# OPEN BOOK DECOMPOSITIONS, AND THE BORDISM OF AUTOMORPHISMS

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AN OPEN book decomposition displays a manifold as the relative mapping torus of an automorphism of a lower dimensional manifold. The main result of this paper is a relative existence theorem for book decompositions. If the dimension of the manifold is odd, a book decomposition always exists. If it is even dimensional, nonsingular bilinear forms appear as obstructions. In contrast to the situations encountered in surgery and knot theory, these forms have no symmetry properties. Consequently the group of equivalence classes of such forms is infinitely generated.

After the statement of the theorem and some historical remarks, the main application is given. The application is a functorial description of the bordism of automorphisms (Z actions) in terms of ordinary bordism and groups of bilinear forms. The proof of the theorem is outlined in 4, and the proof itself occupies 1.

# **§1. STATEMENT OF THE THEOREM**

1.1 THEOREM. Suppose  $M^k$  is a compact manifold (smooth, PL, or topological) which has a book decomposition on the boundary.

- (1) If k is odd  $\geq 5$ , the decomposition extends to M.
- (2) If k is even there is an invariant  $i(M) \in W^{s}(\mathbb{Z}[\pi_{1}M], w_{1}M)$ , which is an invariant of bordisms of M which have a book decomposition on the induced bordism of  $\partial M$ .
- (3) If k is even  $\geq 6$  then i(M) = 0 iff the decomposition on  $\partial M$  extends to M.
- (4) Suppose n is odd  $\geq 5$ ,  $\mathbb{N}^n \times \{0\}$  has a book decomposition given, and  $a \in W^s(\mathbb{Z}[\pi, N], w_1N)$ . Then there is a book on  $N \times \{1\}$  such that  $i(N \times [0, 1]) = a$ .

We define the terms used in the theorem. If  $h: P \to P$  is a map which is the identity on  $\partial_0 P \subset P$ , then the *relative mapping torus*  $t(h, \partial_0 h)$  is

$$P \times [0, 1]/\{(p, 0) \sim (h(p), 1)\} \cup_{\partial_0 P \times S^1} \partial_0 P \times D^2.$$

In forming this union note we have used a canonical identification of  $\partial_0 P \times S^1$  with the mapping torus of the identity of  $\partial_0 P$ .

A book decomposition on a manifold  $M^n$  is an isomorphism  $M \simeq t(h, \partial_0 h)$  where h is an isomorphism of an n-1 manifold P, and  $\partial_0 P$  is a codimension zero submanifold of  $\partial P$ . P is called the page, and  $\partial_0 P$  is the binding.

The terminology was inspired by the following example of H. E. Winkelnkemper: take a thick book and open it until the front and back covers touch. The pages then fan out radially. This illustrates the book decomposition of the 3-disc obtained from the identity on the 2-disc, rel an arc in the boundary. The pages represent the levels  $D^2 \times \{t\}$  in the mapping torus.

The obstruction group is a Witt group of sesquilinear forms. For a ring R with antiinvolution w (i.e. w(ab) = w(b)w(a)) a form  $\lambda(,)$  is sesquilinear if biadditive and  $\lambda(ax, by) = a\lambda(x, y)w(b)$ .

1.2 Definition.  $W^{s}(R,W)$  is generated by forms  $\lambda: R^{k} \times R^{k} \to R$  whose adjoints  $\lambda^{a}: R^{k} \to (R^{k})^{*}$  are simple isomorphisms (the form is then simple nondegenerate). The equivalence relation is defined by setting equal to zero those forms with a "lagran-

gian",  $K \subset \mathbb{R}^k$  which is free and based,  $\lambda(K, K) = 0$ , and  $\lambda^a : \mathbb{R}^k/K \to (K)^*$  is a simple isomorphism.

Note that no symmetry is required of these forms. This unexpected situation is why the theorem does not distinguish dimensions 4k and 4k + 2. It also causes the groups to be rather large. For example  $W^{s}(\mathbb{Z}, 1)$  is  $\mathbb{Z}^{\infty} + (\mathbb{Z}/4)^{\infty} + (\mathbb{Z}/2)^{\infty}$ . A fair amount is known about these groups; we hope to make this the subject of another paper.

The invariant i(M) associated with a particular manifold is obtained from a middle dimensional intersection form. It usually cannot be read off directly (see §6), but the following construction gives the idea.

Suppose  $M^{2n}$  is a manifold,  $P \times I \subset \partial M$  a page in the boundary. Then there is a Poincaré duality isomorphism  $H_n(M, P \times \{0\}) \simeq H_n(M, \partial M - P \times \{0\})$ . However since P is a page in book decomposition, there are isomorphisms  $H^n(M, \partial M - P \times \{0\}) \simeq H^n(M, P \times 1) \simeq H^n(M, P \times \{0\})$ . Together these give an isomorphism

$$H_n(M, P \times \{0\}) \xrightarrow{\sim} H^n(M, P \times \{0\}).$$

If these modules were projective, this would be adjoint to a nonsingular form on  $H_n(M, P \times \{0\})$ . In §6 i(M) will be defined by performing a geometric construction to make these modules with  $Z[\pi_1 M]$  coefficients free. The next proposition shows that some information on i(M) can be obtained without going through the geometry, if a sufficiently good coefficient ring is used.

Suppose R is a notherian ring with the property that a finitely generated module is projective iff it is torsion free. Important examples are Z, and maximal orders of Z  $\pi$  when  $\pi$  is finite[14]. Over such a ring  $H^n(x; R)/\text{torsion} \simeq (H_n(x; R)/\text{torsion})^*$  so we get a nondegenerate form on the projective module  $H_n(M, P \times \{0\}; R)/\text{torsion}$ .

Let  $W^{p}(R, w)$  be defined using forms as in 1.2, but on finitely generated projective modules, and with no simplicity requirement.

1.3 PROPOSITION. Suppose  $M^{2n}$  is a compact manifold with book decomposition on  $\partial M$ , which has page  $P \times I \subset \partial M$ . Suppose  $q: (\mathbb{Z}\pi_1 M, w_1 M) \to (R, w)$  is an antiinvolution preserving ring homomorphism where R is a ring as above. Then the equivalence class of the intersection form on  $H_n(M, P \times \{0\}; R)/torsion$  is the image of iM in  $W^p(R, w)$ .

Notice that this implies rather few of the invariants are realized on closed manifolds. If  $\partial M = 0$ , the intersection form on  $H_n(M; R)/\text{torsion}$  is  $(-1)^n$  symmetric, and rather few classes in  $W^p(R, w)$  are represented by symmetric forms.

#### §2. HISTORICAL REMARKS

Book decompositions were formally introduced in 1973 by H. E. Winkelnkemper [21] and I. Tamura [16]. They have implicitly been in use much longer. The classical 3-dimensional Seifert fibered knots are book decompositions, and J. W. Alexander proved the first book decomposition theorem (every  $M^3$  has one) in this setting in 1923 [2]. Higher dimensional analogs of fibered knots were studied by C. H. Griffin in his 1965 thesis (see [4]).

The provocative terminology is due to Winkelnkemper, as remarked above. Winkelnkemper [21] showed that a closed simply connected *n*-manifold has a book decomposition if *n* is odd  $\geq 7$ , or if *n* is even  $\geq 8$  and the index is zero. Tamura [18] gave an independent proof for the closed odd dimensional simply connected case. More recently Lawson [8] has given a proof for closed odd dimensional manifolds which are not simply connected.

In the present paper the odd dimensional results are extended to a relative setting, and to include 5-manifolds. In even dimensions Proposition 1.4 easily identifies the general obstruction i(M) as the index if  $\pi_1 M = 1$ , and  $\partial M = \emptyset$ . Therefore Winkelnkemper's result is also included.

Applications of book decomposition are fairly numerous. As mentioned above they have been used in the study of knots by Griffin[4], Kaufmann[5].

Book decompositions are closely related to twisted doubles (book $\Rightarrow$ double, and perhaps conversely). Thus our results imply most of the twisted double results of Smale[15], Barden[3], Levitt[9], Winkelnkemper[21], and Alexander[1].

Book decompositions have been used by Neumann and others in studying "cutting and pasting" of manifolds[11].

They have been used by Tamura in [16–18], and by others [7] to study codimension one foliations.

Milnor's fibering theorem [10] produces book decompositions from complex hypersurface singularities.

Winkelkemper has used book decomposition to study the action of the group  $\theta^n$  of homotopy spheres on the set of smooth structures on a manifold, [21, 22], and fibering in a cobordism class [21].

Finally they have been used to study the bordism of diffeomorphisms. This was one of the main motivations behind Winkelnkemper's development of the idea, and this will be the main application made of the results of this paper.

## **§3. BORDISM OF AUTOMORPHISMS**

Let SM denote the category of oriented manifolds ( $\mathcal{M} = \text{diff}, PL$ , or top).

3.1 Definition.  $\Omega_n^{SM}(X; \mathbb{Z}, 1)$  is defined to be bordism classes of 4-tuples (M, f, d, h) where

- (a) M is a compact *n*-dimensional SM manifold,
- (b)  $f: M \to X$  is a map
- (c)  $d: M \to M$  is an orientation preserving isomorphism which is the identity on  $\partial M$ , and
- (d) h is a homotopy  $f \sim f_0 d$  which is constant on  $\partial M$ .

The notation results from considering an automorphism as generating an action of Z. More to the point here, h defines a map on the mapping torus  $H: t(d, \partial d) \rightarrow X$ , which induces an isomorphism of this group with the n + 1 dimensional "book decomposition" bordism of X.

3.2 THEOREM. If  $n \ge 2$  there is a natural exact sequence

$$0 \to \Omega_{2n+1}^{\mathcal{S}\mathcal{M}}(X; \mathbb{Z}, 1) \xrightarrow{i} \Omega_{2n+2}^{\mathcal{S}\mathcal{M}}(X) \xrightarrow{i} W^{s}(\mathbb{Z}[\pi_{1}X]) \xrightarrow{r} \Omega_{2n}^{\mathcal{S}\mathcal{M}}(X; \mathbb{Z}, 1) \longrightarrow \Omega_{2n+1}^{\mathcal{S}\mathcal{M}}(X) \to 0.$$

Here t is the mapping torus, i is the invariant of 1.1, and r is "realization". The realization homomorphism is defined by beginning with a nullcobordant  $(M^{2n}, f, d, h)$ , and applying the realization theorem 1.1(4) to change the book decomposition on r(d) to realize a particular element of  $W^s$ . If  $\pi_1 X$  is finitely presented, we can use any 4-tuple such that  $\pi_1 M \to \pi_1 X$  is an isomorphism. If  $\pi_1 X$  is not finitely presented, the starting 4-tuple varies depending on the entries in a matrix representative for the form to be realized.

The proof of exactness is a straightforward application of 1.1 and 1.2. Further if  $\pi_1 X = 1$ , it follows from 1.4 that the image of *i* is 0 if *n* is odd, and is Z (the index) if *n* is even.

Partial results in this direction have been obtained by Winkelnkemper[21], and Krech[6]. Their work deals with the group  $\Omega_n^{s,\mu}(*; \mathbb{Z})$ , which is defined by 4-tuples as above with empty boundary.

The more complete version is that of Krech. He has calculated  $\Omega_n^{SM}(*, \mathbb{Z}, 1)$ , but with a different invariant from our i(M).

The boundaryless group is related to the bounded one by

$$\Omega_n^{\mathcal{SM}}(X; \mathbb{Z}) \xrightarrow{\text{(forget, underlying manifold)}} \Omega_n^{\mathcal{SM}}(X; \mathbb{Z}, 1) \oplus \Omega_n^{\mathcal{SM}}(X),$$

which is easily seen to be an isomorphism.

## **§4. OUTLINE OF THE PROOF**

In this section we outline the contents of the remaining sections, and indicate how they fit together to yield the main theorem.

We begin with the last step in the proof. Suppose  $P^{n-1} \times I \subset M^n$ , and let  $M^{\uparrow}$  be the closure of  $M - P \times I$ . If both inclusions  $P \times \{0\}, P = \{1\} \subset M^{\circ}$  are simple homotopy equivalences,  $M^{\circ}$  is an s-cobordism between them. Notice that the s-cobordism on the boundary from  $\partial P \times \{0\}$  to  $\partial P \times \{1\}$  is already a product  $P \times I$ . If the dimension of M is  $\ge 6$ , the s-cobordism theorem implies that  $M^{\circ}$  is the mapping cylinder of an automorphism  $P \rightarrow P$  which is 1 on  $\partial P$ .

If we absorb the collar  $P \times I \subset M$  into this mapping cylinder, we get a mapping torus decomposition of M.

The problem is therefore to find good embeddings  $P \times I \subset M$ . Typically we can begin with an [n/2] - 1 skeleton of M as a beginning of P. Then if we can add [n/2]handles in the proper way we get a page.

One consequence of this is that the pages constructed by this process are equivalent to low dimensional CW complexes. We formalize such pages since they have a number of useful properties.

4.1 Definition. A page  $P \times I \subset M$  is almost canonical if P has the homotopy type of an [n/2] complex, and  $P \times I \cap \partial M$  has the homotopy type of an [(n-1)/2]complex.

These pages are called "almost canonical" because they are determined except in one dimension ([n/2]) by M. See [12] for this type of terminology.

# 4.2 The outline

The next section, §5, contains the geometric heart of the result: the study of middle dimensional "ribbon handles". Suppose  $V \times I \subset M^{2n}$  is almost a page, and needs only some *n*-handles. We want the  $\times I$  factor to extend to these handles so we consider handles of the form  $(D^n \times I, S^{n-1} \times I) \subset (M, \partial P \times I)$ . On the boundary this is to be a product with an embedding  $S^{n-1} \subset \partial P$ . We show that a homologically defined form determines whether or not a family of homotopy classes can be represented by disjointly embedded ribbon handles.

§6 associates a nonsingular sesquilinear form to a  $M^{2n}$  with an almost canonical book decomposition on the boundary. This is done by extending the book decomposition to an n-1 skeleton of M, and showing that for what is left the homology modules considered before 1.4 are free. The ribbon handles of §5 are then used to show that if the form contains a lagrangian, the book decomposition can be extended. §7 gives the realization theorem for almost canonical decompositions.

In §8 we discuss sections of homomorphisms. The setting is a pair of module homomorphisms  $B_1 \xleftarrow{f_1} A \xrightarrow{f_2} B_2$ . What we want is to find  $g: B_1 \rightarrow A$  such that  $f_1g$ and  $f_{2g}$  are isomorphisms. Such a thing is called a section. If  $B_1$  and  $B_2$  are free and based we may further ask for a section which gives simple isomorphisms.

In a number of constructions we encounter the need for a section of a pair of homomorphisms. Our main source of sections is Neumann's lemma (8.2) which states that if  $f_1$ ,  $f_2$  have right inverses, and  $B_1$ ,  $B_2$  are abstractly isomorphic, then the diagram has a section after stabilizing by adding  $B_1 \xleftarrow{(1,0)} B_1 \oplus B_1 \xrightarrow{(0,1)} B_1$ .

The proof of the odd dimensional case is given in §9. The proof is quickly reduced to a section problem, and the situation can be stabilized in the proper way to apply Neumann's lemma.

\$10 contains the proof of invariance of the obstruction. Specifically, if  $\partial N^{2n+1} =$  $M \cup M'$  and M' has an almost canonical book decomposition, then the form on M (with  $Z[\pi_1N]$  coefficients) stabilizes to have a lagrangian. The invariance theorem is obtained by applying this to a bordism between two situations where the form is defined.

The only step remaining is a lemma which allows us to eliminate "almost

canonical" from the above. This is provided in \$11 where it is shown that any book decomposition is concordant to an almost canonical one. A concordance is a decomposition on  $M \times I$ .

Now we assemble these pieces to obtain the theorem. To define i(M) glue onto  $\partial M$  a concordance to an almost canonical decomposition and then take the equivalence class of the form defined in §6. Similarly realization and invariance follow from the almost canonical versions of §7 and 10 by glueing on concordances. The odd dimensional case is given in §9.

The even dimensional case is slightly more complex. Suppose  $M^{2n}$  has a book decomposition on  $\partial M$ , and that i(M) = 0. We extend the decomposition to progressively larger pieces of M until we can apply §6.

First extend the decomposition to a collar of  $\partial M$  by a concordance to an almost canonical decomposition (§11). Next extend it to a neighborhood of an n-1 skeleton as in §6. Call what is left U, then U has a book decomposition on the boundary which displays a middle dimensional form  $(A, \lambda)$ . This form represents i(M) by the invariance theorem, and so is equivalent to 0.

Representing 0 in  $W^s(\mathbb{Z}[\pi_1 M])$  means the following: there is another form  $(B, \mu)$  such that both  $(B, \mu)$  and  $(A, \lambda) \perp (B, \mu)$  have lagrangians. It does not seem to follow (at least in a straightforward way) that  $(A, \lambda)$  itself has, or stabilizes to have, a lagrangian.

To avoid this difficulty we partition U using the realization theorem.  $\partial U$  is given with a book decomposition, put this decomposition on each end of a collar  $\partial U \times I$  in U. By the realization theorem we can find a decomposition on  $\partial U \times \{1/2\}$  so that the forms on  $\partial U \times [0, 1/2]$  and  $\partial U \times [1/2, 1]$  are  $(B, -\mu)$ ,  $(B, \mu)$  respectively. If we lump together  $\partial U \times [1/2, 1]$  and the interior of U, we get the form  $(B, \mu) \perp (A, \lambda)$ . Since this and  $(B, -\mu)$  have lagrangians we can apply §6 twice to extend the decomposition from  $\partial U \times \{0\} \cup \partial U \times \{1/2\}$  to all of U.

## 4.3 Low dimensions

The study of books in dimension 5 and 6 is complicated slightly by the failure in the Whitney trick in 4-manifolds. In its place we use the stable Whitney trick of [13] which requires additional algebraic hypotheses, and stabilization of  $M^4$  by  $\#S^2 \times S^2$ . The stabilizations are introduced by connected sum with examples of book decompositions on  $S^5$  and  $S^6$ , which we discuss next.

4.4 The examples. Suppose  $n, k \ge 2$ . Then there are book decompositions on  $S^{n+k+1}$  with pages  $S^n \times D^k \#_{\partial} K^n \times S^k$ , and  $S^n \times D^k \#_{\partial} (S^n \times S^k)_0 \#_{\partial} d^n \times S^k$ , respectively.

The first example results from the standard embedding  $S^n \cup S^k \subset S^{n+k+1}$  with linking number 1. Take the connected sum of regular neighborhoods to get  $(S^n \times D^k \#_{\partial} D^n \times S^k) \times I \subset S^{n+k+1}$ . If  $n + k + 1 \ge 6$ , we can see this is a page by calculating the homology of the complement and applying the *s*-cobordism theorem as at the beginning of the section. Dimension 5 can be verified directly by drawing a picture.

This yields an automorphism of  $S^n \times D^k \#_{\partial} D^n \times S^k$  which is the identity on the boundary. Consider  $S^n \times D^k \#_{\partial} (S^k \times S^n)_0 \#_{\partial} D^n \times S^k$  as two copies of this plumbed together. Then we can put the automorphism on either piece and extend to the rest by the identity. Compose the two automorphisms obtained this way. It is easily seen that the relative mapping torus of this is a homotopy sphere, hence is (or can be modified to be) a sphere. We state the basis for the homology calculation separately.

4.5 Homology of books. Suppose  $h: P \to P$  is the identity on  $\partial_0 P \subset P$ . Then  $1_* - h_*: C_*P \to C_*P$  vanishes on  $C_*\partial_0 P$ . This yields  $(1 - h)_*: H_n(P, \partial_0 P) \to H_*P$ , and a long exact sequence

$$-- \rightarrow H_n(P, \partial_0 P) \xrightarrow{(1-h)_{\bullet}} H_n P \xrightarrow{i_{\bullet}} H_n(t(h, \partial_0 h)) \rightarrow H_{n-1}(P, \partial_0 P) \rightarrow --$$

The sequence results from the sequence of  $(t(h, \partial_0 h), P)$  and the isomorphism  $H_n(t(h, \partial_0 h), P) \simeq H_n(P \times I, P \times \{0, 1\} \cup \partial_0 P \times I) \simeq H_{n-1}(P, \partial_0 P)$ .

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Next we describe book connected sum  $M \#_b N$ . Suppose  $M = t(h, \partial_0 h)$ ,  $h: P \to P$ . By definition (see §1)  $t(h, \partial_0 H) \supset \partial_0 P \times D^2$ . Therefore an embedding  $D^{n-2} \subset \partial_0 P$ specifies  $D^{n-2} \times D^2 \subset M$ . Construct  $D^{n-2} \times D^2 \subset N$  the same way. Then  $M \#_b N$  is defined to be the union of M and N with the interiors of these discs deleted, and with the resulting boundary spheres identified in the evident way.  $M \#_b N$  is then exactly the relative mapping torus of  $h \cup g$  on  $P \#_\partial Q$ . Here  $g: Q \to Q, N = t(g, \partial_0 g)$ , and the boundary sum  $P \#_\partial Q$  identifies the discs in  $\partial_0 P$ ,  $\partial_0 Q$  chosen above.

Notice that the book sum is defined only if the bindings  $\partial_0 P$ ,  $\partial_0 Q$  are both nonempty.

This operation generalizes to apply to intermediate stages of our constructions: Suppose  $P \times I \subset M$  is a codimension 0 embedding, and the closure of  $P \times I - \partial M$  is  $\partial_0 P \times I$ . Let  $D^{n-2} \subset \partial_0 P$ , then a collar of  $\partial_0 P$  in P gives an embedding  $D^{n-2} \times I^2 \subset M$ . Choose a similar  $D^{n-2} \times I^2 \subset Q \times I \subset N$ , and form the connected sum on these discs. This sum contains  $(P \#_{\partial} Q) \times I$  in a standard way.

There is a boundary book sum  $M \#_{\partial b} N$  defined similarly.

The next result is a formalization of the argument given at the beginning of the section. A simple homotopy page is  $P \times I \subset M$  such that  $P \times \{0\}$ ,  $P \times \{1\} \subset M - P \times \{0, 1\}$  are simple equivalences.

4.6 PROPOSITION. Suppose  $P \times I \subset M^n$  is a simple homotopy page which intersects  $\partial M$  in the page of a book decomposition. If  $n \ge 6$  the decomposition extends to one on M with page  $P \times I$ . If  $\partial P \subset \partial M$  and n = 5 the decomposition extends to one on M with page  $P \#_j(S^2 \times D^2 \#_{\partial}(S^2 \times S^2)_0 \#_{\partial}S^2 \times D^2)$  for some j.

It was observed above that the complement gives an s-cobordism from P to itself which is trivial on  $\partial P$ . A trivalization of the s-cobordism extends the decomposition. If  $n \ge 6$  the s-cobordism theorem applies. If n = 5 we use the stable s-cobordism theorem of [13] which requires connected sum along an arc between the two ends with  $S^2 \times S^2 \times I$ . Since book connected sum  $M \#_b S^5$  with the second example of 4.4 has exactly this effect, repetition of this operation produces a trivial s-cobordism.

# **§5. RIBBON HANDLES**

The setting is  $M^{2n}$  with  $Q^{2n-2} \times I \subset \partial M$ . A ribbon handle is a framed embedding

$$(D^n \times I, S^{n-1} \times I) \subset (M, Q \times I)$$

which is a product  $(S^{n-1} \subset Q) \times I$  on the boundary.



A ribbon 1-handle on  $S^1 \times I \subset \partial D^3$ 

# 5.1 The bilinear form

The object is a criterion for representing a set of elements  $\alpha_1, \ldots, \alpha_k \in \pi_n(M, Q \times I)$  by disjointly embedded ribbon handles (lower dimensional ribbons can be embedded by general position.) The first step is to define a sesquilinear form on the

free module

$$A = \mathbb{Z}[\pi_1 M](\alpha_1,\ldots,\alpha_k).$$

Homology classes in either  $(M, Q \times \{0\})$  or  $(M, \partial M - Q \times \{0\})$  can be obtained by pushing  $\partial \alpha_i$  into either  $Q \times \{0\}$ , or into  $Q \times \{1\} \subset \partial M - Q \times \{0\}$ . The form is the composite

$$A \times A \to H_n(M, \partial M - Q \times \{0\}) \times H_n(M, Q \times \{0\}) \xrightarrow{\left[ \left( \cap [M] \right)^{-1} & 0 \\ 1 \right]} \\ H^n(M, Q \times \{0\}) \times H_n(M, Q \times \{0\}) \xrightarrow{\text{evaluation}} \mathbb{Z}[\pi_1 M]$$

Here homology and cohomology are understood to be with  $\mathbb{Z}[\pi_1 M]$  coefficients (see Wall[19]).

5.2 THEOREM. Suppose  $\pi_1 Q \simeq \pi_1 M^{2n}$ , and  $n \ge 4$ . Then  $\alpha_1, \ldots, \alpha_k \in \pi_n(M, Q \times 1)$  can be represented by disjointly embedded ribbon handles iff the form on  $\mathbb{Z}[\pi_1 M](\alpha_1, \ldots, \alpha_k)$  is zero.

5.3 Addendum. Suppose n = 3, so Q is 4 dimensional. If there are classes  $\beta_i \in H_2(Q; \mathbb{Z}[\pi, Q])$  such that  $\beta_i \cap [\partial \alpha_i] = 1$ ,  $\beta_i \cap [\partial \alpha_j] = 0$ ,  $i \neq j$ , and  $(M, Q \times I)$  can be modified in the neighborhood of a disc so that Q changes by  $\#S^2 \times S^2$ , then after some number of such stabilizations the same conclusion is valid.

The proof proceeds by representing the  $\alpha_i$  by framed immersions and then eliminating intersections by a Whitney type device.

5.4. There is a unique regular homotopy class of framed immersions which are products on the boundary, in each homotopy class  $\alpha_i$ .

This is an easy exercise in immersion theory. Since  $D^n$  is contractible, bundles over it are all trivial. This provides an immersion. To get the product structure on the boundary notice that

{framed immersions  $P \rightarrow Q$ }  $\xrightarrow{\times I}$  {framed immersions  $P \times I \rightarrow Q \times I$ }

is always a bijection.

5.5. Intersections of the immersions  $\alpha_i$  can be arranged to be unions of squares in the interior of M, or triangle with one edge on  $Q \times I$ .



In the interior, the ribbons can be approximated so that the center discs  $D^n \times \{1/2\}$  intersect transversely in isolated points. After shrinking the ribbon sufficiently near the center, the ribbons  $D^n \times I$  intersect in squares  $I \times I$ .

The picture above shows ribbons  $D^1 \times I$  in 2-space intersecting in squares. The general situation in 2+2k space is represented locally by the picture crossed with  $D^k \times D^k$ . The first ribbon corresponds to  $D^k \times \{0\} \times (1^{\text{st}} D^1 \times I \text{ ribbon})$ , the second ribbon corresponds to  $\{0\} \times D^k \times (2^{nd} D^1 \times I \text{ ribbon})$ . In particular the intersection is exactly the pictured square in  $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^2$ .

On the boundary we are dealing with immersions  $S^{n-1} \rightarrow Q^{2n-2}$ , so again intersections generically consist of isolated points. When we cross with I we get a line of intersection in  $Q \times I$ . Now consider a collar  $(Q \times I) \times I$  in M. We can deform one

ribbon in a neighborhood of the line of intersection to be parallel to the diagonal  $\{(x, x)\} \subset I \times I$ , crossed with a disc in Q. We can deform the other so that it is parallel to the antidiagonal  $\{(x, x-1)\} \subset I \times I$ , crossed with a complementary disc in Q. The picture shows a boundary intersection of  $D^1 \times I$  ribbons in a 2-manifold, and as with the interior intersections the picture provides a local model for the general case.

We introduce some terminology at this point. At a boundary intersection point the ribbon parallel to the diagonal is said to "lean to the right", and one parallel to the antidiagonal "leans to the left" (see the picture).

## 5.6 Elimination of interior intersections

Suppose  $\alpha_i$  and  $\alpha_j$  have an interior square and a boundary triangle of intersections with opposite signs and such that the resulting loop in M is nullhomotopic. Then if  $n \ge 3$  a minor modification of the standard Whitney trick eliminates the square at the cost of reversing the "direction of lean" on the boundary.



Under the hypotheses of the theorem we can now eliminate all interior intersections. If  $\alpha_i$  and  $\alpha_j$  intersect in the interior, an intersection square defines an element of  $\pi_1 M$ . Since  $\pi_1 M \simeq \pi_1 Q \times I$  this curve deforms into Q. Now deform  $\partial \alpha_i \colon S^{n-1} \to Q$  along an embedded arc representing this curve to introduce a cancelling pair of intersection points. This deformation can be extended to  $\alpha_i$  to introduce a pair of intersection triangles with opposite sign.

By construction the loop between these boundary triangles and the interior square is contractible. One of these triangles has the correct sign to be used as above to remove the square of intersections.

#### 5.7 The intersection form

The form defined in 5.1 can now be interpreted geometrically. Assume there are only boundary type intersections. A review of the definition shows that  $\lambda(\alpha_i, \alpha_j)$  can be calculated by taking the geometric intersection of  $\alpha_i | D^n \times \{0\}$  and  $\alpha_j | D^n \times \{1\}$ .



More precisely we note that associated to a boundary intersection there is an element  $h = \pm g, g \in \pi_1 Q$ . This results from the corresponding intersection point of  $\partial \alpha_i, \partial \alpha_j: S^{n-1} \to Q$ . This triangle of  $\alpha_i \cap \alpha_j$  contributes to  $\lambda(\alpha_i, \alpha_j)$  by

 $(-1)^n h$  if  $\alpha_i$  leans left

0 if 
$$\alpha_i$$
 leans right

In other words  $\lambda(\alpha_i, \alpha_j)$  is the algebraic number of intersections of  $\partial \alpha_i$ ,  $\partial \alpha_j$  at which  $\alpha_i$  leans toward the right in M.

We point out that the ambiguity of self-intersections found in surgery [19] is avoided here. At a boundary intersection point one "sheet" of  $\alpha_i$  leans left, the other

leans right. This makes it possible to distinguish between them and assign an unambiguous fundamental group element to the intersection.

## 5.8 Elimination of boundary intersections

Suppose  $n \ge 4$  and there is a pair of intersection points of  $\alpha_i, \alpha_j: S^{n-1} \rightarrow Q^{2n-2}$ whose intersection numbers cancel algebraically. Then the classical Whitney trick shows how to cancel the intersections geometrically by an isotopy of  $\partial \alpha_i$  across a 2-disc. If we cross with *I*, this extends to  $S^{n-1} \times I \rightarrow Q \times I$ . Finally if in addition  $\alpha_i$  has the same direction of lean in *M* at the two points, the isotopy can be extended to  $\alpha_i$  to cancel the two intersection triangles.

This is the only point where modification is needed for the proof of 5.3. In this case we are concerned with immersions of  $S^2$  in  $Q^4$  and the classical Whitney trick is not available. However if the homology classes can be distinguished by intersections as presumed in 5.3, we can use the stable Whitney trick of [13] after  $\#S^2 \times S^2$ .

The proof of 5.2 can now be completed. As above we arrange the  $\alpha_i: (D^n \times I, S^{n-1} \times I) \to (M, Q \times I)$  to be framed immersions with only boundary intersections. By hypothesis  $\lambda(\alpha_i, \alpha_j) = 0$ , and according to 5.7 this is the algebraic number of intersection triangles in which  $\alpha_i$  leans to the right. These triangles can therefore be arranged in pairs which cancel algebraically, and by the discussion just above these pairs can be cancelled geometrically.

This argument eliminates intersections of  $\alpha_i$ ,  $\alpha_j$  in which  $\alpha_i$  leans to the right. Since also  $\lambda(\alpha_j, \alpha_i) = 0$  the left ones can be cancelled also.

Since all of these modifications take place in neighborhoods of 2-discs, they can be carried out disjointly so that no new intersections are introduced (if  $n \ge 4$ ). Therefore at the end of the process the  $\{\alpha_i\}$  are all framed embedded.

# **§6. THE INVARIANT, AND EVEN DIMENSIONS**

In this section we obtain a form from  $M^{2n}$ , and use ribbon handles to show that if , it has a lagrangian the book decomposition extends.

# 6.1 The nondegenerate form

Let  $P \times I \subset \partial M$  be the page in the boundary, which we suppose is almost canonical. Since P is equivalent to an n-1 complex,  $(P, \partial P)$  is n-2 connected. Therefore we can add handles in M of index  $\leq n-1$  to  $\partial P \times I^2 \subset \partial M$  to expand  $P \times I$ to an n-1 skeleton of M. Let U be the union of these handles and a collar on  $\partial P \times I^2$ in M.

 $(U, \partial P \times I^2)$  is equivalent to an n-1 complex, but has manifold dimension 2n. Therefore the factorization extends to  $(U, \partial P \times I^2) \approx (W, \partial P) \times I^2$ .

This extends the book decomposition on  $\partial M$  to  $\partial M \cup U$ . The objective is to extend it to the rest of M as well.

Let  $M^{\wedge}$  be the closure of M - U.  $\partial M^{\wedge}$  has a book decomposition with page  $(P \cup_{\partial P} W) \times I \subset \partial M^{\wedge}$ . Denote  $P \cup_{\partial P} W$  by  $P^{\wedge}$ . This decomposition is still almost canonical, and in addition the page is an n-1 skeleton of  $M^{\wedge}$ . Therefore

$$H_{j}(M^{\uparrow}, P^{\uparrow} \times I) = 0, j < n$$
  

$$H_{j}(M^{\uparrow}, P^{\uparrow} \times I) \simeq H^{2n-j}(M^{\uparrow}, \partial M^{\uparrow} - P^{\uparrow} \times I) \simeq H^{2n-j}(M^{\uparrow}, P^{\uparrow} \times I)$$
  

$$= 0, j > n.$$

The only vanishing group is  $H_n(M^{\uparrow}, P^{\uparrow} \times I)$ , which is therefore stabily free and based [20]. The sesquilinear form is obtained (as in 1.4 and 5.1) by sliding back and forth in the *I* coordinate, and middle dimensional duality. The adjoint of the form is

$$H_n(M^{\uparrow}, P^{\uparrow} \times \{0\}) \simeq H^n(M^{\uparrow}, \partial M^{\uparrow} - P^{\uparrow} \times \{0\}) \simeq H^n(M^{\uparrow}, P^{\uparrow} \times \{0\})$$
$$\simeq (H_n(M^{\uparrow}, P^{\uparrow} \times \{0\}))^*$$

so it is simply nonsingular.

#### FRANK QUINN

6.2 PROPOSITION. Suppose the form defined above has a lagrangian. If  $n \ge 4$  then the book decomposition on  $\partial M \cup U$  extends to all of M.

Recall from §1 that a lagrangian is a free based submodule  $h: A \to H_n(M^{\uparrow}, P^{\uparrow} \times I)$  such that sequence

$$0 \to A \xrightarrow{h} H_n(M^{\hat{}}, P \times \{1\}) \xrightarrow{h^* \lambda^a} A^* \to 0$$

is simple short exact.

Since  $P^{\wedge} \times I$  is an n-1 skeleton of  $M^{\wedge}, H_n(M^{\wedge}, P^{\wedge} \times I) \simeq \pi_n(M^{\wedge}, P^{\wedge} \times I)$ . Therefore a basis for A can be represented by homotopy elements  $\alpha_1, \ldots, \alpha_k \in \pi_n(M^{\wedge}, P^{\wedge} \times I)$ . Further since  $h^*\lambda^a h = 0$  the form  $\lambda$  vanishes on h(A). Therefore by 5.2, if  $n \ge 4$  the classes  $\alpha_i$  can be represented by disjointly embedded ribbon handles.

Let  $Q \times I \subset M^{\uparrow}$  be the union of a collar on  $P^{\uparrow} \times I$ , and the new ribbon handles. We claim  $Q \times I$  is a page for  $M^{\uparrow}$ .

To see this, let M' be the closure of  $M^{-}Q \times I$ . We want to show that  $H_{*}(M', Q \times \{I\}) = 0$ . From the braid of the 4-ad  $(M^{-}; M', Q \times \{0\}, P^{-} \times \{0\})$ :



By construction  $H_n(Q, P^{\uparrow}) \simeq A$ , with generators represented by the ribbon handles.  $H_n(M^{\uparrow}, M') \simeq H_n(Q, \partial Q - P^{\uparrow} \times I) \simeq H^n(Q, P^{\uparrow} \times I) \simeq A^*$ . The bottom three terms are isomorphic to

$$A \to H_n(M^{\wedge}, P^{\wedge} \times \{0\}) \xrightarrow{h^* \lambda^3} A^*$$

which by hypothesis is short exact (with zero torsion). It follows by diagram chasing that  $(M', Q \times \{0\})$  is acylic (with 0 torsion), so  $Q \times I$  is a page for  $M^{\circ}$  by 4.6.

### 6.3 Dimension 6

A slight modification is required to apply 5.2 if n = 3. First add k copies of the book decomposition  $(S^2 \times D^3 \#_{\partial} D^2 \times S^3) \subset S^6$  of 4.4. Next form M<sup>^</sup> by deleting the n-1 skeleton U, in M, and the pages  $(S^2 \times D^3 \#_{\partial} D^2 \times S^3) \times I$  in the S<sup>6</sup> summands. The boundary no longer has a book decomposition, but the classes  $\alpha_1, \ldots, \alpha_n$  above survive in  $\pi_n(M^{^}, (P^* \#_j(S^2 \times S^2 \# S^1 \times S^3))$ . There also are classes  $a_i$  represented by  $\{*\} \times D^3 \subset S^2 \times D^3$  whose boundaries are distinguished algebraically by products with the classes  $b_i$  represented by  $S^2 \times \{*\} \subset S^2 \times D^3$ . Now, after further stabilization, we can apply 5.2 to represent the classes  $(\alpha_i + \alpha_i)$  by disjoint framed embedded ribbon handles.

The image of these handles alone does not form a page for  $M^{\,}$ . Instead we claim that these handles added to the pages  $j(S^2 \times D^3 \#_{\partial} D^2 \times S^3)$  in the  $S^5$  summands give a homotopy page for the original  $M^{\,}$ . This sort of argument is carried out in detail in §11. In the terminology of 11.6 we began with a lagrangian subsection of a certain singular form. We stabilized it to make it more singular, but did not change the nonsingular part. The subsection was changed slightly by adding things in the singular part of the form, but it is still a subsection, so killing it produces a homotopy page.

Finally further stabilization of this homotopy page produces a genuine book decomposition, by 4.6.

### **§7. REALIZATION OF FORMS**

In this section forms are realized as obstructions to extending book decompositions. 7.1 PROPOSITION. Suppose  $P \times I \subset M^{2n-1}$  is a book decomposition with binding  $\partial_0 P$ . Suppose  $\pi_1 \partial_0 P \to \pi_1 M$  is onto and  $n \ge 3$ . If  $(A, \lambda)$  is any free simply nondegenerate sesquilinear form over  $\mathbb{Z}[\pi_1 M]$ , then there is a submanifold  $Q^{2n}$  in the interior of  $M \times I$ , and a book decomposition on the closure of  $M \times I - Q$  such that

- (1) the book decomposition restricted to  $M \times \{0\}$  is the given one,
- (2) the page of the decomposition restricted to  $\partial Q$  is a n-1 skeleton of Q and
- (3) the sesquilinear form defined on Q is isomorphic to  $(A, \lambda)$ .

Recall that the condition (2) is the setting in which the form is defined in §6. Also note that an almost canonical decomposition satisfies the  $\pi_1$  condition if  $n \ge 3$ .

To begin we observe that a book decomposition with page P and binding  $\partial_0 P$  can be split into 2 parts: a book neighborhood  $\partial_0 P \times D^2$  of the binding with decomposition coming from the radial decomposition of  $D^2$ , and the rest of the manifold. This still has page P, but no binding.

Now suppose M is a manifold as in the proposition, and let  $\partial_0 P \times D^2 \subset M$  be the book neighborhood of the binding. The manifold Q will be  $\partial_0 P \times D^2 \times [1/3, 2/3] \subset M \times [0, 1]$ , when  $n \ge 4$ . This will be stabilized if 2n - 1 = 5. The book decomposition on  $M \times I - Q$  will be the given one except on  $\partial_0 P \times D^2 \times [2/3, 1]$ . The decomposition on this part will be changed by adding ribbon n - 1 handles, which we will study via immersed n handles in Q.



To ease notation a little, relabel the middle segment  $\partial_0 P \times D^2 \times I$ . Let  $J \subset S^1$  be an arc. Then  $\partial_0 P \times J \times I$  is a page for the part of the boundary we want to be standard, namely  $\partial_0 P \times (D^2 \times \{0\} \cup S^1 \times I)$ . After a new page is found for the remaining end  $\partial_0 P \times D^2 \times \{1\}$ , it is to be extended to the right segment by product with I.

Suppose  $(A, \lambda)$  is the form over  $\mathbb{Z}[\pi_1 M]$  to be represented. First lift it to a simply nondegenerate form over  $\mathbb{Z}[\pi_1 \partial_0 P]$ : the adjoint  $\lambda^a: A \to A^*$  is simple, so can be written as a product of elementary matrices. Each elementary matrix can be lifted to an elementary matrix over  $\mathbb{Z}[\pi_1 \partial_0 P]$  because  $\pi_1 \partial_0 P \to \pi_1 M$  is onto. Call the new form  $(A, \lambda)$  also.

Suppose A has rank k. Choose k small trivial ribbon n-1 handles in  $[\partial_0 P \times D^2 \times \{1\}, \partial_0 P \times J \times \{1\})$ . Choose also small disjoint ribbon n handles in  $(\partial_0 P \times D^2 \times I, \partial_0 P \times J \times I)$  which bound the n-1 handles.

The next step is to alter these *n*-handles. We can isotope one of the boundary n-1 handles along an arc to pass through one of the others. This "links" the two n-1 handles. If we extend this map to the *n* handles by the track of the isotopy in a collar it introduces an intersection point. (So the *n* handles are now only immersed). We can do this deformation along an arbitrary arc in  $\partial_0 P$ , and can arrange the new intersection point of the *n* handles to have arbitrary sign and direction of lean (see §5). Therefore by repeating the process we can realize the form  $(A, \lambda)$  as the intersection form of the *n* handles.

The ribbon n-1 handles in  $\partial_0 P \times D^2 \times \{1\}$  are still embedded, since we just passed them through each other. We claim that these ribbon handles now form a page for  $\partial_0 P \times D^2 \times \{1\}$ . More precisely let  $U \times I$  be the union of a collar on  $\partial_0 P \times J \times \{1\}$ , and these ribbon handles. Let N be the closure of  $\partial_0 P \times D^2 \times \{1\} - U \times I$ . Then we claim that  $U \times \{1\} \subset N$  is a simple equivalence.

Consider the exact sequence (with  $Z\pi_1\partial_0 P$  coefficients)

$$\rightarrow H_i(N, U \times \{0\}) \rightarrow H_i(\partial_0 P \times D^2 \times I, U \times \{0\}) \rightarrow H_i(\partial_0 P \times D^2 \times I, N) \rightarrow .$$

- (a)  $H_j(\partial_0 P \times D^2 \times I, U \times \{0\})$  is 0 if  $j \neq n$ , and is freely generated by the classes of the immersed n handles if j = n.
- (b)  $H_j(\partial_0 P \times D^2 \times I, M) \approx H^{2n-j}(\partial_0 P \times D^2 \times I, \partial^0 P \times (D^2 \times \{0\} \cup U \times I))$  $\cong H^{2n-j}(\partial_0 P \times D^2 \times I, U \times I)$ , so this group is also 0 if  $j \neq n$ , and if j = n is generated by classes dual to the immersed *n* handles.

The homomorphism between these two when j = n is adjoint to the intersection form on the *n* handles. Since we have arranged the form to be  $\lambda$ , the homomorphism is a simple isomorphism. Therefore  $H_*(M, U \times \{0\}) = 0$  for all \*, and with zero torsion.

This produces a homotopy page in  $\partial_0 P \times D^2 \times \{1\}$ , even if 2n - 1 = 5. 7.1 now follows by recalling from 4.6 that this gives a real page if 2n - 1 > 5, and stabilizes to a page if 2n - 1 = 5.

It is clear  $(A, \lambda)$  has been realized by this procedure, since the appropriate classes in Q are already represented by immersed ribbon n handles with the appropriate intersections.

# **§8. SECTIONS OF HOMOMORPHISMS**

Suppose  $B_1 \xleftarrow{f_1} A \xrightarrow{f_2} B_2$  is a diagram of homomorphisms (of finitely generated modules over a ring R). We are concerned with the question: when is there a submodule  $L \subseteq A$  such that  $f_1$  and  $f_2: L \rightarrow B_1$  are both isomorphisms? Equivalently we can ask for a right inverse r for  $f_1$  such that  $f_2r$  is an isomorphism. Such a thing will be referred to as a section of the diagram.

The result is that sections can be obtained if we can stabilize the diagram in the proper way. This is a slight sharpening of a lemma attributed to Neumann[21].

8.1 NEUMANN'S LEMMA. Suppose  $a_1, a_2: A \rightarrow B$  both have right inverses. Then the diagram

$$B \oplus B \xleftarrow{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} A \oplus B \oplus B \xrightarrow{\begin{pmatrix} a_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} B \oplus B$$

has a section. Further if B is free, based and even dimensional, then there is a simple section.

To see this, denote the right inverse by  $r_1$ ,  $r_2$  respectively. Then

$$G = \begin{pmatrix} r_1 & r_1 - r_2 a_1 r_1 \\ 1 & -a_2 r_1 \\ 0 & 1 \end{pmatrix} : B \oplus B \to A \oplus B \oplus B$$

is a section. This follows from

$$\begin{pmatrix} a_2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} G = \begin{pmatrix} a_1 r_2 & 1 - a_1 r_2 a_2 r_1 \\ 1 & -a_2 r_1 \end{pmatrix}.$$

This last is an isomorphism (simple if B is free, based and even dimensional) by the identity

$$\begin{pmatrix} x & 1-xy \\ 1 & -y \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -y \end{pmatrix}.$$

#### **§9. ODD DIMENSIONS**

9.1 PROPOSITION. Suppose  $M^{2n+1}$  has a book decomposition on  $\partial M$ , and  $2n + 1 \ge 7$ . Then the decomposition extends to all of M.

Let  $V \times I \subset \partial M$  be a page. Take a handlebody structure for M starting with  $V \times I$ , and let U be the union of a collar on  $V \times I$  and the handles of index  $\leq n$ .

Some notations for this situation are:  $\partial_1 U$  is the closure of  $\partial M - V \times I$ , and  $M^{\uparrow}$  is the closure of M - U.

9.2 The last step

Suppose that  $(K, V \times \{0\}) \subset (\partial_1 U, V \times \{0\})$  is a complex such that both inclusions

$$(M^{\wedge}, V \times \{0\}) \supset (K, V \times \{0\}) \subset (U, V \times \{0\})$$

are simple homotopy equivalences. This determines an extension of the book decomposition: let W be a thickening of K in  $\partial_1 U$  which intersects  $\partial \partial_1 U$  in  $V \times \{0\}$ . In U, extend the collar  $V \times I$  to a collar  $W \times I$ . The hypotheses of simple equivalence easily implies  $W \times I$  is a homotopy page for M. The s-cobordism argument of 4.6 then gives a genuine book decomposition.

# 9.3 The section problem

We proceed to the construction of such a K. Let L be a relative n-1 skeleton of  $(\partial_1 U, V \times \{0\})$ . More precisely take a handlebody decomposition of  $\partial_1 U$  starting with  $V \times \{0\}$ , and let L be the union of handles of index  $\leq n-1$ . Note that L is also an n-1 skeleton for U, and  $M^{\uparrow}$ .

The next observation is that both  $(U, V \times \{0\})$  and  $(M^{\land}, V \times \{0\})$  are equivalent to relative *n*-complexes. U is by definition, and  $(M^{\land}, \partial M - V \times I)$  is because it is the union of handles dual to handles of  $(M, V \times I)$  of index  $\geq n$ . Finally the inclusion  $V \times \{0\} \subset$  closure  $(\partial M - V \times I)$  is a simple equivalence because V is a page for  $\partial M$ .

Now recall that if A is an n complex, B an n-1 skeleton, then  $H_j(A, B) = 0$ ,  $j \neq n$ , and  $H_n(A, B)$  is free and based. Here we use homology of the universal cover, as a  $\mathbb{Z}[\pi_1 M]$  module. Therefore if  $L \subset \partial_1 U$  is an n-1 skeleton, the end terms in the diagram  $H_n(M^{\uparrow}, L) \leftarrow H_n(\partial_1 M, L) \rightarrow H_n(U, L)$  are free and based.

If we can find a simple section (in the sense of §8) of this diagram and  $2n + 1 \ge 7$ , we can construct the complex K. (2n + 1 = 5 will be indicated later.) This goes as follows:  $H_n(\partial_1 U, L) \simeq \pi_n(\partial_1 M, L)$  so we can represent a basis for a section by homotopy classes  $\alpha_1, \ldots, \alpha_k \in \pi_n(\partial_1 U, L)$ . L is the union of handles of index  $\le n - 1$  in  $\partial_1 M$ , so these classes can be represented by disjointly embedded *n*-handles in  $(\partial_1 U, L)$ . Add these *n*-handles to L to obtain K.

## 9.4 Construction of a section

A section of the diagram above will be obtained from Neumann's lemma, so we verify the hypotheses.  $H_n(M^{\uparrow}, \partial_1 U) = H_n(U_1 \partial_1 U) = 0$ , so both homomorphisms are onto. The end modules are free, so the homomorphisms have right inverses. To check that they are isomorphic we show that the ranks are the same, and in this case the rank is the Euler characteristic.

By duality  $2\chi(M^{\circ}) = \chi(\partial M^{\circ}) = \chi(\partial_1 U) + \chi((\partial M^{\circ}) - \chi(\partial (V \times I)))$ . Similarly  $2\chi(U) = \chi(\partial U) = \chi(\partial_1 U) + \chi(V \times I) - \chi(\partial (V \times I))$ . Since  $V \times I$  is a page for  $\partial M, V \times \{0\} \subset (\partial M)^{\circ}$  is an equivalence, and the two ranks are equal.

Now we stabilize. Add a small trivial *n* handle to U in M. M<sup>^</sup> is enlarged by the dual *n*-handle, and  $\partial_1 U$  is changed by connected sum with  $S^n \times S^n$ . If we leave L unchanged, the diagram is therefore enlarged by addition of

$$\mathbf{Z}\pi_1 M \xleftarrow{(1,0)} \mathbf{Z}\pi_1 M \oplus \mathbf{Z}\pi_1 M \xrightarrow{(0,1)} \mathbf{Z}\pi_1 M.$$

If this is done k times, where k is the rank of  $H_n(M^{\uparrow}, L)$ , Neumann's lemma 8.1 applies to give a section of the stabilized diagram. This completes the proof of 9.1 if  $2n + 1 \ge 7$ .

9.5 Dimension 5

The proof above applies except for the last paragraph of 9.3. We are given a section of

$$H_2(M^{\uparrow}, L) \leftarrow H_2(\partial_1 U, L) \rightarrow H_2(U, L)$$

with  $Z\pi_1 M$  coefficients.  $\pi_1 \partial_1 U \approx \pi_1 M$ , but L is a 1-skeleton. Therefore  $\pi_1 L$  must be improved to apply the Hurewicz theorem.

Take connected sum of (M, U) with the first example of 4.4. This enlarges  $\partial_1 U$  by  $S^2 \times S^2 \# S^2 \times S^2$ , which contains  $S^2 \times D^2 \#_{\partial} D^2 \times S^2$  as page in  $S^5$ . Choose loops  $\beta_1, \beta_2$  in L which are trivial in  $\partial_1 U$ . Then there is a map  $L \cup_{\beta_1,\beta_2} 2D^2 \rightarrow \partial_1 U \vee (2S^2)$  which has the same relative homology as  $(\partial_1 U, L)$ . Map into  $\partial_1 U \# 2(S^2 \times S^2)$  taking the  $S^2$  to the 1st and 2nd classes respectively.

The next step is to embed these 2-discs. They are distinguished by products in  $\partial_1 U^{\uparrow} = \partial_1 U$ -(open neighborhood of L), so they can be embedded in  $\partial_1 U^{\uparrow} \# S^2 \times S^2 \# S^2 \times S^2$ , with complement with the same  $\pi_1$  as  $\partial_1 U^{\uparrow}$ [13]. If we stabilize  $\partial_1 U$  this way by adding the book decomposition on  $S^5$  as above, the diagram is increased by adding a piece with a section.

Since the kernel of  $\pi_1 L \to \pi_1 \partial_1 N$  is normally generated by a finite set, we can continue to obtain  $L \subset \partial_1 U$ ,  $\pi_1 L = \pi_1 \partial_1 U = \pi_1 (\partial_1 U - L)$ , and still have a section of the problem above. Let N be a neighborhood of L,  $\partial N = \partial_0 N \cup \partial_1 N$  where  $\partial_o N = \partial N \cap \partial(\partial_1 U)$ . As above  $\partial_1 U^{\uparrow} = \partial_1 U - (N - \partial_1 N)$ . Then  $(\partial_1 U^{\uparrow}, \partial_1 N)$  is 1-connected. Stabilize as above by adding  $(S^2 \times D^2 \#_{\partial} D^2 \times S^2) \times I$  to U, and the pages  $(S^2 \times D^2 \#_{\partial} D^2 \times S^2) \times \{0\}$  to N.  $(\partial_1 U^{\uparrow}, \partial_1 N)$  is changed by  $\#_{\partial} (S^2 \times D^2 \#_{\partial} D^2 \times S^2) \times \{1\}$ . After some number of such operations,  $(\partial_1 U^{\uparrow}, \partial_1 N)$  can be given a handlebody structure with no 1-handles [13].

At this point we return to the proof of 9.3. The basis for the section can be represented by disjoint embedded 2-discs  $(D^2, S^1) \subset (\partial_1 U^2, \partial_1 N)$ . A homotopy page is obtained by adding these to N as in 9.3, and then stabilized to a real page by 4.6.

# **§10. INVARIANCE**

If the special situation used in §6 to define the sesquilinear form on a 2n manifold occurs on the boundary of 2n + 1 manifold, the form is equivalent to one with a lagrangian.

To state this precisely, we suppose  $\partial M$  is split into  $\partial_0 M$  and  $\partial_1 M$ ,  $V \times I \subset \partial_0 M$  is a page which gives a page  $\partial_0 V \times I \subset \partial(\partial_1 M)$ , and the inclusion  $\partial_0 V \times I \subset \partial_1 M$  is n-1 connected.

10.1 PROPOSITION. In the situation above if  $2n \ge 6$  the form  $(H_n(\partial_1 M, \partial_0 V \times \{0\}; \mathbb{Z}_{\pi_1}M), \cap)$  is equivalent to one with a (simple) lagrangian.

The idea of the proof is that the exact sequence

 $H_{n+1}(M, \partial_1 M \cup V \times \{0\}) \rightarrow H_n(\partial_1 M, \partial_1 V \times \{0\}) \rightarrow H_n(M, V \times \{0\})$ 

is "self dual". Therefore the image of the left module should be a lagrangian. In general to make sense of this the sequence must be short exact and free. We will go through a sequence of constructions to arrange this.

If a sufficiently nice coefficient ring is used, a lagrangian can be constructed directly without the geometric preparation. This is shown at the end of the section, and constitutes a proof of 1.3.

We begin the geometric improvement of M.

# 10.2 Eliminate $H_{j}(M; V \times \{0\}), j < n$

Take a handlebody structure for M beginning with  $V \times I$ . Since the page is almost canonical, the attaching maps of handles of index  $\leq n-1$  can be deformed into a book neighborhood of the binding  $\partial_0 V$  (see §6). Let U be a collar on this neighborhood, union with these handles. The handles are in the stable range so the factorization  $\partial_0 V \times D^2$  on the boundary can be extended to all of U.

Now delete the interior of U. This does not change  $\partial_1 M$ , but M,  $\partial_0 M$ ,  $V \times I$  are changed so that  $(M, V \times I)$  has a handlebody structure with no handles of index  $\leq n-1$ .

10.3 Eliminate  $H_n(M, \partial_1 M \cup V \times \{0\})$ 

First note that after 10.2 the only nonzero groups in the sequence of the triple  $(M, \partial_1 M \cup V \times \{0\}, V \times \{0\})$  are

$$0 \to H_{n+1}(M, V \times \{0\}) \to H_{n+1}(M, \partial_1 M \cup V \times \{0\}) \to H_n(\partial_1 M, \partial_1 V \times \{0\})$$
  
$$\to H_n(M, V \times \{0\}) \to H_n(M, \partial_1 M \cup V \times \{0\}) \to 0.$$

Since  $n \ge 3$ ,  $\pi_1 M \simeq \pi_1 V$ . Therefore each element in  $H_n(M, \partial_1 M \cup V \times \{0\})$  is the image of a homotopy class  $(D^n, S^{n-1}) \rightarrow (M, V \times \{0\})$ . The boundary of this map deforms into the binding because V is almost canonical. For dimension reasons it can be approximated by a framed embedding  $(D^n, S^{n-1}) \times D^2 \subset (M, \partial_0 V \times D^2)$  as above.

Now we modify this embedding. Isotope a small disc on the boundary  $S^{n-1}$  to the boundary of the binding, and push off a little into  $\partial_1 M$ . This gives a framed embedding  $(D^n; S_+^{n-1}, S_-^{n-1}) \times D^2 \subset (M; \partial_0 V \times D^2, \partial_1 M)$ , so that the embedding  $(S_-^{n-1}, S^{n-2}) \subset (\partial_1 M, \partial \partial_0 V)$  is a small trivial handle. Let U be a tubular neighborhood of this embedding, then the  $D^2$  factor extends the book decomposition to U.

The geometric operation is to delete the interior of U. Fairly clearly this does eliminate the element in  $H_n(M; \partial_1 M \cup V \times \{0\})$  we started with (by adding it to V). The other hypotheses are also preserved,  $H_*(M, U \times \{0\}) = 0$  for most \*, pages almost canonical, etc.

The effect on  $\partial_1 M$  is to add a small trivial n-1 handle to the binding  $\partial(\partial_1 V) \subset \partial(\partial_1 M)$ , extend the book decomposition to a neighborhood, and delete the interior. It

is easy to see that this "suspends" the form by adding  $\left(Z\pi + Z\pi, \begin{pmatrix} 0 & (-1)^{h} \\ 1 & 0 \end{pmatrix}\right)$ 

Repetition of this process produces  $(M, \partial_1 M, V \times I)$  with  $H_n(M; \partial_1 M \cup V \times \{0\}) = 0$  in addition to the previous improvements.

10.4 The lagrangian

In the situation just above, the modules

 $H_{n+1}(M, \partial_1 M \cup V \times \{0\}) \rightarrow H_n(\partial_1 M, \partial_1 V \times \{0\}) \rightarrow H_n(M, V \times \{0\})$ 

are the only nonzero ones for each respective pair. Therefore [20] they are all stably free and based, and the sequence is simple short exact.

The argument which gives the nonsingular form on the center module easily extends to give a self-duality of the whole sequence. It follows that the left module is a lagrangian for the form.

This completes the justification of 10.1.

10.6 Proof of 1.3. 1.3 follows, as did the invariance part of 1.1 from an odd dimensional vanishing theorem applied to a bordism to the special form of §6.

10.7 PROPOSITION. If R is as in 1.3,  $M^{2n+1}$  has boundary  $\partial_0 M \cup \partial_1 M$ , and  $V \times I \subset \partial_0 M$  is a page, then the form on  $H_n(\partial_1 M, \partial_1 V \times \{0\}; R)$  has a lagrangian.

Begin with the diagram (R coefficients)

The rows are fragments of long exact sequences, the upper isomorphisms are Poincare duality, and the lower ones result from the fact that  $V \times I \subset \partial M$  is a page.

Now divide by torsion. This yields a diagram of the form



The rows are now only exact mod torsion. The lagrangian we seek is ker  $\beta \subset B$ . The fact that  $A \rightarrow \ker \beta$  has torsion cokernel makes it simple to establish the required properties.

# **§11. ALMOST CANONICAL DECOMPOSITIONS**

Here we show that an arbitrary book decomposition is concordant to one which is almost canonical. The proof in some ways is a different approach to the whole theory (see 11.6, 11.11).

11.1 PROPOSITION. Suppose  $\partial M^k = \partial_1 M \cup \partial_0 M$ ,  $\partial_0 M$  has a book decomposition which is almost canonical on  $\partial \partial_0 M$ , and  $\pi_1 \partial_1 M \to \pi_1 M$  is an isomorphism. If  $k \ge 6$ , the book decomposition extends to one on M which is almost canonical on  $\partial_1 M$ .

In particular if  $M = N \times I$  and a decomposition is given on  $N \times \{0\}$ , we get an almost canonical one on  $M \times \{1\}$ .

The idea of the proof is to begin building up a page in M by adding ribbon handles below the middle dimension to the page in  $\partial_0 M$ . Because  $\pi_1 \partial_1 M = \pi_1 M$  we can also represent middle dimensional homotopy classes by embedded ribbon handles. A section problem appears as the obstruction to finding appropriate classes to represent. Sections are found by stabilizing and applying Neumann's lemma.

Let  $P \times I \subset \partial_0 M$  be the given page, with binding  $\partial_0 P$  and induced page  $\partial_1 P \times I \subset \partial_1 M$ .

Take a handlebody structure on  $\partial_1 M$  beginning with  $\partial_1 P \times I \subset \partial_1 M$  and let  $\partial_1 V \times I \subset \partial_1 M$  be the union of the [(k-1)/2] - 1 handles (with the factorization  $\partial_1 P \times I$  extended). Take a handlebody decomposition for  $(M, (\partial_i V \cup P) \times I)$ , and let  $V \times I$  be the union of the [k/2] - 1 handles. Then V is a k - 1 manifold with boundary  $\partial_0 V = P, \partial_1 V$ , and  $\partial_2 V$  which lies inside M.

Define  $M^{\uparrow}$  to be the closure of  $M - V \times I$ ,  $\partial_0 M^{\uparrow} = cl (\partial_0 M - \partial_0 V \times I)$ ,  $\partial_1 M^{\uparrow} = cl (\partial_1 M - \partial_1 V \times I)$ . Then  $\partial M^{\uparrow} = \partial_0 M^{\uparrow} \cup \partial_1 M^{\uparrow} \cup V \times \{0, 1\} \cup \partial_2 V \times I$ .

The proceedings now separate into the even and odd dimensional cases.

# 11.2 Even dimensions

Let k = 2n. Exact sequences and duality reveal that  $H_n(M^{\uparrow}, \partial_1 M \cup V \times \{i\})(i = 0 \text{ or } 1)$  is the only nonzero homology group of the pair. Therefore it is stability free and based. These are the groups we have to eliminate to obtain a page. Ribbon handles, which we want to use for this, represent elements of  $H_n(M^{\uparrow}, \partial_1 M \cup \partial_2 V \times I)$ . This leads to the diagram

$$H_n(M^{\uparrow}, \partial_1 M \cup V \times \{0\}) \leftarrow H_n(M^{\uparrow}, \partial_1 M \cup \partial_2 V \times I) \rightarrow H_n(M^{\uparrow}, \partial_1 M \cup V \times \{1\}).$$

This is a section problem (§8). We see next that a section yields a book decomposition.

11.3 Ribbon handles

Note that  $(M^{\wedge}, \partial_1 M \cup \partial_2 V \times I)$  is n-1 connected, so elements in  $H_n$  can be represented by homotopy classes  $(D^n, S^{n-1}) \rightarrow (M^{\wedge}, \partial_1 M^{\wedge} \cup \partial_2 V \times I)$ . Further  $(\partial_1 M^{\wedge}, \partial_1 M^{\wedge} \cap \partial_2 V \times I)$  is n-2 connected and  $(\partial_2 V \times I, \partial_2 V \times I \cap \partial_1 M)$  is 1-connected (this last from  $\pi_1 \partial_1 M = \pi_1 M$ ). Therefore  $S^{n-1} \rightarrow (\partial_1 M^{\wedge} \cup \partial_2 V \times I)$  can be represented by a map with  $S_+^{n-1} \rightarrow \partial_1 M^{\wedge}$ , and  $S_-^{n-1} \rightarrow \partial_2 V \times I$ .

This map can be represented by a framed immersion  $(D^n; S_+^{n-1}, S_-^{n-1}) \times I \rightarrow$ 

 $(M^{\circ}; \partial_1 M^{\circ}, \partial_2 V \times I)$  which is a product on  $S_{-}^{n-1} + I \rightarrow \partial_2 V \times I$ . As in §5 this immersion can be assumed to have self-intersections triangles on  $\partial_2 V \times I$  and squares in the interior. No intersections appear on  $\partial_1 M^{\circ}$ , since there the immersion is not constrained by a product structure.

The immersed ribbon handle can be regularly homotoped to a framed embedding, by pushing the intersections off  $\partial_1 M$ . In detail, choose arcs  $\alpha_1, \alpha_2$  from an intersection point to  $\partial_1 M^{\uparrow}$ , one along each sheet of the intersection. Since  $\pi_1(M^{\uparrow}, \partial_1 M^{\uparrow}) = 1$  (or  $\pi_1(\partial_2 V \times I, \partial_2 V \times I, \cap \partial_1 M^{\uparrow}) = 1$  for the boundary intersection) the relative homotopy element  $\alpha_1 \alpha_2^{-1}$  can be deformed into  $\partial_1 M^{\uparrow}$ . Approximate the deformation by an embedded 2-disc, and use the disc to construct a Whitney type isotopy which removes the intersection (see 5.8).

This argument requires  $2n \ge 8$  in order to embed the 2-discs. In case 2n = 6, we stabilize and argue as in 6.3.

#### 11.4 Section implies book

Suppose  $\alpha_1, \ldots, \alpha_i \in H_n(M^{\wedge}, \partial_1 M^{\wedge} \cup \partial_2 V \times I)$  form a basis for a simple section of the diagram in 11.2. By 11.3 we can represent the  $\alpha_i$  by disjointly embedded ribbon handles  $(D^n; S_+^{n-1}, S_-^{n-1}) \times I \subset (M^{\wedge}; \partial_1 M^{\wedge}, \partial_2 V \times I)$ . Let  $U \times I$  be  $V \times I$  with these ribbon handles added. We will see that  $U \times I$  is a page for M.

Define  $M^-$ ,  $\partial_0 M^-$ ,  $\partial_1 M^-$ , etc as  $M^-$  was defined, but using  $U \times I$ . There are inclusions  $M^- \supset M^- \supset U \times \{0\} \supset V \times \{0\}$ . Note the inclusion  $U \supset V$  is an equivalence since we have just attached discs on discs in the boundary. This gives an exact sequences

$$\rightarrow H_i(M^-, U \times \{0\}) \rightarrow H_i(M^{\uparrow}, U \times \{0\}) \rightarrow H_i(M^{\uparrow}, M^-) \rightarrow \dots$$

The only nonzero group of  $(M^{\hat{}}, M^{\hat{}})$  is  $H_n(M^{\hat{}}, M^{\hat{}})$ , represented by the duals of the handles  $\alpha_i$ . Similarly  $H_n(M^{\hat{}}, V \times \{0\})$  is dual to  $H_n(M^{\hat{}}, V \times \{1\} \cup \partial_1 M^{\hat{}})$ , and the homomorphism between them is dual to  $\mathbb{Z}[\pi](\alpha_1, \ldots, \alpha_j) \to H_n(M^{\hat{}}, V \times \{1\} \cup \partial_1 M^{\hat{}})$ . This was arranged to be a simple isomorphism, since the  $\alpha_i$  generate a section. Therefore  $H_*(M^{\hat{}}; U \times \{0\}) = 0$  (with zero torsion).

Similarly  $U \times \{1\} \subset M^-$  is a simple equivalence, so  $U \times I$  is a page for M by §4. The page on  $\partial_1 M$  is automatically almost canonical since only handles of index  $\leq n-1$  have been used in the construction.

11.5 Finding a section

The section problem is

$$H_n(M^{\wedge}, \partial_1 M^{\wedge} \cup V \times \{0\}) \leftarrow H_n(M^{\wedge}, \partial_1 M^{\wedge} \cup \partial_2 V \times I) \rightarrow H_n(M^{\wedge}, \partial_1 M^{\wedge} \cup V \times \{1\}).$$

The end modules are stabily free and based. The homomorphisms are onto. Neumann's lemma will therefore apply if we can stabilize the problem properly.

Add a small trivial n-1 handle to  $V \times I \subset M$ . This is the same as connected sum of  $V \times I$  with  $(S^{n-1} \times D^n) \times I$ . Three new classes are added to  $H_n(M^{\uparrow}, \partial_1 M \cup \partial_2 V \times I)$ by this procedure:  $\{*\} \times D^n \times \{0\}, \{*\} \times D^n \times \{1\}$ , and the *n*-disc spanning the  $S^{n-1}$ factor. The first dies in  $H_n(M^{\uparrow}, \partial_1 M \cup V \times \{0\})$ , the second dies on the other side. Therefore the section problem is enlarged by

$$\mathbf{Z}\boldsymbol{\pi} + \mathbf{Z}\boldsymbol{\pi} \xleftarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \mathbf{Z}\boldsymbol{\pi} + \mathbf{Z}\boldsymbol{\pi} + \mathbf{Z}\boldsymbol{\pi} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathbf{Z}\boldsymbol{\pi} + \mathbf{Z}\boldsymbol{\pi}.$$

This is the sum of a trivial suspension and the one we need for Neumann's lemma. Thus iteration of this yields a section.

# 11.6 Another approach to the main theorem

We observe that the same procedure can be followed with  $\partial_1 M = \emptyset$ . This gives a way to try to extend a decomposition on the boundary without assuming that it is almost canonical.

What comes out is a diagram of the form  $A \xleftarrow{\alpha} B \xrightarrow{\beta} A^*$ . The decomposition extends if there is a "lagrangian subsection":  $i: K \to B$  such that  $0 \to K \longrightarrow A \xrightarrow{(\beta_i)^*} K^* \to 0$  is exact. However it is unclear how to assemble these into an obstruction group. Probably there is more data in the problem which must be taken into account.

# 11.7 Odd dimensions

Now let k = 2n + 1, so  $(M^{\uparrow}, V \times \{i\})$  and  $(\partial_1 M^{\uparrow}, \partial_1 V \times \{i\})$  are n - 1 connected. The construction proceeds quite a bit like the proof of invariance: we end up constructing a "lagrangian subsection" in the sense of 11.6.

The first step is to kill off  $H_n(M^{\uparrow}, V \times \{i\} \cup \partial_1 M^{\uparrow})$ . Note that  $H_n(M^{\uparrow}, \partial_2 V \times \{i\}) \rightarrow H_n(M^{\uparrow}, V \times \{i\} \cup \partial_1 M^{\uparrow})$  is onto, and if  $n \ge 3$   $\pi_1 \partial_2 V \simeq \pi_1 M^{\uparrow}$ . Therefore we can represent elements by maps  $(D^n, S^{n-1}) \rightarrow (M^{\uparrow}, \partial_2 V \times \{i\})$ . Represent these by embedded ribbon handles, and isotope a small disc in  $S^{n-1}$  out to  $\partial(\partial_2 V)$ . Push out slightly into  $\partial_1 M^{\uparrow}$ . This gives ribbon handles  $(D^n; S_+^{n-1}, S_-^{n-1}) \times I \subset (M^{\uparrow}; \partial_2 V \times I, \partial_1 M^{\uparrow})$ .

If we add such a ribbon handle to  $V \times I$  we get a new  $V, M^{\uparrow}$ , etc. in which the class in  $H_n(M^{\uparrow}, V \times \{i\} \cup \partial_1 M^{\uparrow})$  has been eliminated. Continue the process until the groups  $H_n(M^{\uparrow}, V \times \{i\} \cup \partial_1 M^{\uparrow})$  are zero.

11.8 The section problem Consider the diagram

The top and bottom rows are stabily free and based. To get a book decomposition on  $\partial_1 M$  we want to kill off the top and bottom center groups. The top and bottom rows provide "lagrangians", but ribbon handles have to be added in the center sequence. We are therefore concerned with the section problem in the left column.

#### 11.9 Section implies book

Suppose  $\alpha_i, \ldots, \alpha_j$  are elements in  $H_{n+1}(M^{\wedge}, \partial_2 V \times I \cup \partial_1 M^{\wedge})$  which go to simple bases in  $H_{n+1}(M^{\wedge}, V \times \{i\} \cup \partial_1 M^{\wedge})$  for i = 0, 1. The boundaries give elements in  $H_n(\partial_1 M^{\wedge}, \partial_{12} V \times I) \simeq \pi_n(\partial_1 M^{\wedge}, \partial_{12} V \times I)$  and so can be represented by immersed ribbon handles (§5).

In order to embed these handles we consider the sesquilinear form of 5.1. The prescription for finding  $\lambda(a, b)$  is to push a up (in the diagram above in 11.8) into  $(\partial_1 M^{\uparrow}, \partial_1 V \times \{0\})$ , push b down into  $(\partial_1 M^{\uparrow}, \partial_1 V \times \{1\})$ , and use the Poincaré duality pairing between these two. However the top and bottom sequences of the diagram are dual. The fact that both elements come from the left side implies that  $\lambda(a, b) = 0$ . From 5.2 we conclude that they can be represented by disjointly embedded ribbon handles.

Add these handles to  $V \times I$  in  $\partial_1 M$ . The exactness properties we have arranged imply that the result is a page for  $\partial_1 M$ . (See the calculations of §6.2). It is almost canonical because only handles  $\leq n$  have been used in the construction.

We now have a book decomposition on all of  $\partial M$ . By 9.1 this extends to a decomposition of M, as required for 11.1.

## 11.10 Finding a section

Add a trivial ribbon n-1 handle to  $\partial_1 V \times I$  in  $\partial_1 M$ , and extend it into M by a

collar. This operation changes the diagram of 11.8 by adding



We have already seen the center column in 11.5. The rest of the diagram is easily determined by seeing what happens to generators in the center column.

This operation suspends the section problem of 11.8 so that we can apply Neumann's lemma. The resulting section gives a solution to the problem by 11.9.

11.11 Remark. The process of 11.7-11.10 is essentially the invariance theorem for the approach of 11.6 to the problem.

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