Assembly maps in bordism-type theories

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Preface

This paper is designed to give a careful treatment of some ideas which have been in use in casual and imprecise ways for quite some time, particularly some introduced in my thesis. The paper was written in the period 1984–1990, so does not refer to recent applications of these ideas.

The basic point is that a simple property of manifolds gives rise to an elaborate and rich structure including bordism, homology, and "assembly maps." The essential property holds in many constructs with a bordism flavor, so these all immediately receive versions of this rich structure. Not everything works this way. In particular, while bundle-type theories (including algebraic K-theory) also have assembly maps and similar structures, they have them for somewhat different reasons.

One key idea is the use of spaces instead of sequences of groups to organize invariants and obstructions. I first saw this idea in 1968 lecture notes by Colin Rourke on Dennis Sullivan's work on the Hauptvermutung ([21]). The idea was expanded in my thesis [14] and article [15], where "assembly maps" were introduced to study the question of when PL maps are homotopic to block bundle projections. This question was first considered by Andrew Casson, in the special case of bundles over a sphere. The use of obstruction spaces instead of groups was the major ingredient of the extension to more general base spaces. The space ideas were expanded in a different direction by Buoncristiano, Rourke, and Sanderson [4], to provide a setting for generalized cohomology theories.

Another application of these ideas was a "homological" description of the surgery sequence. The classical formulation of this sequence describes "normal maps" as a cohomology group—in particular contravariant—while the surgery obstruction is covariant. Applying duality in generalized homology describes the normal map set as a homology group and relates the classical surgery obstruction to an assembly map. This idea was made precise and useful by Andrew Ranicki [19].

The careful development of the material in the generality given here was largely motivated by the work of Lowell Jones [10], [11]. He developed an approach to the classification of piecewise linear actions of cyclic groups as a profound application of surgery theory. This material allows direct recognition of one of Jones' obstructions as a generalized homology class with coefficients in the "fiber of the transfer." The relation to Jones' work is sketched in section 6.4.

I would like to thank Andrew Ranicki for his encouragement over the years to bring this work into the light.

0: Introduction

Let J_n represent a group-valued functor of spaces, for example bordism groups, or Wall surgery groups, or algebraic K-groups (of the fundamental group). In these examples there is an associated generalized homology theory $H_*(X; \mathbf{J})$ and a natural homomorphism

$$H_*(X; \mathbf{J}) \to J_*(X)$$

called the "assembly." These homomorphisms are important for two reasons; they offer a first step in the computation of the functors $J_*(X)$, and some of them arise in geometric situations. For example the assembly map for surgery groups is closely related to surgery obstructions, and the "Novikov conjecture" is equivalent to rational injectivity of the assembly when X is a $K(\pi, 1)$.

The objective is to give two descriptions of these homomorphisms. The first description is very general, in the context of homology with coefficients in a spectrum-valued functor. This yields a wealth of naturality properties and useful elaborations. However it is difficult to see specific elements, particularly homology classes, from this point of view.

The second description is complementary to this. For certain types of theories homology classes can be described explicitly in terms of "cycles." Assembly maps are directly defined by "glueing" (assembling) the pieces in a cycle. This gives an explicit element-by-element view which is good for specific calculations and recognizing homology classes when they occur as obstructions. But the naturality properties become obscure.

The main result is that these two constructions do in fact describe the same groups, spaces, maps, etc. Special cases been used in calculations of surgery groups and obstructions [14], [15], [5], [28], [8]. With this description the assembly map is seen to describe obstructions for certain block bundle problems [15], and constructions of PL regular neighborhoods [10], [11]. The cycle description is well adapted to constructions divided into blocks, like PL regular neighborhoods of polyhedra.

This paper begins with definitions of various types of homology; generalized, twisted, Čech, and spectral sheaf. This logically comes first, but the reader may find it more interesting to begin with the bordism material of section 3.

The second section defines homology with coefficients in a spectrum-valued functor. Assembly maps are part of the functorial structure of these homology theories. The idea is to begin with a map $p\colon E \to X$, apply the functor fiberwise to point inverses of p to get a "spectral sheaf" over X, and take the homology of this. In this setting the usual assembly appears as a morphism induced by a map of data: the constant coefficient homology $H_*(X;\mathbf{J})$ is the $\mathbf{J}$-coefficient homology of the identity map $X \to X$, the groups $J_*(X)$ are the homology of the point map $X \to pt$, and the assembly is induced by the diagram

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \downarrow{=} & & \downarrow \\ X & \longrightarrow & pt \end{array}$$

regarded as a morphism from the identity to the point map.

From this point of view the usual assembly is a small part of a rich structure: there are lots of maps more interesting than the identity and the point map. Special cases were defined by D. W. Anderson [2], and in algebraic K-theory by Loday [12] and Waldhausen [24].

Bordism-type theories are described in section 3. This is a fairly primitive notion, designed so the conditions can be easily verified in examples. These have associated bordism groups, and bordism spectra. The spectrum construction is used to define functors which satisfy the conditions of the second section, so homology with coefficients in these spectra are defined.

This description applies naturally to surgery groups, and bordism groups defined using manifolds, Poincaré spaces, normal spaces, or chain complexes. There is an existence theorem in 3.7 which asserts that one can contrive to obtain any homology theory from a bordism-type theory. However we regard it as a conceptual error to use this result: the theory is designed to take advantage of special structure in a class of examples, and has no special benefits as an approach to general theory.

Roughly speaking the approach applies to theories with classifying spaces which are simplicial complexes satisfying the Kan condition. There is a general-nonsense construction which replaces a space with a Kan complex of the same weak homotopy type, and this gives the existence theorem. However, for example, the natural classifying spaces for algebraic K-theory do not satisfy the Kan condition, so the approach does not naturally or usefully apply to K-theory.

In section 4 “cycles” are introduced as representatives for homology classes in bordism-type theories. These are defined on covers of the space, and associate to an element of the cover a “fragment” of an object, with various “faces.” The prototypical example of such a fragment is a manifold, with its boundary subdivided into submanifolds (this example leads to bordism, hence the title of the paper). The pieces of a cycle fit together; over an intersection of two elements of the cover corresponding faces of the fragments are equal. Over three-fold intersections certain “edges” agree, etc. The assembly map simply glues (assembles) the pieces together using these identifications to get a single object.

Another way to view cycles is in terms of transversality. Suppose X is a finite simplicial complex. The dual cone (or cell) decomposition of X describes X as being assembled from pieces, each a cone on a union of smaller pieces. The boundaries of largest cones are bicollared in X; boundaries of smaller cones are bicollared as subsets of the boundary of the next larger cones. A manifold could therefore be made transverse to all these cones. This breaks the manifold into pieces over each maximal cone, intersecting in faces over the next smaller cones, etc. A cycle is an abstraction of this pattern. Thus a $\mathbf{J}$-cycle may be thought of as a $\mathbf{J}$-object which is transverse to a dual cone decomposition.

There is an associated description of cocycles, representing cohomology classes, given in 4.7. These associate to each simplex of a complex an object with the same pattern of faces as the simplex. Since there is a correspondence between simplices and dual cones, a cycle also associates an object to each simplex. But the objects in a cycle have faces corresponding to faces of the cone dual to the simplex, rather than the simplex itself. So for example in a cocycle dimensions of associated objects increase with dimension of the simplex, while in a cycle the dimension decreases.

Section 5 contains the proof of the main theorem, that cycles represent homology classes.

The final sections presents examples of bordism-type theories, and applications of the representation theorem. In section 6.1 manifolds are shown to form a bordism-type theory. Details are included as a model for verifications in other contexts. In 6.2 this is extended to manifolds with a map to a space. This construction defines a manifold-type theory depending functorially on a space, so leads to a full array of functor-coefficient homology groups, assembly maps, etc.

Transversality is used to show that the assembly maps in the manifold theories are isomorphisms. This is an analog of the classical Pontrjagin-Thom theorem that the bordism groups form a homology theory represented by the Thom spectrum.

This suggests thinking of assemblies in general bordism-type theories in terms of transversality. The fiber of the assembly map, which measures the deviation from isomorphism, then classifies obstructions to transversality.

Poincaré chain complexes are considered in 6.3, and the relation of this development to the work of Ranicki [20] and Weiss [26] is briefly described.

Finally in section 6.4 we sketch a sophisticated application. This begins with the observation that in some circumstances PL regular neighborhoods are equivalent to PL manifold cycles. Then a formulation of surgery in these terms gives a classification of manifold structures on Poincaré cycles. Putting these observations together gives a way to construct PL regular neighborhoods. In particular an obstruction encountered by Jones [10], [11] in the construction of PL group actions is reformulated as a generalized homology class.

Important topics not covered here are applications to surgery classification problems, product structures and duality, and computational aids like spectral sequences.

We mention that there is another class of theories with a description of homology classes and the assembly. These are the controlled theories, which deal with objects with a naturally associated "size," over a metric space. The representation theorem asserts that objects with sufficiently small size represent homology classes. The assembly map simply forgets the size restriction. These theories are well adapted to problems where things cannot be broken into blocks, for example in the study of purely topological neighborhoods [18]. The methods are more those of sheaf theory with things given on overlapping open sets, rather than the articulated fragments of the bordism-type theories.

This paper can be considered a completed version of the author's thesis, where some limited assembly maps for surgery were described, and the term "assembly" was introduced.

1: Homology

Generalized homology spectra (with coefficients in a spectrum) are defined in 1.1, and extended in 1.2 to homology with spectral sheaf coefficients. Twisted homology is discussed in 1.3 as a special case. Finally in 1.4 there is a description of Čech (or "shape") homology, which will be the setting for the general theory.

1.1 Spectra

A "spectrum" is a sequence of based spaces J_n together with based maps $j_n\colon J_n \wedge S^1 \to J_{n+1}$; see Whitehead [27] The spaces in a spectrum will be understood to be compactly generated space with the homotopy type of a CW complex.

We usually require these to be Ω-spectra in the sense that the adjoint of the structure map $J_n \to \Omega J_{n+1}$ is a homotopy equivalence. An arbitrary spectrum has a canonically associated Ω-spectrum with n^{th} space $\operatorname{holim}_{i\to\infty} \Omega^i J_{n+i}$. Generally the homotopy limits used here will be those defined by Bousfield and Kan [3]. In this particular case (a countable ordered direct system) it is just the mapping telescope (union of the mapping cylinders). An Ω-spectrum will be denoted by a boldface character; $\mathbf{J}$.

Given a spectrum $\mathbf{J}$ and a pair (X,Y), the *homology spectrum* $\mathbf{H}_\bullet(X,Y;\mathbf{J})$ is defined to be the Ω-spectrum associated to the spectrum $(X/Y)\wedge J_*$.

Referring to the definition just above of "associated Ω-spectrum" we see that the n^{th} space in the homology spectrum is given by

$$\operatorname*{holim}_{i\to\infty} \Omega^{i-n}(X/Y \wedge J_i).$$

Homology *groups* are defined (by Whitehead [27]) to be the homotopy groups of the homology spectrum.

We can at this point describe the simplest example of an assembly (see also [2]). Suppose $\mathbf{J}$ is a functor from spaces to spectra. Then there is a natural transformation

$$X = \operatorname{maps}(\mathrm{pt}, X) \to \operatorname{maps}(\mathbf{J}(\mathrm{pt}), \mathbf{J}(X)).$$

The adjoint of this is a map $X_+ \wedge \mathbf{J}(\mathrm{pt}) \to \mathbf{J}(X)$. But the left side of this gives the homology spectrum, so this is a map from the homology of X to $\mathbf{J}(X)$.

1.2 Spectral sheaf homology

We think of the homology $\mathbf{H}_\bullet(X,Y;\mathbf{J})$ as homology with coefficients in the constant coefficient system given by $\mathbf{J}$ over each point in X. The twisted coefficient construction extends this to coefficient systems which are "locally constant"; fibered over X. The next step is to generalize to coefficient systems which vary almost arbitrarily. This construction is based on Quinn [16, §8].

The description takes place in the category of "spaces over X" described by James [9]. Fix a base space X, then a space over X is E together with maps $i\colon X \longrightarrow E$ and $p\colon E \longrightarrow X$ whose composition is the identity. Maps in the category are continuous maps $E \longrightarrow F$ which commute with the inclusion of, and projection to, X.

The "suspension" of a space over X is given by $S^k_X E = S^k \times E/\sim$, where the equivalence relation $\sim$ identifies each $S^k \times p^{-1}(x)$ to a point, and $i\colon X \longrightarrow S^k_X E$ takes x to this identification point. A spectrum in this category is therefore a sequence E_n of spaces over X together with maps $E_n \longrightarrow S^1_X E_{n+1}$. Note that over each point the sequence of spaces $p_n^{-1}(x)$ form an ordinary spectrum (except they might violate our convention about having the homotopy type of CW complexes).

We refer to spectra in the category of spaces over X as *spectral sheaves* over X. There are technical connection with ordinary sheaves, but at this point the name is primarily intended to be suggestive.

The simplest examples of these spectral sheaves are products $\mathbf{J} \times X$. Then come the twisted products $\mathbf{J} \times_G \hat{X}$ described in the next section. More elaborate examples will be constructed in section 2.3.

We now define homology with coefficients in a spectral sheaf. As motivation note that in the constant coefficient case we begin with the total space of the product sheaf $\mathbf{J} \times X$, divide out X to get an ordinary spectrum, and pass to the associated Ω-spectrum. More generally, note that identifying the image of X to a point in a suspension over X gives the ordinary suspension; $S^1_X E/i(X) = S^1 E$. Therefore if $\{E_n\}$ is a spectral sheaf over X then an ordinary spectrum is obtained by dividing out X. The homology is the Ω-spectrum associated to this ordinary spectrum:

$$\mathbf{H}_\bullet(X; E) = \operatorname*{holim}_{n \longrightarrow \infty} \Omega^n(E_n/i(X)).$$

Similarly if $Y \subset X$ then the relative homology is defined by dividing out both X and the inverse image of Y; $E_n \bigm/ (i(X) \cup p^{-1}(Y))$.

We caution that we have not included the hypothesis that these "spectra" should have the homotopy type of CW complexes. To ensure the smooth functioning of the machinery of homotopy theory it is important to restrict to cases where this can be verified.

1.3 Twisted homology

This is defined to give a class of examples of the general theory. It will not be used here, so can be skipped by the purposeful reader. This construction does occur in spectral sequences describing general spectral sheaf homology in terms of simpler objects.

Suppose that G is a discrete group (see below for a non-discrete version) which acts on the spectrum $\mathbf{J}$, and $\omega\colon \pi_1 X \to G$ is a homomorphism. We use this data to define a spectrum denoted $\mathbf{H}_\bullet(X;\mathbf{J},\omega)$. Let $\hat{X} \to X$ denote the covering space with G action associated to ω. Then define twisted homology to be the Ω-spectrum associated to the spectrum $(\hat{X}/\hat{Y}) \wedge_G \mathbf{J}$. More explicitly this means take $(\hat{X}/\hat{Y}) \times \mathbf{J}_n$, divide by the diagonal G action, and identify the invariant subset $(\hat{X}/\hat{Y}) \vee \mathbf{J}_n$ to a point.

In the terms of the previous section, $\hat{X} \times_G \mathbf{J}$ is a spectral sheaf over X, and the twisted homology is the homology with coefficients in this sheaf; $\mathbf{H}_\bullet(X, Y; \hat{X} \times_G \mathbf{J})$.

We give another description of this which has better space-level functoriality properties. First, the G action on $\mathbf{J}$ determines fibrations over the classifying space B_G by $\mathbf{J}_n \times_G E_G \to B_G$, where E_G denotes the universal cover of B_G. (This is a fibered spectral sheaf over B_G.) The homomorphism $\pi_1 X \to G$ determines, up to homotopy, a map $\nu\colon X \to B_G$. The G-product $X \times_G \mathbf{J}_n$ is then obtained from the pullback of these two maps to B_G. The G-smash $X \wedge_G \mathbf{J}_n$ is obtained from this by identifying to a point the 0-section and the inverse image of the basepoint in X. The twisted homology is therefore the Ω-spectrum associated to these quotiented pullbacks.

The difference between a map $X \to B_G$ and a homomorphism $\pi_1 X \to G$ is that the first specifies a particular covering space (by pulling back the universal cover of B_G) whereas the second only specifies a cover up to isomorphism. There are also problems with basepoints and disconnected spaces. These are not important for single spaces since changes in basepoints, covers, etc. only change the homology spectrum by homotopy equivalence. The differences become more significant when we consider families of spaces, in section 2.

This point of view is also more general, since B_G can be replaced by the classifying space of a topological monoid (or anything else). We describe an interesting example which can be expressed in these terms. Suppose ν is an oriented vector bundle over X, and let $\Omega_n(X,\nu)$ denote the bordism group of smooth n-manifolds together with a bundle map from the stable normal bundle to ν. These groups occur in the study of intersections and singularities. They have also been used to study surgery normal maps.

To describe this as a twisted theory, let Ω^{fr} be the spectrum classifying framed bordism. The infinite orthogonal group SO acts on this by changing the framing, so defines a bundle over B_{SO} with fiber Ω^{fr}. The oriented vector bundle determines a map $X \to B_{SO}$. The bordism groups defined above are then the SO-twisted homology groups defined by this data;

$$\Omega_n(X,\nu) \simeq H_n(X;\Omega^{\text{fr}},\nu).$$

1.4 Čech homology

Finally we define Čech, or "shape" homology spectra. This is usually thought of as a way to extend homology in a reasonable way to pathological spaces (eg. not locally connected). This is not the motivation here; Čech homology coincides with the usual notion for all the spaces we really care about. Instead, both the definition and the the transversality view of the assembly naturally take place in Čech homology. It can be avoided, but only at the cost of some technical awkwardness.

The discussion here is for constant coefficients. In section 2.3 we will use a Čech version of spectral sheaf homology, which is a straightforward mixture of this section and 1.2.

Suppose $\mathcal{U}$ is a collection of subsets of a space X. We usually think of $\mathcal{U}$ as an open covering, though it is technically convenient to work with more general collections. Also suppose $\mathcal{U}$ has a partial ordering such that any finite number of elements with nonempty common intersection is totally ordered. Then the nerve of the collection, denoted $\text{nerve}(\mathcal{U})$, is a simplicial complex with k-simplices the sets of $k+1$ elements from $\mathcal{U}$ with nonempty intersection. We do not (at this point) require these elements to be distinct, so these sets are partially ordered, and fail to be totally ordered only as a result of duplications. The face operator ∂_j is defined by omission of the j^{th} element, and the degeneracy s_j duplicates the j^{th} entry. (Note that the results are well defined even though the "j^{th} entry" may not be well defined because of the duplications.)

If $Y \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then $\mathcal{U}\cap Y$ is a collection of subsets of Y. The nerve of this is a subcomplex; $\text{nerve}(\mathcal{U}\cap Y) \subset \text{nerve}(\mathcal{U})$.

Given a spectrum $\mathbf{J}$ and a collection $\mathcal{U}$, we can form the homology of the nerve; $\mathbf{H}_\bullet(\,\text{nerve}(\mathcal{U});\mathbf{J})$. Strictly speaking this should be the homology of the realization, and denoted $\mathbf{H}_\bullet(\,\|\text{nerve}(\mathcal{U})\|;\mathbf{J})$, but the simpler notation seems to be clear. More generally if (X,Y) is a pair then we can form the relative homology $\mathbf{H}_\bullet(\,\text{nerve}(\mathcal{U}),\text{nerve}(\mathcal{U}\cap Y);\mathbf{J})$, as an approximation to the homology of (X,Y).

A *morphism* of collections of subsets $\theta\colon \mathcal{U} \to \mathcal{V}$ is a function compatible with the partial orders, and such that $U \subset \theta(U)$. (So $\mathcal{U}$ is a refinement of $\mathcal{V}$.) A morphism induces a simplicial map of nerves $\text{nerve}(\mathcal{U}) \longrightarrow \text{nerve}(V)$, which in turn induces a map of geometric realizations and a map of homology spectra. Note that if $\mathcal{U}$ and $\mathcal{V}$ are two partially ordered collections then the collection obtained from intersections $\mathcal{U} \cap \mathcal{V}$ has natural morphisms to both $\mathcal{U}$ and $\mathcal{V}$.

The partially ordered open covers of X form an inverse system. We define the Čech, or "shape" homology spectrum of (X, Y) to be the homotopy inverse limit of homologies of nerves of this inverse system:

$$\check{\mathbf{H}}_\bullet(X, Y; \mathbf{J}) = \underset{\longleftarrow \mathcal{U}}{\operatorname{holim}} \mathbf{H}_\bullet(\operatorname{nerve}(\mathcal{U}), \operatorname{nerve}(\mathcal{U} \cap Y); \mathbf{J}).$$

We are primarily interested in this as an alternative description of the definition of 1.1, so we show

1.5 Lemma. *If (X, Y) is a metric pair with the homotopy type of a CW pair (K, L), then there is a natural equivalence $\check{\mathbf{H}}_\bullet(X, Y; \mathbf{J}) \simeq \mathbf{H}_\bullet(K, L; \mathbf{J})$.*

Proof. The realization $\|K\|$ of a simplicial complex K has a canonical open collection of subsets consisting of stars of vertices. If v is a vertex the *star*, denoted $\operatorname{star}(v)$, is the union of all open simplices whose closures contain v. An ordered simplicial complex comes equipped with a partial ordering of its vertices so that the vertices of every simplex are totally ordered. This induces a partial ordering of the covering. Further, a finite collection of sets in the cover intersect if and only if the corresponding vertices span a simplex, so this ordering satisfies the hypotheses above. This partially ordered covering of $\|K\|$ is denoted $\operatorname{stars}(K)$.

Now suppose X is a metric space, and $\mathcal{U}$ is a partially ordered open cover. A partition of unity subordinate to $\mathcal{U}$ can be used to construct a map $f\colon X \longrightarrow \|\operatorname{nerve}(\mathcal{U})\|$. Specifically, suppose $h_U\colon X \longrightarrow [0,1]$ are functions with locally finite support, and that the support of h_U is contained in U. Let $x \in X$, and let h_{U_i} for $i = 0, \dots, n$ denote the functions which are nonzero on x. Then $x \in \bigcap_{i=0}^n U_i$, so $(U_0, \dots, U_n)$ defines an n-simplex in the nerve. Represent the simplex Δ^n as the points in real $(n+1)$-space with nonnegative entries with sum 1, then f takes the point x is to the point $(U_0, \dots, U_n) \times \{h_{U_0}(x), \dots, h_{U_n}(x))\} \in \operatorname{nerve}(\mathcal{U})^n \times \Delta^n \subset \|\operatorname{nerve}(\mathcal{U})\|$.

This map induces a morphism from the inverse image of the open star cover of the nerve to the original cover: $\theta_f\colon f^{-1}(\operatorname{stars}(\operatorname{nerve}(\mathcal{U}))) \to \mathcal{U}$. It follows that the inverse system of inverse images of star covers of complexes is cofinal in the system of all covers. Therefore the homotopy inverse limit over maps to complexes is homotopy equivalent to the limit over covers. Explicitly, if X is metric then

$$\check{\mathbf{H}}_\bullet(X, Y; \mathbf{J}) \simeq \underset{(K,L) \longrightarrow (X,Y)}{\operatorname{holim}} \mathbf{H}_\bullet(K, L; \mathbf{J})$$

where (K, L) is a complex pair.

Note there is an analogous definition of "singular" homology obtained by taking the homotopy *direct* limit of homology of complexes mapping *to* (X, Y).

If (X, Y) has the homotopy type of a CW pair then there is a homotopy equivalence to the realization of a pair of simplicial complexes, $(X, Y) \to (K, L)$. The simplicial approximation theorem implies that the subdivisions of (K, L) are cofinal up to homotopy in the inverse system of complexes to which (X, Y) map. Therefore the homotopy inverse limit over the subsystem is homotopy equivalent to the limit over the full system. But homology of the realization of a complex is independent of subdivisions, so the homology is constant on this subsystem. Therefore the inverse limit of the whole system (the Čech homology) is equivalent to $\mathbf{H}_\bullet(K, L; \mathbf{J})$ as required. $\square$

2: Functor coefficient homology

In this section we define the homology of a map with coefficients in a spectrum-valued functor. The functors are discussed in 2.1. A simple special case of the construction, which gives the constant coefficient assembly maps, is described in 2.2. The full construction is then given in detail in 2.3. This construction takes place in Čech homology, which involves a homotopy inverse limit. Proposition 2.4 shows that these limits are unnecessary in some cases.

2.1 Spectrum-valued functors

Suppose that $\mathbf{J}(X)$ is a covariant functor which assigns an Ω-spectrum to a space. In detail this means each space is functorially assigned a sequence of pointed spaces $\mathbf{J}_n(X)$, with natural maps $\mathbf{J}_n(X) \wedge S^1 \to \mathbf{J}_{n+1}(X)$.

A functor is *homotopy invariant* if a homotopy equivalence $X \to Y$ induces a homotopy equivalence $\mathbf{J}(X) \to \mathbf{J}(Y)$. Alternative descriptions of this property are that homotopic maps induce homotopic morphisms of spectra, or that the inclusion $X \times \{0\} \to X \times I$ induces a homotopy equivalence of $\mathbf{J}$ spectra.

A homotopy invariant functor induces a functor on the associated homotopy categories, but we will not use this. For our purposes it is quite important that $\mathbf{J}$ be a functor on maps, and take values in morphisms of spectra, not just homotopy classes.

A slight extension will be required in the applications. Consider the category of pairs (X, ω) where $\omega\colon X \to B$, for some fixed B ($= B_{Z/2}$ in the applications). Morphisms in this category are $X \to Y$ which commute with the maps of B. Then our functors will be defined on this category; $\mathbf{J}(X,\omega)$. In this context "homotopy invariant" means $\mathbf{J}(X,\omega) \to \mathbf{J}(Y,\nu)$ is a homotopy equivalence if $X \to Y$ is a homotopy equivalence. Note this requirement on $X \to Y$ is weaker than homotopy equivalence in the category of spaces over B, since the homotopies are not required to commute with maps to B.

The results of this chapter can be extended to this setting simply by including ω in the notation. Since it plays no essential role we have simplified the notation by omitting it.

2.2 Constant coefficient assembly

Now suppose that $\mathbf{J}$ is a homotopy invariant spectrum-valued functor of pairs. We will define a natural (up to homotopy) morphism of spectra $\mathbf{H}_\bullet(X,\mathbf{J}(pt)) \to \mathbf{J}(X)$.

Heuristically the construction is described as follows: Think of $X \times \mathbf{J}(pt)$ as obtained by applying $\mathbf{J}$ fiberwise to the identity map $X \to X$. Similarly we can obtain $\mathbf{J}(X)$ by applying $\mathbf{J}$ fiberwise to the projection $X \to pt$. Then the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \downarrow{=} & & \downarrow \\ X & \longrightarrow & pt \end{array}$$

maps the first construction into the second. Dividing out X in the first construction and passing to associated Ω-spectra gives $\mathbf{H}_\bullet(X;\mathbf{J}(pt)) \to \mathbf{J}(X)$, as desired.

Rather than literally applying $\mathbf{J}$ fiberwise we do an analogous simplicial construction.

Suppose $X = \|K\|$ is the realization of a simplicial complex K. Regard K as a category, with one objects for each simplex σ and morphisms generated by the face and degeneracy maps ∂_j, s_j. If F is a covariant functor from K to spaces, then $\coprod_{\sigma\in K} F(\sigma)$ is a simplicial space. In particular it has a "geometric realization," defined by:

$$\|F\| = \Big(\coprod_{k,\sigma\in K^k} F(\sigma)\times\Delta^k\Big)/\!\sim$$

where $\sim$ is the equivalence relation generated by: if $x \in F(\partial\sigma)$, $t \in \Delta^{k-1}$, and $u \in \Delta^{k+1}$ then $(x, \partial_j^* t) \in F(\sigma)\times\Delta^k$ is equivalent to $(\partial_j x, t) \in F(\partial_j\sigma)\times\Delta^{k-1}$, and $(x, s_j^* u)$ is equivalent to $(s_j x, u)$. The following properties of the realization are immediate:

2.2A Lemma.

(1) *Realization is natural in K and F,*
(2) *the constant functor $F(\sigma) = X$ has realization $\|K\| \times X$, and*
(3) *suppose $\|F\| \longrightarrow \|K\|$ is defined using the natural transformation $F \longrightarrow pt$ and statements* (1), (2). *If σ is a nondegenerate simplex of K then the inverse image of* int$\|\sigma\| \subset \|K\|$ *is $F(\sigma) \times$ int$\|\sigma\|$.* □

We use this to construct a simplicial version of the assembly.

Consider the covariant functor from K to spaces which takes a simplex to its open star; $\sigma^k \mapsto \mathrm{star}(\sigma)$. Compose this functor with the n^{th} space functor in $\mathbf{J}_n$, to get a functor $\sigma \mapsto \mathbf{J}_n(\mathrm{star}(\sigma))$. Denote the realization of this functor by $\mathbf{J}_n(I_{\|K\|})$. Substituting in the definition above this is

$$\mathbf{J}_n(I_{\|K\|}) = \Big(\coprod_{k, \sigma \in K^k} \mathbf{J}_n(\mathrm{star}(\sigma)) \times \Delta^k \Big) / \sim \ .$$

Next define maps by realizing natural transformations, as in the lemma. The natural transformation $\mathbf{J}_n(\mathrm{star}(\sigma)) \longrightarrow pt$ gives a projection $\mathbf{J}_n(I_{\|K\|}) \longrightarrow \|K\|$. Next, begin with the transformation from star to the point functor. Apply $\mathbf{J}_n$ to this and realize to get a map $\mathbf{J}_n(I_{\|K\|}) \longrightarrow \|K\| \times \mathbf{J}_n(pt)$. This fits with the previous construction to give a commutative diagram

$$\begin{array}{ccc} \mathbf{J}_n(I_{\|K\|}) & \longrightarrow & \|K\| \times \mathbf{J}_n(pt) \\ \downarrow & & \downarrow \\ \|K\| & \xrightarrow{=} & \|K\|. \end{array}$$

It follows from the homotopy invariance of $\mathbf{J}$ that this is a fiber homotopy equivalence over $\|K\|$; according to (3) of the lemma the inverse image of the interior of a nondegenerate simplex σ is $\mathbf{J}_n(\mathrm{star}(\sigma)) \times \mathrm{int}\|\sigma\|$ on the left, and $\mathbf{J}_n(pt) \times \mathrm{int}\|\sigma\|$ on the right. But star(σ) is contractible, so $\mathbf{J}_n(\mathrm{star}(\sigma)) \simeq \mathbf{J}_n(pt)$. Thus $\mathbf{J}_n(I_{\|K\|}) \longrightarrow \|K\| \times \mathbf{J}_n(pt)$ is a homotopy equivalence.

In the other direction, the inclusion star$(\sigma) \subset \|K\|$ gives a natural transformation from the star functor to the constant functor with value $\|K\|$. Applying $\mathbf{J}_n$ and realizing gives $\mathbf{J}_n(I_{\|K\|}) \longrightarrow \|K\| \times \mathbf{J}_n(\|K\|)$. Compose this with the projection to $\mathbf{J}_n(\|K\|)$.

Now consider the spectrum structure of $\mathbf{J}_n$. The spectrum maps give maps $\mathbf{J}_n(I_{\|K\|}) \times S^1 \longrightarrow \mathbf{J}_{n+1}(I_{\|K\|})$, which in fact give $\mathbf{J}(I_{\|K\|})$the structure of a spectral sheaf over $\|K\|$. Divide by the 0-section $i \colon \|K\| \longrightarrow \mathbf{J}_n(I_{\|K\|})$ to get a spectrum, and pass to the associated Ω-spectrum, to get the homology with coefficients in the spectral sheaf. The analogous construction on $\|K\| \times$

$\mathbf{J}_n(pt)$ gives the constant coefficient homology. $\mathbf{J}(\|K\|)$ is already an Ω-spectrum so this construction gives

$$\mathbf{H}_\bullet(\,\|K\|;\mathbf{J}(pt)) \longleftarrow \mathbf{H}_\bullet(\,\|K\|;\mathbf{J}(I_{\|K\|})) \longrightarrow \mathbf{J}(\|K\;|).$$

We have shown that the left of these maps comes from a sequence of homotopy equivalences, so is a homotopy equivalence. Composing with a homotopy inverse gives the desired map $\mathbf{H}_\bullet(\,\|K\|;\mathbf{J}(pt)) \longrightarrow \mathbf{J}(\|K\|)$.

2.3 Functor coefficient homology

In this section we begin with a spectrum-valued functor $\mathbf{J}$, and a map $p\colon E \longrightarrow X$. Roughly, a spectral sheaf $\mathbf{J}(p) \longrightarrow X$ is constructed by applying $\mathbf{J}$ fiberwise to p, generalizing the construction of the previous section. Homology with coefficients in this sheaf is then defined. As pointed out in the introduction, when the construction is done in this generality the assembly does not have to be treated separately; it is functorially induced by the morphism from p to the map which projects E to a point.

The definition takes place in Čech homology. This gives a definition for arbitrary maps, which is useful in naturality arguments. The maps encountered in applications are essentially simplicial, and the definition is shown (in 2.4) to simplify in this case.

Now suppose $p\colon E \longrightarrow X$ is given. If $\mathcal{U}$ is a partially ordered open cover we define a covariant functor from nerve($\mathcal{U}$) to spaces, by $\sigma \mapsto p^{-1}(\cap\sigma)$. Recall that a simplex of the nerve is given by a monotone sequence of elements of $\mathcal{U}$, $\sigma = (U_0, \dots, U_k)$, and $\cap\sigma$ denotes the intersection $\cap\sigma = \cap_i U_i$. Compose this functor with $\mathbf{J}$ to obtain a functor from nerve($\mathcal{U}$) to Ω-spectra. Geometric realization, as in the previous section, defines a spectral sheaf $\mathbf{J}(p,\mathcal{U}) \longrightarrow \|\text{nerve}(\mathcal{U})\|$. The spectra associated with this spectral sheaf have the homotopy type of CW complexes since the total space of the sheaf is defined by geometric realization. We therefore get a homology spectrum $\mathbf{H}_\bullet(\,\text{nerve}(\mathcal{U});\mathbf{J}(p,\mathcal{U}))$.

Next suppose $\theta\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a morphism of partially ordered covers, as considered in 1.4 ($\mathcal{U}$ “refines” $\mathcal{V}$). This induces a simplicial map nerve($\mathcal{U}$) $\longrightarrow$ nerve($\mathcal{V}$), and a natural transformation of inverse image functors $\sigma \mapsto p^{-1}(\cap\sigma)$. Composing with $\mathbf{J}$ gives a natural transformation of spectrum-valued functors. Realizing defines a morphism of spectral sheaves $\mathbf{J}(p,\mathcal{U}) \longrightarrow \mathbf{J}(p,\mathcal{V})$ covering the map $\|\text{nerve}(\mathcal{U})\| \longrightarrow \|\text{nerve}(\mathcal{V})\|$. This in turn induces a morphism of homology spectra;

$$\mathbf{H}_\bullet(\,\text{nerve}(\mathcal{U});\mathbf{J}(p,\mathcal{U})) \longrightarrow \mathbf{H}_\bullet(\,\text{nerve}(\mathcal{V});\mathbf{J}(p,\mathcal{V})).$$

These maps give the homology spectra the structure of an inverse system indexed by the partially ordered covers. We define the ($\mathbf{J}$ coefficient, Čech) homology to be the homotopy inverse limit:

$$\check{\mathbf{H}}_\bullet(X;\mathbf{J}(p)) = \underset{\longleftarrow\,\mathcal{U}}{\operatorname{holim}}\, \mathbf{H}_\bullet(\operatorname{nerve}(\mathcal{U});\mathbf{J}(p,\mathcal{U})).$$

We caution that this homology may not actually be obtained from some spectral sheaf $\mathbf{J}(p)$ on X itself, as the notation suggests. Proposition 2.4 below does imply this when the map p is simplicial.

As an aside we remark that the construction can be simplified to involve only sheaves over simplices, rather than over nerves. If the non-empty intersection requirement is dropped in the definition of the nerve we get a simplex with vertices $\mathcal{U}$. The spectral sheaf $\mathbf{J}(p,\mathcal{U})$ extends to a sheaf over this by the same formula: $\sigma \mapsto \mathbf{J}(p^{-1}(\cap\sigma)) = \mathbf{J}(\phi)$ if σ is not in the nerve. If $\mathbf{J}(\phi)$ is contractible—and $\mathbf{J}$ can always be redefined so this is the case—then the spectral sheaf over the simplex has the same homology as the sheaf over the nerve. This point will be developed further in section 5.3.

2.4 Naturality

Naturality for this definition follows from the naturality of all the ingredients. Specifically, suppose

$$\begin{array}{ccc} F & \xrightarrow{\hat{f}} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array}$$

commutes. If $\mathcal{U}$ is a partially ordered cover of X then $f^{-1}(\mathcal{U})$ is a cover of Y. There is an induced simplicial map (an inclusion in fact) $\operatorname{nerve}(f^{-1}\mathcal{U}) \to \operatorname{nerve}(\mathcal{U})$. Covering this is a natural transformation of inverse image functors; $\hat{f}\colon q^{-1}(\cap f^{-1}(\sigma)) \to p^{-1}(\cap\sigma)$. Compose with $\mathbf{J}$ to get a natural transformation of spectrum-valued functors, and realize to get a morphism of spectral sheaves $\mathbf{J}(q, f^{-1}\mathcal{U}) \to \mathbf{J}(p,\mathcal{U})$ covering the map of realizations of nerves. This induces a morphism of homology spectra,

$$\mathbf{H}_\bullet(\operatorname{nerve}(f^{-1}\mathcal{U});\mathbf{J}(q, f^{-1}\mathcal{U})) \to \mathbf{H}_\bullet(\operatorname{nerve}(\mathcal{U});\mathbf{J}(p,\mathcal{U})).$$

Now take homotopy inverse limits. These are both indexed by the inverse system of covers of X, so there is a natural induced map between the limits. On the right we get homology of X. The homology of Y is obtained by taking the limit of spectra on the left over the larger inverse system of all covers of Y. But there is a natural map from the inverse limit over the larger system to the inverse limit over the subsystem. Composition with the map above gives

$$\check{\mathbf{H}}_\bullet(Y;\mathbf{J}(q)) \to \check{\mathbf{H}}_\bullet(X;\mathbf{J}(p)).$$

We define this to be the morphism functorially associated to $(f,\hat{f})$. The Čech homology is thus a functor of X and p (and $\mathbf{J}$).

Definition

Suppose $\mathbf{J}$ and $p\colon E \longrightarrow X$ are as above. Then the *total assembly map* is defined to be the map $\check{\mathbf{H}}_\bullet(X;\mathbf{J}(p)) \longrightarrow \mathbf{J}(E)$ induced by the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{=} & E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & pt. \end{array}$$

2.5 The long exact sequence of a pair

We can define the relative homology spectrum $\check{\mathbf{H}}_\bullet(X,Y;\mathbf{J}(p))$ to be the cofiber (in the category of spectra) of the natural map $\check{\mathbf{H}}_\bullet(Y;\mathbf{J}(q)) \longrightarrow \check{\mathbf{H}}_\bullet(X;\mathbf{J}(p))$ induced by the inclusion $Y \subset X$. Applying π_* then gives the usual long exact sequence of homology groups.

The same spectrum can be obtained less trivially by taking the homotopy inverse limit of relative homology spectra of nerves:

$$\check{\mathbf{H}}_\bullet(X,Y;\mathbf{J}(p)) \simeq \underset{\longleftarrow\mathcal{U}}{\operatorname{holim}}\, \mathbf{H}_\bullet(\,\|\mathrm{nerve}(\mathcal{U})\|, \|\mathrm{nerve}(\mathcal{U}\cap Y)\|; \mathbf{J}(p,\mathcal{U})).$$

The reason these two constructions agree is that the relative homology spectra for nerves was also defined by taking the cofiber, and homotopy inverse limits preserve cofibers, up to homotopy.

To see this last point, note that cofibers in the category of spectra can also be described as deloopings of homotopy fibers of maps of spaces. But it follows from Bousfield and Kan [3, XI 5.5] that homotopy inverse limits preserve homotopy fibrations.

As an application of the long exact sequence we get a description of the cofiber of the total assembly map defined just above. Let $\hat{p}\colon E\times I \longrightarrow \mathrm{cone}X$ denote the map obtained from $p\times 1$ by dividing out $X\times\{0\}\subset X\times I$. The map $\hat{p}$ fiberwise deformation retracts to $E \longrightarrow pt$, so the homology of the cone is just $\mathbf{J}(E)$. The homotopy fibration for the pair $(\mathrm{cone}X, X)$ therefore gives a homotopy fibration

$$\check{\mathbf{H}}_\bullet(X,\mathbf{J}(p)) \to \mathbf{J}(E) \to \check{\mathbf{H}}_\bullet(\mathrm{cone}X, X;\mathbf{J}(\hat{p})).$$

The long exact sequence of homotopy groups of this homotopy fibration therefore give

$$\cdots \to \check{H}_n(X, \mathbf{J}(p)) \to J_n(E) \to \check{H}_n(\text{cone}X, X; \mathbf{J}(\hat{p})) \\ \to \check{H}_{n-1}(X, \mathbf{J}(p)) \to \cdots .$$

2.6 Simplicial maps

The point here is that when the coefficient map is simplicial, the functor coefficient Čech homology is equivalent to one of the terms in the inverse limit which defines it. This will be used to avoid homotopy inverse limits.

Proposition. *Suppose $p\colon E \to X$, Y is relatively fiber homotopy equivalent to the realization of a simplicial map $q\colon A \to K$, L, with $L \subset K$ a subcomplex. Then the Čech homology is equivalent to the homology of the open cover of $(\|K\|, \|L\|)$ by stars:*

$$\check{\mathbf{H}}_\bullet(X, Y; \mathbf{J}(p)) \xrightarrow{\simeq} \mathbf{H}_\bullet(K, L; \mathbf{J}(q, \mathit{stars}(K))).$$

A "fiber homotopy" of a map between maps p and $\|q\|$ is a commutative diagram

$$\begin{array}{ccc} E \times I & \longrightarrow & \|A\| \\ {\scriptstyle P\times 1}\downarrow & & \downarrow{\scriptstyle \|q\|} \\ X \times I & \longrightarrow & \|K\| \end{array}$$

such that P restricts to p on $E \times \{0\}$ and $\|q\|$ on $E \times \{1\}$. Such a homotopy is "relative" if the image of $Y \times I$ is contained in $\|L\|$. Accordingly two maps are fiber homotopy *equivalent* if there are maps both ways between them and fiber homotopies of the compositions to the identities. Note that this notion of homotopy equivalence preserves the homotopy type of point inverses.

We recall the definition of the star cover (see the proof of 1.5). The star of a vertex $v \in K$ is the union of all open simplices in the realization $\|K\|$ whose closures contain v. More generally, if σ is a simplex of K, then $\text{star}(\sigma)$ is the intersection of the stars of the vertices. Note such intersections also define simplices in the nerve, and in fact the function $\sigma \mapsto \text{star}(\sigma)$ defines an isomorphism of simplicial complexes $K \to \text{nerve}(\text{stars}(K))$.

Proof. First, the homology is homotopy invariant so

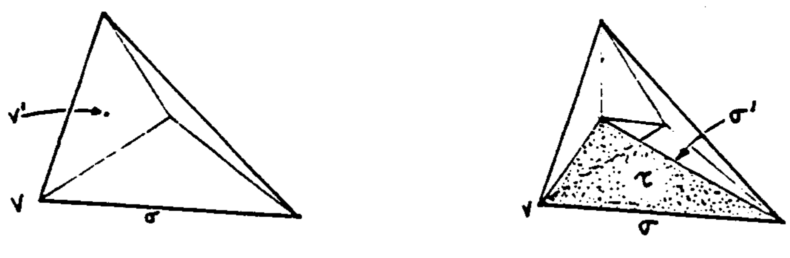
$$\check{\mathbf{H}}_\bullet(X; \mathbf{J}(p)) \to \check{\mathbf{H}}_\bullet(K; \mathbf{J}(q)))$$

is a homotopy equivalence. Next, the star covers of subdivisions of K are cofinal in the system of all covers of $\|K\|$ so it is sufficient to consider the inverse limit over this subsystem. We show that $\mathbf{H}_\bullet$ is constant (up to homotopy) on this subsystem, so they are all homotopy equivalent to the limit.

It is sufficient to consider a subdivision obtained by adding a single vertex; let K' be obtained by adding v'. The choice of ordering for the new vertex determines a simplicial map $K' \to K$. This is covered by a map of spectral sheaves $\mathbf{J}(p, \mathrm{stars}(K')) \to \mathbf{J}(p, \mathrm{stars}(K))$. We will show that this map is a homology equivalence in each degree. This implies that the associated map of homology spectra (obtained by dividing by the bases and passing to associated Ω-spectra) is a homotopy equivalence.

Next suppose $\sigma \in K$ is a nondegenerate simplex, and consider the interior of the realization $\mathrm{int}(\|\sigma\|) \subset \|K\|$. It is sufficient to show that the inverse image of this in $\mathbf{J}(p, \mathrm{stars}(K'))$ maps by homology equivalence to the inverse image in $\mathbf{J}(p, \mathrm{stars}(K))$. To see this is sufficient, consider the filtration of the realization of K by skeleta, and show by induction that the restriction of the spectral sheaves to the skeleta are homotopy equivalent. The induction step follows from the long exact sequence relating skeleta of adjacent dimensions, and the homology equivalence fact for individual simplices.

Now consider inverses of $\mathrm{int}(\|\sigma\|)$, first in the subdivision K'. If σ does not contain the new vertex v' then the inverse is again the simplex σ. If σ contains both v' and its image v under the simplicial map, then the inverse image of the interior is an open simplex in the interior of $\|\sigma\|$. Finally suppose σ contains v but not v'. Then the inverse image is a $(k+1)$-simplex γ of K' with σ as a face and v' as additional vertex. Let σ' denote the face of γ with the same vertices as σ except with v' substituted for v. The map $\|\gamma\| \to \|\sigma\|$ is the linear projection which is the identity on $\|\sigma\|$ and takes v' to v. The inverse image of the interior is therefore the interior of $\|\gamma\|$ union with the interiors of the two faces $\|\sigma'\|$ and $\|\sigma\|$.



To complete the proof we consider inverses in the spectral sheaves. Since

σ is nondegenerate the inverse image of its interior in $\mathbf{J}(p, \mathrm{stars}(K))$ is $\mathbf{J}(p^{-1}(\mathrm{star}(\sigma))) \times \mathrm{int}\|\sigma\|$. (This follows directly from the definition of the spectral sheaf as the realization of a functor; see the lemma in 2.2.) If σ does not contain the new vertex, then the inverse image in $\mathbf{J}(p, \mathrm{stars}(K'))$ is the same, so the condition is satisfied.

Suppose next that σ contains both v' and v. Then the inverse image in K' is a simplex τ in the subdivision of σ, whose realization is taken homeomorphically to the realization of σ. Further, the map $\mathrm{star}(\tau) \longrightarrow \mathrm{star}(\sigma)$ is a homeomorphism. In the spectral sheaves the inverses are given by this homeomorphism times the morphism of spectra induced by the inclusion $q^{-1}(\mathrm{star}(\tau)) \subset q^{-1}(\mathrm{star}(\sigma))$. For any simplex α the set $q^{-1}(\mathrm{star}(\alpha))$ deformation retracts to $q^{-1}(t)$ for any $t \in \mathrm{int}\|\alpha\|$. This implies that the inverses of the stars of both σ and τ deformation retract to the inverse of any point in $\mathrm{int}(\|\tau\|)$, so the inclusion is a homotopy equivalence. According to the homotopy invariance the morphism induced on $\mathbf{J}$ is also a homotopy equivalence, so inverses of σ satisfy the homology equivalence property.

Finally suppose σ contains v but not v'. According to the above, the inverse image of the interior in $\|K'\|$ is the union $\mathrm{int}\|\gamma\| \cup \mathrm{int}\|\sigma\| \cup \|\sigma'\|$. The inverse image in $\mathbf{J}(q, \mathrm{stars}(K'))$ thus has the homotopy type of the union of the mapping cylinders of the morphisms of $\mathbf{J}$ induced by the inclusions

$$q^{-1}(\mathrm{star}(\sigma)) \longleftarrow q^{-1}(\mathrm{star}(\gamma)) \longrightarrow q^{-1}(\mathrm{star}(\sigma')).$$

The rightmost map is a homotopy equivalence: both inverses deformation retract to point inverses in the interior of the respective simplices, but these point inverses are homotopy equivalent since $\mathrm{int}\|\sigma\| \cup \mathrm{int}\|\gamma\|$ lies in the interior of one of the simplices of K. From this we conclude that the union of mapping cylinders deformation retracts to the left end, $\mathbf{J}(q^{-1}(\mathrm{star}(\sigma)))$. Thus the preimage of $\mathrm{int}\|\sigma\|$ in $\mathbf{J}(q, \mathrm{stars}(K'))$ has the homotopy type of the preimage in $\mathbf{J}(q, \mathrm{stars}(K))$, as required. $\square$

3: Bordism-type theories

A "bordism-type theory" consists of a class of objects with faces, indexed by arbitrary sets. The prototype example of oriented manifolds, with faces codimension 0 submanifolds of the boundary, is presented in 3.1. The definition itself is given in 3.2. Bordism groups and spectra are defined in 3.3 and 3.4. Morphisms of these theories are defined in 3.5; these are necessary to define functors taking values in the category of bordism-type theories. The relative theory associated to a morphism is defined in 3.6. Finally in 3.7 the existence theorem is given, which asserts that up to weak homotopy any spectrum can be obtained as a bordism spectrum of some bordism-type theory.

3.1 An example

Before giving the abstraction we describe an example which displays the essential features. This example leads to the bordism theory of oriented manifolds; further examples are given in section 6.

Suppose A is a set. A *manifold A-ad* is a manifold with subsets $\partial_a M \subset \partial M$ for each $a \in A$. We require the union to be ∂M, and allow only finitely many of these to be nonempty. Finally we require each $\partial_a M$ together with the subsets $\partial_b M \cap \partial_a M$ to be a manifold $A - \{a\}$-ad.

Logically speaking this is an inductive definition: we should define A-ads with at most k nonempty faces inductively in k, so that the requirement that the faces be "-ads" is well defined. In any case the faces are codimension 0 submanifolds of ∂M, which intersect in codimension 0 submanifolds of their boundaries, etc.

For example the n-simplex Δ^n with its faces $\partial_i \Delta^n$ is a manifold $[n]$-ad of dimension n. Here we use the notation $[n]$ for the set $\{0, 1, \dots, n\}$.

The "bordism theory" consists of the collection of all manifold -ads, together with some operations on them. More specifically,

(1) for all sets A and integers n, the collections of compact oriented manifold A-ads of dimension n;
(2) face operations ∂_a which take n-dimensional A-ads to $(n-1)$-dimensional $(A - \{a\})$-ads;
(3) reindexing operations which change the labels on the faces and add empty faces;
(4) an involution obtained by reversing the orientations; and
(5) a "Kan" condition wherein a collection of -ads with appropriate incidence relations are assembled to form a single manifold.

The only odd thing which occurs is a sign change in iterated boundaries: when $a \neq b$ then $\partial_a \partial_b M = -\partial_b \partial_a M$, due to the way boundaries work in homology.

Note that the finiteness condition on faces and the reindexing of (3) imply that an arbitrary A-ad is obtained by reindexing a $[k]$-ad, for some $[k]$ and injection $[k] \to A$. It follows that it is logically sufficient to define $[k]$-ads, for each k. However direct definition of general A-ads is no more difficult, and saves a lot of trouble with reindexing.

The thing which gives these theories their characteristic flavor is the addition of empty boundaries in reindexing, i.e. reindexing using injections rather than only bijections.

We now abstract this. The symbol "$\mathcal{J}$" is a script "J", representing a generic theory just as **J** represented a generic functor in the previous section.

3.2 Definition

A *bordism-type theory*, $\mathcal{J}$, consists of :

(1) For every set A a collection $\mathcal{J}_A^n$ of "A-ads of dimension n," with a basepoint denoted $\phi \in \mathcal{J}_A^n$;
(2) for each $a \in A$ a function $\partial_a \colon \mathcal{J}_A^n \longrightarrow \mathcal{J}_{A-a}^{n-1}$ such that $\partial_a \phi = \phi$, and if $M \in \mathcal{J}_A^n$ then $\partial_a M = \phi$ for all but finitely many a;
(3) corresponding to each injection $\theta \colon A \longrightarrow B$ a basepoint-preserving function $\ell_\theta \colon \mathcal{J}_A^n \longrightarrow \mathcal{J}_B^n$ which is natural in θ. Further this satisfies $\partial_{\theta(a)}(\ell_\theta M) = \ell_\theta(\partial_a M)$ and ℓ_θ is a bijection onto $\{M \in \mathcal{J}_B^n \mid \partial_b M = \phi \text{ for all } b \in B - \theta(A)\}$;
(4) there is an involution (-1) on each $\mathcal{J}_A^n$ which commutes with ℓ_θ and ∂_a, and leaves ϕ fixed. Further, if $a \neq b$ in A then $\partial_a \partial_b M = -\partial_b \partial_a M$; and
(5) these satisfy the Kan condition described below.

The most restrictive aspects of this are the bijection hypothesis in (3), and the Kan condition. In manifolds the bijection hypothesis is obvious, and the Kan condition follows from glueing together manifolds along faces in their boundaries. To see these axioms verified in a non-geometric situation look at the proof in 3.7.

Suppose A is a set and $a \in A$ is a fixed element. Then define an n-dimensional Kan (A, a)-cycle in $\mathcal{J}$ to be a function $N \colon A - \{a\} \longrightarrow \mathcal{J}$ so that $N(b) \in \mathcal{J}_{A-\{b\}}^n$, and if $b \neq c$ are in $A - \{a\}$ then $\partial_b N(c) = -\partial_c N(b)$. Also assume only finitely many of the $N(b)$ are different from ϕ,=. Note no object is assigned to a. The principal example is: if M is an object in $\mathcal{J}_A^{n+1}$ then the function $b \mapsto \partial_b M$ for $b \neq a$ is an n-dimensional Kan (A, a)-cycle.

The Kan condition asserts that all Kan cycles arise in this way: if N is an n-dimensional (A, a)-cycle then there is an $(n+1)$-dimensional object M so that $N(b) = \partial_b M$, for all $b \neq a$.

The name is by analogy with the Kan condition for simplicial sets, which requires that a simplicial map defined on all but one of the faces of a simplex, extends to a simplicial map of the whole simplex.

3.3 Bordism groups

Suppose $\mathcal{J}$ is a bordism-type theory. Define $\Omega_n^{\mathcal{J}}$ to be the set of equivalence classes of n-dimensional ϕ-ads (no faces), where the equivalence is defined by $M \sim N$ if there is an $(n+1)$-dimensional $[1]$-ad W with $\partial_0 W = M$, and $\partial_1 W = -N$. (The symbol "$\mathit{\Omega}$" is a slanted version of Ω, used to try to distinguish between bordism groups and loop spaces.)

Proposition. *$\sim$ is an equivalence relation, and the set of equivalence classes $\mathit{\Omega}_n^{\mathcal{J}}$ has a natural abelian group structure.*

"Naturality" will not make sense until we have defined morphisms of bordism-type theories in 3.5.

Proof. The direct proof is elementary but long; we sketch a few pieces of it. The most efficient proof comes from the recognition as homotopy groups of a spectrum, in the next proposition.

We show that $\sim$ is transitive. Suppose W_0 expresses the equivalence $M \sim N$ and W_1 expresses the equivalence $N \sim P$. Then the function $i \mapsto W_i$ for $i = 0, 1$ defines a $([2], \{2\})$-cycle, in the sense defined in the Kan condition. The Kan condition therefore asserts there is an $(n+2)$-dimensional [2]-ad V such that $\partial_0 V = W_0$ and $\partial_1 V = W_1$. Then $\partial_2 V$ is an $(n+1)$-dimensional [1]-ad. Calculations using the axioms reveal that $\partial_0 \partial_2 V = -M$ and $\partial_1 \partial_2 V = P$. Thus $-\partial_2 V$ expresses an equivalence $M \sim P$, and $\sim$ is transitive.

The group structure is defined by: if W is a [2]-ad with $\partial_0 W = M$ and $\partial_1 W = N$, then $[M] + [N] = [-\partial_2 W]$. The Kan condition can be used to show that given M and N such a W exists, and that the equivalence class of $\partial_2 W$ is independent of the choice. This implies the operation is well defined.

The identity element is the equivalence class of the basepoint, $[\phi]$, and inverses come from the involution: $-[M] = [-M]$. To see the inverses, consider M as a $([1], \{1\})$-cycle and apply the Kan condition to get a [1]-ad V with $\partial_0 V = M$. Now define a $([2], \{0\})$-cycle by $2 \mapsto V$ and 1 goes to $-V$ reindexed so $\partial_1 = -\partial_1 V$ and $\partial_2 = -M$. Then apply the Kan condition to get a [2]-ad W with these as faces. The new face, $\partial_1 W$, has faces M and $-M$. Reindex to introduce ϕ as a third face, then it expresses the relation $[M] + [-M] = [\phi]$.

The fact that the group is abelian comes directly from reindexing: let W be as above, expressing $[M] + [N] = [\partial_2 W]$. Let θ be the bijection $[2] \longrightarrow [2]$ which interchanges 0 and 1. Then $\partial_0 \ell_\theta W = M$ and $\partial_1 \ell_\theta W = N$, showing $[N] + [M] = [\partial_2 W]$ and therefore $[M] + [N] = [N] + [M]$. $\square$

For example if SDiff is the theory of oriented smooth manifolds defined as in example 3.1, the bordism groups $\mathit{\Omega}_n^{\mathrm{SDiff}}$ are the classical smooth oriented bordism groups.

3.4 Bordism spectra

The next step is the construction of spectra which serve as classifying spaces for these theories.

We work with Δ-sets, in the sense of Rourke and Sanderson [22]. A Δ-set K is like a simplicial set in having sets $K^{(i)}$ of "i-simplices," the 0-simplices are appropriately partially ordered, and face operators $b_j\colon K^{(i)} \to K^{(i-1)}$ are given for $0 \leq j \leq i$. (We sometimes denote face operators in Δ-sets by ∂_i, but this conflicts somewhat with face operators in the bordism-type theory.) Δ-sets do not have the degeneracy operators of a simplicial set. Geometric realizations are defined for Δ-sets in essentially the same way as for simplicial sets.

Define the Δ-set $\mathit{\Omega}_n^{\mathcal{J}}$ to have k-simplices the $\mathcal{J}$-$[k]$-ads of dimension $k+n$. We also require that the "total intersection" of all faces $\partial_0\partial_1 \cdots \partial_n M$ is the basepoint ϕ. The face operator $b_i M$ is defined by reindexing the $(n-\{i\})$-ad $(-1)^i\partial_i M$ using the order-preserving bijection $[n-1] \to [n] - \{i\}$.

The notation for this Δ-set is the same as for the bordism group. This doubling up of notation seems to be relatively harmless since the group is π_0 of the Δ-set. At any rate it is less harmful than introducing yet another notation. The boldface analog $\mathbf{\Omega}^{\mathcal{J}}$ is reserved for the associated Ω-spectrum.

We define the Ω-spectrum $\mathbf{\Omega}^{\mathcal{J}}$ by geometrically realizing the Δ-set: $\mathbf{\Omega}_n^{\mathcal{J}} = \|\mathit{\Omega}_{-n}^{\mathcal{J}}\|$. Note the minus sign in the index on the Δ-set: this results from an incompatibility in the indexing conventions for bordism and spectra.

Proposition. *The spaces $\mathbf{\Omega}_*^{\mathcal{J}}$ have a natural Ω-spectrum structure with homotopy groups the bordism groups defined above; $\pi_n\mathbf{\Omega}^{\mathcal{J}} = \mathit{\Omega}_n^{\mathcal{J}}$.*

As with bordism groups, the naturality will be considered after morphisms of bordism-type theories are defined.

Proof. First we verify that the simplices described above do in fact give a Δ-set, namely that the face identities $b_j b_i = b_i b_{j+1}$ (if $j \geq i$) hold. The only interesting thing about this is is the role of the sign $(-1)^i$. The point is that when $\partial_i M$ is reindexed to define $b_i M$ the previous faces with index i or higher are all shifted down by one. This shift, with the $(-1)^*$, gives a net change of -1 on these faces. This cancels the -1 in the iterated boundary formula in definition 3.2(3).

Next we observe that the Δ-sets $\mathit{\Omega}_n^{\mathcal{J}}$ satisfy the Kan condition. Let Λ_j^k denote the subcomplex of the k-simplex which consists of all but the j^{th} face. The Kan condition asserts that any Δ-map $\Lambda_j^k \to \mathit{\Omega}_n^{\mathcal{J}}$ extends to a Δ-map $\Delta^k \to \mathit{\Omega}_n^{\mathcal{J}}$. A Δ-map $\Lambda_j^k \to \mathit{\Omega}_n^{\mathcal{J}}$ defines a $([k],\{j\})$-cycle in the sense of 3.2, so the Kan condition in 3.2(4) implies the Kan condition for the Δ-set.

The Kan condition implies a simplicial approximation theorem ([22, §5]) which in turn implies that the homotopy groups are the bordism groups: elements in the homotopy group π_k are represented by maps $\Delta^k \to \|\Omega_n^{\mathcal{J}}\|$ which take $\partial\Delta^k$ to the basepoint. Simplicial approximation asserts that this is homotopic to the realization of a Δ-map, which is exactly a k-simplex with all faces are ϕ. This in turn is obtained by reindexing a $k+n$ dimensional ϕ-ad to get a $[k]$-ad. This ϕ-ad defines an element in the bordism group $\Omega_{n+k}^{\mathcal{J}}$. Similarly homotopies can be interpreted as maps of Δ^{k+1} which take all but two faces to the basepoint. These can be approximated by Δ-maps which can be interpreted as bordisms.

Now we describe the spectrum structure. The cone on a Δ-set K can be described as a Δ-set with k-simplices $K^k \cup \{\mathrm{cone}\,\sigma \mid \sigma \in K^{k-1}\}$. The cone point is put last in the partial ordering, so if σ is a k-simplex $\partial_i\mathrm{cone}(\sigma) = \mathrm{cone}(\partial_i\sigma)$ if $i \leq k$, and $\partial_{k+1}\mathrm{cone}(\sigma) = \sigma$.

Now define a Δ-map $\mathrm{cone}(\Omega_n^{\mathcal{J}}) \to \Omega_{n-1}^{\mathcal{J}}$ by taking both $\Omega_n^{\mathcal{J}}$ and the cone point to the basepoints, and $\mathrm{cone}(M) \mapsto \ell_k(M)$. Here ℓ_θ reindexes the $[k]$-ad M to be a $[k+1]$-ad using the inclusion $[k] \subset [k+1]$. Taking geometric realizations, and dividing out the end $\|\Omega_n^{\mathcal{J}}\|$ of the cone gives a map $\|\Omega_n^{\mathcal{J}}\| \wedge S^1 \to \|\Omega_{n-1}^{\mathcal{J}}\|$. This defines a spectrum structure.

To see this is an Ω-spectrum we show that the adjoint $\|\Omega_n^{\mathcal{J}}\| \to \Omega(\|\Omega_{n-1}^{\mathcal{J}}\|)$ is a homotopy equivalence. For this we use the model of the loop space of a based Kan Δ-set given in [4, p. 36]; the k-simplices of ΩK are defined to be $(k+1)$-simplices $\sigma \in K$ with $b_{k+1}\sigma = \phi = v_{k+1}\sigma$. Here v_{k+1} denotes the $k+1$ vertex, and is obtained by applying all the face maps except b_{k+1}. It is shown in [4] that there is a natural homotopy equivalence $\|\Omega K\| \simeq \Omega\|K\|$. The adjoint of the spectrum structure maps defined above are homotopic to the realizations of $\Omega_n^{\mathcal{J}} \to \Omega(\Omega_{n-1}^{\mathcal{J}})$ defined by ℓ_k on k-simplices. But according to condition (3) of the definition, this is an isomorphism of Δ-sets. It therefore induces a homotopy equivalence on realizations. $\square$

3.5 Morphisms and naturality

Suppose $\mathcal{J}$ and $\mathcal{K}$ are bordism-type theories. A *morphism* $\mathcal{J} \to \mathcal{K}$ is a collection of basepoint-preserving functions $\mathcal{J}_A^n \to \mathcal{K}_A^n$ for all n and sets A, which commute with face functions, the involutions -1, and the reindexing functions. Clearly these can be composed, and form a category.

3.5A Example

If X is a space, define the oriented manifold bordism theory of X to have A-

ads (M, f), where M is an oriented manifold A-ad as in 3.1, and $f\colon M \to X$. The same operations as defined in 3.1 give this the structure of a bordism-type theory. A map of spaces $X \to Y$ induces, by composition, a morphism from the bordism theory of X to that of Y.

3.5B Lemma. *The bordism groups of 3.3 and the bordism spectrum of 3.4 are natural with respect to morphisms of bordism-type theories. Further, for a morphism $F\colon \mathcal{J} \to \mathcal{K}$ the following are equivalent:*

(1) *F induces isomorphisms of bordism groups,*
(2) *F induces homotopy equivalence of bordism spectra, or*
(3) *for every [0]-ad M in $\mathcal{K}$ such that $\partial_0 M = F(N_1)$ for $N_1 \in \mathcal{J}$, there is a [1]-ad W in $\mathcal{K}$ such that $\partial_1 W = M$ and a $\{1\}$-ad $N \in \mathcal{J}$ with $\partial_1 N = N_1$ and $\partial_0 W = F(N)$.*

We say that a morphism which satisfies the conditions of the lemma is a "homotopy equivalence" of theories. The last condition is the one which will be checked in practice; it can be paraphrased as saying a pair in $\mathcal{K}$ with boundary from $\mathcal{J}$ deforms rel boundary into $\mathcal{J}$. It is also equivalent to the vanishing of the relative bordism groups defined in the next section.

Define a functor from spaces to bordism-type theories to be "homotopy invariant" if homotopy equivalences of spaces induce homotopy equivalences of theories. Then the following is immediate from the lemma.

3.5C Corollary. *If $\mathcal{J}$ is a homotopy invariant functor from spaces to bordism-type theories then the associated bordism spectra $\Omega^{\mathcal{J}}$ define a homotopy invariant spectrum-valued functor in the sense of 2.1.*

Proof of the lemma. The naturality is evident from the definitions. The equivalence of (1) and (2) follows from the fact that the bordism groups are the homotopy groups of the spectrum. It remains to show that conditions (1) and (3) are equivalent. We give a direct proof here; a much slicker one comes from the relative bordism groups of 3.6.

Suppose (3) holds. Let $[M]$ represent an element in the group $\Omega_n^{\mathcal{K}}$, and reindex M as a [0]-ad with $\partial_0 M = \phi$. Since $\phi = F(\phi)$ we can apply (3) to find a [1]-ad W with $\partial_1 W = M$, $\partial_0 W = F(N)$, and $\partial_1 N = \phi$. This is a bordism showing $[M] = F_*(-[N])$, so $F_*\colon \Omega_n^{\mathcal{J}} \to \Omega_n^{\mathcal{K}}$ is onto.

Similarly we show F_* is injective by showing $F_*([N_1]) = 0$ implies $[N_1] = 0$, for $[N] \in \Omega_n^{\mathcal{J}}$. The hypothesis implies there is a [0]-ad in $\mathcal{K}$ with $\partial_0 M = F(N_1)$. Condition (3) implies there is a bordism from M to $F(N)$. But $\partial_1 N = N_1$, so $[N] = 0$. Thus (3) implies (1). The other direction is similar. $\square$

4: Cycles

In this section we describe "cycles" which represent functor-coefficient homology classes, when the coefficient functor is obtained from bordism-type theories. Fix a homotopy invariant functor (in the sense of 3.5) $\mathcal{J}$ from spaces to bordism-type theories. The associated spectrum-valued functor is denoted $\Omega^{\mathcal{J}}(X)$. The homology to be described is $H_n(\text{nerve}(\mathcal{U}); \Omega^{\mathcal{J}}(p, \mathcal{U}))$, where $p\colon E \to X$ is a map and $\mathcal{U}$ is a cover of X.

Cycles are described in 4.2, and induced morphisms in 4.3. Groundwork for the main theorem is laid in 4.4 with the development of a bordism-type theory whose objects are themselves cycles. The proof is given in section 5, where the bordism spectrum of this theory is shown to be equivalent to the homology spectrum.

4.1 Δ-nerves

Cycles will be defined using the nerve of a covering. For ease and efficiency we use a more compact model for the nerve than the simplicial complex described in 1.4.

Suppose $\mathcal{U}$ is a set of subsets of a space X, partially ordered as in 1.4. The Δ-nerve, denoted $\text{nerve}_\Delta(\mathcal{U})$, is defined to be the Δ-set with k-simplices the collections of $k+1$ *distinct* elements of $\mathcal{U}$ with nonempty intersection. (We caution that this distinctness is in $\mathcal{U}$, and does not imply that the corresponding subsets of X are distinct.) The face operator b_j is defined by omitting the j^{th} set; this is well defined since a collection with nonempty intersection is totally ordered.

The Δ-nerve is exactly the set of nondegenerate simplices in the simplicial nerve (the degenerate simplices are ones in which some set is repeated). It follows (see [22]) that the geometric realizations of the two nerves are equal.

Some notations are needed involving a k-simplex $\sigma = (U_0, \dots, U_k)$ of the nerve. $\cap\sigma$ denotes, as before, the intersection $\cap_{i=0}^k U_i$. The complement of σ in $\mathcal{U}$ is denoted $\mathcal{U} - \sigma$. And as indicated above $\partial_j\sigma = \{U_i \mid i \neq j\}$.

4.2 Cycles

Suppose $\mathcal{U}$ is a partially ordered cover of X, and $p\colon E \to X$ is given. Then a $\mathcal{J}$-n-cycle in $(X, p; \mathcal{U})$ is a function $N\colon \text{nerve}_\Delta(\mathcal{U}) \to \mathcal{J}$, specifically

(1) if σ is a k-simplex of $\text{nerve}_\Delta(\mathcal{U})$ then $N(\sigma)$ is an $(n-k)$-dimensional $(\mathcal{U} - \sigma)$-ad in $\mathcal{J}(p^{-1}(\cap\sigma))$,
(2) let $\text{incl}_*\colon \mathcal{J}(p^{-1}(\cap\sigma)) \to \mathcal{J}(p^{-1}(\cap b_j\sigma))$ denote the morphism induced by the inclusion, then $\text{incl}_*(N(\sigma)) = (-1)^j \partial_{U_j} N(b_j\sigma)$, and
(3) all but finitely many of the $N(\sigma)$ are ϕ.

For a source of geometric examples, suppose X is the realization of a simplicial complex and $\mathcal{U}$ is the covering by stars, as in 1.5. The dual cones provide a refinement of this cover in which the faces of the cones have collars. Suppose $M \to X$ is a map from a manifold, then M can be made transverse to the cones. This breaks M into pieces which are manifolds with boundary faces indexed by the cones. The function (simplex) $\mapsto$ (inverse image of cone dual to the simplex) defines a cycle in the manifold bordism-type theory. The incidence relations in (2) record how these pieces fit together inside M.

Next define a "homology" between two cycles. This is a function

$$H\colon \mathrm{nerve}_\Delta(\mathcal{U}) \to \mathcal{J}$$

which takes σ to an $(n-k+1)$-dimensional $(\mathcal{U}-\sigma)\amalg[1]$-ad in $\mathcal{J}(p^{-1}(\cap\sigma))$. The disjoint union means $H(\sigma)$ is an -ad with faces ∂_U for U not in σ, and in addition faces ∂_0 and ∂_1. These are required to satisfy the cycle conditions above on the ∂_U faces, and also $\partial_0\partial_1 H = \phi$. It follows that $\partial_0 H$ and $\partial_1 H$ are n-dimensional cycles in the sense above. We then say that $\partial_0 H$ is homologous to $-\partial_1 H$.

In these terms the main theorem can be stated as:

4.2A Theorem. *Suppose $\mathcal{U}$ is a partially ordered cover of X, and $p\colon E \to X$ is given. Then there is a canonical isomorphism from the group of homology classes of $\mathcal{J}$-n-cycles in $(X, p; \mathcal{U})$ and the homology group*

$$H_n(\mathit{nerve}_\Delta(\mathcal{U}); \Omega^{\mathcal{J}}(p, \mathcal{U}))$$

defined in 2.3.

This isomorphism is also natural with respect to the functorially induced functions of cycles defined in section 4.5. This statement will follow from Theorem 5.1.

There is also a relative version of the theorem, and for that we define a relative version of cycles. The idea is that a cycle as defined above is "closed" in the sense that all $(n-1)$-dimensional pieces correspond to intersections $U \cap V$, and consequently occur as faces of the two n-dimensional pieces lying over U and V. We obtain "free boundary" which occurs only once as a face simply by failing to define pieces corresponding to certain subsets. Specifically, if $Y \in \mathcal{U}$ then a relative cycle over $(\mathcal{U}, Y)$ is defined to be a function $(\mathrm{nerve}_\Delta(\mathcal{U}) - \{Y\}) \to \mathcal{J}$ satisfying the conditions (1)–(3) above.

The "Kan cycles" used in the definition of the Kan condition in 3.2 are relative cycles in this sense, taking values in a constant functor. To see these as cycles in a covering, embed the set A as a linearly independent set in a real vector space, and let X be the convex hull. Then X is covered by sets U_a consisting of all points with nonzero a coordinate when expressed as a convex linear combination. An (A, a)-cycle in the sense of the Kan condition is a relative cycle over $(\{U_b \mid b \in A\}, U_a)$.

If N is a relative $\mathcal{J}$-n-cycle in $(X, p; \mathcal{U}, Y)$, then there is a boundary $\partial_Y N$ defined. This is the $(n-1)$-cycle in $(Y, p; (\mathcal{U} - \{Y\}) \cap Y)$, specified by $(\partial_Y N)(\sigma) = \partial_Y(N(\sigma \cup \{Y\}))$.

4.2B Proposition. *There is a canonical isomorphism from the group of homology classes of relative $\mathcal{J}$-n-cycles in $(X, Y, p; \mathcal{U} \cup \{Y\})$ to the relative homology $H_n(X, Y; \Omega^{\mathcal{J}}(p, \mathcal{U}))$. The homomorphism of cycles induced by the boundary operation ∂_Y agrees with the boundary homomorphism in homology.*

4.3 Naturality of cycles

In this section we construct functions of cycles induced by morphisms of data. According to the representation theorem, cycles represent homology classes. According to the general construction in section 2, morphisms of data induce homomorphisms of homology groups, including assembly maps. The objective here is to give cycle-level descriptions of these natural homomorphisms.

The simplest case is the total assembly corresponding to the map $X \longrightarrow pt$, and this is considered first. For this the construction is a simple application of the Kan condition. In general the functoriality of $\mathcal{J}$ is applied to get from a cycle on a cover of X a "multivalued" cycle on a cover of Y. The Kan condition is used to assemble the multiple pieces to get an honest cycle on Y.

Suppose $p\colon E \longrightarrow X$ and $\mathcal{U}$ is a cover of X, as usual, and M is a $\mathcal{J}$-n-cycle in $(X, p; \mathcal{U})$. Let $\mathrm{incl}_*\colon \mathcal{J}(p^{-1}(U)) \longrightarrow \mathcal{J}(E)$ denote the morphism induced by the inclusion. Then $\mathrm{incl}_*(M)$ is a function from $\mathcal{U}$ into $\mathcal{J}(E)$. (Note that once we are in a single theory $\mathcal{J}(E)$ the values of M on higher simplices of the nerve are determined by values on $\mathcal{U}$, so are unnecessary.)

If we add a disjoint element a and reindex the $M(*)$ to add $\partial_a M = \phi$, then this defines a $(\mathcal{U} \cup \{a\}, a)$-cycle in the sense of the Kan condition. Apply the Kan condition to obtain a $(\mathcal{U} \cup \{a\})$-ad N with $\partial_U N = M(U)$ for $U \in \mathcal{U}$. Then define $A(M) = \partial_a N$. Since $\partial_U \partial_a N = \phi$ this is a reindexed n-dimensional ϕ-ad in $\mathcal{J}(E)$.

We can now state the following, which is a special case of 4.3B.

4.3A Proposition. *This construction induces a homomorphism from homology classes of $\mathcal{J}$-n-cycles in $(X, p, \mathcal{U})$ to $\Omega_n^{\mathcal{J}}(E)$. Under the canonical isomorphism with homology this corresponds to the total assembly*

$$H_n(\mathrm{nerve}_\Delta(\mathcal{U}); \Omega^{\mathcal{J}}(p, \mathcal{U})) \to H_n(pt; \Omega^{\mathcal{J}}(E)) \simeq \Omega_n^{\mathcal{J}}(E) \ .$$

Now begin the general construction. Suppose $(f, \hat{f})$ is a map between maps p, q, ie. there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & F \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

Suppose $\mathcal{U}$ is a cover of X, $\mathcal{V}$ a cover of Y, and $\theta \colon \mathcal{U} \to \mathcal{V}$ is a morphism. By this last we mean an order-preserving function so that $f(U) \subset \theta(U)$ for every $U \in \mathcal{U}$.

The construction will be in two parts; first a function from cycles in $(X, p, \mathcal{U})$ to cycles in $(Y, q, \mathcal{V}_\theta)$, where $\mathcal{V}_\theta$ is a cover of Y obtained by introducing multiple copies of the elements of $\mathcal{V}$ indexed by $\mathcal{U}$. The second part goes from cycles in $(Y, q, \mathcal{V}_\theta)$ to cycles in $(Y, q, \mathcal{V})$, by assembling pieces over multiple copies using the Kan condition as above.

Define $\mathcal{V}_\theta$ to be the collection of subsets of Y isomorphic as a partially ordered set with $\mathcal{U}$, by the correspondence $U \mapsto V_U = \theta(U)$. This may not be a cover of Y. There are morphisms of covers (or perhaps just collections of subsets) $\mathcal{U} \to \mathcal{V}_\theta \to \mathcal{V}$, the first defined by $U \mapsto V_U$ and the second by $V_U \mapsto \theta(U)$. Since the first is an isomorphism of partially ordered sets it induces an injective Δ-map $\mathrm{nerve}_\Delta(\mathcal{U}) \to \mathrm{nerve}_\Delta(\mathcal{V}_\theta)$. The second induces a simplicial map of simplicial nerves, as does any morphism, but usually not a Δ-map.

Next suppose M is an n-cycle in $(X, p, \mathcal{U})$, so for σ a k-simplex in $\mathrm{nerve}_\Delta(\mathcal{U})$ we get an $(n-k)$-dimensional $\mathcal{U} - \sigma$-ad $M(\sigma) \in \mathcal{J}(p^{-1}(\cap\sigma)$. Let V_σ denote the image of σ in $\mathrm{nerve}_\Delta(\mathcal{V}_\theta)$. Then $\hat{f}$ induces a function $\hat{f}_* \colon \mathcal{J}(p^{-1}(\cap\sigma)) \to \mathcal{J}(q^{-1}(\cap V_\sigma))$. Therefore we can define a function

$$\hat{f}_* M \colon \mathrm{nerve}_\Delta(\mathcal{V}_\theta) \to \mathcal{J}$$

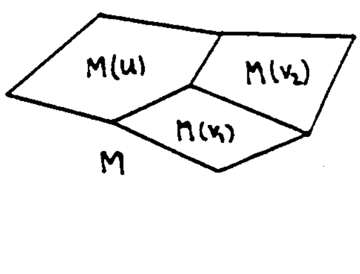
by: if τ is the image of a simplex in $\mathrm{nerve}_\Delta(\mathcal{U})$, so $\tau = V_\sigma$, then $\hat{f}_* M(\tau) = \hat{f}_*(M(\sigma))$. If τ is not in the image define $\hat{f}_* M(\tau) = \phi$.

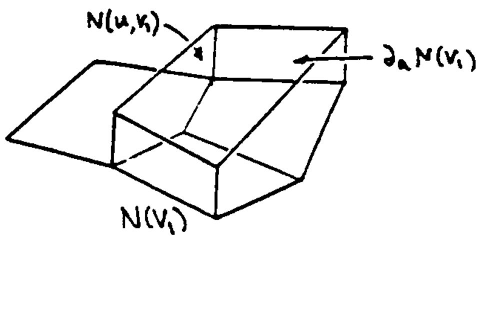
It is immediate that this $\hat{f}_* M$ is an n-cycle in $(Y, q, \mathcal{V}_\theta)$, and further that the same procedure defines a function on homologies. This therefore defines a function on homology classes of cycles.

Now we begin the second step; the passage from $\mathcal{V}_\theta$ to $\mathcal{V}$. This is broken into simpler pieces by a relative version of the θ-construction: suppose $\mathcal{V}' \subset \mathcal{V}$, then define $(\mathcal{V}, \mathcal{V}')_\theta$ to be the collection $\mathcal{V}' \cup \{V_U \mid \theta(U) \notin \mathcal{V}'\}$. There are morphisms $\mathcal{V}_\theta \longrightarrow (\mathcal{V}, \mathcal{V}')_\theta \longrightarrow \mathcal{V}$. By using a sequence of $\mathcal{V}'$ which differ by single sets we get a factorization of $\mathcal{V}_\theta \longrightarrow \mathcal{V}$ into a sequence of morphisms which are bijections except over a single set. It is therefore sufficient to do the construction for such morphisms. Note that the finiteness requirement on cycles implies that for a given cycle only a finite number of such special morphisms are required, even if $\mathcal{V}$ is infinite.

Suppose then that $\mathcal{V}$ is a cover of Y, $V_1 \in \mathcal{V}$, and $V_2, \ldots, V_r$ are additional copies of V_1. Given an n-cycle in $(Y, q, \mathcal{V} \cup \{V_2, \ldots, V_r\})$ we construct an n-cycle in $(Y, q, \mathcal{V})$, by constructing a type of "homology" in which the cover changes. For a given simplex the construction depends on whether or not the simplex contains V_1.

Let M be the n-cycle over $\mathcal{V} \cup \{V_2, \ldots, V_r\}$, and let a, b, c be elements (to be used for indices) not in the cover. On simplices of $\mathrm{nerve}_\Delta(\mathcal{V})$ containing V_1, say $\sigma \cup \{V_1\}$, we think of $M(\sigma \cup \{V_*\})$ as a $(\{V_*, a\}, a)$-cycle and fill in using the Kan condition. Specifically we want a function N so that $N(\sigma \cup \{V_1\})$ is an $(n-k+1)$-dimensional $(\mathcal{V} - \sigma \cup \{V_*\} \cup \{a\})$-ad in $\mathcal{J}(q^{-1}(\cap(\sigma \cup \{V_1\})))$ which is finite, satisfies the face relations as in (2) of the definition of cycles, and $\partial_{V_i} N(\sigma \cup \{V_1\}) = M(\sigma \cup \{V_i\})$ for $1 \le i \le r$.





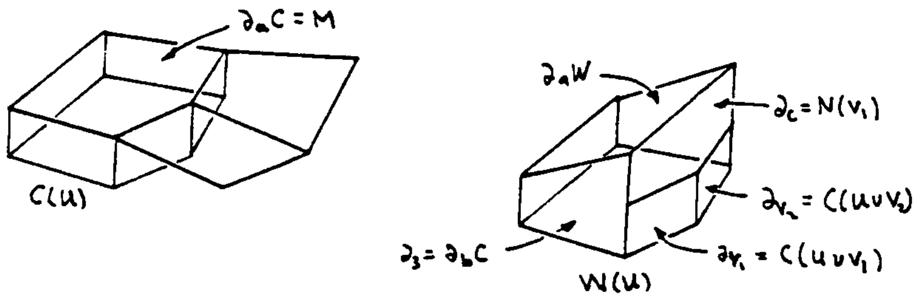
The function N is constructed by induction on dimension, beginning with large dimensions and working down. $M(\tau) = \phi$ for all but finitely many $\tau \in \mathrm{nerve}_\Delta(\mathcal{V} \cup \{V_i\})$, so there is a dimension above which $M = \phi$ and above this dimension we can set $N = \phi$. Now suppose N is defined for simplices of dimension greater than k, and suppose $\sigma \cup \{V_1\}$ has dimension k. Then a Kan-type cycle over $(\mathcal{V} - \sigma \cup \{V_*\} \cup \{a\}, a)$ in $\mathcal{J}(q^{-1}(\cap(\sigma \cup \{V_1\}))$

is defined by $U \mapsto N(\sigma \cup \{V_1, U\})$ and $V_i \mapsto N(\sigma \cup \{V_i\})$ for $1 \leq i \leq r$. Apply the Kan condition to obtain $N(\sigma \cup \{V_1\})$.

We would like to continue applying the Kan condition to extend N over simplices which do not contain V_1. However the pieces $M(\sigma)$ and $N(\sigma \cup \{V_1\})$ do not fit together correctly; the intersection is the cycle $V_i \mapsto M(\sigma \cup \{V_i\})$ rather than a single face. To fix this we introduce some new pieces, which should be thought of as subdividing a collar neighborhood of $\cup M(\sigma \cup \{V_*\})$ in $M(\sigma)$.

Let C be a homology from $\partial_a C = M$ to some other cycle $\partial_b C$. (We think of C as a collar $M \times I$, but construct it by inductive application of the Kan condition as in the construction of N above.) Next we construct a function W on simplices of $\text{nerve}_\Delta(\mathcal{V})$ which do not contain V_1. We want $W(\sigma)$ to be a $(\mathcal{V} - \sigma \cup \{V_*\} \cup \{a, b, c\})$-ad in the same theory as $M(\sigma)$ satisfying

(1) $\partial_U W(\sigma) = W(\sigma \cup \{U\})$ if $U \neq V_1$, and $U \notin \sigma$;
(2) $\partial_{V_i} W(\sigma) = C(\sigma \cup \{V_i\})$; and
(3) $\partial_b W(\sigma) = \partial_b C(\sigma)$ and $\partial_c W(\sigma) = N(\sigma \cup \{V_1\})$.



As with N we define W to be ϕ on high-dimensional simplices and work down by induction. If σ has dimension k and W is defined on higher dimensional simplices then all the faces specified above are defined and form a $(\mathcal{V} - \sigma \cup \{V_*\} \cup \{a, b, c\}, a)$-cycle. Applying the Kan condition yields an -ad which we define to be $W(\sigma)$.

Now define a function on $\text{nerve}_\Delta(\mathcal{V})$ by $\sigma \mapsto \partial_a N(\sigma)$ if $V_1 \in \sigma$, and $\sigma \mapsto \partial_a W(\sigma)$ if $V_1 \notin \sigma$. This defines a $\mathcal{J}$-n-cycle in $(Y, q, \mathcal{V})$, which is defined to be the functorial image of M.

4.3B Proposition. *This construction induces a homomorphism from homology classes of $\mathcal{J}$-n-cycles in $(X, p, \mathcal{U})$ to $\mathcal{J}$-n-cycles in $(Y, q, \mathcal{V})$. Under the canonical isomorphism with homology this corresponds to the induced homomorphism*

$$H_n(\|\text{nerve}(\mathcal{U})\|; \Omega^{\mathcal{J}}(p, \mathcal{U})) \to H_n(\|\text{nerve}(\mathcal{V})\|; \Omega^{\mathcal{J}}(q, \mathcal{V})).$$

4.4 The bordism-type theory of cycles

Fix $p\colon E \longrightarrow X$ and a cover $\mathcal{U}$ of X. In this section we define a bordism-type theory denoted $\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{U})$ whose ϕ-ads are $\mathcal{J}$-cycles in $(X,p,\mathcal{U})$.

Suppose A is a set. An A-ad of dimension n in $\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{U})$ is a function $N\colon \mathrm{nerve}_{\Delta}(\mathcal{U}) \longrightarrow \mathcal{J}$ satisfying exactly the definition of cycles given above, except that it takes values in A-ads. Explicitly,

(1) if σ is a k-simplex of $\mathrm{nerve}_{\Delta}(\mathcal{U})$ then $N(\sigma)$ is an $(n-k)$-dimensional $(\mathcal{U}-\sigma\cup A)$-ad in $\mathcal{J}(p^{-1}(\cap\sigma))$,
(2) let $\mathrm{incl}_*\colon \mathcal{J}(p^{-1}(\cap\sigma)) \longrightarrow \mathcal{J}(p^{-1}(\cap b_j\sigma))$ denote the morphism induced by the inclusion, then $\mathrm{incl}_*(N(\sigma)) = (-1)^j\partial_{U_j}N(b_j\sigma)$, and
(3) all but finitely many of the $N(\sigma)$ are ϕ.

In (2), U_j is the j^{th} element of σ with respect to the partial ordering, as in 4.2.

4.4A Proposition. *$\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{U})$ has the structure of a bordism-type theory. The n-dimensional bordism group of this theory is exactly the set of homology classes of $\mathcal{J}$-n-cycles in $(X,p,\mathcal{U})$.*

Proof. The homologies defined in 4.2 are exactly the type of [1]-ads used to define the equivalence relation in the bordism group, in 2.3, so the assertion that the bordism group is homology classes of cycles is just the definition.

We describe the bordism-type theory structure. The n-dimensional A-ads have been defined. Face operators are defined by $(\partial_a N)(\sigma)) = \partial_a(N(\sigma)$. The reindexing operations ℓ_θ are defined by reindexing all the pieces, and the bijectivity condition in 3.2(3) is immediate. Similarly the involution -1 is defined by applying the involution in $\mathcal{J}$ to each piece.

The remaining ingredient is to see that the Kan condition holds. This construction is similar to a step in the construction of induced maps of cycles in 4.3.

Suppose M is an n-dimensional (A,a)-cycle. This is a function from $A-\{a\}$ to cycles so that $M(b)$ is an n-dimensional $(A-\{b\})$-ad in

$$\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{U}).$$

Unraveling a little further, this is a function $M\colon (A-\{a\})\times\mathrm{nerve}_{\Delta}(\mathcal{U}) \longrightarrow \mathcal{J}$ so that $M(b,\sigma)$ is a $(A-\{b\})\cup(\mathcal{U}-\sigma)$-ad of dimension $n-k$ in $\mathcal{J}(p^{-1}(\cap\sigma))$, where k is the dimension of σ. We can think of this as a cycle of (A,a)-cycles, over $\mathcal{U}$.

What we want is a cycle of A-ads over $\mathcal{U}$, so that for each σ the (A,a)-cycle $N(*,\sigma)$ is obtained from it by taking faces. In more detail this is a function $N\colon \mathrm{nerve}_\Delta(\mathcal{U}) \to \mathcal{J}$ so that $N(\sigma)$ is a $A \cup (\mathcal{U} - \sigma)$-ad of dimension $n - k + 1$ in $\mathcal{J}(p^{-1}(\cap\sigma))$. This should satisfy $M(b, \sigma) = \partial_b N(\sigma)$, and the cycle face relation 4.4(2) relating $N(\sigma)$ and $N(b_j\sigma)$.

We construct N by induction downward on dimensions of simplices in $\mathrm{nerve}_\Delta(\mathcal{U})$. Since $M(b, \sigma) = \phi$ for all but finitely many (b, σ), there is a dimension for σ above which $M = \phi$ and we can set $N(\sigma) = \phi$.

Now suppose M is defined on simplices of dimensions greater than k, and consider a k-simplex σ. Define a Kan-type $(A \cup \mathcal{U} - \sigma, a)$-cycle in $\mathcal{J}(p^{-1}(\cap\sigma))$ by: take $b \in A - \{a\}$ to $M(b, \sigma)$. Take $V \in \mathcal{U} - \sigma$ to $\mathrm{incl}_* N(\sigma \cup V)$, if $\sigma \cup V$ is a simplex in the nerve (ie. $\cap\sigma \cap V \neq 0$), and take it to ϕ otherwise. Note $N(\sigma \cup V)$ is defined since $\sigma \cup V$ is a simplex of dimension greater than k. This formula does in fact form a cycle, so the Kan condition in $\mathcal{J}(p^{-1}(\cap\sigma))$ implies there is a $(A \cup \mathcal{U} - \sigma)$-ad which has this cycle as faces. Select one of these to be $N(\sigma)$, then this satisfies the conditions required for the induction step. $\square$

4.5 Naturality of theories

The constructions of 4.3 give homomorphisms of homology groups of cycles, corresponding to appropriate morphisms of data. The definitions of 4.4 extends the homology groups to entire bordism-type theories. It would be nice to similarly extend the homomorphism construction to give morphisms of bordism-type theories. Such morphisms would by naturality induce maps of the associated bordism spectra. The constructions of 4.3 are not canonical enough to give morphisms of theories, but they do extend directly to the bordism spectra.

Proposition. *Suppose $(f, \hat{f})\colon p \to q$ is a morphism of maps and $\theta\colon \mathcal{U} \to \mathcal{V}$ is a morphism of covers, as in 4.3. Then there is an associated morphism of bordism spectra*

$$(f, \hat{f}, \theta)_*\colon \Omega(\mathit{Cycles}(X, p, \mathcal{U})) \to \Omega(\mathit{Cycles}(Y, q, \mathcal{V}))$$

which is natural up to homotopy, and on homotopy groups induces the homomorphism defined in 4.3.

Proof. We will not give the proof in detail. It is primarily an elaboration on the construction in 4.3, and although the idea behind it can be described easily, the indexing on the various -ads considered gets too complex to be informative. Also relatively little use is made of it here; it is primarily used to replace the word "canonical" in Theorem 4.2A with the word "natural" in the final result (however, see 4.6).

The spaces $\Omega(\text{Cycles}(X, p, \mathcal{U}))$ and $\Omega(\text{Cycles}(Y, q, \mathcal{V}))$ are geometric realizations of Δ-sets (see 3.4), so we get a map between them by realizing Δ-maps.

A k-simplex of $\Omega_n(\text{Cycles}(X, p, \mathcal{U}))$ is by definition a $[k]$-ad in the bordism-type theory $\text{Cycles}(X, p, \mathcal{U})$, which in turn is a function $\text{nerve}_\Delta(\mathcal{U}) \to ([k]\text{-ads in } \mathcal{J})$. If θ is an injection then it induces a Δ-injection of nerves $\text{nerve}_\Delta(\mathcal{U}) \to \text{nerve}_\Delta(\mathcal{V})$. In this case we get a k-simplex of

$$\Omega_n(\text{Cycles}(Y, q, \mathcal{V}))$$

by applying morphisms induced in $\mathcal{J}$ by $(f, \hat{f})$, and then extending the function to $\text{nerve}_\Delta(\mathcal{V})$ simply by defining it to be ϕ on the complement of $\text{nerve}_\Delta(\mathcal{U})$.

As for single cycles this reduces the construction to $X = Y$ and θ a morphism which eliminates duplicate copies of a single set V_1. In this case we define the map by induction on skeleta of $\Omega_n(\text{Cycles}(X, p, \mathcal{U}))$. If M is a 0-simplex then it is a cycle, and we define $(f, \hat{f}, \theta)(M)$ as in 4.3A. Suppose the map is defined on the $(k-1)$-skeleton and M is a k-simplex. The construction from this point is essentially the same as that of 4.3A, except there are more faces. As in 4.3A we proceed by induction downwards on dimension of simplices of $\text{nerve}_\Delta(\mathcal{V})$. The induction is started by setting the values to be ϕ for simplices of sufficiently high dimension.

On simplices containing V_1 we basically want a cycle of solutions to Kan extension problems. For a given simplex we get a Kan cycle with three types of pieces: from the construction on higher simplices, ones of the form $M(\sigma \cup \{V_i\})$, as before, and also pieces from the construction on $b_j M$. According to the induction hypotheses all these are already defined, so the Kan condition can be used to extend the construction to $(\sigma \cup \{V_1\})$.

On simplices not containing V_1 the construction involves first constructing a "collar" to introduce more faces, then finding a cycle of solutions to the resulting Kan problems. Again if this has been done for higher simplices of $\text{nerve}_\Delta(\mathcal{V})$ and also for faces $b_j M$ then we get extension problems whose solutions extend the construction over the simplex.

Finally we indicate why this is well-defined and natural up to homotopy. To see it is well-defined suppose there are two such constructions, and think of these as defined on $\text{nerve}_\Delta(\mathcal{V}) \times \{0, 1\}$. Then use the same proceedure to fill in between these to get a homotopy defined on $\text{nerve}_\Delta(\mathcal{V}) \times [0, 1]$. The only difference is that there are yet more faces, coming from the ends of the homotopy where the construction is already given.

This, together with the Δ-set model for the loop space used in 3.4, also shows that these maps of spaces fit together (up to homotopy) to give maps of spectra.

The proof of naturality is straightforward except for changing covers. For this we need to see that if $\mathcal{V}$ is obtained from $\mathcal{U}$ by eliminating duplicates of two different sets, then the compositions are independent (up to homotopy) of the order in which this is done. Filling in a homotopy between the two compositions proceeds by double induction as before, but the proceedure for each piece is substantially more complicated. It separates into four cases, depending on whether or not the sets being changed are contained in the simplex. The worst case, when the simplex contains neither, seems to require four different applications of the Kan condition. It is not too difficult to guess what to do, but it does seem to be a lot of trouble to verify that the resulting formulae do in fact give Kan-type cycles. □

4.6 Nonsimplicial situations

Cycles are associated with a covering, and describe homology of the nerve of that cover. A homotopy inverse limit is used to define general homology, and although this can be avoided if the situation is simplicial (by 2.6) it is necessary in general. The maps in the inverse system are induced by morphisms of the data, so the proposition gives a description of the system in terms of cycles. This leads to a cycle description of general homology classes.

When the space is reasonable, for example metric, the description of such homotopy inverse limits in terms of arcs can be employed. In these terms a homology class is represented by a half-open arc of cycles: a triangulation of $[0, \infty)$ is given, the vertices are mapped to $(X, p, \mathcal{U})$-cycles, where $\mathcal{U}$ depends on the vertex. Edges are mapped to homologies of the type considered in 4.3 which change the covering. Finally we require that the covers have diameters which go to 0 as we go toward ∞.

Carrying out this algorithm using the constructions as given results in an unpleasantly complicated mess, so we will not do it here. For this to work better it would be helpful to have a direct description of homologies which change covers arbitrarily, not just by duplicating a single subset. Also better naturality constructions would be needed. For this it might be useful to consider more elaborate forms of the Kan condition, for example allowing pieces to intersect in cycles instead of just faces.

4.7 Cocycles and cohomology

There is a representation result for cohomology which is easier than cycles and homology, and we describe this here. They are particularly useful in descriptions of products and duality, but these will not be discussed here. We restrict to the constant coefficient case. The reader who needs the functor coefficient analog should not have trouble working it out.

These representative are basically the same as the unfortunately named "mock bundles" of [4].

Cohomology with coefficients in a spectrum $\mathbf{J}$ is defined, dually to homology, by maps:

$$H^n(X;\mathbf{J}) = \pi_0(\mathrm{maps}(X,\mathbf{J}_n)).$$

When $\mathbf{J}$ is an Ω-spectrum this involves the single space $\mathbf{J}_n$, as written. For general spectra it has to be interpreted in terms of maps of spectra.

Now suppose $X = \|K\|$ is the realization of a Δ-set, and $J_n = \Omega_n^{\mathcal{J}}$ is the bordism spectrum of a bordism-type theory $\mathcal{J}$. According to the simplicial approximation theorem a map of realizations is homotopic to the realization of a Δ-map $f\colon K \longrightarrow \Omega_{-n}^{\mathcal{J}}$. (Note, as in 3.4, the sign difference on the subscripts on the spectrum and Δ-set.) Thus to obtain representatives for cohomology classes we have only to refer to the definition of $\Omega_{-n}^{\mathcal{J}}$ and spell out what such a Δ-map looks like. The result is:

Proposition. *Classes in $H_n(\|K\|;\Omega^{\mathcal{J}})$ are represented by functions f from simplices of K to -ads in $\mathcal{J}$ such that*

(1) *if σ is an k-simplex then $f(\sigma)$ is an $[k]$-ad of dimension $k-n$, and*
(2) *$f(b_j\sigma)$ is obtained from $(-1)^j\partial_j f(\sigma)$ by reindexing by the order-preserving bijection $[k-1] \longrightarrow [k]-\{j\}$.* $\square$

To contrast this with the definition of a cycle we make explicit some of the differences. For this consider an n-cycle M in the star cover associated to the triangulation of $\|K\|$. Since the simplices of the nerve of this cover are indexed by simplices in K, the n-cycle M is also a function from simplices of K to -ads in $\mathcal{J}$. However

(1) the dimension of $M(\sigma)$ is the negative of the dimension of $f(\sigma)$. In particular the dimension of $f(\sigma)$ increases with the dimension of σ while the dimension of $M(\sigma)$ decreases.
(2) the face structures are also dual: $f(\sigma)$ has faces corresponding to the faces of σ, while roughly speaking $M(\sigma)$ has faces corresponding to simplices disjoint from σ.

In its face and dimension structure a cocycle behaves like a product $K \times F$, or more generally like a bundle over K. It can be thought of as a sort of block bundle over K in which the fibers are allowed to change from point to point (hence the term "mock bundle" in [4]).

5: Proof of the representation theorem

We now show that cycles represent homology. This is done on the spectrum level; the bordism spectrum of cycles is equivalent to the homology spectrum.

5.1 Theorem. *Suppose $\mathcal{U}$ is a cover of X, and $p\colon E \to X$. Then there is a homotopy equivalence of spectra,*

$$\Omega\big(\mathit{Cycles}^{\mathcal{J}}(X,p,\mathcal{U})\big) \to \mathbf{H}_\bullet\big(\mathit{nerve}(\mathcal{U}); \Omega^{\mathcal{J}}(p,\mathcal{U})\big)$$

which is natural up to homotopy.

Before giving the formal proof we describe the idea. Given a cycle we want to construct a point in one of the loop spaces in the direct limit defining the homology space. So we seek a map $S^k \to \Omega(p,\mathcal{U})/\mathrm{nerve}_\Delta(\mathcal{U})$, where $\Omega(p,\mathcal{U}) \to \mathrm{nerve}_\Delta(\mathcal{U})$ is the spectral sheaf constructed as in 2.3, and the quotient indicates dividing out the 0-section.

A cycle is a function $M\colon \mathrm{nerve}_\Delta(\mathcal{U}) \to \mathcal{J}$, but not any sort of Δ-map; $M(\sigma)$ has the wrong face structure to be a simplex which is an image of σ. However, assume $\mathcal{U}$ is finite with $n+1$ elements. There is a natural simplicial embedding $\mathrm{nerve}_\Delta(\mathcal{U}) \subset \Delta^{n+1}$, and we can associate to each simplex a dual simplex $D^n(\sigma)$. The cycle has the correct face structure to define a Δ-map $M(\sigma)\colon D^n(\sigma) \to \Omega^{\mathcal{J}}(p^{-1}(\cap\sigma))$.

Geometric realization of this Δ-map gives a map of spaces $\|D^n(\sigma)\| \to \Omega^{\mathcal{J}}(p^{-1}(\cap\sigma)$. This is natural in σ, so defines a natural transformation of functors. Here both $\|D^n(\sigma)\|$ and $\Omega^{\mathcal{J}}$ are regarded as functors from

$$\mathrm{nerve}_\Delta(\mathcal{U})$$

into spaces. The geometric realization of the functor $\sigma \mapsto \Omega^{\mathcal{J}}(p^{-1}(\cap\sigma)$ gives the spectral sheaf $\Omega^{\mathcal{J}}(p,\mathcal{U})$. The central geometric fact in the argument is that the geometric realization of the dual simplex functor $\sigma \mapsto \|D^n(\sigma)\|$ can be canonically identified with S^n. The realization of the natural transformation gives a map between these spaces, and therefore $S^n \to \Omega(p,\mathcal{U})/\mathrm{nerve}_\Delta(\mathcal{U})$, as required.

To reverse the process begin with a map $f\colon S^j \to \Omega(p,\mathcal{U})/\mathrm{nerve}_\Delta(\mathcal{U})$. First f is deformed to be transverse with respect to the simplex coordinates of $\Omega(p,\mathcal{U})$ coming from the construction as a realization. This describes f as coming from the realization of a natural transformation of functors, but defined on some manifold-valued functor usually different from the dual simplex functor. After stabilizing to get into the stable range (for embeddings) we can embed this functor in the dual simplex functor. Extension of the natural transformation to the whole dual simplex functor corresponds to suspension in the spectrum structure. Therefore after suspension the map becomes homotopic to a realization of the desired kind.

5.2 Functors on Δ-sets

Functors on simplicial nerves, regarded as categories, are used in 2.3 to define functor-coefficient homology. Since we are now using the Δ-nerve, we need a Δ-set version of this.

Suppose K is a Δ-set, and regard it as a category with morphisms generated by identity maps and the face operators ∂_j. If F is a covariant functor from K to spaces, then $\coprod_{\sigma \in K} F(\sigma)$ is a Δ-space. The geometric realization of this is defined by:

$$\|F\| = \Big(\coprod_{k, \sigma \in K^k} F(\sigma) \times \Delta^k \Big) / \sim$$

where $\sim$ is the equivalence relation generated by: if $x \in F(\partial\sigma)$, $t \in \Delta^{k-1}$, and $u \in \Delta^{k+1}$ then $(x, \partial_j^* t) \in F(\sigma) \times \Delta^k$ is equivalent to $(\partial_j x, t) \in F(\partial_j \sigma) \times \Delta^{k-1}$.

This construction is natural, with the same properties as the simplicial version in 2.2A. The reason, by the way, that we did not use the Δ version from the beginning is that morphisms of covers defines simplicial maps of simplicial nerves, but not Δ-maps of Δ-nerves. This makes changing covers awkward in the Δ version, and is part of the difficulty encountered in the naturality constructions in section 4.

Now we relate this to the simplicial version. If K is a simplicial set then $\operatorname{core} K \subset K$ is the Δ-set of nondegenerate simplices; the example of particular concern is $\operatorname{nerve}_\Delta(\mathcal{U}) \subset \operatorname{nerve}(\mathcal{U})$.) Then $\operatorname{core} K$ is a subcategory of K. If F is a functor from K to spaces, in the sense of 2.2 then the restriction to $\operatorname{core} K$ is a functor in the sense above. The inclusion $\operatorname{core} K \subset K$ induces a map of realizations.

5.2A Lemma. *Suppose K is a simplicial set and F is a functor from K to spaces. The natural map, from the Δ-realization of the restriction of F to $\operatorname{core} K$, to the simplicial realization of F, is a homeomorphism.*

This is straightforward, basically the same argument as the proof that the Δ-realization of the core is the same as the simplicial realization of K. $\square$

5.3 Extensions to simplices

For convenience we extend both cycles and the spectral sheaves over a simplex containing the nerve. When the cover is finite we will actually work with the extension over $K = \partial\Delta^{\mathcal{U}}$, rather than over the whole simplex.

Suppose $\mathcal{U}$ is a totally ordered collection of subsets of X. Define a Δ-set $\Delta^{\mathcal{U}}$ with k-simplices all collections of $k+1$ distinct elements of $\mathcal{U}$. If $\mathcal{U}$ is finite this is a simplex. With this notation the simplex Δ^n is $\Delta^{[n]}$. The nerve is a sub-Δ-set of this; $\mathrm{nerve}_\Delta(\mathcal{U}) \subset \Delta^{\mathcal{U}}$.

Now suppose $\mathcal{J}$ is a bordism-theory valued functor of spaces, which on the empty set consists only of basepoints. A $\mathcal{J}$-n-cycle over $(X, p, \mathcal{U})$ is defined to be a function from $\mathrm{nerve}_\Delta(\mathcal{U})$ into $\mathcal{J}$ which takes a simplex σ to a $\mathcal{U} - \sigma$-ad in $\mathcal{J}(p^{-1}(\cap\sigma))$. We can consider cycles defined on all of $\Delta^{\mathcal{U}}$ satisfying the same conditions. The value on the additional simplices is determined: if $\sigma \notin \mathrm{nerve}_\Delta(\mathcal{U})$ then $\cap\sigma = \phi$ so $\mathcal{J}(p^{-1}(\cap\sigma)) = \mathcal{J}(\phi)$, which consists only of basepoints. We formalize this as a lemma.

5.3A Lemma. *Suppose $\mathcal{J}(\phi)$ consists only of basepoints, and K is a Δ-set, $\mathrm{nerve}_\Delta(\mathcal{U}) \subset K \subset \Delta^{\mathcal{U}}$. Then restriction defines a bijection from $\mathcal{J}$ cycles defined on K to cycles defined on $\mathrm{nerve}_\Delta(\mathcal{U})$.* $\square$

Spectral sheaves extend in a similar fashion. Suppose $\mathbf{J}$ is a spectrum-valued functor of spaces. The spectral sheaf over $\|\mathrm{nerve}_\Delta(\mathcal{U})\|$ is defined by realizing the functor $\sigma \mapsto \mathbf{J}(p^{-1}(\cap\sigma))$. This functor is evidently defined on all of $\Delta^{\mathcal{U}}$, not just $\mathrm{nerve}_\Delta(\mathcal{U})$. Realization of the extended functor defines a sheaf over $\|\Delta^{\mathcal{U}}\|$, which we continue to denote by $\mathbf{J}(p, \mathcal{U})$.

5.3B Lemma. *Suppose $\mathbf{J}$ is a spectrum-valued functor such that $\mathbf{J}(\phi)$ is contractible, and K is a Δ-set $\mathrm{nerve}_\Delta(\mathcal{U}) \subset K \subset \Delta^{\mathcal{U}}$. Then the induced inclusion of homology spectra*

$$\mathbf{H}_\bullet(\mathit{nerve}_\Delta(\mathcal{U}); \mathbf{J}(p, \mathcal{U})) \to \mathbf{H}_\bullet(K; \mathbf{J}(p, \mathcal{U}))$$

is a homotopy equivalence.

Proof. The homology of $\mathrm{nerve}_\Delta(\mathcal{U})$ comes from the sheaf over $\mathrm{nerve}_\Delta(\mathcal{U})$, divided by $\mathrm{nerve}_\Delta(\mathcal{U})$. Similarly the homology of K comes from the sheaf over K, divided by K. The cofiber of the inclusion is the sheaf over K, divided by K and the sheaf over $\mathrm{nerve}_\Delta(\mathcal{U})$. This is homeomorphic to $(K \times \mathbf{J}(\phi))/(\mathrm{nerve}_\Delta(\mathcal{U}) \times \mathbf{J}(\phi) \cup K \times *)$, which is contractible because $\mathbf{J}(\phi)$ is. Since the cofiber is contractible, the inclusion is a homotopy equivalence. $\square$

Note that if $\mathcal{J}$ is a bordism-theory valued functor as in 5.4A then the associated bordism spectrum is a spectrum-valued functor which satisfies the hypotheses of 5.4B. The bordism spectrum of the empty set is the realization of the Δ-set $\{\phi\}$ with a single simplex in each dimension. This realization is easily seen to be contractible (the fundamental group and homology groups are trivial).

5.4 Dual simplices

Fix an integer n. The "dual simplex functor" is a function $D^n : \partial\Delta^{n+1} \to \partial\Delta^{n+1}$, where $\partial\Delta^{n+1}$ denotes the Δ-set. If σ is a simplex of $\partial\Delta^{n+1}$ then $D^n(\sigma)$ is defined to be the simplex spanned by the vertices of Δ^{n+1} not in σ.

So for example Δ^{n+1} is the join $\sigma * D^n(\sigma)$.

Lemma.

(1) *The function* $\sigma \mapsto \|D^n\|$ *defines a functor from* $\partial\Delta^{n+1}$ *to spaces, and*

(2) *there is a canonical homeomorphism from the realization of this functor to* S^n.

Proof. D^n is contravariant in σ, in the sense that if $\tau \subset \sigma$ then $D^n(\tau) \supset D^n(\sigma)$. The face maps are contravariant with respect to inclusion, so D^n is a covariant functor on the category with morphisms generated by the ∂_*.

The realization of this functor is built of pieces $\|\sigma\| \times \|D^n(\sigma)\|$. Geometrically we think of this as a tubular neighborhood of σ. The homeomorphism to S^n gives the handlebody structure obtained by thickening up the cell decomposition of $\partial\Delta^{n+1}$.

The proof uses some spherical geometry, so we understand S^n to mean the sphere with its usual Riemannian metric. Suppose $X \subset S^n$ is contained in the interior of some hemisphere. Then we define the *convex hull* of X, denoted hull(X), to be the smallest set containing X, contained in the hemisphere, and intersecting each geodesic in a connected set. The hull is also the intersection of all hemispheres containing X, so in particular is independent of any particular hemisphere.

The fact we will use is that certain convex hulls are naturally homeomorphic to simplices. There is, for each set of points $V = (v_0, \dots, v_j)$ in S^n which are equidistant from each other, and lie in the interior of a hemisphere, a homeomorphism $f_V : \Delta^j \to \operatorname{hull} V$. This is continuous in V and natural with respect to faces and isometries; the restriction to the face obtained by omitting the i^{th} vertex is the function associated to the set obtained by omitting v_i, and if $g : S^n \to S^n$ is an isometry then $f_V g$ is $f_{g(V)}$.

Choose points $x_0, \dots, x_n$ in S^n which are equidistant and the maximal distance apart; these are the vertices of an inscribed regular simplex. To make the construction canonical these points should be chosen in some canonical way. The point $-x_k$ is the barycenter of the simplex determined by $\{x_i \mid i \neq k\}$. Define the point $y_{i,k}$ for $i \neq k$ to be the midpoint of the shortest geodesic between x_i and $-x_k$.

Note that given i, j there is an isometry of S^n (in fact a reflection) which interchanges x_i and x_j and leaves the other x_* fixed. Since isometries preserve geodesics this reflection also interchanges $y_{i,k}$ and $y_{j,k}$, and interchanges $y_{k,i}$ and $y_{k,j}$

Next we define, for each $\sigma \in \partial\Delta^{n+1}$, a map $F_\sigma \colon \|\sigma\| \times \|D^n(\sigma)\| \to S^n$. This will be natural in σ and a homeomorphism onto the convex hull of the points $\{y_{i,k} \mid i \in \sigma, \text{ and } k \notin \sigma\}$ (here we have written $i \in \sigma$ if the i^{th} vertex of Δ^{n+1} is in σ). These facts will imply the lemma: the naturality implies these fit together to define a map from the realization of the functor $\|D^n(*)\|$ to S^n, and the homeomorphism statement implies this map is a homeomorphism.

Suppose $(s,t) \in \|\sigma\| \times \|D^n(\sigma)\|$. For any fixed $k \notin \sigma$ the points in the set $y_{\sigma,k} = \{y_{i,k} \mid i \in \sigma\}$ are equidistant from each other. This is because the reflections which interchange the $\{x_i \mid i \in \sigma\}$ also interchange these points, and reflections preserve distances. Therefore the functions $f_{y_{\sigma,k}}$ are defined. Let $V = \{v_k\}$ denote the set obtained by applying these functions to the point s.

The points in the set V are also equidistant from one another. This is because the reflections which interchange x_k with $i \notin \sigma$ interchange the sets $y_{\sigma,k}$ and therefore—by naturality with respect to isometry—the functions $f_{y_{\sigma,k}}$. This implies the reflections also interchange images of a specific point, in this case s. From this we conclude the function f_V is defined, and we define $F_\sigma(s,t) = f_V(t)$.

The function F_σ is continuous because the f_* are continuous in $*$. It is natural in σ because the f_* are. It therefore remains to verify that it is a homeomorphism.

Consider the function f_V again. The image of this intersects σ in a single point, and is perpendicular to σ at that point. This is because the reflections which interchange points of V leave σ invariant; if w is a vector in σ it makes the same angle with each of the geodesics from the intersection point to a vertex of the image of f_V. This angle must therefore be 0. This identifies f_V as the intersection of the linear sphere perpendicular to σ, and the hull of $\{y_{i,k} \mid i \in \sigma, \text{ and } k \notin \sigma\}$.

We can now reverse the construction. If z is a point in the hull then it lies in some sphere perpendicular to σ. Let $V = \{V_k\}$ denote the intersection of this sphere with the hull of $y_{\sigma,k}$, where $k \notin \sigma$. The sphere is invariant under isometries which fix σ, so invariant under the reflections used above. Since these interchange elements of V the same argument used above implies these are equidistant. The point z is therefore $f_V(t)$ for some $t \in \|D^n(\sigma)\|$. Further for each $k \notin \sigma$ the point z_k is $f_{y_{\sigma,k}}(s_k)$, for some $s_k \in \|\sigma\|$. Using the symmetry again we see that all the s_k are equal. This identifies z as $F_\sigma(s_k, t)$.

Explicitly we have explained why F_σ is onto. But the points s and t are easily seen to be uniquely defined, so it is also injective. Therefore it is a homeomorphism. □

5.5 From cycles to homology

Suppose $\mathcal{U}$ is a finite collection of subsets of X. In this section we describe the map, from the bordism spaces of cycles to the homology spaces, both over $\mathcal{U}$. We verify this is a map of spectra in the next section, and consider infinite collections in the section 5.7.

Denote the elements of $\mathcal{U}$ by $U_0, \ldots, U_{n+1}$, so the simplex spanned by this is canonically Δ^{n+1}. Suppose that U_0 is empty. This assumption implies that $\text{nerve}_\Delta(\mathcal{U}) \subset \partial_0\Delta^{n+1}$, and in particular $\text{nerve}_\Delta(\mathcal{U}) \subset \partial\Delta^{n+1}$.

Suppose M is a $\mathcal{J}$-r-cycle in $(X, p, \mathcal{U})$, extended trivially to $\partial\Delta^{n+1}$ as in the previous section. Then $M(\sigma)$ is a $(\mathcal{U} - \sigma)$-ad of dimension $r - j$ in $\mathcal{J}(p^{-1}(\cap\sigma))$, where j is the dimension of σ. Using the canonical bijection $[n - j] \to \mathcal{U} - \sigma$ we can regard $M(\sigma)$ as an $(n - j)$-dimensional simplex of the bordism Δ-set $\Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))$. Or, since $D^n(\sigma)$ is the simplex spanned by $\mathcal{U} - \sigma$, this can be regarded as a Δ-map $D^n(\sigma) \to \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))$

The geometric realization of this Δ-map defines a map of spaces

$$\|D^n(\sigma)\| \to \Omega^{\mathcal{J}}_{n-r}(p^{-1}(\cap\sigma)).$$

(Remember that indices are reversed in forming the bordism spectrum Ω.) This is a natural transformation of functors defined on $\partial\Delta^{n+1}$, in other words

$$\begin{tikzcd}
\|D^n(\partial_i\sigma)\| \arrow[r] \arrow[d] & \Omega^{\mathcal{J}}_{n-r}(p^{-1}(\cap\partial_i\sigma)) \arrow[d] \\
\|D^n(\sigma)\| \arrow[r] & \Omega^{\mathcal{J}}_{n-r}(p^{-1}(\cap\sigma))
\end{tikzcd}$$

commutes. This is condition (2) in the definition of cycles in 4.2. This natural transformation induces a map of realizations of these functors. According to 5.4 the realization of $\sigma \mapsto \|D^n(\sigma)\|$ is S^n. The other realization is the spectral sheaf (over $\partial\Delta^{n+1}$) so we get a map $S^n \to \Omega^{\mathcal{J}}_{n-r}(p, \mathcal{U})$.

The initial hypothesis that $U_0 = \phi$ implies that the basepoint of S^n (the 0^{th} vertex of $\partial\Delta^{n+1}$) maps to the basepoint. Thus this map defines a point in the loop space $\Omega^n(\Omega^{\mathcal{J}}_{n-r}(p,\mathcal{U}))$. Divide by $\partial\Delta^{n+1}$ and include into the homology spectrum to get a point in $\mathbf{H}_{-r}(\partial\Delta^{n+1};\Omega^{\mathcal{J}}(p,\mathcal{U}))$.

This defines a function from the vertices of the bordism spectrum of $\mathcal{J}$-cycles on $(X,p,\mathcal{U})$ to the homology spectrum of $\partial\Delta^{n+1}$. Next we extend this to a map of the whole space of cycles, essentially by adding a Δ^i coordinate to the construction above.

Let M be an i-simplex of the bordism space of r-dimensional cycles. This takes a j-simplex σ of $\partial\Delta^{n+1}$ to a $((\mathcal{U}-\sigma)\cup[i])$-ad of dimension $r-j+i$ in $\mathcal{J}(p^{-1}(\cap\sigma))$. Regard $M(\sigma)$ as an $(n-j+i+1)$-simplex of the associated bordism space. More precisely regard it as a Δ-map of the join simplex $D^n(\sigma) * \Delta^i$ into $\mathit{\Omega}^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap\sigma))$. Realize this to get $\|D^n(\sigma)\| * \|\Delta^i\| \to \Omega^{\mathcal{J}}_{n-r+1}(p^{-1}(\cap\sigma))$.

Again this is a natural transformation of functors of σ, so induces a map of realizations of functors. The realization of the left side is $S^n * \|\Delta^i\|$, and that of the right side is the spectral sheaf, so this gives a map $S^n * \|\Delta^i\| \to \Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U})$. Note $\|D^n(\sigma)\|$ and $\|\Delta^i\|$ are taken to the basepoint: The image of $D^n(\sigma)$ corresponds to $\partial_{[i]}M(\sigma)$, which is ϕ by definition of simplices of the bordism space. The image of Δ^i corresponds to $\partial_{(\mathcal{U}-\sigma)}M(\sigma)$ which is ϕ by condition (3) in the definition of cycles.

Regard the join as $S^n \times \|\Delta^i\| \times I$ with identifications at the ends of the I coordinate. Then the realization of $M(\sigma)$ defines a map $S^n \times \|\Delta^i\| \times I \to \Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U})$. Let $S^n \times I \subset S^{n+1}$ denote the standard embedding, then the map extends by the point map on the complement to give $S^{n+1} \times \|\Delta^i\| \to \Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U})$.

Regard this map as a simplex in the Δ-set of maps from S^{n+1} to $\Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U})$. These also preserve basepoints, so this is a simplex in the loop space. We denote the loop space by $\operatorname{maps}(S^{n+1},\Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U}))$, using "maps" to avoid another Ω. We get one of these for each i-simplex M of the bordism Δ-set of cycles. The naturality of the construction implies these fit together to define a Δ-map

$$R_0\colon \mathit{\Omega}_r\big(\mathrm{Cycles}(X,p,\mathcal{U})\big) \to \operatorname{maps}\big(S^{n+1},\Omega^{\mathcal{J}}_{n-r+1}(p,\mathcal{U})\big).$$

Homology is obtained from the geometric realization of the right side of this by dividing by the image of the 0-section, including into

$$\operatorname{maps}(S^{n+j},\Omega_{n-r+j}),$$

and taking the limit as $j \to \infty$. Therefore realizing R_0 and including the right side in this limit defines maps

$$R\colon \Omega_{-r}(\mathrm{Cycles}(X, p, \mathcal{U})) \to \mathbf{H}_{-r}(\partial\Delta^{n+1}; \Omega^{\mathcal{J}}(p, \mathcal{U})).$$

(Again we note the reversal of the index on the left upon realization of the bordism Δ-set.) The right side of this is equivalent to the homology of $\mathrm{nerve}_\Delta(\mathcal{U})$ by lemma 5.3B, so this gives the desired maps from cycles to homology.

5.6 The spectrum structure

For each $r \in Z$ we have a map from the space of r-cycles to the r^{th} homology space. The next step is to show these form a map of spectra, ie. commute up to homotopy with the spectrum structure maps.

It is sufficient to show that the Δ-map R_0 homotopy commutes with appropriate spectrum structure maps. The appropriate diagram is

$$\begin{array}{ccc}
\mathit{\Omega}_r\big(\mathrm{Cycles}(X, p, \mathcal{U})\big) & \xrightarrow{R_0} & \mathrm{maps}\big(S^{n+1}, \Omega^{\mathcal{J}}_{n-r+1}(p, \mathcal{U})\big) \\
\ell\big\downarrow & & \big\downarrow \\
\Omega\Big(\mathit{\Omega}_{r-1}\big(\mathrm{Cycles}(X, p, \mathcal{U})\big)\Big) & \xrightarrow{\Omega R_0} & \Omega\Big(\ \mathrm{maps}\big(S^{n+1}, \Omega^{\mathcal{J}}_{n-r+2}(p, \mathcal{U})\big)\Big).
\end{array}$$

The outer Ωs in the bottom row denote loop spaces. The right vertical map is induced by the structure map in the spectral sheaf, which roughly speaking is the fiberwise union of the structure maps over points in $\|\partial\Delta^{n+1}\|$. We refine the diagram so we can use a Δ-model for this.

The map R_0 is defined by realizing functors, so we factor it through a Δ-set of functors. Suppose F and G are functors from $\partial\Delta^{n+1}$ to Δ-sets. Then $\mathrm{nat}(F, G)$ will be a space of natural transformations between these (actually we define something closer to the natural transformations from F to the loopspace ΩG). An i-simplex of this space associates to each $\sigma \in \partial\Delta^{n+1}$ a Δ-map $F(\sigma) * \Delta^i \to G(\sigma)$, compatible with the maps induced from the face maps in $\partial\Delta^{n+1}$. We also require that $F(\sigma) * \{\phi\}$ and $\{\phi\} * \Delta^i$ are taken to the basepoint of $G(\sigma)$. Here $K * \Delta^i$ is the Δ-set with simplices $\tau * \sigma$ where $\tau \in K$ and $\sigma \in \Delta^i$. $\tau * \sigma$ denotes a simplex with vertices the union of the vertices of τ and σ, ordered so that those of σ are last.

The construction of the map R_0 proceeds by constructing such natural transformations, from $D^n(\sigma)$ to $\mathit{\Omega}^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap\sigma))$, then geometrically realizing. R_0 therefore factors as

$$\begin{aligned}
\mathit{\Omega}_r\big(\mathrm{Cycles}(X, p, \mathcal{U})\big) \xrightarrow{R_1} \mathrm{nat}\big(D^n(*), \mathit{\Omega}^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))\big) \\
\to\ \mathrm{maps}\big(S^n_{n+1}\Omega^{\mathcal{J}}_{n-r+1}(p, \mathcal{U})\big).
\end{aligned}$$

The second map is obtained by realization (twice; the Δ-sets to get space-valued functors, and then the functors). In fact unraveling all the definitions will show that R_1 is an isomorphism.

Realization preserves spectrum structures, so the part we are concerned with is R_1. It is sufficient to show the following diagram commutes:

$$\begin{array}{ccc}
\varOmega_r\big(\mathrm{Cycles}(X,p,\mathcal{U})\big) & \xrightarrow{R_1} & \mathrm{nat}\big(D^n(*), \varOmega^{\mathcal{J}}_{r-n}(p^{-1}(\cap *))\big) \\
\Big\downarrow{\scriptstyle \ell} & & \Big\downarrow \\
\Omega\Big(\varOmega_{r-1}\big(\mathrm{Cycles}(X,p,\mathcal{U})\big)\Big) & \longrightarrow & \mathrm{nat}\big(D^n(*), \Omega\varOmega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))\big).
\end{array}$$

In these terms we can be more explicit about the right vertical map; this is induced by composition with the natural transformation

$$\ell\colon \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap\sigma)) \to \Omega\varOmega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *)).$$

As in the definition of the spectrum structure in 3.4 we will use the Δ-set model for the loop space: the k-simplices of ΩK are the $(k+1)$-simplices of K with ∂_{k+1} and the opposite vertex both equal to basepoints. (And we denote this opposite vertex by $\partial_0^{k+1}\sigma$.) In this model the map ℓ is defined by reindexing a $[k]$-ad ($=$ a k-simplex) to a $[k+1]$-ad using the natural inclusion $[k] \subset [k+1]$.

We now describe the lower horizontal map. This is a composition of two maps, first

$$\Omega R_1\colon \Omega\big(\varOmega_{r-1}\big(\mathrm{Cycles}(X,p,\mathcal{U})\big)\big) \to \Omega\big(\mathrm{nat}(D^n(*), \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))\big)$$

obtained by restricting R_1 to the models of loopspaces as subsets with face restrictions. Then there is the identification

$$\Omega\big(\mathrm{nat}(D^n(*), \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))\big) \simeq \mathrm{nat}\big(D^n(*), \Omega\varOmega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))\big).$$

We discuss this identification.

A k-simplex of $\Omega\big(\mathrm{nat}\big(D^n(*), \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))\big)\big)$ is a $(k+1)$-simplex of $\mathrm{nat}\big(D^n(*), \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))\big)$ with $\partial_{k+1} = \phi = \partial_0^{k+1}$. This is a natural transformation $D^n(\sigma) * \Delta^{k+1} \to \varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))$, so associates to an $(n-j)$-simplex $\sigma \in \partial\Delta^{n+1}$ a $(j+k+2)$-simplex of $\varOmega^{\mathcal{J}}_{r-n-1}(p^{-1}(\cap *))$. By definition of "nat" the restrictions to $\Delta^j * \{\phi\} = \partial^{k+2}_{j+1}$, and to $\{\phi\} * \Delta^{k+1} = \partial_0^{j+1}$ are both ϕ. Further, the restriction imposed to get the Δ loopspace is $\partial_{j+k+2} = \phi = \partial^{k+1}_{j+1}$.

Similarly a k-simplex of $\mathrm{nat}\big(D^n(*), \Omega\Omega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))\big)$ is a natural transformation which takes an $(n-j)$-simplex $\sigma \in \partial\Delta^{n+1}$ to a $(j+k+1)$-simplex of $\Omega\Omega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))$, again with $\partial^{k+1}_{j+1} = \phi = \partial^{j+1}_0$. This is a $j+k+2$-simplex of $\Omega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))$ with face restrictions $\partial_{j+k+2} = \phi = \partial^{j+k+2}_0$. These are the same conditions as in the previous paragraph, so the two sets are equal.

It is now straightforward to verify that the diagram commutes. Let M be a k-simplex of bordisms of cycles, so it assigns to $\sigma \in \partial\Delta^{n+1}$ a $(\mathcal{U}-\sigma\cup[k])$-ad in $\mathcal{J}(p^{-1}(\cap\sigma))$. Both compositions take this to the natural transformation which takes $D^n(\sigma)*\Delta^k$ to the $(j+k+2)$-simplex of $\Omega^{\mathcal{J}}_{r-n-2}(p^{-1}(\cap *))$ defined by $\ell_{j+k+2}M(\sigma)$, where ℓ_{j+k+2} reindexes by the inclusion $(\mathcal{U}-\sigma\cup[k]) \subset (\mathcal{U}-\sigma\cup[k+1])$.

This completes the verification that the maps defined in the previous section give a map of spectra, from the bordism spectrum of cycles, to homology. □

5.7 Reduction to finite collections

Elsewhere in this section we assume the collection $\mathcal{U}$ of subsets of X is finite. In this section we show this is sufficient for most purposes, and indicate the modifications necessary in the others.

The finiteness condition on cycles implies that any finite subcomplex of the bordism space $\Omega(\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{U}))$ is contained in the image of $\Omega(\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{V}))$ for some finite $\mathcal{V} \subset \mathcal{U}$. In particular homology classes of cycles can be defined solely in terms of finite subsets.

Similarly the homology spectrum is obtained from loop spaces of a quotient $\Omega^{\mathcal{J}}(p,\mathcal{U})/\mathrm{nerve}(\mathcal{U})$. A map of a finite complex into this deforms into the inverse image of a finite subcomplex of $\mathrm{nerve}(\mathcal{U})$ under the projection $\Omega^{\mathcal{J}}(p,\mathcal{U}) \to \mathrm{nerve}(\mathcal{U})$. But this corresponds to the homology spectrum of a finite subset of $\mathcal{U}$. The conclusion is that maps of spheres and homotopies between them lie in homology spaces of finite $\mathcal{V} \subset \mathcal{U}$.

It follows from this and naturality that if the passage from cycles to homology is a homotopy isomorphism for finite $\mathcal{U}$ then it is an isomorphism in general. Also, to define this passage on the group level it is sufficient to consider finite collections. The only ingredient for which this is not sufficient is the definition of the map on the spectrum level, when $\mathcal{U}$ is infinite. Therefore we discuss the construction of the map in this case.

Suppose $\mathcal{V}$ is a finite subset of $\mathcal{U}$. Denote by $\Delta^{\mathcal{V}}$ the simplex with vertices $\mathcal{V}$. In these terms the construction of 5.5 defines a map

$$\Omega(\mathrm{Cycles}^{\mathcal{J}}(X,p,\mathcal{V})) \to \mathrm{maps}(\|\partial\Delta^{\mathcal{V}}\|, \Omega^{\mathcal{J}}(p,\mathcal{V})/\|\partial\Delta^{\mathcal{V}}\|).$$

This construction is natural in $\mathcal{V}$, so forms a direct system of maps.

Suppose $\mathcal{V}$ is enlarged by addition of a copy of the empty set. This does not change the cycles and on the mapping space is equivalent to the suspension (since $\partial\Delta^{\mathcal{U}\cup\{\phi\}} \sim S^1 \wedge \partial\Delta^{\mathcal{U}}$). This means the direct limit used to define the homology can also be obtained as the mapping spaces associated to the sequence of inclusions $\cdots \subset \mathcal{V} \cup n\{\phi\} \subset \mathcal{V} \cup (n+1)\{\phi\} \subset \cdots$.

In general expand $\mathcal{U}$ by adding infinitely many copies of the empty set, and take the direct limit over all finite subsets $\mathcal{V}$ of the map above;

$$\lim_{\mathcal{V}\to} \Omega(\mathrm{Cycles}^{\mathcal{J}}(X, p, \mathcal{V})) \longrightarrow \lim_{\mathcal{V}\to}\ \mathrm{maps}(\|\partial\Delta^{\mathcal{V}}\|, \Omega^{\mathcal{J}}(p, \mathcal{V})/\|\partial\Delta^{\mathcal{V}}\|).$$

It follows from the discussion above that the left side of this is equivalent to the cycle space for $\mathcal{U}$, and the right side is equivalent to the homology space. This therefore defines the desired map when $\mathcal{U}$ is infinite.

5.8 Completion of the proof

Again assume $\mathcal{U}$ is a collection of $n+2$ subsets of X, with the first one empty. The objective is to show that the map defined above, from the bordism spectrum of cycles over $\mathcal{U}$ to the homology with coefficients in the spectral sheaf $\Omega^{\mathcal{J}}(p, \mathcal{U})$ is a homotopy equivalence.

In the previous part of the proof the cycle spectrum has been identified with the Δ-set of natural transformations $\mathrm{nat}(D^n(\sigma), \mathit{\Omega}^{\mathcal{J}}_{r-n+1}(p^{-1}(\cap\sigma)))$, so it is sufficient to show the map from this to homology is a homotopy equivalence.

The first step is to compare the Δ-natural transformations with topological ones. Suppose F and G are topological functors on $\partial\Delta^{n+1}$. Define $\mathrm{nat}_*(F, G)$ to be the Δ-set with k-simplices the topological natural transformations $F(\sigma) * \Delta^k \longrightarrow G(\sigma)$ which take $F(\sigma)$ and Δ^k to the basepoint. The subscript $*$ indicates the use of the join to define the simplex structure (there will be a product version below).

5.8A Lemma. *Realization defines a map of Δ-sets*

$$nat(D^n(\sigma), \mathit{\Omega}^{\mathcal{J}}_{r-n+1}(p^{-1}(\cap\sigma))) \longrightarrow nat_*(\|D^n(\sigma)\|, \Omega^{\mathcal{J}}_{-r+n-1}(p^{-1}(\cap\sigma))).$$

As usual note the reversal of the index upon realization; $\Omega^{\mathcal{J}}_{-r+n-1} = \|\mathit{\Omega}^{\mathcal{J}}_{r-n+1}\|$.

Proof. It is sufficient to show the map induces an isomorphism of homotopy groups. These are both Kan Δ-sets so we can work with single simplices.

Specifically suppose M is a k-simplex of the topological version, $\partial_0 M$ is the realization of N_0, a $(k-1)$-simplex of the Δ version, and $\partial_j M = \phi$ for $j > 0$. Then it is sufficient to construct a (k-simplex N of the Δ version with $\partial_0 N = N_0$ and a homotopy rel ∂_0 of $\|N\|$ to M. This N is to be a functor on $\partial\Delta^{n+1}$, and will be constructed by induction downward on the dimension of σ.

Suppose $N(\sigma)$ is defined for $\sigma \in \partial\Delta^{n+1}$ of dimension greater than j, and suppose τ is a j-simplex. Then N is defined on $D^n(\partial\tau) * \Delta^k \cup D^n(\tau) * \partial\Delta^k = \partial\Delta^{n-j+k+1}$. Realize this, then the induction hypothesis provides a homotopy of this realization to the restriction of M to $\|\partial\Delta^{n-j+k+1}\|$. Regard this homotopy as an extension of $\|N\|$ over a collar of $\|\partial\Delta^{n-j+k+1}\|$, then M provides an extension over the rest of $\|\Delta^{n-j+k+1}\|$.

Now apply the simplicial approximation theorem, [22, Theorem 5.3]. This asserts that a map of a realization into the realization of a Kan Δ-set is homotopic to the realization of a Δ-map. Further, it can be held fixed where it is already a realization. Applying this to the map of $\|\Delta^{n-j+k+1}\|$ constructed above gives an $(n-j+k+1)$-simplex which we define to be $N(\tau)$, together with an extension of the previous homotopies to a homotopy of $\|N(\tau)\|$ to $M(\tau)$. This completes the induction step, and therefore the proof of the lemma. $\square$

The next step is a minor modification, replacing the join by the product in the definition of the spaces "nat." If F, G are topological functors as above, define $\mathrm{nat}_\times(F,G)$ to have k-simplices the natural transformations $F(\sigma) \times \Delta^k \to G(\sigma)$.

Recall the definition of the join $F(\sigma) * \Delta^k$ as the product $F(\sigma) \times I \times \Delta^k$ with identification of the subset with I coordinate 0 with $F(\sigma)$, and identification with Δ^k when the I coordinate is 1. A map of the join defines $F(\sigma) \times I \times \Delta^k \to G(\sigma)$. Use adjointness to shift the I coordinate to G, then this gives a map to the loopspace $F(\sigma) \times \Delta^k \to \Omega G(\sigma)$. This construction defines an isomorphism of Δ-sets $\mathrm{nat}_*(F,G) \simeq \mathrm{nat}_\times(F, \Omega G)$.

When we set G to be a bordism spectrum the loopspace is obtained by shifting the index by one. Putting these definitions and remarks together we get

5.8B Lemma. *The natural morphism*

$$\Omega_r(\mathit{Cycles}^{\mathcal{J}}(X,p,\mathcal{U})) \to \mathit{nat}_\times(\|D^n(\sigma)\|, \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma)))$$

is a homotopy equivalence of spectra. $\square$

The task is now to show the $\mathrm{nat}_\times$ spectrum is equivalent to homology. The realization of a natural transformation $D^n(\sigma) \times \Delta^k \to \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))$

defines a k-simplex of the space of pointed maps $\operatorname{maps}(S^n, \Omega^{\mathcal{J}}_{r-n}(p,\mathcal{U}))$. The mapping space then maps into the homology, which is defined to be the limit $\lim_{j\to\infty} \operatorname{maps}(S^{n+j}, \Omega^{\mathcal{J}}_{r-n-j}(p,\mathcal{U})/S^n)$.

To see this is a homotopy equivalence it is sufficient to see it induces an isomorphism on homotopy. Since these are Kan Δ-set it is sufficient to see that a k-simplex of homology deforms rel boundary to a k-simplex of the natural transformation space. In what follows we do this for 0-simplices, ie. set $k = 0$. The reason this is sufficient is that these are Ω-spectra, so any homotopy group appears as a 0^{th} homotopy group by adjusting the index r. Or we could note that the proof for k-simplices is obtained simply by multiplying everything by Δ^k. In any case this will usefully simplify the notation.

The first step is to deform a point in the homology space to the realization of a natural transformation of functors, but not quite the right functors.

Let $f\colon S^j \to \Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U})/S^n$ represent a point in the homology space $\mathbf{H}_r(X;\Omega^{\mathcal{J}}(p,\mathcal{U}))$. Think of dividing by S^n as adding the cone on S^n, then make f transverse to the $\frac{1}{2}$ level in the cone. This gives a codimension 0 submanifold $W \subset S^j$ and a map $f\colon (W,\partial W) \to (\Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U}), S^n)$. The original f is obtained up to homotopy by dividing $\Omega^{\mathcal{J}}$ by S^n and extending the map to all of S^j by taking $S^j - W$ to the basepoint.

We now will use a transversality construction on W, f to produce

(1) a functor $\sigma \mapsto (W(\sigma), \partial_\alpha W(\sigma))$ from $\partial\Delta^{n+1}$ to pairs of spaces, and a homeomorphism of the realization $\|W(*)\| \simeq W$ taking $\|\partial_\alpha W(*)\|$ to ∂W,
(2) a natural transformation $F\colon (W(\sigma), \partial_\alpha W(\sigma)) \to (\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma), pt)$ of functors of σ, and
(3) a homotopy of maps of pairs from the realization of the transformation $\|F\|$ to the map f.

This construction proceeds by downward induction on dimensions of simplices in $\partial\Delta^{n+1}$. To describe this we need some notation for realization of functors defined on part of $\partial\Delta^{n+1}$.

Suppose $W(\sigma)$ is defined for r-simplices. with $r \geq k$. Define the realization, as in 5.2, to be

$$\|W(*)\|_k = \big(\coprod W(\sigma) \times \|\sigma\|\big)/\sim$$

where the union is over simplices of dimension $\geq k$, and $\sim$ is the equivalence relation generated by: $(x, \partial_r^*(t)) \sim (W(\partial_r)(x), t)$. Here ∂^* denotes the inclusion of the realization of a face in the realization of the whole simplex, and $W(\partial_r)$ denotes the map functorially associated by W to the face map ∂_r. Finally, denote by $\partial_\beta\|W(*)\|_k$ the part of this coming from simplices of dimension less than k; the image of $W(\sigma)\times\|\partial_r\sigma\|$, where σ is a k-simplex.

The induction hypothesis for the construction is, for given k,

(1) a functor $(W(\sigma), \partial_\alpha W(\sigma))$ defined for r-simplices, $r \geq k$, and a homeomorphism $\|W(*)\|_k \to W$ onto a codimension 0 submanifold so that $\|\partial_\alpha W(*)\|_k$ is taken to the intersection of the image with ∂W and $\partial_\beta\|W(*)\|_k$ is taken to the interface (intersection of the image with the closure of its complement),
(2) a natural transformation $F\colon (W(\sigma), \partial_\alpha W(\sigma)) \to (\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma), pt)$, and
(3) a homotopy of maps of pairs from f to f_k, whose restriction to $\|W(*)\|_k$ is the realization of $\|F\|$, and which takes the complement to the part of $\Omega^{\mathcal{J}}_{r-j}(p, \mathcal{U})$ lying over the $(k-1)$-skeleton of $\partial\Delta^{n+1}$.

For the induction step we split off a piece of the complement suitable to be the realization over the $(k-1)$-simplices.

Define W_k to be the closure of the complement of the realization $\|W(*)\|_k$. This has boundary $\partial_\beta\|W(*)\|_k \cup \partial_\alpha W_k$, where the second piece is defined to be $W_k \cap \partial W$. According to (3) the map f_k restricts to a map of this into the part of the spectral sheaf lying over the $(k-1)$-skeleton of $\partial\Delta^{n+1}$, namely $\cup_{\{\sigma^r | r<k\}}\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma)) \times \|\sigma\|$.

Consider the barycenters of the simplices of dimension $k-1$; b_σ. Since the restriction to $\partial_\beta\|W(*)\|_k$ comes from the realization of a natural transformation, this restriction is transverse to $\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma)) \times b_\sigma$. Modify f_k by homotopy fixed on $\partial_\beta\|W(*)\|_k$ to make it transverse to these barycenters on all of W_k. Define the preimage of σ^{k-1} to be $W(\sigma)$.

There is a normal bundle for $W(\sigma^{k-1})$ whose fibers project to concentric copies of $\|\sigma\|$ about b_σ. By a small additional homotopy we may also arrange that on this normal bundle f_k composed with the projection

$$\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma)) \times \|\sigma\| \to \Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma))$$

is constant on fibers. Finally since both these conditions are already satisfied on ∂_β (since it is a realization there) we may assume this normal bundle extends the one given in ∂_β by the realization structure. Then use radial expansion in $\|\sigma\|$ to stretch each fiber out to a homeomorphism to $\|\sigma\|$. This gives a homotopy of f_k to f_{k-1} which takes the complement of these normal bundles to the part of the spectral sheaf lying over the $(k-2)$-skeleton, and is a product over each $k-1$ simplex. This map satisfies the induction hypothesis for $k-1$, and so completes the induction step.

This construction produces manifold-valued functors, specifically $W(\sigma)$ is a manifold $(\mathcal{U}-\sigma\cup\{\alpha\})$-ad of dimension $j-k$, when σ is a k-simplex. Strictly speaking this should have been included in the induction hypothesis, since it was used (for transversality) and does not quite follow from the other hypotheses.

Recall that the original goal was to construct a cycle representing a particular homology class. At this point we can describe the cycle corresponding to the class represented by f. We will not actually use this; after a brief description we return to the more technical goal of constructing an appropriate natural transformation.

Corresponding to a simplex $\sigma \in \mathrm{nerve}_\Delta(\mathcal{U})$ we have a map $W(\sigma) \to \Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma))$. Triangulate $W(\sigma)$ and approximate this map by a Δ-map into the Δ-set $\Omega^{\mathcal{J}}_{j-r}(p^{-1}(\cap\sigma))$. This can be interpreted as a Kan-type cycle and totally assembled to give a single $\mathcal{J}$ object. Since $W(\sigma)$ is a manifold $(\mathcal{U}-\sigma\cup\{\alpha\})$-ad, with $\partial_\alpha W(\sigma)=\phi$, the assembly can be arranged to yield a $(\mathcal{U}-\sigma)$-ad in $\mathcal{J}$. By doing this inductively downwards with respect to the dimension of σ these can be arranged to fit together. The result is a $\mathcal{J}$-cycle in $(p,\mathcal{U})$.

The final step in the proof is to modify the construction to yield a natural transformation defined on the dual simplex functor, and therefore a cycle.

Enlarge the collection $\mathcal{U}$ by adding m copies of the empty set put at the end in the ordering; we denote the result by $\mathcal{U}\cup m\{\phi\}$. This does not affect the homotopy type of either the homology spectrum or the cycles. There is a natural inclusion $\Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U}) \subset \Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U}\cup m\{\phi\})$ covering the inclusion $\partial\Delta^{n+1}\subset\partial\Delta^{n+1+m}$. The homology class represented by the map f is also represented by the composition with this inclusion, $(W,\partial W)\to(\Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U}\cup m\{\phi\}),\partial\Delta^{n+1+m})$. Further, the functor and natural transformation $W(*)$ and F constructed above for f also gives the composition. The point is that by this construction we can adjust n to be arbitrarily large. In particular we can assume $n>2j$.

There is a natural transformation of topological functors on $\partial\Delta^{n+1}$, from $W(*)$ to $D^n(*)$. This is constructed by induction downwards on dimensions of simplices, using collars of boundaries in $W(*)$ and the contractibility of $D^n(*)$. Further for each σ this can be arranged to be an embedding with natural trivial normal bundle $W(\sigma)\times D^{n-j}\subset D^n(\sigma)$. Again we proceed downwards on dimension of σ, using the facts that the dimension of $D^n(\sigma)$ is greater than twice that of $W(\sigma)$. The triviality of the normal bundle comes from the fact that $W(\sigma)$ has trivial normal bundle in S^j.

Next we use the Ω-spectrum structure

$$\Omega^{\mathcal{J}}_{r-j}(p^{-1}(\cap\sigma)) \sim \text{ maps}_0(D^{n-j}, \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))).$$

Here we are using maps_0 to indicate maps of the disk which take the boundary to the basepoint; the $(n-j)$-fold loop space. The adjoints of the maps $W(\sigma) \to \text{ maps}_0(D^{n-j}, \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma)))$ define natural maps $W(\sigma) \times D^{n-j} \to \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))$.

These maps take $W(\sigma) \times \partial D^{n-j} \cup \partial_\alpha W(\sigma) \times D^{n-j}$ to the basepoint. But this is the interface between the embedding in $D^n(\sigma)$ and its complement, so these extend by the basepoint on the complement to give maps $\hat{F}\colon D^n(\sigma) \to \Omega^{\mathcal{J}}_{r-n}(p^{-1}(\cap\sigma))$. These form a natural transformation of the type equivalent to a cycle, so if we show these extended maps represent the same homology class as f then the proof is complete.

Let $S^j \times D^{n-j} \subset S^n$ denote the standard embedding. Part of the direct limit used to define homology is the suspension operation: composition with the spectrum structure map

$$S^j \to \Omega^{\mathcal{J}}_{r-j}(p,\mathcal{U})/\partial\Delta^{n+1} \to \text{ maps}_0(D^{n-j}, \Omega^{\mathcal{J}}_{r-n}(p,\mathcal{U}))/\partial\Delta^{n+1}$$

followed by adjunction to $S^j \times D^{n-j} \to \Omega^{\mathcal{J}}_{r-n}(p,\mathcal{U})/\partial\Delta^{n+1}$, extended by the point map to give $\tilde{f}\colon S^n \to \Omega^{\mathcal{J}}_{r-n}(p,\mathcal{U})/\partial\Delta^{n+1}$. Since this is part of the limit, the maps f and $\tilde{f}$ represent the same homology class.

Recall that the realization of the functor $W(*)$ is S^j, and the realization of $D^n(*)$ gives S^n. The embeddings $W(\sigma) \times D^{n-j} \subset D^n(\sigma)$ thus realize to give an embedding $S^j \times D^{n-j} \subset S^n$. Since $n > 2j$ this embedding is isotopic to the standard embedding. The adjunction construction on the functor level realizes to give the adjunction of maps. Therefore the isotopy between the embeddings defines a homotopy from $\|\hat{F}\|$ to $\tilde{f}$.

This represents the homology class of f by the realization of a transformation from the functor $D^n(*)$, and therefore completes the proof of the representation theorem. $\square$

6: Examples

A few of the main examples of bordism-type theories are described here, along with their special features. In section 6.1 the prototype examples of manifolds are described. Then in 6.2 these are extended to functors from spaces to bordism-type theories, by including a map in the data. From these the general machinery developed earlier gives bordism spectra, functor-coefficient homology, and assembly maps.

Applying the representation theorem gives manifold cycles which represent homology classes in these theories. Cycles are identified with maps transverse to dual cones, so the total assembly is seen to be just a matter of forgetting that a map is transverse. The fact that maps of manifolds can be made transverse to bicollared subsets is then used to reverse this: an arbitrary map can be made transverse to dual cones, thus can be realized as a cycle. This shows that assembly maps are isomorphisms for these categories. This is an analog of the classical Pontrjagin-Thom theorem which asserts that manifold bordism is a homology theory with coefficients the appropriate Thom spectra.

The second class of examples are constructed from chain complexes. The Poincaré chain complexes developed by Mishchenko, Ranicki, Weiss, and others fit into this framework, giving assembly maps described by glueing cycles. This is related to the papers of Ranicki [20] and Weiss [26] on algebraic assemblies.

6.1 Manifolds

We begin with the definition of manifold A-ads, adding precision to the sketch in 3.1. Let $S\mathcal{M}$ denote one of the categories of oriented manifolds, TOP, DIFF or PL.

The definition is inductive in the number of elements in A, or equivalently after reindexing, the number of nonempty faces. To begin the induction, suppose A is empty and define an A-ad to be a compact oriented $S\mathcal{M}$ manifold without boundary. Define the involution by letting $-M$ be the same manifold with the opposite orientation. The basepoint is the empty manifold ϕ.

An -ad has an underlying manifold (forgetting the face structure) which for the purposes of the definition we denote by $|M|$. If A is empty define $|M| = M$.

Now suppose -ads with k faces have been defined, and A has $k+1$ elements. Then a manifold A-ad is a compact oriented manifold with boundary $|M|$ together with an $(A-a)$-ad $\partial_a M$ for each $a \in A$ such that

(1) $|\partial_a M| \subset \partial M$ as an oriented codimension 0 submanifold, and $\cup_a |\partial_a M| = \partial M$,
(2) if $a \neq b$ then $|\partial_a \partial_b M| = |\partial_a M| \cap |\partial_b M|$, and further
(3) $\partial_a \partial_b M = -\partial_b \partial_b M$ as $(A-a-b)$-ads.

The involution $-M$ is defined to have underlying manifold $|M|$ with opposite orientation, and face structure $\partial_a(-M) = -\partial_a M$.

By induction this defines A-ads for all finite A. If A is infinite then an A-ad is defined to be a B-ad for some finite subset $B \subset A$, and $\partial_a M = \phi$ if

$a \notin B$.

A manifold -ad has dimension n if its underlying manifold has dimension n. In the notation of section 3 this completes the definition of classes $S\mathcal{M}^n_A$, of A-ads of dimension n. These objects can be reindexed via an injection $\theta\colon A \to B$ simply by defining $\partial_{\theta(a)}\ell_\theta M = \partial_a M$, and $\partial_b \ell_\theta M = \phi$ if b is not in the image of θ.

This definition needs to be refined in the smooth category. Strictly speaking we need manifolds with "corners" so that three or more can fit together around lower-dimensional face to give a smooth structure. The iterated codimension-1 approach used here can be made to work in the smooth category using the "straightening the angle" device to change face angles when necessary. We have chosen this approach because it requires less detail on the structure of cone complexes, and it emphasizes that only the simplest type of transversality—to trivial 1-dimensional bundles —is needed. The more direct approach would be required if we were considering more rigid objects, like manifolds with a Riemannian metric, or a conformal or affine structure.

We briefly describe the more direct approach, which gives a technically better way to approach the topic in any category. The basic idea is to consider -ads as objects modeled on specific examples of -ads, just as manifolds with boundary are modeled on disks.

To get an appropriate model suppose A is a collection of points in $\mathbf{R}^n$ equidistant from each other (so the number of points is no greater than $n+1$). Let R_a denote the points in the space whose distance from a is less than or equal to the distance to the other points. This has faces $R_a \cap R_b$ lying in the $(n-1)$-plane orthogonal to the center of the edges joining a and b. Similarly an iterated intersection $\cap_{a\in S} R_a$ lies in the affine subspace orthogonal to the center of the simplex spanned by S.

A smooth manifold -ad of dimension n should have coordinate charts modeled on open sets in some R_a, so that faces in the -ad correspond to faces of R_a. The model establishes particular angles at which faces meet. This particular model is chosen so that when pieces of a Kan cycle of smooth -ads are glued together the result has an obvious natural smooth structure.

Lemma. *The collections $S\mathcal{M}^n_A$ together with the reindexing operations form a bordism-type theory.*

Proof. The reindexing hypothesis is clear, that reindexing defines a bijection from $S\mathcal{M}^n_A$ to $\{M \in S\mathcal{M}^n_B \mid \partial_b = \phi \text{ if } b \notin \theta(A)\}$.

The other thing to check is the Kan condition. Suppose $N\colon (A-a) \to$

$S\mathcal{M}$ is a Kan cycle, in the sense of 3.2. Let $\cup_b N(b)$ denote the union of these, then this is a manifold (assuming the interiors are disjoint; see the appendix) with boundary $\cup_b \partial_a N(b)$. The fact it is a manifold can be seen by inductively adding one piece at a time to the union, and observing that the union is over codimension 0 submanifolds of the boundary. Or, thinking of the pieces $N(b)$ as locally modeled on convex regions $R_b \subset \mathbf{R}^n$ as above, then the union is a manifold because the union of the model regions is a manifold.

We define an A-ad M of dimension $n+1$ by: the underlying manifold is $\cup_b N(b) \times I$, and the faces are $\partial_b M = N(b) \times \{0\}$ for $b \neq a$. Finally $\partial_a M$ has underlying manifold $(\cup N) \times \{1\} \cup \partial(\cup N) \times I$. The face structure of $\partial_a M$ is specified by $\partial_b \partial_a M = \partial_a N(b) \times \{0\}$.

This A-ad satisfies the conclusion of the Kan condition, so the $S\mathcal{M}$ are bordism-type theories. $\square$

The bordism groups associated to these theories by 3.3 are exactly the classical manifold bordism groups (see [23]). The bordism spectrum of 3.4 is similarly homotopy equivalent to the Thom spectrum, whose homotopy groups are identified with bordism groups by the Pontrjagin-Thom construction.

6.2 Manifolds over spaces

We augment the construction above with a map to a space, to obtain a (bordism-type theory) valued functor. Then we show that assembly maps in the associated homology theory are isomorphisms.

If $S\mathcal{M}$ is a category of oriented manifolds as in the previous section, and X is a topological space, then define $S\mathcal{M}^n_A(X)$ to be the collection of (M, f), where M is an n-dimensional A-ads in $S\mathcal{M}$, and $f\colon M \to X$. Define $\partial_a(M, f)$ to be $(\partial_a M, f|\partial_a M)$, and define the involution and reindexing using these operations on M, without changing f.

6.2A Lemma. *The collections $S\mathcal{M}^n_A(X)$ together with these operations are bordism-type theories, natural in X. The resulting (bordism-type theory)-valued functors of spaces are homotopy invariant, in the sense of 3.5.*

Proof. The only part of the bordism-type theory structure which might need comment is the Kan condition. Since the maps are part of the structure, the maps on pieces of a Kan cycle N fit together to define a map on the union used in the previous proof, $\cup_b N(b) \to X$. A suitable map $M \to X$ for the solution to the problem is obtained by projecting on the first factor $M = \cup_b N(b) \times I \to \cup_b N(b)$, and composing with the map on the union.

To check the homotopy invariance we use the criterion in Lemma 3.5B (3). Also, rather than general homotopy equivalences it is sufficient to check invariance under inclusions $Y \subset X$ which are deformation retracts. (Because both spaces in a homotopy equivalence embed as deformation retracts in a mapping cylinder.)

Suppose, then, that X deformation retracts to Y by a deformation $H\colon X \times I \to X$. Suppose M is a [0]-ad in $S\mathcal{M}(X)$, so a pair (M, f) with M a manifold with boundary, which is the face $\partial_0 M$, and $f/ : M \to X$. Suppose $\partial_0 M$ comes from $S\mathcal{M}(Y)$, which means $f(\partial M) \subset Y$. We deform M into $S\mathcal{M}(Y)$ rel $\partial_0 M$. Define $W = M \times I$ as a [1]-ad with $\partial_1 W = M \times \{0\}$ and $\partial_0 W$ the rest of the boundary. Define a map to X by $H(f \times \mathrm{id})$. Then since H is a deformation retraction the restriction of this to $M \times \{0\}$ is f, and the rest of the boundary maps into Y. Therefore $\partial_0 W$ is an element of $S\mathcal{M}(Y)$, as required. □

According to Corollary 3.5C the bordism spectra of these theories define homotopy invariant spectrum-valued functors of spaces. The notation established in Section 3 for these spectra is $\Omega^{S\mathcal{M}}(X)$. The constructions of section 2 define homology with coefficients in these functors.

6.2 B Proposition. *Suppose $S\mathcal{M}$ is one of the manifold theory functors defined above, and $p\colon E \to X$ is fiber homotopy equivalent to the realization of a simplicial map. Then the assembly $H_n(X; \Omega^{S\mathcal{M}}(p)) \to H_n(pt, \Omega^{S\mathcal{M}}(E)) = \Omega_n^{S\mathcal{M}}(E)$ is an isomorphism.*

This is a version of the classical result that bordism groups form a homology theory. It also identifies the coefficient spectrum of the theory as the bordism spectrum of a point, which is therefore equivalent to the appropriate Thom spectrum.

Proof. The proof uses the transversality to dual cones referred to several times, and here we describe the process in some detail. There are three stages to the discussion: first define the dual cone decomposition and transversality to it, second observe that manifold cycles are exactly manifolds transverse to the dual cones, and finally show that any manifold can be made transverse.

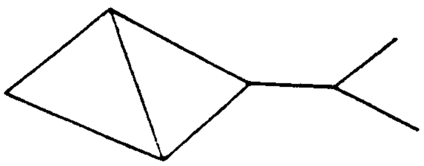
Suppose K is a simplicial complex. Take the first barycentric subdivision of the realization. If σ is a simplex of K define the dual $D(\sigma)$ to be the union of all simplices of the subdivision which intersect σ in exactly the barycenter. The dual of a vertex is the closure of the star used in 1.5.

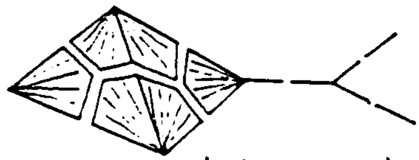
It is not hard to see (eg. in [6]) that

(1) $D(\sigma)$ is the cone on $\cup D(\tau)$, where the union is over τ which contain σ as a face, and

(2) the boundary of $D(\sigma)$ is bicollared in the boundary of $D(\partial_i\sigma)$ (or in $\|K\|$ if σ is a vertex).
(3) the boundary of $D(\sigma)$ is naturally equivalent as a union of cones to the dual cone decomposition of the link of σ.

For example we give a picture of a complex K and its dual cones:





A little more information about the collaring in (2) is necessary. $\partial D(\sigma)$ separates $\partial D(\partial_i\sigma)$ (or $|K|$ if σ is a vertex) into two pieces: the cone and the exterior. There is an obvious collar on the cone side given by the cone parameter. On the outside the collar is also radial. In consequence it respects intersections with other cones: if $\partial D(\sigma) \cap D(\tau) = \partial D(\tau)$ then the intersection of $D(\tau)$ with the collar is a collar on $\partial D(\tau)$. Further, the collar mapping is transverse to the interior of $D(\tau)$.

Now if M is a manifold then we say $f\colon M \longrightarrow |K|$ is *transverse to the cone structure* (or "trans-simplicial" [6]) if for each σ the restriction of f to $f^{-1}(D(\partial_i\sigma)) \to D(\partial_i\sigma)$ (or $\to |K|$ if σ is a vertex) is transverse to the bicollared subset $\partial D(\sigma)$.

This should be understood inductively: if v is a vertex then f is transverse to the bicollared subset $\partial D(v) \subset K$. Therefore $f^{-1}(\partial D(v)) \longrightarrow \partial D(v)$ is a manifold. Next, if τ is an edge with vertex v then $f^{-1}(\partial D(v))$ is transverse to the bicollared subset $\partial D(\tau) \subset \partial D(v)$, and so on. Note this is all codimension 1 transversality (to trivial line bundles) so no sophisticated theory of normal bundles is necessary.

Finally some technical adjustments should be made in the smooth case, along the lines of the comments following the definition of -ads. Namely, rather than iterated codimension 1 situations the local structure around $\partial D(\sigma)$ should be recognized as a product with some $\mathbf{R}^s$, which the other cones intersect in the pattern described in the earlier comment. Smooth transversality to this gives -ads with face structure with the correct angles, etc.

The next step identifies cycles and transverse maps as being essentially the same. More precisely we show a transverse map naturally determines a cycle, and that any cycle is homotopic to one obtained this way.

6.2C Lemma. *Suppose $p\colon |E| \longrightarrow |K|$ is the realization of a simplicial map, and $f\colon M \longrightarrow |E|$ is a map such that pf is transverse to the dual cones in $|K|$. Then the function $\sigma \mapsto pf^{-1}(D(\sigma))$ defines an $\mathcal{SM}$ cycle in E in the inverse of the star cover of $|K|$. Conversely, any such cycle is homotopic to one obtained in this way.*

There is an important space version of this, namely there is a Δ-set, and even a bordism-type theory, of transverse maps defined similarly to the cycle theory in 4.4. In this language the lemma asserts that there is a natural inclusion of theories from the transverse maps into cycles, and the corresponding inclusion of bordism spectra is a deformation retraction.

Proof. Recall that a cycle is a function on the nerve of the cover, and the nerve of the star cover is K. The cover itself is indexed by the vertices of K, which we denote K^0. Similarly denote the vertices of σ by σ^0, then the vertices of K not in σ are $K^0 - \sigma^0$. With this notation the definition 4.2 becomes: a $\Omega^{\mathcal{SM}}$-cycle of dimension n in $(|K|, p, \mathrm{stars}(K))$ is a function $N\colon K \longrightarrow \Omega^{\mathcal{SM}}$ such that

(1) if σ is a k-simplex then $N(\sigma)$ is an $(n-k)$-dimensional $(K^0-\sigma^0)$-ad in $\Omega^{\mathcal{SM}}(p^{-1}(\sigma))$,
(2) let $\mathrm{incl}_*\colon \Omega^{\mathcal{SM}}(p^{-1}(\sigma)) \longrightarrow \Omega^{\mathcal{SM}}(p^{-1}(b_j\sigma))$ denote the morphism induced by the inclusion, then $\mathrm{incl}_*(N(\sigma)) = (-1)^j \partial_{U_j} N(b_j\sigma)$, and
(3) all but finitely many of the $N(\sigma)$ are empty.

The function $\sigma \mapsto pf^{-1}(D(\sigma))$ does satisfy these conditions. If σ is a k-simplex then $pf^{-1}(D(\sigma))$ is the result of k layers of codimension-1 transversality, so has codimension k in M, therefore dimension $n-k$. The faces of $pf^{-1}(D(\sigma))$ correspond to the faces of $D(\sigma)$, therefore to simplices τ which have σ as a face. Such simplices are determined by their vertices not in σ^0, so $pf^{-1}(D(\sigma))$ is naturally a $(K^0-\sigma^0)$-ad. Finally, since M is compact only finitely many of these inverse images can be nonempty.

Now for the converse suppose N is a cycle. $N(\sigma)$ is a $\mathcal{SM}$-ad with a map to $p^{-1}(\mathrm{star}\sigma)$, and its faces $N(\tau)$ map to subsets $p^{-1}(\mathrm{star}\,\tau) \subset p^{-1}(\mathrm{star}\,\sigma)$. But the inclusions $p^{-1}(D(\sigma)) \subset p^{-1}(\mathrm{star}\,\sigma)$ are homotopy equivalences (both spaces deformation retract to the inverse image of the barycenter of σ). Therefore the reference maps are (coherently) homotopic to maps $N(\sigma) \rightarrow p^{-1}(D(\sigma))$.

Next take the union of the pieces $N(\sigma)$ to get a manifold M with a map $f\colon M \longrightarrow |E|$. (See the proof of the Kan condition to see that this is a manifold. (M, f) also represents the total assembly of the cycle N, in $\Omega_n^{S\mathcal{M}}(|E|)$.) This map has the property that the inverse images $f^{-1}(D(\sigma)) = N(\sigma)$ are manifolds, but may need a little modification to actually be transverse.

Since $p\colon |E| \to |K|$ is transverse to the dual cones, the inverse image $p^{-1}(\partial D(\sigma))$ is collared in $p^{-1}(D(\sigma))$. But $N(\partial\sigma) = fp^{-1}(\partial D(\sigma))$ is the boundary of the manifold $fp^{-1}(D(\sigma))$ so is also collared. Thus the map from the second to the first can be change by homotopy rel boundary to preserve collars. f is then "transverse on one side" to $p^{-1}(\partial D(\sigma))$.

To arrange transversality use this construction inductively beginning with the largest simplices (smallest dual cones) over which N is nonempty. Suppose S is a collection of cones, so that for each $D(\sigma) \in S$ the map

$$pf^{-1}(\partial D(\sigma)) \to p^{-1}(\partial D(\sigma))$$

is transverse to the inverse images of the cones in $\partial D(\sigma)$. Change f rel all these boundaries so that it preserves collars of boundaries of $D(\tau)$ for $D(\tau) \in S$. Then f is transverse with respect to the larger collection obtained by adding to S the c ones whose boundaries lie in S.

In the smooth category a little more precision is appropriate. The interior of each cone in $|K|$, thus the inverses in $|E|$, have neighborhoods canonically isomorphic to the cone crossed with one of the smooth models described in 6.1. Neighborhoods of pieces of cycles also have such structures. Rather than working with collars inductively one works directly with the models, arranging the maps to be the identity on the model coordinate near the center stratum. $\square$

Since transverse maps are the same as cycles, we can complete the proof of the proposition by showing that any map $f\colon M \to E$ with M a manifold, is homotopic to one such that pf is transverse to the dual cones in $|K|$.

The basic idea is that since the boundaries of duals of vertices are bicollared in $|K|$, ordinary transversality can be used to make f transverse to them. The inverse image $f^{-1}(\partial D(v)) \to \partial D(v)$ is again a map of a manifold to a complex with dual cones, but the dimension (of both the manifold and the complex) is smaller. Therefore this serves as the induction step in obtaining transversality by induction on dimension.

In more detail first note that since $p\colon |E| \to |K|$ is transverse to the dual cones, the inverse images $p^{-1}(\partial D(\sigma))$ have the same collaring properties as the boundaries themselves. Next suppose f is transverse over an open set $U \subset |K|$, and let v be a vertex. Then there is a homotopy fixed over a closed set slightly smaller than U to a new f which is also transverse to

$p^{-1}(\partial D(v))$. Then $pf^{-1}(\partial D(v)) \to p^{-1}(\partial D(v))$ is a map of a manifold to a complex over the dual cone decomposition of the link of v.

By induction on dimension we can assume this is homotopic, fixed over a closed set slightly smaller than $U \cap \partial D(v)$, to a map transverse to the inverse images of the cones. Use this homotopy to modify f to a map which restricts to the new one on the inverse image. Since the collar on $p^{-1}(\partial D(v))$ is transverse to the inverse images of the other cones, this new f is transverse to all the cones over a neighborhood of $\partial D(v)$. Add this to the set U. By induction the links of all vertices of K can be added to U, at which point f is transverse to all cones.

This completes the proof of Proposition 6.2B. □

6.3 Chain complexes

A chain complex together with a chain equivalence with its dual serves as an algebraic analog of a manifold. This idea and elaborations have been developed by Mishchenko, Ranicki, Weiss, and others as a powerful tool for the investigation of surgery theory.

Assembly maps have been important in the algebraic theory: we mention particularly the total surgery obstruction of Ranicki [19], [20] which lies in a fiber of an assembly map, and the visible theory of Weiss [26], for which an assembly map is an isomorphism and provides a calculation.

In this section we describe the constructions of Ranicki and Weiss, roughly and with little detail, and relate them to the approach taken in this paper. Specifically the first subsection describes the theory, and the way in which cycles appear in it. Section 6.3B describes how chain A-ads are defined, thereby giving bordism-type theories to which this paper applies. Then 6.3C extends additive and algebraic bordism categories to be functors of spaces, thereby defining (bordism-type theory)-valued functors. This leads to functor-coefficient assembly maps, etc. resulting from the general theory. Finally in 3.6D there are some remarks about an analog for "bounded" chain complexes over a metric space.

6.3A Ranicki's construction

These constructions take place in an additive category, rather than the usual setting of modules over a ring. There are substantial benefits to working in this generality, as will be pointed out later.

An algebraic bordism category Λ is defined by Ranicki [20, §3] to be a triple $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$. In this $\mathbb{A}$ is an additive category with chain duality [20, 1.1]: the model is the category of modules over a commutative ring, with the functor which takes a module to its hom dual. (Make this a "chain" duality by thinking of M^* as a very short chain complex.) $\mathbb{C} \subset \mathbb{B}$ are subcategories of the chain complexes in $\mathbb{A}$. The models for these are: $\mathbb{B}$ is finitely generated free chain complexes, with morphisms chain homotopy equivalences, and $\mathbb{C}$ is the full subcategory of contractible complexes.

We describe the way this data is used. Consider complexes from $\mathbb{B}$ together with a duality structure, roughly a chain map $C \to C^*$, whose mapping cone is in $\mathbb{C}$. Depending on the type of duality structure used one gets symmetric or quadratic Poincaré complexes, with bordism groups denoted by $L^n(\Lambda)$ and $L_n(\Lambda)$. Adjusting $\mathbb{B}$ gives variations: finitely generated projective complexes gives the L_n^p groups, finitely generated free with homotopy equivalences the L_n^h, and free based complexes with simple equivalences gives L_n^s. In all these cases $\mathbb{C}$ consists of the contractible complexes. Other variations are obtained by changing this: contractible over some other ring gives the Cappell-Shaneson Γ-groups, and $\mathbb{C} = \mathbb{B}$ gives the "normal" bordism groups.

Ranicki's next step is to construct new bordism categories $\Lambda_*(K)$ and $\Lambda^*(K)$ depending on the original bordism category and a simplicial complex K ([20, §5]). The two versions, distinguished by the position of the $*$, correspond to cycles and cocycles in K. Applying the previous construction gives symmetric or quadratic Poincaré objects in these categories. These, it turns out, represent homology or cohomology classes of K , with coefficients in an appropriate spectrum $\mathbf{L}(\Lambda)$.

When Λ is the bordism category of modules over a ring R, then a glueing construction called "universal assembly" defines a morphism from the bordism category $\Lambda_*(K)$ to the bordism category of modules over the ring $R[\pi_1 K]$. Naturality then gives morphisms of L-groups,

$$L(\Lambda_*(K)) \to L(R[\pi_1 K]).$$

Using the identification of the L-groups of $\Lambda_*(K)$ as homology then gives an assembly map

$$H_n(|K|; \mathbf{L}(R)) \to L_n(R[\pi_1 |K|]).$$

6.3B Poincaré chain -ads

In order to engage the machinery of this paper in the chain complex context we need -ads, and there are two ways to approach this. The low-tech way is to observe that Poincaré pairs are defined, and appropriate glueings are possible. Thus a definition of Λ-Poincaré A-ads can be pieced together inductively as was done with manifolds in 6.1. The high-tech approach is to use Ranicki's machinery, and obtain n-ads of dimension m as Poincaré objects of dimension $m-n$ in the algebraic bordism category associated to the n-simplex, $\Lambda^*(\Delta^n)$ (see [20, 5.4]). Then reindex the faces to get arbitrary A-ads.

These -ads can be reindexed in obvious ways, and they satisfy the Kan condition (Weiss [26,1.10]), so they form bordism-type theories in the sense of §3. Denote by $\mathcal{L}_*(\Lambda)$ and $\mathcal{L}^*(\Lambda)$ respectively the bordism-type theory of quadratic and symmetric -ads in Λ. There are then bordism spectra, homology, cycles, etc. associated to this theory. We extend this to a functor of spaces to get a full version in the next section, but first explain how this is related to the Ranicki constructions.

The theory of Poincaré chain complexes may be thought of as being obtained in three stages: first one has the category of modules over a ring R, with the duality functor which sends a module to its dual. Next one forms the category of chain complexes over R, again with a duality operation. Finally symmetric, quadratic, etc. Poincaré complexes are obtained as chain complexes together with some sort of elaboration of a chain homotopy equivalence with the dual complex. The L-groups appear as bordism groups of these Poincaré complexes.

We could think of the formation of cycles as a fourth stage in this development, using Poincaré chain A-ads. However the cycle construction "commutes" with the other constructions. If $\mathbb{A}$ is an additive category one can basically think of Ranicki's category $\mathbb{A}_*(K)$ as the category of cycles of $\mathbb{A}$-objects. Chain complexes in this cycle category are cycles of $\mathbb{A}$-chain complexes. The duality operation becomes a little more complex, which is why "chain duality" is introduced [20, 1.1]. Finally given a bordism category the formal approach defines Poincaré objects in the chains-of-cycles category, and these are exactly cycles of Poincaré complexes. Therefore by doing the Poincaré chain constructions in general additive categories with chain duality, cycles are obtained as a special case.

From our point of view the key to being able to see assemblies by this approach is the functoriality of glueing. In the bordism-type theories of section 3, pieces are glued together by application of the Kan condition: the result is known to exist but not naturally or canonically. In the algebra it is given by a natural formula (for manifolds too; see the proof of the lemma in 6.1). Thus it works out that (in a sense) glueings of modules lead to glueings of chain complexes, and glueings of the chain complexes underlying Poincaré complexes lead to glueings of the Poincaré complexes. Because of this, assemblies of Poincaré complexes can be obtained by naturality from module-level assemblies.

6.3C Categories over spaces

Our machinery is set up to produce functor-coefficient homology and assemblies from functors of spaces. Accordingly we extend the algebra along the lines of [17] to incorporate a space.

Suppose $\mathbb{A}$ is an additive category, and X a space. Define a new additive category $\mathbb{A}_X$ with objects (M, S, i), where

(1) S is a set, and $i\colon S \to X$ a function which is locally finite,
(2) $M\colon S \to \text{objects}\mathbb{A}$ is a function.

Morphisms in this category are equivalence classes of paths in X together with morphisms in $\mathbb{A}$. Specifically a morphism $(M, S, i) \to (M', S', i')$ is a collection $(\rho_j, 0_j, 1_j, f_j)$, where

(1) $0_j \in S$, $1_j \in S'$, and ρ_j is a path in X from $i(0_j)$ to $i'(1_j)$,
(2) $f_j\colon M(0_j) \to M'(1_j)$ is a morphism in $\mathbb{A}$, and
(3) j runs over some index set, and each elem ent of S (respectively S') occurs only finitely many times as 0_j, (respectively 1_j).

The equivalence relation on morphisms is generated by:

(1) the paths can be changed by homotopy in X holding the ends fixed,
(2) if for indices j, k the endpoints and paths are the same, then the data for these indices can be replaced in the collection by $(\rho_j, 0_j, 1_j, f_j + f_k)$, and
(3) if $f_j = 0$ then the datum $(\rho_j, 0_j, 1_j, f_j)$ can be deleted from the collection.

For example, if R is a ring and $\mathbb{A}$ is the category of finitely generated free R-modules, and X is compact, then $\mathbb{A}_X$ is equivalent to the category of free finitely generated $R[\pi_1 X]$ modules [17].

If X is a space then at least for the standard choices of subcategories $\mathbb{B}$, $\mathbb{C}$ there are standard ways to lift to subcategories $\mathbb{B}_X$, $\mathbb{C}_X$ of chain complexes in the category $\mathbb{A}_X$. (We will not try to mechanize this in general). Therefore given an appropriate algebraic bordism category Λ there is a functor from the category of spaces to the category of algebraic bordism categories; $X \mapsto \Lambda_X$. The definition of Poincaré -ads in an algebraic bordism category functorially associates bordism-type theories $\mathcal{L}_*(\Lambda_X)$ and $\mathcal{L}^*(\Lambda_X)$, as explained above.

This now makes contact with the earlier development. Applying the bordism spectrum functor gives spectrum-valued functors $X \mapsto \Omega(\mathcal{L}_*(\Lambda_X))$ and $X \mapsto \Omega(\mathcal{L}^*(\Lambda_X))$. Denote these functors more compactly by $\mathbf{L}_*^\Lambda(X)$ and $\mathbf{L}^*_\Lambda(X)$. Associated to these functors are functor coefficient homology, assembly maps, etc. For example taking $p\colon E \to X$ to the point map gives the total assembly

$$\mathbf{H}_\bullet(X; \mathbf{L}_*^\Lambda(p)) \to \mathbf{L}_*^\Lambda(E)$$

(and similarly for the symmetric case $\mathbf{L}^*$.)

The main theorems of this paper identify the functor-coefficient homology as represented by cycles, and describe assembly maps in terms of glueing cycles together. If p is the identity map of K then unraveling the definitions shows that cycles over the star cover of $|K|$ are the same as Poincaré objects in Ranicki's category $\Lambda_*(K)$. Further the algebraic assembly described in [20, §9] is the same as the glueing via the Kan condition used here, since the algebraic assembly is the mechanism by which the Kan condition is verified. Putting these together, we see that the algebraic assembly of Ranicki coincides with the constant coefficient spectrum assembly. More generally the straightforward generalization of Ranicki's construction to variable coefficients using the algebraic bordism categories $\Lambda_{p^{-1}(*)}$ coincides with the associated spectrum-functor assembly.

Weiss [26] shows that the assembly for "visible hyperquadratic" Poincaré complexes is an isomorphism. These occur as the relative theory relating quadratic and finite, or "visible" symmetric complexes. The isomorphism theorem provides a calculation, particularly as the coefficient spectrum is a product of Eilenberg-MacLane spectra of 8-torsion groups. It also has important theoretical consequences for example in the structure of Ranicki's total surgery obstruction [20,§17].

6.3D Bounded algebra

Ferry and Pedersen [7] have described a bounded version of surgery, expanding on analogous K-theory work by Pedersen and Weibel [13] , and controlled surgery by Yamasaki [2]. The constructions in this section are so formal that much of it can be applied to the bounded theory.

The constructions of categories over spaces in the previous section can easily be modified to give additive categories of bounded homomorphisms over metric spaces, see [17]. Then using the machinery of Ranicki one can consider chain complexes in these categories. Bordism groups of Poincaré quadratic chain complexes in these categories give the obstruction groups for bounded surgery. These Poincaré complexes also define algebraic bordism categories, so bordism-type theories, cycles, assemblies, etc.

In some significant special cases an assembly map from homology into bounded surgery groups is an isomorphism. This occurs for the $L^{-\infty}$ over control space an open cone. On the chain complex level it is proved using a transversality theorem of Yamasaki [28], beginning with a global object and using transversality to divide it up into a cycle in exactly the same way as was done with manifolds in section 6.2. See also the second appendix in [20].

6.4 An application to group actions

In this section we briefly sketch an application to the construction of PL group actions, slightly reformulating work by Lowell Jones. We begin with an outline of Jones' construction, then use the machinery of this paper to formulate the obstruction.

The problem is: given $K \subset U$ a subcomplex of a PL manifold, and a prime p, when is there a PL Z/p action on U with K as fixed set? This breaks into two pieces: construction of an action on a neighborhood of K, and the extension to the rest of U. Our focus is on the first part, so we will assume U is a regular neighborhood of K. For the next step we note (see 6.4D, below) that a regular neighborhood is the "mapping cylinder" of a PL manifold cycle over K. The problem is therefore to construct a free Z/p action on such a cycle. More precisely, let M denote the cycle corresponding to the boundary of the regular neighborhood. Suppose N is another manifold cycle over K with a homomorphism from π_1 of the assembly (glued up total space) to Z/p, and suppose there is an isomorphism of cycles from M to the associated Z/p cover $\hat{N}$ of N. This gives an isomorphism of mapping cylinders. The mapping cylinder of $\hat{N}$ has an action of Z/p with fixed set exactly K. Since the mapping cylinder of M is U this provides the desired action on U.

This reformulates the problem to: construct a free Z/p action on a PL manifold cycle M over K. The next step in Jones' program is to construct a *homotopy* action. This is a cycle of Poincaré spaces, together with a homomorphism from π_1 of the assembly to Z/p, and a *homotopy equivalence* of cycles from M to the associated Z/p cover. There are obstructions to this. The first come from Smith theory: K must be a mod p homology manifold. The solution of the "homotopy fixed point conjecture" shows that the remaining obstructions to finding a homotopy action are rational. Jones avoids them by assuming that a PL action is already given on the cycle over $\partial_0 K \subset K$, and the rational homology $H_*(K, \partial_0 K; Q)$ is trivial. This is a very important special case, and under these conditions there is a unique homotopy action on M extending the PL action given on $\partial_0 M$. Another obstruction argument gives a *normal* structure on this Poincaré cycle. This is a reduction of the stable normal bundle to the structure group PL, whose Z/p cover agrees with the PL normal bundle of M.

6.4A The standard situation

The data is now close to a surgery situation. We have a Poincaré cycle X

over K, together with a manifold structure on the restriction to $\partial_0 K$ and a PL structure on the stable normal bundle. There is a homomorphism from the total fundamental group of X to Z/p, an equivalence of the Z/p cover with a manifold cycle M, and a PL isomorphism of normal bundles covering this equivalence. The problem is to extend the manifold structure over $\partial_0 K$ to a manifold structure on X with the specified normal bundle and Z/p cover.

The objective is to extract an obstruction from this, whose vanishing implies that there is a solution to the "problem" and therefore a Z/p action on the cycle M.

We review some surgery theory. A "surgery problem" is a Poincaré space Y with a manifold structure on part of it's boundary, say $\partial_0 Y$, and an extension of the PL normal bundle of $\partial_0 Y$ to a PL bundle structure on the stable normal bundle of Y. If we are given a homomorphism $\pi' \to \pi$ then the surgery obstruction group $L(\pi, \pi')$ is the bordism group of such surgery problems together with homomorphisms

$$\begin{array}{ccc} \pi_1(\partial_1 Y) & \longrightarrow & \pi' \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi \end{array}$$

Here $\partial_1 Y$ denotes the (closure of) the complement of $\partial_0 Y$ in ∂Y. The fundamental theorem of surgery states that if the obstruction is trivial in $L(\pi_1 Y, \pi_1 \partial_1 Y)$ (and the dimension is at least 6) then there is a "solution" to the surgery problem: a manifold homotopy equivalent to Y with the given ∂_0 and PL normal bundle.

We modify the definition of "surgery problem" to include the covering information in the standard situation. Suppose (A, B) is a pair with a homomorphism $\rho\colon \pi_1 A \to Z/p$. A "standard problem" over (A, B) is a Poincaré triad $(Y, \partial_0 Y, \partial_1 Y)$ with $\partial_0 Y$ a PL manifold, an extension of the normal bundle of $\partial_0 Y$ to a PL structure on the stable normal bundle of Y, a map $(Y, \partial_1 Y) \to (A, B)$, and a homotopy equivalence of triads $(M, \partial_0 M, \partial_1 M) \to (\hat{Y}, \partial_0 \hat{Y}, \partial_1 \hat{Y})$. Here $\hat{Y}$ is the Z/p cover of Y induced by the homomorphism ρ, M is a PL manifold, the equivalence is a PL isomorphism on ∂_0, and the resulting isomorphism of PL normal bundles over ∂_0 extends to a PL isomorphism over Y refining the natural bundle homotopy equivalence.

This is a lot of data, but it can be managed by recalling where it came from. There is an empty standard problem, and it is straightforward—if tedious—to define ads of standard problems and verify that the Kan condition is satisfied. This therefore forms a bordism-type theory which we denote by $\mathcal{L}$. This theory has bordism groups, spaces, etc. The bordism spaces can be described in terms of traditional surgery problems. Recall that $\Omega^{\mathcal{L}}(A, B, \rho)$ denotes the bordism space of standard problems over (A, B, ρ). Then there is a homotopy fibration

$$\Omega^{\mathcal{L}}(A, B, \rho) \to \mathbb{L}(A, B) \to \mathbb{L}(\hat{A}, \hat{B})$$

where $(\hat{A}, \hat{B})$ is the cover induced by the homomorphism ρ, $\mathbb{L}$ is the surgery space (the bordism space of surgery problems) and the second map is the transfer. (The transfer is defined on the simplex level by taking induced covers). As a consequence of this description the space $\Omega^{\mathcal{L}}$ is often called "the fiber of the transfer."

Now return to the standard situation in 6.4A. Assembling the cycle X over K gives a map $p\colon |X| \to K$. The rest of the data gives a cycle of "standard problems" mapping to $|X|$, subordinate to the cover of K by stars of simplices. This cycle is constructed from the boundary of a regular neighborhood in the original manifold M, so the dimension is $m-1$. Applying the main theorem 4.2A identifies the homology class of this cycle as an element in the functor-coefficient homology group defined in 2.3,

$$H_{m-1}(K, \partial_0 K; \Omega^{\mathcal{L}}(p, \text{ star } (K))).$$

In brief, this is a homology class with coefficients in the the "fiber of the transfer."

Again according to the main theorem this homology class vanishes if and only if the cycle is homologous to the empty cycle. Applying the fundamental theorem of surgery to a nullhomology shows that the original problem can be "solved" and there is a Z/p action.

6.4B Vague Proposition. *In the "standard situation" of 6.4A there is an obstruction in the $(m-1)$-dimensional homology of $(K, \partial K)$ with coefficients in the fiber of the transfer, applied fiberwise to $p\colon |X| \to K$, ie. $\Omega^{\mathcal{L}}(p, \text{ star } (K))$. If the dimensions are sufficiently high then there is a Z/p action on M with K as fixed set and link quotient in the quotient homotopic to X, if and only if this homology class vanishes.*

6.4C More precision

"Dimensions sufficiently high" means that no manifold encountered in the cycle should have dimension less than 5. This happens if the codimension of

K in M at least 6, and can be arranged if $(K, \partial_0 K)$ is 6-connected. We have neglected several issues in the discussion. One is orientation, though that is easily incorporated by making the notation more complicated. A more significant omission is discussion of simple homotopy issues. Since a PL isomorphism is desired at the end, and this has to come from application of the s-cobordism theorem, we want to work with Poincaré spaces, homotopy equivalences, etc. with torsions lying in the kernel of the transfer to the fundamental group of the fragments of the cycle M. This is only really a problem if the embedding is "locally knotted," and this can only happen in codimensions 1 and 2.

In fact it is usual in this problem to assume that the embedding $K \subset M$ has codimension at least 4, so it is locally 1-connected. This simplifies the situation a great deal: the coefficient functor becomes constant, and equal to the fiber of the transfer $\mathbb{L}^h(Z/p) \to \mathbb{L}(1)$. The obstruction therefore lies in an standard constant-coefficient homology group. In Jones' treatment the obstruction is not directly recognized as a homology class. Rather the characteristic variety theorem is used to derive invariants from it, and these derived classes are shown to characterize the obstruction and also define a homology class. Directly recognizing the obstruction as a homology class allows a simpler treatment of parts of the construction.

6.4D Cycles and regular neighborhoods

This section explains the equivalence between PL cycles and PL regular neighborhoods. For more detail on this construction see Akin [1].

Suppose U is a regular neighborhood of a polyhedron K, and suppose it is compact to avoid finiteness and subdivision problems. Then U can be described as the mapping cylinder of a map $\partial U \to K$ which is simplicial with respect to an appropriate triangulation. This map is transverse to the dual cones of the triangulation of K, so it defines a function on the nerve of the covering, as in 4.2. Assume in addition that $U - K$ is a PL n-manifold, then this function defines a PL manifold $(n-1)$-cycle. This gives a construction going from regular neighborhoods with $U - K$ a manifold, to PL manifold cycles over K. More precisely the output is a $(K, \mathrm{id}, \ \mathrm{star}\ (K))$ cycle in the sense of 4.2.

There is a converse to this construction. If we begin with a PL manifold cycle over the star cover of a triangulation of K then there is an associated map from the pieces of the cycle to the dual cones. This is not well-defined, but there is a standard construction which is well-defined up to PL cell-like automorphisms of the cycle. The mapping cylinder of this map gives a regular neighborhood U of K, with $U - K$ a manifold, and this neighborhood is well-defined up to isomorphism rel K by the original cycle.

These constructions are inverses:

6.4E Proposition. *Fix a triangulation of K in which stars of simplices are contractible. These constructions give a bijection between isomorphism classes of PL manifold cycles over the dual cells of the triangulation of K, and isomorphism classes rel K of regular neighborhoods $K \subset U$ with $U - K$ a manifold, and with embedding simplicial with respect to the triangulation.*

A proof can be extracted from Akin [1] in a reasonably straightforward way, though it is not stated explicitly. Here "isomorphism" of cycles means the following: cycles are functions from the nerve of the star cover to PL manifolds. Two such are isomorphic if for each simplex in the nerve there is a PL isomorphism of the corresponding manifolds, and all these isomorphisms commute with the boundary relations in the definition of a cycle. Note that isomorphism is a much stronger relation than homology, and the associated regular neighborhood is definitely not an invariant of the homology class of the cycle.

The notion of "isomorphism" of cycles over K can be elaborated to allow for subdivision of the triangulation. This gives a statement that isomorphism classes of cycles correspond to isomorphism classes of regular neighborhoods, with no reference to a particular triangulation. This refinement is not needed here.

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