \to^{j} M$ be the injections, $k: N^{\perp} \to N^{\perp}/N$ the canonical surjection and $\bar{h}$ the induced form on $N^{\perp}/N$. It suffices to show the exactness of

$$0 \longrightarrow N^{\perp} \xrightarrow{\binom{j}{k}} M \oplus N^{\perp}/N \xrightarrow{(j^*h, -k^*\bar{h})} N^{\perp *} \longrightarrow 0.$$

Exactness at $N^{\perp}$ and $N^{\perp *}$ is obvious. Moreover, $(j^*h, \; -k^*\bar{h})\binom{j}{k} = j^*hj - k^*\bar{h}k = 0$ by the definition of $\bar{h}$. Hence

$$N^{\perp} \hookrightarrow K := \ker(j^*h, -k^*\bar{h}).$$

On the other hand, we have for $\binom{j'}{k'}: K \to M \oplus N^{\perp}/N$ the equation $(ji)^*hj' = (ki)^*\bar{h}k' = 0$ and hence $K \hookrightarrow \ker((ji)^*h) = N^{\perp}$.

(2) The assertion follows from the exact sequence

$$0 \longrightarrow N_1 \oplus N_2^* \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & j^* \end{pmatrix}} N \oplus N^* \xrightarrow{\begin{pmatrix} 0 & i^* \\ \epsilon j & 0 \end{pmatrix}} N_1^* \oplus N_2 \longrightarrow 0. \quad \blacksquare$$

5.4 LEMMA. (1) *If $N = N^{\perp}$ is a direct summand of M, then $(M, h) \cong \mathbb{H}^{\epsilon}(N, a)$ for some ϵ-hermitian $a: N^* \to N$. If h is even, then $(M, h) \cong \mathbb{H}^{\epsilon}(N)$.*

(2) *There exists an isometry*

$$F: \mathbb{H}(N) \perp \mathbb{H}(N, -a) \cong \mathbb{H}(N, a) \perp \mathbb{H}(N, -a)$$

Proof. The usual proof carries over. F is given by

$$F = \begin{pmatrix} 1 & 0 & -1 & \epsilon a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad F^{-1} = \begin{pmatrix} 1 & \epsilon a & 1 & -\epsilon a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \quad \blacksquare$$

We have the Grothendieck groups $KH_i^{\epsilon}(\mathscr{M}) = K_i(H^{\epsilon}(\mathscr{M}))$, $i = 0, 1$ of the category with product $H^{\epsilon}(\mathscr{M})$. We define the *Grothendieck group* $GH_0^{\epsilon}(\mathscr{M})$ as the factor group of $KH_0^{\epsilon}(\mathscr{M})$ modulo the subgroup generated by all

$$[M, h] - [\mathbb{H}^{\epsilon}(N)], \qquad ((M, h), N) \text{ metabolic pair.}$$

This group is the analogue of the Grothendieck group $G_0(\mathcal{M})$ which is defined with respect to all exact sequences [2, Chap. VIII]. By 5.3 (2) the hyperbolic functor induces a homomorphism

$$\mathbb{H}_0^\epsilon \colon G_0(\mathcal{M}) \to GH_0^\epsilon(\mathcal{M})$$

the cokernel of which is the *Witt group* $GW_0^\epsilon(\mathcal{M})$. This construction is due to Quillen [18].

5.5 COROLLARY. *If N is a totally isotropic submodule of (M, h), the following equation holds in $GH_0^\epsilon(\mathcal{M})$:*

$$[M, h] = [\mathbb{H}(N)] + [N^\perp/N, \bar{h}_{N^\perp}].$$

Proof. 5.3(1) and 5.4(2). ∎

We consider now the K_1 and W_1 groups. By definition, we have

$$KH_1^\epsilon(\mathcal{M}) = KH_0^\epsilon(\mathrm{Aut}(\mathcal{M}))/R,$$

where R is the subgroup generated by all elements

$$[M, h, ff'] - [M, h, f] - [M, h, f']; f, f' \in \mathrm{Aut}(M, h).$$

Let $\bar{R}$ denote the image of R under the canonical epimorphism $KH_0^\epsilon(\mathrm{Aut}(\mathcal{M})) \to GH_0^\epsilon(\mathrm{Aut}(\mathcal{M}))$. We define

$$GH_1^\epsilon(\mathcal{M}) := GH_0^\epsilon(\mathrm{Aut}(\mathcal{M}))/\bar{R}.$$

The hyperbolic functor induces homomorphisms $G_0(\mathrm{Aut}(\mathcal{M})) \to GH_0^\epsilon(\mathrm{Aut}(\mathcal{M}))$ and $\mathbb{H}_1^\epsilon \colon G_1(\mathcal{M}) \to GH_1^\epsilon(\mathcal{M})$. We define the *Witt group*

$$GW_1^\epsilon(\mathcal{M}) = \mathrm{coker}(\mathbb{H}_1^\epsilon).$$

5.6 THEOREM. *If $\mathcal{M}$ is a semisimple category (i.e. every short exact sequence splits), then the canonical epimorphisms*

$$KH_i^\epsilon(\mathcal{M}) \to GH_i^\epsilon(\mathcal{M}),\ i = 0, 1$$

are isomorphisms.

Proof. For $i = 0$ this is an immediate consequence of 5.4. Let $(N, f_N) \in \mathrm{Aut}(\mathcal{M})$ be a Lagrangian submodule of $(M, f, h) \in H^\epsilon(\mathrm{Aut}(\mathcal{M}))$. To prove the theorem we have to show

$$[M, f, h] = [\mathbb{H}(N), \mathbb{H}(f_N)] \text{ in } KH_1^\epsilon(\mathcal{M}).$$

Using 5.4 we can assume

$$(M, h) = \mathbb{H}^{\epsilon}(N, a) \qquad \text{and} \qquad f = \begin{pmatrix} f_N & b \\ 0 & f_N^{*-1} \end{pmatrix}$$

where $b\colon N^* \to N$ satisfies

$$a = f_N^{-1} a f_N^{*-1} + \epsilon f_N^{-1} b + (f_N^{-1} b)^*.$$

We stabilize with the isometry

$$f' = \begin{pmatrix} f_N & -b \\ 0 & f_N^{*-1} \end{pmatrix}$$

of $\mathbb{H}^{\epsilon}(N, -a)$. Using the isometry F of 5.4 we get

$$F^{-1}(f \oplus f')F = (\mathbb{H}(f_N) \oplus f')X, \qquad X = \begin{pmatrix} 1 & 0 & 0 & \epsilon(f_N^{-1}b)^* \\ 0 & 1 & 0 & 0 \\ 0 & -f_N^{-1}b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This yields in $KH_1^{\epsilon}(\mathscr{M})$ the equation

$$\begin{aligned}[]&[\mathbb{H}(N, a), f] + [\mathbb{H}(N, -a), f'] \\ &\quad = [\mathbb{H}(N), \mathbb{H}(f_N)] + [\mathbb{H}(N, -a), f'] + [\mathbb{H}(N) \perp \mathbb{H}(N, -a), X].\end{aligned}$$

It remains to prove that X vanishes in $KH_1^{\epsilon}(\mathscr{M})$. We identify $\mathbb{H}(N) \perp \mathbb{H}(N, -a)$ with $\mathbb{H}(N \oplus N, A)$, $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -a \end{smallmatrix}\right)$. Then X corresponds to

$$X_+(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \epsilon b^* f_N^{*-1} \\ -f_N^{-1}b & 0 \end{pmatrix}.$$

As in [3, II, Sect. 5] one proves that this represents the trivial element of $KH_1^{\epsilon}(\mathscr{M})$. ∎

5.7 Example. Let A be a ring with involution. The last result applies in particular to the subcategory $\mathscr{P}(A)$ of all finitely generated projective modules in the abelian category of all modules. More generally, one could consider for a scheme X the subcategory of all locally free $\mathscr{O}_X$-modules in the category of all quasi-coherent sheaves. (See Knebusch [12].).

6. The Jordan–Hölder Theorem

In this section $\mathscr{M}$ will be an abelian category with a duality functor *. It is easy to see that * is an exact functor. First we establish hermitian versions of the isomorphism theorems and the Zassenhaus lemma, and then prove a hermitian Jordan-Hölder theorem. (M, h) will always be an object of $H^\epsilon(\mathcal{M})$.

6.1 LEMMA. *Let N and L be submodules of M.*

(1) *If $N \subset L$, then $L^\perp \subset N^\perp$ and $N^\perp/L^\perp \cong (L/N)^*$.*

(2) *For $N \cap L := \ker(M \to M/N \oplus M/L)$ and $N + L := \operatorname{im}(N \oplus L \to M)$ one has*

$$(N \cap L)^\perp = N^\perp + L^\perp \qquad \textit{and} \qquad (N + L)^\perp = N^\perp \cap L^\perp.$$

This can be proved easily using 5.2. ∎

For a submodule N let $\bar{h} = \bar{h}_N$ denote the induced form on $\bar{N} := N/(N \cap N^\perp)$ (see 5.1).

6.2 PROPOSITION. (First isometry theorem). *Let N, N_0 be submodules such that $N^\perp = N + N_0$. Then $N_0 \cap N_0^\perp = N_0 \cap N$, and there is a canonical isometry*

$$(\bar{N}_0, \bar{h}) \xrightarrow{\cong} (\bar{N}^\perp, \bar{h}).$$

Proof. By the last lemma we have $N_0 \cap N = N_0 \cap (N + N_0)^\perp = N_0 \cap N^\perp \cap N_0^\perp = N_0 \cap N_0^\perp$. From the commutative diagram

$$\begin{array}{ccc} N_0 & \xrightarrow{\quad i \quad} & N + N_0 = N^\perp \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} \\ N_0/(N_0 \cap N) & \xrightarrow[j]{\cong} & (N + N_0)/N \end{array}$$

and 5.1 we get

$$k^*\bar{h}_{N_0}k = h\,|_{N_0} = i^*h\,|_{N^\perp}\, i = i^*l^*\bar{h}_{N^\perp}l^*i^*$$
$$= k^*j^*\bar{h}_{N^\perp}jk.$$

Since k is an epimorphism and k^* a monomorphism, we have $\bar{h}_{N_0} = j^*\bar{h}_{N^\perp} j$, and hence the canonical isomorphism j is an isometry. ∎

6.3 PROPOSITION (Zassenhaus lemma). *If N, L are totally isotropic submodules of M, then*

$$N_1 := N + (N^\perp \cap L), \qquad L_1 := L + (N \cap L^\perp)$$

are totally isotropic and there is a canonical isometry

$$(N_1^\perp/N_1, \bar{h}) \cong (L_1^\perp/L_1, \bar{h}).$$

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Proof. $N \subset N^\perp$ implies $N_1^\perp = N^\perp \cap (N + L^\perp) = N + (N^\perp \cap L^\perp) \supset N_1$. Defining $N_0 := N^\perp \cap L^\perp$ we have $N_1 + N_0 = N + N_0 = N_1^\perp$. Using the first isometry theorem, we get $(\bar{N}_1^\perp, \bar{h}) \simeq (\bar{N}_0, \bar{h})$. Since N_0 is symmetric in N and L, we get the assertion. ▮

6.4 COROLLARY. (1) *If N is a maximal totally isotropic submodule, then $(\bar{N}^\perp, \bar{h})$ is unique up to isometry.*

(2) *If (M_1, h_1) and $(M_1, h_1) \perp (M_2, h_2)$ are metabolic, then (M_2, h_2) is metabolic.*

(3) $[M, h] = 0$ *in $GW_0^\epsilon(\mathscr{M})$, if and only if (M, h) is metabolic.*

Proof. (1) follows immediately from 6.3 and (3) follows from (2). Concerning (2), let N, L be Lagrangian submodules of M_1, $M = M_1 \oplus M_2$, respectively. For $N \subset N^\perp = N \oplus M_2$ in M we consider $N_1 = N + (N^\perp \cap L)$ and get from 6.3 that $N_1^\perp = N_1$. Hence, we can replace L by N_1; that is, we can assume $N \subset L$. Consequently $L = L^\perp \subset N^\perp = N \oplus M_2$ and hence $L = N \oplus (M_2 \cap L)$. Therefore $M_2 \cap L$ is a Lagrangian submodule of M_2. ▮

6.5 PROPOSITION. (Second isometry theorem). *Let N, L be submodules of M such that $N \subset L \subset N^\perp$. Then*

(1) $(L/N)^\perp = L^\perp/N$ *in $(\bar{N}^\perp, \bar{h})$. Every totally isotropic submodule $\bar{L}$ of $\bar{N}^\perp$ lifts to a totally isotropic submodule L of M such that $N \subset L$ and $\bar{L} = L/N$.*

(2) *For $L \subset L^\perp$ there is a canonical isometry $j: L^\perp/L \to (L^\perp/N)/(L/N)$.*

Proof. (1) Let $i: L \hookrightarrow N$, $\bar{i}: L/N \hookrightarrow N^\perp/N$ be the canonical maps. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\perp & \longrightarrow & N^\perp & \xrightarrow{i^*h} & L^* & & \\ & & & & \downarrow & & \uparrow & & \\ 0 & \longrightarrow & (L/N)^\perp & \longrightarrow & N^\perp/N & \xrightarrow{\bar{i}^*\bar{h}} & (L/N)^* & \longrightarrow & 0. \end{array}$$

This gives a surjection $L^\perp \to (L/N)^\perp$ and hence $L^\perp/N = (L/N)^\perp$ in $\bar{N}^\perp = N^\perp/N$. If we lift $\bar{L}$ to a submodule L with $N \subset L \subset N^\perp$, we get $N \subset L^\perp \subset N^\perp$ and $L/N \subset L^\perp/N$; hence $L \subset L^\perp$.

(2) If $l: L^\perp \to L^\perp/L$ is the canonical epimorphism, one sees easily $l^*\bar{h}_{L^\perp} l = l^*j^*\bar{h}_{(L/N)^\perp} jl$, and hence $\bar{h}_{L^\perp} = j^*\bar{h}_{(L/N)^\perp} j$ so that j is an isometry. ▮

We consider filtrations. Let $\mathscr{M}_0$ be a full subcategory of the abelian category $\mathscr{M}$. Assume that $\mathscr{M}_0$ is closed with respect to the duality and to taking sub- and factor-modules. A *hermitian $\mathcal{M}_0$-filtration* of (M, h) consists of a totally isotropic submodule N such that $N^{\perp}/N \in \mathcal{M}_0$ and a finite $\mathcal{M}_0$-filtration (N_i) of N

$$0 = N_0 \subset N_1 \subset \cdots \subset N_r = N, \qquad N_i/N_{i-1} \in \mathcal{M}_0 .$$

Because of $N_{i-1}^{\perp}/N_i^{\perp} \cong (N_i/N_{i-1})^* \in \mathcal{M}_0$ we have a $\mathcal{M}_0$-filtration of M:

$$0 = N_0 \subset \cdots \subset N_r \subset N_r^{\perp} \subset \cdots \subset N_0^{\perp} = M.$$

A second filtration $(L_j)_{j=1,\ldots,r}$ is called *hermitian equivalent* to (N_i) if $(\bar{N}_r^{\perp}, \bar{h}) \cong (\bar{L}_r^{\perp}, \bar{h})$ and $N_i/N_{i-1} \cong L_{\sigma(i)}/L_{\sigma(i)-1}$ for a suitable permutation σ. The Zassenhaus lemma implies a refinement theorem, just as in the linear theory. Using the Zassenhaus lemma the proof is an obvious adaption of the proof in the linear case.

6.6 THEOREM (Jordan-Hölder-Schreier). *Any two hermitian $\mathcal{M}_0$-filtrations of $(M, h) \in H^{\epsilon}(\mathcal{M})$ admit hermitian equivalent refinements.* ∎

Using the Jordan-Hölder theorem 6.6 one can compute the Grothendieck groups by "dévissage". The following considerations are modelled on Bass [2, Chap. VIII, Sect. 3]. We make the following assumptions:

(A_0) Every $N \in \mathcal{M}$ has a $\mathcal{M}_0$-filtration.

(B_0) For every (M, h) there is a totally isotropic submodule N with $N^{\perp}/N \in \mathcal{M}_0$.

It follows that every (M, h) has a hermitian $\mathcal{M}_0$-filtration. Under the assumption (A_0) the inclusion $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ induces an isomorphism $G_0(\mathcal{M}_0) \cong G_0(\mathcal{M})$ [2, VIII, 3.3].

6.7 THEOREM. *If (A_0) and (B_0) are satisfied, the inclusion $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ induces an isomorphism*

$$g\colon GH_0^{\epsilon}(\mathcal{M}_0) \to GH_0^{\epsilon}(\mathcal{M}).$$

Proof. We shall construct an inverse homomorphism g'. For (M, h), choose a hermitian $\mathcal{M}_0$-filtration (N_i). We claim

$$g'(M, h) := \sum_{i=1}^{r} [\mathbb{H}(N_i/N_{i-1})] + [\bar{N}_r^{\perp}, \bar{h}]$$

in $GH_0^{\epsilon}(\mathcal{M}_0)$ is independent of the choice of the filtration. By the refinement theorem it suffices to prove this for a filtration where one submodule L is added to the submodules N_i. Two cases can occur: If ... $N_i \subset L \subset N_{i+1}, \ldots$, we get the claim from 5.3 (2) applied to $0 \to L/N_i \to N_{i+1}/N_i \to N_i/L \to 0$. If $N_r \subset L \subset L^{\perp} \subset N_r^{\perp}$, then by the second isometry theorem, L/N_r is totally isotropic in

$(\bar{N}_r^{\perp}, \bar{h})$, and the claim follows from the second isometry theorem and the relation in 5.5:

$$[\bar{N}_r^{\perp}, \bar{h}] = [\mathbb{H}(L/N_r)] + [(L/N_r)^{\perp}/(L/N_r), \bar{h}]$$
$$= [\mathbb{H}(L/N_r)] + [L^{\perp}/L, \bar{h}].$$

Now, it follows easily that $gg'(M, h) = [M, h]$ in $GH_0^{\epsilon}(\mathscr{M})$. Since $g'g = id$ will be obvious it remains to prove that g' induces a homomorphism for the Grothendieck groups. The compatibility with orthogonal sums is obvious. It remains to show that $g'(M, h) = g'(\mathbb{H}(N))$ for a metabolic pair $((M, h), N)$. One chooses a $\mathscr{M}_0$-filtration for $N = N^{\perp}$, and it follows that

$$g'(M, h) = \sum [\mathbb{H}(N_i/N_{i-1})] = g'(\mathbb{H}(N)). \quad \blacksquare$$

To prove an analogous result for GH_1 one needs stronger assumption:

(A_1) Every module $N \in \mathscr{M}$ has a *characteristic* $\mathscr{M}_0$-filtration, i.e. an $\mathscr{M}_0$-filtration (N_i) such that $f(N_i) = N_i$ for all $f \in \mathrm{Aut}(N)$.

(B_1) For every (M, h) there is a totally isotropic submodule N such that $N^{\perp}/N \in \mathscr{M}_0$ and $f(N) = N$ for all isometries.

6.8 THEOREM. *If* (A_1) *and* (B_1) *are satisfied, the inclusion* $\mathscr{M}_0 \hookrightarrow \mathscr{M}$ *induces isomorphisms*

$$g_0 \colon GH_0^{\epsilon}(\mathrm{Aut}(\mathscr{M}_0)) \to GH_0^{\epsilon}(\mathrm{Aut}(\mathscr{M}))$$
$$g_1 \colon GH_1^{\epsilon}(\mathscr{M}_0) \to GH_1^{\epsilon}(\mathscr{M}).$$

Proof. The statement concerning g_0 follows from the last theorem applied to $\mathrm{Aut}(\mathscr{M})$. Let

$$g_0' \colon GH_0^{\epsilon}(\mathrm{Aut}(\mathscr{M})) \to GH_0^{\epsilon}(\mathrm{Aut}(\mathscr{M}_0))$$

be the inverse homomorphism as defined in the proof of 6.7. One has to prove

$$g_0'[M, h, ff'] = g_0'[M, h, f] + g_0'[M, h, f'] \quad \text{in} \quad GH_1^{\epsilon}(\mathscr{M}_0).$$

Choose N and a characteristic filtration N_i, $i = 1,..., r$, as in (B_1), (A_1). By definition of g_0' and the relations in $GH_1^{\epsilon}(\mathscr{M}_0)$ we get, in fact,

$$g_0'[M, h, ff'] = [\bar{N}_r^{\perp}, \bar{h}, \overline{ff'}] + \sum [\mathbb{H}(N_i/N_{i-1}), \mathbb{H}(\overline{ff'})]$$
$$= [\bar{N}_r^{\perp}, \bar{h}, \bar{f}] + [\bar{N}_r^{\perp}, \bar{h}, \bar{f}']$$
$$+ \sum [\mathbb{H}(N_i/N_{i-1}), \mathbb{H}(\bar{f})] + \sum [\mathbb{H}(N_i/N_{i-1}), \mathbb{H}(\bar{f}')]$$
$$= g_0'[M, h, f] + g_0'[M, h, f']. \quad \blacksquare$$

Since we used the assumptions (A) only for hyperbolic modules, we get:

6.9 COROLLARY. *If* (B_i) *is satisfied, the inclusion* $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ *induces isomorphisms*

$$GW_i^\epsilon(\mathcal{M}_0) \to GW_i^\epsilon(\mathcal{M}), \qquad i = 0, 1.$$

We assume now that all modules in $\mathcal{M}$ are of *finite length*. Let $\mathcal{M}_0$ be the subcategory of *semisimple* objects. $\mathcal{M}_0$ satisfies our assumptions.

6.10 THEOREM. *The assumptions* (A_1) *and* (B_1) *are satisfied for* $\mathcal{M}_0 \hookrightarrow \mathcal{M}$. *Hence there are canonical isomorphisms*

$$GH_i^\epsilon(\mathcal{M}_0) \cong GH_i^\epsilon(\mathcal{M}).$$

Proof. A characteristic filtration of N is obtained by taking for N_1 the socle $s(M)$, i.e. the sum of all simple submodules. N_i is defined inductively by $N_i/N_{i-1} = s(M/N_{i-1})$. To prove (B_1) one uses the *hermitian socle*

$$s'(M, h) = s(M) \cap s(M)^\perp.$$

By 6.5 we can find totally isotropic submodules N_i such that

$$N_i/N_{i-1} = s'(N_{i-1}^\perp/N_{i-1}, \bar{h}), \qquad N_0 = 0.$$

Since M is of finite length, $N_{r+1} = N_r$ for some r. By the following lemma $N_r{}^\perp/N_r$ is semisimple. Obviously, the N_i are stable under all isometries. ∎

6.11 LEMMA. $s'(M, h) = 0$ *if and only if* M *is semisimple.*

Proof. Assume $s'(M, h) = 0$, and consider $N := s(s(M)^\perp)$. Since N is semisimple we have $N \subset s(M)$, hence $N = 0$. It follows that $s(M)^\perp = 0$, that is $M = s(M)$ semisimple. ∎

The computation of $GH_i^\epsilon(\mathcal{M})$ is now trivially reduced to skew-fields: Let (N_j) be a system of representatives of the *simple* modules, and let $D_j = \mathrm{End}(N_j)$. Since every $M \in \mathcal{M}_0$ is canonically a direct sum of isotypic submodules we have

$$\mathcal{M}_0 \cong \bigoplus_j \mathcal{M}_0 \mid N_j \cong \bigoplus_j \mathcal{P}(D_j).$$

One has to distinguish the cases $N_j \cong N_j^*$ and the hyperbolic case $N_j \not\cong N_j^*$, and as in Section 3 the reduction to the skew-fields D_j is obvious.

From Lemma 6.11 we can also conclude the following final result concerned with an individual ϵ-hermitian module rather than the Grothendieck group.

6.12 THEOREM. *Assume that all modules in* $\mathcal{M}$ *are of finite length. Then every (possible singular)* ϵ*-hermitian module* (M, h) *contains a maximal totally isotropic* *submodule N. The module* $(\overline{N}^{\perp}, \bar{h})$ *is uniquely determined up to isometry; it is nonsingular and anisotropic.* $\overline{N}^{\perp}$ *is semisimple.*

Proof. Because of 5.1 and 6.4 it remains to prove that $\overline{N}$ is semisimple. This follows from 6.11. ∎

Problems. (1) It would be interesting to know whether a suitable embedding theorem can be formulated and proved for an abelian category with duality.

(2) In most concrete situations, the duality is represented by some object. Can this be formulated in a categorical way?

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