over *all* of $\psi_Y^{-1}(V)$. Therefore, using the additivity axiom, we obtain

$$\begin{aligned} i(K(f) \times K(g), \psi_X^{-1}(U) \times \psi_Y^{-1}(V)) \\ &= \sum_{j=1}^{r} i(K(f) \times K(g), \psi_X^{-1}(U) \times C_j) \\ &= \sum_{j=1}^{r} i(K(f)_{y_j'}, \psi_X^{-1}(U)) \cdot i(K(g), C_j) \qquad \text{for } y_j' \in \bar{C}_j \\ &= i(K(f)_{y'}, \psi_X^{-1}(U)) \sum_{j=1}^{r} i(K(g), C_j) \\ &= i(K(f)_{y'}, \psi_X^{-1}(U)) \cdot i(K(g), \psi_Y^{-1}(V)) \end{aligned}$$

for any $y' \in \psi_Y^{-1}(\bar{V})$. Consequently, equations (1), (2), and (3) imply that

$$i(f \times g, U \times V) = i(f_y, U) \cdot i(g, V)$$

for any $y \in \bar{V}$. ||

10. The Index on Acyclic Spaces [Section 80]

Theorem 10.1 [28]. *Let X be a compact ANR which is acyclic, i.e., $H^0(X) \cong \mathbf{Q}$ and $H^p(X) = 0$ for $p \neq 0$, and let $f : X \to X$ be a map such that (X, f, U) is an admissable triple. If $g : \bar{U} \to X$ is a map such that $g(x) = f(x)$ for all $x \in \partial U$, then $i(f, U) = i(g, U)$.*

Proof. Define $G : X \to X$ by

$$G(x) = \begin{cases} g(x) & \text{if } x \in \bar{U}, \\ f(x) & \text{if } x \in X - U. \end{cases}$$

By the additivity axiom

$$L(f) = i(f, U) + i(f, X - \bar{U}),$$
$$L(G) = i(G, U) + i(G, X - \bar{U}).$$

Now $L(f) = L(G) = 1$ because X is acyclic. Since $G = f$ on $X - U$, we have $i(f, X - \bar{U}) = i(G, X - \bar{U})$. Consequently,

$$i(f, U) = i(G, U) = i(g, U). \quad ||$$

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Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*

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In this paper I give new proofs of the structure theorems for the unoriented cobordism ring [12] and the complex cobordism ring [8, 13]. The proofs are elementary in the sense that no mention of the Steenrod algebra or Adams spectral sequence is made. In fact, the only result from homotopy theory which is used in an essential way is the Serre finiteness theorem in order to know that the complex cobordism group of a given dimension is finitely generated.

The technique used here capitalizes on the fact that there are two rather different approaches to defining operations in the complex and unoriented cobordism generalized cohomology theories. The first proceeds via characteristic classes and leads to the Landweber–Novikov operations [6, 9], while the second is the analog of the Steenrod power method due to tom Dieck [14]. Using the technique of "localization at the fixpoint set" (Atiyah-Segal [1], tom Dieck [15, 16]), it is possible to derive an equation expressing the Steenrod operation in terms of the Landweber–Novikov operations in which the Steenrod operation is zero modulo terms of high filtration. One thereby obtains nontrivial relations involving the action of the Landweber–Novikov operations on the cobordism ring which can be used to show that the cobordism ring is generated by the coefficients of the formal group law expressing the behavior of cobordism Euler classes of line bundles under tensor product. From this, Lazard's results [7] on formal group laws can be applied to neatly prove that the two cobordism rings are polynomial rings.

The paper also contains two new results of interest. The main theorem of the paper shows that the reduced complex cobordism $\tilde{U}^*(X)$ of a finite complex is generated by its elements of positive degree as a module over the complex cobordism ring. By duality this implies that

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the complex bordism of a finite complex of dimension r is generated by its elements of degree $\leqslant 2r$ as a module over the complex cobordism ring, which answers a question posed by Conner and Smith [3]. The other result is the construction of a canonical ring isomorphism (announced in [10])[1]

$$N^*(X) \cong N^*(pt) \otimes H^*(X; Z_2),$$

where N^* is the oriented cobordism theory. In addition, it is shown how the formal group law of N^* furnishes a distinguished system of polynomial generators for the unoriented cobordism ring.[1]

The first two sections contain a review of complex cobordism theory, cobordism characteristic classes, and the Landweber–Novikov operations. I have been strongly influenced by Grothendieck's theory of motives in algebraic geometry (see [4] for some aspects of this theory) and like to think of a cobordism theory as a universal contravariant functor on the category of C^∞ manifolds endowed with Gysin homomorphism for a class of proper "oriented" maps, instead of as the generalized cohomology theory given by a specific Thom spectrum. One will find a precise assertion in Proposition 1.10 which suffices for the needs of this paper but which is far from being systematic. The third section is devoted to a review of the "localization at the fixpoint set" formalism and to the derivation of the basic formula (3.17) relating the Steenrod and Landweber–Novikov operations. The proof of the main theorem occupies the fourth and fifth sections and the theory of formal group laws is brought in at the end. I have included in Section 6 an exposition of Lazard's theorem, more intelligible to topologists than the one in [7], which was given by Adams at the 1969 Arbeitstagung.[2]

I would like to acknowledge the benefit of a year's study with A. Grothendieck at the Institut des Hautes Études Scientifiques, and also the influence of the papers of T. tom Dieck, who kindly provided me with copies of his work. I discovered how to use Steenrod operations in conjunction with formal group laws to handle the unoriented cobordism ring while visiting the Mathematics Institute of Aarhus University during August, 1969, and I am very grateful for the hospitality shown to me by everyone there. The extension to the complex cobordism ring was worked out later, and has been done independently by tom Dieck.

[1] After writing this paper, I discovered that these results are contained in an old (1967) unpublished paper of J. M. Boardman.

[2] Adams' own account is now available [17] and is better than the one here.

1. Geometric Interpretation of $U^*(X)$

By a manifold we mean a C^∞ manifold which can be embedded as a closed C^∞ submanifold of some Euclidean space. Maps of manifolds will always be C^∞.

Given a topological space X, let $U^*(X)$ and $U_*(X)$ be the complex cobordism and bordism, respectively, of X, i.e., the generalized cohomology and homology of X with values in the Thom spectrum MU. One knows very well how to interpret elements of $U_*(X)$ as bordism classes of maps $f : M \to X$ where M is a closed weakly-complex manifold. It will be convenient to have a similar geometric picture for cobordism elements. For this it will be necessary to suppose that X is a manifold; however, this assumption does not represent much loss of generality since any finite complex is of the homotopy type of a manifold, viz., a regular neighborhood of an embedding into Euclidean space.

Let us recall what is meant by a *complex orientation* for a map of manifolds $f : Z \to X$, this being a generalization of a weakly-complex structure on Z when X is a point. Suppose first that at each point z of Z the dimension of f, defined to be (dim Z at z) − (dim X at $f(z)$), is even. Then by a complex orientation of f we mean an equivalence class of factorizations of f

$$Z \xrightarrow{i} E \xrightarrow{p} X, \tag{1.1}$$

where $p : E \to X$ is a complex vector bundle over X and where i is an embedding endowed with a complex structure on its normal bundle ν_i. The factorization 1.1 is considered to be equivalent to another one denoted by primes, if E and E' can be embedded as subvector-bundles of an E'', such that, in E'', i and i' are isotopic compatibly with the normal complex structure, that is, the isotopy is given by an embedding $i'' : X \times I \to E'' \times I$ over I endowed with a complex structure on its normal bundle which matches to that of i and i' in E'' at the ends. Given a factorization 1.1, where the dimension of E is sufficiently large, the standard embedding and isotopy theorems imply that one obtains each complex orientation of f from exactly one homotopy class of complex structures on ν_i. A complex orientation for a map of odd dimension will be defined as one for the map $(f, \epsilon) : Z \to X \times \mathbf{R}$, where $\epsilon(Z) = 0$ or, equivalently, an equivalence class of factorizations of the form 1.1 but with E replaced by $E \times \mathbf{R}$. For a general map, f we define a complex orientation to be one for $f' : Z' \to X$ and $f'' : Z'' \to X$, where $Z = Z' \coprod Z''$ and f' (resp., f'') is the even (resp., odd) dimensional part of f.

It is clear that if $f : Z \to X$ is a complex-oriented map and if $g : Y \to X$ is a map which is transversal to f, then the pull-back $Y \times_X Z \to Y$ has an induced complex orientation. Let us call two proper (inverse image of any compact set is compact) complex-oriented maps $f_i : Z_i \to X$, $i = 0, 1$, *cobordant* if there is a proper complex-oriented map $b : W \to X \times \mathbf{R}$ such that the map $\epsilon_i : X \to X \times \mathbf{R}$, $\epsilon_i(x) = (x, i)$, is transversal to b and such that the pull-back of b by ϵ_i is isomorphic with the induced complex orientation to f_i for $i = 0, 1$. Cobordism is an equivalence relation and there is the following generalization of Thom's celebrated theorem [12] expressing cobordism groups as homotopy groups.

PROPOSITION 1.2. *For a manifold X, $U^q(X)$ is canonically isomorphic to the set of cobordism classes of proper complex-oriented maps of dimension $-q$.*

The proof follows closely that of Thom's theorem, which is the case when X is a point and is left to the reader. At the same time, one can check that the structure of $U^*(X)$ admits the following description in terms of cobordism classes.

1.3. *Contravariant variance.* Let $g : Y \to X$ be a map of manifolds, and let $f : Z \to X$ be a proper complex-oriented map. By Thom's transversality theorem, g may be moved by a homotopy until it is transversal to f. The cobordism class of the pull-back $Y \times_X Z \to Y$ depends on the cobordism class of f, and this gives the map

$$g^* : U^q(X) \to U^q(Y)$$

for each q.

1.4. *Covariant variance (the Gysin homomorphism).* A proper complex-oriented map $g : X \to Y$ of dimension d induces a map

$$g_* : U^q(X) \to U^{q-d}(Y)$$

which sends the cobordism class of $f : Z \to X$ into the class of $gf : Z \to Y$.

1.5. *Addition.* The sum of the maps $f_i : Z_i \to X$, $i = 1, 2$, is the class of the map $Z_1 \coprod Z_2 \to X$ with components f_i. The negative of the cobordism class of $f : Z \to X$ is the cobordism class of f endowed with the negative complex orientation, which is defined for f of even dimension as follows. Let the orientation of f be represented by a factorization $Z \xrightarrow{i} \mathbf{C}^n \times X \to X$ with complex structure on ν_i; then the negative orientation is represented by the same factorization, with the same complex structure on ν_i, but with the new complex structure on $\mathbf{C}^n$ given by $i(z_1, \ldots, z_n) = (iz_1, \ldots, iz_{n-1}, -iz_n)$.

1.6. *Products.* The external product $x_1 \otimes x_2 \in U^*(X_1 \times X_2)$, where x_i is the cobordism class of $f_i : Z_i \to X_i$, is the class of the map $f_1 \times f_2 : Z_1 \times Z_2 \to X_1 \times X_2$. The ring structure of $U^*(X)$ is given by $x_1 \cdot x_2 = \Delta^*(x_1 \otimes x_2)$, where $\Delta : X \to X \times X$ is the diagonal.

The two variances 1.3 and 1.4 can be used to characterize the functor U^* on the category of manifolds as we shall now describe. Let h be a contravariant functor from the category of manifolds to the category of sets with $g^* : h(X) \to h(Y)$ denoting the induced map corresponding to a map $g : Y \to X$. Suppose also that for each proper complex-oriented map $f : Z \to X$, there is given a map $f_* : h(Z) \to h(X)$ such that the following conditions are satisfied:

1.7. Assume that

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & X \end{array}$$

is a cartesian square of manifolds, where g is transversal to f and suppose that f is proper and complex-oriented and f' is endowed with the pull-back of the complex orientation of f. Then

$$g^* f_* = f_*' g'^* : h(Z) \to h(Y).$$

1.8. If $f_0, f_1 : Y \to X$ are homotopic maps, then $f_0^* = f_1^*$.

1.9. If $f : Z \to X$ and $g : X \to Y$ are proper complex-oriented maps, and if gf is endowed with the composite complex orientation, then $(gf)_* = g_* f_*$.

PROPOSITION 1.10. *Given an element a of $h(pt)$, there is a unique morphism $\theta : U^* \to h$ of functors commuting with Gysin homomorphisms and such that $\theta 1 = a$, where $1 \in U^0(pt)$ is the cobordism class of the identity map.*

Let $\pi_X : X \to pt$ and let $x \in U^*(X)$ be the cobordism class of the manifold X represented by a proper complex-oriented map $f : Z \to X$ (note that x and f may have components of different dimension). Then $x = f_* \pi_Z^* 1$ in the notation of 1.4, 1.5. Hence θ on this class must be

$$\theta(x) = f_* \pi_Z^* a \quad \text{in} \quad h(X),$$

which proves the uniqueness of θ. For the existence it is necessary to show that the right side depends only on x. Let $u : W \to X \times \mathbf{R}$ be a proper complex-oriented map which is transversal to $\epsilon_i : X \to X \times \mathbf{R}$ and is such that $f_i : Z_i \to X$ is the pull-back of u by ϵ_i, where $f = f_0$ and $i = 0, 1$. Then in $h(X)$ we have

$$
\begin{aligned}
f_*\pi_{Z_0}^*1 &= \epsilon_0{}^*u_*\pi_W{}^*1 && \text{(by 1.7)}\\
&= \epsilon_1{}^*u_*\pi_W{}^*1 && \text{(by 1.8)} \qquad (1.11)\\
&= f_1{}^*\pi_{Z_1}^*1 && \text{(by 1.7)}
\end{aligned}
$$

showing θ is well-defined. The proof that θ commutes with f^*, f_* is straightforward from the definitions.

The universal property of U^* expressed by 1.10 will be used later in constructing operations. It is possible to characterize the ring structure of U^* by a similar universal property by adding more conditions to 1.8–1.10.

It would have been almost possible to write this paper without ever mentioning the Thom spectrum MU and homotopy theory by defining $U^*(X)$ in the above geometric way. For unoriented cobordism theory $N^*(X)$ this would in fact have been possible. However, we need the following basic result from homotopy theory which does not as yet have a geometric cobordism-type proof.

Proposition 1.12. *If X is of the homotopy type of a finite complex, then $U^q(X)$ is a finitely generated abelian group.*

2. Characteristic Classes in U^*

In this section, we review the construction of characteristic classes and operations in U^* [6, 9]. As in the preceding section, we shall suppose X is a manifold. Vector bundles are assumed to be complex.

Let E be a vector bundle of dimension n over X and let $i : X \to E$ be the zero-section. The element $i^*i_*1 \in U^{2n}(X)$, where $1 \in U^0(X)$ is the cobordism class of the identity, is called the *Euler class* of E and will be denoted $e(E)$.

Proposition 2.1. *Let $f : \mathbf{P}E \to X$ be the projective bundle of lines in E, let $\mathcal{O}(1)$ be the canonical line bundle on $\mathbf{P}E$, whose fiber at $l \subset E$ is the linear*

functions on l, and let $\xi = e(\mathcal{O}(1))$. Then f^* makes $U^*(\mathbf{P}E)$ into a free $U^*(X)$-module with basis $1, \xi, \ldots, \xi^{n-1}$, where $n = \dim E$. Moreover, if $E = L_1 \oplus \cdots \oplus L_n$, where the L_i are line bundles, then

$$\prod_{i=1}^{n} (\xi - e(L_i)) = 0.$$

For a proof, see [2]. Note that this result does not assume anything about the structure of $U^*(pt)$, and in fact an analogous result holds for symplectic cobordism and quaternionic bundles even though the symplectic cobordism ring is unknown.

Let $t_1, t_2, \ldots$ be a sequence of indeterminates with degree $t_i = -2i$. Using 2.1, it is well-known how to associate to a bundle E over X an element $c_t(E)$ of $U^*(X)[\mathbf{t}]$ in a natural way such that

$$
\begin{aligned}
c_t(E \oplus E') &= c_t(E) \cdot c_t(E'),\\
c_t(L) &= \sum_{j \geq 0} t_j e(L)^j, \qquad t_0 = 1,
\end{aligned}
\tag{2.2}
$$

where L is a line bundle. In fact, we have the formula

$$c_t(E) = \text{Norm}\left(\sum_{j \geq 0} t_j e(\mathcal{O}(1))^j\right),$$

where the norm of α is the determinant of the linear transformation of multiplication by α, and this is well-defined since 2.1 implies that $U^*(\mathbf{P}E)[\mathbf{t}]$ is an algebra which is projective and finitely-generated as a $U^*(X)[\mathbf{t}]$-module. Letting $\alpha = (\alpha_1, \alpha_2, \ldots)$ range over all sequences of nonnegative integers with all terms but a finite number equal to zero, we have

$$c_t(E) = \sum_{\alpha} t^\alpha c_\alpha(E) \tag{2.3}$$

where $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots$ and $c_\alpha(E) \in U^{2|\alpha|}(X)$ with $|\alpha| = \sum j\alpha_j$.

If $f : Z \to X$ is a complex-oriented map of even dimension whose orientation is represented by a factorization $Z \xrightarrow{i} E \to X$ as in 1.1, then the difference $f^*E - \nu_i$ represents an element ν_f of $K(Z)$, the Grothendieck group of complex vector bundles on Z, which depends only on the complex orientation of f. When f is of odd dimension with orientation represented by $Z \xrightarrow{i} E \times \mathbf{R} \to X$, let $\nu_f = f^*E - \nu_i$ in $K(Z)$. The Landweber–Novikov operation

$$s_t = \sum t^\alpha s_\alpha : U^*(X) \to U^*(X)[\mathbf{t}] \tag{2.4}$$

is the operation which sends the cobordism class f_*1 of a proper complex-oriented map $f : Z \to X$ into

$$s_t(f_*1) = f_*c_t(\nu_f). \tag{2.5}$$

To see s_t is well-defined, one can use the reasoning 1.11. In fact, if one introduces a new Gysin homomorphism $f_!$ on the functor $X \mapsto U^*(X)[\mathbf{t}]$ by the formula

$$f_!(x) = f_*(c_t(\nu_f) \cdot x),$$

then s_t is the map θ of 1.10 which is compatible with the new Gysin homomorphism, and, moreover, we have the Riemann–Roch type formula

$$s_t(f_*x) = f_*(c_t(\nu_f) \cdot s_t x)$$

for any proper complex-oriented map f.

Recall that a power series $F(T_1, T_2)$ with coefficients in a commutative ring R is said to be a *formal* (commutative) *group law* if the identities

$$\begin{aligned} F(0, T) &= F(T, 0) = 0, \\ F(T_1, F(T_2, T_3)) &= F(F(T_1, T_2), T_3), \\ F(T_1, T_2) &= F(T_2, T_1) \end{aligned} \tag{2.6}$$

hold.

PROPOSITION 2.7. *There is a unique series* $F(T_1, T_2) = \sum_{i,j \geq 0} c_{ij} T_1{}^i T_2{}^j$ *with* $c_{ij} \in U^{2-2i-2j}(pt)$ *such that*

$$e(L_1 \otimes L_2) = F(e(L_1), e(L_2)) \tag{2.8}$$

for any two lines bundles over the same manifold X. Morever, F is a formal group law.

From 2.1 it follows that

$$U^*(\mathbf{C}P^n \times \mathbf{C}P^n) = U^*(pt)[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$$

where z_i is the Euler class of $pr_i{}^*\mathcal{O}(1)$, hence there are unique elements c_{ij}^n such that

$$e(pr_1{}^*\mathcal{O}(1) \otimes pr_2{}^*\mathcal{O}(1)) = \sum_{i,j \leq n} c_{ij}^n z_1{}^i z_2{}^j.$$

One checks that c_{ij}^n does not change as $n \to \infty$ and so one gets a well-defined power series $F(T_1, T_2)$ with coefficients in $U^*(pt)$. Since any line bundle is induced from $\mathcal{O}(1)$ by a map to $\mathbf{C}P^n$ for some n, 2.8 holds. The associativity identity is proved by evaluating the Euler class of $pr_1{}^*\mathcal{O}(1) \otimes pr_2{}^*\mathcal{O}(1) \otimes pr_3{}^*\mathcal{O}(1)$ over $\mathbf{C}P^n \times \mathbf{C}P^n \times \mathbf{C}P^n$ in two ways and letting $n \to \infty$. The other identities are proved in similar fashion.

3. The Fixpoint Formula and Steenrod Operations

In this section, we review the technique of localization at the fixpoint set [1, 15] and use it to derive a basic formula (3.17) expressing the Steenrod power operations [14] in terms of the Landweber–Novikov operations.

Let Y, Z be closed submanifolds of X which intersect *cleanly*, that is, $W = Y \cap Z$ is a submanifold of X and at each point x of W the tangent space of W at x is the intersection of the tangent spaces of Y and Z. Let F be the *excess* bundle of the intersection, i.e., the vector bundle over W which is the quotient of the tangent bundle of X by the sum of the tangent bundles of Y and Z restricted to W. Thus $F = 0$ if and only if Y and Z intersect transversally. If the relevant inclusion maps are denoted

$$\begin{array}{ccc} W & \xrightarrow{j'} & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i}, \\ Y & \xrightarrow[j]{} & X \end{array} \tag{3.1}$$

then F fits into an exact sequence

$$0 \to \nu_{i'} \to j'^*\nu_i \to F \to 0. \tag{3.2}$$

Suppose that $\nu_{i'}$, ν_i, and F are endowed with complex structures compatible with this exact sequence. Then there are Gysin–Thom isomorphisms

$$i_* : U^*(Z) \xrightarrow{\sim} U^{*+a}(X, X - Z)$$
$$i_*' : U^*(W) \xrightarrow{\sim} U^{*+b}(Y, Y - W),$$

where a, b are the real dimensions of ν_i and $\nu_{i'}$, respectively. (When A is a nice closed subset of a manifold X, i.e., A is a strong deformation retract of some neighborhood, then 1.2 generalizes to show that the group

$U^*(X, X - A)$ can be identified with cobordism classes of proper complex-oriented maps $Z \to X$ with image contained in A.) We have the following "clean intersection" formula:

PROPOSITION 3.3. *If $z \in U^*(Z)$, then*

$$j^* i_* z = i_*'(e(F) \cdot j'^* z)$$

in $U^{*+a}(Y, Y - W)$.

For the proof it is clear that we can replace X by a tubular neighborhood of W. Thus we may suppose that 3.1 is of the form

$$\begin{array}{ccc} W & \xrightarrow{j'} & E_1 \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ E_2 & \xrightarrow[j]{} & E_1 \oplus E_2 \oplus F \end{array},$$

where E_1 is a real vector bundle over W with zero section j', E_2 is a complex vector bundle with zero section i', and i and j are the obvious inclusions. Let $i_\epsilon : E_\epsilon \to E_1 \oplus E_2$, $\epsilon = 1, 2$ and $k : E_1 \oplus E_2 \to E_1 \oplus E_2 \oplus F$ be the inclusion maps. Then

$$j^* i_* z = i_2^* k^* k_* i_{1*} z = i_2^*(e(\nu_k) \cdot i_{1*} z).$$

Since $i_2^*(\nu_k) = \pi^* F$ where $\pi : E_2 \to X$ is the projection, this last term can be written

$$\begin{aligned} \pi^*(e(F)) \cdot i_2^* i_{1*} z &= \pi^*(e(F)) \cdot i_*' j'^* z \qquad \text{(by 1.7)} \\ &= i_*'(i'^* \pi^* e(F) \cdot j'^* z) \\ &= i_*'(e(F) \cdot j'^* z) \end{aligned}$$

which proves the proposition.

Let G be a compact Lie group and let $i : Z \to X$ be an embedding of G-manifolds. Then the fixpoint submanifold X^G and Z intersect cleanly, and we get a diagram

$$\begin{array}{ccc} Z^G & \xrightarrow{r_Z} & Z \\ {\scriptstyle i^G}\downarrow & & \downarrow{\scriptstyle i} \\ X^G & \xrightarrow[r_X]{} & X \end{array} \tag{3.4}$$

like 3.1 except that the whole situation is equivariant for G. As $r_Z^*(\nu_i)$ is a G-bundle over a trivial G-space, there is a decomposition

$$r_Z^*(\nu_i) = \nu_{iG} \oplus \mu_i, \tag{3.5}$$

where μ_i is the sum of the eigenbundles corresponding to the nontrivial irreducible representations of G. From 3.2, one sees that μ_i is the excess bundle of the intersection.

Suppose that ν_i is endowed with an equivariant complex structure, so that 3.5 is a direct sum of complex G-bundles. Let h be a multiplicative equivariant cohomology theory for G-spaces with Thom classes for complex G-bundles, such as equivariant K_G theory or the theory U_G^* of tom Dieck [15, 16]. We have in mind the theory $X \mapsto U^*(Q \times_G X)$, where Q is given a principal G-bundle over a manifold B, and where the Gysin homomorphism $f_* : h(X) \to h(Y)$ associated to a proper G-map with equivariant complex orientation (defined just as in Section 1) is the Gysin homomorphism $U^*(Q \times_G X) \to U^*(Q \times_G Y)$ associated to the map $Q \times_G f$. We let $e(\mu_i) \in h(Z^G)$ be the Euler class in the theory h. Exactly the same reasoning used in proving 3.3 works equivariantly, and in fact for $U^*(Q \times_G ?)$ one can apply 3.3 directly to the square obtained by applying $Q \times_G ?$ to 3.4. One obtains the following "restriction to the fixpoint" formula.

PROPOSITION 3.6. *If $z \in h(Z)$, then*

$$r_X^* i_* z = i_*^G(e(\mu_i) \cdot r_Z^* z)$$

in $h(X^G, X^G - Z^G)$.

This formula for an embedding generalizes to an arbitrary proper complex-oriented G-map $f : Z \to X$ by the "Riemann–Roch" argument. Suppose for simplicity that f is of even dimension and that the complex orientation is represented by a factorization

$$Z \xrightarrow{i} E \xrightarrow{p} X, \tag{3.7}$$

where E is a complex G-bundle over X and i is an embedding with an equivariant complex structure on its normal bundle. Let $\mu(E)$ be the sum of the eigen bundles of $r_X^* E$ corresponding to the nontrivial irreducible representations of G, where, as before, r_X is the inclusion of the fixpoint submanifold X^G in X.

PROPOSITION 3.8. *If $z \in h(Z)$, then*

$$e(\mu(E)) \cdot r_X{}^* f_* z = f^G{}_*(e(\mu_i) \cdot r_Z{}^* z)$$

in $h(X^G)$.

To see this, recall that $f_* : h(Z) \to h(X)$ is the composition

$$h(Z) \xrightarrow{i_*} h(E, E - DE) \xrightarrow{p_*} h(X),$$

where DE is the disk bundle of E for some Riemannian metric chosen such that $i(Z)$ is contained in the interior of DE. Here p_* is the inverse of the Thom isomorphism $j_* : h(X) \simeq h(E, E - DE)$. Applying 3.6 to i, j we obtain

$$r_E{}^* i_* z = i^G{}_*(e(\mu_i) \cdot r_Z{}^* z),$$
$$r_E{}^* j_* f_* z = j^G{}_*(e(\mu(E)) \cdot r_X{}^* f_* z).$$

Since $j_* f_* = i_*$, $j^G{}_* f^G{}_* = i^G{}_*$ and $j^G{}_*$ is an isomorphism, the proposition follows.

Remark 3.9. In order to have a formula independent of the choice of the factorization (3.7), it is convenient to form the localized theory $S^{-1}h$ (see [15, 16], where S is the set of Euler classes in $h(pt)$ of the nontrivial irreducible representations of G. Then the formula of the proposition can be written as a Riemann–Roch type formula

$$r_X{}^* f_* z = f^G{}_*(e(\mu_f) \cdot r_Z{}^* z),$$

where $e(\mu_f) = e(\mu_i) \cdot e(\mu(E))^{-1}$ is now well-defined since S has been inverted.

We are going to apply 3.8 to the Steenrod operations in U^* [14]. Let G be a group acting on the set $\{1, 2, \ldots, k\}$ and let h be a G-equivariant theory as above. The *external Steenrod operation*

$$P_{\text{ext}} : U^{-2q}(X) \to h^{-2qk}(X^k) \tag{3.10}$$

is defined by the formula

$$P_{\text{ext}}(f_* 1) = f^k{}_* 1.$$

Here $f : Z \to X$ is a proper complex-oriented map of *even* dimension $2q$ and $f^k : Z^k \to X^k$ is its k-fold product regarded as a G-map, where G

permutes the factors, and where f^k has a natural equivariant complex orientation since the dimension of f is even. To see P_{ext} is independent of the choice of the map f, one can use the argument 1.11. Composing with the diagonal $\Delta : X \to X^k$ we obtain the (internal)*Steenrod operation*

$$\begin{aligned} P : U^{-2q}(X) &\to h^{-2qk}(X), \\ P(f_* 1) &= \Delta^* f^k{}_* 1. \end{aligned} \tag{3.11}$$

The analogy of this definition with Steenrod's construction in ordinary cohomology [11] is especially apparent when $h = U^*(Q \times_G ?)$, where Q is a principal G-bundle over B. In this case, $P(f_* 1)$ is represented by the map g in a pull-back diagram

$$\begin{array}{ccc} W & \longrightarrow & Q \times_G Z^k \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (id, f^k)_G} \\ B \times X & \xrightarrow[d]{} & Q \times_G X^k \end{array},$$

where d, the analog of the "diagonal approximation", is homotopic to $(id, \Delta)_G$ and transversal to $(id, f^k)_G$.

PROPOSITION 3.12. *Suppose G acts transitively on $\{1, \ldots, k\}$, and let ρ denote the corresponding representation of G on the subspace of $(z_1, \ldots, z_k)$ in $\mathbf{C}^k$ such that $\sum z_i = 0$, where G permutes the coordinates. Suppose $f : Z \to X$ is a proper complex-oriented map of dimension $2q$ and that m is an integer larger than the dimension of Z, so that $m\epsilon + \nu_f$ is a vector bundle over Z, well-defined up to isomorphism, where ϵ is the trivial complex line bundle. Then*

$$e(\rho)^m P(f_* 1) = f_* e(\rho \otimes (m\epsilon + \nu_f)) \tag{3.13}$$

in $h^{2m(k-1)-2qk}(X)$.

Since m is large, the complex orientation of f can be represented by a factorization $Z \xrightarrow{i} m\epsilon \xrightarrow{p} X$ together with a complex structure on ν_i, and we have that $\nu_i = m\epsilon + \nu_f$ in the notation of the proposition. Let us apply 3.8 to the equivariant map $f^k = p^k i^k : Z^k \to X^k$. Then $\mu_{(i^k)} = \rho \otimes \nu_i$, $\mu((m\epsilon)^k) = \rho \otimes m\epsilon$ and $\Delta : X \to X^k$ is the fixpoint submanifold for the G-action, since G acts transitively. Hence 3.8 yields the formula 3.13 and the proposition is proved.

Let $G = \mathbf{Z}_k$ be the cyclic group of order k acting cyclically on $\{1, \ldots, k\}$ and let η denote the representation of $\mathbf{Z}_k$ on $\mathbf{C}$, where the generator multiplies by $\exp(2\pi i/k)$. Let $F(T_1, T_2)$ be formal group law of 2.7 and let C be the subring of $U^{ev}(pt)$ generated by the coefficients of F. If i is an integer let $[i]_F(T) \in C[[T]]$ be the operation of "multiplication by i" for the formal group, so that we have the following formulas

$$\begin{aligned} [i]_F(T) &= F(T, [i-1]_F(T)), \\ [1]_F(T) &= T, \\ [i]_F(T) &= iT + \text{higher terms}. \end{aligned} \tag{3.14}$$

In 3.12 we take h to be the equivariant theory $U^*(Q \times_G ?)$. If L is a line bundle over Z on which G acts trivially, then in our notation $e(\rho \otimes L)$ is the Euler class in $U^*(B \times Z)$ of the bundle over $B \times Z$ which is the tensor product of the bundle induced from ρ and the bundle L. Thus setting $v = e(\eta) \in U^2(B)$ we have an expression of the form

$$\begin{aligned} e(\rho \otimes L) &= \prod_{i=1}^{k-1} e(\eta^i \otimes L), \\ &= \prod_{i=1}^{k-1} F([i]_F(v), e(L)), \\ &= w + \sum_{j \geq 1} a_j(v)\, e(L)^j, \end{aligned} \tag{3.15}$$

where $a_j(T) \in C[[T]]$ and

$$w = e(\rho) = (k-1)!\, v^{k-1} + \sum_{j \geq k} b_j v^j \tag{3.16}$$

with $b_j \in C$. If $E = L_1 \oplus \cdots \oplus L_r$ is a sum of line bundles, then with the notation 2.3 one computes that

$$\begin{aligned} e(\rho \otimes E) &= \prod_{i=1}^{r} e(\rho \otimes L_i) \\ &= \sum_{l(\alpha) \leq r} w^{r-l(\alpha)} (a(v))^\alpha\, c_\alpha(E), \end{aligned}$$

where $l(\alpha) = \sum \alpha_j$. By the splitting principle this formula holds for any vector bundle of dimension r. Putting this in the right side of 3.13 and using the definition of the operation s_α (2.5) we obtain the following.

PROPOSITION 3.17. *Let $Q \to B$ be a principal $\mathbf{Z}_k$-bundle and let*

$$P : U^{-2q}(X) \to U^{-2qk}(B \times X)$$

be the Steenrod k-th power operation. Let v be the Euler class of the line bundle over B induced from the character sending the generator to $\exp(2\pi i/k)$, and let w be the Euler class of the bundle induced from the reduced regular representation ρ. Then the Steenrod operation is related to the Landweber–Novikov operations by the formula

$$w^{n+q} Px = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^\alpha\, s_\alpha(x) \tag{3.18}$$

where $x \in U^{-2q}(X)$. Here n is any integer sufficiently large with respect to the dimension of X and q, and the $a_j(T)$ (see 3.15) are power series with coefficients in the subring C of $U^{ev}(pt)$ generated by the coefficients of the formal group law F of 2.7.

4. A Technical Lemma

In order to be able to use the formula 3.18 we need a result (4.4) which is derived in this section.

Fix a positive integer k and let

$$\begin{aligned} \emptyset(T) &= \frac{[k]_F(T)}{T} \\ &= k + d_1 T + d_2 T^2 + \cdots, \end{aligned} \tag{4.1}$$

where $d_j \in C$ and we use the notation of 3.14 and 3.17.

PROPOSITION 4.2. *Let $f : Q \to B$ be a principal $\mathbf{Z}_k$-bundle and let $L = Q \times_{\mathbf{Z}_k} \mathbf{C}$ be the line bundle associated to the character η. Then $f_*1 = \emptyset(e(L))$ in $U^0(B)$.*

Let $j : Q \to L$ be the obvious embedding, so that elements of L may be expressed as products $zj(q)$ with $z \in \mathbf{C}$ and $q \in Q$ modulo the equivalence relation $(\zeta z) j(q) = zj(q\sigma)$, where σ is the generator of $\mathbf{Z}_k$ and $\zeta = \exp(2\pi i/k)$. Let i be the zero-section of L and let $g : L \to B$ be the projection. Then the line bundle $g^*L (= L \times_B L$ with projection $pr_1)$ comes with a tautological section s, which is transversal to zero and vanishes on $i(B)$. Thus g^*L with the trivialization off $i(B)$ furnished by s

extends to a line bundle M over the one-point compactification $L \cup \{\infty\}$, where for simplicity we suppose B is compact, and $e(M) = i_*1$, where

$$i : U^q(B) \cong U^{q+2}(L \cup \{\infty\}, \{\infty\})$$

is the Thom isomorphism. Now the bundle $g^*L^{\otimes k}$ trivialized off $i(B)$ by the section $s^{\otimes k}$ extends to the bundle $M^{\otimes k}$. Consider the section t of $g^*L^{\otimes k}$ given by $t(zj(q)) = (zj(q), z^k j(q)^{\otimes k} - j(q)^{\otimes k})$. This section extends to a section of $M^{\otimes k}$, which is smooth off ∞ and is transversal to zero with zero-set $j(Q)$. Thus

$$\begin{aligned} j_*1 &= e(M^{\otimes k}) \\ &= [k]_F(i_*1) \\ &= i_*1 \cdot \emptyset(i_*1) \\ &= i_*\emptyset(i^*i_*1). \end{aligned}$$

Since $i^*i_*1 = e(L)$, $i_*f_* = j_*$ and i_* is an isomorphism, we conclude that $f_*1 = \emptyset(e(L))$, proving the proposition.

Let Y be a fixed manifold and let h be the theory $h^*(X) = U^*(X \times Y)$. To explain what is happening in the rest of this section, suppose that we are willing to work with the cobordism of infinite complexes such as $B\mathbf{Z}_k$, the classifying space of $\mathbf{Z}_k$. Then there is an exact sequence

$$h^q(pt) \xrightarrow{\emptyset(v)} h^q(B\mathbf{Z}_k) \xrightarrow{v} h^{q+2}(B\mathbf{Z}_k), \tag{4.3}$$

where v is the Euler class of the line bundle associated to η and where the maps are multiplications by the indicated elements. Indeed this follows from the commutative diagram

$$\begin{array}{ccccc} \tilde{h}^{q+1}(S^\infty \times_{\mathbf{Z}_k} S^1) & \to & h^q(B\mathbf{Z}_k) & \xrightarrow{v} & h^{q+2}(B\mathbf{Z}_k) \\ \uparrow \simeq & & \uparrow \emptyset(v) & & \\ \tilde{h}^{q+1}(S^1/\mathbf{Z}_k) & \cong & h^q(pt), & & \end{array}$$

where the top row comes from the Gysin sequence of η, where the bottom arrow is the suspension isomorphism, and where the vertical isomorphism is the map induced by the projection $pr_2 : S^\infty \times_{\mathbf{Z}_k} S^1 \to S^1/\mathbf{Z}_k$, which is a homotopy equivalence since the fiber is contractible. Although the steps in this argument with infinite complexes can be justified,

we have preferred to derive a slightly less conceptual variant of 4.3 using only manifolds and, therefore, more in the spirit of this paper.

Let $\mathbf{Z}_k$ act on $S^{2n-1} \subset \mathbf{C}^n$ with the generator multiplying by $\exp(2\pi i/k)$, and let $v_n \in h^2(S^{2n-1}/\mathbf{Z}_k)$ be the Euler class of the line bundle induced from η. Let $j_n : S^{2n-1}/\mathbf{Z}_k \to S^{2n+1}/\mathbf{Z}_k$ be induced by the inclusion of $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$. The variant of 4.3 that we shall prove is the following.

PROPOSITION 4.4. *Let $x \in h^q(S^{2n+1}/\mathbf{Z}_k)$ satisfy $x \cdot v_{n+1} = 0$. Then there exists an element $y \in h^q(pt)$ such that $y \cdot \emptyset(v_n) = j_n{}^*x$ in $h^q(S^{2n-1}/\mathbf{Z}_k)$.*

Recall that if E is a complex vector bundle of dimension n over X and $\pi : SE \to X$ is its sphere bundle for some Riemannian structure, then there is an exact Gysin sequence

$$h^{q-2n}(X) \xrightarrow{e} h^q(X) \xrightarrow{\pi^*} h^q(SE) \xrightarrow{\pi_*} h^{q-2n+1}(X),$$

where the first map is multiplication by the Euler class of E. We can consider the map $p_n : S^{2n-1} \times_{\mathbf{Z}_k} S^1 \to S^1/\mathbf{Z}_k$ induced by the projection on the second factor as the sphere bundle of the bundle over $S^1/\mathbf{Z}_k$ induced from the representation $n\eta$. Hence there is a diagram of Gysin sequences

$$\begin{array}{ccccccc} \xrightarrow{v_1^{n+1}} & h^q(S^1/\mathbf{Z}_k) & \xrightarrow{p_{n+1}^*} & h^q(S^{2n+1} \times_{\mathbf{Z}_k} S^1) & \xrightarrow{p_{n+1,*}} & h^{q-2n-1}(S^1/\mathbf{Z}_k) & \\ & \downarrow id & & \downarrow j_n'^* & (*) & \downarrow v_1 & \\ \xrightarrow{v_1^n} & h^q(S^1/\mathbf{Z}_k) & \xrightarrow{p_n^*} & h^q(S^{2n-1} \times_{\mathbf{Z}_k} S^1) & \xrightarrow{p_{n,*}} & h^{q-2n+1}(S^1/\mathbf{Z}_k), & \quad (4.5) \end{array}$$

where j_n' is induced by the inclusion of $\mathbf{C}^n$ in $\mathbf{C}^{n+1}$. The commutativity of the diagram is clear except for the square (*), which is commutative by the following:

LEMMA 4.6. *Let E, F be complex vector bundles over X, and let $f : S(E \oplus F) \to X$, $g : SE \to X$ be the associated sphere bundles. If $j : SE \to S(E \oplus F)$ is the inclusion, then*

$$g_*j^*z = e(F) \cdot f_*z$$

for any $z \in h^*(S(E \oplus F))$.

In effect, the projection $p : S(E \oplus F) \to F$ is transversal to the zero-section $s : X \to F$ and the pull-back of s by p is isomorphic to j, hence

$j_*1 = p^*s_*1 = f^*s^*s_*1 = f^*e(F)$, where we have used that p and sf are homotopic. Therefore,

$$g_*j^*z = f_*j_*j^*z = f_*(j_*1 \cdot z) = f_*(f^*e(F) \cdot z) = e(F) \cdot f_*z,$$

proving the lemma.

The element $v_1 \in h^2(S^1/\mathbf{Z}_k)$ comes from an element of $U^2(S^1/\mathbf{Z}_k)$ which is zero for dimensional reasons. Thus $v_1 = 0$ in the diagram (4.5). If $\pi_{n+1} : S^{2n+1} \times_{\mathbf{Z}_k} S^1 \to S^{2n+1}/\mathbf{Z}_k$ is the map induced by the projection on the first factor, then π_{n+1} is the sphere bundle of the line bundle induced from η, so there is an exact Gysin sequence,

$$h^{q+1}(S^{2n+1} \times_{\mathbf{Z}_k} S^1) \xrightarrow{\pi_{n+1,*}} h^q(S^{2n+1}/\mathbf{Z}_k) \xrightarrow{v_{n+1}} h^{q+2}(S^{2n+1}/\mathbf{Z}_k).$$

Let x be as in the proposition. Then $x = \pi_{n+1,*}z$ for some z, so $j_n^*x = \pi_{n*}j_n'^*z$; by 4.5 $j_n'^*z = p_n^*z'$ for some $z' \in h^{q+1}(S^1/\mathbf{Z}_k)$. Let $i : pt \to S^1/\mathbf{Z}_k$ be the inclusion induced by the isomorphism of $\mathbf{Z}_k$ with the k-th roots of unity. Then $z' = y' \cdot 1 + y \cdot i_*1$, where $y' \in h^{q+1}(pt)$ and $y \in h^q(pt)$. Suppose we have proved the formulas

$$\begin{aligned} \pi_{n*}p_n^*1 &= 0, \\ \pi_{n*}p_n^*i_*1 &= \emptyset(v_n). \end{aligned} \tag{4.7}$$

Then it follows that $j_n^*x = \pi_{n*}p_n^*z' = y \cdot \emptyset(v_n)$ proving the proposition.

It remains to prove 4.7. The first equation follows from the Gysin sequence since $1 = \pi_n^*1$. For the second, we compute in U^* using cobordism classes and use that the canonical map

$$pr_1^* : U^*(\,?\,) \to U^*(\,?\, \times Z) = h^*(\,?\,)$$

commutes with Gysin homomorphisms, Euler classes, etc. Now i_*1 is the cobordism class of the map $\mathbf{Z}_k/\mathbf{Z}_k \hookrightarrow S^1/\mathbf{Z}_k$, so $p_n^*i_*1$ is the cobordism class of the composite $S^{2n-1} \cong S^{2n-1} \times_{\mathbf{Z}_k} \mathbf{Z}_k \hookrightarrow S^{2n-1} \times_{\mathbf{Z}_k} S^1$, hence $\pi_{n*}p_n^*i_*1$ is the cobordism class of the projection map $S^{2n-1} \to S^{2n-1}/\mathbf{Z}_k$. By 4.2, this is $\emptyset(v_n)$. This completes the proof of Proposition 4.4.

5. The Main Theorem

Let $\tilde{U}^*(X)$ denote the ideal in $U^*(X)$ consisting of elements which vanish when restricted to any point of X. Recall that $C \subset U^{ev}(pt)$ is

the subring (with unit) generated by the coefficients of the formal group law F of 2.7.

Theorem 5.1. *If X is of the homotopy type of a finite complex, then*

$$U^*(X) = C \cdot \sum_{q \geq 0} U^q(X),$$

$$\tilde{U}^*(X) = C \cdot \sum_{q > 0} U^q(X).$$

It is trivial to show that $U^0(pt) = \mathbf{Z}$ and $U^q(pt) = 0$ for $q > 0$ using 1.2, so the theorem implies the following:

Corollary 5.2. *$U^{ev}(pt) = C$ and $U^{odd}(pt) = 0$.*

For the proof of the theorem it suffices to show that

$$\tilde{U}^{ev}(X) = C \cdot \sum_{q>0} U^{2q}(X), \tag{5.3}$$

since there are suspension isomorphisms

$$\begin{aligned} U^{2j-1}(X) &\simeq \tilde{U}^{2j}(S^1 \times X/\{*\} \times X), \\ U^{2j}(X) &\simeq \tilde{U}^{2j+2}(S^2 \times X/\{*\} \times X), \\ \tilde{U}^{2j-1}(X) &\simeq \tilde{U}^{2j}(S^1 \times X/\{*\} \times X \cup S^1 \times \{x_0\}), \end{aligned} \tag{5.4}$$

where $*$ and x_0 are basepoints and X is assumed connected. Let R be the right side of 5.3. It suffices to show that $R_{(p)} = \tilde{U}^{ev}(X)_{(p)}$ for any given prime number p and where the subscript denotes the localization at the prime ideal (p) in $\mathbf{Z}$. Let us assume as induction hypothesis that $R_{(p)}^{-2j} = \tilde{U}^{-2j}(X)_{(p)}$ for $j < q$, this equality being clear if $q = 0$. Let $x \in \tilde{U}^{-2q}(X)$. By the key formula 3.18, for some large integer n we have

$$w^{n+q}Px = \sum_{l(\alpha) \leq n} w^{n-l(\alpha)} a(v)^\alpha s_\alpha x \tag{5.5}$$

in $U^{2n-2q}(S^{2m+1}/\mathbf{Z}_p \times X)$ for all m, where $a_j(T) \in C[[T]]$, where v ($=$ the v_{m+1} of 4.4) is the Euler class of the line bundle induced from η, and where w is a power series in v with coefficients in C and leading term $(p-1)!\, v^{p-1}$ by 3.16. As p is a prime number, $(p-1)!$ becomes a unit in $\mathbf{Z}_{(p)}$, so $v^{p-1} = w \cdot \theta(v)$ where θ is a power series with coefficients in $C_{(p)}$. For $\alpha > 0$, $s_\alpha x \in R$ by induction hypothesis, hence 5.5 yields equation of the form

$$v^m(w^q Px - x) = \psi(v) \quad \text{in} \quad U^*(S^{2m+1}/\mathbf{Z}_p \times X)_{(p)}, \tag{5.6}$$

where $\psi(T) \in R_{(p)}[[T]]$. Suppose m is the least integer $\geqslant 1$ for which there is an equation such as this. Let $i^* : U^*(S^{2m+1}/\mathbf{Z}_p \times X) \to U^*(X)$ be the ring homomorphism induced by the inclusion of a point in $S^{2m+1}/\mathbf{Z}_p$. Applying i^* to both sides of 5.6, we see that $\psi(0) = 0$, hence $\psi(T) = T\psi_1(T)$ with $\psi_1 \in R_{(p)}[[T]]$ and

$$v(v^{m-1}(w^q Px - x) - \psi_1(v)) = 0.$$

By 4.4 there is a $y \in U^*(X)_{(p)}$ of degree $2(m-1) - 2q$ such that

$$v^{m-1}(w^q Px - x) = \psi_1(v) + y\theta(v) \quad \text{in} \quad U^*(S^{2m-1}/\mathbf{Z}_p \times X)_{(p)}. \tag{5.7}$$

Restricting this equation from X to its basepoint, we obtain $y'\theta(v) = 0$, where y' is the component of y in $U^*(pt)_{(p)}$. Subtracting y' from y, we can suppose $y \in \tilde{U}^*(X)_{(p)}$. If $m > 1$, then $y \in R_{(p)}$ by induction hypothesis, and the right side of 5.7 is in $R_{(p)}[[v]]$, contradicting the minimality of m. Thus $m = 1$, so applying i^* again we obtain

$$\begin{aligned} -x &= \psi_1(0) + py && \text{if } q > 0, \\ x^p - x &= \psi_1(0) + py && \text{if } q = 0, \end{aligned} \tag{5.8}$$

in $\tilde{U}^{-2q}(X)_{(p)}$. If $q > 0$, then as x is arbitrary, it follows that $U^{-2q}(X) \subset R^{-2q}_{(p)} + pU^{-2q}(X)_{(p)}$, whence $U^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$ as $U^{-2q}(X)$ is a finitely generated abelian group by homotopy theory (1.12). If $q = 0$, then it follows that $x \mapsto x^p - x$ kills $\tilde{U}^0(X)/(R^0 + p\tilde{U}^0(X))$. But the ideal $\tilde{U}^0(X)$ is nilpotent, so $x \mapsto x^p$ is a nilpotent endomorphism of $\tilde{U}^0(X)/p\tilde{U}^0(X)$. Thus in either case, $\tilde{U}^{-2q}(X)_{(p)} = R^{-2q}_{(p)}$, which completes the induction and finishes the proof of the theorem.

The theorem can be used to answer a question posed by Conner and Smith[3].

COROLLARY 5.9. *Let X be a finite complex which can be embedded in a weakly complex manifold M of dimension n. Then $U_*(X)$ is generated as a $U^*(pt)$-module by elements of degree $\leqslant n$, and even $< n$ if none of the components of M are compact.*

If N is a closed regular neighborhood of X in M then by duality $U_*(X) \simeq U^{n-*}(N, \partial N)$ as $U^*(pt)$-modules. If none of the components of N are closed, then $N/\partial N$ is connected and $U^*(N, \partial N) \simeq \tilde{U}^*(N/\partial N)$, so $U_*(X)$ is generated by elements of degree $< n$ by the theorem. Otherwise, one needs elements of degree $\leqslant n$, so the corollary follows.

COROLLARY 5.10. *If X is a finite complex of dimension r, then $U_*(X)$ is generated as a $U^*(pt)$-module by its elements of degrees $\leqslant 2r$.*

Since X embeds in $\mathbf{R}^{2r+1}$, this follows from 5.9.

6. STRUCTURE OF $U^*(pt)$

In this section we show how the known structure theorem for $U^*(pt)$ follows from 5.2 and a theorem of Lazard about formal group laws.

Consider the functor which associates to a commutative graded ring $R = \oplus R_q$, $q \in \mathbf{Z}$, the set of formal group laws (2.6) $\sum a_{ij} T_1{}^i T_2{}^j$ with $a_{ij} \in R_{i+j-1}$. This functor is obviously representable; we let Laz be a ring representing it and let F_{univ} be the universal group law over Laz. We shall need the following theorem of Lazard [7]:

PROPOSITION 6.1. *Laz is a polynomial ring over $\mathbf{Z}$ with one generator of degree q for each $q > 0$.*

Let $\epsilon : U^*(X) \to H^*(X)$ be the Thom homomorphism from complex cobordism to ordinary integral cohomology. In terms of 1.10, it is the unique natural transformation compatible with Gysin homomorphisms. Let

$$\beta : U^*(X) \to H^*(X)[t]$$

be the Boardman map; it is the Landweber–Novikov operation s_t followed by ϵ and satisfies the formula

$$\beta(f_* z) = f_*(c_t{}^H(\nu_f) \cdot \beta z), \tag{6.2}$$

for a proper complex-oriented map $f : Z \to X$, where, to avoid confusion, we let $c_t{}^H$ resp., $c_t{}^U$) denote the characteristic classes in H^* (resp., U^*) constructed in the manner of 2.2. When X is a point, this formula shows that βx is the polynomial whose coefficients are the Chern numbers of x.

It is clear from 6.2 and 2.2 that

$$\beta e^U(L) = \sum_{j \geq 0} t_j (e^H(L))^{j+1}, \qquad t_0 = 1,$$

for any line bundle L. Hence putting $L_1 \otimes L_2$ in for L in this formula and arguing universally as in the proof of 2.7, we obtain the formula

$$(\beta F)(\theta_t(T_1), \theta_t(T_2)) = \theta_t(T_1 + T_2), \qquad \text{where} \quad \theta_t(T) = \sum_{j \geq 0} t_j T^{j+1}. \quad (6.3)$$

Therefore, there are ring homomorphisms

$$\begin{array}{c} \mathrm{Laz} \xrightarrow{\delta} U^*(pt) \xrightarrow{\beta} \mathbf{Z}[\mathbf{t}] \\ F_{\mathrm{univ}} \mapsto F \mapsto \theta_t{}^*(T_1 + T_2), \end{array} \qquad (6.4)$$

where δ is the homomorphism sending the universal law to F (note that $\delta\, \mathrm{Laz}_q \subset U^{-2q}(pt)$) and where $*$ denotes conjugation of a group law by a power series.

THEOREM 6.5. *The homomorphism δ is an isomorphism and the homomorphism β of 6.4 is injective. Consequently $U^*(pt)$ is a polynomial ring over $\mathbf{Z}$ with one generator of degree $-2q$ for each $q > 0$, and any element of $U^*(pt)$ is determined by the set of its Chern numbers.*

By 5.2, the map δ is surjective. On the other hand, the composition $\beta\delta$ induces an isomorphism $\mathbf{Q} \otimes \mathrm{Laz} \to \mathbf{Q}[\mathbf{t}]$. To see this consider the morphism of functors represented by $\beta\delta$. A map $u : \mathbf{Z}[\mathbf{t}] \to R$ may be identified with the power series $\theta_u = \sum u(t_j)\, T^{j+1}$, and the composite $u\beta\delta$ may be identified with the formal group law $\theta_u{}^*(T_1 + T_2)$. If R is a $\mathbf{Q}$-algebra, one knows by formal Lie theory [5, p. 96] that any formal group law over R is of the form $\theta_u{}^*(T_1 + T_2)$ for a unique θ_u, the so-called logarithm of the law. Thus for $\mathbf{Q}$-algebras R, $\beta\delta$ induces a one–one correspondence between maps $\mathbf{Z}[\mathbf{t}] \to R$ and maps $\mathrm{Laz} \to R$, which implies that $\mathbf{Q} \otimes (\beta\delta)$ is an isomorphism. By Lazard's theorem (6.1), the ring Laz is torsion-free, hence $\beta\delta$ is injective. Consequently, δ is an isomorphism and β is injective, so the theorem is proved.

For the benefit of topologists, we shall indicate what is involved in the proof of Lazard's theorem.[3]

[3] The exposition follows a talk by J. F. Adams at the Arbeitstagung, 1969.

Let us denote by $\alpha : \mathrm{Laz} \to \mathbf{Z}[\mathbf{t}]$ the composition $\beta\delta$ of 6.4 and adopt the algebraic grading, so that $\deg t_q = q$. We compute the induced homomorphism

$$Q_q(\mathrm{Laz}) \to Q_q(\mathbf{Z}[\mathbf{t}]), \qquad q > 0, \qquad (6.6)$$

on indecomposable elements. A homomorphism from $Q_q(\mathrm{Laz})$ to an abelian group A may be identified with a formal group law over the ring $\mathbf{Z} \oplus A\epsilon_q$, where ϵ_q is an element of dimension q such that $\epsilon_q{}^2 = 0$. Such a law is of the form $T_1 + T_2 + G(T_1, T_2)\, \epsilon_q$, where G is a homogeneous polynomial of degree $q + 1$ with coefficients in A satisfying the identities

$$G(0, T_2) = G(T_1, 0) = 0,$$
$$G(T_1, T_2) = G(T_2, T_1),$$
$$G(T_2, T_3) - G(T_1 + T_2, T_3) + G(T_1, T_2 + T_3) - G(T_1, T_2) = 0.$$

KEY LEMMA 6.7. *There is a unique element $a \in A$ such that*

$$G(T_1, T_2) = a \cdot \frac{1}{\gamma_q} [(T_1 + T_2)^{q+1} - T_1^{q+1} - T_2^{q+1}],$$

where $\gamma_q = p$ if $q + 1 = p^a$ for some prime number p and $a > 1$, and $\gamma_q = 1$ otherwise.

For an efficient proof see [5, p. 62], or better, [17]. This lemma implies that $Q_q(\mathrm{Laz}) \simeq \mathbf{Z}$ for each $q > 0$. On the other hand, a homomorphism from $Q_q(\mathbf{Z}[\mathbf{t}])$ to the abelian group A may be identified with a power series $T + b\epsilon_q T^{q+1}$ with $b \in A$, and the map 6.6 induced by α sends this power series to the group law

$$(T + b\epsilon_q T^{q+1})^*(T_1 + T_2) = T_1 + T_2 + b\epsilon_q[(T_1 + T_2)^{q+1} - T_1^{q+1} - T_2^{q+1}].$$

In other words, we have proved the following:

PROPOSITION 6.8. *The homomorphism 6.6 is isomorphic to multiplication by $\gamma_q : \mathbf{Z} \to \mathbf{Z}$.*

Choose an element $x_q \in \mathrm{Laz}_q$ whose image in $Q_q(\mathrm{Laz})$ is a generator. Then the homomorphism $\mathbf{Z}[X_1, X_2, \ldots] \to \mathrm{Laz}$ sending X_q to x_q is surjective. However, it is also injective since on tensoring with $\mathbf{Q}$ the element $\alpha(x_q)$ form a system of polynomial generators for $\mathbf{Q}[\mathbf{t}]$ by 6.8. This proves 6.1.

7. Unoriented Cobordism Theory

In this section we sketch the modifications needed for unoriented cobordism theory.

Let $N^*(X)$ be the unoriented cobordism ring of X. When X is a manifold, an element of $N^q(X)$ may be identified with a cobordism class of proper maps $f : Z \to X$ of dimension $-q$. For a real n-dimensional vector bundle E over X there is an Euler class $e(E) \in N^n(X)$ and the projective bundle theorem analogous to 2.1 holds so that characteristic classes and operations

$$c_t(E) = \sum t^\alpha c_\alpha(E), \qquad c_\alpha(E) \in N^{|\alpha|}(X),$$
$$s_{\mathbf{t}} : N^*(X) \to N^*(X)[\mathbf{t}]$$

can be defined by the same method. There is a formal group law $F(T_1, T_2) = \sum c_{ij} T_1{}^i T_2{}^j$ with $c_{ij} \in N^{1-i-j}(pt)$ giving the behavior of the Euler class of a line bundle under tensor product. Since the square of a real line bundle is trivial, we have the identity

$$F(T, T) = 0. \tag{7.1}$$

Theorem 7.2. *The unoriented cobordism ring $N^*(pt)$ is generated by the coefficients of the formal group law F.*

This is proved in exactly the same way as 5.1, using the Steenrod squaring operation

$$P : N^q(X) \to N^{2q}(\mathbf{R}P^n \times X)$$

defined as in 3.11. Here things are simpler, since by the projective bundle theorem

$$N^*(\mathbf{R}P^n \times X) \simeq N^*(X)[[v]]/(v^{n+1}),$$

where v is the Euler class of the real line bundle induced from the nontrivial character of $\mathbf{Z}_2$. Put another way, the power series $\emptyset$ of 4.1 is identically zero by 7.1, so that when one gets to 5.8 there is no py term. Consequently it is not necessary to suppose known that $N^q(pt)$ is finitely generated and the whole argument can be carried out independently of homotopy theory.

It is also possible to prove a statement about a finite complex X analogous to 5.1, but as we shall see, a better assertion can be proved using the classification of formal group laws satisfying 7.1. The relevant fact is the following:

Proposition 7.3. *Let R be a commutative ring of characteristic p where p is a prime number, and let $F(T_1, T_2) = \sum r_{ij} T_1{}^i T_2{}^j$ be a formal group law with coefficients in R such that $[p]_F(T) = 0$ (notation as in 3.14). Then there is a unique power series $l(T) = T + \sum_{j \geq 1} a_j T^{j+1}$ such that*

(i) $l(F(T_1, T_2)) = l(T_1) + l(T_2)$

(ii) $a_j = 0 \quad$ if $\; j = p^i - 1 \quad$ for some $\; i > 0$.

Furthermore if $R = \oplus R_j$, $j \in \mathbf{Z}$, is graded and $r_{ij} \in R_{i+j-1}$, then $a_j \in R_j$.

By a successive approximations argument using essentially only the key Lemma 6.7 [5, p. 67], one constructs a series $l(T) = T + \sum a_j T^{j+1}$ which is a logarithm for F, i.e., it satisfies condition (i). Now $(T + \alpha T^{p^n}) \cdot l(T)$ is another logarithm whose coefficient of T^{p^n} is $a_{p^n-1} + \alpha$. From this one sees that $l(T)$ can be modified until all the a_j with $j = p^i - 1$ for some $i > 0$ are zero. This proves the existence of l. For uniqueness, note that if l_1 is another such, then the series $u(T) = l_1(l^{-1}(T))$ satisfies $u(T_1 + T_2) = u(T_1) + u(T_2)$; hence u has only terms of degree p^n. If $u(T) = T + \alpha T^{p^n} +$ higher terms and $\alpha \neq 0$, then the coefficient of T^{p^n} in l_1 is α, contradicting the assumption that l_1 satisfies (ii). Thus $u(T) = T$ and uniqueness is proved. The last assertion follows from the uniqueness and the observation that if a_j is replaced by its homogeneous component of degree j then one obtains another series such as l. Thus the proposition is proved.

The series $l(T)$ will be called the *canonical logarithm* of F. Let Γ be a ring of characteristic p which represents the functor assigning to an R the set of formal group laws F as in the above proposition, and let F_{univ} be the universal such law over Γ. It is clear from the proposition that to give such a law is the same as giving the coefficients of its canonical logarithm, hence Γ is a graded ring which is a polynomial ring with one generator of every degree not of the form $p^i - 1$, and where generator in degree j is the coefficient of T^{j+1} in the canonical logarithm of the universal law.

For the applications to N^* we take $p = 2$ and change the signs of the grading according to the familiar rule $\Gamma_j = \Gamma^{-j}$. We are going to show that the map $\Gamma \to N^*(pt)$ corresponding to the formal group law of N^* is an isomorphism.

Given a graded (commutative) ring R over $\mathbf{Z}_2$ let $\mathscr{F}(R)$ be the following category. Its objects are the formal group laws F as in the proposition with $r_{ij} \in R^{1-i-j}$. Such laws are in one-one correspondence with graded ring homomorphisms $u : \Gamma \to R$; we let F_u denote the image of F_{univ} under u. A morphism in the category $\mathscr{F}(R)$ from F_u to F_v is defined to be a power series $\theta(T) = \sum r_j T^{j+1}\ j \geqslant 0,\ r_0 = 1,\ r_j \in R^{-j}$ such that

$$F_u(\theta(T_1), \theta(T_2)) = \theta(F_v(T_1, T_2)),$$

i.e., $F_u = \theta * F_v$ in the notation of the preceding section. Composition is given by composition of power series; it is clear that every morphism in this category is an isomorphism.

If $u : \Gamma \to R$ is a homomorphism, let

$$h_u{}^*(X) = R \otimes_\Gamma N^*(X) \tag{7.4}$$

be the corresponding base extension. Although $h_u{}^*$ is not *a priori* a generalized cohomology theory because of the exactness axiom, it is a contravariant functor on manifolds to graded rings over R which inherits Gysin homomorphisms for proper maps and characteristic classes from N^*. We use a superscript u to avoid confusion with the characteristic classes of N^*, e.g., $e^u(E) = 1 \otimes e(E)$. Note that by construction the formal group law of $h_u{}^*$ giving the behavior of Euler classes of line bundles under tensor product is $F_u(T_1, T_2)$. Let $\theta(T) = \sum r_j T^{j+1}$ be a morphism in the category $\mathscr{F}(R)$ from F_u to F_v. The composition

$$N^*(X) \xrightarrow{s_t} N^*(X)[\mathbf{t}] \xrightarrow{g} h_v{}^*(X),$$

where $g(x) = 1 \otimes x$ if $x \in N^*(X)$ and $g(t_j) = r_j$, sends $e(L)$ into $\theta(e^v(L))$ and is a ring homomorphism, hence it carries the formal group law F of N^* into the law $\theta * F_v = F_u$. By definition (7.4) this composition induces an R-linear natural ring homomorphism.

$$\hat{\theta} : h_u{}^*(X) \to h_v{}^*(X). \tag{7.5}$$

We claim that $\hat{\theta}$ is characterized by the fact that it is R-linear, multiplicative, and, on Euler classes of line bundles, is given by

$$\hat{\theta} e^u(L) = \theta(e^v(L)). \tag{7.6}$$

Indeed given another such operation ψ, ψ and $\hat{\theta}$ coincide on Euler classes of vector bundles by multiplicativity and the splitting principle, hence ψ and $\hat{\theta}$ coincide on Thom classes which are examples of Euler classes. This implies that ψ and $\hat{\theta}$ coincide on elements coming from $N^*(X)$, and hence ψ and $\hat{\theta}$ are equal by R-linearity. Using this characterization, one sees immediately that $(\theta_1\theta)^\wedge = \hat{\theta}_1\hat{\theta}$, i.e., that $u \mapsto h_u{}^*$ is a functor on $\mathscr{F}(R)$. Since all morphisms of $\mathscr{F}(R)$ are isomorphisms, it follows that

$\hat{\theta}$ is a natural ring isomorphism. Of course, $\hat{\theta}$ is not compatible with Gysin homomorphisms, but it does commute with the suspension homomorphism since s_t does.

We apply these considerations in the following situation. Let $R = \Gamma$, let v be the identity map of Γ, and let u be the homomorphism corresponding to the law $T_1 + T_2$ over Γ. For θ we take the canonical logarithm of the universal law. Note that u factors into maps $\Gamma \to \mathbf{Z}_2 \to \Gamma$, and therefore we obtain a natural isomorphism of Γ-algebras

$$\hat{\theta} : \Gamma \otimes_{\mathbf{Z}_2} (\mathbf{Z}_2 \otimes_\Gamma N^*(X)) \xrightarrow{\sim} N^*(X) \tag{7.7}$$

compatible with the suspension homomorphism. Taking X to be a point and using $\mathbf{Z}_2 \otimes_\Gamma N^*(pt) \simeq \mathbf{Z}_2$, by 7.2, we obtain the following more precise version of Thom's theorem on the structure of the unoriented cobordism ring.

THEOREM 7.8. *The homomorphism $\Gamma \to N^*(pt)$ given by the formal group law F of N^* is an isomorphism, and hence F is a universal law over a ring of characteristic two satisfying 7.1. Furthermore $N^*(pt)$ is a polynomial ring over $\mathbf{Z}_2$ with one generator a_j of degree $-j$ for each positive integer j not of the form $2^i - 1$, where a_j is the coefficient of T^{j+1} in the canonical logarithm of F.*

On the other hand, 7.7 implies that $\mathbf{Z}_2 \otimes_\Gamma N^*(X)$ is a direct summand of N^* and is, therefore, a generalized cohomology theory. Hence the map

$$\mathbf{Z}_2 \otimes_\Gamma N^*(X) \xrightarrow{\sim} H^*(X, \mathbf{Z}_2) \tag{7.9}$$

given by the Thom homomorphism from N^* to ordinary cohomology mod 2 is a map of generalized cohomology theories inducing an isomorphism for $X = pt$ by 7.2, and which, therefore, is an isomorphism by the Eilenberg–Steenrod uniqueness theorem. Combining this with the definition of the isomorphism 7.7 we obtain the following.

THEOREM 7.10. *There is a unique natural ring homomorphism from $H^*(X, \mathbf{Z}_2)$ to $N^*(X)$ which sends an element x of degree one to $l(e(L))$, where L is the real line bundle classified by x and where l is the canonical logarithm of the formal group law of N^*. This homomorphism extends to an isomorphism of $N^*(pt)$-algebras*

$$N^*(pt) \otimes_{\mathbf{Z}_2} H^*(X, \mathbf{Z}_2) \xrightarrow{\sim} N^*(X).$$

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On the Reducibility of Π_1^1 Sets*

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DEDICATED TO W. T. MARTIN

1. Introduction

Although our purpose below is to study some rather technical, metarecursion-theoretic questions raised by G. Kreisel, it is possible to give a sample of our principal construction in the familiar language of descriptive set theory. Let T be a member of 2^ω, and let $\mathscr{B}$ be a lightface Δ_1^1 subset of 2^ω. We say *T is generic with respect to $\mathscr{B}$* if there exists a hyperarithmetically encodable,[1] perfect closed subset P of 2^ω such that $T \in P$ and either $P \subseteq \mathscr{B}$ or $P \subseteq 2^\omega - \mathscr{B}$. Corollary 3.5 provides a $\Pi_1^1 T$ such that T is generic with respect to every lightface Δ_1^1 subset of 2^ω.

The proof of 3.5 combines the methods of forcing [1] and priority [2, 3] (cf. Hinman [4]). The priority method is an essential tool for the construction of nontrivial, recursively enumerable sets. Its use in 3.5 is appropriate for two reasons: T is a Π_1^1 subset of ω if and only if T is metarecursively enumerable; the forcing relation employed in it is metarecursively enumerable. Forcing with finite conditions will not suffice for the proof of this corollary. This last remark is made precise in 2.12, but for now consider the following intuitive argument. Suppose T is Cohen-generic; i.e. T is constructed by means of a forcing argument involving finite conditions and some ramified language $\mathscr{L}$ strong enough to define all the lightface Δ_1^1 sets. Then T will be generic with respect to all lightface Δ_1^1 sets in the sense of generic defined above. T will be infinite, since the collection of all finite sets is lightface Δ_1^1 but contains no perfect closed subset. Suppose T is Π_1^1; then T must contain some infinite hyperarithmetic set H. The set of all T's containing H is lightface Δ_1^1, so the language $\mathscr{L}$ must have the power to express the fact that H is

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[1] A closed subset P of 2^ω is said to be hyperarithmetically encodable if the set of all finite initial segments of members of P is hyperarithmetic.