

Ends of Maps, II

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Abstract. Versions of the finiteness obstruction and simple homotopy theory "within ε over X" are developed. This provides a setting for obstructions to the map analogs of the end and s-cobordism theorems for manifolds. These are applied to study equivariant mapping cylinder neighborhoods in topological group actions, triangulations of locally triangulable spaces, and block bundle structures on approximate fibrations.

Introduction

This paper continues the investigation of completions of ends of maps begun in Part I. The principal question is: given $f: M \to X$ with M a manifold and fnot proper (i.e. preimages of compact sets may not be compact), when can boundary be added to M and the map extended to a proper map on the completed manifold?

In Part I we saw that f can be completed in this way if the end of f is tame (a necessary homotopy condition), the local fundamental group is locally constant, and satisfies $Wh(\pi \times \mathbb{Z}^n) = 0$ for all n. Here we explore the consequences of relaxing the fundamental group hypotheses. As in the X = * case (Siebenmann's thesis) there is an obstruction, which can be thought of as a finiteness obstruction of a dominated space (ε , over X). Similarly the obstruction for the *h*-cobordism problem is a simple homotopy obstruction. The major part of the paper is concerned with development of the obstructions.

To illustrate the theory we consider an example. Suppose a finite group G acts topologically on a manifold M. Let M^* denote the singular set: $M^* = \{x | gx = x \text{ for some } g \neq 1\}$, and suppose the quotient M^*/G is an ANR. Then there is a closed neighborhood of M^*/G in M/G which retracts to it; let $r: U \rightarrow M^*/G$ be such a retraction. Then $M/G - M^*/G$ is a manifold, and $r(U - M^*)/G \rightarrow M^*/G$ has an end. As with the example in the introduction to Part I, this end has a completion if and only if M^* has an equivariant mapping cylinder neighborhood in M. However the results of Part I do not apply. First

the local fundamental group of this end over $x \in M^*/G$ is the isotropy subgroup G_x , so is usually not locally constant. Second and more serious is the fact that usually $Wh(G \times \mathbb{Z}^n) \neq 0$.

In dealing with the first problem we note that M^*/G has a finite closed filtration $X_i \supset X_{i-1} \supset ...$ (determined by conjugacy classes of isotropy subgroups) such that the local group of the end is locally constant over $X_j - X_{j-1}$, and the pieces fit together in a nice way. This is a "stratified system of groups". Essentially we are able to piece together locally constant theorems to obtain results in this case. Such an extension of some of the material of Part I has been worked out by Farrell and Hsiang [15].

The second problem leads to the "finite structure spectrum" $\mathscr{S}(\pi)$, designed to measure problems related to the existence and uniqueness of finite complex structures on spaces. It is also closely related to pseudoisotopies, and in fact we use a pseudoisotopy description of it. Applying \mathscr{S} fiberwise to a stratified system of group yields a "stratified system of spectra" which can be used to define homology groups. The end obstructions lie in one of these groups.

Returning to the group action example, the mapping cylinder obstruction lies in the locally finite stratified spectrum homology group $H_0^{lf}(M^*/G; \mathscr{S}(G_x))$. An Atiyah-Hirzebruch spectral sequence (8.7) and the vanishing theorem of Carter [7] relates this to the sheaf homology group $H_0^{lf}(M^*/G; \tilde{K}_0(\mathbb{Z}G_x))$ and $H_1^{lf}(M^*/G; K_{-1}(\mathbb{Z}G_x))$. These are quite practical to deal with. This example is explored in more detail in Sect. 2.1.

Organization of the Paper

Section 1 contains statements of results, the principal of which are the end and h-cobordism Theorems 1.1 and 1.2. There are ε versions of C.T.C. Wall's finiteness obstruction (1.3), and of J.H.C. Whitehead's simple homotopy theory (1.4). There are also approximate versions of the principal results, and formulae relating various of the invariants.

Section 2 is devoted to applications. In 2.1 the group action question given above is worked out. Proposition 2.1.4 provides examples with nontrivial obstructions. These are locally smooth actions of finite groups on discs, which are not even equivariantly homotopy equivalent to PL actions on compact polyhedra. In 2.2 block neighborhoods of polyhedra are considered. An ndimensional polyhedron P has a subcomplex L (the intrensic n-1 shelelon) of dimension less than n, such that P-L is a PL manifold. Further a neighborhood of L is a mapping cylinder of a PL map. Now suppose $X \supset L$ is a space, L a polyhedron, and X-L is a (topological) manifold. The obstruction to finding a mapping cylinder neighborhood is given by the end theorem. We find further obstructions to breaking the map up into blocks over L analogous to a *PL* map. This block structure is related to triangulation question for Xbecause if X-L is triangulable then the triangulation is isotopic to one in which the mapping cylinder neighborhood is PL. These results are applied to the triangulation of finite group actions (2.2.5, see 2.2.6 and the notes following), and locally triangulable spaces (2.2.7).

As the final application, in 2.3 we discuss obstructions to approximating approximate fibrations by block bundles.

The remainder of the paper is devoted to proofs. Section 3 describes the algebraic obstructions, essentially a codification of the material of Part I. Section 4 gives the stability theorem for these obstructions. This states roughly that for small enough ε the obstructions to ε problems are independent of ε . The proof is rather formal, and is used again in Sect. 5 for pseudoisotopies.

Section 5 develops the finite structure spectrum. This is defined using pseudoisotopies, continuing the work of Hatcher [17, 18], Anderson-Hsiang [1], and many others.

Sections 6 and 7 apply these developments to prove the main results. Section 8 is an appendix on homology groups and spectra.

The most complex part of the paper is the development of the spectrum \mathscr{S} . We actually use rather little of it (dimensions ≤ 1) but are unable to avoid developing the whole thing. Further, it would be preferable (at least asthetically) to have a finite complex description of it along the lines of Hatcher [17], rather than using pseudoisotopies. There are great technical difficulties in this, however. For example, to use contractible pairs and contractible maps it seems to be necessary to work with pairs with a fixed retraction. This changes the theory considerably. Waldhausen [34] encountered a similar difficulty, and required some fairly complex category constructions to repair the damage. Needless to say, category theory does not adapt gracefully to situations with ε estimates.

Metric Conventions

Suppose X is a metric space. We continue the convention of Part I that if $K \subset X$, $\varepsilon > 0$ then K^{ε} is the ε -neighborhood of K, and $K^{-\varepsilon} = X - (X - K)^{\varepsilon}$. If $H: Y \times I \to X$ is a homotopy, $\delta: X \to (0, \infty)$, then H has radius $<\delta$ if for every y, $d(H(y, 0), H(y, t)) < \delta$. This is slightly different from, and more convenient than, the diameter notion used in Part I. There are similar modifications in the notion of size for geometric group phenomena, etc.

1. Main Results

The section begins with some background material on stratified systems of groups, and on the homology groups in which the obstructions lie. Homology groups in general are discussed in more detail in the appendix, Sect. 8. The main results of the paper are the end and *h*-cobordism Theorems 1.1 and 1.2. Theorems 1.3 and 1.4 give the ε analogs of Wall's finiteness theorem and Whitehead's simple homotopy theory. Theorems 1.5 and 1.6 are approximate versions of the main results. Technically they are stronger than the exact versions in several important respects. However they are also more complicated, so should be thought of as refinements. Finally there are formulae relating various invariants. 1.7 and 1.8 are analogs of well-known composition

and boundary formulae. 1.9 and 1.10 give global versions of the duality formulae for finiteness and Whitehead torsion invariants of Poincaré spaces.

Recall that a groupoid is a disjoint union of groups, and is the appropriate notion of fundamental group for spaces which are not connected. A *stratified* system of groupoids on X consists of

1) a closed filtration $X \supset X_i \supset X_{i-1} \dots \supset X_0$,

2) neighborhoods U_i of $X_i - X_{i-1}$

3) a locally constant system of groupoids $A_i \rightarrow U_i$ for each *i*, and

4) for each i > j a homomorphism $\theta_{ij}: A_i \to A_j$ over $U_i \cap U_j$, such that if i > j > k then $\theta_{ik} = \theta_{ik} \theta_{ij}$ over $U_i \cap U_j \cap U_k$.

Given a map $f: M \to X$ it is a little awkward (because of basepoint problems) to define directly a fundamental group system of X. However given a stratified system of groupoids $\rho = (X_*, A_*, \theta_*)$ we can define what it means for f to have fundamental group system equal to ρ (within some $\delta > 0$, say). One way is to pressume the existence of regular covers on which the groups $A_i(x)$ act. This is the approach used in Ends I. Here we will find it technically more convenient to apply the classifying space functor B fiberwise to obtain $B\rho \to X$. If the X_i are ANRs then this is a stratified system of fibrations in the sense of 8.2. Having fundamental group system ρ within δ is equivalent to a δ -factorization of f as $M \xrightarrow{f^{\wedge}} B\rho \to X$ such that f^{\wedge} is $(\delta, 1)$ connected. By mild abuse of

notation, if $p: E \to X$ has a stratified fundamental group system then we will denote the system by $\pi_1(p)$.

In the statements we often express the invariants in terms of a stratified system of fibrations $p: E \rightarrow X$. This is mostly a notational convenience. It is the fundamental group system which is important, since the passage to the associated system of groups $E \rightarrow B(\pi_1 p)$ induces an isomorphism of obstruction groups.

Given a spectrum valued functor of spaces, $\mathscr{S}(Y)$, and a stratified system of fibrations, $p: E \to X$, we describe in Sect. 8 how to define the "spectral sheaf" homology $H_*(X; \mathscr{S}(p))$. For reasonable functors there is a spectral sequence of Atiyah-Hirzebruch type (8.7) which relates this group to ordinary (sheaf) homology groups:

$$H_i(X; \pi_j \mathscr{S}(p)) \Rightarrow H_{i+j}(X; \mathscr{S}(p)).$$

Notice in particular that $H_0(X; \mathscr{G}(p))$ involves the lower homotopy of the spectrum, $\pi_i \mathscr{G}(p), j \leq 0$.

The particular spectrum we use is the "finite structure spectrum" $\mathscr{S}(Y) = \mathscr{S}(*; Y \to *)$ defined in Section 5. The first space in the spectrum is Hatcher's space $\mathscr{S}(Y)$, though for technical reasons we use a pseudoisotopy description of it. This space has received a lot of attention recently. With the exception of Anderson and Hsiang [1], most of this attention has been centered on the higher homotopy groups. As explained above it is the *lower* homotopy groups of the spectrum which interest us. These groups are $\pi_1 \mathscr{S}(Y) \cong Wh(\pi_1(Y))$, $\pi_{-j} \mathscr{S}(Y) = \tilde{K}_{-j}(\mathbb{Z}\pi_1 Y)$ for $j \ge 0$.

In 1.1 and 1.2 we assume X is a locally compact metric ANR. Tame ends are defined in Ends I, 1.1.

1.1. End Theorem. Suppose M is a manifold and $f: M \rightarrow X$ has a tame end with ANR stratified fundamental group system (denoted $\pi_1 \partial(Cf)$). Then

a) there is an invariant $q_0(f) \in H_0^{lf}(X; \mathscr{S}(B\pi_1 \partial (Cf)))$ which, if dim $M \ge 6$, vanishes if and only if f has a completion.

b) If dim $M \ge 5$ then any two completions are h-cobordant in the sense that there is a completion of $M \times I \rightarrow X$ which restricts to the given ones on $M \times \{0, 1\}$. (The boundary of such a completion is an h-cobordism over X.) Conversely given a completion and an h-cobordism of the boundary, the h-cobordism occurs as the boundary of a completion of $M \times I$ extending the given one on $M \times \{0\}$.

c) given two completions, then the invariant $q_1 \in H_1^{lf}(X; \mathscr{G}(B\pi_1 \partial (Cf)))$ of the associated h-cobordism is zero if and only if for every $\varepsilon: X \to (0, \infty)$ and neighborhood U of the end there is an ε automorphism of M which extends to an isomorphism of the completions and is the identity outside U.

An *h*-cobordism over X is a manifold $(W; \partial_0 W, \partial_1 W) \rightarrow X$ which ε deformation retracts to either $\partial_0 W$ or $\partial_1 W$, for any $\varepsilon: X \rightarrow (0, \infty)$.

1.2. *h*-cobordism Theorem. Suppose $f: M \rightarrow X$ is proper and has ANR stratified local fundamental group system.

a) If $F: (W; M, M') \rightarrow X$ is an h-cobordism over X and F/M = f, then there is an invariant $q_1(F, f) \in H_1^{lf}(X; \mathscr{G}(B\pi_1 f))$.

b) If dim $M \ge 5$ and $\alpha \in H_1^{lf}(X; \mathscr{G}(B\pi_1 f))$ then there is an h-cobordism over X with $q_1(F, f) = \alpha$.

c) If dim ≥ 5 then two h-cobordisms over X beginning with M are ε isomorphic for every $\varepsilon: X \to (0, \infty)$ if and only if they have the same q_1 invariant.

We remark that if the local fundamental groups are locally constant, the geometric group material of Ends I allows relaxation of the conditions on X in 1.1 and 1.2 to locally compact locally 1-connected metric space. In this case the obstructions lie in the Čech homology groups $\check{H}_{*}^{lf}(X_{i}\mathcal{S}(B\pi))$.

In Theorems 1.3 and 1.4 we assume X is a locally compact metric ANR, $p: E \rightarrow X$ is a stratified system of fibrations, and $Y \subset X$ is a closed p-NDR subset (see 8.2) so that $(X - Y)^-$ is compact. We will be considering phenomena on X - Y with hypotheses which break down near Y. The conditions on Y are to enable us to get obstructions in $H_*(X, Y; \mathscr{G}(p)) (= H_*^{lf}(X - Y; \mathscr{G}(p)))$. More complicated and general statements can easily be obtained from these.

A space $Z \xrightarrow{f} E$ is said to be δ dominated by a proper polyhedron over $C \subseteq X$ if there is a polyhedron $K \xrightarrow{j} Z$ such that $K \rightarrow X$ is proper, and a map $i: (pf)^{-1}(C) \rightarrow K$ such that ji is δ homotopic to the inclusion $(pf)^{-1}(C) \rightarrow Z$.

1.3. Existence of Finite Complex Structures. There is $\delta > 0$ such that if $f: Z \to E$ is δ dominated over $X - Y^{\delta}$ by a proper polyhedron, then an invariant $q_0(Z) \in H_0(X, Y; \mathcal{S}(p))$ is defined. Given n and sufficiently small $\varepsilon > 0$ there is $\delta(n, \varepsilon)$ such that if f is $(\delta, 1)$ connected and δ dominated over $X - Y^{\delta}$ by a complex of dimension $\leq n$, then Z is ε equivalent to a proper polyhedron over $X - Y^{\varepsilon}$ if and only if $q_0(Z) = 0$.

1.4. Uniqueness of Finite Complex Structures. There is $\delta > 0$ so that if $f: K_1 \rightarrow K_2$ is a δ homotopy equivalence of proper polyhedra over $X - Y^{\delta}$, then there is an invariant $q_1(f) \in H_1(X, Y; \mathscr{S}(p))$. Given n and sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that if f is a δ equivalence and K_2 is $(\delta, 1)$ connected over $X - Y^{\delta}$, and the dimensions of K_1, K_2 are less than n, then $q_1(f) = 0$ if and only if there is a proper polyhedron K_3 and an ε commutative diagram



such that g_1, g_2 are PL and contractible over $X - Y^{\epsilon}$.

We recall that a map is contractible over C if point inverses over C are nonempty and contractible. These are the straightforward generalizations of the classical existence and uniqueness theorems of Wall [35] and Whitehead (see Cohen [13]). We point out the dramatic difference between the ε and fibered version of simple homotopy theory. Hatcher [17] obtains invariants in $H^1(X; \mathscr{S}(p))$ (our notation), so they are *contravariant* and involve the *higher* homotopy of \mathscr{S} .

Notice that although the invariants are well defined generally, a dimension restriction is necessary for a geometrical conclusion. We can replace the restrictions on K by the assumption that X is finite dimensional, but so far cannot omit some such hypothesis. This will not be a hinderence in our applications, but is certainly an asthetic blemish.

We note a refinement of 1.3. If $K \underset{j}{\xleftarrow{f}} Z$ is an ε domination, then K can be given the structure of a *finite complex projection*. This is a map $r: K \to K$, a homotopy $h: r \sim r^2$, and a symmetry condition on $h; rh \sim h(1 \times r)$ releads. The invariant q_0 is most naturally defined for finite complex projections (see 6.5), and $q_0(Z) = q_0(K, r, ...)$.

The reason this is a refinement is that unlike the traditional X = pt. case not every projection comes from a dominated space. The usual construction is the infinite mapping telescope of r, but this does not satisfy any ε estimates. There are finite approximations to this construction (the CW analog of the end theorems) but the size of these is estimated in terms of the dimension of K, an undesirable dependence.

The reason this refinement is useful is that the invariant in the end theorem is q_0 of a finite complex projection constructed very easily from a tame structure (6.6).

Suppose $p: E \to X$, $Y \subset X$ satisfy the conditions given above 1.3. Tame structures are as defined in Ends I, 2.4, except that we substitute for the regular cover hypothesis (2) the condition

2') the maps $(U_{j-1} - W_{j+1}, U_j - W_j) \rightarrow (E, E)$ and $(U_{j-1} - U_{j+2}, U_j - U_{j+1}) \rightarrow (E, E)$ are relatively $(\delta, 1)$ connected.

1.5. Approximate End Theorem. If $n \ge 6$ is given, $\varepsilon > 0$, and $D \subset X$, then there is $\delta > 0$ with the following property: Suppose $f: M^n \to E$ is a manifold with $a(\delta, j(n))$ tame structure U_*, W_* over $X - Y^{\delta}$ which is parallel to $a(\delta, W_0, W_1)$ approximate completion given on ∂M and over D^{ε} . Suppose the invariant $q_0(f) \in H_0(X - D^{\varepsilon/2}, Y: \mathscr{S}(p))$ of the finite complex projection associated to the tame structure vanishes. Then there is an approximate $(\varepsilon, W_0, W_{j(n)})$ completion over $X - Y^{\varepsilon}$ which agrees with the given one over D and on ∂M , and which lies between U_0 and $U_{j(n)}$.

Here, as in Ends I, we can take $j(n) = 2^{(n+1)(n+2)/2}$. See 6.6 for the "associated finite complex projection".

1.6. Approximate *h*-cobordism Theorems. Suppose $n \ge 6$, $\varepsilon > 0$, and $D \subseteq X$ are given. Then there is $\delta > 0$ such that if $f: (M, \partial_0 M) \to E$ is a (δ, h) -cobordism over $X - Y^{\delta}$, $f: M \to E$ is $(\delta, 1)$ connected over $X - Y^{\delta}$, a δ product structure is given over D^{ε} , and the homotopy equivalence invariant $q_1(M, \partial_0 M) \in H_1(X - D^{\varepsilon/2}, Y; \mathscr{S}(p))$ vanishes, then there is an ε product structure over $X - Y^{\varepsilon}$ which agrees with the given one over D. Further, given $N \to E$ $(\delta, 1)$ connected over $X - Y^{\delta}$, N a manifold of dimension ≥ 5 proper over X, and given $\alpha \in H_1(X - D^{\varepsilon}, Y; \mathscr{S}(p))$ then there is an (ε, h) -cobordism (M, N) over $X - Y^{\varepsilon}$ with a product structure over $D^{\varepsilon/2}$ whose invariant in $H_1(X - D^{\delta}, Y; \mathscr{G}(p))$ is the image of α .

We now turn to formulae relating various invariants. Some of these, such as naturality with respect to p, are too straightforward to justify a formal statement. Similarly although we give a composition formula (1.7) we omit the various union formulae.

1.7. Proposition. Suppose $K_1 \xrightarrow{f} K_2 \xrightarrow{g} K_3$ are δ homotopy equivalences as in 1.4. Then $q_1(gf) = q_1(g) + q_1(f)$. In particular if (W; M, M') and (W'; M', M'') are (δ, h) cobordisms as in 1.6, then $q_1(W \bigcup_{M'} W', M) = q_1(W, M) + q_1(W', M')$. Further, if N is a finite complex, $q_1(g \times 1_N)$ is $\chi(N)$ times the image of $q_1(g)$ in $H_1(X, Y; \mathcal{G}(1_N \times p))$.

The next proposition has an approximate version stated in 6.8.

1.8. Proposition. Suppose $f: K \to E \times [0, \infty)$ is a polyhedron proper over $X \times [0, \infty)$, and the end of $pf: K \to X$ is tame.

a) K is ε dominated over X for all $\varepsilon > 0$, and $q_0(K) = q_0(pf)$.

b) If h: $K' \to K$ is proper and a $(\delta, 1/t)$ homotopy equivalence over $X \times (a, \infty)$, then for sufficiently small δ the end of pfh: $K' \to X$ is also tame, and $q_0(pfh) = q_0(pf) - \partial q_1(h)$. In other words $\partial q_1(h) = q_0(K) - q_0(K')$.

c) If N is a finite complex, then $N \times K \xrightarrow{\pi} K \xrightarrow{f} E \times [0, \infty)$ also satisfies the

hypotheses and $q_0(f\pi)$ is $\chi(N)$ times the image of $q_0(f)$ in $H_0(X, Y; \mathscr{S}(1_N \times \mathbf{p}))$.

The next results concern Poincaré duality. If $M, \partial M$ is a manifold, and $(U, \partial_1 U) \supset (M, \partial M)$ is a regular neighborhood in $\mathbb{R}^{n-1} \times [0, \infty)$, then $\partial_0 U \rightarrow M$ is a sphere bundle of some kind. By analogy we say that $(K, L) \rightarrow X$ is ε *Poincaré over* X if there is a Euclidean regular neighborhood $(U, \partial_1 U) \supset (K, L)$,

a spherical fibration $\xi: S\xi \to K$, and an ε homotopy equivalence $b: (U, \partial_0 U, \partial_1 U) \to (D\xi, S\xi, D(\xi|L))$. Here $D\xi$ denotes the disc bundle (mapping cylinder) of the sphere fibration.

There is a homological characterization of ε Poincaré in terms of Poincaré duality over little pieces of X, like the homological characterization of tameness in Ends I, 1.6. There is also a definition for dominated spaces, or more generally finite complex projections (see 7.7).

In order to study the invariants of Poincaré spaces some involutions are necessary. If $p: E \to X$ is a stratified system of fibrations and $\omega: \pi_1 E \to \mathbb{Z}/2(=$ $\{-1, 1\})$ then we define an involution $\overline{\omega}: H_i(X; \mathscr{S}(p)), i \leq 1$, in 7.4. The involution $\overline{1}$ corresponding to the trivial homomorphism is related to Spanier-Whitehead duality. Generally $\overline{\omega} = \overline{1}[\omega]$, where $[\omega]$ is a geometric operation defined using the line bundle over E corresponding to ω . On $E_{i,j}^2$ $= H_i(X; \tilde{K}_j(\mathbb{Z}\pi_1 p))$ $(j \leq 1)$ it is the involution induced by the standard antiinvolution of the group ring, $\overline{\omega}(\Sigma n_g g) = \Sigma n_g \omega(g) g^{-1}$.

1.9. Proposition. Given n there is $\delta > 0$ such that if $(K, L) \rightarrow E$ is δ dominated Poincaré pair of dimension $k \leq n$ over $X - Y^{\delta}$ such that the first Stiefel-Whitney class of the bundle of (K, L) factors through $\omega: \pi_1 E \rightarrow \mathbb{Z}/2$, then $q_0(L) = q_0(K) + (-1)^k \bar{\omega} q_0(K)$ in $H_0^{lf}(X - Y; \mathscr{S}(p))$.

For example suppose $f: M \to X$ is a manifold with tame end with stratified fundamental group. Then we can define a homotopy completion $Cf:(CM, \partial CM) \to X$ by attaching the homotopy inverse limit of neighborhoods of the end, ∂CM (see 7.8). $(CM, \partial CM)$ is a dominated Poincaré pair. Restricting to a small manifold neighborhood of the end and applying 1.8, 1.9 gives $q_0(f) + (-1)^m \bar{\omega} q_0(f) = q_0(\partial CM)$. This gives a duality formula for $q_0(f)$ in situations where ∂CM can be shown to be finite (see 2.2.3).

Now suppose (K, L) is a finite δ Poincaré pair over p. We will see in 7.3 that there is an essentially unique finite complex structure on a spherical fibration, so $b: (U, \partial_0 U) \rightarrow (D\xi, S\xi)$ is an ε homotopy equivalence of finite pairs. The torsion $\tau(K, L)$ of a Poincaré pair is essentially $q_1(b)$ (exactly, $\tau(K, L) = (-1)^s \bar{1} q_1(b)$, where s is the fiber dimension of $D\xi$. See 7.5). Since a manifold has b a homeomorphism, its torsion is 0.

A Poincaré triad is a triad $(K; \partial_0 K, \partial_1 K)$ such that $(K, \partial_0 K \cup \partial_1 K)$, $(\partial_i K, \partial_0 K \cap \partial_1 K)$ are all Poincaré.

1.10. Proposition. Given n there is $\delta > 0$ such that the following hold. Poincaré means δ Poincaré over $X - Y^{\delta}$, dimensions are assumed $\leq n$, and bundles are assumed to have w_1 factoring through $\omega: \pi_1 E \to \mathbb{Z}/2$. Invariants all lie in $H_1^{lf}(X - Y; \mathscr{G}(p))$.

a) (duality) if (K, L) is Poincaré of dimension k, then $\tau(L) = \tau(K, L) + (-1)^k \overline{\omega} \tau(K, L)$.

b) (unions) If $(K; \partial_0 K, \partial_1 K)$ and $(K'; \partial_0 K', \partial_1 K')$ are Poincaré triads and $\partial_0 K = \partial_0 K'$, then $(K \cup K', \partial_1 K \cup \partial_1 K')$ is Poincaré and $\tau(K \cup K'; \partial_1 K \cup \partial_1 K') = \tau(K, \partial K) + \tau(K', \partial K') - \tau(\partial_0 K, \partial_0 \partial_0 K).$

c) (h-cobordisms) If $(K; \partial_1 K, \partial_2 K)$ is a finite Poincaré triad of dimension k which is a (δ, h) -cobordism, then

d) (homotopy equivalences) If $f: (K_0, \partial K_0) \rightarrow (K_1, \partial K_1)$ is a homotopy equivalence of finite k-dimensional Poincaré pairs, then

$$\tau(K_0, \partial K_0) - \tau(K_1, \partial K_1) = \overline{\omega} q_1(f) + (-1)^k q_1(f, \partial f).$$

e) (products) If (K, L) is Poincaré over $p: E \rightarrow X$ and $(N, \partial N)$ is finite orientable Poincaré (over a point) then $(K \times N, K \times \partial N \cup L \times N)$ is Poincaré, and has torsion $X(N, \partial N) \cdot \tau(K, L)$.

We note that (c) specializes to the duality formula for manifold h-cobordisms.

2. Applications

2.1. Finite Group Actions. Suppose G is a finite group which acts continuously on a manifold M. We are concerned with equivariant mapping cylinder neighborhoods of the singular set, continuing the example of the introduction.

Some notation will be helpful. If $x \in M$, G_x is the isotropy subgroup $\{g \in G | gx = x\}$. If $H \subseteq G$ then $M\{H\}$ is the subset of points whose isotropy subgroup is a conjugate of H. $M\{\supseteq H\}$ denotes the union of $M\{J\}$ for $J \supseteq H$. Finally the singular set $M\{\supset 0\}$ will be denoted more briefly by M^* .

An action is said to be ANR if $M\{\supseteq H\}/G$ is an ANR for each $H \subseteq G$. It is *effective* if $M^* \neq M$.

For the first result we note that for an ANR action the isotropy subgroups form a stratified system of groups over M. This system is G equivariant, so defines a system over M/G, which we will denote by G_* .

2.1.1. Theorem. Suppose the action of G on M is effective and ANR, that $M^* \subset M$ is 1-LC embedded, that the dimension of M is ≥ 6 , and that an equivariant mapping cylinder neighborhood is given for a neighborhood of $M^* \cap \partial M$. Then there is an invariant in $H_1^U(M^*/G; \mathscr{G}(G_*))$ which is zero if and only if there is an equivariant mapping cylinder neighborhood of M^* which agrees with the given one near $\partial M \cap M^*$.

Proof. Since the action is ANR there is a closed neighborhood N of M^*/G with a retraction $e: N \to M^*/G$, and such that $N - M^*/G$ is a manifold. Since $M^* \subset M$ is 1-LC embedded, the end of $(N - M^*/G) \to M^*/G$ has fundamental group system G_* . In fact there is an equivariant retraction e^{\sim} of the inverse image $N^{\sim} \subset M$ to M^* . Then since $N^{\sim} - M^*$ is free, e^{\sim} factors $(N^{\sim} - M^*) \to E_G \times M^* \to M^*$. This gives a factorization $(N - M^*/G) \xrightarrow{f} (E_G \times M^*)/G \xrightarrow{p} M^*/G$. f is locally 1-connected at the end, and p is a stratified system of fibrations with fiber $E_G/G_x \simeq B_{G_*}$ over $[x] \in M^*/G$.

Next, also since M^* is 1-LC embedded, the end of $N - M^*/G$ is homologically tame (see Ends I, 3.1.1) therefore tame (Ends I, 5.5).

Finally since an equivariant mapping cylinder neighborhood is equivalent to a completion of the end of $N - M^*/G$, the theorem follows from the End Theorem 1.1.

We note that the uniqueness aspect of 1.1 shows that if a mapping cylinder neighborhood exists, then different ones are classified up to ambient isotopy by $H_1^{lf}(M^*/G; \mathscr{G}(G_*))$.

Notice that there is a natural map of group systems from G_* over M^*/G to G over a point. This defines a homomorphism $H_0^{lf}(M^*/G; \mathscr{G}(G_*)) \to \tilde{K}_0(\mathbb{Z}G)$.

2.1.2. Proposition. Suppose M is as in 2.1.1 and is compact. Then $(M - M^*)/G$ is dominated by a finite complex, and the image of the (Wall) finiteness obstruction is the image of the mapping cylinder obstruction under the homomorphisms

$$H_0^{lf}(M^*/G; \mathscr{S}(G_x)) \to \tilde{K}_0(\mathbb{Z}G) \leftarrow \tilde{K}_0(\mathbb{Z}\pi_1((M-M^*)/G)).$$

Proof. This is because the end obstruction is the finiteness obstruction of a neighborhood of the end (1.5), and the complement of the neighborhood is already finite.

Another view of this piece of the invarient is that it is equivalent to the obstruction to (M, M^*) being equivariantly homotopy equivalent to (K, M^*) , where K is obtained from M^* by attaching cells which are freely permuted by G. Such obstructions have been encountered by Oliver [26, 27] Oliver and Petrie [28], and Quinn [30]. More precisely the finiteness obstruction of $(M/G, M^*/G)$ is $(-1)^m \overline{\omega}$ applied to that of $(M-M^*)/G$ (by Poincaré duality, still assuming as in 2.1.1 that $\partial M \cap M^*$ has a mapping cylinder neighborhood so that $(\partial M - M^*)/G$ is finite).

The next remark is that Carter [7] has shown that for a finite group $K_{-j}(\mathbb{Z}G)=0$ for $j \ge 2$. The bottom corner of the spectral sequence therefore gives an exact sequence

$$H_{2}^{lf}(M^{*}/G; K_{-1}(\mathbb{Z}G_{*})) \xrightarrow{a_{2}} H_{0}^{lf}(M^{*}/G; \tilde{K}_{0}(\mathbb{Z}G_{*})) \rightarrow H_{0}^{lf}(M^{*}/G; \mathscr{S}(G_{*}))$$
$$\rightarrow H_{1}^{lf}(M^{*}/G; K_{-1}(\mathbb{Z}G_{*})) \rightarrow 0.$$

2.1.3. Corollary. Suppose M is as in 2.1.1 and is compact. If the fixed point set $M\{G\}$ is nonempty, and $M\{\supseteq H\}$ is 1-acyclic (i.e. $H_i(; \mathbb{Z}) = 0$ for i = 0, 1) for all isotropy subgroups $H \neq 1$, then M^* has an equivariant mapping cylinder neighborhood extending the one given near ∂M if and only if $(M - M^*)/G$ has the homotopy type of a finite complex.

Proof. The acyclicity hypothesis reduces the exact sequence to

$$\xrightarrow{a_2} \tilde{K}_0(\mathbb{Z}G) \to H_0(M^*/G; \mathscr{S}(G_*)) \to 0.$$

The construction of 2.1.2 shows that this is an isomorphism and identifies the invariants. Then 2.1.1 connects the invariant to mapping cyclinders, and 1.3 (or Wall [35], since we are over a point) connects it to finiteness. This completes 2.1.3.

We also note that if a fixed point is deleted from M^*/G then the obstruction group vanishes entirely. Therefore there is a mapping cylinder neighborhood of the complement of any fixed point.

The 1-acyclic hypothesis on the sets $M{H}$ in 2.1.3 imply that the singular set is 1-acyclic. This is a fairly strong assumption, but it may well be a context for reasonably general constructions of actions (see Quinn [30]). The main

reason for imposing the requirement here is to avoid K_{-1} . Since Carter [8] has explicitly calculated $K_{-1}(\mathbb{Z}G)$, these obstructions can surely be understood in more general circumstances.

The final result provides some examples with nontrivial obstruction.

2.1.4. Proposition. There are locally smooth actions of finite groups on discs which have nonvanishing mapping cylinder obstructions, and in particular are not equivariantly homotopy equivalent to finite contractible G complexes with cellular actions.

More precisely there is an action of $G = \langle a, b | a^{15} = b^4 = 1$, $b a b^{-1} = a^2 \rangle$, with fixed point set an arbitrary finite complex homotopy type with Euler characteristic congruent to 1 or 3 mod 4. By contrast Oliver [27, Part II] has shown that a cellular action of this group on a finite contractible complex must have fixed point set with Euler characteristic congruent to 1 mod 4. This group is metacyclic 2-hyperelementary, and has order 60 (Oliver [27, Part II], p. 263).

In relating these examples to 2.1.1 allowance must be made for the boundary, since in general the action on the boundary sphere does not have a mapping cylinder neighborhood.

Proof. Begin with a G-resolution X in the sense of Oliver [26], with nonempty fixed point set. A resolution is a finite G complex with a single nonzero (reduced) homology group, which is projective as a $\mathbb{Z}G$ module. Equivariantly PL embed X into a linear representation of G with fixed point set of dimension at least 2, and with singular set of codimension greater than j. Let M be an equivariant regular neighborhood. If $H_j(X)$ were free, then we could attach sets of (j+1)-handles to ∂M on which G acts freely, to kill $H_j(X)$. This new manifold would be a disc with a G action. This is the construction of Oliver.

If $H_j(X)$ is not free, we will describe how to "wrinkle" ∂M in a neighborhood of a fixed point. This changes $M - M^*$ by adding an ε, h -cobordism, so the effect on the end invariant is given by 1.8(b). The specific construction changes $H_j(M)$ by adding a complementary projective module. Therefore the homology can be killed by adding free handles as above, giving a disc.

Let K be a finite connected 1-complex on which G acts freely. Let cK denote the cone $K \times [0, 1]/K \times \{0\}$, and let $h: cK \rightarrow \partial M$ be an equivariant PL embedding. The image of the cone point is a fixed point, which we denote by x.

Next let $p: A \to A$ be a projection (i.e. $p^2 = p$) on a finitely generated free $\mathbb{Z}G$ module with $im(p) = H_j(X)$. Let q be the complementary projection (q = 1 - p).

Now let $k \in K$, and for each $i \ge 1$ trivially attach *j*-handles to ∂M representing a basis for *A*, in a very small neighborhood of $h\left(Gk \times \left\{\frac{1}{i}\right\}\right)$.



This is topologized so that these handles converge to x.

The next step is for each *i* add *j*+1 handles, again corresponding to a basis of *A*. These handles are to be added in a small neighborhood of $h\left(K \times \left[\frac{1}{i}, \frac{1}{i+1}\right]\right)$ and to the *j*-handles near $h(Gk \times \{i\})$ and $h(Gk \times \{i+1\})$. This can be done so that the boundary homomorphism to the *i*th set of *j* handles is *p*, and to the $(i+1)^{\text{st}}$ is *q*.



Again these handles are topologized to converge to x.

The result of this construction is a smooth manifold in the complement of x. The homology is that of M except H_j which is $H_j(M) \oplus A/p(A)$. Since p was chosen so that this is free, free sets of handles may be added to make it contractible. Call the result D. D is a contractible G-space which is smooth in the complement of x, and whose boundary is 1-connected. It remains to see that D is a manifold, and the action is locally linear at x. This will be done by a sort of engulfing.

Since the representation was assumed to have fixed point set of dimension at least 2, there is a neighborhood of x in M of the form $V \times [0, 1)$, where V is a half space in a representation of G and x corresponds to $(0, \frac{1}{2})$. Let W be the neighborhood in D obtained by adding to $V \times [0, 1)$ the j and j+1 handles which intersect it. Since only the free part is changed, $W^* = V^* \times [0, 1)$. Further, the end of $(W-W^*)/G \rightarrow (V^*/G) \times [0, 1)$ is changed only over x, and there it is not changed homotopically. Therefore the end is still tame, with the same fundamental group system.



Now we apply the end theorem, first on the boundary. $\partial(W-W^*)/G \rightarrow (\partial V - V^*)/G \times [0, 1)$ has a completion over $(\partial V - V^*)/G \times [0, \frac{1}{3}]$; the product completion of the corresponding end in $V \times [0, 1)$. The obstruction to extending this completion from over $[0, \frac{1}{4}]$ to over all of $(\partial V - V^*)/G \times [0, 1)$ lies in $H_1^{if}((\partial V - V^*)/G \times [\frac{1}{4}, 1); \mathscr{S}(G_*)) = 0$. Therefore there is such a completion. Similarly this extends to a completion of the whole end which agrees with the given one over $(V - V^*)/G \times [0, \frac{1}{4})$. Let $f: N \rightarrow (V - V^*)/G \times [0, 1)$ denote the new boundary in this completion.

The next step is to change f by h-cobordism. Then end in $V \times [0, 1)$ has completion a product $P \times [0, 1] \rightarrow (V - V^*)/G \times [0, 1)$. Since the two ends are homotopically the same, the projection $N \rightarrow [0, 1)$ is an approximate fibration, and is $P \times [0, \frac{1}{4}]$. The restriction of $N \times I \rightarrow V^*/G \times [0, 1) \times I$ to $\{(y, s, t)|s < 1$ $-3/4t\}$ gives an h-cobordism from f (over $V^*/G \times [0, 1) \times \{0\}$) to the product P $\times [0, \frac{1}{4}) \times \{1\} \rightarrow V^*/G \times [0, \frac{1}{4}) \times \{1\}$. According to 1.1(b) a map h-cobordant to the boundary of a completion can be realized as a completion of the same end. Therefore the end in W has a completion which is a product over [0, 1). It follows that a neighborhood of x in D is a product, so D is a manifold and the action is locally linear.

We have shown that any G-resolution can be thickened and completed to a locally linear action on a disc. But Oliver has shown that these often have nonzero finiteness obstruction, and that for a few groups the wrong choice of Euler characteristic for the fixed point set of X actually forces the finiteness obstruction to be nonzero. The proof of 2.1.4 is therefore complete.

2.2. Block Neighborhoods of Polyhedra. Suppose X is a space, L a polyhedron closed in X, and that X - L is a topological manifold. The question is, when can a neighborhood of L be decomposed into blocks in a manner analogous to the decomposition of a PL regular neighborhood? If the end of X-L is assumed to have an appropriate homotopy structure, then various completion obstructions are encountered. The main result is 2.2.4. This should be regarded as preliminary, because the obstructions are not yet effectively organized. The application to triangulation of finite group actions in 2.2.5 is reasonably satisfactory because Carter's vanishing theorem avoids most of the inefficiency of the general case. Finally 2.2.6 contains a comparison with the triangulation work of Anderson and Hsiang [2].

The first step is to recall the dual cone decomposition of a polyhedron (Cohen [13]). Choose a triangulation of L, take the first barycentric subdivision, and then consider the stars of the original vertices. These are cones (on the links), their union is L, and the intersection of two is a cone contained in the boundary of each. In fact any intersection of cones has a natural cone structure, and is contained in the boundary of the larger cones.



Next we note that if $N \rightarrow L$ is simplicial with respect to the first triangulation of L, then N decomposes into mapping cylinders over the cones in L. This is essentially Hatcher's [19] iterated mapping cylinder decomposition. The first consequence is that point inverses in N change in a very controlled way.

2.2.1. Lemma. Suppose $f: N \rightarrow L$ is proper and PL. Then there is a stratified system of fibrations with PL filtration $p: E \rightarrow L$ and an approximate homotopy equivalence over L, $N \rightarrow E$. Further, all of the fibers of E have the homotopy type of finite complexes.

An approximate homotopy equivalence is a map which commutes with the maps to L, which is an ε homotopy equivalence for every $\varepsilon > 0$. Alternatively it is a map whose restriction $f^{-1}(U) \rightarrow p^{-1}(U)$ is a homotopy equivalence for every open set $U \subset L$.

The next observation about the cone decomposition is that boundaries of maximal cones are bicollared. Further, smaller cones have boundaries bicollared in the boundaries of the next larger cones. Therefore the intersection of k distinct maximal cones has a D^k product neighborhood (or perhaps more accurately Δ^k , since it comes from a simplex in the original triangulation). If $f: N \rightarrow L$ is a manifold it therefore makes sense to speak of N being transverse to the dual cones. If f were simplicial, as above, then f is automatically transverse to the cones. The next lemma is in a sense a converse to this observation.

2.2.2. Lemma. Let $f: N \rightarrow L$ be a proper map, N a manifold, and f approximately homotopy equivalent to a stratified system of fibrations $p: E \rightarrow L$ with PL filtration. Suppose f is transverse to the cones dual to a triangulation in which the filtration of p is simplicial. Then f is s-cobordant over L to a map $f': N \rightarrow L$ which decomposes as mapping cylinders over the cone decomposition. Finally if dim $N \ge \dim L + 5$, then any combinatorial PL structure on N is isotopic to one in which f' is PL.

We recall that the significance of s-cobordism over L is that s-cobordant maps have homeomorphic mapping cylinders (1.1). This therefore gives a criterion for splitting up, or triangulating, a mapping cylinder.

Proof. Since f is transverse to the boundary of a maximal cone cB, the inverse of a small collar is a product

$$f^{-1}(B \times [1-\varepsilon, 1]) = f^{-1}(B) \times [1-\varepsilon, 1] \rightarrow B \times [1-\varepsilon, 1].$$

If we compose f with the map $cB \times I \rightarrow cB$ which squeezes the subcone $B \times [0, 1 - \varepsilon]/B \times \{0\}$ to the cone point, then we get $F: N \times I \rightarrow L$ so that $F_0 = f$, and F_1 is a mapping cylinder over cB. Compose p with the same squeeze to obtain a commutative diagram



which is an approximate homotopy equivalence over L. But p is a stratified system of fibrations, and this squeeze preserves the filtration. Therefore use of the homotopy lifting property in the pieces gives an approximate homotopy

equivalence



Composing with the map $N \times I \rightarrow E \times I$ shows that $F: N \times I \rightarrow L$ is an *h*-cobordism. It is an *s*-cobordism because it can be broken up into little segments with ε trivializations, (hence trivial q_1 invariant) and the invariant is additive.

Now proceed by induction going down on the size of the cones. A deformation over a small cone can be extended to the larger cones using the mapping cylinder structures.

Finally suppose N is a PL manifold. The topological Caurns-Hirsh Theorem (Kirby and Siebenmann [23]) shows that the structure may be deformed to make a codimension 1 submanifold PL, provided the dimension is ≥ 6 . Using this inductively on inverse images of cones, we see that the structure can be deformed until the map f is PL transverse to the cones. The construction of f' then displays N as the union of PL mapping cylinders (with the usual cautions about these, so the projection to the cone complex is PL.

This completes the proof of Lemma 2.2.2.

The final clue is that transversality is an end problem.

2.2.3. Lemma. Suppose $f: N \to L$ is approximately homotopy equivalent to a stratified system of fibrations $p: E \to L$ with PL filtration and with fibers equivalent to finite complexes, and suppose that $L_0 \subset L$ is bicollared subcomplex transverse to the filtration. If N is a manifold of dimension $n \ge 6$ then f is s-cobordant to a map transverse to L_0 if and only if the end of one side of $N - f^{-1}(L_0)$ has a completion over L_0 . The end invariant satisfies $q_0 = (-1)^n \overline{\omega}(q_0)$, where $\overline{\omega}$ is the involution induced by the first Stiefel-Whitney class of the normal bundle of $N, \omega: \pi_1 E \to \mathbb{Z}/2$. Finally, if there is a completion then any completion can be realized as the transverse inverse image.

Proof. First note that since L_0 is transverse to the filtration we can choose a neighborhood $L_0 \times [-1,1]$ so that p over this neighborhood is equivalent to the product $p_0 \times 1_{[-1,1]}$, where p_0 is the restriction of p to $E_0 = p^{-1}(L_0 \times \{0\})$. It follows from this that the end of $f^{-1}(L_0 \times \{0,1])$ over L_0 is tame and has homotopy completion equivalent to p_0 .

To show the only if part, suppose $W \to L_0$ is an *h*-cobordism with $\partial_0 W = N$ and invariant $q_1 \in H_1^{lf}(L; \mathscr{S}(p))$. Then 1.8b and 1.10c show that the restrictions of $\partial_0 W$ and $\partial_1 W$ to $L_0 \times (0, 1]$ have end invariants which differ by $\partial(q_1 + (-1)^n \bar{\omega} q_1) \in H_0^{lf}(L_0; \mathscr{S}(p_0))$. In particular they are equal if W is an *s*-cobordism. Therefore if f is *s*-cobordant to a transverse map, the original end has a completion.

The next step is to construct a finite complex approximation to E. This is a proper polyhedron $r: W \rightarrow L$ and a map $W \rightarrow E$ which commutes with projection to L and is an approximate homotopy equivalence over L. Take as usual a

cone decomposition dual to a triangulation in which the filtration of p is simplicial. We construct W inductively over the "skeleta" $L_j = \text{union}$ of cones of dimension $\leq j$. Suppose $r_j: W_j \rightarrow L_j$ is defined, and $W_j \rightarrow p^{-1}(L_j)$ is an approximate homotopy equivalence. Suppose $(cB, B) \subset (L_{j+1}, L_j)$ is a cone of dimension j+1. Since p has the homotopy lifting property with respect to PL homotopies which move points in a monotone decreasing way between strata, and because the stratification of p is simplicial with respect to the triangulation, we can lift $r_j^{-1}(B) \times [0,1] \rightarrow cB$ into E so that $r_j^{-1}(B) \times \{1\} \rightarrow p^{-1}(B)$ is the given equivalence. Choose a homotopy equivalence $p^{-1}(c) \rightarrow Z, Z$ a finite complex. Then define W_{j+1} over cB to be the mapping cylinder of the composition $r_j^{-1}(B)$ $\times \{0\} \rightarrow p^{-1}(c) \rightarrow Z$. Define the projection r_{j+1} by $(x,t) \in r_j^{-1}(B) \times [0,1]$ goes to $(r_j(x), 2t-1) \in B \times [0,1]$ for $t \geq \frac{1}{2}$, and goes to c for $t \leq \frac{1}{2}$. Then a homotopy inverse for $p^{-1}(c) \rightarrow Z$, and the lift found above, can be used to lift r_{j+1} into Ein an appropriate way.

The first consequence is the duality formula. $r^{-1}(L_0)$ is a polyhedron proper over L_0 , and is approximately homotopy equivalent to $p^{-1}(L_0)$, which in turn is equivalent to the boundary of the homotopy completion of $f^{-1}(L_0 \times (0,1]) \rightarrow L_0$. This gives a manifold neighborhood of the end the structure of an ε dominated Poincaré space over L_0 , with finite boundary. The duality then follows from 1.9. (See the comments following 1.9.)

The next consequence of the finite complex approximation is that if $(a, b) \subset (-1, 1)$ then the invariant of one end of $f^{-1}(L_0 \times (a, b)) \rightarrow L_0$ is the negative of the invariant of the other end. This is because the invariants are finiteness invariants of neighborhoods of the ends, so gluing the neighborhoods together we see that the sum of the invariants is the finiteness invariant of $f^{-1}(L_0 \times (a, b)) \rightarrow L_0$. This is equivalent to $r^{-1}\left(L \times \left\{\frac{a+b}{2}\right\}\right)$ which is finite, so the sum is zero.

Applying this conclusion to $(a,b) = (-\frac{1}{2},0), (-\frac{1}{2},\frac{1}{2})$ and $(0,\frac{1}{2})$ shows that the invariants of the ends of $f^{-1}(L_0 \times (0,1]) \rightarrow L_0$ and $f^{-1}(L_0 \times [-1,0]) \rightarrow L_0$ are negatives of each other. In particular if one has a completion so does the other.

Now suppose we are given a completion of $f^{-1}(L_0 \times (0, 1])$ over $L_0 \times \{0\}$. Then by the argument above, there is also a completion of $f^{-1}(L_0 \times [-1, 0])$ over $L_0 \times \{0\}$. Consider these as a completion on the boundary of the end of $f^{-1}(L_0 \times [-1, 1]) \times [0, 1] - f^{-1}(L_0 \times \{0\}) \times \{1\}$.



The obstruction to extending this boundary completion to the whole end is the finiteness obstruction of a neighborhood of the end. This is zero because the

original description as a subset of $f^{-1}(L_0 \times [-1, 1]) \times [0, 1]$ gives a completion (though not a relative one). This completion of the interior gives an *h*-cobordism of $f^{-1}(L_0 \times [-1, 1])$ which is a product a little outside $L_0 \times \{0\}$. We can change the completion of the interior by an *h*-cobordism (by 1.1) to get an *s*cobordism of *f*. Finally by pushing everything over $L_0 \times \{0\}$ except the original completion of $f^{-1}(L_0 \times (0, 1])$ a little toward -1, we obtain an *s*-cobordism to a transverse map with this as inverse image.

This completes the proof of Lemma 2.2.3.

2.2.4. Theorem. Suppose L is a polyhedron, $L \subset X$ is closed, and X - L is a manifold with dimension $r \ge \dim L + 6$. Suppose the end of X - L over L is tame and the boundary of the homotopy completion is a stratified system of fibrations $p: E \rightarrow L$ with PL filtration and fibers equivalent to finite complexes. Then there are obstructions

- a) the end invariant $q_0(X-L) \in H_0^{lf}(L; \mathscr{S}(p))$, which satisfies $q_0 = (-1)^r \overline{\omega}(q_0)$.
- b) a sequence $t_j \in \sum_{i \ge j} H_i^{lf}(L; H^{i+j+r}(\mathbb{Z}/2; \tilde{K}_{-j}(\mathbb{Z}\pi_1 p), \bar{\omega}))$ for $j \ge 0$, such that

 t_{j-1} is defined if t_j is defined and is zero. c) if the obstructions in (b) vanish then a sequence

 $u_i \in H^{r+j+1}(\mathbb{Z}/2; H^{lf}_{-i}(L(j); \mathscr{S}(p)), \bar{\omega}), \quad j \ge 0.$

If all of these vanish, then L has a mapping cylinder neighborhood whose map decomposes as iterated mapping cylinders over a dual cone decomposition of L.

Note: The obstructions in (b) and (c) are 2-torsion: if K is a group with involution $\overline{\omega}$ then $H^p(\mathbb{Z}/2; K, \overline{\omega})$ is a fancy notation for the subquotient $\{k \in K | k = (-1)^p \overline{\omega}(k)\}/\{k + (-1)^p \overline{\omega}(k)\}$. This group has exponent 2, so the obstructions in (b) are ordinary homology classes with coefficients in a sheaf whose stalks are $\mathbb{Z}/2$ vector spaces. In (c) $L_{(j)}$ denotes the union of dual cones of L of codimension > j. Finally we may allow X - L to have boundary if the boundary is already properly decomposed over L.

Proof. The plan is this: Suppose the end of X-L has a completion with boundary $N^{r-1} \rightarrow L$. Since the circle T^1 has Euler characteristic 0, multiplying by T^1 kills end obstructions (and therefore by 2.2.3 transversality obstructions). Therefore (roughly) we can make $N \times T^1 \rightarrow L$ transverse to codimension 1 cones, $N \times T^j \rightarrow L$ transverse to cones of codimension *j*, etc. Eventually we get a map $N \times T^n \rightarrow L$ which is properly decomposed into blocks. The obstructions in (b) come from trying to remove T^n factor from each block (by unwinding one coordinate at a time and completing blockwise). If this is successful we get $M_0 \rightarrow L$ decomposed, such that $M_0 \times T^n$ is *h*-cobordant to $N \times T^n$. The obstruction in (c) arise in trying to construct an *h*-cobordism from M_0 to N (again by unwinding and completing). Since *h*-cobordant maps have homeomorphic open mapping cylinders, this would give the desired conclusion.

Since the union of the codimension 1 cones is not itself codimension 1, we begin with a modification of the cone decomposition. A relative cone decomposition of a polyhedral pair $(L, \partial L)$ is a presentation $L = \partial L \cup cB_i$, such that any intersection of maximal cones cB_i is again a cone whose interior is disjoint from ∂L , and is contained in the boundary of larger cones.

Fix a triangulation of L in which the filtration of $p: E \to L$ is simplicial. Now for each $j \ge 0$ we define $S_j \subset L$ with ∂S_j (defined to be $S_j \cap (L-S_j)^-$) bicollared in L, and such that S_j intersects the dual cone decomposition of L in a relative cone decomposition. For $n = \dim L$, define $S_{n+1} = L$. If S_j is defined, there is a disjoint collection of embeddings, one for each j-simplex σ , of the form $\sigma \times cB \subseteq S_j$. Here B is the link of σ . The embedding intersects ∂S_j in $\sigma \times B$, and intersects the filtration of S_j in (the cone on the filtration of $\partial \sigma$ by skeleta) $\times B$. Define S_{j-1} by deleting the images of $\sigma \times (\operatorname{int} cB)$.



Now we make $N \times T^j$ transverse to something. $(S_1, \partial S_1)$ consists only of maximal cones. By 2.2.3 we can find an s-cobordism of $N \times T^1 \to L$ to $f_1: N \times T^1 \to L$ which is transverse to ∂S_1 , therefore to the relative cone decomposition of S_1 . Now suppose $f_j: N \times T^j \to L$ is transverse to the cones and boundary of S_j . S_{j+1} is obtained from this by adding things of the form $\sigma \times cB$, where σ is a (j+1) simplex. The restriction to the boundary $(\sigma \times B, \partial \sigma \times B) \subseteq (L - int(S_j), \partial S_j)$ is bicollared, and $\partial \sigma \times B \subset \partial S_j$ is transverse to the filtration of ∂S_j by cones. This gives a lot of finiteness obstructions: in each stratum of ∂S_j the corresponding stratum in $\partial \sigma \times B$ is bicollared. Take the finiteness obstruction of the end of f_j lying over one side of the complement of $\partial \sigma \times B$. Crossing with T^1 kills all these obstructions. Therefore we can find an s-cobordism of $f_j \times 1$ through map transverse to filtration of S_j to $g: N \times T^{j+1} \to L$ which is transverse to $\sigma \times B$ as well. For this we use 2.2.3 inductively beginning in the lowest stratum of ∂S_j and working up. At each step the finiteness obstruction is interpreted as the obstruction to making the next stratum transverse.

The region lying over $\sigma \times cB$ is now an *h*-cobordism with respect to any face $\partial_i \sigma \times cB$. The final step in constructing f_{j+1} is absorb these *h*-cobordisms into the part of g lying over a maximal cone, and adjust the remainder to match up with the stratification of S_{j+1} .



By induction we obtain an *h*-cobordism of $f_0 \times 1_n$ to $f_n: N \times T^n \to L$ which is transverse to the cone decomposition of L.

The next step is to unwind the circle factors in the blocks of f_n . Suppose $g_j: M_j \rightarrow L$ is transverse to the cones of L, and $M_j \rightarrow E \times T^j$ is an appropriate homotopy equivalence. Define M_j^{\sim} to be the cover of M_j corresponding to $E \times T^{j-1} \times \mathbb{R}$. Then $M_j^{\sim} \rightarrow L$ has two ends. Choose a manifold neighborhood W of one end such that ∂W is transverse to the inverse images of the cone decomposition. The transversality splits W into "half-open" pieces over cones in L, each of which is dominated. If all the finiteness obstructions varnish then there is a completion of M_j^{\sim} which is also transverse. The boundary of this completion would be the next stage, $M_{i-1} \sim E \times T^{j-1}$.

We organize the finiteness obstructions of the pieces of W into a homology class as in (b). This will be done using bordism of dominated Poincaré spaces.

Suppose Y is a space and $\omega: \pi_1 Y \to \mathbb{Z}/2$. We define a Δ -set $\Omega_j^{p,h}(Y, \omega)$ by: a k-simplex is dominated Poincaré (k+3)-ad of dimension j+k, $(V; \partial_0 V, \partial_1 V, \dots, \partial_k V, \partial_+ V)$, such that $\partial_+ V$ is a finite complex, together with a map $V \to Y$, and such that ω is the first Strefel-Whitney class of the normal fibration of V. It is easily seen (see eg. Quinn [31]) that these form a spectrum: $\Omega(\Omega_j^{p,h}(Y,\omega)) \cong \Omega_{j+1}^{p,h}(Y,\omega)$. Next, a small amount of Poincaré surgery (Quinn [32]) shows that the homotopy groups of the spectrum are

$$\pi_i(\Omega^{p,h}(Y,\omega)) = H^i(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}[\pi_1 Y]), \bar{\omega}), \quad i \ge 4.$$

This is because a map $S^i \to \Omega^{p,h}(Y, \omega)$ corresponds to an *i*-dimensional dominated Poincaré pair with finite boundary $(V, \partial_+ V) \to Y$. The finiteness obstruction in $\tilde{K}_0(\mathbb{Z}\pi_1 Y)$ satisfies the duality formula $q_0 = (-1)^i \bar{\omega}(q_0)$, and changing V by a bordism with finite boundary changes the obstruction by the symmetrization of the finiteness of the bordism. It is therefore well defined in $H^i(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}\pi_1 Y), \bar{\omega})$). The surgery is required to reduce $\pi_1 V$ to $\pi_1 Y$, and in case $q_0(V, \partial_+ V) = a + (-1)^i \bar{\omega}(a)$, to construct a bordism with invariant a.

The spectrum $\Omega^{p,h}$ (or at least its homotopy groups) are familiar from surgery theory, where it appears in the Rothenberg sequence relating L^p and L^h .

Next note that $\Omega^{p,h}$ has a natural module structure over Ω^{so} , the smooth oriented bordism spectrum (by cartesian product). Further since the homotopy of $\Omega^{p,h}$ is 2-torsion (above dimension 5), it is a module over the localization $(\Omega^{SO})_{(2)}$. But the orientation homomorphism $(\Omega^{SO})_{(2)} \rightarrow (B^*(\mathbb{Z}))_{(2)}$ splits, so in a standard way it follows that $\Omega^{p,h}$ is a product of Eilenberg-MacLane spectra (B is used here to denote the E-M spectrum, to avoid confusion with K-theory). This fact globalizes: We can apply $\Omega^{p,h}$ fiberwise to a stratified system of fibrations $p: E \to L$ to obtain $\widehat{\Omega}^{p,h}(p,\overline{\omega}) \to L$ a stratified system of spectra (Sect. 8). Since the Ω^{SO} module structure is natural, this system of spectra is Eilenberg-MacLane systems $\pi_i B^i$ equivalent the product of to $(H^{i}(\mathbb{Z}/2; \tilde{K}_{0}(\mathbb{Z}\pi_{1}p), \bar{\omega}))$ (for dimension $i \geq 4$). Therefore we get homology classes as in (b) from homology with coefficients in the system of spectra $\Omega^{p,h}(p,\omega)$.

A cohomology class, in H^k , corresponds to a section of the projection $\Omega^{p,h}_{-k}(p,\omega) \rightarrow L$. This means an assignment which takes a *j*-simplex σ to a dominated Poincaré (j+3)-ad $(V_{\sigma}; \partial_0 V, ..., \partial_j V, \partial_+ V)$ of dimension j-k, with $\partial_+ V$ a finite complex.

Now return to the situation $M_j \cong N^{r-1} \times T^j$ and neighborhood of the end W in M_j^\sim . We can form the homotopy completion of W (7.7) blockwise over the cones of L. Further, since the fibers are equivalent to finite complexes, the boundary of the completion is approximately homotopy equivalent to a finite complex decomposed as iterated mapping cylinders over the dual cones of L (see the proof of 2.2.3). Replacing the boundary of the completion by this gives a dominated Poincaré pair over L, with finite boundary, transverse to the dual cones.

The next step is to interpret this as a homology class. Choose a simplicial embedding of L in \mathbb{R}^m , and let U be a regular neighborhood. Then the dual cones of U intersect L in the dual cones of L. Since U is a manifold, the dual cones are cells. They can therefore be subdivided into simplices. If this is done linearly, then generically L is transverse to the subdivision and is subdivided also (though not into cones). Finally the Poincaré spaces lying over the cones of L can be subdivided (by adding collars on parts of the boundary) to be transverse to the subdivision of L. Composing, we get a Poincaré space mapping to U transverse to a triangulation and disjoint from ∂U . Since the total dimension is j+r-1 this gives a cohomology class (as explained above) in $H^{m-(r+j-1)}(U, \partial U; \Omega^{p,h}(p^* \times T^{j-1}, \omega))$. Here p^* is the pullback of p to U over the collapse $U \to L$. The Poincaré dual of this is a class in

$$H_{r+i-1}^{lf}(U; \Omega^{p,h}(p^* \times T^{j-1}, \omega)) \cong H_{r+i-1}^{lf}(L; \Omega^{p,h}(p \times T^{-1}, \omega)).$$

The hypothesis that $r \ge \dim L + 6$ ensures that the low dimensional problems in $\Omega^{p,h}$ do not arise, so this defines an element in

$$\sum_{i} H_{i}^{lf}(L; H^{r+j+i-1}(\mathbb{Z}/2; \tilde{K}_{0}(\pi_{1}p \times \mathbb{Z}^{j-1}), \bar{\omega})).$$

The final modification of the invariant is to observe that since we will unwind the T^{j-1} factors anyway, we can permit changing the invariant by passing to finite covers in these coordinates. Recall that the definition of $K_{1-j}(\mathbb{Z}\pi)$ in Bass [4] is the summand of $\tilde{K}_0((\pi \times \mathbb{Z}^{j-1}))$ consisting of elements invariant under finite transfers in the \mathbb{Z}^{j-1} factor. It essentially follows from this that if the image in this submodule is zero, then the whole invariant vanishes in some finite cover. Therefore we define t_{j-1} to be the image in

$$\sum_{i} H_{i}^{lf}(L; H^{r+i+j-1}(\mathbb{Z}/2; \tilde{K}_{i-j}(\mathbb{Z}\pi_{1}\,p), \bar{\omega}))$$

Now suppose $t_{j-1} = 0$. Then in some cover the dominated Poincaré pieces are bordant to finite Poincaré complexes. But by 1.10(c) and 1.8(b) an *h*-cobordism of M_j can be used to achieve the same effect on the finiteness invariant as an arbitrary bordism. Therefore there is a (transverse) *h*-cobordism of $M_j \rightarrow L$ to a map whose cover over $T^{j-1} \times R$ can be completed to be transverse to the cones of *L*. The completion is the next manifold, M_{j-1} .

For the obstructions in (c), assume that M_j is *h*-cobordant to $N \times \hat{T}^j$. Then $t_{j-1}=0$ implies that there is a transverse *h*-cobordism of an appropriate cover of M_j to M'_j whose infinite cyclic cover over the last T^1 coordinate has a

transverse completion, with boundary M_{j-1} . The corresponding cover of the *h*-cobordism has an end which if completed gives an *h*-cobordism from M_{j-1} to $N \times T^{j-1}$. Since the boundary of the homotopy completion is finite (as in 2.2.3) the obstruction satisfies $q_0 = (-1)^{r+j} \overline{\omega}(q_0)$. It can be changed by *h*-cobordism rel ends, and this changes q_0 by a symmetrization. This gives a class in $H^{r+j}(\mathbb{Z}/2; H_0^{lf}(L; \mathcal{S}(p \times T^{j-1})), \overline{\omega})$. As above by passing to finite covers in the T^{j-1} coordinate (essentially up to $\mathcal{S}(\mathbb{R}^{j-1}; p \times 1) \cong \mathcal{S}_{j-1}(p)$) the significant part of the obstruction is seen to be the image in $H^{r+j}(\mathbb{Z}/2; H_{j-1}^{lf}(L; \mathcal{S}(p)), \overline{\omega})$. Define this image to be u_{j-1} .

The final refinement is to recall that in the process of splitting up N by crossing with T^1 , at the j^{th} stage it is split over cones down to codimension j. Keeping track of this restricts the obstruction t_j to H_i for $i \ge j$, and u_j to homology of the "skeleton". Details of this are omitted.

This completes the proof of 2.2.4.

2.2.5. Corollary. Suppose G is a finite group with an action on a compact manifold $(M; \partial_0 M, \partial, M)$ of dimension r which satisfies

1) $\partial_0 M \cup M^*$ has a PL structure in which the action is PL.

2) The pair $(M \{ \supseteq H \}, \partial_1 M \{ \supseteq H \})$ is 1-acyclic for all isotropy subgroups $H \neq 1$.

3) M^* has codimension ≥ 6 , is 1-LC embedded, $\partial_1 M^* \subset \partial_1 M$ is 1-LC embedded, and the homotopy link of M^* at each point has the homotopy type of a finite G-complex.

Then there are obstructions

a) The Kirby-Siebenmann triangulation obstruction in

$$H^4((M-M^*)/G, \partial_0(M-M^*)/G; Z/2).$$

b) The mapping cylinder obstruction in $H_0(M^*/G; \mathscr{G}(G_*))$

c) Splitting obstructions

$$\begin{split} t_1 &\in \sum_i H_i(M^*/G, \partial_0 M^*/G; \ H^{i+r+1}(\mathbb{Z}/2; K_{-1}(\mathbb{Z}G_*), \bar{\omega})), \quad and \ if \quad t_1 = 0, \\ t_0 &\in \sum_i H_i(M^*/G, \partial_0 M^*/G; \ H^{i+r}(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}G_*), \bar{\omega})). \end{split}$$

If these all vanish, then there is a combinatorial triangulation of M extending the one given on $\partial_0 M \cup M^*$, in which the action is PL.

For a simpler corollary see 2.2.6. The "homotopy link" in (3) is the fiber of the boundary of the homotopy completion of the end of the complement. This condition is satisfied if the action is locally triangulable, but the cone on (boundaries of, or doubles of) examples given in 2.1.4 shows that it may not be in general. Notation is that of Sect. 2.1.

Proof. By the analysis of the end of the complement given in Sect. 2.1, Theorem 2.2.4 (in a relative version) applies to $(\partial_0 M \cup M^*)/G \subset M/G$. Carter's [7] vanishing theorem for $K_{-j}(\mathbb{Z}G)$ reduces the obstructions in 2.2.4(b) to the ones in (c) above. The vanishing and the acyclicity hypothesis (2) shows that

$$H_{-i}(\partial_1 M^*/G; \mathscr{S}(BG_*)) \to H_{-i}(M^*/G; \mathscr{S}(BG_*))$$

is an isomorphism for $j \ge 0$. This and the duality formula for boundaries resulting from the fact that the boundary of the homotopy completion of the end is finite shows that the mapping cylinder obstruction for $\partial_1 M^*$ is determined by that for M^* . Further it follows that the obstructions in 2.2.4(c) (there is only one) can be avoided by changing the *h*-cobordism of the boundary.

Therefore if the obstructions (b) and (c) vanish, the conclusion of 2.2.4 applies. Suppose that the triangulation obstruction (a) also is zero. Then according to 2.2.2 the resulting *PL* structure on $(M - M^*)/G$ is isotopic to one in which the mapping cylinder is *PL*. It remains only to see that the resulting equivariant triangulation of M is combinatorial (is a *PL* manifold). For this we need to show that the link in M of $x \in M^*$ is a sphere. Take an equivariant link of the orbit of x, then the link of x itself has a natural action of the normalizer N(Gx). This link is a homology manifold which is a manifold in the complement of the singular set, and is a homotopy sphere ($\pi_1 = 1$ by the 1-LC hypothesis (3)). If it is a manifold it must be a sphere by the Poincaré conjecture. So the triangulation is combinatorial at x.

We can now proceed by induction. To show M is a PL manifold it is sufficient to show that a link, which satisfies the same hypothesis but has smaller dimensions, is a PL manifold. Since the codimension of M^* is ≥ 6 , the induction starts with links of dimension ≥ 5 , so the Poincaré conjecture (the only dimension-dependent ingredient) is available. Therefore M is a PL manifold.

2.2.6. Corollary. Suppose G is a finite group containing an element of order 2. Suppose there is a semifree action of G on $M \times I$, M a compact PL manifold, which satisfies

1) The action is PL on $M \times \{0\} \cup \partial M \times I$ and the triangulation extends to a triangulation of $M \times \{0\} \cup \partial M \times I \cup (M \times I)^*$.

2) The fixed point set $(M \times I)^*$ has codimension ≥ 6 , is 1-LC embedded, and $(M \times \{1\})^* \subset M \times \{1\}$ is 1-LC embedded.

3) $(M \times I)^*$ and $(M \times \{1\})^*$ are 1-connected.

Then there is an isotopy of the action rel $M \times \{0\} \cup \partial M \times I$ to a PL action if and only if $(M \times I - (M \times I)^*)/G$ has the homotopy type of a finite complex, and the homotopy link of each point of $(M \times I)^*$ in $(M \times I)/G$ has the homotopy type of a finite complex.

Notes. The question addressed here is roughly "if a topological action is concordant to a PL one, can it be triangulated?" The answer is nearly yes for the actions specified above.

The role of the element of order 2 in G is to conclude from Smith's theory that $H_i(M \times I)^*$, $M \times \{1\}^*$; $\mathbb{Z}/2 = 0$. This assumption can be substituted for the one on the group.

The setting is similar to that of Jones [22], where it is shown that given the action on $M \times \{0\}$ and a *PL* potential fixed set $(M \times I)^*$ satisfying the Smith theorems, (and a few other conditions) there exists a *PL* concordance with fixed set $(M \times I)^*$. If the $\mathbb{Z}/2$ acyclicity hypothesis is dropped, Jones [21] has

announced that there are $\mathbb{Z}/2$ homology obstructions to constructing a *PL* concordance. Comparing with 2.2.5 shows that there are $\mathbb{Z}/2$ homology obstructions to triangulating a topological concordance. One might expect that some of these are the same: there probably usually is topological concordance, and the *PL* concordance obstruction is part of the triangulation obstruction. Note the analogy with the relation of 2.1.4 with Oliver's work.

Finally note that in some cases one can get information about the finiteness obstructions (e.g. nilpotent actions Quinn [30], Jones [22]). The link obstructions are trivial if the action is locally triangulable.

Proof. Corollary 2.2.5 applies, but as remarked in the notes Smith theory implies that $H_*((M \times I)^*, (M \times 1)^*; \mathbb{Z}/2) = 0$. Therefore the obstructions in 2.2.5(c) are zero. It also follows that the complements are $\mathbb{Z}/2$ acyclic, so the Kirby-Siebenmann obstruction is zero. Finally, the mapping cylinder obstruction is identified with the finiteness of the complement as in 2.1.2 and 2.1.3.

2.2.7. Locally Triangulable Spaces. Theorem 2.2.4 applies directly to the triangulation question for locally triangulable spaces. After a discussion of this we compare the conclusions with the work of Anderson and Hsiang [2].

A locally triangulable space is stratified by the topological intrensic skeleta: two points are in the same stratum if there is an ambient isotopy carrying one to the other. These strata are manifolds, so we may apply 2.2.4 inductively to triangulate. Suppose X is locally triangulable, $X_i \subset X$ is a closed union of strata, and Y is a stratum such that $X_i \cup Y$ is closed. If X_i is triangulated then $(X_i \cup Y, X_i)$ satisfies the hypothesis of 2.2.4 except for the codimension condition dim $Y \ge \dim X_i + 6$. If this holds also then we get a list of obstructions: (a), (b), (c) from 2.2.4, and the Kirby-Siebenmann obstruction to triangulating Y. If these all vanish then the triangulation can be extended to $X_i \cup Y$. Note that no restrictions are put on the intrensic skeleton of the triangulation. In particular it need not agree with the topological skeleton.

A few direct conclusions can be drawn from this. For instance if the skeleta are $\mathbb{Z}/2$ acyclic then the worst of the obstructions vanish. Also product formules for the obstructions show that if X is locally triangulable and satisfies the codimension condition, then X is triangulable if and only if $X \times CP^2$ is.

The most complete previous treatment of this topic is that of Anderson and Hsiang [2]. Their approach is dual to the one given here in several senses. First they consider the problem of extending triangulations to a neighborhood of a single (manifold) stratum, rather than the whole lower skeleton. However they allow this neighborhood to be stratified rather than just a manifold as in our case.

Restrict to a situation where both approaches apply, namely two triangulated manifold strata, one in the closure of the other, and consider the problem of triangulating the union. Under a fairly stringent form of the local triangulability hypothesis they show (Theorem A) that there is a lifting problem for the classifying map of a "tangent microbundle". If there is a lift then there is a global triangulation which agrees with the ones given on the strata, and further the lower stratum is the PL intrensic skeleton of the triangulation. Our approach begins with less data: a weak (homotopy) form of local triangulability. There is then a list of obstructions whose vanishing implies a global triangulation. The conclusion is also less stringent in that no restrictions are put on the PL intrensic skeleta.

Next we compare the obstructions. Anderson and Hsiang [2], page 227 describe a spectral sequence for the homotopy groups involved in their lifting problem. The lifting obstruction can therefore be considered as a sequence of cohomology classes. These classes are for the most part Poincaré dual to the classes in 2.2.4(b). The differences are: they have an extra set of Wh obstructions on their q=0 line, which come from requiring the lower stratum to be the intrensic skeleton of the result (see 2.3). We have obstructions on their p+q=0 line coming from the weaker local triangulability hypotheses. Finally the mapping cylinder obstruction 2.2.4(a) shows up on their p=m line. Notice that our obstructions 2.2.4(c) do not appear. Presumably this means that they are redundant. This illustrates the final difference: their obstructions are well organized (as a lifting problem) and efficient in the sense that there is a realization theorem. Our obstructions are more functorial and defined in more primitive circumstances, but are not as well organized and are not efficient.

Finally we remark that our techniques, especially 2.2.3, can be used to find the relative homotopy groups of Anderson and Hsiang's classifying spaces.

2.3. Approximate Fibrations. Suppose M is a topological manifold without boundary, L a polyhedron, and $f: M \rightarrow L$ an approximate fibration. If the homotopy fiber has dimension ≥ 5 and satisfies $Wh(\pi_1 \times \mathbb{Z}^k) = 0$ for all k, then by Ends I, 3.3.2, f is concordant to a topological block bundle projection. The result here is that if the Whitehead group hypothesis is relaxed then obstructions are encountered. The obstructions are almost the same as those for block neighborhoods in 2.2.4. In particular they are preliminary to the same degree: there is no realization theorem, and in fact there are indications that some of them are redundant (see 2.2.7). Chapman and Ferry [11] have constructed examples realizing some of these obstructions.

2.3.1. Theorem. Suppose $f: M \to L$ is an approximate fibration of a manifold without boundary over a polyhedron. Suppose dim $M = m \ge \dim L + 5$. Then there are obstructions

1) a sequence $t_j \in \sum_{i \ge j} H_i^{lf}(L; H^{i+j+m+1}(\mathbb{Z}/2; \tilde{K}_{-j}(\mathbb{Z}\pi_1 f), \bar{\omega}))$ for $j \ge -1$, such

that t_{j-1} is defined if t_i^{-1} is defined and is zero.

2) If the obstructions (1) are zero, then a sequence

$$u_i \in H^{m+j}(\mathbb{Z}/2; H^{lf}_{-i}(L(j); \mathscr{S}(f)), \bar{\omega}), \quad j \ge 0$$

such that u_{j-1} is defined if u_i is defined and is zero, and then

 $u_{-1} \in H_1^{lf}(L; \mathscr{S}(f)) / \{a + (-1)^{m+1} \bar{\omega}(a) | a \in H_1^{lf}\}.$

If all of these vanish, then for every $\varepsilon > 0$ there is an approximate fibration $M \times I \rightarrow L \times I$ which is f over $L \times \{0\}$, is a topological block bundle over $L \times \{1\}$, and has radius $< \varepsilon$ as a homotopy.

The group $\tilde{K}_1(\mathbb{Z}\pi)$ which occurs in (1) for j = -1 is understood to be $Wh(\pi)$. L(j) in (2) is the "coskeleton," union of dual cones of codimension $\geq j$, as in 2.2.4. Appropriate analogs of variations 1-3 of Ends I, 3.3.2 are valid.

Proof. The proof is essentially that of 2.2.4: N^{r-1} is replaced by M^m , and is multiplied by circles until it is transverse to the dual cones. The obstructions t_j , $j \ge 0$ are encountered in deleting circle factors from the blocks. If these vanish we get a map transverse to the triangulation. Inverse images of cones are then *h*-cobordisms with respect to top dimensional faces. The obstruction t_{-1} is the obstruction to changing these to be *s*-cobordisms. It is constructed the same way as t_0 , except that we use the bordism spectrum $\Omega^{h,s}$ of finite Poincaré complexes with simple ($\tau = 0$) boundary. Lemma 4.1 of Ends I shows that a transverse map with cone inverses *s*-cobordism is concordant to a block bundle.

The obstructions u_i , $i \ge 0$ also arise as in 2.2.4. If these are zero then M is h-cobordant to a block bundle. The obstruction u_{-1} is q_1 of this h-cobordism, taken in the quotient of H_1 because the h-cobordism can be changed by h-cobordism rel ends.

3. The Algebraic Obstructions

In this section the obstruction for *h*-cobordisms with constant local fundamental groups are expressed "algebraically" in terms of geometric groups. This is essentially a summary of the material of Ends I. Then a "local cancellation of inverses" procedure is developed for ε isomorphisms. This procedure is a main ingredient of the stability theorem of the next section.

Suppose π is a group and X is metric space. The definitions of geometric $\mathbb{Z}[\pi]$ module, ε isomorphism, deformation, etc. are given in Ends I, Sect. 8. The only modification we make is that radius will be used instead of diameter. Thus $h: G_1 \rightarrow G_2$ has radius $<\varepsilon$ if for each generator x of $G_1, \underline{h}(x) \subseteq x^{\varepsilon}$ (\underline{h} the underlying set function). Similarly a deformation has radius $<\varepsilon$ if the composition of the underlying set functions has radius $<\varepsilon$, both for the deformation and its inverse deformation. Finally let h be a homomorphism of radius $<\varepsilon$ of geometric modules on X. Then an ε deformation of h over X - Y is a homomorphism of the form $H_1 h H_2$, where H_1, H_2 are compositions of an ε deformation, and a geometric (basis preserving) isomorphism of radius $<\varepsilon$.

3.1. Theorem. Suppose X is a metric space, $Y \subseteq X$, $e: (M^n, \partial_0 M; \partial_1 M) \to X$ is proper, and $\hat{M} \to M$ is a regular cover with group π .

(1) Suppose further that $(M, \partial_0 M)$ is an (ε, h) -cobordism over X - Y. Then a choice of ε handlebody structure and ε deformation retraction of M to $\partial_0 M$ over X - Y defines a geometric $\mathbb{Z}\pi$ module isomorphism over $X - Y^{3\varepsilon}$ of radius 3ε .

(2) Changing the choice of handle structure and deformation changes this isomorphism by stabilization and $2n\varepsilon$ deformation over $X - Y^{2n\varepsilon}$.

(3) Suppose X is locally 1-connected in a neighborhood of X - Y, X - Y is compact, and $n \ge 6$ and $\varepsilon > 0$ are given. Then there is $\delta > 0$ such that if ε as above is a (δ, h) cobordism over X - Y, $M^{\wedge} \rightarrow X$ is $(\delta, 1)$ connected over X - Y and the

isomorphism of (1) can be stably δ -deformed over X - Y to one induced by a bijection of bases, then $(M, \partial_0 M)$ has a product structure $(\partial_0 M \times I, \partial_0 M \times \{0\})$ over $X - Y^{\varepsilon}$ of radius $< \varepsilon$.

Proof. (1) Choose an ε handlebody structure on $(M, \partial_0 M)$: Then this defines a chain complex of $\mathbb{Z}\pi$ module on $X, C_* = C_*(M^{\wedge}, \partial_0 M^{\wedge})$ with boundary homomorphisms of radius $<\varepsilon$. A deformation retraction defines a contracting homotopy $s: C_* \to C_{*+1}$, i.e. $s\partial + \partial s = 1$ over X - Y. Then $s\partial s + \partial s\partial$: $\sum_{j \text{ even}} C_j \to \sum_{j \text{ odd}} C_j$ is a 3ε isomorphism over $X - Y^{3\varepsilon}$. In fact the same formula from odd to even gives the inverse.

(2) Changing the deformation retraction changes the isomorphism by something which raises degree, so is "upper triangular," hence a deformation. Two handle decompositions are related by introduction (or deletion) of cancelling pairs, isotopy between layers of handles, and handle additions. The first operation stabilizes (or removes stabilization in) the chain complex. The second operation does not change the complex. The third changes boundary homomorphisms, therefore the isomorphism, by multiplication by elementary matrices.

In the PL case the fact that two handlebody structures are related by such operations results from the fact that any two triangulations have common subdivisions. This also makes it clear that the deformations can be arranged to have small radius. See Hatcher and Wagoner [19] for the smooth case.

The third statement is the result proved in Ends I, Sect. 6. This completes the proof of 3.1.

There is a similar statement for approximate completions, and projections on geometric modules, which is essentially proved in Ends I, Sect. 7. This statement is omitted since it is a bit elaborate, and not used very strongly.

3.2. Local Cancellation of Inverses. If A is an isomorphism there is a matrix identity

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A^{-1} & I \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The first and last three terms on the left side are canonically products of elementary homomorphism, so this is a deformation from

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \text{ to } \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

This is the cancellation of inverses.

Now suppose X is a metric space, $Y \subseteq X$, and we have an ε homomorphism on a geometric module on X which over X - Y has the form

$$\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

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The goal is to chancel it away from Y, but leave it unchanged over Y. This is done by factoring the deformation above as $E_1 E_0 \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} F_0 F_1$, where E_0, F_0 are constant on Y, and E_1, F_1 are constant on $X - Y^{3\varepsilon}$. Applying E_0, F_0 then cancels the inverses over $X - Y^{6\varepsilon}$.

The deformation on the right is easy to factor. If we set

$$F_0 = \begin{cases} F \text{ over } X - Y^{\varepsilon} \\ \text{constant over } Y^{\varepsilon}, & \text{and} & F_1 = \begin{cases} \text{constant over } X - Y^{\varepsilon} \\ F \text{ over } Y^{\varepsilon} \end{cases}$$

then F_0 , F_1 are deformations and F_0 , $F_1 = F$. If $W = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$, define

$$W^{\delta} = \begin{bmatrix} I & B/Y^{\delta} \\ 0 & I \end{bmatrix}, \text{ and}$$
$$W^{-\delta} = \begin{bmatrix} I & B/X - Y^{\delta} \\ 0 & I \end{bmatrix}.$$

Similarly for $W = \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$. Notice that $W = W^{\delta} W^{-\delta}$, and that if V, W both have radius $<\varepsilon$ then W^{δ} and $V^{-\delta-\varepsilon}$ commute (since one is the identity where the other is not). Now we can factor the left hand deformation WVW by $(W^{3\varepsilon} W^{-3\varepsilon}) (V^{2\varepsilon} V^{-2\varepsilon}) (W^{\varepsilon} W^{-\varepsilon}) = (W^{3\varepsilon} V^{2\varepsilon} W^{\varepsilon}) (W^{-3\varepsilon} V^{-2\varepsilon} W^{-\varepsilon}).$

Notice that this defines a function on the set of ε homomorphisms which are of the form $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}$ over X - Y. It is important to note that this function is responsive to the size of the input in the sense that if the homomorphism is actually a δ homomorphism, $\delta < \varepsilon$, then the result is a 4δ homomorphism. In fact if the data has radius $<\delta$ on $K^{3\delta} \subset X$, then the result has radius $<3\delta$ on K.

The next object is to extend this cancellation function to deformations. The reader can skip this part without jeopardizing his understanding of the rest of the proof.

Suppose RAS is a deformation of A. The cancellation process above gives deformations

$$E_0\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} F_0$$
 and $E'_0\begin{bmatrix} RAS & 0 \\ 0 & S^{-1}A^{-1}R^{-1} \end{bmatrix} F'_0$,

so there is a composite deformation between the results of the process. The results are both $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ over $X - Y^{3\varepsilon}$ however, and we would like to modify this deformation to be constant off some neighborhood of Y.

Over $X - Y^{3\epsilon}$ the composite deformation is

$$\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$$
(1)

on the right, and on the left is

$$\begin{bmatrix} I & RAS \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -S^{-1}A^{-1}R^{-1} & I \end{bmatrix} \begin{bmatrix} I & RAS \\ 0 & I \end{bmatrix}$$

$$\cdot \begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ A^{-1} & I \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}.$$
(2)

The idea is to write these as products of elementary matrices, and use identities to replace some sequences in the product by different sequences.

Begin with the right side, (1). Define $W = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$, then the deformation is $W \begin{bmatrix} S & 0 \\ 0 & R^{-1} \end{bmatrix} W^{-1}$. The identity we want to use is

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} W^{-1} = W^{-1} \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}$$

R and S are deformations, so $R = \prod_{i=0}^{m} R_i$, $S = \prod_{i=0}^{n} S_j$, with elementary factors.

$$\begin{bmatrix} S & 0 \\ 0 & R^{-1} \end{bmatrix} = \prod_{i=1}^{n} \begin{bmatrix} S_{i} & 0 \\ 0 & I \end{bmatrix} \prod_{i=1}^{m} \begin{bmatrix} I & 0 \\ 0 & R_{m-i}^{-1} \end{bmatrix}$$

so consider the piece $\left(\prod_{0}^{m} \begin{bmatrix} I & 0 \\ 0 & R_{m-i}^{-1} \end{bmatrix}\right) W^{-1}$.

Change the deformation as follows: suppose it has the form

$$\prod_{0}^{k} \begin{bmatrix} I & 0 \\ 0 & R_{m-i}^{-1} \end{bmatrix} W^{-1} \prod_{k+1}^{m} \begin{bmatrix} R_{m-i}^{-1} & 0 \\ 0 & I \end{bmatrix}$$

off of the inverse image of $Y^{3\epsilon}$ under the composition of the underlying set functions for $\prod_{k+1}^{m} \begin{bmatrix} R_{m-i}^{-1} & 0\\ 0 & I \end{bmatrix}$. Then leave the deformation unchanged on this inverse image, and replace it by

$$\prod_{0}^{k-1} \begin{bmatrix} I & 0 \\ 0 & R_{m-i}^{-1} \end{bmatrix} W^{-1} \prod_{k}^{m} \begin{bmatrix} R_{m-i}^{-1} & 0 \\ 0 & I \end{bmatrix} \text{ elsewhere.}$$

Do the same for the $\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}$ part. This process gradually deforms the deformation to one unchanged over $Y^{2\epsilon}$, but of the form

$$WW^{-1}\begin{bmatrix} R^{-1} & 0\\ 0 & S \end{bmatrix} \quad \text{over } X - Y^{4\varepsilon}.$$

We refer to this process as "substituting the identity over $X - Y^{3\epsilon}$ ". Similarly substitute the identity $WW^{-1} = I$ over $X - Y^{4\epsilon}$, to end up with $\begin{bmatrix} R^{-1} & 0 \\ 0 & S \end{bmatrix}$ over $X - Y^{4\epsilon}$ on the right side.

On the left side, (2), the middle three factors are

$$\begin{bmatrix} I & RAS \\ 0 & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \quad \text{over } X - Y^{3\varepsilon}.$$

Substitute the identities

$$\begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -RAS \\ 0 & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} I & RAS \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -RAS \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \text{ over } X - Y^{3\varepsilon}.$$

This leaves

$$\begin{bmatrix} I & RAS \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -S^{-1}A^{-1}R^{-1} & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ A^{-1} & I \end{bmatrix} \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$$

over $X - Y^{5\varepsilon}$. Substitution of two similar identities leaves $\begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix}$ over X $-Y^{9\varepsilon}$.

Putting them together, the deformation of $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ now has the form

$$\begin{bmatrix} R & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & S \end{bmatrix} \quad \text{over } X - Y^{9\varepsilon}.$$

Substitution using the cancellation of inverses leaves just the constant deformation over $X - Y^{10\varepsilon}$.

Again we note that the construction is responsive in the sense that if the original data was small, the result is small. Notice also that these manipulations of deformation are essentially calculations in the Steinberg group. For example the process used in reducing (1) corresponds to a special case of Lemma 9.2 of Milnor [25].

4. The Stability Theorem

The main result of the section is that many geometric objects with ε control are stable with respect to ε , in the sense that they are equivalent to δ objects for any smaller δ . The proof is presented in the special case of ε isomorphisms of geometric modules. This is done to avoid the unpleasantness of an abstraction general enough to encompass all the situations which will arise. However the proof is quite formal, and the properties used are discussed in 4.2.

4.1. Theorem. (absolute stability theorem for ε isomorphisms). Suppose X is a locally compact metric ANR. Then there is $\varepsilon_0: X \to (0, \infty)$ such that

1) for any ε ($\varepsilon_0 > \varepsilon > 0$) there is δ ($\varepsilon > \delta > 0$) such that ε deformation is an equivalence relation on (locally finite) δ isomorphisms of geometric $\mathbb{Z}[\pi]$ modules any group π , and

2) the set of equivalence classes forms an abelian group, which is naturally isomorphic to the inverse limit of such groups as $\varepsilon \rightarrow 0$.

The fact that makes (1) nontrivial is that generally ε deformation is not transitive: the composition of two ε deformations may only be a 2ε deformation. The remedy for this is a "shrinking function" which takes ε deformations (of δ isomorphisms) to ones of radius $\langle \varepsilon/2 \rangle$. These can then be composed to yield an ε deformation. Similarly the statement (2) results from a shrinking function defined on objects.

Theorem 4.1 can be extended to stratified coefficient systems using the relative version 4.5. However in this setting the function ε_0 depends on the structure of the coefficient system.

4.2. Properties of ε Isomorphisms. There are four that we will use: functoriality, pullbacks, naive homotopy, and the local cancellation of inverses.

Functionial images are defined for proper maps. Suppose $f: X \to Y$ is proper, $\varepsilon: Y \to (0, \infty)$ and $\delta: X \to (0, \infty)$ are such that if $d(x_1, x_2) < \delta(x_1)$ then $d(f(x_1), f(x_2)) < \varepsilon(f(x))$, and suppose $D \subseteq Y$. Then a homomorphism $A: M_1 \to M_2$ of modules of X which is a δ isomorphism over $X - f^{-1}(D)$, has a functorial image $f_*A: f_*M_1 \to f_*M_2$ which is an ε isomorphism over Y - D. If $M = \mathbb{Z}\pi[x_i]$, then $f_*M_* = \mathbb{Z}\pi[f(x_i)]$, and the homomorphisms are defined in the evident way.

Pullbacks are defined for local homeomorphisms. Suppose $f: X \to Y$ is a local homeomorphism, with $D \subset X$, $\varepsilon: X \to (0, \infty)$ and $\delta: Y \to (0, \infty)$ such that f is injective on any ball of radius ε , and if $x \in X - D$ then the image $f(x^{\varepsilon})$ contains the ball $(f(x))^{\delta}$. (This is a uniform continuity type estimate on f^{-1} .) Then a δ isomorphism over Y - C, $A: M_1 \to M_2$, has a pullback $f^*A: f^*M_1 \to f^*M_2$ which is an ε isomorphism over $X - (f^{-1}(C) \cup D)$. For modules it is defined by $f^*\mathbb{Z}\pi[\{y_i\}] = \mathbb{Z}\pi[f^{-1}(\{y_i\})]$. If $A: \mathbb{Z}\pi[y_i] \to \mathbb{Z}\pi(y_j]$ then define $f^*A(x)$ for $x \in f^{-1}(y_i)$ by: let $A(y_i) = \Sigma n_j y_j$, then $f^*A(x) = \Sigma n_j x_j$, where $x_j \in f^{-1}(y_j)$, and $d(x_i, x) < \varepsilon$. Such an x_i may not exist, but if it does it is unique.

Naive homotopy is a natural deformation between functorial images of homotopic maps. Suppose $f: X \times I \to Y$ is a proper homotopy, and suppose $A: M_1 \to M_2$ is an isomorphism of geometric modules. Then there is a deformation $(f_0)_* A \sim (f_1)_* A$: let $B_i(f_0)_* M_i \to (f_1)_* M_i$ be the isomorphism induced by the bijection of bases $f_0(x_j) \to f_1(x_j)$. Then B_i is an isomorphism and $(f_1)_* A = B_2((f_0)_* A)B_1^{-1}$. If the homotopy f has radius $<\varepsilon$, then the deformation also has radius $<\varepsilon$.

The cancellation of inverses involves the additive structure. There is a sum operation (direct sum), a natural inverse function, and a natural cancellation (deformation to 0) of the sum of an object and its inverse. More specifically if A is an ε isomorphism over X - Y then A^{-1} is also, and the cancellation is a stable 3ε deformation which takes $A \oplus A^{-1}$ to a homomorphism on a geometric module which is trivial over $X - Y^{3\varepsilon}$.

This cancellation can be localized in the sense that if $W \subset X$ then the deformation $A \oplus A^{-1} \sim 0$ can be factored canonically into two deformations. The first deformation is trivial over $X - W^{3e}$, but cancels $A \oplus A^{-1}$ over W. The second deformation is trivial over W, and finishes cancelling the sum over the rest of X. Finally this localization procedure extends in an appropriate way to deformations. The cancellation of isomorphisms is discussed in detail in Sect. 3.

We also use the fact that two ε deformations can be composed to yield a 2ε deformation.

The canonical nature of these operations, and the consequent constructions, is important in several ways. Many of the applications of this theorem involve Δ -sets rather than just sets (see 5.6). So the construction must be canonical for simplices of various dimensions to fit together. But even with sets of isomorphisms our approach will be to use the canonical nature to define functions which are "responsive" in the sense that the size of the output can be estimated in terms of the size of the input. These a priori estimates are necessary for the construction as it is set up here.

Outline of the Proof. We begin with the homotopy lemma, which is useful in other situations. The main body begins with a precise statement of a relative form, 4.5. There is a series of reductions, ending with a handle in an *n*-manifold. Here we use a Kirby type torus argument; pull back over an immersion of a punctured torus, use a completion lemma to fill in the puncture, pass to the universal cover. The desired reduction in radius is then achieved by shrinking radially in \mathbb{R}^n .

4.3. Lemma. There is a responsive function C_Y defined on ε homomorphisms which are ε isomorphisms over X - Y, and ε deformations of these, with values in

6ε isomorphisms and deformations, such that $C_{Y}(A)$ is equal to $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}$ over X

 $-Y^{20\varepsilon}$, and is 0 over Y. C_Y acts similarly on deformations, and the standard deformation

 $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad over \ X - Y^{20\varepsilon}$

extends canonically to a deformation $C_{\gamma}(A) \sim I$.

The point is that $C_Y(A)$ is an isomorphism everywhere, not just over X - Y.

Proof. Form $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}$. Hold it fixed over $Y \cup X - Y^{20\varepsilon}$, and apply the local cancellation of inverses to cancel over $Y^{11\varepsilon} - Y^{9\varepsilon}$. The resulting homomorphisms and deformations are then trivial over $Y^{11\varepsilon} - Y^{9\varepsilon}$. This "band" separates $X - Y^{11\varepsilon}$ from $Y^{9\varepsilon}$, so we can discard the part over $Y^{10\varepsilon}$. This removes the part which failed to be an isomorphism. What is left is $C_Y(A)$.

This construction is represented pictorially as follows:



The next lemma states that if f is homotopic to 1_x , then the functorial image f_*A can be stabily deformed to A by a deformation much shorter than the one given by the naive homotopy property. A more elaborate statement (for pseudoisotopies, but the appropriate analog holds) is given in 5.7.

4.4. Homotopy Lemma. Suppose $F: X \times I \to X$ has $F^{-1}(Y) = Y \times I$, $F_0 = 1_X$, and F satisfies the Lipschitz condition with constant r. Let $\varepsilon > \gamma > 0$. Then there is a function \hat{F} defined on ε homomorphisms which are ε isomorphisms over X - Y, and ε deformations of these. If A is one such, then

1) $\widehat{F}(A) = A$ over Y.

2) $\hat{F}(A) = (F_1)_* A \text{ over } X - Y^{20re}$.

3) \hat{F} is responsive in the sense that if A is a δ isomorphism (deformation) over X - Y, $\varepsilon > \delta > \gamma$, then $\hat{F}(A)$ is a $\delta r \delta$ isomorphism (deformation) over X - Y. Further there is a canonical stable 13r δ deformation $A \sim \hat{F}(A)$ which is constant over Y.

Proof. Fortunately the proof is little longer than the statement. Choose $0 = t_0 < t_1 \dots < t_n = 1$ such that $d(F_{t_1}, F_{t_{1+1}}) < \gamma$. Suppose A is a δ isomorphism over X - Y. Then there is a stable $6r\delta$ deformation

$$A \sim A \oplus \sum_{i=1}^{n} C_Y((F_{t_i})_* A).$$

Over $X - Y^{6r\delta}$ this is

$$A \oplus \sum_{i=1}^{n} (F_{t_i})_* \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

Rearrange as $\left[\sum_{i=1}^{n} (F_{t_{i-1}})_* A \oplus (F_{t_i})_* A^{-1}\right] \oplus (F_1)_* A$, and use the naive homotopy property to γ deform $(F_{t_{i-1}})_* A \sim (F_{t_i})_* A$. This gives

$$\left[\sum_{i=1}^{n} (F_{t_i})_* (A \oplus A^{-1})\right] \oplus (F_1)_* A,$$

so we can apply the local cancellation $A \oplus A^{-1} \sim 0$ over $X - Y^{13r\delta}$. This gives a stable $7r\delta$ deformation, constant over $Y^{6r\delta}$ to something which is $(F_1)_*A$ over $X - Y^{13r\delta}$. Define the result of this last deformation to be $\hat{F}(A)$.

We represent this pictorially: (the homotopy F pushes things straight up).



(i) the data



(ii) introduce the $C_{\gamma}(F_i A)$



(iii) cancel adjacent inverses to yield $\hat{F}(A)$.

The responsiveness of \hat{F} follows from that of C_y . However it is bounded by γ because of the spacing chosen between the F_{t_x} .

The main body of the proof begins with a relative version of the theorem.

To state this, presume X is a locally compact metric ANR, $X \supset W_1 \supset W_2$, $X \supset Y, Z$ all closed. Suppose that the closure $(X - Y)^-$ is compact, and Y is disjoint from $(Z - W_2)^-$.



The goal is to define a "shrinking function" which takes an isomorphism over X - Y which is small over W_1 , and deforms it to one which is also small over Z. The statement is complicated by the fact that while shrinking over Z it expands everywhere else. Clearly the rate of expansion has to be very carefully controlled if we are to get any benefit from the process.

4.5. Theorem. In the situation above there is $\varepsilon > 0$ and a homeomorphism $r: [0, \infty) \rightarrow [0, \infty)$ with $r(t) \ge t$ such that for every $\gamma > 0$, $\gamma < \varepsilon$, there is a "shrinking function" S_{γ} defined on endomorphisms of geometric modules on X which are ε isomorphisms over X - Y, and ε deformations of these. This function satisfies the following:

1) $S_{y}(A) = A$ over Y.

2) If $\varepsilon > \alpha > \beta \ge \gamma$ and A has radius $<\beta$ over W_1 and $<\alpha$ over X, then $S_{\gamma}(A)$ has radius $< r(\beta)$ on $W_2 \cup Z$, $< r(\alpha)$ on X. S_{γ} acts similarly on deformations.

3) If the bounds of (2) hold, there is a stable deformation from A to $S_{\gamma}A$ constant over Y, of radius $\langle r(\beta) \text{ over } W_2 \text{ and } \langle r(\alpha) \text{ over } X.$

Proof. The proof begins with a sequence of reductions. We indicate only the salient details.

First isometrically embed a neighborhood of $(X - Y)^-$ in the Hilbert cube. Since X is an ANR, neighborhoods of the image retract to it. Note that if the theorem holds for a neighborhood which retracts properly to the image (with appropriate W_i , etc.), then the theorem holds for X: just project the solution in a neighborhood down to X. The control function r changes by uniform continuity estimates on the retractions. This is why we need a retraction and not just nearby approximation as in part I.

Next note that (just because of the topology on the cube I^{∞}) the neighborhood can be chosen of the form $V \times I^{\infty - k}$, where $V \subset I^k$ is a codimension 0 submanifold. The subsets similarly can be chosen to be of the form $W_i \times I^{\infty - k}$ for $W_i \subset V$, etc. Now we claim that if the theorem holds for compact manifolds it holds for $V \times I^{\infty}$, $W_i \times I^{\infty}$, $Y \times I^{\infty}$, $Z \times I^{\infty}$. The shrinking function is defined by using the homotopy lemma and the standard deformation to $V \times \{0\}$ to deform things on $V \times I^{\infty}$ rel $Y \times I^{\infty}$ so that a little way out from $Y \times I^{\infty}$ they lie over V

 \times {0}. Then apply a shrinking function in $V \times$ {0}. The use of the homotopy lemma here (and elsewhere) introduces the lower bound γ on the amount of shrinking which can be achieved.

The final reduction is to note that if it holds for handles, it holds for compact manifolds. In piecing together shrinking functions in a handlebody it is quite important to know ahead of time (via the control function r) how much peripheral swelling there will be. It can then be allowed for by a little extra shrinkage in earlier steps.

Therefore let

$$X = D^j \times D^k, \quad Y = \partial (D^j \times D^k), \quad W_1 = W_2 = D^j \times (D^k - \frac{1}{2}D^k), \quad Z = (\frac{1}{2}D^j) \times D^k.$$

The first step in shrinking things on Z is simple. Consider the linear homotopy on D^{j} from the identity to the radial map which is the identity on the boundary and takes $\frac{1}{2}D^{j}$ to 0. Cross with the identity on D^{k} . The result has Lipschitz constant 2, so the homotopy lemma gives a function which compresses the part of an isomorphism over $\frac{1}{2}D^j \times \frac{1}{2}D^k$ to $\{0\} \times \frac{1}{2}D^k$, and increases radius by less than a factor of 12.



(ii) the first step

Call the resulting homomorphism A.

A is good everywhere except over $\{0\} \times \frac{1}{2}D^k$, and even there the radius is sufficiently small in the D^{j} coordinates. It remains to shrink in the D^{k} coordinate over $\{0\} \times \frac{1}{2}D^k$. This is achieved with a Kirby type torus argument.

Let $f: (T^k - *) \rightarrow D^k$ be a smooth immersion which is the identity on a neighborhood of $\frac{1}{2}D^k$. Since it is smooth there are d>1 such that if $0 < \delta < d$ and $t \in T^k$ such that $d(t, *) > r\delta$, then $f | t^{r\delta}$ is a homeomorphism onto an open set which contains the ball $(f(t))^{\delta}$.

Suppose $12\varepsilon < \delta$. Then $12\alpha < \delta$ also, so there is a well defined pullback (1) $(x f)^* A$ over $D^j \times (T^k - *)$, which is at least an $r\delta$ isomorphism over $D^j \times T^k$ $-(S^{j-1}\times T^k\cup D^k\times (*)^{r\delta}).$

The next step is a lemma which extends the isomorphism across the puncture in the torus.

4.6. Completion Lemma. Given (X, Y), and $\varepsilon > 0$, there is a completion function D defined on ε isomorphisms over $X \times I - (Y \times I \cup X \times \{0\})$, and ε deformations of these. This function satisfies

1) D(A) is a 6 ε isomorphism or deformation over $X \times I - Y^{6\varepsilon} \times I$, and is equal to A over $X \times (20\varepsilon, 1]$.

2) D is responsive in that if A is a δ isomorphism then D(A) is a $\delta\delta$ isomorphism.

Proof. Let B be the image of A^{-1} under the retraction $X \times I \to X \times [0, 20\varepsilon]$ which takes [20 ε , 1] to 20 ε . Then over $X \times (0, 20\varepsilon) A \oplus B$ is $A \oplus A^{-1}$. Cancel in the center of this strip as in 4.3, and discard the part over $X \times [0, 10\varepsilon]$. The result is D(A).

We resume the proof of 4.5. Let $g: T^k \to [0, 1]$ be distance from $*^{r\delta}$, and project $(1 \times f)^* A$ to $D^j \times [0, 1]$ by $1 \times g$. Apply the completion lemma to the result. We obtain an isomorphism over (at least) $(1 - 6r\delta) D^j \times I$, which agrees with A over $D^j \times (20r\delta, 1]$. Since it agrees we can lift it back up to $D^j \times T^k$ by putting the new stuff over $D^j \times \{*\}$. This yields a homomorphism on $D^j \times T^k$ which is an isomorphism over $(1 - 6r\delta) D^j \times T^k$.

Pull this homomorphism up to the universal cover $D^j \times \mathbb{R}^k$, and call the result B. We list the properties of B.

1) *B* is an isomorphism over $(1-6r\delta)D^j \times \mathbb{R}^k$, and agrees with *A* over $\frac{1}{2}D^j \times \frac{1}{2}D^k$.

2) In the D^{j} coordinate B has radius $< 6 \cdot 12\beta$ (the only expansion coming from the completion lemma).

3) In the \mathbb{R}^k coordinate *B* has radius $< 6 \cdot 12r\beta$ except over $0 \times \mathbb{R}^k$, where it is $< 6 \cdot 12r\alpha$. (Expansion coming from both *f* and the completion lemma.)

Next project B to $D^j \times D^k$ by the identity on D^j , the identity on $\frac{1}{2}D^k$, and by radial compression $\mathbb{R}^k - \frac{1}{2}D^k \rightarrow D^k - \frac{1}{2}D^k$. This gives a homomorphism which satisfies the estimates of B, but in addition gets very very small near $D^j \times S^{k-1}$ (where it also fails to be locally finite). Call this C. Then we can cancel $C \oplus C^{-1}$ as in 4.3 near $S^j \times D^k$, and over $D^j \times (D^k - tD^k)$ where t is very close to 1. This yields a global isomorphism (now locally finite) over $D^j \times D^k$, with a small natural deformation to the identity.

The penultimate step is to add this to A. Over $\frac{1}{2}D^j \times \frac{1}{2}D^k$ this has the form $A \oplus A^{-1} \oplus A$. Cancel A and the copy of A^{-1} in the new piece over $\frac{1}{2}D^j \times \frac{1}{2}D^k$. The result still looks like A over $\frac{1}{2}D^j \times D^k$, but this part is embedded in a copy of C (cancelled near the edges). We can apply the homotopy lemma to this piece, using the homotopy obtained from multiplication $[s, 1] \times \mathbb{R}^k \to \mathbb{R}^k$ by conjugating by the projection $\mathbb{R}^k \to B^k$ used above. If t is chosen sufficiently near 1 and s sufficiently near 0, then the homomorphism over $0 \times \frac{1}{2}D^k$ will be shrunk to radius $< \gamma$ in the D^k coordinate by this operation, and expanded at most by a factor of 6 elsewhere.



This completes the proof of Theorem 4.5, and the stability Theorem 4.1 can easily be deduced from this.

5. Finite Structure Spectra

Given a map $p: E \to X$ we define a spectrum $\mathscr{S}(X; p)$ designed to measure obstructions to existence and uniqueness of finite complex structures with ε control in X. The main theorem of the section is that for reasonable p these are homology spectra: \mathscr{S} can be applied fiberwise to give a "spectrum over X", $\mathscr{S}(p) \to X$, and $\pi_* \mathscr{S}(X; p) = H_*^{lf}(X; \mathscr{S}(p))$. The actual identification of these as the end obstruction groups takes place in the next section.

These spectra extend the "Whitehead spaces" defined by Hatcher [17], although for technical reasons we use a pseudoisotopy description of them. Pseudoisotopy spaces have been considered by Anderson and Hsiang [1], Burghelea and Lashoff [6], Hatcher [17, 18], and many others.

5.1. Definition. Fix a locally compact metric space X, $p: E \to X$. If $C \subseteq X$ is compact, $\varepsilon > 0$, and K a polyhedron, then a *family over K with support C, radius* ε and dimension n consists of (U, r, θ) , where

(a) U is a codimension 0 submanifold of \mathbb{R}^n , $r: U \to E$ is continuous and pr is proper.

(b) θ is a topological embedding, θ : $(pr)^{-1}(C) \times I \times K \to U \times I \times K$ which satisfies

(1) θ commutes with projection to K,

(2) θ restricted to

 $[(\mathbf{pr})^{-1}(U) \times \{0\} \times K] \cup [(\partial U \cap (\mathbf{pr})^{-1}(C)) \times I \times K]$

is the inclusion,

(3) $\theta((\operatorname{pr})^{-1}(C) \times \{1\} \times K) \subseteq U \times \{1\} \times K$,

(4) the image of θ contains $(pr)^{-1}(C^{-\varepsilon}) \times I \times K$, and

(5) pr θ and pr θ^{-1} (where defined) have radius $<\varepsilon$ when considered as homotopies of $(pr)^{-1}(C) \times \{0\} \times K$ in X.

Although not displayed, the support and radius are part of the data of the family.

To understand this definition, consider some special cases. First let X = point. Then this is a family (parametrized by K) of pseudoisotopies of a compact manifold; isomorphisms $U \times I \rightarrow U \times I$ which are the identity on $U \times \{0\}$ and $\partial U \times I$. The map to E serves to control the fundamental group.

Next consider a family of pseudoisotopies with a reference map to E, and suppose $p: E \to X$ is nontrivial. Then (b5) is the straightforward way to introduce an ε condition into the situation. Finally the general situation (with the conditions involving the subset C) formalizes what one gets by restricting an ε family of pseudoisotopies to $(pr)^{-1}(C^{\varepsilon})$.
5.2. Operations on Families. Suppose (U, r, θ) is a family over K with support C and radius ε .

1. Reduction. Suppose $D \subseteq C$ is compact, and $\delta \geq \varepsilon$. If $U' \subset U$ is a closed codimension 0 submanifold which contains $\theta((pr)^{-1}(D) \times I \times K)$, and r', θ' are restrictions, then (U', r', θ') is a family with support D and radius δ . This is called a *reduction* of (U, r, θ) to D.

2. Deletion. Suppose that $V \subseteq U$ is a closed codimension 0 submanifold such that $\theta | [(U-V)^- \cap pr^{-1}(C)] \times I \times K$ is the inclusion. Denote by r', θ' the restrictions of r, θ to V, $V \times I \times K$. Then (V, r', θ') is a family over K with support C and radius ε . This is said to be obtained by deleting the region U-V, where θ is the identity.

3. Suspension. Consider $I^j \times U$. (id) $\times \theta$ satisfies the axiom 5.1 except for b2: it is not the identity on $(\partial I^j) \times U \times I \times K$. To remedy this, let $S_j: I^j \times I \to I^j \times I$ be an isomorphism which carries $\partial I^j \times I \cup I^j \times \{0\}$ to $I^j \times \{0\}$. Define an isomorphism on $I^j \times U \times I \times K$ (also denoted S_j) by using S_j on the first and third factors, the identity on the second and forth. Then the suspension of (U, r, θ) is $(I^j \times U, rq, S_j^{-1}((\operatorname{id}) \times \theta)S_j)$, where $q: I^j \times U \to U$ is the projection. We will also denote this by $\Sigma^j(U, r, \theta)$.

4. Inverse. Condition b4 ensures that there is an inverse for θ on $(pr)^{-1}(C^{-\epsilon}) \times I \times K$. Condition b5 implies that (U, r, θ^{-1}) is a family over K with support $C^{-\epsilon}$ and radius ϵ . This is the *inverse* of (U, r, θ) .

Notice that the suspension operation is not quite canonically defined because the isomorphism S_j is not. The reason for this ambiguity is that we want iterated suspensions to be suspensions, and there are no choices of S_j which satisfy all the necessary identities. However the space of such S_j is contractible, so the construction is well defined in a "homotopy everything" sense.

The next step is to define a space of pseudoisotopies. The usual approach is to use the Δ -set with k-simplices the families over Δ^k , and boundaries the restrictions to $\partial_j \Delta^k$. However, we wish to "identify" families which differ by the first three operations of 5.2. This is achieved by incorporating the operations into the boundary operations. If (U, r, θ) is a family over Δ^k , $\partial_j(U, r, \theta)$ will denote the restriction to $\partial_i \Delta^k \subseteq \Delta^k$.

5.3. Definition. Suppose $p: E \to X$, X locally compact metric. The pseudoisotopy space $\mathscr{P}(X, p)$ is the Δ -set with simplices defined inductively: a 0-simplex is a family over a point (with unrestricted support and radius). A k-simplex σ consists of a family over Δ^k , $(U_{\sigma}, V_{\sigma}, \theta_{\sigma})$, together with k+1 (k-1)-simplices $\partial_0 \sigma, \ldots, \partial_k \sigma$. We require these to satisfy the usual $\partial_i \partial_j$ identities, and in addition require that the suspension of a reduction of the underlying family of $\partial_i \sigma$ be obtained by deleting something from the family $\partial_i (U_{\sigma}, V_{\sigma}, \theta_{\sigma})$.

We define the support and radius of a simplex of \mathscr{P} to be that of the underlying family. The space \mathscr{P} will be a convenient place to work, but is not what we want because no restrictions have been placed on radius and support. Roughly speaking what we want is to index subspaces of \mathscr{P} on support C and radius ε , and take the (homotopy) inverse limit as $C \to X$ and $\varepsilon \to 0$. The next definition has the limit built into it.

5.4. Definition. Suppose X is locally compact metric, $p: E \to X$, and $Y \subseteq X$. Then a k simplex of $\mathscr{P}_{-2}(X, Y, p)$ is defined to be a simplical map $\Delta^k \times [0, \infty) \to \mathscr{P}(X; p)$ which satisfies the following conditions: First there is a sequence of compact sets C_i with $C_i \subseteq (C_{i+1})^0$, $\bigcup C_i \supseteq X - Y$, and C_i is contained in the support of simplices in the image of $\Delta^k \times [i, \infty)$. Secondly there is a sequence ε_i of numbers monotone decreasing to 0, such that the radius of simplices in the image of $\Delta^k \times [i, \infty)$ is less than ε_i .

As yet the subscript -2 has no significance. Eventually it will be the index in the spectrum structure.

5.5. The Kan Condition. Before we do anything substantial with these spaces we discuss a basic awkwardness: they fail to satisfy the Kan condition. The main value of this discussion is to give the reader some sample manipulations of simplices.

Consider for example two 1-simplices σ_0, σ_1 of $\mathscr{P}(X; p)$ with $\partial_0 \sigma_0 = \partial_0 \sigma_1$. The Kan condition would assert that there is a 2-simplex τ with $\partial_0 \tau = \sigma_0, \partial_1 \tau = \sigma_1$. The basic idea for constructing such a family over Δ^2 is to take the union of the underlying families of σ_0, σ_1 to get a family over $\partial_0 \Delta^2 \cup \partial_1 \Delta^2$. Next cross with I to get a family over $(\partial_0 \Delta^2 \cup \partial_1 \Delta^2) \times I$, and then use an isomorphism $\Delta^2 \cong (\partial_0 \Delta^2 \cup \partial_1 \Delta^2) \times I$. The problem is that the two families are not equal on the overlap $\partial_0 \Delta^2 \cap \partial_1 \Delta^2$, but are related by operations.

Let the underlying family of σ_i be (U_i, r_i, θ_i) , and let that of $\partial_0 \sigma_i$ be (V, s, ψ) . Deletions of $\partial_0(U_i, r_i, \theta_i)$ are equal to reductions of suspensions of (V, s, ψ) . As a first step suspend (V, s, ψ) and one of the (U_i, r_i, θ_i) so they are all the same dimension. They are then related by reductions and deletions. Reduce the support of all of them to the intersection of the supports of σ_0, σ_1 . They are then related by deletions: $V \subseteq U_i$, and $\partial_0 \theta_i$ is the identity on $U_i - V$. If $U_0 - V$, $U_1 - V$ have disjoint closures, then the union is a manifold and we can construct a family $(U_0 \cup U_1, r, \Theta)$ over $\partial_0 \Delta^2 \cup \partial_1 \Delta^2$ by: over $\partial_0 \Delta^2 \Theta$ is θ_0 on U_0 and the identity on $(U_0 \cup U_1) - U_0 = U_1 - V$. Over $\partial_1 \Delta^2$, Θ is θ_1 on U_1 and the identity on $(U_0 \cup U_1) - U_0 = U_1 - V$. Crossing with I as indicated above gives the desired Kan extension.

The problem occurs when $U_0 - V$, $U_1 - V$ do not have disjoint closures so the union is not a manifold. We remedy this by suspending once more, and using the new *I* coordinate to separate them; we use $I \times V \cup [0, \frac{1}{3}] \times U_1 \cup [\frac{2}{3}, 1] \times U_2$.



The formula above gives a family over Δ^2 of pseudoisotopies of this manifold. To relate this back to σ_0 for example, consider the restriction to $\partial_0 \Delta^2$. After deleting $[\frac{2}{3}, 1] \times (U_1 - V)$ we get essentially a suspension of a reduction of the underlying family of σ_0 , so we can use this to define a 2-simplex of \mathscr{P} with ∂_0 essentially σ_0 .

This construction gives a 2-simplex with boundaries isomorphic to σ_0, σ_1 in a straightforward standard way. They are not actually equal, however. Technically the way to think of this is as a preliminary deformation (simplicial homotopy) of σ_0, σ_1 .

More generally given a simplicial map $K \to \mathscr{P}$, these ideas can be used to construct a simplicial homotopy (by making deleted pieces disjoint) to a map such that any Kan extension problem $\Lambda_j^k \to K \to \mathscr{P}$ has an extension. (This simplicial homotopy also gives time to iron out problems with the non-canonical nature of suspensions, which we suppressed in the discussion above.) This method can be used to obtain satisfactory versions of most of the consequences of the Kan condition.

The most succinct way to summarize this is that it is safe to pretend that these spaces do satisfy the Kan condition. We will proceed on this basis.

The first result with substance is a version of the stability theorem. This is also the prototype for other "space level" stability theorems.

To state the theorem we need some notation. Suppose $C \subseteq X$ is compact and $\varepsilon > 0$. Then $\mathscr{P}(X, C; p, \varepsilon)$ is the subset of $\mathscr{P}(X, p)$ of simplices with support containing C and radius less than ε . If $C \subseteq X - Y$ then there is a restriction function $\mathscr{S}(X, Y; p) \rightarrow \mathscr{P}(X, C; p, \varepsilon)$ defined by: a simplex of \mathscr{S} is a simplicial map $\rho: \varDelta^k \times [0, \infty) \rightarrow \mathscr{P}(X, p)$. The restriction is $\rho(n)$, where n is such that the image of $\varDelta^k \times [n, \infty)$ has support containing C and radius $<\varepsilon$. Simplices with different choices of n can be patched together using the map on the interval between the choices. Technically this idea is used to define a map on a subdivision of \mathscr{S} , and then the "Kan condition" is applied to obtain a simplicial map of the original space.

5.6. Theorem. Suppose $p: E \to X$ is a stratified system of fibrations over a locally compact metric ANR X, and suppose $Y \subset X$ is a p-NDR subset such that X - Y has compact closure. Then there is a compact set $C_0 \subseteq X - Y$, and $\varepsilon_0 > 0$ such that for any $C_0 \subseteq C \subseteq X - Y$ and any $\varepsilon_0 \ge \varepsilon > 0$ there is a function S: $\mathscr{P}(X, C, p, \varepsilon) \to \mathscr{S}_{-2}(X, Y, p)$ such that the following composition is homotopic to the identity:

$$\mathscr{S}_{-2}(X,Y;p) \xrightarrow{\operatorname{restriction}} \mathscr{P}(X,C,p,\varepsilon) \xrightarrow{S} \mathscr{S}_{-2}(X,Y;p).$$

The proof shows that a sufficiently small simplex $\Delta^k \to \mathcal{P}$ has an essentially unique extension to $\Delta^k \times [0, \infty) \to \mathcal{P}$, a simplex of \mathcal{S} . It follows that in some appropriate sense the composition $\mathcal{P} \to \mathcal{S} \to \mathcal{P}$ is also homotopic to the identity. We will also have a little bit of control on dimensions of images of S, see 5.8 (4).

Proof. The major part of the proof is an appeal to the methods of Sect. 4. We therefore begin by verifying the ingredients of these methods. This yields the

pseudoisotopy analog of 4.5. This is used to prove a somewhat more global "shrinking lemma", 5.8. Finally 5.8 is used to prove the stability Theorem 5.6.

Functorial images are defined by composition: suppose there is a commutative diagram



and let (U, r, θ) be a p_1 family over a polyhedron K. If f is proper, then (U, Fr, θ) is a p_2 family over K. Radii and supports behave as in 4.2.

Pullbacks are formed by taking topological pullbacks: Suppose there is a diagram as above, F is a local homeomorphism, and (U, r, θ) is a p_2 family over K. Then the pullback U' in the diagram



is a manifold and θ pulls back to a pseudoisotopy θ' of U'. Thus (U', r', θ') is a p_2 family over K. Again radii and supports behave as in 4.2.

The third property is naive homotopy of functorial images. Suppose there is a commutative diagram of homotopies



We need a natural homotopy from $(f_0)_*$ to $(f_1)_*$, with radius the same as that of F. Suppose (U, r, θ) is a p_1 family over K. We describe a natural homotopy over $I \times X_1$ from $(U, \{0\} \times r, \theta)$ to $(U, \{1\} \times r, \theta)$. The desired homotopy of images is then the image of this homotopy.

Define a $p_1 \times id$ family over $K \times [0, 1]$ by $([0, 3] \times U, \alpha \times r, \theta_t)$ where: (a) α : $[0, 3] \rightarrow [0, 1]$ takes [0, 1] to $\{0\}$, [2, 3] to $\{1\}$, and is linear on [1, 2], (b) θ_t is defined for $t \in [0, 1]$ by $\theta_t = (+2t) S^{-1} (id \times \theta) S(-2t)$, the S denoting the suspension homeomorphism and $\pm 2t$ translation in \mathbb{R} . (c) θ_t is the identity where (b) is undefined (i.e. off of $[2t, 1+2t] \times U$). Then when t=0, θ_0 is the suspension on $[0, 1] \times U$ and the identity on [1, 3]. Similarly θ_1 is the suspension of θ on [2, 3] and the identity on [0, 2]. The identity pieces on the ends can be deleted to give suspensions of $(U, i \times r, \theta)$. Recalling that suspensions and delations are allowable operations in boundaries in \mathcal{P} , we see that after subdivision this defines a natural simplical homotopy in \mathcal{P} from $(f_0)_*$ to $(f_1)_*$.

To describe the cancellation of inverses we must first discuss the additive structure. Addition in \mathcal{P} , \mathcal{S} is by disjoint union, after suspending and reducing to make the dimensions, supports, and radii equal. As with the Kan condition there is a problem with the disjointness (e.g. $U \cap U \neq \emptyset$) so the addition may not always be defined. As with the Kan condition this can be delt with by a preliminary deformation to make things disjoint. Again the details are not interesting enough to write down, and the best way to proceed is to pretend everything is disjoint from everything, including itself.

The inverse of a family over Δ^k , (U, r, θ) , is defined in 5.2 to be (U, r, θ^{-1}) . The cancellation is given by a family over $\Delta^k \times I$ which over $\Delta^k \times \{0\}$ can be deleted to give the suspension of $(U, r, \theta) \sqcup (U, r, \theta^{-1})$, and over $\Delta^u \times \{1\}$ is the identity and so can be deleted to give \emptyset .

A notation will be useful: recall the homeomorphism $S: I \times I \rightarrow I \times I$ used in 5.2 to define the suspension. Let S_{ab} be the same homeomorphism defined on $[a,b] \times I$.

Now consider the union $(U, r, \theta) \perp (U, r, \theta^{-1})$. The suspension of this is canonically isomorphic to $([0, 1] \times U, rq, S_{01}^{-1}(1 \times \theta) S_{01}) \perp ([2, 3] \times U, rq, S_{23}^{-1}(1 \times \theta^{-1}) S_{23})$. This we can think of as obtained by deleting $(1, 2) \times U$ from $([0, 3] \times U, rq, AB)$, where A is defined to be $S_{01}^{-1}(1 \times \theta) S_{01}$ on $[0, 1] \times U$ and the identity elsewhere, and B is defined similarly using θ^{-1} over [2, 3]. Extend this to a family over $\Delta^k \times [0, 1]$ by defining

$$A_t = (1+2t) S_{01}^{-1} (1 \times \theta) S_{01} \left(\frac{1}{1+2t}\right)$$

on $[0, 1+2t] \times U$ and $A_t = id$ on $[1+2t, 3] \times U$. Similarly define B_t by expanding B from [2,3] out to [0,3]. The family is then $([0,3] \times U, rq, A_tB_t)$ for $t \in [0,1]$. Let C be the support of the family. Then over $\Delta^k \times \{1\}$ and support $C^{-\epsilon}$ the family is $([0,3] \times U, rq, (S_{03}^{-1}(1 \times \theta) S_{03})(S_{03}^{-1}(1 \times \theta^{-1}) S_{03})) = ([0,3] \times U, rq$, id). Therefore the whole thing may be deleted to give the empty family.

This construction commutes with deletions, reductions and suspensions, so is well defined on simplices of \mathscr{P} . Subdividing $\varDelta^k \times I$ in the standard way gives an essentially canonical simplicial homotopy from (id) \amalg (inverse): $\mathscr{P} \to \mathscr{P}$ to the constant map at \emptyset . Finally if a family has support C and radius ε then the cancellation homotopy has support $C^{-\varepsilon}$ and radius $<2\varepsilon$. The homotopy therefore extends to one defined on \mathscr{S} .

The next step is to localize the cancellation procedure. Suppose $W \subseteq X$. Suspend $(U, r, \theta) \perp (U, r, \theta^{-1})$ as above, but instead of adding $(1, 2) \times U$ add a close manifold approximation to $(1, 2) \times r^{-1}(W^{3\epsilon})$. Let $\rho = \min\left(1, \frac{1}{\epsilon} \text{ (distance to } X - W^{2\epsilon})\right)$, so $\rho = 0$ on $X - W^2$, and is 1 on W^{ϵ} . Then define a family over $\Delta^k \times I$ by $([0, 3] \times U, rq, A_{s \cdot \rho r}, B_{s \cdot \rho r})$ over $\Delta^k \times \{s\}$. This is the identity near X $-W^{3\varepsilon}$, so extends by the identity outside $W^{3\varepsilon}$. Finally the family over W with s=1 is ([0,3] $\times U, rq$, id), and so can be deleted as above. This defines a deformation of the family which is (essentially) constant outside $W^{3\varepsilon}$, and cancels it over W.

This concludes the verification of properties analogous to 4.2. The first application is a generalization of the homotopy Lemma 4.4.

5.7. Homotopy Lemma. Suppose there is a commutative diagram



with G a proper map. Suppose X, Y are metric, $W \subseteq X$, and $\varepsilon > \gamma > 0$. Then there is a homotopy

$$G_{\varepsilon,\gamma}^{\sim}: \mathscr{P}(X,\emptyset,p,\varepsilon) \to \mathscr{P}(Y,q)$$

such that

1. $(G_{\epsilon, \gamma})(\tau, 1) = (G_0)_* \tau$,

2. if $Z \subseteq Y$ such that the projection of $(G)^{-1}(Z)$ to X is in the support of τ , then Z is in the support of $(G_{\epsilon, \gamma}^{\sim})(\{\tau\} \times I)$,

3. if $Z \subseteq Y$ satisfies $G^{-1}(Z) \subseteq W \times I$, then $(G_{\varepsilon, \gamma}^{\sim})(\tau, 1) = (G_1)_*(\tau)$ over Z, and

4. if $r: [0, \infty)$ is a homeomorphism such that $d(G(x, t), G(x'; t')) < r(\max\{|t - t'|, d(x, x')\})$, then if τ has radius $<\delta$, $G_{\varepsilon, \gamma}^{\sim}\{\{\tau\} \times I\}$ has radius $<2r(2\delta) + \gamma$.

Proof. Suppose τ is a simplex with radius $\delta < \varepsilon$. Define $C_{\varepsilon}(\tau)$ to be $\tau \perp \tau^{-1}$, cancelled over $W^{9_{\varepsilon}} - W^{7_{\varepsilon}}$, and the part outside $W^{8_{\varepsilon}}$ discarded. The remainder of the cancellation of the inverses gives a 2δ deformation $\emptyset \sim C_{\varepsilon}(\tau)$.

Next choose $t_i \in I$, $t_0 = 0$, $t_n = 1$ such that $G|X \times [t_j, t_{j+1}]$ is a homotopy of radius $\leq \delta$. Applying G to the deformation above gives an $r(2\delta)$ deformation $(G_0)_* \tau \sim (G_0)_* (\tau) \perp \prod_{i=1}^n (G_{t_i})_* (C_W \tau)$. The part of this family which comes from $W^{4\varepsilon}$ is also the image of

$$(G_0)_* \tau \sqcup \coprod_{n=1}^n ((G_{t_1})_*(\tau) \sqcup (G_{t_1})_*(\tau^{-1})).$$

the naive homotopy induced by $G|[t_{i-1}, t_i]$ gives a deformation of radius γ from $(G_{t_i})_* \tau^{-1}$ to $(G_{t_{i-1}})_* \tau^{-1}$ over $W^{3\epsilon}$. Over $W^{3\epsilon}$ we can then rearrange this to get

$$\left[\prod_{i=0}^{n-1} (G_{t_i})_* (\tau \sqcup \tau^{-1})\right] \sqcup (G_1)_* \tau.$$

Finally we can cancel the inverses over W by an $r(2\delta)$ deformation to leave only $(G_1)_* \tau$ over W.

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The composition of these deformations gives one of radius $\langle 2r(\delta) + \delta$. Viewed as a family over $\Delta^{u} \times I$ this defines the homotopy of the lemma. The properties are verified easily.

This concludes the proof of the homotopy lemma.

The proof of Sect. 4 also provides a stability theorem analogous to 4.5, as a consequence of these formal properties. The properties used are actually for "constant coefficients"; $p: X \times F \rightarrow X$ the projection. The stability theorem also applies to this context. The next statement is midway between 4.5 and our present goal, 5.6.

5.8. Shrinking Lemma. Suppose $p: E \to X$, $Y \subseteq X$ are as in 5.6, and $C \subseteq X - Y$ is a compact set such that a neighborhood of $(X - C)^-$ p-deformation retracts to Y in X. Then there is $\delta > 0$, a homeomorphism $r: [0, \infty)$ with $r(t) \ge t$, and an integer n such that for every $\gamma > 0$ there is a homotopy

$$S: \mathscr{P}(X, C, p, \delta_0) \times I \to \mathscr{P}(X, p)$$

such that

1. $S | \mathcal{P} \times \{0\}$ is the inclusion,

2. S $\mathscr{P} \times \{1\}$ lies in $\mathscr{P}(X, X - Y^{\delta}, p, \delta)$,

3. if τ is a simplex with support $\supseteq X - Y^{\alpha}$ and radius $\leq \alpha$, for $\alpha \geq \delta$, then $S(\tau)$ has support $\supseteq X - Y^{r(\alpha)}$ and radius $\leq r(\alpha)$, and

4. if τ has dimension k, then $S(\tau)$ has dimension $\leq n+k$.

S shrinks families of pseudoisotopies, r bounds the amount of expansion which takes place in the shrinking process.

Proof of 5.8. The first step is to extend the analog of 4.5 from constant coefficient systems to stratified systems. A version which applies to fibrations (stratified systems with a single stratum) easily follows from 4.5 by the same piecing process used to globalize 4.5. (The fibration must be specified before ε and r can be chosen, however).

Suppose as an induction hypothesis that the statement 4.5 is valid for stratified systems with fewer than k strata and let $p: E \to X$ be a system with k strata. Let $f: X_0 \to X$ be the inclusion of the smallest stratum, and let X_f be the mapping cylinder. Since X_0 is a p-NDR subset of X there is a homotopy $h: X \times I \to X_f$ which is the inclusion on $X \times \{0\}$ and outside a neighborhood of $X_0 \times I$, takes $X_0 \times \{1\}$ to $X_0 \times \{1\}$ in the mapping cylinder, and is covered by a homotopy of E. Construct a shrinking function as follows: Shrink outside a neighborhood of X_0 using the hypothesized extension of 4.5 to k-1 strata. Use h and the homotopy Lemma 5.7 to obtain a function into $\mathcal{P}(X_f)$ which is small except near $X_0 \times \{1\}$. Now apply the single stratum result to shrink near $X_0 \times \{1\}$ while holding things fixed near $X \subset X_f$. Finally project back to X.

There is some expansion involved in the use of the homotopy lemma and the second application of 4.5. However this expansion can be estimated beforehand, so can be allowed for in the earlier steps.

The next problem is the support. By hypothesis there is a p-homotopy h: $X \times I \rightarrow X$ which contracts a neighborhood U of $(X - C)^-$ to Y, and holds Y fixed. Use a shrinking function as constructed above to shrink things on X - U

while holding them fixed on X - C. Then apply the homotopy lemma with h (and W = X) to the result. The end of this is $(h_1)_*$, which is small except over Y.

The support and radius estimates follow from a straightforward combination of the estimates of 4.5 and 5.7.

To verify conclusion 4, the estimate on dimension, we must briefly review the proof. First note that of the basic operations, taking inverses, images, and pullbacks do not change dimensions. The cancellation of inverses and naive homotopy both raise dimensions by 1. An application of the homotopy Lemma 5.7 may therefore raise dimensions by 2. Next note that in deducing 5.8 from 4.5 we used the homotopy lemma once, and a shrinking function from 4.5 once, for each stratum of X. Therefore if there is an estimate for the single stratum case (4.5), there is one in general. The proof of 4.5 begins with a reduction to the manifold case, using a single application of the homotopy lemma. Suppose the manifold has dimension k. Then the theorem for the manifold follows by k applications of the theorem to handles in a handlebody. An estimate for the whole will therefore also follow from estimates for handles.

The proof for 4.5 for a handle begins with an application of the homotopy lemma, which may raise dimension by 2. Pulling back over a torus does not change dimensions. The completion lemma, which is proved via cancellation of inverses, raises dimension by 1. The pullback to the universal cover $D^j \times \mathbb{R}^k$ does not change dimension. The "penultimate step" is a cancellation which raises dimension by 1, and the final step an application of the homotopy lemma which may add 2. Putting them together, we see that the shrinking function on a handle raises dimension by 6 or less. We conclude therefore that there is an estimate for suitable general X.

Proof of 5.6. Since $(X - Y)^-$ is compact and Y is a p-NDR subset of X, there is a homotopy covered by a homotopy of p, rel Y, from the identity of X to a map which retracts a neighborhood of Y into Y. Let C be a compact set in X - Y whose complement is retracted into Y.

Next choose a sequence of numbers ε_i as follows: Let $\varepsilon_{-1} = 1$. Then by 5.8 there is $\delta_0 > 0$, and control function r_0 for shrinking functions for pseudoisotopies with support C. Let ε_0 be small enough so that $\varepsilon_0 < \delta_0$, and $r_0(\varepsilon_0) < \varepsilon_{-1}$. Choose ε_1 so that $r_0(\varepsilon_1) < \varepsilon_0$. Suppose ε_i , $i \le 2k-1$ has been chosen, and there are control functions (for shrinking functions) r_j , j < k. Let δ_k , r_k be data for shrinking functions on $X - Y^{\varepsilon_{2k-1}}$. Then choose ε_{2k} , ε_{2k+1} so that

1.
$$\varepsilon_{2k} < \delta_k, r_k(\varepsilon_{2k}) < \varepsilon_{2k-1}$$

2.
$$r_k(\varepsilon_{2k+1}), r_{k-1}(\varepsilon_{2k+1}) < \varepsilon_{2k}$$
, and

3. $r_k(\varepsilon_{2k+1}) < \varepsilon_{2k}$.

Now we can define shrinking functions S_k as in 5.8, with control function r_k , which shrink things on $X - Y^{\varepsilon_{2k-1}}$ down to size ε_{2k+4} .

The function S: $\mathscr{P}(X, C, p, \varepsilon_0) \to \mathscr{S}_{-2}(X, Y; p)$ is defined by iterating and concatenating the S_k . Suppose τ is a family over Δ^j . Define $S(\tau) | \Delta^j \times [0, 1]$ to be $S_0(\tau)$, and inductively

$$S(\tau) | \Delta^j \times [k, k+1] = S_k(S(\tau) | \Delta^j \times \{k\}).$$

The restrictions on the ε_i ensure that this is defined, and is a simplex of \mathscr{G}_{-2} .

Most of the care in the choices is needed to construct the homotopy of

$$\mathscr{S}_{-2} \xrightarrow{\text{restriction}} \mathscr{P} \xrightarrow{S} \mathscr{S}_{-2}$$

to the identity. To do this, suppose $\rho: \Delta^j \times [0, \infty) \to \mathscr{P}$ is a simplex of \mathscr{P}_{-2} , and suppose (after reparameterization in $[0, \infty)$ if necessary) that $\rho | [k, \infty)$ has radius $\langle \varepsilon_{2k+1}$, and support $\supseteq X - Y^{\varepsilon_{2k+1}}$. We want $R: \Delta^j \times [0, \infty) \times [0, 1] \to \mathscr{P}$ such that when the last coordinate is 0 we get ρ , and when it is 1 we get $S(\rho | \Delta^j \times \{0\})$. Define $R | \Delta^j \times \{0\} \times I$ to be $(\rho | \{0\}) \times I$. Next suppose R is defined on $\Delta^j \times [0, k] \times I$, and $R | \Delta^j \times \{k\} \times I$ has support containing $X - Y^{\varepsilon_{2k}}$, and radius $\langle \varepsilon_{2k}$. Then we can apply S_k to $R | \Delta^j \times \{k\} \times I \cup \rho | \Delta^j \times [k, k+1] \times \{0\}$ to obtain a family over this polyhedron $\times I$. Reparametrize the parameter spaces as $\Delta^k \times [k, k+1] \times I$, with $R | D^j \times \{k\} \times I$ and $\rho | \Delta^j \times [k, k+1] \times \{0\}$ in the indicated places, and $S_k(R | \Delta^j \times \{k\} \times \{1\})$ over $\Delta^j \times [k, k+1] \times \{1\}$.



It is easily seen that the induction hypotheses $(R|\Delta^j \times \{k+1\} \times I \text{ has radius} < \varepsilon_{2k+2}$ and support $\supseteq X - Y^{\varepsilon_{2k+2}}$ are satisfied, that $R|D^j \times [k, k+1] \times I$ is $< \varepsilon_{2k+1}$, and that this process builds up $S(\rho | \Delta^j \times \{0\})$ on the top. The desired homotopy R is therefore defined by induction.

This completes the proof of the stability theorem.

The next result describes the spectrum structure, and is a decendent of Hatcher [19], Proposition 10.2. See also Anderson and Hsiang [1], Burghelea and Lashoff [6].

5.9. Theorem. Suppose X is a locally compact metric ANR, and $p: E \to X$ is a stratified system of fibrations. Then there is a natural homotopy equivalence T: $\mathscr{G}_{-2}(X;p) \to \Omega \mathscr{G}_{-2}(X \times \mathbb{R}; p \times 1).$

Proof. First define an intermediate function T_1 . Suppose (U, r, θ) is an (X, p) family over K with support C and radius ε . For every integer n we can suspend this family using the interval [n, n+1] to obtain $([n, n+1] \times U, r, S^{-1}(1 \times \theta) S)$. Since these agree on overlaps (they are identity maps) they fit together to give a family of pseudoisotopies of $\mathbb{R} \times U$. We denote these pseudoisotopies by $T_1 \theta$. This family has radius $<\varepsilon$ in the X coordinate, <1 in the v coordinate.

dinate. We therefore get an $(\mathbb{R} \times X, 1 \times p)$ family of radius ε by $(\mathbb{R} \times U, (\frac{1}{\varepsilon}) \times r, T_1 \theta)$.

The construction T_1 is periodic in the **R** coordinate. Denote translation by t by (+t): $\mathbb{R} \to \mathbb{R}$. Then $((+1) \times id) T_1 \theta = T_1 \theta((+1) \times id)$. We use this to describe a family over $K \times I$,

$$T\theta: \mathbb{R} \times (pr^{-1}(C^{-\varepsilon})) \times I \times K \times I \to \mathbb{R} \times U \times I \times K \times I$$

by:

$$T\theta | \mathbb{R} \times (\mathrm{pr}^{-1}(C^{-\varepsilon})) \times I \times K \times \{t\}$$

is

$$(-t \times id)(T_1 \theta)(+t \times id)(T_1 \theta^{-1}).$$

Notice that because of the periodicity this is the identity when t=0 or 1.

Finally we define T on $\mathscr{G}_{-2}(X; p)$. A k-simplex is a map $\rho: \Delta^k \times [0, \infty) \to \mathscr{P}(X; p)$, with estimates on supports and radii. Think of the radius estimates as given by a function $\varepsilon: \Delta^k \times [0, \infty) \to (0, \infty)$. Then for a simplex $\sigma \in \Delta^k \times [0, \infty)$ with underlying family (U, r, θ) use $(\mathbb{R} \times U, 1/\varepsilon \times r, T\theta)$. Since T commutes with reduction, deletion and suspension, these fit together to give $T\rho: \Delta^k \times I \times [0, \infty) \to \mathscr{P}(\mathbb{R} \times X, \operatorname{id} \times p)$. Since the T pseudoisotopies are the identity over $\{0, 1\} \subset I$, they may be deleted to be the empty family there. $T\rho$ can therefore be considered as a k-simplex of the loop space $\Omega(\mathscr{G}_{-2}(\mathbb{R} \times X, \operatorname{id} \times p), \{\emptyset\})$. This defines the Δ -map of the theorem.

The theorem asserts that T is a homotopy equivalence. For this we need to show it maps into every component, and that all relative homotopy groups vanish. For clarity we will do this in two steps. First we present the basic idea, but without sufficient attention paid to the ε estimates. Then we indicate the contortions required to supply the estimates. We will also supress mention of supports.

An element in the relative homotopy $\pi_j(T)$ is represented by a map $\sigma: \Delta^j \times [0, \infty) \times I \to \mathscr{P}(\mathbb{R} \times X; \text{ id } \times p)$ such that $\sigma |\partial_i \Delta^j = \emptyset$ for i < j, and $\sigma |\partial_j \Delta^j = T\rho$ for $\rho: \Delta^{j-1} \times [0, \infty) \to \mathscr{P}(X, p)$. We need a deformation of σ rel $\partial_j \sigma$ to a map in the image of T. We give the construction for a single level $\Delta^j \times \{t\} \times I$, and will justify this later by appeal to the stability theorem.

A map $\sigma: \Delta^j \times I \to \mathscr{P}(\mathbb{R} \times X, \operatorname{id} \times p))$ consists of families over simplicies of $\Delta^j \times I$ which are suitably compatible. These families can be fitted together (by the unions used in the "Kan condition" 5.5) to give a single family over $\Delta^j \times I$. Over $\partial_j \Delta^j$ since it is T of something, something is deleted to give a family on a manifold of the form $\mathbb{R} \times U$. We can arrange that the whole family is on a manifold $\mathbb{R} \times M$ by suspending using $I \times M \subset \mathbb{R} \times M$ and interchanging the two \mathbb{R} coordinates near $\partial_j \Delta^j$. Finally we may proceed as if M were compact. This is because \mathscr{S} is defined using compact supports, but in the interest of comprehensible notation we are not making the support explicit. Denote this family by $(\mathbb{R} \times M, \frac{1}{2} \times r, \psi)$. The restriction to $\partial_i \Delta^j \times I$ is $T(M, r, \theta)$, and the restriction

by $\left(\mathbb{R} \times M, \frac{1}{\varepsilon} \times r, \psi\right)$. The restriction to $\partial_j \Delta^j \times I$ is $T(M, r, \theta)$, and the restriction to $\partial_i \Delta^j \times I$ is the identity if i < j.

The first step is to consider the restriction of ψ to the 0 level in **R**.



Over $\partial_j \Delta^j \times I$, $\psi = T\theta = (-t \times id)(T_1 \theta)(+t \times id)(T_1 \theta^{-1})$. $T_1 \theta^{-1}$ is the identity on $\{0\} \times M \times \partial_j \Delta^j$, so the restriction is a translation of $T_1 \theta$. Define $T_2 \theta$ on $\mathbb{R} \times M \times I \times \partial_j \Delta^j$ by $\Sigma \theta$ on $[0,1] \times M \times I \times \partial_j \Delta^j$ and the identity on the complement. Let $\partial_j F$ denote the family on $\mathbb{R} \times M$ over $\partial_j \Delta^j \times I$ defined by $(-t \times id)(T_2 \theta)$ $(+t \times id)$ on $\mathbb{R} \times M \times I \times \partial_j \Delta^j \times \{t\}$. Extend this by the identity to a family ∂F over the rest of $\partial \Delta^j$.

The next step is to apply the isotopy extension theorem. The family ∂F agrees with $\partial \psi$ when restricted to $\{0\} \times M$. Let ∂F_+ be the restriction of ∂F to $[0, \infty) \times M$, then we have a family of embeddings $\psi_0: \{0\} \times M \times I \to \mathbb{R} \times M \times I$, over $\Delta^j \times I$. These embeddings are the identity over $\Delta^j \times \{1\}$, and we have an extension over $\partial \Delta^j \times I$ to proper embeddings $\partial F_+: [0, \infty) \times M \times I \to \mathbb{R} \times M \times I$ which are the identity over $\partial \Delta^j \times \{1\}$, and on $[1, \infty) \times M \times I$. Finally (by the ε estimates) the image of ψ_0 lies in $[-1, 1] \times M \times I$. The isotopy extension theorem of Edwards and Kirby [14], and Lees [24] implies that there is an extension of $\partial F_+, \psi_0$ to a family of proper embeddings $F_+: [0, \infty) \times M \times I \to \mathbb{R} \times M \times I \to \mathbb{R} \times M \times I \to \mathbb{R} \times M \times I$, which is the identity on $[1, \infty) \times M \times I$ and over $\Delta^j \times \{1\}$.

Consider $F_+ | [0,1] \times M \times I \times \Delta^k \times \{0\} \to \mathbb{R} \times M \times I \times \Delta^k \times \{0\}$. By construction this is the identity on $\{1\} \times M \times I$, and agrees with ψ_0 on $\{0\} \times M \times I$. However ψ_0 is also the identity when the last coordinate is 0. Therefore this defines a family of automorphisms of $[0,1] \times M \times I$ which are the identity on $\{0,1\} \times M \times I$. Call this family Θ , and notice that over $\partial_j \Delta^j$, $\partial_j \Theta = \Sigma \theta$. To complete the proof of the theorem (i.e. deform the relative homotopy class into $\mathscr{S}_{-2}(X;p)$) it is sufficient to show that $T\Theta \sim \Sigma \psi$, rel $\partial \Delta^j \times I$.

First recall that the *T* construction begins by suspending to obtain [n, n+1] coordinates. Θ however is already defined on $I \times M$, so we may juxtapose copies of Θ to get Θ_{∞} : $\mathbb{R} \times M \times I \times \Delta^{j} \to \mathbb{R} \times M \times I \times \Delta^{j}$. Note that (by rotating *I* factors) there is a deformation $\Sigma \Theta_{\infty} \sim T_{1} \Theta$ through families which are the identity on $I \times \mathbb{Z} \times M \times I$. Composing with the same construction for Θ^{-1} gives a deformation $T\Theta \sim \Sigma[(-t \times id)\Theta_{\infty}(+t \times id)\Theta_{\infty}^{-1}]$. Denote $(-t \times id)\Theta_{\infty}(+t \times id)\Theta_{\infty}^{-1}$ by Θ^{-} . It will be sufficient to describe a deformation $\Theta^{-} \sim \psi \operatorname{rel} \partial \Delta^{j}$, in $\mathcal{S}_{-2}(\mathbb{R} \times X, p)$.

The restriction of Θ^{\sim} to $\{0\} \times M \times I$ is $(-t \times id) \Theta(+t \times id)$. F_+ defines a deformation of families of embeddings $\psi_0 \sim \Theta_0^{\sim}$ (rel $\partial \Delta^i$) so this can be used to modify ψ to agree with Θ^{\sim} in a neighborhood of $\{0\} \times M \times I$.

By the cancellation of inverses, it is sufficient to describe a deformation Θ^{\sim} $\perp \psi^{-1} \sim \emptyset$. On a neighborhood of $\{0\} \times M \times I$ we have arranged $\psi^{-1} = (\Theta^{\sim})^{-1}$, so the inverses can be cancelled there to get a deformation to a family which is the union of two pieces, one over $[0, \infty) \times X$ and one over $(-\infty, 0] \times X$. Each of these may be deformed to the empty family, by using a slight modification of the homotopy lemma, and the homotopies which push things out toward $\pm \infty$.

This concludes the description of the basic idea. It remains to describe how to add the ε estimates to this argument. The basic difficulty is that we do not have very good ε control on the extensions given by the isotopy extension theorem.

The first step is to observe that it is sufficient to prove the theorem for X = K - L, K a compact manifold, L a closed submanifold such that K - L has compact closure. This follows from the homotopy invariance and fibration axioms verified in the proof of the next theorem. Next choose $\varepsilon > 0$ and $C \subseteq K - L$ compact which satisfy the conclusion of the stability Theorem 5.6. Let k be the dimension of K, and choose a handle decomposition of K - L of diameter $< \varepsilon/2^{k+1}$.

Now consider a relative homotopy class, represented as above by $\sigma: \Delta^j \times [0, \infty) \times I \to \mathscr{P}(\mathbb{R} \times (K-L); p)$. Restrict to a level $\Delta^j \times \{t\} \times I$ such that $\sigma | \Delta^j \times [t, \infty) \times I$ has radius $\langle \varepsilon/2^{k+1}$, and support containing a compact handlebody containing C. Go through the construction as above to obtain the family of embeddings ψ_0 of $\{0\} \times M \times I$ in $\mathbb{R} \times M \times I$. Now we can extend ψ_0 to the family F_+ in steps: assume there is an extension defined on the inverse image of the handles of dimension $\leq i$ in K, of radius $\langle 2^i/2^{k+1}\varepsilon$. Over each i+1 handle we get relative extension problem which stays inside a submanifold of diameter $\langle 2^{i+1}/2^{k+1}$ in K. There is therefore an extension which stays in the submanifold.

This process yields an extension F_+ with radius $<\varepsilon/2$. The argument can easily be completed to give a deformation $\psi \sim T\Theta$ of radius $<\varepsilon$. Finally apply the shrinking function of the stability theorem. The deformation at the ε level implies that there is a deformation of the entire families defining the homotopy class in \mathcal{S} .

This completes the proof of 5.9.

5.10. Definition. The spectrum $\mathscr{S}(X;p)$ is defined to have j^{th} space $\mathscr{S}_{-2}(\mathbb{R}^{j+2} \times X; \text{ id} \times p)$ if $j \ge -2$, $\Omega^{2-j} \mathscr{S}_{-2}(X;p)$ if $j \le -2$, and structure maps the natural maps T of Theorem 5.9.

The next theorem is the main result of the section. Homology spectra are defined in the appendex, Sect. 8.

5.11. Theorem. Suppose X is a locally compact metric ANR and $p: E \rightarrow X$ is a stratified system of fibrations. Then there is a natural homotopy equivalence of spectra

$$A: \operatorname{I\!H}^{lf}(X; \mathscr{S}(p)) \to \mathscr{S}(X; p).$$

Proof. This is a consequence of the characterization Theorem 8.5, so we verify the axioms for \mathcal{S} .

The limit axiom (3) is essentially built in to the Definition 5.4 by the requirements on supports. In more detail, notice that since X is σ -compact we can reduce the limit to a countable well ordered inverse subsystem, $\mathscr{S}(X)$

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 $-Y_i$; p). The holim can then be described explicitly as maps

$$\rho_i: [i, \infty) \to \mathscr{S}(X - Y_i, p)$$

which agree under restrictions. Referring to 5.4, we see that this is just a map $[0, \infty) \times [0, \infty) \rightarrow \mathscr{P}(X; p)$, so a map from the limit to $\mathscr{S}(X; p)$ can be obtained by restricting to the diagonal. This is easily seen to be a homotopy inverse for the natural map induced by restrictions.

Now consider the restriction axiom (1). The restriction map is defined by the reduction operation of 5.2. It is sufficient to show

$$\mathscr{S}_{-2}(Y;p) \to \mathscr{S}_{-2}(X;p) \to \mathscr{S}_{-2}(X-Y;p)$$

is a homotopy fibration, because the other spaces in the spectrum are defined to be \mathscr{S}_{-2} of some other spaces.

Let F denote the homotopy fiber of $\mathscr{G}_{2}(X;p) \to \mathscr{G}_{2}(X-Y;p)$. Recall that a simplex of this fiber is defined to be a simplex of $\mathscr{G}_{2}(X;p)$, together with a homotopy of the image in $\mathscr{G}_{2}(X-Y;p)$ to the basepoint (empty family). Since the restriction of a family in $\mathscr{G}_{2}(Y;p)$ is already empty, a simplex together with the constant homotopy defines a map $\mathscr{G}_{2}(Y;p) \to F$. We will show that the relative homotopy of this map vanishes.

By the Kan condition, an element in the jth relative homotopy group is represented by a simplicial map $\rho: \Delta^j \times [0, \infty) \to \mathscr{P}(X; p)$ whose boundary lies in $\mathscr{P}(Y; p)$, and a simplicial map $\sigma: \Delta^j \times I \times [0, \infty) \to \mathscr{P}(X - Y; p)$ so that $\sigma | \Delta^j \times \{0\} \times \{0, \infty)$ is the restriction of ρ to X - Y, and $\sigma | \partial \Delta^j \times I \times [0, \infty) \cup \Delta^j \times \{1\} \times [0, \infty)$ is empty. Consider a level $n \in [0, \infty)$. Then ρ_n is a family over Δ^j which over $\partial \Delta^j$ is the identity over X - Y (because it must delete to give something over Y). Similarly σ_n is an isotopy of the reduction of this family to some compact $C_n \subset X - Y$ to the identity family (rel $\partial \Delta^j$). This isotopy can be used to isotope the family ρ_n rel $\partial \Delta^j$ to a family which is the identity over a slightly smaller compact, C_{n-1} . Delete to obtain a family over $X - C_n$.

Now recall that Y is a *p*-NDR subset of X. Therefore if C_{n-1} is large enough there is a homotopy which pulls its complement into Y. Use this and the homotopy lemma to deform the family obtained above to one over Y. By arranging the rates of convergence of radii and supports properly to allow for expansion in the homotopy lemma, this argument extends to the whole maps ρ , σ .

This shows that the relative homotopy element is trivial, so the restriction sequence is a homotopy fibration. This verifies axiom (1).

Next consider the continuity axiom (2). First of all since we are working with Δ -sets the axiom must be interpreted in this context. For mor (p_1, p_2) we use the Δ -set with k-simplices diagrams



This changes the problem from one of continuity to one of definition: the functoriality of \mathscr{S} defines a map on 0-simplices which must be extended to the higher simplices. We construct the adjoint

$$\mathscr{S}(X_1; p_1) \times \operatorname{mor}(p_1, p_2) \to \mathscr{S}(X_2; p_2).$$

As a final modification we note that since these spaces are Kan we can use the categorical product of Δ -sets (*n*-simplices are products of *n*-simplices, see Rourke and Sanderson [33]).

An *n*-simplex of $\mathscr{G}_{j}(x_{1}; p_{1})$ is a map $\rho: \Delta^{n} \times [0, \infty) \to \mathscr{P}(\mathbb{R}^{j+2} \times X_{1}; 1 \times p_{1})$. For simplicity consider first the restriction of ρ to a level $\Delta^{n} \times \{t\}$, which is a family (U, r, θ) over Δ^{n} . Let (F, f) be an *n*-simplex of mor (p_{1}, p_{2}) . Then form the composition

$$U \times \Delta^n \xrightarrow{r \times 1} \mathbb{R}^{j+2} \times E_1 \times \Delta^n \xrightarrow{1 \times F} \mathbb{R}^{j+2} \times E_2,$$

and denote it by r^{\sim} . Then $(U \times \Delta^n, r^{\sim}, \theta \times 1)$ is a family except that $\theta \times 1$ is not the identity on $U \times \partial \Delta^n \times I$. The remedy is as in the definition of suspension in 5.2: let S be a homeomorphism S: $\Delta^n \times I$ \supset which takes $\partial \Delta^k \times I \cup \Delta^k \times \{0\}$ to Δ^k $\times \{0\}$, and then use $(U \times \Delta^k, r^{\sim}, S^{-1}(\theta \times 1)S)$. To obtain a family with small radius, use an S which is small in the Δ^k coordinate. This construction commutes with suspension, deletion, and reduction (except for the nonuniqueness of S), so fits together to define a map $\Delta^n \times [0, \infty) \to \mathscr{P}(\mathbb{R}^{j+2} \times X_2; 1 \times p_2)$. If the obvious care is taken with radii, this gives an *n*-simplex of $\mathscr{S}_j(X_2; p_2)$. This defines the map required for the axiom.

6. Proofs of the Results

This section assembles the proofs of the theorems of Sect. 1, except for the duality Theorems 1.9 and 1.10 which will be proved in Sect. 7. We begin by defining q_1 of a homotopy equivalence in 6.1. 6.2 contains the proofs of 1.4 and 1.6. 6.3 contains the proof of the existence of finite structures, 1.3. The approximate end theorem is proved in 6.4, and the main results 1.1 and 1.2 are proved in 6.5. Finally 6.6 contains proofs of the formulas 1.7 and 1.8.

6.1. Definition of q_1 **of a Homotopy Equivalence.** Suppose $p: E \to X$ is a stratified system of fibrations, and $Y \subseteq X$ is a *p*-NDR subset with $(X - Y)^-$ compact. It will be sufficient to define small pseudoisotopies over $X \times R$ with support an appropriate $C \times R$: the stability Theorem 5.6 provides a map

S:
$$P(X \times \mathbb{R}, C \times \mathbb{R}, p \times 1, \varepsilon) \rightarrow \mathscr{S}_{2}(X \times \mathbb{R}, Y \times \mathbb{R}; p \times 1),$$

and

$$\pi_0 \mathscr{S}_{-2}(X \times \mathbb{R}, Y \times \mathbb{R}; p \times 1) \xrightarrow{\sim} 5.10 \pi_1 \mathscr{S}_0(X, Y; p) \xrightarrow{\sim} 5.11 H_1(X, Y; \mathscr{S}(p)).$$

Suppose r: $(K,L) \to X$ is the mapping cylinder of a map of compact polyhedra, which is an ε homotopy equivalence over $C \subseteq X$. Then there is an ε retraction d: $(r^{-1}(C) \cup L) \times I \to K$ which is the inclusion on $r^{-1}(C) \times \{0\} \cup L \times I$, and $d(r^{-1}(C) \times \{1\}) \subseteq L$. (This is a contractible pair, see 6.3). Let $n \ge 2 \dim(k)$

+3, and embed $(K, L) \subset \mathbb{R}^n \times [0, \infty)$, $\mathbb{R}^n \times \{0\}$ with regular neighborhood $(U, \partial_0 U)$. Extend r to U by composing with the collapse $U \to K$, and define $\partial_1 U = (\partial U - \partial_0 U)^-$.

The first step is to construct an inverse for $(U, \partial_0 U)$ as an *h*-cobordism. By general position we can extend the embedding of $(r^{-1}(C) \cup L) \times \{0\} \subset U$ to an embedding of the mapping cylinder of *d*. By pushing along the mapping cylinder coordinate we get a 2ε ambient isotopy of *U* to $U' \subset U$ such that $U' \cap r^{-1}(C^{-2\varepsilon})$ lies inside a collar on $\partial_0 U$: $\partial_0 U \times I \subset U$. Let $V = (\partial_0 U \times I \cap r^{-1}(C^{-4\varepsilon}) - U')^{-}$.



Then from the definition of V, and from the ambient isotopy, we get 4ε isomorphisms over $C^{-4\varepsilon}$: $U \cup_{\partial_1} V \cong \partial_0 U \times I$, and $V \cup_{\partial_0} U = \partial_1 U \times I$. The next step is to find a 2ε embedding α : $\partial_0 U \times I \subset U$ which is the

The next step is to find a 2ε embedding α : $\partial_0 U \times I \subset U$ which is the inclusion on $\partial_0 U \times \{0\}$, and $\alpha(\partial_0 U \times \{1\}) \subset \partial_1 U$. Again this can be done by general position. A collar $\partial_0 U \times I \subset U$ is a regular neighborhood of $L \times I$. Make $L \times \{1\}$ disjoint from K, then it deforms into $\partial_1 U$. The deformation can be approximated by an embedding of $\partial_0 U \times I$ by general position.



Now rotate this around the bottom copy of $\partial_1 U$ to get a copy of $\partial_0 U \times I$ embedding in $\partial_1 U \times I^2$.



The embedding of $\partial(\partial_0 U) \times I$ can be extended to collars with edges in the top of $\partial_1 U \times I^2$.



Now push away from $\partial_1 U \times I \times I \cup \partial_1 U \times I \times \{0\}$ to get an embedding β : $\partial_0 U \times I \rightarrow \partial_1 U \times I \times I$ such that interior goes to interior, β : $\partial_0 U \times \{0\} \rightarrow \partial_1 U \times \{0\} \times I$ is equal to β : $\partial_0 U \times \{1\} \rightarrow \partial_1 U \times \{1\} \times I$, and $\beta(\partial_0 U \times (0, 1)) \subset \partial_1 U \times (0, 1) \times \{1\}$. Next we string copies of this together to get a periodic embedding β^{\wedge} : $\partial_0 U \times \mathbb{R} \rightarrow \partial_1 U \times \mathbb{R} \times I$. Taking a collar of this embedding gives a periodic decomposition $\partial_1 U \times \mathbb{R} \times I \cong A \cup \partial_0 U \times \mathbb{R} \times I \cup B$, B being the upper pieces.

Finally we define the pseudoisotopy. Define $\theta: \partial_1 U \times \mathbb{R} \times I \supset \text{by}: \theta$ is the identity on A, is the shift by 1 on B, and the evident isotopy between these two on $\partial_0 U \times \mathbb{R} \times I$. Mapping to $E \times R$ by $(r, 10\varepsilon)$ gives a pseudoisotopy defined over $C^{-10\varepsilon} \times \mathbb{R}$ with radius <10 ε . This defines an element in $H_1(X, Y; \mathcal{S}(p))$ if ε is small enough, as indicated above. Define $q_1(K, L)$ to be the negative of the element.

We will need a more careful description of the decomposition $A \cup B$. Decompose $\partial_1 U \times I \times I$ as shown,



where the arrow indicates the direction from ∂_0 to ∂_1 , and the transverse direction is the $\times I$ direction.

The embeddings are shown in half the picture:



In particular we see that B is obtained by glueing together copies of $U \times [0,3] \cup_{\hat{e}_1} V \times ([0,1] \cup [2,3])$ by their ends.

6.2. Thin *h*-cobordisms, and Contractible Maps. Suppose $f: (M, \partial_0 M) \to E$ is a (δ, h) cobordism satisfying the conditions of 1.6, and that $q_1(M, \partial_0 M) = 0$. This means that there is a path in $\mathscr{S}(X \times \mathbb{R}; p \times 1)$ from the construction of 6.1 to the empty family. A path is a map to the pseudoisotopy space $I \times [0, \infty) \to P(X \times \mathbb{R}; p \times 1)$. Restricting to $\{0\} \times [0, t] \cup I \times \{t\}$ gives a path in $P(X \times \mathbb{R}; p \times 1)$ from θ to the empty family. (We will be more specific about the choice of t later). Using the union method of 5.5 we can join these to get a single family over I. This can be assumed to be of the form $(Z \times \mathbb{R}, f \times 1, \eta)$. (The argument is in the proof of 5.9), and is such that over $\{0\}$ a deletion is a suspension of θ , and over $\{1\}, \eta$ is the identity.

To get back to h-cobordisms we construct an "inverse" to the construction of 6.1. This inverse will be called the V construction.

Suppose $(U \times \mathbb{R}, r \times 1, \theta)$ is a pseudoisotopy of radius $\langle \gamma \text{ over } p \times 1 \rangle$: $E \times \mathbb{R} \to X \times \mathbb{R}$, with support containing $C \times [-\gamma, 7\gamma]$. Then $V_{\theta} = U \times [0, \infty) \times I \cap \theta(U \times (-\infty, 5\gamma] \times I)$ is an *h*-cobordism from $\partial_0 V_{\theta} = U \times [0, 5\varepsilon] \times \{0\} \cup U \times \{0\} \times I \cup \theta(U \times \{5\varepsilon\} \times I)$ to $\partial_1 V_{\theta} = U \times \mathbb{R} \times \{1\} \cap V_{\theta}$. More generally if $(U \times \mathbb{R}, r \times 1, \theta)$ is a family over Δ^k , then this construction gives $V_{\theta} \to \Delta^k$ which is a locally trivial fibration with *h*-cobordism fibers.

The first step in the construction of $q_1(M, \partial_0 M)$ is to embed $(M, \partial_0 M)$ in $\mathbb{R}^n \times [0, \infty)$, and take a regular neighborhood. The result is the disc bundle of the normal bundle, $(D v_M^k, D v_{\partial_0 M}^k)$. Applying the construction gives a pseudoisotopy $(D v_{\partial_1 M} \times \mathbb{R}, r \times 1/\gamma, \theta)$. Applying the V construction to this gives by the description of B given in 6.1 (after deleting cancelling inverses) exactly $D v \times [0,3] \cup V \times ([0,1] \cup [2,3])$, $(D v \cup V) \times \{0,1\} \cup D v_{\partial_0 M} \times [0,3]$. Therefore if we apply the V construction to the family $(Z \times \mathbb{R}, f \times 1, \eta)$ obtained above, we get a locally trivial fibration of h-cobordisms $V_{\eta} \rightarrow I$ which over $\{0\}$ restricts to a suspension of this h-cobordism and which over $\{1\}$ is a product.

Now we use the algebraic obstruction theory of Sect.3 to see that an *h*-cobordism which is "concordant" to a trivial one in this way, itself has a product structure. The general stratified π_1 case follows from the unstratified one by induction, so suppose the fundamental group locally constant. According to 3.1 the obstruction can be thought of as a isomorphism of geometric modules obtained from a handlebody structure on $(M, \partial_0 M)$. $D v \times [0, 3]$, $D v \times \{0, 1\} \cup D v_{\partial_0 M} \times [0, 3]$ has a handlebody structure with the same geometric intersections, algebraically changed by an involution of the ring. Therefore one is a product if and only if the other is. It is similarly unaltered by addition of the V collars, and by suspension. $V_{\eta}|_{\{0\}}$ restricts to this, and by restriction here we mean deletion of a product, so it also has the same invariant. Finally since $V_{\eta} \rightarrow I$ is a locally trivial fibration $V_{\eta}|_{\{0\}}$ and $V_{\eta}|_{\{1\}}$ have equivalent isomorphisms, so since $V_{\eta}|_{\{1\}}$ has a product structure $(M, \partial_0 M)$ does also.

We comment on the last step, that $V_{\eta}|\{1\}$ and $V_{\eta}|\{0\}$ define equivalent invariants. The obvious thing to do is use the fact that a locally trivial fibration over I is trivial, so $V_{\eta}|\{0\} \cong V_{\eta}|\{1\}$. It is a little tricky to do this with small radius. It can be done by going back to the V construction and using the isotopy extension theorem on $\eta(Z \times \{5\varepsilon\} \times I)$. The proof of 5.9 explains how to obtain sufficient control.

A more natural approach to this problems is to generalize slightly the invariance statement 3.1(2). Since V_{η} is locally trivial we can construct a small 1-parameter family of handlebody structures on it, and then apply the argument used for $M \times I \rightarrow I$ in 3.1.

Finally we come to the radius estimates for this proof. The problem is that some of the estimates are dimension dependent. We seek an ε product structure on $(M, \partial_0 M)$. There is a function $\delta(n)$ so that if the fibration V_η has dimension *n* we need for it to have radius $<\delta(n)$ to apply the invariance argument (see 3.1(2)). Going back one step, there is $\gamma(n)$ such that if the path in $P(X \times \mathbb{R}, p \times 1)$ from θ the empty family consists of families of dimension < nand radius $<\gamma(n)$, for some *n*, then we get the desired product structure. To arrange this coincidence we use the stability theorem.

In defining $q_1(f, \partial_0 f)$ we required the radius be small enough so that θ is small enough so that the stability Theorem 5.6 applies. Therefore $q_1 \sim 0$ implies that there is a path in $P(X \times \mathbb{R}, p \times 1)$ small enough as that the shrinking Lemma 5.8 applies. (This is the restriction on t alluded to in the beginning of the proof.) Let the maximum dimension in the path be k. The shrinking lemma asserts that there is an integer m and a shrinking homotopy which shrinks smaller than $\gamma(m+k)$ and increases dimension by at most m. This gives a new path to ϕ ; first the shrinking homotopy applied to θ , and then the shrinking function applied to the original path. In the first part we have control on the dimension ($\leq \dim M + m$), and in the second we have sufficient control on the size ($\langle \gamma(m+k) \rangle$) to make up for the indeterminate part of the dimension (k). We can therefore apply the invariance argument to this path.

The characterization of the obstructions in terms of geometric group isomorphisms makes the realization of obstructions simple: use the isomorphisms to construct a handlebody on M with 2 and 3 handles, and use the homotopy theory of Ends I to see that the result is an *h*-cobordism. Finally it is also simple to use the realization theorem and the duality formula for the obstructions to show that the invariant characterizes h-cobordisms: given A, A', then

$$A' = 1A' = (AA^{-1})A' = A(A^{-1}A') = A1 = A.$$

This concludes the proof of 1.6. It also proves half of 1.3: If $f: K_1 \to K_2$ has $q_1(f)=0$ then we have seen that the regular neighborhood $(U, \partial_0 U) \supset ((K_2)_f, K_1)$ is a product (assuming the appropriate conditions on size and dimension). It therefore collapses to $\partial_0 M$ which collapses to K_1 . But since it is a regular neighborhood U also collapses to $(K_2)_f$ which collapses to K_2 .

We observe that we have the dimension estimate $\dim K_3 \leq 2 \max (\dim K_1, \dim K_2) + 4$.

For the invariance statement we show that $q_1(f)$ is unchanged by composition with contractible maps. Suppose $g: K_2 \to L$ is contractible and let $(U, \partial_0 U)$ be the (*PL*) regular neighborhood of $((K_2)_f, K_1)$ in Euclidean space. Let $(U', \partial_0 U')$ be obtained by identifying point inverses in K_2 to points. This is a regular neighborhood of (L_{gf}, K_1) in a *PL CE* image of Euclidean space, hence is a regular neighborhood in Euclidean space. Further, the quotient $(U, \partial_0 U) \to (U', \partial_0 U')$ is *CE* so can be approximated by homeomorphisms. The starting points for the construction of $q_1(f)$ and $q_1(gf)$ are therefore the same, so the invariants are equal. Compositions fg behave similarly.

The effect of this is to establish a "bijection" between H_1 and contractible pairs. Begin with a contractible pair (K, L) and construct the pseudoisotopy θ of 6.1. Then apply the V construction to it. The result is equivalent to $(K \times I, K \times \{0; 1\} \cup L \times I)$ by a composition of contractible maps and deletions of trivial pieces. This suspension is the inverse for contractible pairs so $V(\theta) \sim (K, L)^{-1}$. However we defined $q_1 = -[\theta]$, so $V(q_1(K, L)) \sim (K, L)$. Similarly given a pseudoisotopy α , $q_1(V(\alpha)) \cong [\alpha]$, providing the size and dimension estimates are satisfied.

This correspondence will be used to obtain formulas in H_1 by manipulating contractible pairs. The only qualification is that it must be possible to predict in advance the effect of the manipulation on dimensions and sizes.

6.3. Proof of 1.3. We derive 1.3 from 1.4. For this it is helpful to recall some notation for homotopies. Consider a homotopy as a map $K \times I \rightarrow L \times I$ which in the *I* coordinate is just projection. If $a, b: K \times I \rightarrow L \times I$ let $a \cdot b$ denote the track sum, a_t the map $a(, t): K \rightarrow L$, and $T: K \times I \rightarrow K \times I$ the map T(k, t) = (k, 1 - t). If $a, b: K \times I \rightarrow K \times I$ note that there are homotopies with ends fixed

$$a \circ b \sim ((a_0 \times 1_I) \circ b) \cdot (a \circ (b_1 \times 1_I)) \sim (a \circ (b_0 \times 1)) \cdot ((a_1 \times 1) \circ b).$$

Now suppose we have a finite δ projection over p. This is $s: K \to E$, $r: K \to K$, $h: r \sim r^2$, $j: (r \times 1_1) \circ h \sim h \circ (r \times 1_1)$ rel ends, all having radius $<\delta$ and all commuting with s up to δ homotopy. Let M be the mapping telescope $M = \prod_n [n, n+1] \times K/\approx$, where $(n, k) \in [n-1, n] \times K$ is equivalent to $(n, r(k)) \in [n, n+1] \times K$. Define $H: M \to M$ to be $T(h \cdot ((r \times 1) \circ h)): [n, n+1] \times K \to [-n-1, -n] \times K$ for each n.



This is a 10δ homotopy equivalence



H itself is a homotopy inverse: on $[n, n+1] \times K, H^2$ is

$$T(h \cdot ((r \times 1) \circ h))) T(h \cdot ((r \times 1) \circ h))$$

= $(T \circ ((r \times 1) \circ h) \circ T \circ h) \cdot (T \circ h \circ T \circ (r \times 1) \circ h).$

But

$$ThTh \sim (T \circ h \circ T \circ (r \times 1)) \cdot (T \circ (r \times 1) \circ T \circ h)$$
$$\stackrel{1 \cdot j}{\sim} (r \times 1)(Th \ T \cdot h) \sim r^3 \times 1$$

 $(ThT \cdot h \sim r^2 \times 1)$ is the traditional cancellation of inverses). Therefore the first piece of H^2 is $r^4 \times 1$. Similarly the second piece is homotopic to $r^4 \times 1$, so H^2 is homotopic to $r^4 \times 1$. This is homotopic to $r^5 \times (+1)$ by pushing one stage to the right $(+1: \mathbb{R} \to \mathbb{R})$ is addition of 1). This is homotopic to $r \times (+1)$. Finally $r \times (+1)$ is homotopic to the identity by pushing one stage to the left.

Define $q_0(K, r, h, j) = \partial_+ q_1(H)$ in

$$H_0(X, Y; \mathscr{G}(p)) \xleftarrow{} H_1(X \times \mathbb{R}, Y \times \mathbb{R}; \mathscr{G}(p \times 1)),$$

where ∂_+ denotes the boundary at the $+\infty$ end. The next step is to define an inverse construction from contractible pairs over $X \times \mathbb{R}$ to projections.

Suppose s: $(K, L) \rightarrow E \times \mathbb{R}$ is an ε contractible pair, with retraction t: $K \times I \rightarrow K$. For $a < b - 10\varepsilon$ consider the complex

$$N = s^{-1}(E \times [a, b]) \cup s^{-1}(E \times [a, b+3\varepsilon]) \cap L$$

Let $\alpha: R \to [0, 1]$ satisfy $\alpha(x) = 0$ for $x < a + 3\varepsilon$, $\alpha(x) = 1$ for $x > b - 3\varepsilon$, and define $r: N \to N$ by $r(k) = t(k, \alpha(sk))$. This map is the identity near the *a* end, and is the retraction into *L* near the *b* end. It is a projection because $r^2(k)$

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= $t(t(k, \alpha(sk)), \alpha(r(k, \alpha(sk)))$, which is homotopic to r(k) by multiplying the last coordinate by β and letting β go from 1 to 0. Similarly there is a symmetry homotopy.

This construction shows that q_0 is well defined, and will be useful in studying the obstruction. To complete 1.3 we show that if $q_0=0$ then the projection "splits", i.e. r factors $K \xrightarrow{a} L \xrightarrow{b} K$, $ba \sim r$, $ab \sim 1_L$. There is a diagram of contractible maps



L is the intersection of inverse images of partial stages of the telescope in each piece.



The map b is defined by projecting to a copy of K in the telescope, a is obtained by finding a small homotopy inverse for one of these maps, on one of the intermediate copies of K. This construction is due to Ferry [16].

6.4. Proof of 1.5. We start with the construction of a finite complex projection from a tame structure. Suppose $e: M \to X$ has a (W, 4) tame structure (refer to Ends I, except delete the covering space conditions. We will keep the same notation and not reproduce the definition). Let the deformations be denoted S_i , and let $K_i = M - U_i$. We may assume K_i is a polyhedron. Then S_0 is a homotopy from the inclusion $M - W_0 \to M$ to a map into K_2 . We can use S_1 , S_2 to modify S_0 so that S_0 keeps K_2 in K_3 and K_3 in K_4 . Denote S_0 by S so that we can use the S_i notation for the restrictions of S to $(M - W_0) \times \{t\}$.

To obtain a projection, let $K = K_4$, $r = S_1$. $h = S(S_1 \times 1)$ is a homotopy from r to r^2 in $M - W_0$, and in fact is a homotopy in K_3 because $S_1(K) \subset K_2$. Finally we want $(r \times 1 \circ h \sim h \circ (r \times 1))$. $\#(r \times 1)h = (S_1 \times 1) \circ S \circ (S_1 \times 1)$ and $h(r \times 1) = S \circ (S_1 \times 1) \circ (S_1 \times 1)$, so it is sufficient to show that $(S_1 \times 1) \circ S \sim S \circ (S_1 \times 1)$ on K_2 .

$$(S_1 \times 1) \circ S \sim (TST) \cdot (S \cdot (S_1 \times 1) \circ S) \sim (TST) \cdot (S \circ S)$$

$$\sim (TST) \cdot (S \cdot S \circ (S_1 \times 1)) \sim S \circ (S_1 \times 1),$$

and on K_2 this homotopy takes place in K_4 .

The invariant q_0 of a tame structure of length ≥ 4 and sufficiently small is defined to be q_0 of a finite complex projection defined in this way. It is easy to see that changing the tame structure by interpolation, etc. changes the projection by a deformation, so does not change q_0 . Finally, it is also not difficult to see a sufficiently long and small manifold tame structure has a completion. Follow Ends I up to 7.4, where the end obstruction is interpreted as a projection on a geometric group arising from a very nice tame structure. Interpret $q_0=0$ as giving a deformation from the complex projection associated to this tame structure, to the empty projection. Then this geometric data implies that the geometric group projection can be deformed to a geometric one, and an approximate completion exists.

6.5. Proof of the End and *h*-cobordism Theorems. Suppose we have the data of 1.1(a). Then by Ends I, 5.5, there are tame structures on the end so that 1.5 applies. If the invariant $q_0(f)=0$ then there is an approximate completion, $N \subseteq M$. The existence part of the theorem follows from the assertion that a sufficiently small approximate completion extends to a completion. By this we mean that there is a completion in which the approximate completion is a collar neighborhood of the boundary.

First, since $q_0(f) = 0$ then for any sequence of $\delta_i > 0$ there is a sequence $N_i \supset N_{i+1} \ldots$ of δ_i approximate completions. Let $\partial_0 N_i$ denote $N_i \cap (M - N_i)^-$, then $(N_i - N_{i+1})^-$ is an *h*-cobordism between $\partial_0 N_i$ and $\partial_0 N_{i+1}$. By the realization part of 1.6 there is a decomposition of $\partial_0 N_i \times I$ as a union of *h*-cobordisms $V_i \cup W_i$, so that

$$q_1(V_i, \hat{\sigma}_0 N_i) = -\sum_{j=0}^{i-1} q_1(N_j - N_{j+1})^-, \hat{\sigma}_0 N_j).$$

Consider these as lying in collars of $\partial_0 N_i$.



Then formulae 1.7 and 1.10 show that the *h*-cobordisms $W_i(N_i - N_{i+1}) V_{i+1}$ have trivial invariants, so are products. By fitting these product structures together we get an open collar structure on the end.

In order for this collar structure to give a completion, the collar arcs must be Cauchy in X. This will follow from appropriate estimates on the product structures on $W_i(N_i - N_{i+1})^- V_{i+1}$. These follow from estimates on the size of the *h*-cobordism structures on the pieces. We have direct control on $(N_i - N_{i+1})^-$ through the δ_i . The W_i, V_i come from the realization part of 1.6, which shows that the size may be estimated from the 1-connectivity of $\partial_0 N_i \rightarrow E$. This connectivity can also be controlled because $M \rightarrow E$ is 1-connected at the end. Therefore the collar can be arranged to converge in X, and defines a completion. This finishes 1.1 (a).

Next suppose $U_1 \subset U_2 \subset M$ are two approximate completions of a tame end. Then we can extend these to an approximate completion $M \times I \supset U_2 \times [0, \frac{1}{2}] \cup U_1 \times [\frac{1}{2}, 1]$:



We use this in two places. If we begin with two completions and extend the approximate completion to a completion rel the ends, then we get an *h*-cobordism of the two completions. If one is the standard completion of $N \times I$ and the other is a thin *h*-cobordism $A \subset A \cup B \cong N \times I$, then extending to a completion rel the one end gives an *h*-cobordism ε isomorphic to the approximate one. Combining this with the realization of approximate *h*-cobordisms in 1.6 gives the realization of 1.2.

Finally suppose as in 1.1(b) that $M' \xrightarrow{f} X$ is a completion and $(W, \partial M', N) \xrightarrow{g} X$ is an *h*-cobordism over *X*. We want to realize *W* as a completion of $f: M \times I \to X$. Choose a small collar $\partial M' \times [-\infty, \infty) \subset M'$. Use a very small retraction of $W \times [-\infty, \infty)$ to $W \cup \partial M' \times [-\infty, \infty)$ to extend *g* union with $\partial M' \times [-\infty, \infty) \subset M' \xrightarrow{f} X$ to all of $W \times [-\infty, \infty)$. Let *F* be this extension. Then $(F, 1): W \times [-\infty, \infty) \to X \times [-\infty, \infty)$ is a (δ, h) -cobordism which is very close to $g \times 1$. Now consider the interior $W \times \mathbb{R} \to X \times \mathbb{R}$. Again this is close to $g \times 1$, so it has $q_1 = 0$. Choose an isomorphism $\theta: W \times \mathbb{R} \cong \partial M' \times \mathbb{R} \times I$ of radius $\langle (\varepsilon, e^t)$ on *X*. Then *f* extends to $W \times [-\infty, \infty) \cup_{\theta} M \times I$ to be g on $W \times \{-\infty\}$.

This completes the proofs of 1.1 and 1.2.

6.6. Proofs of 1.7 and 1.8. As in 6.2 we translate into contractible pairs. First note that the pair $((K_3)_{gf}, K_1)$ is a contractible image of $((K_3)_g \cup (K_2)_f, K_1)$. Denote $((K_3)_g \cup (K_2)_f, (K_2)_f, K_1)$ by (K, L, M), then (K, M) has a contraction $r: K \times I \to K$ which restricts to give the contraction of (L, M). r defines a map from the mapping cylinder of $r_1 | L \to M$ to $K; r: (M)_{r_1 | L} \to K$. This lies over a 2-simplex whose vertices we label as shown.



Consider the pair consisting of the whole thing rel $M \times \Delta^2$ union with the copy of L over the vertex 0. This maps contractibly to (K, L) by collapsing the mapping cylinder of r. It also is a relative mapping cylinder from the 0,2 edge to the other two edges, so it collapses to

$$(M_{r|L} \cup_{M \times \{1\}} M \times [1,2] \cup_{M \times \{2\}} K, L \cup M \times [0,2])$$

This is equivalent to $(K, M) \perp (M_{r|L}, L \cup M \times I)$ by deleting $M \times (1, 2)$, on which the retraction is the identity. A similar argument shows $(M_{r|L}, L \cup M \times I)$ $\perp (L, M)$ is "concordant" to (\emptyset, \emptyset) , so we see $(K; L) \perp (L, M) \sim (K, M)$. Applying q_1 gives the formula.

We begin 1.8 with a construction for an approximate version. Suppose that $s: K_2 \to E \times [0, \infty)$ is proper over $X \times [0, \infty)$, and suppose that K_2 has a $(\delta, 2k)$ tame structure with $U_i = s^{-1}(E \times (a_i, \infty))$ for $a_{i+1} > a_i + 10\delta$ $(a_{-1} \ge 1)$, and suppose $W_i \supseteq s^{-1}(E \times [b, \infty))$ for some very large b. Finally suppose that h: $K_1 \to K_2$ is a proper map which is a δ homotopy equivalence over $(X - Y^{\delta}) \times [a_0 - 10\delta, b + 10\delta]$. Then K_1 has a $(3\delta, k)$ tame structure with $U_i = (sh)^{-1}(E \times [a_{2i} + 5\delta, \infty))$: let g be a homotopy inverse for h, then the i^{th} retraction for this tame structure is obtained by homotoping

$$1_{K_1} \sim a = \begin{cases} 1 & \text{on } (sh)^{-1} (E \times [0, a_{2i} + 7\delta)) \\ gh & \text{on } (sh^{-1}) (E \times [a_{2i} + 9\delta, \infty)) \end{cases}$$

and then inserting between g and h the retraction for the 2i+1 set in K_2 .

To see the obstruction, suppose the retraction r_1 in K_2 has been improved as in 6.4. Using the homotopy $1_{K_2} \sim gh$ construct $t: (K_1)_g \to (K_1)_g$ which is the identity over $E \times [0, a_0]$ and maps into K_1 over $E \times [a_0 + 3\delta, \infty)$. The projection in K_1 is obtained by composing this with the projection r_1 in K_2 , and projecting back to K_1 by the mapping cylinder collapse.



 $(K_1, \operatorname{proj}(r_1, t)) \sim ((K_1)_g, r_1 t)$. The subcomplex $[(K_1)_g \text{ over } E \times [0, a_0 + 3\delta]] \cup K_2 \times \{1\}$ contains the image of the retraction so it defines the same q_0 (the mapping telescope of 6.3 changes by a countable map). Further we may delete $s^{-1}(E \times (a_0 + 5\delta, a_0 + 7\delta))$ because the retraction is the identity there (delete an identity piece from the map of mapping telescopes). Therefore $q_0(K_1) = q_0([(K_1)_g \text{ over } E \times [0, a_0 + 3\delta] \cup s^{-1}(E \times [0, a_0 + 5\delta]) \times \{1\}, t) + q_0(s^{-1}(E \times [a_0 + 7\delta, \infty)), r_1).$

ference to the construction of a projection over V from

Reference to the construction of a projection over X from a homotopy equivalence over $X \times R$ shows that the $q_0(t)$ piece is exactly $\partial q_1(h)$. This gives an approximate version of 1.8 which is easily seen to imply 1.8.

7. Poincaré Duality

In this section the duality formulae of 1.9 and 1.10 are derived. The Spanier-Whitehead duality involution $\overline{1}$ is defined in 7.1 and 7.2. Finite complex quasifibrations are developed in 7.3. The torsion invariant τ is defined in 7.5. Theorems 1.9 and 1.10 are proved in 7.6, 7.7, and finally homotopy completions of ends are investigated in 7.8.

Throughout this section we assume $p: E \rightarrow X$ is stratified system of fibrations, $Y \subset X$ is a p-NDR subset, with $(X - Y)^-$ compact, and δ is small enough that the Theorems 1.1-1.8 apply as long as dimensions don't get too big.

7.1. Definition of $\overline{\mathbf{I}}$. Fix $p: E \to X$ a stratified system of fibrations, and suppose (U^n, r, θ) is a family of pseudoisotopies over K with support C and radius δ . Let θ_1 denote θ restricted to $(pr)^{-1}(C) \times \{1\} \times K \to U \times \{1\} \times K$, and let T be the involution $(\operatorname{id}) \times (1 - \operatorname{id}) \times (\operatorname{id})$ on $U \times I \times K$. Then we define

$$(-1)^n \overline{1}(U,r,\theta) = (U,r,T\theta T(\theta_1^{-1} \times 1)).$$

What this does is first turn the pseudoisotopy upside down. It is no longer a pseudoisotopy because then it starts with θ_1 at {0}. This is repaired by composing with θ_1^{-1} .

The operation $(-1)^n \overline{1}$ is an involution on families (except for a slight loss of radius), and commutes with reduction and deletion. Up to isotopy it anticommutes with suspension. A specific isotopy

$$\left[(-1)^{n+1}\overline{1}\right]\sum (U,r,\theta) \sim -\sum \left[(-1)^{n}\overline{1}\right](U,r,\theta)$$

is described in the appendix of Burghelea and Lashoff [1]. Therefore $\overline{1}$ (defined to be $(-1)^n[(-1)^n\overline{1}]$) commutes with suspension up to isotopy. Here (-1) denotes the inverse for families, $-(U, r, \theta) = (U, r, \theta^{-1})$.

This defines an involution, also denoted $\overline{1}$, on $H_*^{lf}(X, Y; \mathscr{S}(p))$. The homology group is a homotopy group whose elements can be represented by families, by 5.11, 5.6, and the Kan condition. The operation defined by applying $\overline{1}$ to such a family is well defined because $\overline{1}$ commutes up to isotopy with the changes encountered in a deformation of families.

Although this is satisfactory for our purposes, it falls short of what might be hoped for. Ideally one would construct a natural "homotopy everything" involution on the spectra $\mathscr{S}_*(X, Y; p)$. This would induce an involution on the homology spectrum, which presumably would commute with the characterization 5.11. The most direct consequence would be to extend the involution to the spectral sequence 8.7. This can be done directly in the special cases were we actually need it (dimensions ≤ 1).

With some effort \overline{I} on simplices can be pieced together to define a map on $\mathscr{S}(X, Y; p)$ whose square is homotopic to the identity. Current technology seems to be inadequate to construct involutions in a strong enough sense to pass to homology. Presumably this would be much easier if we had a direct finite complex description of \mathscr{S} .

Our point of entry to the proofs of the duality results is the case of flat h-cobordisms.

7.2. Proposition. Suppose $(U; \partial_0 U, \partial_1 U)$ is a codimension 0 submanifold of \mathbb{R}^n , $Y \subseteq X$ as usual, and $r: U \to E$ makes U a (δ, h) -cobordism over X - Y. If δ is small enough, then $q_1(U, \partial_1 U) = (-1)^n \overline{1} q_1(U, \partial_0 U)$ in $H_1^{lf}(X - Y; \mathscr{S}(p))$.

Proof. We resume the notation of 6.1. There is a periodic decomposition $\partial_1 U \times \mathbb{R} \times I \cong A \cup \partial_0 U \times \mathbb{R} \times I \cup B$, and the pseudoisotopy θ defining $q_1(U, \partial_0 U)$ is defined to be the identity on A, shifts by +1 on B, and is $1 \times (+t) \times \{t\}$ on $\partial_0 U \times \mathbb{R} \times \{t\}$. Reference to the "more careful description" before 6.2 shows that $T\theta T(\theta_1^{-1} \times 1)$ can be described as: rotate $\partial_1 U \times I$, α about $\partial_1 U$ to get a copy of $\partial_0 U \times I$ embedded near the *top* of $\partial_1 U \times I^2$. Extend the edges up to the top and string together to get:



 $T\theta T(\theta_1^{-1} \times 1)$ is the identity on the lower piece, and shifts the upper piece by (-1).

Since $q_0(U, \partial_0 U)$ is defined to be the class of the inverse of the pseudoisotopy θ , $(-1^{n+1} \overline{I}) q_0(U, \partial_0 U)$ is represented by the pseudoisotopy which shifts the top part of the picture by (+1).

Next apply the V construction of 6.2. After deleting products, this yields the contractible pair $(V \times I, \partial_0 U \times I)$ (where V is the inverse *h*-cobordism for U). The sum formula 1.7 shows that $q_1(V, \partial_0 U) = -q_1(U, \partial_1 U)$, so since the V construction is an inverse for q_1 , we get $(-1)^{n+1} \prod q_1(U, \partial_0 U) = -q_1(U, \partial_1 U)$, as required.

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This proof also provides a contractible pair description of $\overline{1}$: if (K, L) is contractible, embed in a neighborhood in \mathbb{R}^n , $(K, L) \subset (U, \partial_0 U)$, then $\overline{1}(q_1(K, L)) = (-1)^n q_1(U, \partial_1 U)$.

7.3. Finite Complex Quasifibrations. To progress to the non-flat case we must make allowance for normal bundles. For Poincaré spaces this usually means a spherical fibration, but here we need a finite complex structure. The appropriate compromise seems to be finite quasifibrations.

If $f: X \to Y$ is a map, let $X^f \to Y$ denote the associated path fibration $(X^f = \{\theta: I \to Y, x \in X | \theta(0) = f(x)\})$. Then for our purposes a quasifibration is a map $g: E \to Y$ such that the natural inclusion $E \to E^g$ is a homotopy equivalence on inverses of each point in Y.

Now suppose K is a polyhedron, and g: $E \to K$ is a quasifibration whose fiber is homotopy equivalent to a finite complex. Then there is a proper PL map $L \to K$ and a fiber map $L \to E^g$ which is an equivalence on fibers. Further given two such, there is a third and a PL commutative diagram



which are equivalences on fibers. The construction of such an approximation is indicated in the proof of 2.2.3 (for stratified systems of fibrations).

Several extensions will be useful. First, we recall that a map has the (n, ε) lifting property if homotopies of polyhedra of dimensions $\leq n$ can be lifted within ε (Ends I, 3.3). Now suppose $L \to K$ is a map of polyhedra which has the (ε, n) lifting property for $n > \dim L$, and sufficiently small ε . Lifting a contraction of a neighborhood of a point in K gives a "projection" on the inverse image of the neighborhood, in the manner of 6.4. The direct limit of copies of this gives the fiber of the map. It is not hard to see that if n, ε, K are given, there is a $\delta > 0$ such that if $L \to K$ has the (n, δ) lifting property, dim $L + \dim K < n$, and the "fiber" is equivalent to a finite complex, then the map is ε homotopy equivalent to a finite complex quasifibration.

It is also useful to note that there is a homological characterization of the (ε, n) lifting property (using the eventual Hurewicz Theorem 5.2 of Ends I, and the map $L \rightarrow L^{f}$). As an application one has the existence of inverse sphere quasifibrations: if $L \rightarrow K$ has fiber $\sim S^{j}$, choose an approximately fiberwise *PL* embedding $L \subset K \times S^{N}$. Duality discloses that the complement of an open regular neighborhood satisfies the homological characterization, so can be approximated by a finite quasifibration.

Finally we remark that if $p: L \to K$ is a quasifibration and $H_1^{lf}(K; \mathscr{S}(p)) = 0$ (e.g. if the fiber is S^k) then any two finite quasifibration approximations are related by small contractible maps (1.4). This implies that finite complex construction made with these are well defined.

We use this development to define the operation $(+\xi)$ on $H_i^{lf}(X; \mathscr{G}(p))$, $i \leq 1$.

Suppose ξ is a homotopy S^{j-1} fibration over E, $p: E \to X$ a stratified system of fibrations. Represent an element $\alpha \in H_1^{lf}(X; \mathscr{S}(p))$ by a finite contractible pair $(K, L) \to E$ (6.2). Pull ξ back to K, then $(D \xi_K, D \xi_L \cup S \xi_K)$ is again contractible (D, S) denote disc and sphere bundles respectively). According to the discussion above it also has a well defined finite complex structure over X. Therefore we can define

$$(+\xi)(\alpha) = (-1)^j q_1(D\xi_K, D\xi_L \cup S\xi_K).$$

The sign $(-1)^{j}$ makes this depend only on the stable fiber homotopy type of ξ ; $(+S^{r} \oplus \xi)(\alpha) = (+S^{r})((+\xi)(\alpha))$ is the (r+1)-fold suspension of $(+\xi)(\alpha)$. But suspension corresponds to (-1) in H_{1} .

7.4. Lemma. Suppose ξ is a spherical fibration over E, and $p: E \to X$ is a stratified system of fibrations. Then $(+\xi)$ depends only on $w_1 \xi: \pi_1 E \to \mathbb{Z}/2$. Further $\overline{1}(+\xi) = (-\xi) \overline{1}$, so $\overline{\xi} = \overline{1}(+\xi)$ is an involution on $H_1^{lf}(X; \mathscr{S}(p))$. By $(-\xi)$ we mean addition of the inverse bundle.

Proof. For the first part, let $w = w_1 \xi$. Let $E \times_w B_G$ denote the product of the 2fold covers, divided by $\mathbb{Z}/2$. Let p_w denote the projection $E \times_w B_G \to X$. $E \times_w B_G$ is the classifying space for maps $h: Y \to E$, with a spherical fibration η over Ysuch that $w_1(\eta) = w_1(\xi) \circ \pi_1 h$. In particular a bundle η over E corresponds to a map $E \to E \times_w B_G$ which is the identity when projected to E, so is a morphism of stratified systems of fibrations $p \to p_w$. This defines a homomorphism $H_1^{lf}(X; \mathscr{S}(p)) \to H_1^{lf}(X; \mathscr{S}(p_w))$. Since $p \to p_w$ is an isomorphism on π_1 of each fiber (it is a section of $p^{-1}(x) \times_w B_G \to p^{-1}(x)$ which is a fibration with fiber B_{SG} , and this is 1-connected) it induces an isomorphism on H_1^{lf} . Finally since η is the pullback of the universal bundle γ over $E \times_w B_G$, the homomorphisms $(+\eta)$ and $(+\gamma)$ agree.

For the second part we note that the remark just before 7.3 identifies \overline{I} as being essentially Spanier-Whitehead duality. With the identification the formula $\overline{I}(+\xi) = (-\xi)\overline{I}$ is just the Atiyah [3] formula for the Spanier-Whitehead dual of a Thom space.

The next topic is the torsion of a Poincaré space. As described in Sect. 1, a proper polyhedral pair (K, L) over X is Poincaré if there is a spherical fibration ξ with fiber $\sim S^{n-1}$, a Euclidean neighborhood $(K, L) \subset (U, \partial_1 U)$ of dimension n+k, and an ε homotopy equivalence $b: (U; \partial_0 U, \partial_1 U) \rightarrow (D\xi; S\xi, D\xi|L)$.

7.5. Definition. If (K, L) is Poincaré with data as described above, then $\tau(K, L) = (-1)^n \bar{1} q_1(b, \partial_0 b)$.

As in the definition of $\overline{1}$, the sign $(-1)^n$ is included to make it invariant under stabilization.

7.6. Proof of 1.10a. Suppose as above that $(K, L), \xi, b$ are an ε Poincaré space of dimension k over X. The object is to show that $\tau(K, L) + (-1)^k \overline{\xi} \tau(K, L) = \tau(L)$. Let the dimension of $(U; \partial_0 U, \partial_1 U)$ be k+n.

The first step is to find an expression for $\overline{1} q_1(b, \partial_0 b)$. Pull ξ back to the Euclidean neighborhood U of K, and approximate the inclusion $(S \xi, S \xi | \partial_1 U) \subset (U \times S^{j-1}, \partial_1 U \times S^{j-1})$ (very large j) by an embedding. Denote regular neigh-

borhoods of the image by $(S \xi, S \xi | \partial_1 U)$ again. Notice that the complement is equivalent to the complementary sphere bundle, so we denote $(U \times S^{j-1} - S \xi)^- = S(-\xi)$.

There is a homotopy from the inclusion $\partial_0 U \times D^j$ to the map $\partial_0 b$: $\partial_0 U \to S \xi$, in $U \times S^{j-1}$. Presuming that j is very large, we can use general position to approximate this by an ambient isotopy: α : $U \times D^j \times I \to U \times D^j$ such that α_0 is the identity, $\alpha_t(\partial_1 U \times D^j) = \partial_1 U \times D^j$, and $\alpha_1(\partial_0 U \times D^j) \subset S \xi$ is homotopic to $\partial_0 b$. This displays $q_1(b, \partial_0 b)$ as $q_1(U \times D^j \times I, V)$, where V is $U \times D^j \times \{0\}$ union with the mapping cylinder of α_1 : $\partial_0 U \times D^j \to S \xi$. According to 7.2

$$(-1)^{n+k+j+1} \bar{1} q_1(b,\partial_0 b) = q_1(U \times D^j \times I, (\partial (U \times D^j \times I) - V)).$$

The complement is the mapping cylinder of the inverse map

$$\alpha_1^{-1}: (U \times D^j, S(-\xi) \cup \partial U \times D^j) \to (U \times D^j, U \times S^{j-1} \cup \partial_1 U \times D^j)$$

(using the fact that $\alpha_1(\partial_1 U \times D^j) = \partial_1 U \times D^j$). Using the composition formula 1.7, q_1 of this is

$$q_1(\alpha_1^{-1}: (U \times D^j, (-\xi) \cup \partial_0 U \times D^j) \to (U \times D^j, U \times S^{j-1})) -q_1(\partial_1 \alpha_1^{-1}: (\partial_1 U \times D^j, (-\xi) \cup \partial_1 U \times D^j \cup \partial_{01} U \times D^j) \to (\partial_1 U \times D^j, \partial_1 U \times S^{j-1})).$$

The next step is an expression for $(+\xi)q_1(\alpha_1^{-1})$. We can form the pullback of ξ by crossing with D^j , and restricting to the part of the boundary denoted $S\xi$. In these terms ξ pulled back over

$$\mathfrak{x}_1^{-1}$$
: $(U \times D^j, S(-\xi) \cup \partial_0 U \times D^j) \to (U \times D^j, U \times S^{j-1})$

is

$$(D^{j} \times U \times D^{j}, (S \xi) \times D^{j} \cup D^{j} \times S(-\xi) \cup D^{j} \times \partial_{0} U \times D^{j})$$
$$\xrightarrow{-1 \oplus \alpha_{1}^{-1}} (D^{j} \times U \times D^{j}, (S \xi) \times D^{j} \cup D^{j} \times U \times S^{j-1}).$$

The diagonal

is an ε equivalence with $q_1 = 0$ (this is the cancellation of inverse bundles). Then $(1 \oplus \alpha_1^{-1}) \varDelta = (1, \alpha_1^{-1})$ can be composed on the right with the isotopy α . This gives a homotopy of $(1, \alpha_1^{-1})$ to

$$\alpha \times 1: (U, \partial_0 U) \times (D^j, S^{j-1}) \to (D^j \times U, S(\xi)) \times (D^j, S^{j-1}).$$

Therefore

$$q_1(\xi \oplus \alpha_1^{-1}) = (-1)^j q_1(\alpha_1) = (-1)^j q_1(b, \partial_0 b)$$

Referring to the definition $((+\xi)q_1(\alpha_1^{-1}) = (-1)^n q_1(\xi \oplus \alpha_1^{-1}))$ we see $(+\xi)q_1(\alpha_1^{-1}) = (-1)^{j+n}q_1(b,\partial_0 b)$, or alternatively $q_1(\alpha_1^{-1}) = (-1)^{j+1}(-\xi)q_1(b,\partial_0 b)$. The same is true for $\alpha_1^{-1}|(\partial_1 U \times D^j)$.

Putting these expressions for $q_1(\alpha_1^{-1})$ together gives 1.10a.

7.7. Proofs of 1.9 and 1.10. The first topic is 1.10c (*h*-cobordisms). Let $(K; \partial_1 K, \partial_2 K) \subset (U; \partial_1 U, \partial_2 U)$ be an *h*-cobordism, and a neighborhood in \mathbb{R}^{n+k} , and let $b: (U; \partial_0 U, \partial_1 U \cup \partial_2 U) \rightarrow (D\xi; S\xi, D\xi | \partial_1 U \cup \partial_2 U)$ be the structure map. The proof in 7.6 expressed $\tau(K, \partial K)$ as $(-1)^{k+j+1} q_1(\alpha_1^{-1})$, where

$$\alpha_1^{-1}: (U \times D^j, S(-\xi) \cup \partial U \times D^j) \to (U \times D^j, U \times S^{j-1} \cup (\partial_1 U \cup \partial_2 U) \times D^j).$$

The composition formula shows that q_1 of this is

$$\begin{aligned} q_1(\alpha_1^{-1}: & (U \times D^j, S(-\xi) \cup (\partial_0 U \cup \partial_1 U) \times D^j) \to (U \times D^j, U \times S^{j-1} \cup \partial_1 U \times D^j)) \\ &- q_1(\partial_2 \alpha_1^{-1}: (\partial_2 U \times D^j, S(-\xi) \cap \partial_2 U \times D^j \cup \partial(\partial_2 U) \times D^j)) \\ &\to (\partial U \times D^j, \partial_2 U \times S^{j-1} \cup \partial_{12} U \times D^j). \end{aligned}$$

In 7.6 the second term was identified as $(-1)^{j} \overline{\xi} \tau(\partial_{2} K, \partial \partial_{2} K)$. Since K is an *h*-cobordism, the pairs in the first map are already contractible. Therefore q_{1} of the map is the difference

$$q_1(U \times D^j, U \times S^{j-1} \cup \partial_1 U \times D^j) - q_1(U \times D^j, S(-\xi) \cup (\partial_0 U \cup \partial_1 U) \times D^j).$$

The first of these terms is $(-1)^j q_1(K, \partial_1 K)$, and by 7.1 the second is

$$(-1)^{n+k+j}\overline{1} q_1(U \times D^j, S(\xi) \cup (\partial_2 U) \times D^j) = (-1)^{k+j}\overline{\xi} q_1(K, \partial_2 K).$$

Putting these together yields the formula 1.10(c).

The union formula 1.10(b) is a simple consequence of the union version of 1.7.

The homotopy equivalence formula 1.10(d) results from the fact that the mapping cylinder of f is a Poincaré *h*-cobordism, and two applications of 1.10(c).

The product formula 1.10(e) follows easily from the product formula in 1.7.

Finally we come to 1.9. Suppose (K, L) is a dominated Poincaré pair, dominated by $(M, N) \rightarrow (K, L)$. Then the composition $(M, N) \xrightarrow{r} (M, N)$ is a pair of finite complex projections. Since (K, L) is Poincaré, the mapping telescope used in 6.3 is a finite Poincaré pair over $X \times \mathbb{R}$. (In fact this is the most general definition of "dominated Poincaré"). The formula 1.9 results from the homotopy equivalence formula 1.10(d) applied to the map H of mapping telescopes, recalling that $q_0(K)$ was defined as $\partial_+ q_1(H)$.

7.8. Homotopy Completions. Suppose M, X are metric spaces, and $f: M \to X$ is not proper. A completion of the end of f is equivalent to a closed neighborhood of the end of the form $N \times [0, 1)$, such that each arc in X obtained by $\{n\} \times [0, 1) \to M \to X$ converges; extends to a map $[0, 1] \to X$. As a homotopy analog we define $\partial C(f)$ to be the space of proper maps $[0, 1] \to M$ such that the composition $[0, 1] \to M \to X$ extends to a map $[0, 1] \to X$.

The set $\partial C(f)$ is topologized as follows: choose a map $M \to [0, 1)$ so that the product $M \to X \times [0, 1)$ is proper. Then take the union of (1) the compactopen topology on Map ([0, 1), M), (2) the inverse of the standard topology in the projection $\partial C(f) \to \text{Map}$ ([0, 1], [0, 1]), and (3) the inverse of the topology of X under evaluation at 1, $\partial C(f) \to X$. This is actually a metric topology. We may regard any neighborhood of the end, U, as a pair $(U, \partial C(f))$. The map $\partial C(f) \rightarrow U$ is obtained by constructing a function $\rho: \partial C(f) \rightarrow [0, 1)$ such that for each θ , $\theta([\rho(\theta), 1)) \subset U$, and then evaluating at $\rho(\theta)$. Any two such are canonically homotopic. By further restricting ρ we can obtain ε homotopies (ε measured in X).

If the end of M over X is tame and has stratified fundamental group, and if M is a finite dimensional polyhedron (or manifold), then there are maps of neighborhoods of the end to $\partial C(f)$. More specifically, given a neighborhood U_1 and $\varepsilon > 0$, then there is a neighborhood U_2 , and a map $U_2 \rightarrow \partial C(f)$ such that the composition $U_2 \rightarrow \partial C(f) \rightarrow U_1$ is ε homotopic to the inclusion. Such maps are obtained as follows: First, for each U_1 and $\varepsilon > 0$, there is $U_2 \subset U_1$ such that for every $V \subset U_2$ there is an ε homotopy of U_2 in U_1 beginning with the inclusion, and ending up inside V.



Given this we can construct $U_1 \rightarrow \partial C(f)$ by repeatedly pulling it toward the end, by a sequence of homotopies the i^{th} of which has radius $<1/2^i$. The homotopies are obtained from the approximate Hurewicz theorem Ends I, 5.1. Exactly such homotopies are involved in the definition of an approximate tame structure (I, 2.4), except that they are going the wrong way: pulling things in away from the end. However the definition of a homological tame structure (before I, 5.3) is symmetric, replacing U_i, W_i by $M - W_i, M - U_i$. Therefore making this replacement in I, 5.3 gives homotopies pulling things out toward the end.

Repeated use of these homotopies shows that under the conditions assumed above, $\partial C(f)$ is naturally ε homotopy equivalent to the homotopy inverse limit of neighborhoods of the end. It then follows easily from the constructions in the proof of the approximate end theorem, I Sect. 7, that $\partial C(f)$ is ε dominated by sets of the form $(U-V)^-$, U, V, PL or manifold neighborhoods of the end, and that for appropriate $U, (U, \partial U \cup \partial C(f))$ is ε dominated Poincaré.

8. Appendix

Homology with Stratified Coefficients

The purpose is to briefly discuss homology with "twisted" spectrum coefficients. The main points are the definition, a characterization theorem, and an Atiyah-Hirzebruch type spectral sequence relating it to ordinary homology.

Suppose \mathscr{S} is a covariant functor from the category of spaces to spectra. Suppose $p: E \to |K|$ is a map, K a simplicial complex. Then we apply \mathscr{S} "blockwise" to p. Define $\mathscr{S}(p)$ to be $\prod_{\sigma \in K} \mathscr{S}(p^{-1}(\sigma)) \times \sigma/\approx$, where the equivalence relation is generated by: $(\rho, t) \in \mathscr{S}(p^{-1}(\partial_j \sigma)) \times \partial_j \sigma$ is

equivalent to its image in $\mathscr{S}(p^{-1}(\sigma)) \times \sigma$. Denote by p_* the natural projection of $\mathscr{S}(p)$ to the realization |K|.

Since a spectrum is a sequence of spaces \mathscr{S}_j , the construction above defines a sequence of spaces $\mathscr{S}_j(p)$, with projections $p_*: \mathscr{S}_j(p) \to |K|$, and sections $i: |K| \to \mathscr{S}_j(p)$ defined by fitting together the basepoints of the pieces. Further the structure maps of the $\mathscr{S}_j(\sigma)$ fit together to make $\mathscr{S}(p)$ an "ex-spectrum over |K|" (see the remarks below 8.8). If $L \subset K$ is a subcomplex, these structure maps define a map

$$\mathscr{S}_{i}(p)/i(K) \cup (p_{*})^{-1}(L) \to \Omega(\mathscr{S}_{i+1}(p)/i(K) \cup (p_{*})^{-1}(L)).$$

8.1. Definition. The homology spectrum $IH(K, L; \mathcal{S}(p))$ is the Ω -spectrum defined by

 $\mathbf{IH}(K,L;\mathscr{S}(p)) = \lim \Omega^{j}(\mathscr{S}_{i}(p)/i(K) \cup (p_{\star})^{-1}(L)).$

The homology groups are the homotopy groups of this spectrum, but we will find it convenient to work directly with the spectrum. Some background for this definitions is given after the statements of the theorems.

In this generality the homology usually depends strongly on the triangulation used. We will give definitions for homology of spaces, but will eventually impose enough restrictions on p and \mathcal{S} so that the general definition is equivalent to 8.1.

Suppose $p: E \to X$. Let S(X) denote the singular complex of X, and let $p^*: E^* \to |S(X)|$ denote the pullback of E over the natural map $|S(X)| \to X$. If U is a neighborhood of the diagonal $\Delta \subset X \times X$, define $S_U(X)$ to be the subcomplex of S(X) of singular simplices whose square lies inside U. Then we define

$$\mathbb{H}(X; \mathscr{S}(p)) = \operatorname{holim} \mathbb{H}(S_{U}(X); \mathscr{S}(p)),$$

where the homotopy inverse limit is taken in the category of spectra, indexed by the neighborhoods U of Δ .

Homotopy inverse limits are defined and discussed in Bousfield-Kan [5]. We will not make explicit use of the properties of these limits, since in practice we will use the inverse limit-free Definition 8.1. Therefore, none but the hardiest reader need try to make sense of this theory. Further, in practice we can usually use a simpler description.

Suppose $Y_0 \leftarrow \frac{f_1}{f_1} Y_1 \leftarrow \frac{f_2}{f_2} Y_2$ is a countable well ordered inverse system. Define Y_∞ to be the space

of collections of maps $\theta_i: [i, \infty) \to Y_i$, such that $f_i \theta_i = \theta_{i-1} | [i, \infty)$. Then there are maps $Y_{\infty} \to Y_i$ for all *i* (evaluation $\theta(i)$) which homotopy commute with the f_i . This induces a natural map $Y_{\infty} \to$ holim $Y_{\mathbf{x}}$, which is a homotopy equivalence.

In these terms the reader will recognize the inverse limit built into the definition of \mathscr{S} in 5.4. Similarly if X is a metric space then the inverse system used in the definition of IH can be replaced by the system of 1/n neighborhoods of the diagonal. The holim can then be described explicitly, as above.

Note if X is a subspace of I^{∞} , and $p: E \to V$, V a neighborhood of X in I^{∞} , then a Čech-type theory can be defined by letting $S_{\epsilon}(X)$ be the space of singular simplices in V of diameter $<\epsilon$, and (distance from X) $<\epsilon$.

The next step is to define a nicely behaved class of maps p. We first modify a standard term. If $p: E \to X$ and $X \supseteq Y$, then Y is a p-NDR (neighborhood deformation retract) subset of X if there is a neighborhood U of Y, and homotopies $H: U \times I \to X$, $H^{\wedge}: p^{-1}(U) \times I \to E$ such that H is the identity on $U \times \{0\}$ and $Y \times I$, $H(U \times \{1\}) \subseteq Y$, H^{\wedge} is the identity on $p^{-1}(U) \times \{0\}$ and $p^{-1}(Y) \times I$, and the diagram



commutes.

8.2. Definition. A stratified system of fibrations on a space X consists of a map $p: E \to X$ and a closed (finite) filtration of X, $X = X_n \supseteq X_{n-1} \ldots \supseteq X_0$, such that each X_j is a p-NDR subset of X, and each $p: p^{-1}(X_j - X_{j-1}) \to X_j - X_{j-1}$ is a fibration.

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A pleasant example of this arises when $X_j \subset X$ and $p^{-1}(X_j) \subset E$ have mapping cylinder neighborhoods on which p is induced from a commutative diagram of maps. The neighborhood deformation retraction can then be obtained by collapsing the mapping cylinder coordinate. In particular the iterated mapping cylinder decomposition of a PL map described by Hatcher [17] shows that any PL map can be filtered to be a stratified system of fibrations.

Stratified systems of fibrations can be understood locally, up to homotopy. To take advantage of this we need a homotopy invariance hypothesis on the functor \mathcal{S} .

8.3. Definition. A functor \mathscr{S} from spaces to spectra is said to be homotopy invariant provided a homotopy equivalence of spaces $X \to Y$ induces a homotopy equivalence of spectra $\mathscr{S}(X) \to \mathscr{S}(Y)$.

8.4. Proposition. Suppose \mathscr{S} is a homotopy invariant functor from spaces to spectra. Then $\mathbb{H}(:\mathscr{S})$ is a homology theory on the category of spaces with stratified systems of fibrations. Further if (K, L) is a simplical pair, and $p: E \to |K|$ is simplicially stratified, then there is an equivalence with the spectrum of 8.1,

$$\mathbb{H}(K,L;\mathscr{S}(p)) \xrightarrow{\sim} \mathbb{H}(|K|,|L|;\mathscr{S}(p)).$$

By a "homology theory" we mean a functor which satisfies appropriate formulations of the axioms for homology. The locally finite versions of these axioms are given in 8.5. Some remarks on the proof of 8.4 are given after 8.7.

Locally finite homology is defined for locally compact spaces X and maps $p: E \to X$ by

$$\mathbb{H}^{lf}(X; \mathscr{G}(p) = \operatorname{holim} \mathbb{H}(X, Y; \mathscr{G}(p)).$$

Here "holim" means homotopy inverse limit in the category of spectra, taken over $Y \subset X$ such that $(X - Y)^{-}$ is compact. Note that if X is σ -compact we can replace the inverse system by a countable well ordered one, and use the simpler version of holim given above.

8.5. Characterization Theorem. Suppose \mathscr{S} is a covariant spectrum valued functor on the category of locally compact and σ -compact ANRs with stratified systems of fibrations. Suppose \mathscr{S} satisfies

1. (restriction) if $X \supset W$ is open, then there is a natural restriction map $\mathscr{S}(X;p) \rightarrow \mathscr{S}(W;p/W)$ such that if $Y \subset X$ is a closed p-NDR subset then $\mathscr{S}(Y;p/Y) \rightarrow \mathscr{S}(X;p) \xrightarrow{\text{restriction}} \mathscr{S}(X-Y;p/X-Y)$ has composition the point map, and is a homotopy fibration.

2. (continuity) if p_1, p_2 are stratified systems of fibrations, then the function

$$\operatorname{mor}(p_1, p_2) \to \operatorname{mor}(\mathscr{S}(X_1; p_1), \mathscr{S}(X_2; p_2))$$

is continuous, and

3. (inverse limit) the restrictions define a homotopy equivalence $\mathscr{G}(X;p) \xrightarrow{\sim} \operatorname{holim} \mathscr{G}(X-Y;p)$,

the inverse limit being over Y closed in X such that $(X - Y)^{-1}$ is compact.

Then there is a homotopy equivalence of spectra

$$A: \operatorname{I\!H}^{lf}(X; \mathscr{S}(p)) \to \mathscr{S}(X; p)$$

which is natural up to homotopy.

Remarks. The functor of spaces used to construct $\mathscr{S}(p)$ is $\mathscr{S}(E) = \mathscr{S}(pt; E \to pt)$. In the second axiom, for morphisms $p_1 \to p_2$ we will use commutative diagrams with the compact-open topology. Similarly a map of spectra is a degreewise map of the sequence of spaces, together with homotopies for the diagram with the structure maps. When the theorem is applied in 5.11 we will need to interpret the continuity axiom in a Δ -set context, but the proof is essentially the same.

Sketch of the proof. The proof is in two parts; producing the transformation A, and then showing it is an equivalence. The first part generalizes a fairly common construction in algebraic k-theory.

8.6. Lemma. Suppose \mathscr{S} is a functor which satisfies the continuity and inverse limit axioms of 8.5. Then there is a transformation $A: \mathbb{H}^{lf}(X; \mathscr{S}(p)) \to \mathscr{S}(X; p)$ natural up to homotopy, which is the identity if X is a point.

Proof of 8.5 from 8.6. Notice that the continuity axiom implies homotopy invariance for \mathscr{G} . This shows that A is an equivalence for fibrations $E \to D^n$. Restriction for $D^n \supset S^{n-1} \supset D^{n-1}$ and in-

duction on dimension extends this to fibrations over \mathbb{R}^n , and then to finite unions of copies of \mathbb{R}^n . The simplicially stratified case follows from the restriction axiom applied to the skeletal filtration of the base simplicial complex. Finally the general case follows from this, homotopy invariance, and the inverse limit axiom.

Proof of 8.6. The inverse limit axiom and homotopy invariance reduces the construction of A to situations $p: E \to |K| - |L|$, where K is a finite simplicial complex and is simplicially stratified. In this situation

$$\mathbb{H}^{lf}(|K| - |L|; \mathscr{S}(p)) \cong \mathbb{H}(|K|, |L|; \mathscr{S}(p)) \cong \mathbb{H}(K, L; \mathscr{S}(p)),$$

this last being the spectrum of Definition 8.1. Therefore we need

$$A_{i}: \underline{\lim} \ \Omega^{k}(\mathscr{S}_{i+k}(p)/i K \cup p_{*}^{-1} L) \to \mathscr{S}_{i}(|K| - |L|; p).$$

It is sufficient to define suitably related maps $\mathscr{G}_{j}(p) \rightarrow \mathscr{G}_{j}(|K|, |L|p)$ (and then take the limit of loops of these). For this we alter the definition of $\mathscr{G}_{j}(p)$ a little.

Since $p: E \to |K|$ is simplicially stratified, it is fiber homotopy equivalent to a map with an iterated mapping cylinder decomposition. Define $\mathscr{G}'_j(p)$ to be the realization of system of maps obtained by applying \mathscr{G}_j to the maps in the decomposition of p. In detail, there is a partial order on the vertices of K, and for each 1-simplex there is a map $p^{-1}(\partial_0 \tau) \to p^{-1}(\partial_1 \tau)$ such that the diagram corresponding to a simplex of K commutes. Let v_σ denote the smallest vertex of the simplex σ . Then the realization is $E = \prod_{\sigma \in K} p^{-1}(v_\sigma) \times |\sigma|/\approx$, where a point in $p^{-1}(v_\sigma) \times |\partial_j \sigma|$ is identified with its image in $p^{-1}(v_{\sigma,\sigma}) \times |\partial_j \sigma|$. Therefore

$$\mathscr{S}'(p) = \coprod_{\sigma \in K} \mathscr{S}(p^{-1}(v_{\sigma})) \times |\sigma| / \approx$$

Define $\mathscr{G}'(p) \to \mathscr{G}(p)$ as follows: the piece of the mapping cylinder decomposition



can be regarded as a map $\sigma \to \operatorname{mor}([p^{-1}(v_{\sigma}) \to pt], p||\sigma|)$. The continuity axiom therefore gives $\mathscr{G}_{j}(p^{-1}(v_{\sigma})) \times \sigma \to \mathscr{G}_{j}(p^{-1}(\sigma))$. The union over all σ gives a map $\mathscr{G}'(p) \to \mathscr{G}(p)$. This is a homotopy equivalence on each $p_{*}^{-1}(\sigma)$, so \mathscr{G}' can be substituted for S in the definition of IH.

Next we define $\mathscr{G}'_j(p) \to \mathscr{G}_j(|K| - |L|; p)$. As above regard the mapping cylinder decomposition as a map $|\sigma| \to mor([p^{-1}(v_{\sigma}) \to pt], p)$ to define

$$\mathscr{S}_{i}(p^{-1}(v_{\sigma})) \times |\sigma| \to \mathscr{S}_{i}(|K| - |L|; p).$$

This assembles to give a map

$$\mathscr{S}'_{i}(p) \to \mathscr{S}_{i}(|K|; p) \to \mathscr{S}_{i}(|K| - |L|; p).$$

Since maps of spectra preserve basepoint, i(K) is taken to the basepoint. Further since $p_*^{-1}(L)$ maps to $\mathscr{G}_i(|L|; p)$ it goes to the basepoint in $\mathscr{G}_i(|K| - |L|; p)$. This therefore gives the required map

$$\mathscr{S}'_{i}(p)/i(K) \cup p_{*}^{-1}(L) \to \mathscr{S}'_{i}(|K| - |L|; p).$$

The next topic is the spectral sequence. For this we need the notion of a stratified system of groups. (See Sect. 1.)

Suppose X is filtered by closed subsets X_i . A stratified system of groups over X consists of neighborhoods U_i of $X_i - X_{i-1}$, local coefficient systems A_i over U_i , and for i > j a homomorphism θ_{ij} : $A_i \rightarrow A_j$ over $U_i \cap U_j$, such that if i > j > k then $\theta_{jk}\theta_{ij} = \theta_{ik}$ over $U_i \cap U_j \cap U_k$.

If A is a stratified system of abelian groups, we can apply the Eilenberg-MacLane space functor fiberwise. We denote this by $B_X^n A$ rather than K(A, n) to avoid confusion with algebraic Ktheory. This gives fibrations $B^n(A(x)) \to B_{X_1}^n(A_i) \to U_i$, and the homomorphisms θ_{ij} define fiber maps over $U_i \cap U_j$. These maps can be used to define a topology on the union $B_X^n(A) = \coprod B_{X_1}^n(A_i)$. If each X_i is an ANR then $B_X^n(A) \to X$ is a stratified system of fibrations, and they fit together to give a spectrum over X; $B_X^n(A)$. Define $\mathbb{H}_{*}^{lf}(X; A) = \mathbb{H}_{*}^{lf}(X; B_X^n(A))$, and as usual the homology groups are

$$H^{lf}_{I}(X;A) = \pi_{I} \operatorname{IH}^{lf}_{*}(X;A).$$

8.7. Theorem. (Atiyah-Hirzebruch type spectral sequence). Suppose \mathscr{S} is a homotopy invariant spectrum valued functor of spaces, and $p: E \to X$ is an ANR stratified system of fibrations. Then the homotopy groups $\pi_k \mathscr{S}(p^{-1}(x))$ form stratified systems of groups over X (denoted by $\pi_k \mathscr{S}(p)$) and there is a (homological) spectral sequence with $E^2 = H_i^{lf}(X; \pi_k(\mathscr{S}(p)))$ which abuts to $H_{l+k}^{lf}(X; \mathscr{S}(p))$.

It seems inappropriate to include detailed proofs of these facts here; they are fairly straighforward extensions of arguments long understood. We will give some discussion of the setting for these arguments, partly as motivation for some of the constructions used.

Suppose \mathscr{S} is a spectrum. Whitehead [36] defines the homology groups $H_n(X, Y; \mathscr{S})$ to be π_n of the spectrum $X/Y \wedge \mathscr{S}$. Our point of view is to work directly with the spectrum, defining $\operatorname{IH}_*(X, Y; \mathscr{S}) = \lim \Omega^k(X/Y \wedge \mathscr{S}_k)$. The maps in the direct system are induced from the structure maps $\mathscr{S}_n \to \Omega \mathscr{S}_{n+1}$ in the spectrum. This defines the 0th space of an Ω spectrum; the n^{th} space is $\lim \Omega^{k-n}(X/Y \wedge \mathscr{S}_k)$.

We define an ex-spectrum over X in the context of I.M. James' [20] ex-homotopy theory, motivated by the structure of $X \times \mathscr{S}$. Recall that an ex-space over X is $X \xrightarrow{i} E \xrightarrow{p} X$ with pi = id. The loop space over X, denoted $\Omega_x(E)$, is the space of maps $p: X \times I \rightarrow E$ which commute with projection to X, and which agree with *i* when restricted to $X \times \{0\}$ or $X \times \{1\}$. $\Omega_x E$ is again an exspace over X. Roughly speaking the fibers of $\Omega_x E \rightarrow X$ are loop spaces of fibers of $E \rightarrow X$. An exspectrum over X is a sequence of ex-spaces E_i , and morphisms $\alpha_i: E_i \rightarrow \Omega_x E_{i+1}$.

We now recognize the construction $\mathscr{S}(p) \to |K|$ at the beginning of the section (blockwise application of a spectrum-valued functor to $p: E \to |K|$) as giving an ex-spectrum over |K|. Similarly application of the B_X^* functor to a stratified system of groups as used in 8.7, gives an ex-spectrum.

If E_* is an ex-spectrum over X, and if $Y \subset X$, define the homology spectrum by

$$\operatorname{IH}_{*}(X, Y; E) = \lim \Omega^{k}(E_{k}/p^{-1}(Y) \cup i(X)).$$

The maps in the limit are loops applied to

$$E_{i}/p^{-1}(Y) \cup i(X) \xrightarrow{\alpha_{j}} (\Omega_{X}E_{j+1})/p^{-1}(Y) \cup i(X) \xrightarrow{\text{incusion}} \Omega(E_{j+1}/p^{-1}(Y) \cup i(X))$$

.....

As before this is the 0^{th} space of an Ω -spectrum.

The proof of 8.4 can be obtained by suitably extending the arguments of G. Whitehead [36] to ex-spectra.

The (constant coefficient) Atiyah-Hirzebruch spectral sequence is derived in G. Whitehead [37]. An approach perhaps more suitable to our context is the following. A spectrum has a natural k-connected "cover" $\mathscr{S}(k) \to \mathscr{S}$. Applying this construction fiberwise to an ex-spectrum (when this can be reasonably defined, e.g. for $\mathscr{S}(p) \to |K|$) gives a sequence of ex-spectra $\ldots \to \mathscr{S}_{(k)}(p) \to \mathscr{S}_{(k-1)}(p) \to \ldots$ which we can regard as a filtration of $\mathscr{S}(p)$. Applying Π_* to this filtration gives a spectral sequence which abouts to $H_*(X; \mathscr{S}(p))$. The E^2 terms are as described in 8.7 because the fiber over X of $\mathscr{S}_{(k)}(p) \to \mathscr{S}_{(k-1)}(p)$ is the ex-spectrum $\Omega_x^{-k-1} B_x^*(\Pi_k \mathscr{S}(p))$.

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